Robotic Systems I

Lecture 3: Orientation Representations and Lie Groups Introduction

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How can we represent orientations?

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- Rotation Matrices

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- Rotation Matrices
- Euler Angles
- Axis-Angle
- Quaternions

Axis-Angle:

$$\boldsymbol{\theta} = \boldsymbol{r}\theta$$

where $\boldsymbol{r} \in \mathbb{R}^3$ is the "axis of rotation" unit vector, and $\theta \in \mathbb{R}$ is the rotation angle.

- **Singularity** at $\theta = \pm\pi$
- $\boldsymbol{\theta}_{wb} = (\log \boldsymbol{R}_{wb})^\vee$
- $\boldsymbol{R}_{wb} = \exp(\boldsymbol{\theta}_{wb}^\wedge)$

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- $\boldsymbol{R}_{wb} = \exp(\boldsymbol{\theta}^\wedge_{wb})$
- We can compute the above more efficiently than regular exp/log matrix operations

Euler Angles:

$$\begin{aligned} \boldsymbol{R} &= \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_3 \\ &= \underbrace{\boldsymbol{R}_{\alpha_1}(\theta_1)}_{\boldsymbol{R}_1}\underbrace{\boldsymbol{R}_{\alpha_2}(\theta_2)}_{\boldsymbol{R}_2}\underbrace{\boldsymbol{R}_{\alpha_3}(\theta_3)}_{\boldsymbol{R}_3} \end{aligned}$$

where α_i are the axes of rotation and θ_i the corresponding angles of rotation.

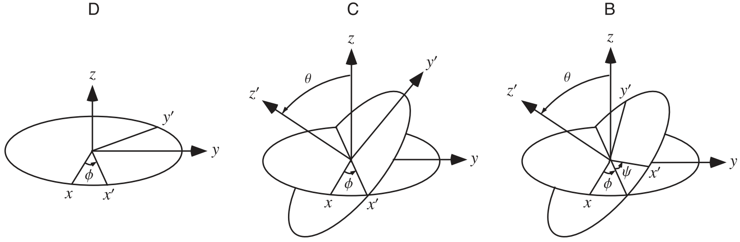

Figure: Taken from WolframAlpha

Euler Angles (2)

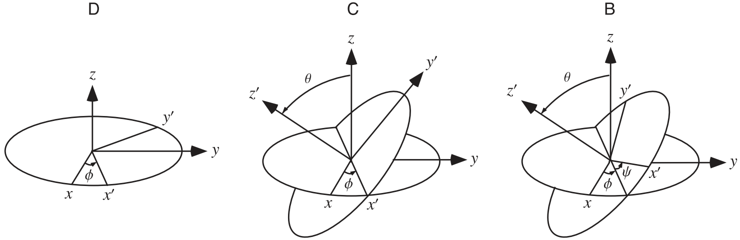

Figure: Taken from WolframAlpha

- There are many conventions: *ZYX* (yaw-pitch-roll) and *XYZ* (roll-pitch-yaw) the most popular ones
  - $\theta_1 \in [-\pi, \pi], \theta_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}], \theta_3 \in [-\pi, \pi]$
  - **Singularity** at $\theta = \pm\frac{\pi}{2}$
- We can convert from/to rotation matrices

Axis-Angle:

$$
\begin{aligned}
\boldsymbol{\theta} &= \boldsymbol{r}\theta \\
\boldsymbol{R} &= \exp(\boldsymbol{\theta}^{\wedge}_{wb}) \\
&= \boldsymbol{I} + \sin\theta\hat{\boldsymbol{r}} + (1-\cos\theta)\hat{\boldsymbol{r}}^2,\ \textit{(Rodrigues formula)} \Rightarrow \\
\dot{\boldsymbol{R}} &= \dot{\theta}\cos\theta\hat{\boldsymbol{r}} + \sin\theta\dot{\hat{\boldsymbol{r}}} + \dot{\theta}\sin\theta\hat{\boldsymbol{r}}^2 + (1-\cos\theta)(\dot{\hat{\boldsymbol{r}}}\hat{\boldsymbol{r}} + \hat{\boldsymbol{r}}\dot{\hat{\boldsymbol{r}}}) \Rightarrow \\
&\dots \\
&\Rightarrow \boldsymbol{\omega}_w = \dot{\theta}\boldsymbol{r} + \sin\theta\dot{\boldsymbol{r}} + \hat{\boldsymbol{r}}\dot{\boldsymbol{r}}(1-\cos\theta) \\
&\Rightarrow \boldsymbol{\omega}_b = \dot{\theta}\boldsymbol{r} + \sin\theta\dot{\boldsymbol{r}} - \hat{\boldsymbol{r}}\dot{\boldsymbol{r}}(1-\cos\theta)
\end{aligned}
$$

ZYX Euler Angles:

$$\boldsymbol{\omega}_b = \begin{bmatrix} -\sin\theta_2 & 0 & 1 \\ \cos\theta_2\sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_3\cos\theta_2 & -sin\theta_3 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$\boldsymbol{\omega}_w = \begin{bmatrix} 0 & -\sin\theta_1 & \cos\theta_2\cos\theta_1 \\ 0 & \cos\theta_1 & \cos\theta_2\sin\theta_1 \\ 1 & 0 & -\sin\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

XYZ Euler Angles:

$$\vdots$$

$$\vdots$$

Let's start with the **Axis-Angle:**

$$\boldsymbol{\theta} = \boldsymbol{r}\theta$$

A **unit quaternion** describing the same orientation/rotation is given by:

$$\boldsymbol{q} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \boldsymbol{r}\sin\frac{\theta}{2} \end{bmatrix} \in \mathbb{R}^4$$

- $\cos\frac{\theta}{2}$ is the **scalar** part, and $\boldsymbol{r}\sin\frac{\theta}{2}$ is the **vector** part
- For $\boldsymbol{q}$ to be a valid orientation/rotation, we also need $\|\boldsymbol{q}\| = 1$
- $\boldsymbol{q}$ and $-\boldsymbol{q}$ describe the **same orientation**, but *different rotations!* (double cover)

Quaternion Math

- $\boldsymbol{q}_{wb}$ is a rotation from "body frame" to "world frame"
- $\boldsymbol{q}_{wa} = \boldsymbol{q}_{wb} \odot \boldsymbol{q}_{ba}$

$$\boldsymbol{q}_{wa} = \begin{bmatrix} s_{wb} \\ \boldsymbol{v}_{wb} \end{bmatrix} \odot \begin{bmatrix} s_{ba} \\ \boldsymbol{v}_{ba} \end{bmatrix} = \begin{bmatrix} s_{wb}s_{ba} - \boldsymbol{v}_{wb}^T\boldsymbol{v}_{ba} \\ s_{wb}\boldsymbol{v}_{ba} + s_{ba}\boldsymbol{v}_{wb} + \boldsymbol{v}_{wb} \times \boldsymbol{v}_{ba} \end{bmatrix}$$

- **Quaternion Conjugate** (Inverse):

$$\boldsymbol{q}_{wb}^{\dagger} = \begin{bmatrix} s_{wb} \\ -\boldsymbol{v}_{wb} \end{bmatrix}$$

- $\boldsymbol{q}_{wb}^{\dagger} \odot \boldsymbol{q}_{wb} = \boldsymbol{q}_0,\ \boldsymbol{q}_0 = \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix}$
- $\begin{bmatrix} 0 \\ \boldsymbol{x}_w \end{bmatrix} = \boldsymbol{q}_{wb} \odot \begin{bmatrix} 0 \\ \boldsymbol{x}_b \end{bmatrix} \odot \boldsymbol{q}_{wb}^{\dagger}$, *(rotating a vector/point)*

Time derivative of Quaternion:

$$\begin{aligned}\dot{\boldsymbol{q}} &= \frac{1}{2}\boldsymbol{q} \odot \begin{bmatrix} 0 \\ \boldsymbol{\omega}_b \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 0 \\ \boldsymbol{\omega}_w \end{bmatrix} \odot \boldsymbol{q}\end{aligned}$$

- We can now integrate quaternions

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- We can now integrate quaternions
- **Beware!** Integrating naively with $\dot{\boldsymbol{q}}$ (e.g. with Euler) results in a non unit-norm quaternion!

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- We can now integrate quaternions
- **Beware!** Integrating naively with $\dot{\boldsymbol{q}}$ (e.g. with Euler) results in a non unit-norm quaternion!
- **Lots of “custom” functions to use the quaternions!**

Quaternion Magic

$$\boldsymbol{q}_{wa} = \begin{bmatrix} s_{wb} \\ \boldsymbol{v}_{wb} \end{bmatrix} \odot \begin{bmatrix} s_{ba} \\ \boldsymbol{v}_{ba} \end{bmatrix} = \begin{bmatrix} s_{wb}s_{ba} - \boldsymbol{v}_{wb}^T\boldsymbol{v}_{ba} \\ s_{wb}\boldsymbol{v}_{ba} + s_{ba}\boldsymbol{v}_{wb} + \boldsymbol{v}_{wb} \times \boldsymbol{v}_{ba} \end{bmatrix}$$

How can we write the above such that we do it via matrix multiplication?

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$$\boldsymbol{q}_{wa} = \begin{bmatrix} s_{wb} \\ \boldsymbol{v}_{wb} \end{bmatrix} \odot \begin{bmatrix} s_{ba} \\ \boldsymbol{v}_{ba} \end{bmatrix} = \underbrace{\begin{bmatrix} s_{wb} & -\boldsymbol{v}_{wb}^T \\ \boldsymbol{v}_{wb} & s_{wb}\boldsymbol{I} + \hat{\boldsymbol{v}}_{wb} \end{bmatrix}}_{L(\boldsymbol{q}_{wb})} \begin{bmatrix} s_{ba} \\ \boldsymbol{v}_{ba} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} s_{ba} & -\boldsymbol{v}_{ba}^T \\ \boldsymbol{v}_{ba} & s_{ba}\boldsymbol{I} - \hat{\boldsymbol{v}}_{ba} \end{bmatrix}}_{R(\boldsymbol{q}_{ba})} \begin{bmatrix} s_{wb} \\ \boldsymbol{v}_{wb} \end{bmatrix}$$

Quaternion Magic (2)

A few interesting relations:

$$\begin{aligned}\boldsymbol{q}_{wb} \odot \boldsymbol{q}_{ba} &= L(\boldsymbol{q}_{wb})\boldsymbol{q}_{ba} \\ &= R(\boldsymbol{q}_{ba})\boldsymbol{q}_{wb}\end{aligned}$$

$$L(\boldsymbol{q}^\dagger) = L(\boldsymbol{q})^T = L(\boldsymbol{q})^{-1}$$
$$R(\boldsymbol{q}^\dagger) = R(\boldsymbol{q})^T = R(\boldsymbol{q})^{-1}$$

Quaternion Magic (3)

We can use the same logic in all quaternion operations:

- $\boldsymbol{q}^{\dagger} = \begin{bmatrix} s \\ -\boldsymbol{v} \end{bmatrix} = \boldsymbol{T}\boldsymbol{q} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -\boldsymbol{I} \end{bmatrix}}_{\boldsymbol{T}} \begin{bmatrix} s \\ \boldsymbol{v} \end{bmatrix}$
- Rotate a vector/point:

$$\begin{bmatrix} 0 \\ \boldsymbol{x}_w \end{bmatrix} = \boldsymbol{q}_{wb} \odot \begin{bmatrix} 0 \\ \boldsymbol{x}_b \end{bmatrix} \odot \boldsymbol{q}_{wb}^{\dagger}$$

$$\underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{H} \in \mathbb{R}^{4\times 3}} \boldsymbol{x}_w = \boldsymbol{q}_{wb} \odot \boldsymbol{H}\boldsymbol{x}_b \odot \boldsymbol{q}_{wb}^{\dagger}$$

$$\begin{aligned} \boldsymbol{H}\boldsymbol{x}_w &= L(\boldsymbol{q}_{wb})(R(\boldsymbol{q}_{wb})^T \boldsymbol{H}\boldsymbol{x}_b) \\ \boldsymbol{x}_w &= \underbrace{\boldsymbol{H}^T L(\boldsymbol{q}_{wb}) R(\boldsymbol{q}_{wb})^T \boldsymbol{H}}_{\boldsymbol{R}_{wb}} \boldsymbol{x}_b \end{aligned}$$

Quaternion Magic (4)

We can use the same logic in all quaternion operations:

$$\begin{aligned}
&\vdots \\
\dot{\boldsymbol{q}} &= \frac{1}{2}\boldsymbol{q} \odot \begin{bmatrix} 0 \\ \boldsymbol{\omega}_b \end{bmatrix} \\
&= \frac{1}{2} L(\boldsymbol{q}) \boldsymbol{H} \boldsymbol{\omega}_b
\end{aligned}$$

Rodrigues Parameters:[1]

$$\boldsymbol{\phi} = \boldsymbol{r}\tan\frac{\theta}{2} \in \mathbb{R}^3$$

where $\boldsymbol{r} \in \mathbb{R}^3$ is the "axis of rotation" unit vector, and $\theta \in \mathbb{R}$ is the rotation angle. $\boldsymbol{\phi}$ is referred to as "Rodrigues" or "Gibbs" vector.

Quaternions and Rodrigues Parameters:

$$\boldsymbol{q} = \varphi(\boldsymbol{\phi}) = \frac{1}{\sqrt{1+\boldsymbol{\phi}^T\boldsymbol{\phi}}}\left(\begin{bmatrix}1\\ \boldsymbol{\phi}\end{bmatrix}\right)$$
$$\boldsymbol{\phi} = \varphi^{-1}(\boldsymbol{q}) = \frac{\boldsymbol{v}}{s}$$

[1] https://en.wikipedia.org/wiki/Rotation_formalisms_in_three_dimensions

Orientation Representations - Code Example

jupyter 2-orientations Last Checkpoint: 4 minutes ago

File Edit View Run Kernel Settings Help

Code

```
theta = 1.
axis = np.array([[0., 0., 1.]]).T

# quaternion
q = np.block([[np.cos(theta/2.)], [axis * np.sin(theta/2)]])

# rotation matrix
R = quat_to_rotation_matrix(q)

omega_b = np.array([[1., 0., 0.]]).T
```

Orientation from a Geometric Perspective

- The **Special Orthogonal Group**, representing orientations and rotations, is simply the set of valid rotation matrices:

$$SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I},\ det(\boldsymbol{R}) = 1\}$$

- The set of all 3×3 matrices is a vectorspace. What does it mean?

Orientation from a Geometric Perspective

- The **Special Orthogonal Group**, representing orientations and rotations, is simply the set of valid rotation matrices:

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- The set of all 3×3 matrices is a vectorspace. What does it mean?

- $SO(3)$ is not:
  1. The zero matrix $\mathbf{0}$ is not part of the space!
  2. $\boldsymbol{R}_1, \boldsymbol{R}_2 \in SO(3) \not\Rightarrow \boldsymbol{R}_1 + \boldsymbol{R}_2 \in SO(3)$

Groups

What is a Group? A set non-empty $\mathcal{G}$ along with a binary operator $\circ$ that satisfies four properties:

1. **Closure:** $\forall \mathcal{X}, \mathcal{Y} \in \mathcal{G}, \mathcal{X} \circ \mathcal{Y} \in \mathcal{G}$
2. **Associativity:** $\forall \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{G}, (\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z})$
3. **Identity:** $\exists \varepsilon \in \mathcal{G}, \forall \mathcal{X} \in \mathcal{G}, \varepsilon \circ \mathcal{X} = \mathcal{X} \circ \varepsilon = \mathcal{X}$
4. **Inverse:** $\forall \mathcal{X} \in \mathcal{G}, \exists \mathcal{X}^{-1} \in \mathcal{G}, \mathcal{X} \circ \mathcal{X}^{-1} = \mathcal{X}^{-1} \circ \mathcal{X} = \varepsilon$

The operator $\circ$ is not necessarily commutative!

Manifolds

What is a Manifold? A Manifold, $\mathcal{M}$, is a topological space that locally resembles Euclidean space, $\mathbb{R}^n$, near each point $\mathcal{X}$:

- The neighborhood of $\mathcal{X}$ lies on the tangent space $T_{\mathcal{X}}\mathcal{M}$ of the manifold $\mathcal{M}$ at point $\mathcal{X}$
- The neighborhood of $\mathcal{X}$ and the corresponding tangent space are homeomorphic to an open subset of $\mathbb{R}^n$: $T_{\mathcal{X}}\mathcal{M} \cong \mathbb{R}^n$
- A smooth manifold is infinitely differentiable at each point

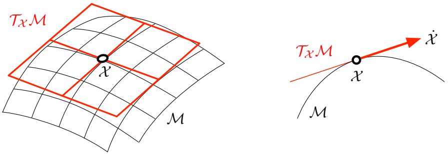


Figure: *Sola, Joan, Jeremie Deray, and Dinesh Atchuthan. "A micro lie theory for state estimation in robotics." arXiv:1812.01537 (2018).*

Lie Groups

What is a Lie Group? A Lie Group is a group that is also a smooth manifold:

- The group operations are smooth: $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$, $(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \circ \mathcal{Y}$
- The inverse operation is smooth: $\mathcal{G} \rightarrow \mathcal{G}$, $\mathcal{X} \mapsto \mathcal{X}^{-1}$
- The identity element is smooth: $\mathcal{G} \rightarrow \mathcal{G}$, $\mathcal{X} \mapsto \varepsilon$

Why are Lie Groups important? They are the natural setting for the study of continuous symmetries of differential equations.

$SO(3)$ is a Matrix Lie Group

$SO(3)$ forms a *matrix Lie group* ($\circ$ is the regular matrix multiplication, ε is the identity matrix):

- $\boldsymbol{R}_1, \boldsymbol{R}_2 \in SO(3) \Rightarrow \boldsymbol{R}_1\boldsymbol{R}_2 \in SO(3)$
- $(\boldsymbol{R}_1\boldsymbol{R}_2)\boldsymbol{R}_3 = \boldsymbol{R}_1(\boldsymbol{R}_2\boldsymbol{R}_3) = \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_3$
- $\boldsymbol{R}, \boldsymbol{I} \in SO(3)$ and $\boldsymbol{R}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{R} = \boldsymbol{R}$
- $\boldsymbol{R}^{-1} = \boldsymbol{R}^T \in SO(3)$

Lie Group Actions

Given a lie group $\mathcal{M}$ and a set $\mathcal{V}$, a *Lie group action*, $\cdot$, is a map $\cdot : \mathcal{M} \times \mathcal{V} \to \mathcal{V}$ that satisfies:

1. **Identity:** $\varepsilon \cdot \boldsymbol{v} = \boldsymbol{v}$
2. **Compatibility:** $\mathcal{X} \cdot (\mathcal{Y} \cdot \boldsymbol{v}) = (\mathcal{X} \circ \mathcal{Y}) \cdot \boldsymbol{v}$

where $\mathcal{X}, \mathcal{Y} \in \mathcal{M}$ and $\boldsymbol{v} \in \mathcal{V}$

Example: The rotation of a vector $\boldsymbol{v}$ by a rotation matrix $\boldsymbol{R}$:

$$\boldsymbol{R} \cdot \boldsymbol{v} = \boldsymbol{R}\boldsymbol{v}$$

Tangent Spaces of a Lie Group

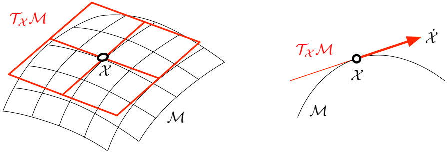


Figure: *Sola, Joan, Jeremie Deray, and Dinesh Atchuthan. "A micro lie theory for state estimation in robotics." arXiv:1812.01537 (2018).*

Given a lie group $\mathcal{M}$ and a point $\mathcal{X} \in \mathcal{M}$, the *tangent space* at $\mathcal{X}$ is written as $T_\mathcal{X}\mathcal{M}$.

- Let's assume a trajectory $\mathcal{Y}(t)$ in $\mathcal{M}$ that passes through the point $\mathcal{X}$ at some t_0 (from now on, $t_0 = 0$ for simplicity)
- $\dot{\mathcal{Y}}(t_0) = \dot{\mathcal{X}}(t_0)$ must be in the tangent space $T_\mathcal{X}\mathcal{M}$
- The tangent space is constructed by the velocities $\dot{\mathcal{Y}}(t_0)$ of all possible trajectories $\mathcal{Y}(t)$ that pass through $\mathcal{X}$ at $t = t_0$
- The tangent space is a vectorspace, and the elements of $T_\mathcal{X}\mathcal{M}$ are called *tangent vectors*

Lie Algebra

The tangent space at the identity element ε is called the *Lie algebra* of the Lie group. A *Lie algebra* consists of a vectorspace, $\mathbb{V}$, over some field, $\mathbb{F}$, together with a binary operation, $[\cdot, \cdot]$, called the *Lie bracket* (of the algebra) and that satisfies four properties:

- closure: $[\boldsymbol{X}, \boldsymbol{Y}] \in \mathbb{V}$
- bilinearity:

$$
\begin{aligned}
[a\boldsymbol{X} + b\boldsymbol{Y}, \boldsymbol{Z}] &= a[\boldsymbol{X}, \boldsymbol{Y}] + b[\boldsymbol{Y}, \boldsymbol{Z}] \\
[\boldsymbol{Z}, a\boldsymbol{X} + b\boldsymbol{Y}] &= a[\boldsymbol{Z}, \boldsymbol{X}] + b[\boldsymbol{Z}, \boldsymbol{Y}]
\end{aligned}
$$

- alternating: $[\boldsymbol{X}, \boldsymbol{X}] = \mathbf{0}$
- Jacobi identity: $[\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]] + [\boldsymbol{Y}, [\boldsymbol{Z}, \boldsymbol{X}]] + [\boldsymbol{Z}, [\boldsymbol{X}, \boldsymbol{Y}]] = \mathbf{0}$

for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{V}$ and $a, b \in \mathbb{F}$.

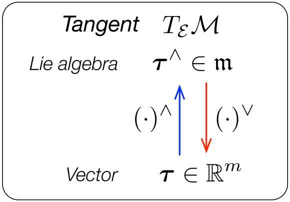
Lie Algebra Representations

There exist two isomorphic representations of the tangent space:

- The Lie algebra $\mathfrak{m} = T_{\mathcal{X}}\mathcal{M}$ ($\mathfrak{m}$ has m dimensions)
- The corresponding Cartesian space $\mathbb{R}^m$: $\mathfrak{m} \cong \mathbb{R}^m$
- We can go from one representation to the other using two mutually inverse linear maps called "hat" and "vee":

$$\boldsymbol{\tau}^\wedge \in \mathfrak{m} \quad \text{“hat”}$$

$$\boldsymbol{\tau} = (\boldsymbol{\tau}^\wedge)^\vee \in \mathbb{R}^m \quad \text{“vee”}$$



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$SO(3)$'s Lie Algebra:

$$\begin{aligned}\textbf{vectorspace:}\quad & \mathfrak{so}(3) = \{\boldsymbol{\Phi} = \phi^\wedge | \phi \in \mathbb{R}^3\}\\ \textbf{field:}\quad & \mathbb{R}\\ \textbf{Lie bracket:}\quad & [\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2] = \boldsymbol{\Phi}_1\boldsymbol{\Phi}_2 - \boldsymbol{\Phi}_2\boldsymbol{\Phi}_1\end{aligned}$$

where $\phi^\wedge$ is the skew symmetric matrix operator:

$$\phi^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

We also use $\phi^\vee$ as the inverse operator:

$$\boldsymbol{\Phi} = \phi^\wedge \Rightarrow \phi = \boldsymbol{\Phi}^\vee$$

In Lie Groups, we can get the structure of the tangent space (or the Lie algebra) by differentiating the inverse group constraint. For groups with multiplicative binary operator, we get:

$$\boldsymbol{\tau}^{\wedge} = \mathcal{X}^{-1}\dot{\mathcal{X}} = -\dot{\mathcal{X}^{-1}}\mathcal{X}$$

For $SO(3)$ specifically, we get:

$$\phi^{\wedge} = \boldsymbol{R}^T\dot{\boldsymbol{R}} = -\dot{\boldsymbol{R}^T}\boldsymbol{R}$$

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What does this reminds us of?

$$\boldsymbol{\omega}_w^{\wedge} = \dot{\boldsymbol{R}}_{wb}\boldsymbol{R}_{wb}^T$$
$$\boldsymbol{\omega}_b^{\wedge} = \boldsymbol{R}_{wb}^T\dot{\boldsymbol{R}}_{wb}$$

!!!!

Relation between Lie Algebra and the Manifold

For groups with multiplicative group binary operator:

$$\boldsymbol{v}^{\wedge} = \mathcal{X}^{-1}\dot{\mathcal{X}}$$
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This is a linear differential equation. We can solve it:

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$$\mathcal{X}(t) = \mathcal{X}(0)\exp(\boldsymbol{v}^{\wedge}t)$$

But $\mathcal{X}(0) = \varepsilon = \boldsymbol{I}$! Why?

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But $\mathcal{X}(0) = \varepsilon = \boldsymbol{I}$! Why? Setting $\boldsymbol{\tau}^{\wedge} = \boldsymbol{v}^{\wedge}t$, we get:

$$\mathcal{X} = \exp(\boldsymbol{\tau}^{\wedge})$$

Now, we can go back to the manifold using the exponential map! We can do the inverse operation using the logarithm map:

$$\boldsymbol{\tau}^{\wedge} = \log(\mathcal{X})$$

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- And back?

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- What does this reminds us of?
- **Angle-Axis!**
- We can compute the above more efficiently than regular exp/log matrix operations

Exp/Log Map for Rotations

We can compute the exp/log operations efficiently:

$$
\begin{aligned}
\exp(\boldsymbol{\phi}^\wedge) = \exp(\phi \boldsymbol{a}^\wedge) &= \sum_{n=0}^{\infty} \frac{1}{n!} (\boldsymbol{\phi}^\wedge)^n \\
&= \mathbf{1} + \phi \boldsymbol{a}^\wedge + \frac{1}{2!} \phi^2 \boldsymbol{a}^\wedge \boldsymbol{a}^\wedge + \cdots \\
&= \cdots \\
&= \boldsymbol{I} \cos\phi + (1 - \cos\phi) \boldsymbol{a}\boldsymbol{a}^T + \sin\phi \boldsymbol{a}^\wedge \\
&= \boldsymbol{I} + \sin\phi \boldsymbol{a}^\wedge + (1 - \cos\phi) \boldsymbol{a}^\wedge \boldsymbol{a}^\wedge \text{ (Rodrigues formula)} \\
&= \boldsymbol{R}
\end{aligned}
$$

Similarly:[1]

$$
\begin{aligned}
\log(\boldsymbol{R})^\vee &= \boldsymbol{\phi} = \phi \boldsymbol{a} \\
\boldsymbol{a}^\wedge &= \frac{1}{2\sin\phi} (\boldsymbol{R} - \boldsymbol{R}^T), \phi = \cos^{-1}(\frac{\mathrm{tr}(\boldsymbol{R}) - 1}{2})
\end{aligned}
$$

[1] We also need to take care of some special cases!

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Thank you

- **Any Questions?**
- **Office Hours**:
  - **Tue-Wed** *(10:00-12:00)*
  - 24/7 by email (`costashatz@upatras.gr`, subject: *ECE_RSI_AM*)
- **Material and Announcements**

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