Robotic Systems II

Lecture 7: Optimization and Optimal Control on Manifolds

Konstantinos Chatzilygeroudis - costashatz@upatras.gr

Department of Electrical and Computer Engineering
University of Patras

Template made by Panagiotis Papagiannopoulos

Optimization with Orientations

- When we optimize, we have steps like: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta\boldsymbol{x}$
- What happens if my parameters $\boldsymbol{x}$ are orientations?

Optimization with Orientations

- When we optimize, we have steps like: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta\boldsymbol{x}$
- What happens if my parameters $\boldsymbol{x}$ are orientations?
- Orientations do not work with addition:

$\boldsymbol{R}_{k+1} = \boldsymbol{R}_k + \delta\phi$ **does not mean anything!**

Optimization with Orientations

- When we optimize, we have steps like: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta\boldsymbol{x}$
- What happens if my parameters $\boldsymbol{x}$ are orientations?
- Orientations do not work with addition:

  $\boldsymbol{R}_{k+1} = \boldsymbol{R}_k + \delta\phi$ **does not mean anything!**

- We need to multiply for orientations: $\boldsymbol{R}_{k+1} = \boldsymbol{R}_k \exp(\delta\phi^\wedge)$, where $\delta\phi$ is a 3D rotation difference

Optimization with Orientations

- When we optimize, we have steps like: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta\boldsymbol{x}$
- What happens if my parameters $\boldsymbol{x}$ are orientations?
- Orientations do not work with addition:

  $\boldsymbol{R}_{k+1} = \boldsymbol{R}_k + \delta\phi$ **does not mean anything!**

- We need to multiply for orientations: $\boldsymbol{R}_{k+1} = \boldsymbol{R}_k \exp(\delta\phi^\wedge)$, where $\delta\phi$ is a 3D rotation difference
- Or use the correct plus: $\boldsymbol{R}_{k+1} = \boldsymbol{R}_k \oplus \delta\phi$

Optimization with Orientations

- When we optimize, we have steps like: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta \boldsymbol{x}$
- What happens if my parameters $\boldsymbol{x}$ are orientations?
- Orientations do not work with addition:

  $\boldsymbol{R}_{k+1} = \boldsymbol{R}_k + \delta\phi$ **does not mean anything!**

- We need to multiply for orientations: $\boldsymbol{R}_{k+1} = \boldsymbol{R}_k \exp(\delta\boldsymbol{\phi}^\wedge)$, where $\delta\boldsymbol{\phi}$ is a 3D rotation difference
- Or use the correct plus: $\boldsymbol{R}_{k+1} = \boldsymbol{R}_k \oplus \delta\boldsymbol{\phi}$
- We should also be able to compute derivatives/Jacobians wrt orientations!!

Derivatives on Lie Group $SO(3)$

The Jacobian of a function $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$ is defined as:

$$\boldsymbol{J} = \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} = \lim_{\delta \boldsymbol{x} \rightarrow 0} \frac{f(\boldsymbol{x} + \delta \boldsymbol{x}) - f(\boldsymbol{x})}{\delta \boldsymbol{x}}$$

Derivatives on Lie Group $SO(3)$

The Jacobian of a function $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$ is defined as:

$$\boldsymbol{J} = \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} = \lim_{\delta \boldsymbol{x} \rightarrow 0} \frac{f(\boldsymbol{x} + \delta \boldsymbol{x}) - f(\boldsymbol{x})}{\delta \boldsymbol{x}}$$

When dealing with functions that accept $SO(3)$ inputs and outputs, we can use a similar definition (using the right $\oplus, \ominus$ operators):

$$\begin{aligned}\boldsymbol{J} = \frac{{}^{\boldsymbol{R}}Df(\boldsymbol{R})}{\partial \boldsymbol{R}} &= \lim_{\delta \boldsymbol{\phi} \rightarrow 0} \frac{f(\boldsymbol{R} \oplus \delta \boldsymbol{\phi}) \ominus f(\boldsymbol{R})}{\delta \boldsymbol{\phi}} \\ &= \left.\frac{\partial \log(f(\boldsymbol{R})^{-1} f(\boldsymbol{R}) \exp(\delta \boldsymbol{\phi}^{\wedge}))}{\partial \delta \boldsymbol{\phi}}\right|_{\delta \boldsymbol{\phi} = 0}\end{aligned}$$

We call this the *right Jacobian*.

Derivatives on Lie Group $SO(3)$ - (2)

Continuing the previous, and using the left $\oplus, \ominus$ operators, we have:

$$\boldsymbol{J} = \frac{{}^{\boldsymbol{I}}Df(\boldsymbol{R})}{\partial \boldsymbol{R}} = \lim_{\delta\boldsymbol{\phi}\to 0} \frac{f(\delta\boldsymbol{\phi} \oplus \boldsymbol{R}) \ominus f(\boldsymbol{R})}{\delta\boldsymbol{\phi}}$$
$$= \frac{\partial \log(\exp(\delta\boldsymbol{\phi}^{\wedge})f(\boldsymbol{R})f(\boldsymbol{R})^{-1})}{\partial\delta\boldsymbol{\phi}}\bigg|_{\delta\boldsymbol{\phi}=0}$$

We call this the *left Jacobian*. Unless otherwise stated, we will use the **right Jacobians** and differentiation.
The chain rule applies! Let's assume that we have $\boldsymbol{R}_{k+1} = f(\boldsymbol{R}_k)$ and $\boldsymbol{R}_{k+2} = g(f(\boldsymbol{R}_k))$. We have:

$$\frac{Dg}{D\boldsymbol{R}_k} = \frac{Dg}{D\boldsymbol{R}_{k+1}}\frac{Df}{\boldsymbol{R}_k} \text{ or } \boldsymbol{J}^{g}_{\boldsymbol{R}_k} = \boldsymbol{J}^{g}_{\boldsymbol{R}_{k+1}}\boldsymbol{J}^{f}_{\boldsymbol{R}_k}$$

Derivatives on Lie Group $SO(3)$ - Example

Using the same mechanism we can compute derivatives of any function involving $SO(3)$ components. For example, let's assume that we have f as follows:

$$f(\boldsymbol{R}) = \boldsymbol{R}\boldsymbol{p}$$

where $\boldsymbol{R} \in SO(3), \boldsymbol{p} \in \mathbb{R}^3$.

$$\boldsymbol{J} = \frac{Df(\boldsymbol{R})}{\partial \boldsymbol{R}} = \lim_{\delta\phi \to 0} \frac{f(\boldsymbol{R} \oplus \delta\phi) \ominus f(\boldsymbol{R})}{\delta\phi}$$

$$= \lim_{\delta\phi \to 0} \frac{(\boldsymbol{R} \oplus \delta\phi)\boldsymbol{p} - \boldsymbol{R}\boldsymbol{p}}{\delta\phi} = \lim_{\delta\phi \to 0} \frac{\boldsymbol{R}\exp(\delta\phi^\wedge)\boldsymbol{p} - \boldsymbol{R}\boldsymbol{p}}{\delta\phi}$$

$$\overset{\exp(\delta\phi^\wedge) \approx \boldsymbol{I} + \delta\phi^\wedge}{\approx} \lim_{\delta\phi \to 0} \frac{\boldsymbol{R}(\boldsymbol{I} + \delta\phi^\wedge)\boldsymbol{p} - \boldsymbol{R}\boldsymbol{p}}{\delta\phi} = \lim_{\delta\phi \to 0} \frac{\boldsymbol{R}\boldsymbol{p} + \boldsymbol{R}\delta\phi^\wedge\boldsymbol{p} - \boldsymbol{R}\boldsymbol{p}}{\delta\phi}$$

$$= \lim_{\delta\phi \to 0} \frac{\boldsymbol{R}\delta\phi^\wedge\boldsymbol{p}}{\delta\phi} \overset{\boldsymbol{a}\boldsymbol{b}^\wedge = -\boldsymbol{b}^\wedge\boldsymbol{a}}{=} \lim_{\delta\phi \to 0} \frac{-\boldsymbol{R}\boldsymbol{p}^\wedge\delta\phi}{\delta\phi} = -\boldsymbol{R}\boldsymbol{p}^\wedge$$

Left and Right Jacobians ($\theta = \|\phi\|$):

- $\boldsymbol{J}_r(\phi) = \boldsymbol{J}_{\phi}^{\exp(\phi^\wedge)} = \boldsymbol{I} - \frac{1-\cos\theta}{\theta^2}\phi^\wedge + \frac{\theta-\sin\theta}{\theta^3}\phi^\wedge\phi^\wedge$
- $\boldsymbol{J}_r^{-1}(\phi) = \boldsymbol{I} + \frac{1}{2}\phi^\wedge + (\frac{1}{\theta^2} - \frac{1+\cos\theta}{2\theta\sin\theta})\phi^\wedge\phi^\wedge$
- $\boldsymbol{J}_l(\phi) = \boldsymbol{J}_{\phi}^{\exp(\phi^\wedge)} = \boldsymbol{I} + \frac{1-\cos\theta}{\theta^2}\phi^\wedge + \frac{\theta-\sin\theta}{\theta^3}\phi^\wedge\phi^\wedge$
- $\boldsymbol{J}_l^{-1}(\phi) = \boldsymbol{I} - \frac{1}{2}\phi^\wedge + (\frac{1}{\theta^2} - \frac{1+\cos\theta}{2\theta\sin\theta})\phi^\wedge\phi^\wedge$
- $\boldsymbol{J}_l = \boldsymbol{J}_r^T, \boldsymbol{J}_l^{-1} = \boldsymbol{J}_r^{-T}$
- $\boldsymbol{J}_r(-\phi) = \boldsymbol{J}_l(\phi)$
- $\boldsymbol{Ad}_{\exp(\phi^\wedge)} = \boldsymbol{J}_l(\phi)\boldsymbol{J}_r^{-1}(\phi)$

Left and Right Jacobians of $SO(3)$ - Properties

A few interesting properties for small $\delta\phi$:

- $\exp(\phi + \delta\phi) \approx \exp(\phi)\exp(\boldsymbol{J}_r(\phi)\delta\phi)$
- $\exp(\phi)\exp(\delta\phi) \approx \exp(\phi + \boldsymbol{J}_r^{-1}(\phi)\delta\phi)$
- $\log(\exp(\phi)\exp(\delta\phi)) \approx \phi + \boldsymbol{J}_r^{-1}(\phi)\delta\phi$

- $\exp(\phi + \delta\phi) \approx \exp(\boldsymbol{J}_l(\phi)\delta\phi)\exp(\phi)$
- $\exp(\delta\phi)\exp(\phi) \approx \exp(\phi + \boldsymbol{J}_l^{-1}(\phi)\delta\phi)$
- $\log(\exp(\delta\phi)\exp(\phi)) \approx \phi + \boldsymbol{J}_l^{-1}(\phi)\delta\phi$

Examples of Jacobians of $SO(3)$

A few *elementary Jacobians:*

- $\boldsymbol{J}_{\boldsymbol{R}}^{\boldsymbol{R}^{-1}} = -\boldsymbol{Ad}_{\boldsymbol{R}} = -\boldsymbol{R}$
- $\boldsymbol{J}_{\boldsymbol{R}_1}^{\boldsymbol{R}_1\boldsymbol{R}_2} = \boldsymbol{Ad}_{\boldsymbol{R}_2}^{-1} = \boldsymbol{R}_2^T$, $\boldsymbol{J}_{\boldsymbol{R}_2}^{\boldsymbol{R}_1\boldsymbol{R}_2} = \boldsymbol{I}$
- $\boldsymbol{J}_{\boldsymbol{R}}^{\boldsymbol{R}\boldsymbol{p}} = -\boldsymbol{R}\boldsymbol{p}^\wedge$, $\boldsymbol{J}_{\boldsymbol{p}}^{\boldsymbol{R}\boldsymbol{p}} = \boldsymbol{R}$
- $\boldsymbol{J}_{\boldsymbol{R}}^{\log(\boldsymbol{R})} = \boldsymbol{J}_r^{-1}(\phi)$, where $\phi = log(\boldsymbol{R})$
- $\boldsymbol{J}_{\boldsymbol{R}}^{\boldsymbol{R}\oplus\delta\phi} = \boldsymbol{Ad}_{\exp(\delta\phi^\wedge)}^{-1}$, $\boldsymbol{J}_{\delta\phi}^{\boldsymbol{R}\oplus\delta\phi} = \boldsymbol{J}_r(\delta\phi)$
- Given $\boldsymbol{R} = \boldsymbol{R}_1^T\boldsymbol{R}_2, \phi = \boldsymbol{R}_2 \ominus \boldsymbol{R}_1 = \log \boldsymbol{R}$:
  - $\boldsymbol{J}_{\boldsymbol{R}_1}^{\boldsymbol{R}_2\ominus\boldsymbol{R}_1} = -\boldsymbol{J}_r^{-1}(\phi)\boldsymbol{R}^T$
  - $\boldsymbol{J}_{\boldsymbol{R}_2}^{\boldsymbol{R}_2\ominus\boldsymbol{R}_1} = \boldsymbol{J}_r^{-1}(\phi)$

Taylor Series with Orientations

Let's assume $f(\boldsymbol{R}) : SO(3) \rightarrow \mathbb{R}^d$. The Taylor expansion looks as follows:

$$f(\boldsymbol{R}) = f(\bar{\boldsymbol{R}} \oplus \delta\boldsymbol{\phi}) \approx f(\bar{\boldsymbol{R}}) + \boldsymbol{J}_{\boldsymbol{R}}^{f(\boldsymbol{R})}\delta\boldsymbol{\phi}$$

where $f(\bar{\boldsymbol{R}}) \in \mathbb{R}^{d\times 1}, \boldsymbol{J}_{\boldsymbol{R}}^{f(\boldsymbol{R})} \in \mathbb{R}^{d\times 3}, \delta\boldsymbol{\phi} = \boldsymbol{R} \ominus \bar{\boldsymbol{R}} \in \mathbb{R}^{3\times 1}$.

Now let's assume $f(\boldsymbol{R}) : SO(3) \rightarrow SO(3)$. The Taylor expansion looks as follows:

$$f(\boldsymbol{R}) = f(\bar{\boldsymbol{R}} \oplus \delta\boldsymbol{\phi}) \approx f(\bar{\boldsymbol{R}}) \oplus \boldsymbol{J}_{\boldsymbol{R}}^{f(\boldsymbol{R})}\delta\boldsymbol{\phi}$$

where $f(\bar{\boldsymbol{R}}) \in SO(3), \boldsymbol{J}_{\boldsymbol{R}}^{f(\boldsymbol{R})} \in \mathbb{R}^{3\times 3}, \delta\boldsymbol{\phi} \in \mathbb{R}^{3\times 1}$.

Taylor Series with Orientations (2)

Finally let's assume $f(\boldsymbol{x}) : \mathbb{R}^d \rightarrow SO(3)$. The Taylor expansion looks as follows:

$$f(\boldsymbol{x}) = f(\bar{\boldsymbol{x}} + \Delta\boldsymbol{x}) \approx f(\bar{\boldsymbol{x}}) \oplus \boldsymbol{J}_{\boldsymbol{x}}^{f(\boldsymbol{x})}\Delta\boldsymbol{x}$$

where $f(\bar{\boldsymbol{x}}) \in SO(3), \boldsymbol{x} \in \mathbb{R}^{d\times 1}, \boldsymbol{J}_{\boldsymbol{x}}^{f(\boldsymbol{x})} \in \mathbb{R}^{3\times d}, \Delta\boldsymbol{x} \in \mathbb{R}^{d\times 1}$.

Taylor Series with Orientations – Examples

- $f(\boldsymbol{R}) : SO(3) \rightarrow \mathbb{R}^3$ case ($\boldsymbol{p} \in \mathbb{R}^3$):

$$\begin{aligned} f(\boldsymbol{R}) &= \boldsymbol{R}\boldsymbol{p} \\ f(\boldsymbol{R}) &= f(\bar{\boldsymbol{R}} \oplus \delta\phi) \approx f(\bar{\boldsymbol{R}}) + \boldsymbol{J}_{\boldsymbol{R}}^{f(\boldsymbol{R})}\delta\phi \\ &= f(\bar{\boldsymbol{R}}) - \boldsymbol{R}\boldsymbol{p}^{\wedge}\delta\phi \end{aligned}$$

- $f(\boldsymbol{R}) : SO(3) \rightarrow SO(3)$ case ($\boldsymbol{R}_x \in SO(3)$):

$$\begin{aligned} f(\boldsymbol{R}) &= \boldsymbol{R}\boldsymbol{R}_x \\ f(\boldsymbol{R}) &= f(\bar{\boldsymbol{R}} \oplus \delta\phi) \approx f(\bar{\boldsymbol{R}}) \oplus \boldsymbol{J}_{\boldsymbol{R}}^{f(\boldsymbol{R})}\delta\phi \\ &= f(\bar{\boldsymbol{R}}) \oplus (\boldsymbol{R}_x^T\delta\phi) \end{aligned}$$

$$\min_{\boldsymbol{R}} f(\boldsymbol{R}), f(\boldsymbol{R}) : SO(3) \to \mathbb{R}$$

repeat until convergence

$$\boldsymbol{g} = \left.\frac{\partial f}{\partial \boldsymbol{R}}\right|^{T}_{\boldsymbol{R}=\boldsymbol{R}_k}$$

$$\boldsymbol{G} = \left.\frac{\partial^2 f}{\partial \boldsymbol{R}^2}\right|_{\boldsymbol{R}=\boldsymbol{R}_k}$$

Regularization of $\boldsymbol{G}$

$$\delta\boldsymbol{\phi} = -\boldsymbol{G}^{-1}\boldsymbol{g}$$

$$\alpha = \text{LineSearch}$$

$$\boldsymbol{R}_{k+1} = \boldsymbol{R}_k \oplus \alpha\delta\boldsymbol{\phi}$$

Almost identical to normal Newton's method!

Optimization with Orientations - Code Example

$$\min_{\boldsymbol{R}} f(\boldsymbol{R}) = \frac{1}{2}\sum_{i=1}^{N}\|\boldsymbol{x}_{wi} - \boldsymbol{R}\boldsymbol{x}_{bi}\|^2 = \frac{1}{2}r(\boldsymbol{R})^T r(\boldsymbol{R})$$

where

$$r(\boldsymbol{R}) = \begin{bmatrix} r_1 = \boldsymbol{x}_{w1} - \boldsymbol{R}\boldsymbol{x}_{b1} \\ r_2 = \boldsymbol{x}_{w2} - \boldsymbol{R}\boldsymbol{x}_{b2} \\ \vdots \\ r_N = \boldsymbol{x}_{wN} - \boldsymbol{R}\boldsymbol{x}_{bN} \end{bmatrix} \in \mathbb{R}^{3N\times 1}$$

$$\boldsymbol{J}^r_{\boldsymbol{R}} = \begin{bmatrix} \boldsymbol{J}^{r_1}_{\boldsymbol{R}} \\ \boldsymbol{J}^{r_2}_{\boldsymbol{R}} \\ \vdots \\ \boldsymbol{J}^{r_N}_{\boldsymbol{R}} \end{bmatrix} \in \mathbb{R}^{3N\times 3}$$

Optimization with Orientations - Code Example (2)

$$\boldsymbol{J}^{r}_{\boldsymbol{R}} = \begin{bmatrix} \boldsymbol{J}^{r_1}_{\boldsymbol{R}} \\ \boldsymbol{J}^{r_2}_{\boldsymbol{R}} \\ \vdots \\ \boldsymbol{J}^{r_N}_{\boldsymbol{R}} \end{bmatrix}$$

$$\begin{aligned} r_i &= \boldsymbol{x}_{wi} - \boldsymbol{R}\boldsymbol{x}_{bi} \\ \boldsymbol{J}^{r_i}_{\boldsymbol{R}} &= -(\boldsymbol{J}^{\boldsymbol{R}\boldsymbol{x}_{bi}}_{\boldsymbol{R}}) = \boldsymbol{R}\boldsymbol{x}^{\wedge}_{bi} \\ \frac{\partial f}{\partial \boldsymbol{R}} = \boldsymbol{J}^{f}_{\boldsymbol{R}} &= \boldsymbol{J}^{\frac{1}{2}r(\boldsymbol{R})^T r(\boldsymbol{R})}_{\boldsymbol{R}} = \frac{1}{2}\Big((\boldsymbol{J}^{r}_{\boldsymbol{R}} r(\boldsymbol{R}))^T + r(\boldsymbol{R})^T \boldsymbol{J}^{r(\boldsymbol{R})}_{\boldsymbol{R}}\Big) \\ &= r(\boldsymbol{R})^T \boldsymbol{J}^{r}_{\boldsymbol{R}} \end{aligned}$$

$$\frac{\partial f}{\partial \boldsymbol{R}} = r(\boldsymbol{R})^T \boldsymbol{J}^r_{\boldsymbol{R}} = {r_1}^T \boldsymbol{J}^{r_1}_{\boldsymbol{R}} + {r_2}^T \boldsymbol{J}^{r_2}_{\boldsymbol{R}} + \cdots + {r_N}^T \boldsymbol{J}^{r_N}_{\boldsymbol{R}} = \sum_{i=1}^{N} {r_i}^T \boldsymbol{J}^{r_i}_{\boldsymbol{R}} \in \mathbb{R}^{1\times 3}$$

$$\frac{\partial^2 f}{\partial \boldsymbol{R}^2} = \left(\boldsymbol{J}^r_{\boldsymbol{R}}\right)^T \boldsymbol{J}^r_{\boldsymbol{R}} + \ldots \text{ we throw away higher order derivatives}$$

We proceed with the Gauss-Newton method as usual with care for updating our estimate of orientation (aka multiplying).

Optimization with Orientations - Code Example (4)

$$\frac{\partial^2 f}{\partial \boldsymbol{R}^2} = \frac{\partial \sum_{i=1}^{N} \left(\boldsymbol{x}_{wi}^T \boldsymbol{R} \boldsymbol{x}_{bi}^{\wedge} - \boldsymbol{x}_{bi}^T \boldsymbol{x}_{bi}^{\wedge} \right)}{\partial \boldsymbol{R}}$$

$$= \sum_{i=1}^{N} \frac{\partial \left(\boldsymbol{x}_{wi}^T \boldsymbol{R} \boldsymbol{x}_{bi}^{\wedge} - \boldsymbol{x}_{bi}^T \boldsymbol{x}_{bi}^{\wedge} \right)}{\partial \boldsymbol{R}}$$

$$= \sum_{i=1}^{N} \frac{\partial \left(\boldsymbol{x}_{wi}^T \boldsymbol{R} \boldsymbol{x}_{bi}^{\wedge} \right)}{\partial \boldsymbol{R}}$$

Left and Right Jacobians of $SE(3)$

Left and Right Jacobians ($\theta = \|\phi\|$):

- $\boldsymbol{J}_l(\boldsymbol{\xi}) = \boldsymbol{J}_l(\phi, \boldsymbol{\rho}) = \begin{bmatrix} \boldsymbol{J}_l(\phi) & \mathbf{0} \\ \boldsymbol{Q}(\phi, \boldsymbol{\rho}) & \boldsymbol{J}_l(\phi) \end{bmatrix}$
- $\boldsymbol{J}_l^{-1}(\boldsymbol{\xi}) = \boldsymbol{J}_l^{-1}(\phi, \boldsymbol{\rho}) = \begin{bmatrix} \boldsymbol{J}_l^{-1}(\phi) & \mathbf{0} \\ -\boldsymbol{J}_l^{-1}(\phi)\boldsymbol{Q}(\phi, \boldsymbol{\rho})\boldsymbol{J}_l^{-1}(\phi) & \boldsymbol{J}_l^{-1}(\phi) \end{bmatrix}$
- $\boldsymbol{J}_r(\boldsymbol{\xi}) = \boldsymbol{J}_r(\phi, \boldsymbol{\rho}) = \boldsymbol{J}_l(-\phi, -\boldsymbol{\rho}) = \boldsymbol{J}_l(-\boldsymbol{\xi})$
- $\boldsymbol{J}_r^{-1}(\boldsymbol{\xi}) = \boldsymbol{J}_r^{-1}(\phi, \boldsymbol{\rho}) = \boldsymbol{J}_l^{-1}(-\phi, -\boldsymbol{\rho}) = \boldsymbol{J}_l^{-1}(-\boldsymbol{\xi})$
- $\boldsymbol{Q}(\phi, \boldsymbol{\rho}) = \frac{1}{2}\boldsymbol{\rho}^\wedge + \frac{\theta - \sin\theta}{\theta^3}(\phi^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\phi^\wedge + \phi^\wedge\boldsymbol{\rho}^\wedge\phi^\wedge) - \frac{1 - \frac{\theta^2}{2} - \cos\theta}{\theta^4}(\phi^\wedge\phi^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\phi^\wedge\phi^\wedge - 3\phi^\wedge\boldsymbol{\rho}^\wedge\phi^\wedge) - \frac{1}{2}\left(\frac{1 - \frac{\theta^2}{2} - \cos\theta}{\theta^4} - \frac{\theta - \sin\theta - \frac{\theta^3}{6}}{\theta^5}\right)\left(\phi^\wedge\boldsymbol{\rho}^\wedge\phi^\wedge\phi^\wedge + \phi^\wedge\phi^\wedge\boldsymbol{\rho}^\wedge\phi^\wedge\right)$

Examples of Jacobians of $SE(3)$

A few *elementary Jacobians:*

- $\boldsymbol{J}_{\boldsymbol{T}}^{\boldsymbol{T}^{-1}} = -\boldsymbol{Ad}_{\boldsymbol{T}} = -\begin{bmatrix} \boldsymbol{R} & 0 \\ \boldsymbol{t}^{\wedge}\boldsymbol{R} & \boldsymbol{R} \end{bmatrix}$
- $\boldsymbol{J}_{\boldsymbol{T}_1}^{\boldsymbol{T}_1\boldsymbol{T}_2} = \boldsymbol{Ad}_{\boldsymbol{T}_2}^{-1}$, $\boldsymbol{J}_{\boldsymbol{T}_2}^{\boldsymbol{T}_1\boldsymbol{T}_2} = \boldsymbol{I}$
- $\boldsymbol{J}_{\boldsymbol{T}}^{\boldsymbol{T}\boldsymbol{p}} = \begin{bmatrix} -\boldsymbol{R}\boldsymbol{p}^{\wedge} & \boldsymbol{R} \end{bmatrix}$, $\boldsymbol{J}_{\boldsymbol{p}}^{\boldsymbol{T}\boldsymbol{p}} = \boldsymbol{R}$
- $\boldsymbol{J}_{\boldsymbol{T}}^{\log(\boldsymbol{T})} = \boldsymbol{J}_r^{-1}(\boldsymbol{\xi})$, where $\boldsymbol{\xi} = log(\boldsymbol{T})$
- $\boldsymbol{J}_{\boldsymbol{T}}^{\boldsymbol{T}\oplus\delta\boldsymbol{\xi}} = \boldsymbol{Ad}_{\exp(\delta\boldsymbol{\xi}^{\wedge})}^{-1}$, $\boldsymbol{J}_{\delta\boldsymbol{\xi}}^{\boldsymbol{T}\oplus\delta\boldsymbol{\xi}} = \boldsymbol{J}_r(\delta\boldsymbol{\xi})$
- Given $\boldsymbol{T} = \boldsymbol{T}_1^T\boldsymbol{T}_2, \boldsymbol{\xi} = \boldsymbol{T}_2 \ominus \boldsymbol{T}_1 = \log \boldsymbol{T}$:
  - $\boldsymbol{J}_{\boldsymbol{T}_1}^{\boldsymbol{T}_2\ominus\boldsymbol{T}_1} = -\boldsymbol{J}_r^{-1}(\boldsymbol{\xi})\boldsymbol{T}^{-1}$
  - $\boldsymbol{J}_{\boldsymbol{T}_2}^{\boldsymbol{T}_2\ominus\boldsymbol{T}_1} = \boldsymbol{J}_r^{-1}(\boldsymbol{\xi})$

LQR for Non-Linear Problems?

- How can we control non linear systems?
- We linearize around a fixed point or trajectory and use LQR!
- **What does it mean we "linearize around a trajectory"?**

$$\bar{\boldsymbol{x}}_{k+1} = f_{\text{discrete}}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k) \qquad \text{trajectory points}$$

LQR for Non-Linear Problems?

- How can we control non linear systems?
- We linearize around a fixed point or trajectory and use LQR!
- **What does it mean we "linearize around a trajectory"?**

$$\bar{\boldsymbol{x}}_{k+1} = f_{\text{discrete}}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k) \qquad \text{trajectory points}$$

- Taylor Series around $\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k$:

$$\boldsymbol{x}_{k+1} = f_{\text{discrete}}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k) + \underbrace{\frac{\partial f_{\text{discrete}}}{\partial \boldsymbol{x}}\bigg|_{\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k}}_{\boldsymbol{A}_k}(\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k) + \underbrace{\frac{\partial f_{\text{discrete}}}{\partial \boldsymbol{u}}\bigg|_{\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k}}_{\boldsymbol{B}_k}(\boldsymbol{u}_k - \bar{\boldsymbol{u}}_k)$$

We can set $\boldsymbol{x}_k = \bar{\boldsymbol{x}}_k + \Delta\boldsymbol{x}_k, \boldsymbol{u}_k = \bar{\boldsymbol{u}}_k + \Delta\boldsymbol{u}_k$ and:

$$\bar{\boldsymbol{x}}_{k+1} + \Delta\boldsymbol{x}_{k+1} = f_{\text{discrete}}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k) + \boldsymbol{A}_k\Delta\boldsymbol{x}_k + \boldsymbol{B}_k\Delta\boldsymbol{u}_k$$
$$\Delta\boldsymbol{x}_{k+1} = \boldsymbol{A}_k\Delta\boldsymbol{x}_k + \boldsymbol{B}_k\Delta\boldsymbol{u}_k$$

- We have a **time-varying linear system**

Linearizing Orientations

Consider a system with state: $\boldsymbol{x} = \begin{bmatrix}\boldsymbol{R} & \boldsymbol{\omega}_b\end{bmatrix}^T$, $\boldsymbol{R} \in SO(3), \boldsymbol{\omega}_b \in \mathbb{R}^{3\times 1}$.
How do we proceed in linearizing around a trajectory?

Linearizing Orientations

Consider a system with state: $\boldsymbol{x} = \begin{bmatrix}\boldsymbol{R} & \boldsymbol{\omega}_b\end{bmatrix}^T$, $\boldsymbol{R} \in SO(3), \boldsymbol{\omega}_b \in \mathbb{R}^{3\times 1}$.
How do we proceed in linearizing around a trajectory?

There are two main differences with respect to normal vectors:

- $\boldsymbol{A}_k = \frac{\partial f_{\text{discrete}}}{\partial \boldsymbol{x}}$ and $\boldsymbol{B}_k = \frac{\partial f_{\text{discrete}}}{\partial \boldsymbol{u}}$ need to use the derivatives of the $SO(3)$
- We write:

$$\begin{aligned}
\boldsymbol{x}_k &= \bar{\boldsymbol{x}}_k \oplus \Delta\boldsymbol{x}_k \Rightarrow \Delta\boldsymbol{x}_k = \boldsymbol{x}_k \ominus \bar{\boldsymbol{x}}_k \\
\boldsymbol{u}_k &= \bar{\boldsymbol{u}}_k + \Delta\boldsymbol{u}_k \\
\Delta\boldsymbol{x}_{k+1} &= \boldsymbol{A}_k\Delta\boldsymbol{x}_k \oplus \boldsymbol{B}_k\Delta\boldsymbol{u}_k
\end{aligned}$$

Linearizing Poses

Consider a system with state: $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{T} & \mathcal{V}_b \end{bmatrix}^T$, $\boldsymbol{T} \in SE(3), \mathcal{V}_b \in \mathbb{R}^{6\times 1}$.
How do we proceed in linearizing around a trajectory?

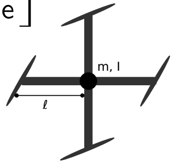
There are two main differences with respect to normal vectors:

- $\boldsymbol{A}_k = \frac{\partial f_{\text{discrete}}}{\partial \boldsymbol{x}}$ and $\boldsymbol{B}_k = \frac{\partial f_{\text{discrete}}}{\partial \boldsymbol{u}}$ need to use the derivatives of the $SE(3)$
- We write:

$$\begin{aligned}
\boldsymbol{x}_k &= \bar{\boldsymbol{x}}_k \oplus \Delta\boldsymbol{x}_k \Rightarrow \Delta\boldsymbol{x}_k = \boldsymbol{x}_k \ominus \bar{\boldsymbol{x}}_k \\
\boldsymbol{u}_k &= \bar{\boldsymbol{u}}_k + \Delta\boldsymbol{u}_k \\
\Delta\boldsymbol{x}_{k+1} &= \boldsymbol{A}_k\Delta\boldsymbol{x}_k \oplus \boldsymbol{B}_k\Delta\boldsymbol{u}_k
\end{aligned}$$

State:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{r}_w \in \mathbb{R}^3, \text{ position in world frame} \\ \boldsymbol{R}_{wb} \in SO(3), \text{ orientation in world frame} \\ \boldsymbol{v}_b \in \mathbb{R}^3, \text{ velocity in body frame} \\ \boldsymbol{\omega}_b \in \mathbb{R}^3, \text{ angular velocity in body frame} \end{bmatrix}$$



Equations of Motion:

$$\dot{\boldsymbol{r}}_w = \boldsymbol{v}_w = \boldsymbol{R}_{wb}\boldsymbol{v}_b$$

$$\dot{\boldsymbol{v}}_b = \boldsymbol{R}_{wb}^T\dot{\boldsymbol{v}}_w - \boldsymbol{\omega}_b \times \boldsymbol{v}_b = \frac{1}{m}\boldsymbol{F}_b - \boldsymbol{\omega}_b \times \boldsymbol{v}_b$$

$$\dot{\boldsymbol{\omega}}_b = \mathcal{I}^{-1}\left(\boldsymbol{\tau}_b - \boldsymbol{\omega}_b \times \mathcal{I}\boldsymbol{\omega}_b\right)$$

Integrate orientation with: $\boldsymbol{R}'_{wb} = \boldsymbol{R}_{wb} \oplus \boldsymbol{\omega}_b dt$

m is the mass of the body, and $\mathcal{I}$ is the **Inertia matrix**.

3D Quadrotor (2)

Each motor i generates one "force" and some "torque":

$$F_i = K_f u_i$$
$$\tau_i = K_\tau u_i$$

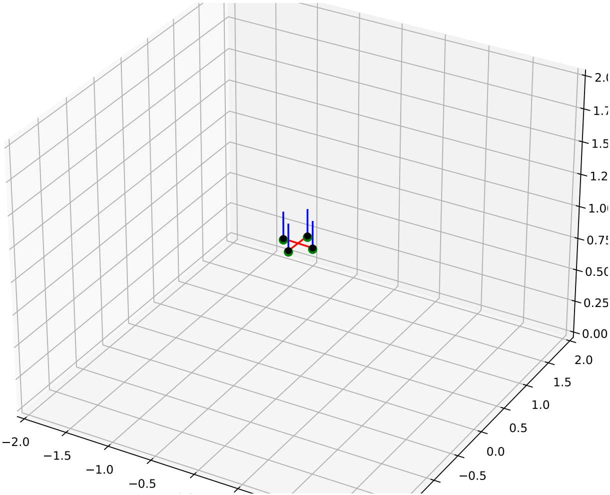
where $\boldsymbol{u} \in \mathbb{R}^4$ are the motor commands. So we have:

$$\boldsymbol{F}_b = \boldsymbol{R}_{wb}^T \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ K_f & K_f & K_f & K_f \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{\tau}_b = \begin{bmatrix} \ell K_f (u_2 - u_4) \\ \ell K_f (u_3 - u_1) \\ K_\tau (u_1 - u_2 + u_3 - u_4) \end{bmatrix}$$

ℓ is the length of the quadrotor and g is the gravity.

LQR with 3D Quadrotor - Code Example



Thank you

- **Any Questions?**
- **Office Hours**:
  - **Tue & Thu** *(09:00-11:00)*
  - 24/7 by email (`costashatz@upatras.gr`, subject: *ECE_RSII_AM*)
- **Material and Announcements**

LAR

Laboratory of Automation & Robotics

Laboratory of Automation & Robotics