Robotic Systems I

Lecture 5: Kalman Filters

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Template made by Panagiotis Papagiannopoulos

Probability Density Function (PDF):

$$\int_a^b p(x)dx = 1$$

Probability:

$$p(c \leqslant x \leqslant d) = \int_c^d p(x)dx$$

Conditional Variable: $p(x|y)$ is a PDF over $x \in [a, b]$ conditioned on $y \in [r, s]$ such that

$$(\forall y) \int_a^b p(x|y)dx = 1$$

Joint Probability: $p(x, y)$
Law of Total Probability:

$$p(x) = \int_y p(x|y)p(y)dy$$

Bayes Rule & Inference

Bayes Rule/Theorem:

$$p(\boldsymbol{x}, \boldsymbol{y}) = p(\boldsymbol{x}|\boldsymbol{y})p(\boldsymbol{y}) = p(\boldsymbol{y}|\boldsymbol{x})p(\boldsymbol{x})$$

We some rearrangements:

$$p(\boldsymbol{x}|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\boldsymbol{x})p(\boldsymbol{x})}{p(\boldsymbol{y})}$$

Marginalization:

$$\begin{aligned}
p(\boldsymbol{y}) &= p(\boldsymbol{y}) \int p(\boldsymbol{x}|\boldsymbol{y}) d\boldsymbol{x} \\
&= \int p(\boldsymbol{x}|\boldsymbol{y}) p(\boldsymbol{y}) d\boldsymbol{x} \\
&= \int p(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{x} \\
&= \int p(\boldsymbol{y}|\boldsymbol{x}) p(\boldsymbol{x}) d\boldsymbol{x}
\end{aligned}$$

Mean and Covariance

First moment or mean of a PDF:

$$\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{x}] = \int \boldsymbol{x} p(\boldsymbol{x}) d\boldsymbol{x}$$

Second moment or covariance of a PDF:

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\boldsymbol{xx}} = \text{Cov}[\boldsymbol{x}, \boldsymbol{x}] = \mathbb{E}[(\boldsymbol{x} - \boldsymbol{\mu})(\boldsymbol{x} - \boldsymbol{\mu})^T]$$

We draw a sample from a PDF: $\boldsymbol{x}_s \sim p(\boldsymbol{x})$

Sample mean:

$$\boldsymbol{\mu}_{\text{sample}} = \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{x}_{s_i}$$

Sample covariance:

$$\boldsymbol{\Sigma}_{\text{sample}} = \frac{1}{N-1} \sum_{i=1}^{N} (\boldsymbol{x}_{s_i} - \boldsymbol{\mu}_{\text{sample}})(\boldsymbol{x}_{s_i} - \boldsymbol{\mu}_{\text{sample}})^T$$

Statistical Independence

If two random variables $\boldsymbol{x}, \boldsymbol{y}$ are statistically independent, then:

$$p(\boldsymbol{x}, \boldsymbol{y}) = p(\boldsymbol{x})p(\boldsymbol{y})$$

If two random variables $\boldsymbol{x}, \boldsymbol{y}$ are *uncorrelated*, then:

$$\mathbb{E}[\boldsymbol{x}^T\boldsymbol{y}] = \mathbb{E}[\boldsymbol{x}]^T\mathbb{E}[\boldsymbol{y}]$$

and

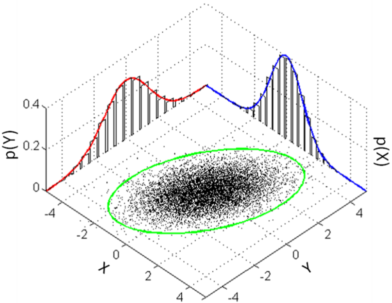
$$\text{Cov}[\boldsymbol{x}, \boldsymbol{y}] = \mathbb{E}[(\boldsymbol{x} - \boldsymbol{\mu}_{\boldsymbol{x}})(\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{y}})^T] = \mathbf{0}$$

Gaussian Distribution

A multivariate Gaussian PDF over a random variable $\boldsymbol{x} \in \mathbb{R}^N$ is defined by its mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$:

$$p(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^N \det(\boldsymbol{\Sigma})}} \exp\Big(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\Big)$$

We write $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.



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Joint Gaussians:

$$p(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_{\boldsymbol{x}} \\ \boldsymbol{\mu}_{\boldsymbol{y}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{xx}} & \boldsymbol{\Sigma}_{\boldsymbol{xy}} \\ \boldsymbol{\Sigma}_{\boldsymbol{yx}} & \boldsymbol{\Sigma}_{\boldsymbol{yy}} \end{bmatrix} \right)$$

Conditional:

$$p(\boldsymbol{x}|\boldsymbol{y}) = \mathcal{N}\Big(\boldsymbol{\mu}_{\boldsymbol{x}} + \boldsymbol{\Sigma}_{\boldsymbol{xy}}\boldsymbol{\Sigma}_{\boldsymbol{yy}}^{-1}(\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{y}}), \boldsymbol{\Sigma}_{\boldsymbol{xx}} - \boldsymbol{\Sigma}_{\boldsymbol{xy}}\boldsymbol{\Sigma}_{\boldsymbol{yy}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{yx}}\Big)$$

Marginalization:

$$\boldsymbol{x} \sim \mathcal{N}\Big(\boldsymbol{\mu}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{xx}}\Big)$$
$$\boldsymbol{y} \sim \mathcal{N}\Big(\boldsymbol{\mu}_{\boldsymbol{y}}, \boldsymbol{\Sigma}_{\boldsymbol{yy}}\Big)$$

Linear Combinations of Gaussian Distributions

Let's assume a random variable $\boldsymbol{x} \in \mathbb{R}^N$ such that:

$$\boldsymbol{x} \sim \mathcal{N}\Big(\boldsymbol{\mu}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{xx}}\Big)$$

And we have another random variable $\boldsymbol{y} = \boldsymbol{Ax}$. What is the distribution of $\boldsymbol{y}$?

Linear Combinations of Gaussian Distributions

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$$\boldsymbol{x} \sim \mathcal{N}\Big(\boldsymbol{\mu}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{xx}}\Big)$$

And we have another random variable $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. What is the distribution of $\boldsymbol{y}$? It is a Gaussian with:

$$\boldsymbol{y} \sim \mathcal{N}\Big(\boldsymbol{A}\boldsymbol{\mu}_{\boldsymbol{x}}, \boldsymbol{A}\boldsymbol{\Sigma}_{\boldsymbol{xx}}\boldsymbol{A}^T\Big)$$

Passing Gaussians through Non-linearities

What happes if $p(\boldsymbol{y}|\boldsymbol{x}) = \mathcal{N}\Big(g(\boldsymbol{x}), \boldsymbol{R}\Big)$? We do not get a Gaussian!

What happes if $p(\boldsymbol{y}|\boldsymbol{x}) = \mathcal{N}\Big(g(\boldsymbol{x}), \boldsymbol{R}\Big)$? We do not get a Gaussian! **We can linearize!**

$$\overline{g}(\boldsymbol{x}) = \boldsymbol{\mu}_{\boldsymbol{y}} + \boldsymbol{J}(\boldsymbol{x} - \boldsymbol{\mu}_{\boldsymbol{x}})$$

where $\boldsymbol{J} = \frac{\partial g}{\partial \boldsymbol{x}}\Big|_{\boldsymbol{x}=\boldsymbol{\mu}_{\boldsymbol{x}}}$.

And thus:

$$\boldsymbol{y} \sim \mathcal{N}\Big(g(\boldsymbol{\mu}_{\boldsymbol{x}}), \boldsymbol{R} + \boldsymbol{J}\boldsymbol{\Sigma}_{\boldsymbol{xx}}\boldsymbol{J}^T\Big)$$

State Estimation Problem

Estimate the state $\boldsymbol{x}$ given observations $\boldsymbol{z}$ and control inputs $\boldsymbol{u}$.
The goal is to estimate:

$$p(\boldsymbol{x}|\boldsymbol{z}, \boldsymbol{u})$$

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Estimate the state $\boldsymbol{x}$ given observations $\boldsymbol{z}$ and control inputs $\boldsymbol{u}$.
The goal is to estimate:

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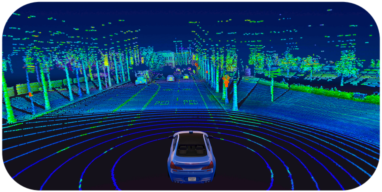
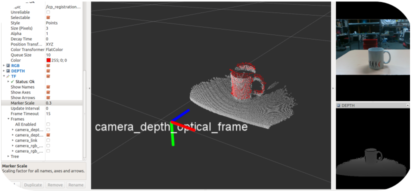
How can we do that?

Sensors and Observations





Bayes Filtering

$\text{bel}(\boldsymbol{x}_t) = p(\boldsymbol{x}_t | \boldsymbol{z}_{1:t}, \boldsymbol{u}_{1:t})$

Bayes Filtering

$$\begin{aligned}\text{bel}(\boldsymbol{x}_t) &= p(\boldsymbol{x}_t|\boldsymbol{z}_{1:t}, \boldsymbol{u}_{1:t}) \\ &= \underbrace{\eta p(\boldsymbol{z}_t|\boldsymbol{x}_t, \boldsymbol{z}_{1:t-1}, \boldsymbol{u}_{1:t})p(\boldsymbol{x}_t|\boldsymbol{z}_{1:t-1}, \boldsymbol{u}_{1:t})}_{\text{Bayes Rule}}\end{aligned}$$

Bayes Filtering

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&= \quad \underbrace{\eta p(\boldsymbol{z}_t|\boldsymbol{x}_t)}_{\text{Markov Assumption}} \quad p(\boldsymbol{x}_t|\boldsymbol{z}_{1:t-1}, \boldsymbol{u}_{1:t})
\end{aligned}
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&= \eta p(\boldsymbol{z}_t|\boldsymbol{x}_t) \underbrace{\int_{\boldsymbol{x}_{t-1}} p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{z}_{1:t-1}, \boldsymbol{u}_{1:t}) p(\boldsymbol{x}_{t-1}|\boldsymbol{z}_{1:t-1}, \boldsymbol{u}_{1:t}) d\boldsymbol{x}_{t-1}}_{\text{Law of Total Probability}}
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&= \eta p(\boldsymbol{z}_t|\boldsymbol{x}_t)\underbrace{\int_{\boldsymbol{x}_{t-1}} p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1},\boldsymbol{u}_t)p(\boldsymbol{x}_{t-1}|\boldsymbol{z}_{1:t-1},\boldsymbol{u}_{1:t-1})d\boldsymbol{x}_{t-1}}_{\text{Markov Assumption}}
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&= \eta p(\boldsymbol{z}_t|\boldsymbol{x}_t) \int_{\boldsymbol{x}_{t-1}} p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{u}_t) \mathrm{bel}(\boldsymbol{x}_{t-1}) d\boldsymbol{x}_{t-1}
\end{aligned}
$$

Prediction Step

$$\overline{\text{bel}}(\boldsymbol{x}_t) = \int_{\boldsymbol{x}_{t-1}} p(\boldsymbol{x}_t | \boldsymbol{x}_{t-1}, \boldsymbol{u}_t)\text{bel}(\boldsymbol{x}_{t-1}) d\boldsymbol{x}_{t-1}$$

Correction Step

$$\text{bel}(\boldsymbol{x}_t) = \eta p(\boldsymbol{z}_t | \boldsymbol{x}_t)\overline{\text{bel}}(\boldsymbol{x}_t)$$

Motion and Observation Model

Prediction Step

$$\overline{\text{bel}}(\boldsymbol{x}_t) = \int_{\boldsymbol{x}_{t-1}} \underbrace{p(\boldsymbol{x}_t | \boldsymbol{x}_{t-1}, \boldsymbol{u}_t)}_{\text{Motion Model}} \text{bel}(\boldsymbol{x}_{t-1}) d\boldsymbol{x}_{t-1}$$

Correction Step

$$\text{bel}(\boldsymbol{x}_t) = \underbrace{\eta p(\boldsymbol{z}_t | \boldsymbol{x}_t)}_{\text{Observation Model}} \overline{\text{bel}}(\boldsymbol{x}_t)$$

Typical Motion Models for 2D Navigation

- **Odometry Model:** Wheeled robots! The robot moves from (x, y, θ) to (x', y', θ'). Equations:

$$\begin{aligned}
\delta_{\text{trans}} &= \sqrt{(x' - x)^2 + (y' - y)^2} \\
\delta_{\text{rot}_1} &= \text{atan2}(y' - y, x' - x) - \theta \\
\delta_{\text{rot}_2} &= \theta' - \theta - \delta_{\text{rot}_1}
\end{aligned}$$

- **Velocity Model:** Non-wheeled robots! The robot moves from (x, y, θ) with velocity (v, ω, γ). Equations:

$$\begin{aligned}
x' &= x + \frac{v}{\omega}\Big(\sin(\theta + \omega dt) - \sin\theta\Big) \\
y' &= y + \frac{v}{\omega}\Big(\cos\theta - \cos(\theta + \omega dt)\Big) \\
\theta' &= \theta + \omega dt + \gamma dt
\end{aligned}$$

We add noise to the velocities/commands!

What other models?

What other models?

The models basically define the differences in 2D poses! We can use our analytical models to get this information!

- **Differential Drive Robot**
- **Omnidirectional Robot**

Typical Observation Models

Laser Scanner!

- We have K laser beams: $\boldsymbol{z}_t = \left[z_t^1, z_t^2, \cdots, z_t^K\right]^T$
- We assume that each beam is independent given the robot position (m is the environment):

$$p(\boldsymbol{z}_t|\boldsymbol{x}_t, m) = \prod_{i=1}^{K} p(z_t^i|\boldsymbol{x}_t, m)$$

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- OK! How do I compute those individual probabilities?

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- OK! How do I compute those individual probabilities?
- Many models! We will see a few in the lab!

Kalman Filter

- Bayes filter!
- Assumes a linear motion model (control affine)!
- Assumes a linear observation model!
- Everything is a Gaussian!

Kalman Filter

- Bayes filter!
- Assumes a linear motion model (control affine)!
- Assumes a linear observation model!
- Everything is a Gaussian! **And it stays Gaussian!**
- Optimal for linear models and Gaussian distributions!

Kalman Filtering in Two Steps

Prediction Step

$$\overline{\mathrm{bel}}(\boldsymbol{x}_t) = \int_{\boldsymbol{x}_{t-1}} p(\boldsymbol{x}_t | \boldsymbol{x}_{t-1}, \boldsymbol{u}_t) \mathrm{bel}(\boldsymbol{x}_{t-1}) d\boldsymbol{x}_{t-1}$$

Correction Step

$$\mathrm{bel}(\boldsymbol{x}_t) = \eta p(\boldsymbol{z}_t | \boldsymbol{x}_t) \overline{\mathrm{bel}}(\boldsymbol{x}_t)$$

- $p(\boldsymbol{x}_t | \boldsymbol{x}_{t-1}, \boldsymbol{u}_t)$ is Gaussian
- $p(\boldsymbol{z}_t | \boldsymbol{x}_t)$ is Gaussian
- $\mathrm{bel}(\boldsymbol{x}_0)$ is Gaussian

Kalman Filtering in Two Steps

Prediction Step

$$\overline{\text{bel}}(\boldsymbol{x}_t) = \int_{\boldsymbol{x}_{t-1}} p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{u}_t)\text{bel}(\boldsymbol{x}_{t-1})d\boldsymbol{x}_{t-1}$$

Correction Step

$$\text{bel}(\boldsymbol{x}_t) = \eta p(\boldsymbol{z}_t|\boldsymbol{x}_t)\overline{\text{bel}}(\boldsymbol{x}_t)$$

- $p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{u}_t)$ is Gaussian
- $p(\boldsymbol{z}_t|\boldsymbol{x}_t)$ is Gaussian
- $\text{bel}(\boldsymbol{x}_0)$ is Gaussian
- $\text{bel}(\boldsymbol{x}_t)$ **stays Gaussian!**

$$\boldsymbol{x}_t = \boldsymbol{A}_t \boldsymbol{x}_{t-1} + \boldsymbol{B}_t \boldsymbol{u}_t + \boldsymbol{\epsilon}_t^m$$
$$\boldsymbol{z}_t = \boldsymbol{C}_t \boldsymbol{x}_t + \boldsymbol{\epsilon}_t^o$$

where $\boldsymbol{\epsilon}_t^m \sim \mathcal{N}(\mathbf{0}, \boldsymbol{R}_t), \boldsymbol{\epsilon}_t^o \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q}_t)$.

Linear models with zero mean Gaussian noise!

Kalman Filter - Linear Models (2)

BUT this is not a probability!

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$$p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{u}_t) = \frac{1}{\sqrt{(2\pi)^N \det(\boldsymbol{R}_t)}} \exp\Big(-\frac{1}{2}(\boldsymbol{x}_t - \boldsymbol{A}_t\boldsymbol{x}_{t-1} - \boldsymbol{B}_t\boldsymbol{u}_t)^T\boldsymbol{R}_t^{-1}(\boldsymbol{x}_t - \boldsymbol{A}_t\boldsymbol{x}_{t-1} - \boldsymbol{B}_t\boldsymbol{u}_t)\Big)$$

where $\boldsymbol{R}_t$ is the motion noise variance!

BUT this is not a probability!

$$p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1},\boldsymbol{u}_t) = \frac{1}{\sqrt{(2\pi)^N \det(\boldsymbol{R}_t)}} \exp\Big(-\frac{1}{2}(\boldsymbol{x}_t - \boldsymbol{A}_t\boldsymbol{x}_{t-1} - \boldsymbol{B}_t\boldsymbol{u}_t)^T \boldsymbol{R}_t^{-1}(\boldsymbol{x}_t - \boldsymbol{A}_t\boldsymbol{x}_{t-1} - \boldsymbol{B}_t\boldsymbol{u}_t)\Big)$$

where $\boldsymbol{R}_t$ is the motion noise variance!

$$p(\boldsymbol{z}_t|\boldsymbol{x}_t) = \frac{1}{\sqrt{(2\pi)^M \det(\boldsymbol{Q}_t)}} \exp\Big(-\frac{1}{2}(\boldsymbol{z}_t - \boldsymbol{C}_t\boldsymbol{x}_t)^T \boldsymbol{Q}_t^{-1}(\boldsymbol{z}_t - \boldsymbol{C}_t\boldsymbol{X}_t)\Big)$$

where $\boldsymbol{Q}_t$ is the observation/measurement noise variance!

Kalman Filter - Algorithm

1. Input: $\boldsymbol{\mu}_{t-1}, \boldsymbol{\Sigma}_{t-1}, \boldsymbol{u}_t, \boldsymbol{z}_t$
2. $\overline{\boldsymbol{\mu}}_t = \boldsymbol{A}_t \boldsymbol{\mu}_{t-1} + \boldsymbol{B}_t \boldsymbol{u}_t$
3. $\overline{\boldsymbol{\Sigma}}_t = \boldsymbol{A}_t \boldsymbol{\Sigma}_{t-1} \boldsymbol{A}_t^T + \boldsymbol{R}_t$
4. $\boldsymbol{K}_t = \overline{\boldsymbol{\Sigma}}_t \boldsymbol{C}_t^T \left(\boldsymbol{C}_t \overline{\boldsymbol{\Sigma}}_t \boldsymbol{C}_t^T + \boldsymbol{Q}_t \right)^{-1}$
5. $\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \boldsymbol{K}_t (\boldsymbol{z}_t - \boldsymbol{C}_t \overline{\boldsymbol{\mu}}_t)$
6. $\boldsymbol{\Sigma}_t = (\boldsymbol{I} - \boldsymbol{K}_t \boldsymbol{C}_t) \overline{\boldsymbol{\Sigma}}_t$
7. Output: $\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t$

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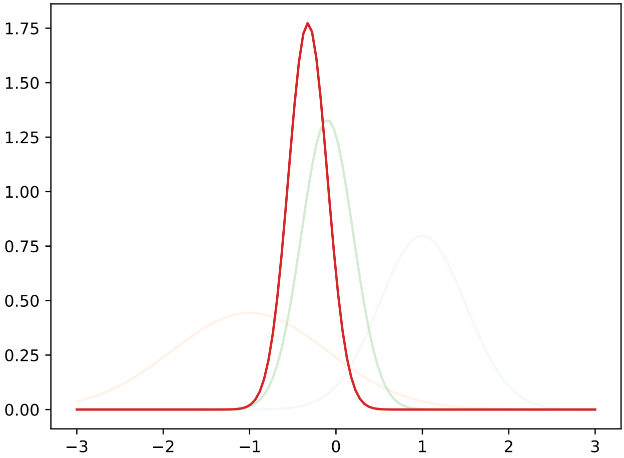
2. $\overline{\boldsymbol{\mu}}_t = \boldsymbol{A}_t\boldsymbol{\mu}_{t-1} + \boldsymbol{B}_t\boldsymbol{u}_t$
3. $\overline{\boldsymbol{\Sigma}}_t = \boldsymbol{A}_t\boldsymbol{\Sigma}_{t-1}\boldsymbol{A}_t^T + \boldsymbol{R}_t$
4. $\boldsymbol{K}_t = \overline{\boldsymbol{\Sigma}}_t\boldsymbol{C}_t^T\left(\boldsymbol{C}_t\overline{\boldsymbol{\Sigma}}_t\boldsymbol{C}_t^T + \boldsymbol{Q}_t\right)^{-1}$
5. $\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \boldsymbol{K}_t(\boldsymbol{z}_t - \boldsymbol{C}_t\overline{\boldsymbol{\mu}}_t)$
6. $\boldsymbol{\Sigma}_t = (\boldsymbol{I} - \boldsymbol{K}_t\boldsymbol{C}_t)\overline{\boldsymbol{\Sigma}}_t$
7. Output: $\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t$

Steps 2 and 3 are the *Prediction Step*

Steps 4, 5, 6 and 7 are the *Correction Step*

Kalman Filter - Code Example



Bibliography

Chapters 2 and 3 from **State Estimation for Robotics**, *Timothy D. Barfoot*, 2022, Cambridge University Press. ebook

Sections 2.1 - 2.5, 3.1, 3.2, 5.1 - 5.4, and 6.3 from **Probabilistic Robotics**, *Sebastian Thrun, Wolfram Burgard, Dieter Fox*, 2005, The MIT Press. online

Thank you

- **Any Questions?**
- **Office Hours**:
  - **Tue-Wed** *(10:00-12:00)*
  - 24/7 by email (`costashatz@upatras.gr`, subject: *ECE_RSI_AM*)
- **Material and Announcements**

Laboratory of Automation & Robotics