ΠΑΝΕΠΙΣΤΗΜΙΟ ΠΑΤΡΩΝ
UNIVERSITY OF PATRAS

Robotic Systems II

Lecture 6: Matrix Lie Groups and 3D Poses

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How can we represent orientations?

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- Rotation Matrices

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- Rotation Matrices
- Euler Angles
- Axis-Angle
- Quaternions

What is orientation?

The transformation that transforms a vector/point, $\boldsymbol{v}_b$, expressed in "body frame" to the same vector/point, but expressed in the "fixed/world frame" orientation, $\boldsymbol{v}_w$.

$$\boldsymbol{v}_w = \boldsymbol{R}_{wb}\boldsymbol{v}_b$$

where $\boldsymbol{R_{wb}} \in \mathcal{SO}(3)$ is a **rotation matrix** that rotates the "fixed/world frame" such that it matches the "body frame". We use this rotation (transformation) matrix to rotate a vector from "body frame" to the "world frame".

Frames and Rotation Matrices

A frame is basically a set of 3 unit-norm vectors, that are orthogonal to each other:

$$\boldsymbol{w}_x \cdot \boldsymbol{w}_y = 0$$
$$\boldsymbol{w}_x \cdot \boldsymbol{w}_z = 0$$
$$\boldsymbol{w}_y \cdot \boldsymbol{w}_z = 0$$

where $\mathcal{W} = (\boldsymbol{w}_x, \boldsymbol{w}_y, \boldsymbol{w}_z)$ is the world frame. If we also define a body frame as $\mathcal{B}$, then we have:

$$\boldsymbol{R}_{wb} = \begin{bmatrix} (\boldsymbol{w}_x \cdot \boldsymbol{b}_x) & (\boldsymbol{w}_x \cdot \boldsymbol{b}_y) & (\boldsymbol{w}_x \cdot \boldsymbol{b}_z) \\ (\boldsymbol{w}_y \cdot \boldsymbol{b}_x) & (\boldsymbol{w}_y \cdot \boldsymbol{b}_y) & (\boldsymbol{w}_y \cdot \boldsymbol{b}_z) \\ (\boldsymbol{w}_z \cdot \boldsymbol{b}_x) & (\boldsymbol{w}_z \cdot \boldsymbol{b}_y) & (\boldsymbol{w}_z \cdot \boldsymbol{b}_z) \end{bmatrix}$$

We refer to this version of derivation of the rotation matrix as the **Direction Cosine Method/Matrix** (DCM).

Rotation Matrices

- If $\boldsymbol{R}_1, \boldsymbol{R}_2$ are rotation matrices, then $\boldsymbol{R}_1\boldsymbol{R}_2$ is also a rotation matrix
- $(\boldsymbol{R}_1\boldsymbol{R}_2)\boldsymbol{R}_3 = \boldsymbol{R}_1(\boldsymbol{R}_2\boldsymbol{R}_3)$
- $\boldsymbol{R}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{R} = \boldsymbol{R}$ ($\boldsymbol{I}$ is the identity matrix)
- $\boldsymbol{R}^{-1}\boldsymbol{R} = \boldsymbol{I}$
- $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$, aka $\boldsymbol{R}_{wb}^T = \boldsymbol{R}_{bw}$
- $\det(\boldsymbol{R}) = 1$
- $\boldsymbol{R}_{az} = \boldsymbol{R}_{ab}\boldsymbol{R}_{bc}\boldsymbol{R}_{cd}\ldots\boldsymbol{R}_{yz}$

Axis-Angle:

$$\boldsymbol{\theta} = \boldsymbol{r}\theta$$

where $\boldsymbol{r} \in \mathbb{R}^3$ is the "axis of rotation" unit vector, and $\theta \in \mathbb{R}$ is the rotation angle.

- **Singularity** at $\theta = \pm\pi$
- $\boldsymbol{\theta}_{wb} = (\log \boldsymbol{R}_{wb})^\vee$
- $\boldsymbol{R}_{wb} = \exp(\boldsymbol{\theta}_{wb}^\wedge)$

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- **Singularity** at $\theta = \pm\pi$
- $\boldsymbol{\theta}_{wb} = (\log \boldsymbol{R}_{wb})^{\vee}$
- $\boldsymbol{R}_{wb} = \exp(\boldsymbol{\theta}_{wb}^{\wedge})$
- We can compute the above more efficiently than regular exp/log matrix operations

Euler Angles:

$$\begin{aligned} \boldsymbol{R} &= \boldsymbol{R}_1 \boldsymbol{R}_2 \boldsymbol{R}_3 \\ &= \underbrace{\boldsymbol{R}_{\alpha_1}(\theta_1)}_{\boldsymbol{R}_1} \underbrace{\boldsymbol{R}_{\alpha_2}(\theta_2)}_{\boldsymbol{R}_2} \underbrace{\boldsymbol{R}_{\alpha_3}(\theta_3)}_{\boldsymbol{R}_3} \end{aligned}$$

where α_i are the axes of rotation and θ_i the corresponding angles of rotation.

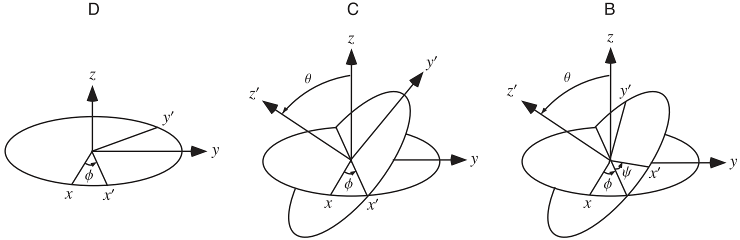

Figure: Taken from WolframAlpha

Euler Angles (2)

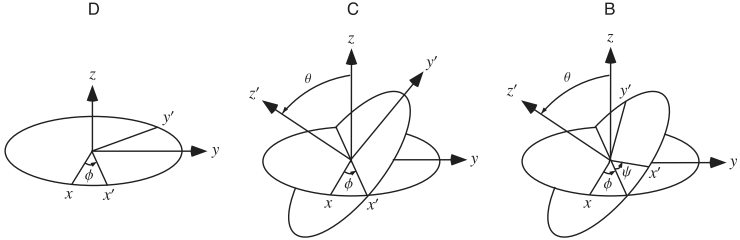

Figure: Taken from WolframAlpha

- There are many conventions: *ZYX* (yaw-pitch-roll) and *XYZ* (roll-pitch-yaw) the most popular ones
  - $\theta_1 \in [-\pi, \pi], \theta_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}], \theta_3 \in [-\pi, \pi]$
  - **Singularity** at $\theta = \pm\frac{\pi}{2}$
- We can convert from/to rotation matrices

Axis-Angle:

$$\begin{aligned}
\boldsymbol{\theta} &= \boldsymbol{r}\theta \\
\boldsymbol{R} &= \exp(\boldsymbol{\theta}_{wb}^{\wedge}) \\
&= \boldsymbol{I} + \sin\theta\hat{\boldsymbol{r}} + (1-\cos\theta)\hat{\boldsymbol{r}}^2, \ \textit{(Rodrigues formula)} \Rightarrow \\
\dot{\boldsymbol{R}} &= \dot{\theta}\cos\theta\hat{\boldsymbol{r}} + \sin\theta\dot{\hat{\boldsymbol{r}}} + \dot{\theta}\sin\theta\hat{\boldsymbol{r}}^2 + (1-\cos\theta)(\dot{\hat{\boldsymbol{r}}}\hat{\boldsymbol{r}} + \hat{\boldsymbol{r}}\dot{\hat{\boldsymbol{r}}}) \Rightarrow \\
&\dots \\
&\Rightarrow \boldsymbol{\omega}_w = \dot{\theta}\boldsymbol{r} + \sin\theta\dot{\boldsymbol{r}} + \hat{\boldsymbol{r}}\dot{\boldsymbol{r}}(1-\cos\theta) \\
&\Rightarrow \boldsymbol{\omega}_b = \dot{\theta}\boldsymbol{r} + \sin\theta\dot{\boldsymbol{r}} - \hat{\boldsymbol{r}}\dot{\boldsymbol{r}}(1-\cos\theta)
\end{aligned}$$

Euler Angles and Angular Velocity

ZYX Euler Angles:

$$\boldsymbol{\omega}_b = \begin{bmatrix} -\sin\theta_2 & 0 & 1 \\ \cos\theta_2\sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_3\cos\theta_2 & -sin\theta_3 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$\boldsymbol{\omega}_w = \begin{bmatrix} 0 & -\sin\theta_1 & \cos\theta_2\cos\theta_1 \\ 0 & \cos\theta_1 & \cos\theta_2\sin\theta_1 \\ 1 & 0 & -\sin\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

XYZ Euler Angles:

$$\vdots$$

$$\vdots$$

Quaternions

Let's start with the **Axis-Angle:**

$$\boldsymbol{\theta} = \boldsymbol{r}\theta$$

A **unit quaternion** describing the same orientation/rotation is given by:

$$\boldsymbol{q} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \boldsymbol{r}\sin\frac{\theta}{2} \end{bmatrix} \in \mathbb{R}^4$$

- $\cos\frac{\theta}{2}$ is the **scalar** part, and $\boldsymbol{r}\sin\frac{\theta}{2}$ is the **vector** part
- For $\boldsymbol{q}$ to be a valid orientation/rotation, we also need $\|\boldsymbol{q}\| = 1$
- $\boldsymbol{q}$ and $-\boldsymbol{q}$ describe the **same orientation**, but *different rotations!* (double cover)

Quaternion Math

- $\boldsymbol{q}_{wb}$ is a rotation from "body frame" to "world frame"
- $\boldsymbol{q}_{wa} = \boldsymbol{q}_{wb} \odot \boldsymbol{q}_{ba}$

$$\boldsymbol{q}_{wa} = \begin{bmatrix} s_{wb} \\ \boldsymbol{v}_{wb} \end{bmatrix} \odot \begin{bmatrix} s_{ba} \\ \boldsymbol{v}_{ba} \end{bmatrix} = \begin{bmatrix} s_{wb}s_{ba} - \boldsymbol{v}_{wb}^T \boldsymbol{v}_{ba} \\ s_{wb}\boldsymbol{v}_{ba} + s_{ba}\boldsymbol{v}_{wb} + \boldsymbol{v}_{wb} \times \boldsymbol{v}_{ba} \end{bmatrix}$$

- **Quaternion Conjugate** (Inverse):

$$\boldsymbol{q}_{wb}^{\dagger} = \begin{bmatrix} s_{wb} \\ -\boldsymbol{v}_{wb} \end{bmatrix}$$

- $\boldsymbol{q}_{wb}^{\dagger} \odot \boldsymbol{q}_{wb} = \boldsymbol{q}_0$, $\boldsymbol{q}_0 = \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix}$
- $\begin{bmatrix} 0 \\ \boldsymbol{x}_w \end{bmatrix} = \boldsymbol{q}_{wb} \odot \begin{bmatrix} 0 \\ \boldsymbol{x}_b \end{bmatrix} \odot \boldsymbol{q}_{wb}^{\dagger}$, *(rotating a vector/point)*

Quaternions and Angular Velocity

Time derivative of Quaternion:

$$
\begin{aligned}
\dot{\boldsymbol{q}} &= \frac{1}{2}\boldsymbol{q} \odot \begin{bmatrix} 0 \\ \boldsymbol{\omega}_b \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 0 \\ \boldsymbol{\omega}_w \end{bmatrix} \odot \boldsymbol{q}
\end{aligned}
$$

- We can now integrate quaternions

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- We can now integrate quaternions
- **Beware!** Integrating naively with $\dot{\boldsymbol{q}}$ (e.g. with Euler) results in a non unit-norm quaternion!

Quaternions and Angular Velocity

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- We can now integrate quaternions
- **Beware!** Integrating naively with $\dot{\boldsymbol{q}}$ (e.g. with Euler) results in a non unit-norm quaternion!
- **Lots of "custom" functions to use the quaternions!**

$$\boldsymbol{q}_{wa} = \begin{bmatrix} s_{wb} \\ \boldsymbol{v}_{wb} \end{bmatrix} \odot \begin{bmatrix} s_{ba} \\ \boldsymbol{v}_{ba} \end{bmatrix} = \begin{bmatrix} s_{wb}s_{ba} - \boldsymbol{v}_{wb}^T\boldsymbol{v}_{ba} \\ s_{wb}\boldsymbol{v}_{ba} + s_{ba}\boldsymbol{v}_{wb} + \boldsymbol{v}_{wb} \times \boldsymbol{v}_{ba} \end{bmatrix}$$

How can we write the above such that we do it via matrix multiplication?

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$$\boldsymbol{q}_{wa} = \begin{bmatrix} s_{wb} \\ \boldsymbol{v}_{wb} \end{bmatrix} \odot \begin{bmatrix} s_{ba} \\ \boldsymbol{v}_{ba} \end{bmatrix} = \underbrace{\begin{bmatrix} s_{wb} & -\boldsymbol{v}_{wb}^T \\ \boldsymbol{v}_{wb} & s_{wb}\boldsymbol{I} + \hat{\boldsymbol{v}}_{wb} \end{bmatrix}}_{L(\boldsymbol{q}_{wb})} \begin{bmatrix} s_{ba} \\ \boldsymbol{v}_{ba} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} s_{ba} & -\boldsymbol{v}_{ba}^T \\ \boldsymbol{v}_{ba} & s_{ba}\boldsymbol{I} - \hat{\boldsymbol{v}}_{ba} \end{bmatrix}}_{R(\boldsymbol{q}_{ba})} \begin{bmatrix} s_{wb} \\ \boldsymbol{v}_{wb} \end{bmatrix}$$

Quaternion Magic (2)

A few interesting relations:

$$\begin{aligned} \boldsymbol{q}_{wb} \odot \boldsymbol{q}_{ba} &= L(\boldsymbol{q}_{wb})\boldsymbol{q}_{ba} \\ &= R(\boldsymbol{q}_{ba})\boldsymbol{q}_{wb} \end{aligned}$$

$$L(\boldsymbol{q}^\dagger) = L(\boldsymbol{q})^T = L(\boldsymbol{q})^{-1}$$
$$R(\boldsymbol{q}^\dagger) = R(\boldsymbol{q})^T = R(\boldsymbol{q})^{-1}$$

We can use the same logic in all quaternion operations:

- $\boldsymbol{q}^{\dagger} = \begin{bmatrix} s \\ -\boldsymbol{v} \end{bmatrix} = \boldsymbol{T}\boldsymbol{q} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -\boldsymbol{I} \end{bmatrix}}_{\boldsymbol{T}} \begin{bmatrix} s \\ \boldsymbol{v} \end{bmatrix}$
- Rotate a vector/point:

$$\begin{bmatrix} 0 \\ \boldsymbol{x}_w \end{bmatrix} = \boldsymbol{q}_{wb} \odot \begin{bmatrix} 0 \\ \boldsymbol{x}_b \end{bmatrix} \odot \boldsymbol{q}_{wb}^{\dagger}$$

$$\underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{H} \in \mathbb{R}^{4\times 3}} \boldsymbol{x}_w = \boldsymbol{q}_{wb} \odot \boldsymbol{H}\boldsymbol{x}_b \odot \boldsymbol{q}_{wb}^{\dagger}$$

$$\boldsymbol{H}\boldsymbol{x}_w = L(\boldsymbol{q}_{wb})(R(\boldsymbol{q}_{wb})^T \boldsymbol{H}\boldsymbol{x}_b)$$

$$\boldsymbol{x}_w = \underbrace{\boldsymbol{H}^T L(\boldsymbol{q}_{wb}) R(\boldsymbol{q}_{wb})^T \boldsymbol{H}}_{\boldsymbol{R}_{wb}} \boldsymbol{x}_b$$

Quaternion Magic (4)

We can use the same logic in all quaternion operations:

$$\vdots$$

$$\begin{aligned}\dot{\boldsymbol{q}} &= \frac{1}{2}\boldsymbol{q}\odot\begin{bmatrix}0\\ \boldsymbol{\omega}_b\end{bmatrix}\\ &= \frac{1}{2}L(\boldsymbol{q})\boldsymbol{H}\boldsymbol{\omega}_b\end{aligned}$$

Orientation from a Geometric Perspective

- The **Special Orthogonal Group**, representing orientations and rotations, is simply the set of valid rotation matrices:

$$SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I},\ det(\boldsymbol{R}) = 1\}$$

- The set of all 3×3 matrices is a vectorspace. What does it mean?

Orientation from a Geometric Perspective

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- The set of all 3×3 matrices is a vectorspace. What does it mean?

- $SO(3)$ is not:
  1. The zero matrix $\mathbf{0}$ is not part of the space!
  2. $\boldsymbol{R}_1, \boldsymbol{R}_2 \in SO(3) \not\Rightarrow \boldsymbol{R}_1 + \boldsymbol{R}_2 \in SO(3)$

Groups

What is a Group? A set non-empty $\mathcal{G}$ along with a binary operator $\circ$ that satisfies four properties:

1. **Closure:** $\forall \mathcal{X}, \mathcal{Y} \in \mathcal{G}, \mathcal{X} \circ \mathcal{Y} \in \mathcal{G}$
2. **Associativity:** $\forall \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{G}, (\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z})$
3. **Identity:** $\exists \varepsilon \in \mathcal{G}, \forall \mathcal{X} \in \mathcal{G}, \varepsilon \circ \mathcal{X} = \mathcal{X} \circ \varepsilon = \mathcal{X}$
4. **Inverse:** $\forall \mathcal{X} \in \mathcal{G}, \exists \mathcal{X}^{-1} \in \mathcal{G}, \mathcal{X} \circ \mathcal{X}^{-1} = \mathcal{X}^{-1} \circ \mathcal{X} = \varepsilon$

The operator $\circ$ is not necessarily commutative!

What is a Manifold? A Manifold, $\mathcal{M}$, is a topological space that locally resembles Euclidean space, $\mathbb{R}^n$, near each point $\mathcal{X}$:

- The neighborhood of $\mathcal{X}$ lies on the tangent space $T_{\mathcal{X}}\mathcal{M}$ of the manifold $\mathcal{M}$ at point $\mathcal{X}$
- The neighborhood of $\mathcal{X}$ and the corresponding tangent space are homeomorphic to an open subset of $\mathbb{R}^n$: $T_{\mathcal{X}}\mathcal{M} \cong \mathbb{R}^n$
- A smooth manifold is infinitely differentiable at each point

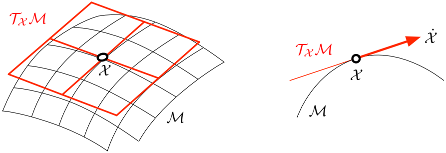


Figure: *Sola, Joan, Jeremie Deray, and Dinesh Atchuthan. "A micro lie theory for state estimation in robotics." arXiv:1812.01537 (2018).*

Lie Groups

What is a Lie Group? A Lie Group is a group that is also a smooth manifold:

- The group operations are smooth: $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \circ \mathcal{Y}$
- The inverse operation is smooth: $\mathcal{G} \to \mathcal{G}$, $\mathcal{X} \mapsto \mathcal{X}^{-1}$
- The identity element is smooth: $\mathcal{G} \to \mathcal{G}$, $\mathcal{X} \mapsto \varepsilon$

Why are Lie Groups important? We can effectively model rotations and full poses.

$SO(3)$ is a Matrix Lie Group

$SO(3)$ forms a *matrix Lie group* ($\circ$ is the regular matrix multiplication, ε is the identity matrix):

- $\boldsymbol{R}_1, \boldsymbol{R}_2 \in SO(3) \Rightarrow \boldsymbol{R}_1\boldsymbol{R}_2 \in SO(3)$
- $(\boldsymbol{R}_1\boldsymbol{R}_2)\boldsymbol{R}_3 = \boldsymbol{R}_1(\boldsymbol{R}_2\boldsymbol{R}_3) = \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_3$
- $\boldsymbol{R}, \boldsymbol{I} \in SO(3)$ and $\boldsymbol{R}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{R} = \boldsymbol{R}$
- $\boldsymbol{R}^{-1} = \boldsymbol{R}^T \in SO(3)$

Lie Group Actions

Given a lie group $\mathcal{M}$ and a set $\mathcal{V}$, a *Lie group action*, $\cdot$, is a map $\cdot : \mathcal{M} \times \mathcal{V} \rightarrow \mathcal{V}$ that satisfies:

1. **Identity:** $\varepsilon \cdot \boldsymbol{v} = \boldsymbol{v}$
2. **Compatibility:** $\mathcal{X} \cdot (\mathcal{Y} \cdot \boldsymbol{v}) = (\mathcal{X} \circ \mathcal{Y}) \cdot \boldsymbol{v}$

where $\mathcal{X}, \mathcal{Y} \in \mathcal{M}$ and $\boldsymbol{v} \in \mathcal{V}$

Example: The rotation of a vector $\boldsymbol{v}$ by a rotation matrix $\boldsymbol{R}$:

$$\boldsymbol{R} \cdot \boldsymbol{v} = \boldsymbol{R}\boldsymbol{v}$$

Tangent Spaces of a Lie Group

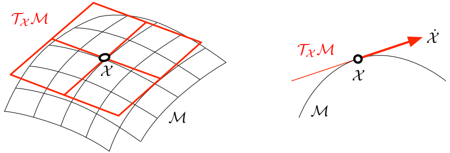


Figure: *Sola, Joan, Jeremie Deray, and Dinesh Atchuthan. "A micro lie theory for state estimation in robotics." arXiv:1812.01537 (2018).*

Given a lie group $\mathcal{M}$ and a point $\mathcal{X} \in \mathcal{M}$, the *tangent space* at $\mathcal{X}$ is written as $T_{\mathcal{X}}\mathcal{M}$.

- Let's assume a trajectory $\mathcal{Y}(t)$ in $\mathcal{M}$ that passes through the point $\mathcal{X}$ at some t_0 (from now on, $t_0 = 0$ for simplicity)
- $\dot{\mathcal{Y}}(t_0) = \dot{\mathcal{X}}(t_0)$ must be in the tangent space $T_{\mathcal{X}}\mathcal{M}$
- The tangent space is constructed by the velocities $\dot{\mathcal{Y}}(t_0)$ of all possible trajectories $\mathcal{Y}(t)$ that pass through $\mathcal{X}$ at $t = t_0$
- The tangent space is a vectorspace, and the elements of $T_{\mathcal{X}}\mathcal{M}$ are called *tangent vectors*

Lie Algebra

The tangent space at the identity element ε is called the *Lie algebra* of the Lie group. A *Lie algebra* consists of a vectorspace, $\mathbb{V}$, over some field, $\mathbb{F}$, together with a binary operation, $[\cdot,\cdot]$, called the *Lie bracket* (of the algebra) and that satisfies four properties:

- closure: $[\boldsymbol{X},\boldsymbol{Y}] \in \mathbb{V}$
- bilinearity:

$$[a\boldsymbol{X} + b\boldsymbol{Y}, \boldsymbol{Z}] = a[\boldsymbol{X},\boldsymbol{Y}] + b[\boldsymbol{Y},\boldsymbol{Z}]$$
$$[\boldsymbol{Z}, a\boldsymbol{X} + b\boldsymbol{Y}] = a[\boldsymbol{Z},\boldsymbol{X}] + b[\boldsymbol{Z},\boldsymbol{Y}]$$

- alternating: $[\boldsymbol{X},\boldsymbol{X}] = \mathbf{0}$
- Jacobi identity: $[\boldsymbol{X},[\boldsymbol{Y},\boldsymbol{Z}]] + [\boldsymbol{Y},[\boldsymbol{Z},\boldsymbol{X}]] + [\boldsymbol{Z},[\boldsymbol{X},\boldsymbol{Y}]] = \mathbf{0}$

for all $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z} \in \mathbb{V}$ and $a, b \in \mathbb{F}$.

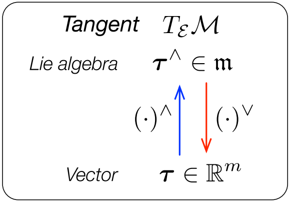
Lie Algebra Representations

There exist two isomorphic representations of the tangent space:

- The Lie algebra $\mathfrak{m} = T_{\mathcal{X}}\mathcal{M}$ ($\mathfrak{m}$ has m dimensions)
- The corresponding Cartesian space $\mathbb{R}^m$: $\mathfrak{m} \cong \mathbb{R}^m$
- We can go from one representation to the other using two mutually inverse linear maps called "hat" and "vee":

$$\boldsymbol{\tau}^{\wedge} \in \mathfrak{m} \text{ “hat”}$$

$$\boldsymbol{\tau} = (\boldsymbol{\tau}^{\wedge})^{\vee} \in \mathbb{R}^m \text{ “vee”}$$



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$SO(3)$'s Lie Algebra

$SO(3)$'s Lie Algebra:

$$\begin{aligned} \textbf{vectorspace:}\ & \mathfrak{so}(3) = \{\mathbf{\Phi} = \phi^\wedge | \phi \in \mathbb{R}^3\} \\ \textbf{field:}\ & \mathbb{R} \\ \textbf{Lie bracket:}\ & [\mathbf{\Phi}_1, \mathbf{\Phi}_2] = \mathbf{\Phi}_1\mathbf{\Phi}_2 - \mathbf{\Phi}_2\mathbf{\Phi}_1 \end{aligned}$$

where $\phi^\wedge$ is the skew symmetric matrix operator:

$$\phi^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

We also use $\phi^\vee$ as the inverse operator:

$$\mathbf{\Phi} = \phi^\wedge \Rightarrow \phi = \mathbf{\Phi}^\vee$$

Lie Algebra structure

In Lie Groups, we can get the structure of the tangent space (or the Lie algebra) by differentiating the inverse group constraint. For groups with multiplicative binary operator, we get:

$$\boldsymbol{\tau}^{\wedge} = \mathcal{X}^{-1}\dot{\mathcal{X}} = -\dot{\mathcal{X}^{-1}}\mathcal{X}$$

For $SO(3)$ specifically, we get:

$$\phi^{\wedge} = \boldsymbol{R}^{T}\dot{\boldsymbol{R}} = -\dot{\boldsymbol{R}^{T}}\boldsymbol{R}$$

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What does this reminds us of?

$$\boldsymbol{\omega}_w^\wedge = \dot{\boldsymbol{R}}_{wb}\boldsymbol{R}_{wb}^T$$
$$\boldsymbol{\omega}_b^\wedge = \boldsymbol{R}_{wb}^T\dot{\boldsymbol{R}}_{wb}$$

!!!!

Relation between Lie Algebra and the Manifold

For groups with multiplicative group binary operator:

$$\boldsymbol{v}^{\wedge} = \mathcal{X}^{-1}\dot{\mathcal{X}}$$
$$\dot{\mathcal{X}} = \mathcal{X}\boldsymbol{v}^{\wedge}$$

This is a linear differential equation. We can solve it:

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$$\mathcal{X}(t) = \mathcal{X}(0)\exp(\boldsymbol{v}^{\wedge} t)$$

But $\mathcal{X}(0) = \varepsilon = \boldsymbol{I}$! Why?

Relation between Lie Algebra and the Manifold

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$$\mathcal{X} = \exp(\boldsymbol{\tau}^\wedge)$$

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But $\mathcal{X}(0) = \varepsilon = \boldsymbol{I}$! Why? Setting $\boldsymbol{\tau}^{\wedge} = \boldsymbol{v}^{\wedge}t$, we get:

$$\mathcal{X} = \exp(\boldsymbol{\tau}^{\wedge})$$

Now, we can go back to the manifold using the exponential map!
We can do the inverse operation using the logarithm map:

$$\boldsymbol{\tau}^{\wedge} = \log(\mathcal{X})$$

Relation between $SO(3)$ and $\mathfrak{so}(3)$

- How can we go from $\mathfrak{so}(3)$ to $SO(3)$?

Relation between $SO(3)$ and $\mathfrak{so}(3)$

- How can we go from $\mathfrak{so}(3)$ to $SO(3)$?
- $\boldsymbol{R} = \exp(\phi^\wedge)$, exp is the matrix exponential!
- This is a many to one relationship: aka we can generate the same $\boldsymbol{R}$ from multiple ϕ
- And back?

Relation between $SO(3)$ and $\mathfrak{so}(3)$

- How can we go from $\mathfrak{so}(3)$ to $SO(3)$?
- $\boldsymbol{R} = \exp(\phi^\wedge)$, exp is the matrix exponential!
- This is a many to one relationship: aka we can generate the same $\boldsymbol{R}$ from multiple ϕ
- And back?
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- What does this reminds us of?
- **Angle-Axis!**
- We can compute the above more efficiently than regular exp/log matrix operations

Exp/Log Map for Rotations

We can compute the exp/log operations efficiently:

$$
\begin{aligned}
\exp(\boldsymbol{\phi}^\wedge) = \exp(\phi \boldsymbol{a}^\wedge) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{\phi}^\wedge)^n \\
&= \mathbf{1} + \phi \boldsymbol{a}^\wedge + \frac{1}{2!}\phi^2 \boldsymbol{a}^\wedge \boldsymbol{a}^\wedge + \cdots \\
&= \cdots \\
&= \boldsymbol{I}\cos\phi + (1-\cos\phi)\boldsymbol{a}\boldsymbol{a}^T + \sin\phi \boldsymbol{a}^\wedge \\
&= \boldsymbol{I} + \sin\phi \boldsymbol{a}^\wedge + (1-\cos\phi)\boldsymbol{a}^\wedge \boldsymbol{a}^\wedge \ \text{(Rodrigues formula)} \\
&= \boldsymbol{R}
\end{aligned}
$$

Similarly:[1]

$$
\begin{aligned}
\log(\boldsymbol{R})^\vee &= \boldsymbol{\phi} = \phi \boldsymbol{a} \\
\boldsymbol{a}^\wedge &= \frac{1}{2\sin\phi}(\boldsymbol{R} - \boldsymbol{R}^T), \phi = \cos^{-1}\left(\frac{\mathrm{tr}(\boldsymbol{R}) - 1}{2}\right)
\end{aligned}
$$

[1] We also need to take care of some special cases!

Interesting Properties of the exp Operator

- $\exp((a+b)\boldsymbol{\tau}^\wedge) = \exp(a\boldsymbol{\tau}^\wedge)\exp(b\boldsymbol{\tau}^\wedge)$

- $\exp(a\boldsymbol{\tau}^\wedge) = \exp(\boldsymbol{\tau}^\wedge)^a$

- $\exp(-\boldsymbol{\tau}^\wedge) = \exp(\boldsymbol{\tau}^\wedge)^{-1}$

- $\exp(\mathcal{X}\boldsymbol{\tau}^\wedge\mathcal{X}^{-1}) = \mathcal{X}\exp(\boldsymbol{\tau}^\wedge)\mathcal{X}^{-1}$

We also usually write $\operatorname{Exp}(\boldsymbol{\tau}) \triangleq \exp(\boldsymbol{\tau}^\wedge)$ and $\operatorname{Log}(\mathcal{X}) \triangleq \log(\mathcal{X})^\vee$ to simplify notation.

Right (local) operators

$$\mathcal{Y} = \mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau} = \mathcal{X} \circ \mathrm{Exp}({}^{\mathcal{X}}\boldsymbol{\tau}) \in \mathcal{M}$$
$${}^{\mathcal{X}}\boldsymbol{\tau} = \mathcal{Y} \ominus \mathcal{X} = \mathrm{Log}(\mathcal{X}^{-1} \circ \mathcal{Y}) \in T_{\mathcal{X}}\mathcal{M}$$

Left (global) operators

$$\mathcal{Y} = {}^{\varepsilon}\boldsymbol{\tau} \oplus \mathcal{X} = \mathrm{Exp}({}^{\varepsilon}\boldsymbol{\tau}) \circ \mathcal{X} \in \mathcal{M}$$
$${}^{\varepsilon}\boldsymbol{\tau} = \mathcal{Y} \ominus \mathcal{X} = \mathrm{Log}(\mathcal{Y} \circ \mathcal{X}^{-1}) \in T_{\varepsilon}\mathcal{M}$$

Adjoint Operator

We have:

$$\mathcal{Y} = \mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau}$$
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$$\begin{aligned}
\mathrm{Exp}({}^{\varepsilon}\boldsymbol{\tau}) \circ \mathcal{X} &= \mathcal{X} \circ \mathrm{Exp}({}^{\mathcal{X}}\boldsymbol{\tau}) \\
\exp({}^{\varepsilon}\boldsymbol{\tau}^{\wedge}) &= \mathcal{X} \exp({}^{\mathcal{X}}\boldsymbol{\tau}^{\wedge})\mathcal{X}^{-1} = \exp(\mathcal{X}{}^{\mathcal{X}}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}) \\
{}^{\varepsilon}\boldsymbol{\tau}^{\wedge} &= \mathcal{X}{}^{\mathcal{X}}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}
\end{aligned}$$

We now define the **adjoint** ($\mathrm{Ad}_{\mathcal{X}} : \mathfrak{m} \to \mathfrak{m}$):

$$\mathrm{Ad}_{\mathcal{X}}(\boldsymbol{\tau}^{\wedge}) = \mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1}$$

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Some interesting properties:

$$\begin{aligned}
\mathrm{Ad}_{\mathcal{X}}(a\boldsymbol{\tau}^{\wedge} + b\boldsymbol{\sigma}^{\wedge}) &= a\mathrm{Ad}_{\mathcal{X}}(\boldsymbol{\tau}^{\wedge}) + b\mathrm{Ad}_{\mathcal{X}}(\boldsymbol{\sigma}^{\wedge}) \\
\mathrm{Ad}_{\mathcal{X}}(\mathrm{Ad}_{\mathcal{Y}}(\boldsymbol{\tau}^{\wedge})) &= \mathrm{Ad}_{\mathcal{X}\mathcal{Y}}(\boldsymbol{\tau}^{\wedge})
\end{aligned}$$

Adjoint Matrix

The Adjoint matrix $\mathbf{Ad}_{\mathcal{X}} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined as:

$$
\begin{aligned}
{}^{\varepsilon}\boldsymbol{\tau} &= \mathbf{Ad}_{\mathcal{X}}{}^{\mathcal{X}}\boldsymbol{\tau} \\
\mathbf{Ad}_{\mathcal{X}}\boldsymbol{\tau} &= (\mathcal{X}\boldsymbol{\tau}^{\wedge}\mathcal{X}^{-1})^{\vee}
\end{aligned}
$$

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Some interesting properties:

$$
\begin{aligned}
\mathcal{X} \oplus \boldsymbol{\tau} &= (\mathbf{Ad}_{\mathcal{X}}\boldsymbol{\tau}) \oplus \mathcal{X} \\
\mathbf{Ad}_{\mathcal{X}}^{-1} &= \mathbf{Ad}_{\mathcal{X}^{-1}} \\
\mathbf{Ad}_{\mathcal{X}\mathcal{Y}} &= \mathbf{Ad}_{\mathcal{X}}\mathbf{Ad}_{\mathcal{Y}}
\end{aligned}
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\end{aligned}
$$

For $SO(3)$, we have:

$$
\mathbf{Ad}_{\boldsymbol{R}} = \boldsymbol{R}
$$

Orientation Errors

In other words, I am at $\boldsymbol{R}_{wc}$ and I want to end up at $\boldsymbol{R}_{wt}$. What is the orientation error?

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$$
\begin{aligned}
\boldsymbol{R}_{ct} &= \boldsymbol{R}_{tc}^T \\
&= (\boldsymbol{R}_{tw}\boldsymbol{R}_{wc})^T \\
&= (\boldsymbol{R}_{wt}^T\boldsymbol{R}_{wc})^T \\
\Rightarrow \boldsymbol{R}_{ct} &= \boldsymbol{R}_{wc}^T\boldsymbol{R}_{wt}
\end{aligned}
$$

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- This is the orientation error expressed in *body frame*
- How can I get the error in *world frame*?

$$\begin{aligned}\boldsymbol{R}_{ct}^{w}\boldsymbol{R}_{wc} &= \boldsymbol{R}_{wt} \\ \Rightarrow \boldsymbol{R}_{ct}^{w} &= \boldsymbol{R}_{wt}\boldsymbol{R}_{wc}^T\end{aligned}$$

Orientation Errors and Lie Algebra

In general, the *rotation difference* can be described by:

$$\phi_{12} = \log(\boldsymbol{R}_1^T \boldsymbol{R}_2)^\vee = \mathrm{Log}(\boldsymbol{R}_1^T \boldsymbol{R}_2) \;\; _{\textit{Right} \ominus}$$
$$\phi_{21} = \log(\boldsymbol{R}_2 \boldsymbol{R}_1^T)^\vee = \mathrm{Log}(\boldsymbol{R}_2 \boldsymbol{R}_1^T) \;\; _{\textit{Left} \ominus}$$

Also the *difference angle* is given by:

$$\phi_{12} = \sqrt{\phi_{12}^T \phi_{12}} = |\phi_{12}|$$
$$\phi_{21} = \sqrt{\phi_{21}^T \phi_{21}} = |\phi_{21}|$$

We cannot do:

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Why?

We cannot do:

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We CAN do:

$$\boldsymbol{R} = (\boldsymbol{R}_2\boldsymbol{R}_1^T)^\alpha \boldsymbol{R}_1 = \boldsymbol{R}_1(\boldsymbol{R}_1^T\boldsymbol{R}_2)^\alpha$$

Why?

$$\boldsymbol{R}_2\boldsymbol{R}_1^T = \exp(\phi_{21}^\wedge) \Rightarrow (\boldsymbol{R}_2\boldsymbol{R}_1^T)^\alpha = \exp(\alpha\phi_{21}^\wedge)$$

Thus:

$$\boldsymbol{R} = (\alpha\phi_{21}) \oplus \boldsymbol{R}_1 = \Big(\alpha\,\underbrace{(\boldsymbol{R}_2 \ominus \boldsymbol{R}_1)}_{\text{Left }\ominus}\Big) \oplus \boldsymbol{R}_1$$

$$\boldsymbol{R} = \boldsymbol{R}_1 \oplus (\alpha\phi_{12}) = \boldsymbol{R}_1 \oplus \Big(\alpha\,\underbrace{(\boldsymbol{R}_2 \ominus \boldsymbol{R}_1)}_{\text{Right }\ominus}\Big)$$

Integrating Orientations with Angular Velocity

$$\boldsymbol{R}_{k+1} = \boldsymbol{R}_k \exp(\boldsymbol{\omega}_b^\wedge dt) = \boldsymbol{R}_k \oplus (\boldsymbol{\omega}_b dt)$$
$$= \exp(\boldsymbol{\omega}_w^\wedge dt)\boldsymbol{R}_k = (\boldsymbol{\omega}_w dt) \oplus \boldsymbol{R}_k$$

Note that:

- "Right" multiplication is *local*, $\boldsymbol{\omega}_b \in T_{\mathcal{X}}\mathcal{M}$
- "Left" multiplication is *global/world*, $\boldsymbol{\omega}_w \in T_{\varepsilon}\mathcal{M}$

Full Poses from a Geometric Perspective

- The **Special Euclidean Group**, representing full transformation matrices is the set:

$$SE(3) = \{\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix} \in \mathbb{R}^{4\times4} \mid \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I},\ det(\boldsymbol{R}) = 1\}$$

- The set of all 4×4 matrices is a vectorspace. What does it mean?

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- The set of all 4×4 matrices is a vectorspace. What does it mean?

- $SE(3)$ is not:
  1. The zero matrix $\boldsymbol{0}$ is not part of the space!
  2. $\boldsymbol{T}_1, \boldsymbol{T}_2 \in SE(3) \not\Rightarrow \boldsymbol{T}_1 + \boldsymbol{T}_2 \in SE(3)$

$SE(3)$ is a Matrix Lie Group

$SE(3)$ forms a *matrix Lie group* ($\circ$ is the regular matrix multiplication, ε is the identity matrix):

- $\boldsymbol{T}_1, \boldsymbol{T}_2 \in SE(3) \Rightarrow \boldsymbol{T}_1\boldsymbol{T}_2 \in SE(3)$
- $(\boldsymbol{T}_1\boldsymbol{T}_2)\boldsymbol{T}_3 = \boldsymbol{T}_1(\boldsymbol{T}_2\boldsymbol{T}_3) = \boldsymbol{T}_1\boldsymbol{T}_2\boldsymbol{T}_3$
- $\boldsymbol{T}, \boldsymbol{I} \in SE(3)$ and $\boldsymbol{T}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{T} = \boldsymbol{T}$
- $\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix} \in SE(3)$

Full Poses - $SE(3)$

Recap for the **Special Euclidean Group**(3), $SE(3)$:

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}, \boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}, \boldsymbol{Ad}_{\boldsymbol{T}} = [Ad_{\boldsymbol{T}}] = \begin{bmatrix} \boldsymbol{R} & 0 \\ \boldsymbol{t}^\wedge \boldsymbol{R} & \boldsymbol{R} \end{bmatrix}$$

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\rho} \end{bmatrix} \in \mathbb{R}^6, \exp(\boldsymbol{\xi}^\wedge) = \boldsymbol{T}, \log(\boldsymbol{T})^\vee = \boldsymbol{\xi}$$

$$\boldsymbol{\xi}_{12} = \mathrm{Log}(\boldsymbol{T}_1^{-1}\boldsymbol{T}_2), \xi_{12} = \sqrt{\boldsymbol{\xi}_{12}^T\boldsymbol{\xi}_{12}} = |\boldsymbol{\xi}_{12}|$$

$$\boldsymbol{\xi}_{21} = \mathrm{Log}(\boldsymbol{T}_2\boldsymbol{T}_1^{-1}), \xi_{21} = \sqrt{\boldsymbol{\xi}_{21}^T\boldsymbol{\xi}_{21}} = |\boldsymbol{\xi}_{21}|$$

$$\boldsymbol{T} = (\boldsymbol{T}_2\boldsymbol{T}_1^{-1})^\alpha \boldsymbol{T}_1 = \boldsymbol{T}_1(\boldsymbol{T}_1^{-1}\boldsymbol{T}_2)^\alpha$$

$$\begin{aligned} \boldsymbol{T}_{k+1} &= \exp([\mathcal{V}_w]dt)\boldsymbol{T}_k = ({}^{\varepsilon}\boldsymbol{\xi}dt) \oplus \boldsymbol{T}_k \\ &= \boldsymbol{T}_k \exp([\mathcal{V}_b]dt) = \boldsymbol{T}_k \oplus ({}^{\boldsymbol{T}_k}\boldsymbol{\xi}dt) \end{aligned}$$

where $\boldsymbol{\xi}^\wedge = [\boldsymbol{\xi}] = \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ \boldsymbol{0} & 0 \end{bmatrix}, \boldsymbol{\phi}, \boldsymbol{\rho} \in \mathbb{R}^3$.

Exp/Log Map for $SE(3)$

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\rho} \end{bmatrix} \in \mathbb{R}^6, \boldsymbol{\phi} = \phi \boldsymbol{r}$$

$$\exp(\boldsymbol{\xi}^\wedge) = \boldsymbol{T} = \begin{bmatrix} \exp(\boldsymbol{\phi}^\wedge) & \boldsymbol{J}_l(\boldsymbol{\phi})\boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\log(\boldsymbol{T})^\vee = \boldsymbol{\xi} = \begin{bmatrix} \mathrm{Log}(\boldsymbol{R}) \\ \boldsymbol{J}_l^{-1}(\boldsymbol{\phi})\boldsymbol{t} \end{bmatrix}$$

where

$$\boldsymbol{J}_l(\boldsymbol{\phi}) = \boldsymbol{I} + \frac{1 - \cos\phi}{\phi^2}\boldsymbol{\phi}^\wedge + \frac{\phi - \sin\phi}{\phi^3}\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge$$

$$\boldsymbol{J}_l^{-1}(\boldsymbol{\phi}) = \boldsymbol{I} - \frac{1}{2}\boldsymbol{\phi}^\wedge + (\frac{1}{\phi^2} - \frac{1 + \cos\phi}{2\phi\sin\phi})\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge$$

Manifolds and Lie Algebra - Code Example

jupyter 3-so3 Last Checkpoint: 6 minutes ago

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Code

```
def angular_velocity_local_to_deriv_rotation(omega_b, R):
    return R @ hat(omega_b)

# Some testing
R = RotX(1.)

omega_b = np.array([[1., 0., 0.]]).T

# Integrate naively for a few steps
dt = 0.01
R1 = np.copy(R)
R2 = np.copy(R)
R3 = np.copy(R)
R4 = np.copy(R)

# Try increasing the steps!
```

Thank you

- **Any Questions?**
- **Office Hours**:
  - **Tue & Thu** *(09:00-11:00)*
  - 24/7 by email (`costashatz@upatras.gr`, subject: *ECE_RSII_AM*)
- **Material and Announcements**

Laboratory of Automation & Robotics