



ΠΑΝΕΠΙΣΤΗΜΙΟ
ΠΑΤΡΩΝ
UNIVERSITY OF PATRAS

Robotic Systems I

Lecture 6: Non-linear Filters and SLAM

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Template made by Panagiotis Papagiannopoulos



Bayes Filtering

$$\begin{aligned}\text{bel}(\mathbf{x}_t) &= p(\mathbf{x}_t | \mathbf{z}_{1:t}, \mathbf{u}_{1:t}) \\&= \underbrace{\eta p(\mathbf{z}_t | \mathbf{x}_t, \mathbf{z}_{1:t-1}, \mathbf{u}_{1:t}) p(\mathbf{x}_t | \mathbf{z}_{1:t-1}, \mathbf{u}_{1:t})}_{\text{Bayes Rule}} \\&= \underbrace{\eta p(\mathbf{z}_t | \mathbf{x}_t)}_{\text{Markov Assumption}} p(\mathbf{x}_t | \mathbf{z}_{1:t-1}, \mathbf{u}_{1:t}) \\&= \eta p(\mathbf{z}_t | \mathbf{x}_t) \underbrace{\int_{\mathbf{x}_{t-1}} p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{z}_{1:t-1}, \mathbf{u}_{1:t}) p(\mathbf{x}_{t-1} | \mathbf{z}_{1:t-1}, \mathbf{u}_{1:t}) d\mathbf{x}_{t-1}}_{\text{Law of Total Probability}} \\&= \eta p(\mathbf{z}_t | \mathbf{x}_t) \underbrace{\int_{\mathbf{x}_{t-1}} p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_t) p(\mathbf{x}_{t-1} | \mathbf{z}_{1:t-1}, \mathbf{u}_{1:t-1}) d\mathbf{x}_{t-1}}_{\text{Markov Assumption}} \\&= \eta p(\mathbf{z}_t | \mathbf{x}_t) \int_{\mathbf{x}_{t-1}} p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_t) \text{bel}(\mathbf{x}_{t-1}) d\mathbf{x}_{t-1}\end{aligned}$$

Bayes Filtering in Two Steps

Prediction Step

$$\overline{\text{bel}}(\mathbf{x}_t) = \int_{\mathbf{x}_{t-1}} p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_t) \text{bel}(\mathbf{x}_{t-1}) d\mathbf{x}_{t-1}$$

Correction Step

$$\text{bel}(\mathbf{x}_t) = \eta p(\mathbf{z}_t | \mathbf{x}_t) \overline{\text{bel}}(\mathbf{x}_t)$$

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$$\text{bel}(\mathbf{x}_t) = \eta p(\mathbf{z}_t | \mathbf{x}_t) \overline{\text{bel}}(\mathbf{x}_t)$$

- $p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_t)$ is Gaussian
- $p(\mathbf{z}_t | \mathbf{x}_t)$ is Gaussian
- $\text{bel}(\mathbf{x}_0)$ is Gaussian
- **$\text{bel}(\mathbf{x}_t)$ stays Gaussian!**

Kalman Filter - Linear Models

$$\begin{aligned} \mathbf{x}_t &= \mathbf{A}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \boldsymbol{\epsilon}_t^m \\ \mathbf{z}_t &= \mathbf{C}_t \mathbf{x}_t + \boldsymbol{\epsilon}_t^o \end{aligned}$$

where $\boldsymbol{\epsilon}_t^m \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t)$, $\boldsymbol{\epsilon}_t^o \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t)$.

Linear models with zero mean Gaussian noise!

Kalman Filter - Algorithm

- 1 Input: $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$
- 2 $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
- 3 $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$
- 4 $K_t = \bar{\Sigma}_t C_t^T \left(C_t \bar{\Sigma}_t C_t^T + Q_t \right)^{-1}$
- 5 $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
- 6 $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$
- 7 Output: μ_t, Σ_t

Steps 2 and 3 are the *Prediction Step*

Steps 4, 5, 6 and 7 are the *Correction Step*

What happens if the motion or observation model is not linear?

$$\begin{aligned}\mathbf{x}_t &= f(\mathbf{x}_{t-1}, \mathbf{u}_t) + \boldsymbol{\epsilon}_t^m \\ \mathbf{z}_t &= h(\mathbf{x}_t) + \boldsymbol{\epsilon}_t^o\end{aligned}$$

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What can we do?

Extended Kalman Filter

When in doubt, linearize!

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$$f(\mathbf{x}_{t-1}, \mathbf{u}_t) \approx f(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t) + \underbrace{\frac{\partial f}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\boldsymbol{\mu}_{t-1}} (\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1})}_{\mathbf{F}_t}$$

$$h(\mathbf{x}_t) \approx h(\bar{\boldsymbol{\mu}}_t) + \underbrace{\frac{\partial h}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\bar{\boldsymbol{\mu}}_t} (\mathbf{x}_t - \bar{\boldsymbol{\mu}}_t)}_{\mathbf{H}_t}$$

When in doubt, linearize!

$$f(\mathbf{x}_{t-1}, \mathbf{u}_t) \approx f(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t) + \frac{\partial f}{\partial \mathbf{x}} \Big|_{\mathbf{x} = \boldsymbol{\mu}_{t-1}} (\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1})$$
$$\underbrace{\quad\quad\quad}_{\mathbf{F}_t} \quad \mathbf{u} = \mathbf{u}_t$$

$$h(\mathbf{x}_t) \approx h(\overline{\boldsymbol{\mu}}_t) + \frac{\partial h}{\partial \mathbf{x}} \Big|_{\mathbf{x} = \overline{\boldsymbol{\mu}}_t} (\mathbf{x}_t - \overline{\boldsymbol{\mu}}_t)$$
$$\underbrace{\quad\quad\quad}_{\mathbf{H}_t}$$

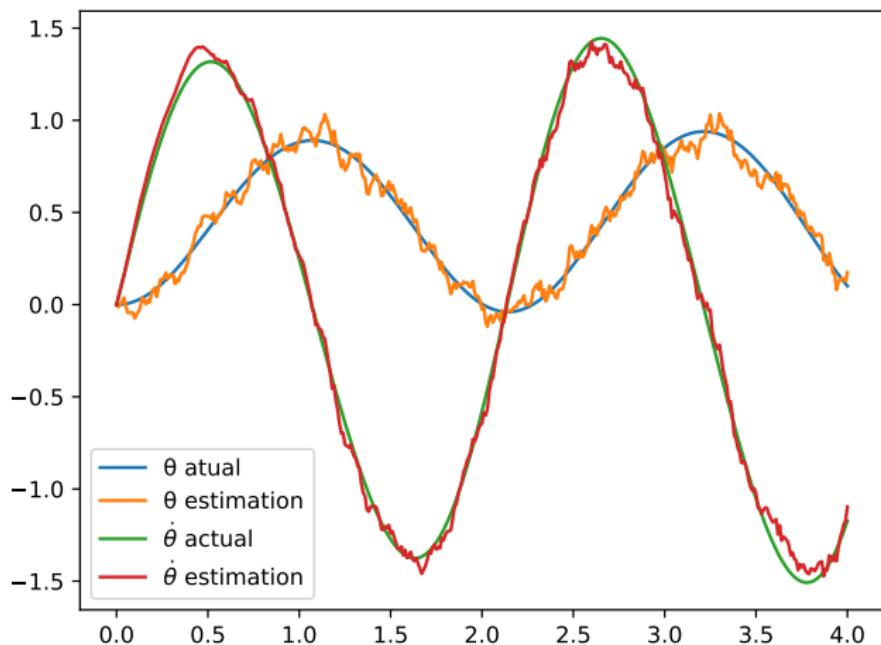
How does this affect our algorithm?

Extended Kalman Filter - Algorithm

- 1 Input: $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$
- 2 $\bar{\mu}_t = f(\mu_{t-1}, u_t)$
- 3 $\bar{\Sigma}_t = F_t \Sigma_{t-1} F_t^T + R_t$
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- 7 Output: μ_t, Σ_t

- We use the non-linear functions
- We use F_t instead of A_t
- We use H_t instead of C_t

Extended Kalman Filter - Code Example



Odometry

- *Odometry* is the process of estimating the robot base body pose \mathbf{x} from the joint motions

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- In more concrete terms, we are given \mathbf{x}_t , Δw_i (change of wheel angle between steps) and we want to estimate \mathbf{x}_{t+1}
- **How can we do this?**

Omnidirectional Mobile Robots - Reminder!

We have:

$$h_i(\theta) = \frac{1}{r_i \cos \gamma_i} \begin{bmatrix} x_i \sin(\beta_i + \gamma_i) - y_i \cos(\beta_i + \gamma_i) \\ \cos(\beta_i + \gamma_i + \theta) \\ \sin(\beta_i + \gamma_i + \theta) \end{bmatrix}^T$$

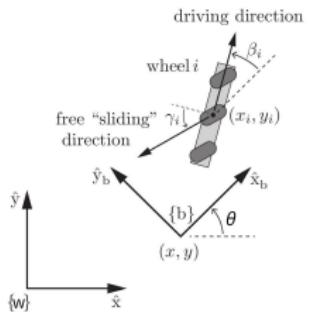
If we have m wheels, we can write something as follows:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \mathbf{H}(\theta) \dot{\mathbf{x}} = \begin{bmatrix} h_1(\theta) \\ \vdots \\ h_m(\theta) \end{bmatrix} \dot{\mathbf{x}}$$

Also:

$$\mathbf{u} = \mathbf{H}(0) \mathcal{V}_b = \begin{bmatrix} h_1(0) \\ \vdots \\ h_m(0) \end{bmatrix} \begin{bmatrix} \omega \\ v_x \\ v_y \end{bmatrix}$$

Independent of θ !



Source: Modern Robotics:
Mechanics, Planning, and Control,
Kevin M. Lynch and Frank C. Park, 2017, Cambridge University Press.

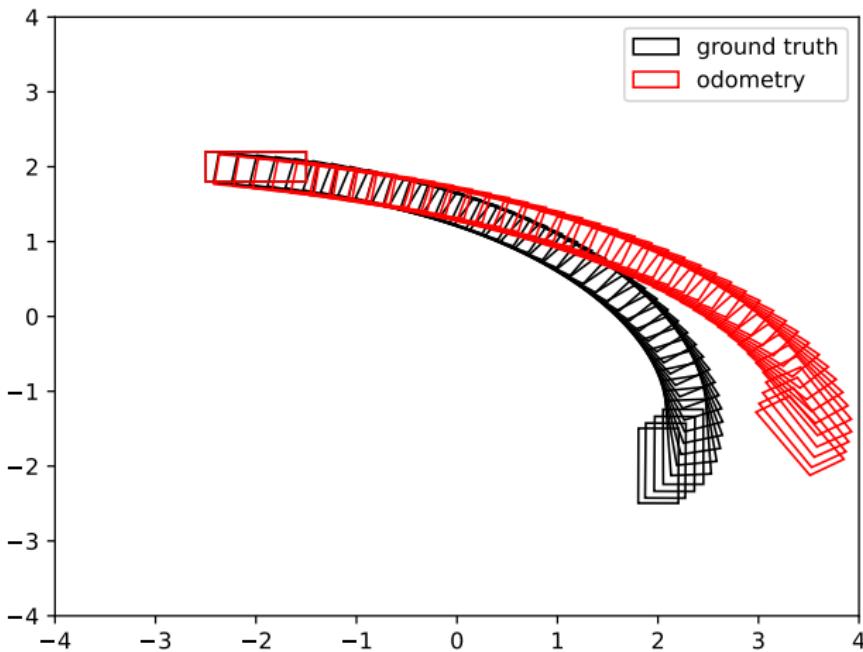
- u_i are the wheel velocities
- We only have access to the wheel encoder measurements!
- In other words, we have access to the wheel position at time t , w_i^t , and at time $t + 1$, w_i^{t+1}
- **How can we find the next body pose x_{t+1} ?**

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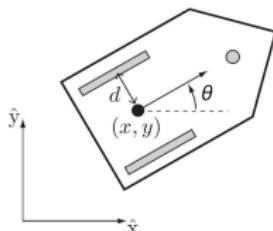
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- Now we can integrate using the previous x_t and $\bar{\mathcal{V}}_b$
- $H(0)$ is a $m \times 3$ matrix! Not always invertible! We use the pseudo-inverse: $H(0)^\dagger$

Odometry - Code Example



Differential Drive Robots - Reminder!



Source: Modern Robotics: Mechanics, Planning, and Control, *Kevin M. Lynch and Frank C. Park*, 2017, Cambridge University Press.

Kinematics:

The state of a differential drive robot is $\mathbf{x} = [\theta, x, y, \theta_L, \theta_R]^T$. And the kinematic equations are given by:

$$\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \\ \dot{\theta}_L \\ \dot{\theta}_R \end{bmatrix} = \begin{bmatrix} -\frac{r}{2d}\theta & \frac{r}{2d} \\ \frac{r}{2}\cos\theta & \frac{r}{2}\cos\theta \\ \frac{r}{2}\sin\theta & \frac{r}{2}\sin\theta \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_L \\ u_R \end{bmatrix}$$

where (θ, x, y) is the pose of the robot in world space, r are the wheel radii, d is the distance of the wheels from the robot center, θ_L, θ_R are the wheel rotation angles (values from encoder of the motors) and u_L, u_R are the wheel angular velocities (commands).

Odometry for Differential Drive Robots

- u_L, u_R are the wheel angular velocities (commands)
- We only have access to the wheel encoder measurements!
- $u_i \approx \frac{\theta_i^{t+1} - \theta_i^t}{dt}$ for $i \in \{L, R\}$
- **Thus:** $\bar{\mathcal{V}}_w = \begin{bmatrix} -\frac{r}{2d} & \frac{r}{2d} \\ \frac{r}{2} \cos \theta & \frac{r}{2} \cos \theta \\ \frac{r}{2} \sin \theta & \frac{r}{2} \sin \theta \end{bmatrix} \begin{bmatrix} u_L \\ u_R \end{bmatrix}$
- But this is in world space (**why?**)! How can we transform this in body space?

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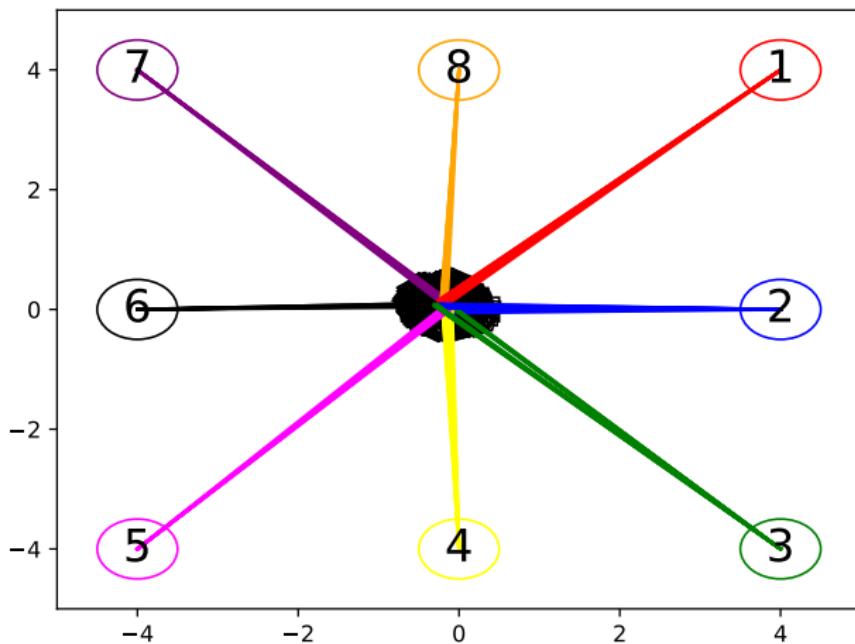
- We set $\theta = 0$ (**why?**) and thus $\bar{\mathcal{V}}_b = \begin{bmatrix} -\frac{r}{2d} & \frac{r}{2d} \\ \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_L \\ u_R \end{bmatrix}$



- **Sensors:**

- Range sensors
- LIDARs
- ...
- We have K beams: $\mathbf{z}_t = [z_t^1, z_t^2, \dots, z_t^K]^T$
- $z_t^i = [r_t^i, \phi_t^i]^T$
- r_t^i is distance to the object, ϕ_t^i relative angle to the sensor/robot

Range-Bearing Sensors - Code Example

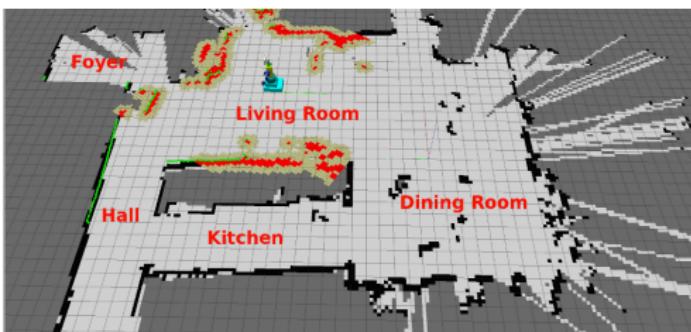


Simultaneous Localization and Mapping (SLAM):

- Build a map and locate the robot in the map at the same time

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- Build a map and locate the robot in the map at the same time
- We are given control inputs $u_{1:K}$ and observations $z_{1:K}$
- We want to estimate the *map of the environment* m and the trajectory of the robot $x_{0:K}$



Source: <https://sifrobot.com/clinical-apps/simultaneous-localization-and-mapping-slam/>

EKF SLAM:

- What if we include the map inside the state \mathbf{x} ?
- For example, $\mathbf{x} = [\theta, x, y, m_x^1, m_y^1, \dots, m_x^N, m_y^N]^T$

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- N landmarks
- A landmark is an “*important*” place/descriptor of the environment
- Now we can use EKF *almost as is!!!*
- **Assumption:** We know which observation corresponds to which landmark!
- **Belief representation:**

$$\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \mathcal{N}\left(\begin{bmatrix} \mathbf{x}_R \\ \mathbf{m} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{x}_R \mathbf{x}_R} & \boldsymbol{\Sigma}_{\mathbf{x}_R \mathbf{m}} \\ \boldsymbol{\Sigma}_{\mathbf{m} \mathbf{x}_R} & \boldsymbol{\Sigma}_{\mathbf{m} \mathbf{m}} \end{bmatrix}\right)$$

We basically perform EKF! We can make it faster/better by exploiting the problem structure:

- The *Prediction Step* only affects the robot variables!
- The *Correction Step* initializes new landmarks and/or updates already visited ones!

High-Level Steps

- 1 Predict next state
- 2 Predict observations (only for visible landmarks!)
- 3 Compare predicted observations with real ones (*data association*)
- 4 Update EKF

High-Level Pseudocode

- 1 Compute $\bar{\mu}_t$ only for the robot states given the motion model
- 2 Compute $\bar{\Sigma}_t$ as

$$\bar{\Sigma}_t = \begin{bmatrix} F_t \Sigma_{x_R x_R} F_t^T + R_t & F_t \Sigma_{x_R m} \\ \Sigma_{m x_R} F_t^T & \Sigma_{mm} \end{bmatrix}$$

- 3 For each visible landmark, compute *expected* observation:
 $h(\bar{\mu}_t)$
- 4 Compute H_t and proceed with the rest of EKF!

What if the landmarks are unknown!?

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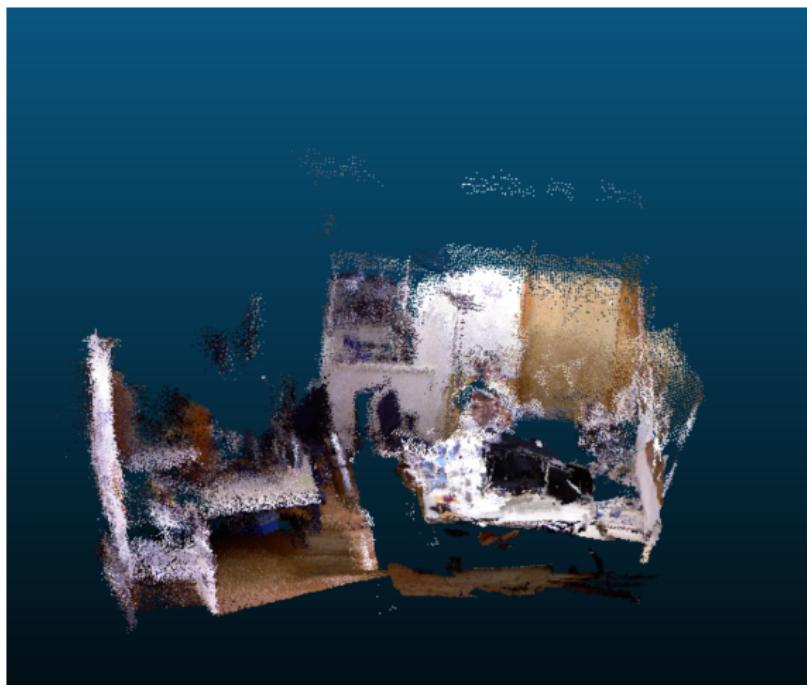
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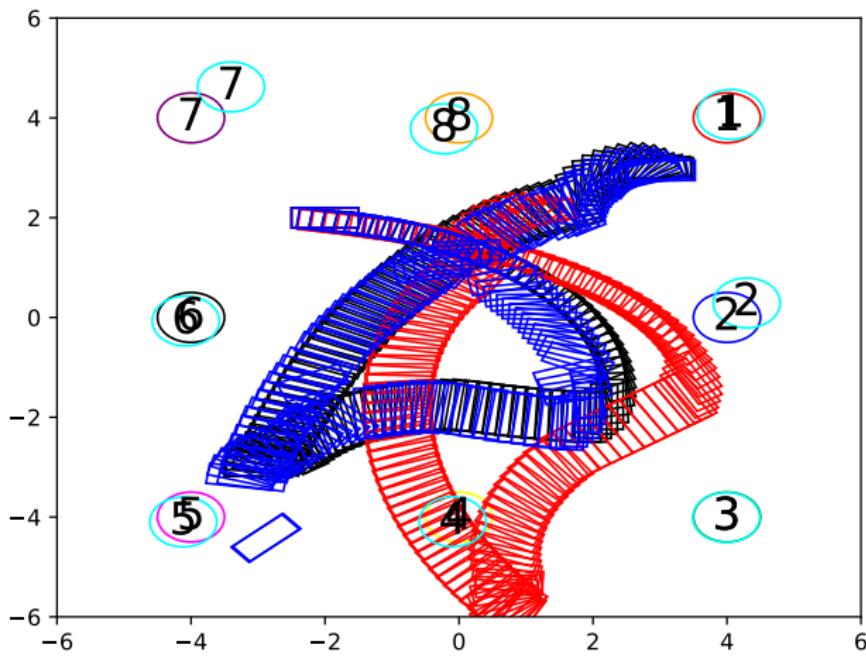
- 1 We need to keep a “dictionary” of landmark descriptions
- 2 We add a new description for new landmarks that have very low probability of belonging to an existing one

EKF SLAM - Results

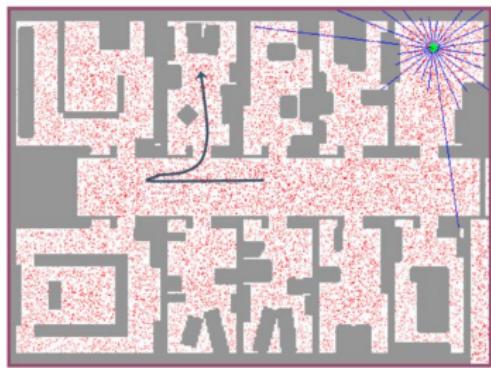


Source: <https://github.com/sjnarmstrong/ekf-slam>

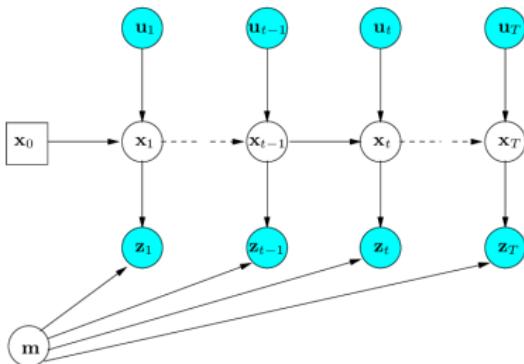
EKF SLAM - Code Example



Other solutions to SLAM?



Particle Filters



Graph-based Optimization¹

¹Source: Grisetti, G., Kümmerle, R., Stachniss, C. and Burgard, W., 2010. A tutorial on graph-based SLAM. *IEEE Intelligent Transportation Systems Magazine*.

State-of-the-art SLAM



Universidad
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Instituto Universitario de Investigación
en Ingeniería de Aragón
Universidad Zaragoza

ORB-SLAM2: an Open-Source SLAM System for Monocular, Stereo and RGB-D Cameras

Raúl Mur-Artal and Juan D. Tardós

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Bibliography

Chapter 4 and Section 8.2 from **State Estimation for Robotics**, *Timothy D. Barfoot*, 2022, Cambridge University Press. ebook

Sections 3.3, 4.2, 6.3, 6.6, and 10.2 from **Probabilistic Robotics**, *Sebastian Thrun, Wolfram Burgard, Dieter Fox*, 2005, The MIT Press. online

Thank you

- Any Questions?

- Office Hours:

- Tue-Wed (10:00-12:00)

- 24/7 by email (costashatz@upatras.gr, subject: *ECE_RSI_AM*)

- Material and Announcements



Laboratory of Automation & Robotics