3 PERFORMANCE LIMITATIONS IN SISO SYSTEMS

3.1 Input-Output Controllability

"Control" is not only controller design and stability analysis. Three important questions:

I. How well can the plant be controlled?

II. What control structure should be used?
(What variables should we measure, which variables should we manipulate, and how are these variables best paired together?)

III. How might the process be changed to improve control?

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Definition 1 (Input-output) controllability *is the ability to achieve acceptable control performance; that is, to keep the outputs (y) within specified bounds displacements from their references (r), in spite of unknown but bounded variations, such as disturbances (d) and plant changes (including uncertainties), using available inputs (u) and available measurements (y_m or d_m).*

Note: controllability is independent of the controller, and is a property of the plant (or process) alone.

It can only be affected by:

- changing the apparatus itself, e.g. type, size, etc.
- relocating sensors and actuators
- adding new equipment to dampen disturbances
- adding extra sensors
- adding extra actuators
- changing the control objectives

3.1.1 Feedback Control

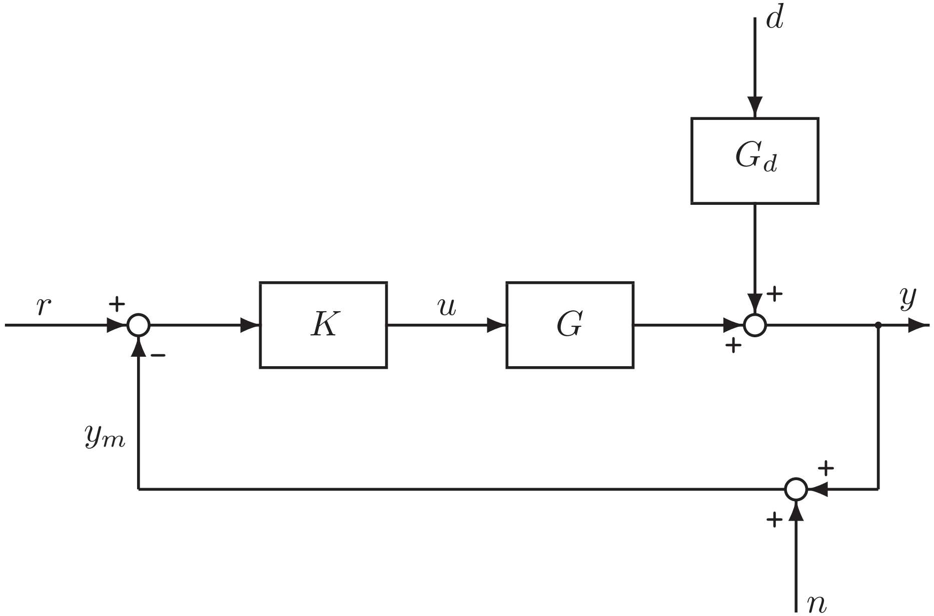


Figure 1: Closed-loop system

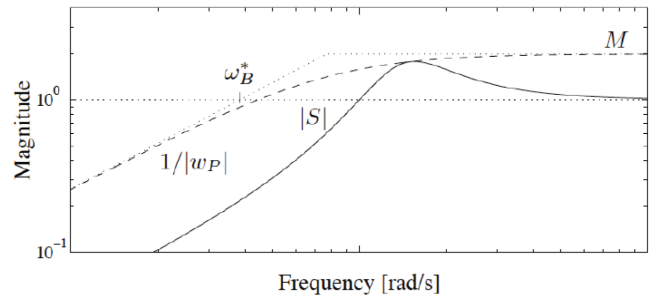
Control error:

$$e = -Sr + SG_d d + Tn \tag{3.1}$$

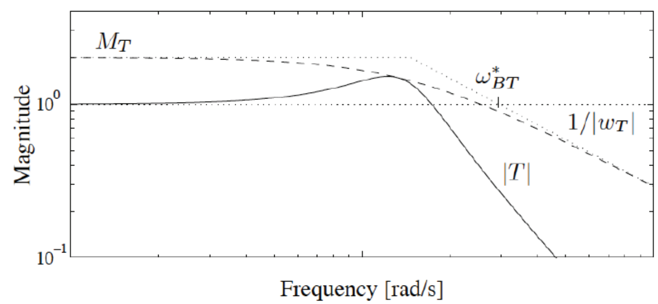
Control objective: design K so that

- $|S(j\omega)|$ is small for frequencies where d and r are important
- $|T(j\omega)|$ is small for frequencies where n is large

3.1.2 Shaping S and T



$$|S| < 1/|w_p| \ \forall\omega \ \Leftrightarrow \ \|w_P S\|_\infty < 1$$



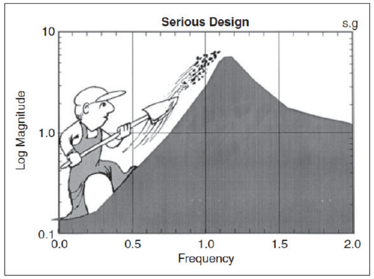
$$|T| < 1/|w_T| \ \forall\omega \ \Leftrightarrow \ \|w_T T\|_\infty < 1$$

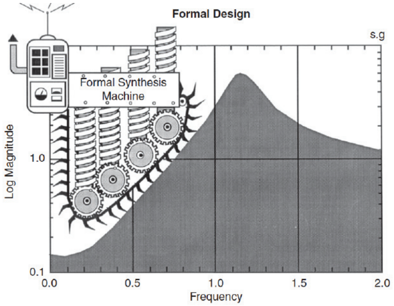
Figure 2: shaping S and T

- **Question:** can we shape S and T independently, i.e., choose any weights w_P and w_T?
- **Answer:** NO! there exist a number of fundamental constraints/limitations.

3.1.3 Limitations/Constraints

- Algebraic constraints
  - $S + T = 1$
  - Interpolation constraints
- Analytic constraints
  - RHP-poles and zeros
  - Bode sensitivity integral
- Practical constraints
  - Input bounds
  - Delay measurement





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3.2 Fundamental limitations on Sensitivity

3.2.1 S plus T is one

$$S + T = 1 \tag{3.2}$$

$\Longrightarrow$ at any frequency $|S(j\omega)| \geq 0.5$ or $|T(j\omega)| \geq 0.5$

3.2.2 The waterbed effects (sensitivity integrals)

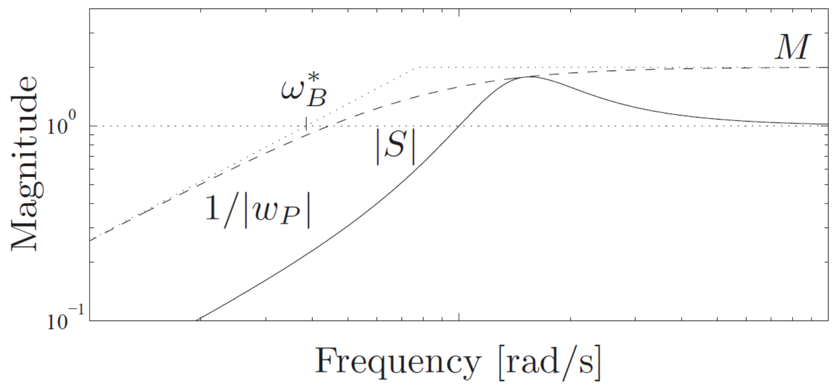

Figure 3: Plot of typical sensitivity, $|S|$, with upper bound $1/|w_P|$

Note: $|S|$ has peak greater than 1; we will show that this is unavoidable in practice.

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Pole excess of two: First waterbed formula

When $L(s)$ has a relative degree of two or more, then for some ω the distance between L and -1 is less than one.

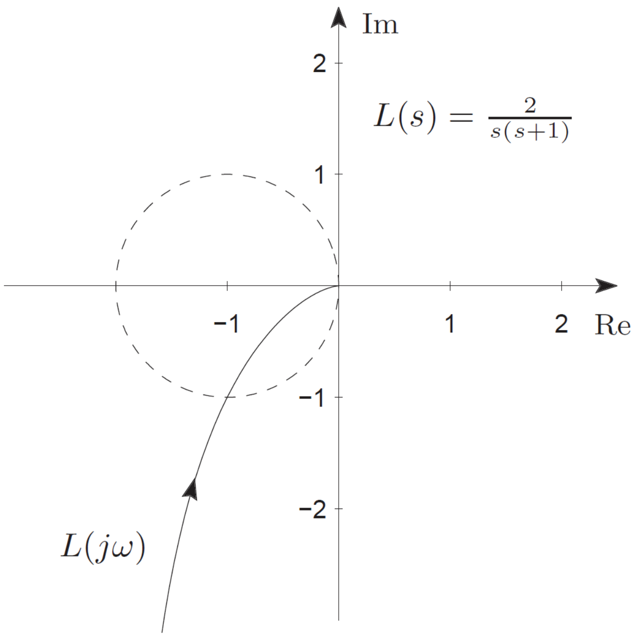

Figure 4: $|S| > 1$ whenever the Nyquist plot of L is inside the circle

Theorem 1 Bode Sensitivity Integral (first waterbed formula).

Suppose that the open-loop transfer function $L(s)$ is rational and has at least two more poles than zeros (relative degree of two or more).

Suppose also that $L(s)$ has N_p RHP-poles at locations p_i.

Then for closed-loop stability the sensitivity function must satisfy

$$\int_0^\infty \ln |S(j\omega)| d\omega = \pi \cdot \sum_{i=1}^{N_p} Re(p_i) \qquad (3.3)$$

where $Re(p_i)$ denotes the real part of p_i.

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RHP-zeros: Second waterbed formula

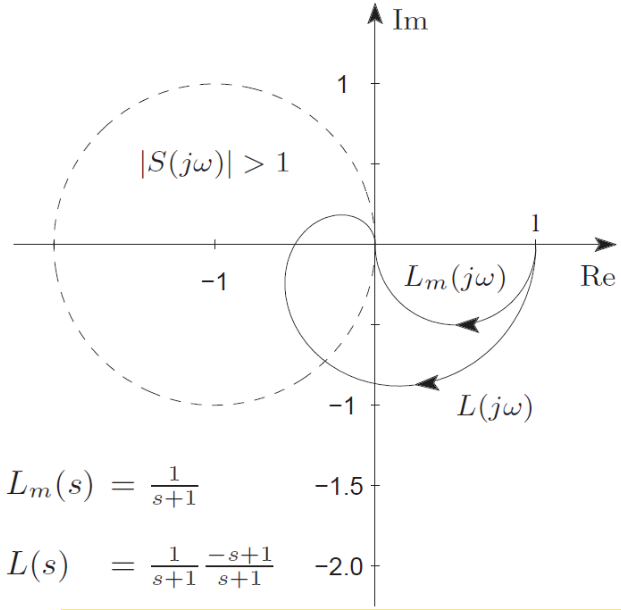

Figure 5: Additional phase lag contributed by RHP-zero causes $|S| > 1$

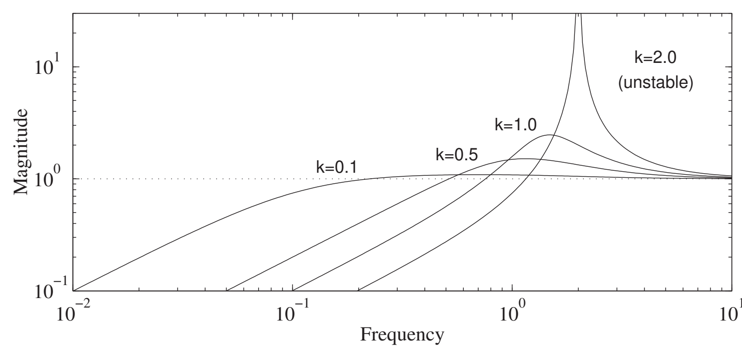

Figure 6: Effect of increased controller gain on $|S|$ for system with RHP-zero at $z = 2$, $L(s) = \frac{k}{s}\frac{2-s}{2+s}$

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Theorem 2 Weighted sensitivity integral (second waterbed formula). *Suppose that $L(s)$ has a single real RHP-zero z and has N_p RHP-poles, p_i. Then for closed-loop stability the sensitivity function must satisfy*

$$\int_0^\infty \ln |S(j\omega)| \cdot w(z,\omega) d\omega = \pi \cdot \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z}{p_i - z} \right| \quad (3.4)$$

where:

$$w(z,\omega) = \frac{2z}{z^2 + \omega^2} = \frac{2}{z} \frac{1}{1 + (\omega/z)^2} \quad (3.5)$$

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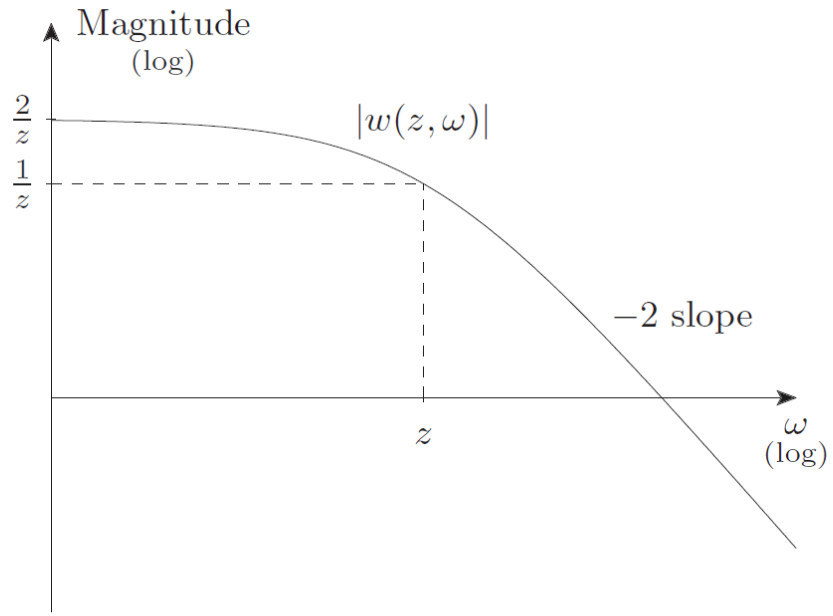

Figure 7: Plot of weight $w(z,\omega)$ for case with real zero at $s = z$

Weight $w(z,\omega)$ "cuts off" contribution of $ln|S|$ at frequencies $\omega > z$. Thus, for a stable plant:

$$\int_0^z \ln|S(j\omega)|d\omega \approx 0 \quad (\text{ for } |S| \approx 1 \text{ at } \omega > z) \qquad (3.6)$$

The waterbed is finite, and a large peak for $|S|$ is unavoidable when we reduce $|S|$ at low frequencies (Figure 6).

Note also that when $p_i \to z$ then $\frac{p_i+z}{p_i-z} \to \infty$.

3.3 Fundamental limitations: bounds on peaks

If p is a RHP-pole of $L(s)$ then

$$T(p) = 1, \quad S(p) = 0 \tag{3.7}$$

Similarly, if z is a RHP-zero of $L(s)$ then

$$T(z) = 0, \quad S(z) = 1 \tag{3.8}$$

3.3.1 Minimum peaks for S and T

Maximum modulus principle. *Suppose $f(s)$ is stable (i.e. $f(s)$ is analytic in the complex RHP). Then the maximum value of $|f(s)|$ for s in the right-half plane is attained on the region's boundary, i.e. somewhere along the $j\omega$-axis. Hence, we have for a stable $f(s)$*

$$\|f(j\omega)\|_\infty = \max_\omega |f(j\omega)| \geq |f(s_0)| \quad \forall s_0 \in \text{RHP} \tag{3.9}$$

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Imagine $|f(s)|$ as 3D-plot over complex variable s
$\Rightarrow$ $|f(s)|$ has "peaks" at its poles and "valleys" at its zeros

Suppose $f(s)$ has LHP-poles but no RHP-poles
$\Rightarrow$ $|f(s)|$ slopes down from LHP to RHP
$\Rightarrow$ (3.9)

The results below follow from (3.9) with
$f(s) = w_P(s)S(s)$
$f(s) = w_T(s)T(s)$
for weighted sensitivity and weighted complementary sensitivity.

Theorem 3 Weighted sensitivity peak

Suppose that $G(s)$ has a RHP-zero z and let $w_P(s)$ be any stable weight function.

Then for closed-loop stability the weighted sensitivity function must satisfy

$$\|w_P S\|_\infty \geq |w_P(z)S(z)| = |w_P(z)| \tag{3.10}$$

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Theorem 4 Weighted complementary sensitivity peak

Suppose that $G(s)$ has a RHP-pole p and let $w_T(s)$ be any stable weight function.

Then for closed-loop stability the weighted complementary sensitivity function must satisfy

$$\|w_T T\|_\infty \geq |w_T(p)T(p)| = |w_T(p)| \tag{3.11}$$

Derivation of additional penalty if the plant has RHP-poles:

1. Factor out the RHP-zeros of S into all-pass S_a

$$S = S_a S_m, \quad S_a(s) = \prod_i \frac{s - p_i}{s + \bar{p}_i} \tag{3.12}$$

   $|S_a(j\omega)| = 1$ at all frequencies. S_m is the "minimum-phase version" of S with all RHP-zeros mirrored into the LHP.

2. Consider a RHP-zero located at z, for which we get from the maximum modulus principle

$$\max_\omega |w_P S(j\omega)| = \max_\omega |w_P S_m(j\omega)| \geq |w_P(z) S_m(z)|$$

   where $S_m(z) = S(z)S_a(z)^{-1} = 1 \cdot S_a(z)^{-1}$.

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Theorem 5 Combined RHP-poles and RHP-zeros.

Suppose that $G(s)$ has N_z RHP-zeros z_j, and N_p RHP-poles p_i.

Then for closed-loop stability the weighted sensitivity function must satisfy for each *RHP-zero z_j*

$$\|w_P S\|_\infty \geq c_{1j}|w_P(z_j)|, \quad c_{1j} = \prod_{i=1}^{N_p} \frac{|z_j + \bar{p}_i|}{|z_j - p_i|} \geq 1 \tag{3.13}$$

and the weighted complementary sensitivity function must satisfy for each *RHP-pole p_i*

$$\|w_T T\|_\infty \geq c_{2i}|w_T(p_i)|, \quad c_{2i} = \prod_{j=1}^{N_z} \frac{|\bar{z}_j + p_i|}{|z_j - p_i|} \geq 1 \tag{3.14}$$

For $w_P = w_T = 1$:

$$\|S\|_\infty \geq \max_j c_{1j}, \quad \|T\|_\infty \geq \max_i c_{2i} \tag{3.15}$$

$\Longrightarrow$ Large peaks for S and T are unavoidable if a RHP-zero and a RHP-pole are close to each other.

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3.3.2 Bandwidth limitation

Performance requirement:

$$|S(j\omega)| < 1/|w_P(j\omega)| \quad \forall\omega \quad \Leftrightarrow \quad \|w_P S\|_\infty < 1 \tag{3.16}$$

However, from (3.10) we have that
$\|w_P S\|_\infty \geq |w_P(z)|$,

so the weight must satisfy

$$\boxed{|w_P(z)| < 1} \tag{3.17}$$

For performance weight

$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A} \tag{3.18}$$

and a real zero at z we get:

$$\omega_B^*(1 - A) < z\left(1 - \frac{1}{M}\right) \tag{3.19}$$

e.g. $A = 0, M = 2$:

$$\omega_B^* < \frac{z}{2}$$

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3.3.3 Limitations imposed by RHP-poles

Specification:

$$|T(j\omega)| < 1/|w_T(j\omega)| \quad \forall\omega \quad \Leftrightarrow \quad \|w_T T\|_\infty < 1 \tag{3.20}$$

However, from (3.11) we have that:

$$\|w_T T\|_\infty \geq |w_T(p)| \tag{3.21}$$

so the weight must satisfy

$$\boxed{|w_T(p)| < 1} \tag{3.22}$$

For:

$$w_T(s) = \frac{s}{\omega_{BT}^*} + \frac{1}{M_T} \tag{3.23}$$

we get:

$$\boxed{\omega_{BT}^* > p\frac{M_T}{M_T - 1}} \tag{3.24}$$

e.g. $M_T = 2$:

$$\omega_{BT}^* > 2p$$

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3.3.4 Combined RHP-poles and RHP-zeros

RHP-zero:

$$\omega_c < z/2$$

RHP-pole:

$$\omega_c > 2p$$

RHP-pole and RHP-zero:

$z > 4p$ for acceptable performance and robustness.

Sensitivity peaks.

From Theorem 5 for a plant with a single real RHP-pole p and a single real RHP-zero z, we always have:

$$\boxed{\|S\|_\infty \geq c, \ \|T\|_\infty \geq c, \quad c = \frac{|z+p|}{|z-p|}} \tag{3.25}$$

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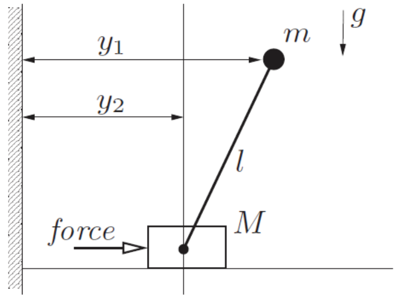
Example 1 Balancing a rod. *The objective is to keep the rod upright by movement of the cart, based on observing the rod either at its far end (output y_1) or the cart position (output y_2).*



l [m] = length of rod

m [kg] = mass of rod

M [kg] = mass of hand

$g \approx 10\ m/s^2$ = acceleration due to gravity.

The linearized transfer functions for the two cases are

$$G_1(s) = \frac{-g}{s^2\left(Mls^2 - (M+m)g\right)};$$

$$G_2(s) = \frac{ls^2 - g}{s^2\left(Mls^2 - (M+m)g\right)}$$

Poles: $p = 0, 0, \pm\sqrt{\frac{(M+m)g}{Ml}}$. For output $y_1(G_1(s))$ stabilization requires minimum bandwidth (3.24). For output $y_2(G_2(s))$ zero at $z = \sqrt{\frac{g}{l}}$

- *For light rod $m << M$, pole $\approx$ zero $\rightarrow$ "impossible" to stabilize*
- *For heavy rod (m large) difficult to stabilize because $p > z$*

Example: $m/M = 0.1 \Rightarrow \|S\|_\infty \geq 42$; $\|T\|_\infty \geq 42 \Rightarrow$ poor control

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3.4 Limitations imposed by time delays

For plant with delay:

$$S = 1 - T = 1 - e^{-\theta s} \tag{3.26}$$

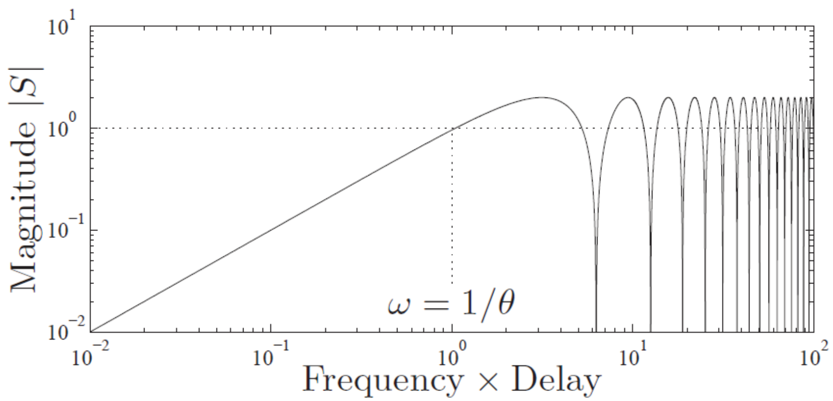

Figure 8: Sensitivity function (3.26) for a plant with delay

$|S(j\omega)|$ crosses 1 at $\frac{\pi}{3}\frac{1}{\theta} = 1.05/\theta$.

Because here $|S| = 1/|L|$, we have:

$$\omega_c < 1/\theta \tag{3.27}$$

Pade approximation of time delay:

$$e^{-\theta s} \approx \frac{1 - \frac{\theta}{2}s}{1 + \frac{\theta}{2}s} \rightarrow z = \frac{2}{\theta} \ (\omega_c < z/2) \tag{3.28}$$

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3.5 Limitations imposed by input constraints

The input required to achieve perfect control ($e = 0$) is

$$u = G^{-1}r - G^{-1}G_d d \tag{3.29}$$

Disturbance rejection. $r = 0$, $|d(\omega)| = 1$; $|u(\omega)| < 1$ implies

$$|G^{-1}(j\omega)G_d(j\omega)| < 1 \quad \forall\omega \tag{3.30}$$

Command tracking. $d = 0$, $|r(\omega)| = R\forall\omega < \omega_r$ $|u(\omega)| < 1$ implies:

$$|G^{-1}(j\omega)R| < 1 \quad \forall\omega \leq \omega_r \tag{3.31}$$

For *acceptable control* (namely $|e(j\omega)| < 1$) requirements change to:

$$\boxed{|G| > |G_d| - 1} \quad \text{at frequencies where} \quad |G_d| > 1 \tag{3.32}$$

$$\boxed{|G| > |R| - 1 < 1} \quad \forall\omega \leq \omega_r \tag{3.33}$$