7 LIMITATIONS ON PERFORMANCE IN MIMO SYSTEMS

In a MIMO system, disturbances, the plant, RHP-zeros, RHP-poles and delays each have directions associated with them. A multivariable plant may have a RHP-zero and a RHP-pole at the same location, but their effects may not interact.

7.1 Algebraic Limitation I

From the identity $S + T = I$ we get

$$|1 - \bar{\sigma}(S)| \leq \bar{\sigma}(T) \leq 1 + \bar{\sigma}(S) \tag{7.1}$$

$$|1 - \bar{\sigma}(T)| \leq \bar{\sigma}(S) \leq 1 + \bar{\sigma}(T) \tag{7.2}$$

$\Rightarrow$ S and T cannot be small simultaneously; $\bar{\sigma}(S)$ is small if and only if $\bar{\sigma}(T)$ is large.

$\Rightarrow \bar{\sigma}(T) \gg 1 \Leftrightarrow \bar{\sigma}(S) \gg 1$.

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7.2 Algebraic Limitation II

- If $G(s)$ has a RHP zero at z with output direction y_z, then for internal stability we require

$$y_z^H T(z) = 0 \ ; \quad y_z^H S(z) = y_z^H$$

  - follows from

$$y_z^H L(z) = 0 \quad \Rightarrow \quad y_z^H T(z) = 0 \quad \Rightarrow \quad y_z^H (I - S(z)) = 0$$

  Thus,

  - $T(s)$ must retain any RHP zero and zero direction in $G(s)$
  - implies that $\bar{\sigma}(S(z)) \geq 1$

- If $G(s)$ has a RHP pole at p with output direction y_p, then for internal stability we require

$$S(p)y_p = 0 \ ; \quad T(p)y_p = y_p$$

  - follows from $L^{-1}(p)y_p = 0$ and $S = TL^{-1}$

  Thus,

  - $S(s)$ must have RHP zeros where $G(s)$ has RHP poles
  - implies that $\bar{\sigma}(T(p)) \geq 1$

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7.3 Analytical Constraint I

Minimum peaks for S and T from RHP poles and zeros

Theorem 1 Weighted sensitivity. *Suppose the plant $G(s)$ has a RHP-zero at $s = z$. Let $w_P(s)$ be any stable scalar weight. Then for closed-loop stability the weighted sensitivity function must satisfy*

$$\|w_P(s)S(s)\|_\infty \geq |w_P(z)| \tag{7.3}$$

Theorem 2 Weighted complementary sensitivity. *Suppose the plant $G(s)$ has a RHP-pole at $s = p$. Let $w_T(s)$ be any stable scalar weight. Then for closed-loop stability the weighted complementary sensitivity function must satisfy*

$$\|w_T(s)T(s)\|_\infty \geq |w_T(p)| \tag{7.4}$$

Combined RHP poles and zeros

Consider a rational $G(s)$ with N_z distinct RHP zeros and N_p distinct RHP poles, with corresponding normalized output directions $y_{z,i}$ and $y_{p,i}$, respectively. Then the following tight lower bounds apply

$$\min_K \|S\|_\infty = \min_K \|T\|_\infty = \sqrt{1 + \bar{\sigma}^2 \left(Q_z^{-1/2} Q_{zp} Q_p^{-1/2} \right)}$$

where

$$[Q_z]_{ij} = \frac{y_{z,i}^H y_{z,j}}{z_i + \bar{z}_j}, \quad [Q_p]_{ij} = \frac{y_{p,i}^H y_{p,j}}{\bar{p}_i + p_j}, \quad [Q_{zp}]_{ij} = \frac{y_{z,i}^H y_{p,j}}{z_i - p_j}$$

- computable **tight bound** for any number of RHP poles and zeros
- minimum peaks depend on distance between z and p as well as the alignment of their directions (no interference if orthogonal directions)

Single RHP pole and zero

For a system $G(s)$ with one RHP pole p and one RHP zero z

$$\min_K \|S\|_\infty = \min_K \|T\|_\infty = \sqrt{sin^2\phi + \frac{|z+p|^2}{|z-p|^2} cos^2\phi}$$

where $\phi = cos^{-1}|y_z^H y_p|$

Example:

$$G(s) = \frac{1}{s+1} \begin{pmatrix} \frac{s+1}{s-1} & s+7 \\ 1 & s+1 \end{pmatrix} ; \quad z = 2,\ y_z = \begin{pmatrix} -0.32 \\ 0.95 \end{pmatrix},\ p = 1,\ y_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

this yields $\phi = 1.25\ rad$, and

$$\min_K \|S\|_\infty = \min_K \|T\|_\infty = 1.1$$

For SISO plant with $z = 2$ and $p = 1$ we get
$\min \|S\|_\infty = \min \|T\|_\infty = 1.73$ $(\phi = 0)$

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7.4 Analytical Constraint II

Sensitivity Integral

Assume the loop-gain $L(s)$ has entries with pole excess at least 2, and N_P RHP poles at p_i. Then, for closed-loop stability the sensitivity function must satisfy

$$\int_0^\infty \ln|\det S(j\omega)|d\omega = \pi \sum_{i=1}^{N_P} Re(p_i)$$

- essentially, $\det S(s)$ is a sensitivity function with $\det S(\infty) = 1$. The rest then follows from *Cauchy integral theorem*

- for any square matrix

$$|det(S)| = \prod_i \sigma_i(S)$$

hence, the sensitivity integral can be written

$$\sum_i \int_0^\infty \ln \sigma_i\,(S(j\omega))\,d\omega = \pi \sum_{i=1}^{N_P} Re(p_i)$$

  - interpretation: must make trade-off between frequencies as well as between system directions.

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7.5 Performance requirements from disturbances

Recall

$$y = Gu + G_d d \quad \Rightarrow \quad e = SG_d d = Sg_{d_1} d_1 + Sg_{d_2} d_2 + \dots$$

where d_i are scalar disturbances

- performance requirement $\|e(\omega)\|_2 < 1$ implies, for each disturbance d_i

$$\bar{\sigma}(Sg_{d_i}) \leq 1 \ \forall\omega \quad \Leftrightarrow \quad \|Sg_{d_i}\|_\infty \leq 1$$

- define *disturbance direction*

$$y_{d_i} = \frac{g_{d_i}}{\|g_{d_i}\|_2}$$

- requirement becomes

$$\bar{\sigma}(Sy_{d_i}) \leq \frac{1}{\|g_{d_i}\|_2} \ \forall\omega$$

  thus, requirement on S is only in the disturbance direction y_{d_i}

Disturbances and directions of S

Consider SVD of *given* sensitivity function, $S = U\Sigma V^H$

$$S\bar{v} = \bar{\sigma}(S)\bar{u}\,, \quad S\underline{v} = \underline{\sigma}(S)\underline{u}$$

- Case 1: disturbance alligned with high-gain direction of S

$$y_{d_i} = \bar{v} \quad \Rightarrow \quad \bar{\sigma}(S) \leq \frac{1}{\|g_{d_i}\|_2} \forall\omega$$

  i.e., requirement is on $\bar{\sigma}(S)$
- Case 2: disturbance alligned with low-gain direction of S

$$y_{d_i} = \underline{v} \quad \Rightarrow \quad \underline{\sigma}(S) \leq \frac{1}{\|g_{d_i}\|_2} \forall\omega$$

  i.e., requirement is on $\underline{\sigma}(S)$

Disturbances and RHP zeros

If $G(s)$ has a RHP zero z, then $y_z^H S(z) = y_z^H$ and

$$\|Sg_{d_i}\|_\infty \geq \|y_z^H S g_{d_i}\|_\infty \geq |y_z^H g_{d_i}(z)|$$

- hence, must require

  $$|y_z^H g_{d_i}(z)| < 1$$

  recall, for SISO $|g_{d_i}(z)| < 1$
- requirements depend on allignment of y_z and y_{d_i}:
  - if $y_z \perp y_{d_i}$ then $y_z^H g_{d_i} = 0$, i.e., no interference between disturbance and RHP zero
  - if $y_z \parallel y_d$ then $y_z^H g_{d_i} = \|g_d(z)\|_2$ and we require $\|g_{d_i}(z)\|_2 < 1$, as in SISO case

Example

$$G(s) = \frac{1}{0.1s+1}\begin{pmatrix} \frac{s+1}{s-1} & s+7 \\ 1 & s+1 \end{pmatrix} \;;\quad G_d = \frac{1}{0.1s+1}\begin{pmatrix} -0.6 & 50 \\ 1.8 & 16 \end{pmatrix}$$

Zero at $s = 2$ with $y_z^H = \begin{pmatrix} -0.31 & 0.95 \end{pmatrix}$

- For disturbance d_1

  $$|y_z^H g_{d_1}(2)| = 1.58$$
- For disturbance d_2

  $$|y_z^H g_{d_2}(2)| = 0.25$$

Thus, attenuation of disturbance d_1 not feasible

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Limitations imposed by input constraints

Perfect disturbance attentuation

$$y = Gu + g_{d_i} d_i \overset{y=0}{\Rightarrow} u = -G^{-1} g_{d_i} d_i$$

- with $|d_i| \leq 1 \; \forall\omega$ the condition $\|u(\omega)\|_2 < 1 \; \forall\omega$ implies

$$\bar{\sigma}(G^{-1} g_{d_i}) < 1 \; \forall\omega \quad \Rightarrow \quad \|G^{-1} g_{di}\|_\infty < 1$$

- similar for reference tracking, with $G_d = R$

Disturbances and setpoint changes closely alligned with weak output direction $\underline{u}$ of G most difficult

Example

$$G = \begin{pmatrix} 10 & -11 \\ 11 & -10 \end{pmatrix} \; ; \quad G_d = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$$

inputs for perfect disturbance attenuation

$$u = G^{-1} G_d = \begin{pmatrix} 0.14 & -2 \\ -0.14 & -2 \end{pmatrix}$$

- disturbance d_2 requires largest inputs despite $\|g_{d_2}\|_2 < \|g_{d_1}\|_2$