5 ELEMENTS OF LINEAR SYSTEM THEORY

5.1 System descriptions

5.1.1 State-space representation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5.1}$$

$$y(t) = Cx(t) + Du(t) \tag{5.2}$$

or:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{5.3}$$

or:

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \tag{5.4}$$

(5.1)–(5.3) is *not* a unique description of the input-output behaviour of a linear system.

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Define new states $\widetilde{x} = Sx$, i.e. $x = S^{-1}\widetilde{x}$.

Equivalent state-space realization (i.e. with same input-output behaviour): Similarity Transformation

$$\widetilde{A} = SAS^{-1}, \quad \widetilde{B} = SB, \quad \widetilde{C} = CS^{-1}, \quad \widetilde{D} = D \tag{5.5}$$

Dynamical system response $x(t)$ for $t \geq t_0$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \tag{5.6}$$

For a system with disturbances d and measurement noise n:

$$\dot{x} = Ax + Bu + B_d d \tag{5.7}$$

$$y = Cx + Du + D_d d + n \tag{5.8}$$

5.1.2 Transfer function representation – Laplace transforms

Laplace transforms of (5.1) and (5.3) become for $x(t = 0) = 0$

$$sx(s) = Ax(s) + Bu(s) \quad \Rightarrow$$

$$\Rightarrow \quad x(s) = (sI - A)^{-1}Bu(s) \tag{5.9}$$

$$y(s) = Cx(s) + Du(s) \quad \Rightarrow$$

$$\Rightarrow \quad y(s) = \underbrace{\left(C(sI - A)^{-1}B + D\right)}_{G(s)} u(s) \tag{5.10}$$

where $G(s)$ is the transfer function matrix. Equivalently,

$$G(s) = \frac{1}{\det(sI - A)}\left[C\mathrm{adj}(sI - A)B + D\det(sI - A)\right] \tag{5.11}$$

5.2 State controllability and state observability

Definition

State controllability. The dynamical system $\dot{x} = Ax + Bu$, or equivalently the pair (A, B), is said to be state controllable if, for any initial state $x(0) = x_0$, any time $t_1 > 0$ and any final state x_1, there exists an input $u(t)$ such that $x(t_1) = x_1$. Otherwise the system is said to be state uncontrollable.

1. The pair: (A, B) is state controllable if and only if the controllability matrix

$$\mathcal{C} \triangleq [\, B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B \,] \qquad (5.12)$$

has rank n (full row rank). Here n is the number of states.

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2. From (5.6) one can verify that for $x(t_1) = x_1$

$$u(t) = -B^T e^{A^T(t_1-t)} W_c(t_1)^{-1}(e^{At_1}x_0 - x_1) \tag{5.13}$$

where $W_c(t)$ is the Gramian matrix at time t,

$$W_c(t) \triangleq \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau \tag{5.14}$$

Thus (A, B) is state controllable if and only if $W_c(t)$ has full rank (and thus is positive definite) for any $t > 0$. For a stable system (A is stable) check only $P \triangleq W_c(\infty)$,

$$P \triangleq \int_0^\infty e^{A\tau} B B^T e^{A^T\tau} d\tau \tag{5.15}$$

P may also be obtained as the solution to the Lyapunov equation

$$AP + PA^T = -BB^T \tag{5.16}$$

3. Let p_i be the i'th eigenvalue of A and q_i the corresponding left eigenvector, $q_i^H A = p_i q_i^H$. Then the system is state controllable if and only if $q_i^H B \neq 0, \forall i$.

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Example:

$$A = \begin{bmatrix} -2 & -2 \\ 0 & -4 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ C = [1 \quad 0], \ D = 0$$

The transfer function

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s+4}$$

has only one state.

1. The controllability matrix has two linearly dependent rows:

$$\mathcal{C} = [\,B \quad AB\,] = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}.$$

2. The controllability Gramian is singular

$$P = \begin{bmatrix} 0.125 & 0.125 \\ 0.125 & 0.125 \end{bmatrix}$$

3. $p_1 = -2$ and $p_2 = -4$, $q_1 = [0.707 \quad -0.707]^T$ and $q_2 = [0 \quad 1]^T$.

$$q_1^H B = 0, \quad q_2^H B = 1$$

the first mode (eigenvalue) is not state controllable.

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Controllability is a system-theoretic concept important for computation and realizations; but no practical insight:

1. It says nothing about how the states behave, e.g. it does not imply that one can *hold* (as $t \to \infty$) the states at a given value.

2. Required inputs may be very large with sudden changes.

3. Some states may be of no practical importance.

4. Existence result which provides no “degree of controllability”.

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Definition State observability. The dynamical system $\dot{x} = Ax + Bu$, $y = Cx + Du$ (or the pair (A, C)) is said to be state observable if, for any time $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input $u(t)$ and the output $y(t)$ in the interval $[0, t_1]$. Otherwise the system, or (A, C), is said to be state unobservable.

1. (A, C) is state observable if and only if the observability matrix

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{5.17}$$

has rank n (full column rank).

2. For a stable system the observability Gramian

$$Q \triangleq \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau \tag{5.18}$$

must have full rank n (and thus be positive definite). Q can also be found as the solution to the following Lyapunov equation

$$A^T Q + QA = -C^T C \tag{5.19}$$

3. Let p_i be the i'th eigenvalue of A and t_i the corresponding eigenvector, $At_i = p_i t_i$. Then the system is state observable if and only if $Ct_i \neq 0, \forall i$.

Observability is a system theoretical concept but may not give practical insight.

5.3 Stability

Definition

A system is **(internally) stable** if none of its components contains hidden unstable modes and the injection of bounded external signals at any place in the system results in bounded output signals measured anywhere in the system. "internal", i.e. <u>all</u> the states must be stable not only inputs/outputs.

Definition

State stabilizable, state detectable and hidden unstable modes. A system is state stabilizable if all unstable modes are state controllable. A system is state detectable if all unstable modes are state observable. A system with unstabilizable or undetectable modes is said to contain hidden unstable modes.

5.4 Poles

Definition

Poles. The poles p_i of a system with state-space description (5.1)–(5.2) are the eigenvalues $\lambda_i(A), i = 1, \ldots, n$ of the matrix A. The pole or characteristic polynomial $\phi(s)$ is defined as $\phi(s) \triangleq \det(sI - A) = \prod_{i=1}^{n}(s - p_i)$. Thus the poles are the roots of the characteristic equation

$$\phi(s) \triangleq \det(sI - A) = 0 \tag{5.20}$$

5.4.1 Poles and stability

Theorem 1 *A linear dynamic system* $\dot{x} = Ax + Bu$ *is stable if and only if all the poles are in the open left-half plane (LHP), that is,* $\mathrm{Re}\{\lambda_i(A)\} < 0, \forall i$. *A matrix A with such a property is said to be "stable" or Hurwitz.*

5.4.2 Poles from transfer functions

Theorem 2 MacFarlane and Karcanias *The pole polynomial* $\phi(s)$ *corresponding to a minimal realization of a system with transfer function* $G(s)$*, is the least common denominator of all non-identically-zero minors of all orders of* $G(s)$.

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Example:

$$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix} \quad (5.21)$$

The minors of order 1 are the four elements all have $(s+1)(s+2)$ in the denominator.

Minor of order 2

$$\det G(s) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)} \quad (5.22)$$

Least common denominator of all the minors:

$$\phi(s) = (s+1)(s+2) \quad (5.23)$$

Minimal realization has two poles: $s = -1$; $s = -2$.

Example: Consider the 2×3 system, with 3 inputs and 2 outputs,

$$G(s) = \frac{1}{(s+1)(s+2)(s-1)} *$$

$$* \begin{bmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{bmatrix} \quad (5.24)$$

Minors of order 1:

$$\frac{1}{s+1},\ \frac{s-1}{(s+1)(s+2)},\ \frac{-1}{s-1},\ \frac{1}{s+2},\ \frac{1}{s+2} \quad (5.25)$$

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Minor of order 2 corresponding to the deletion of column 2:

$$M_2 = \frac{(s-1)(s+2)(s-1)(s+1)+(s+1)(s+2)(s-1)^2}{((s+1)(s+2)(s-1))^2} =$$

$$= \frac{2}{(s+1)(s+2)} \qquad (5.26)$$

The other two minors of order two are

$$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2}, \quad M_3 = \frac{1}{(s+1)(s+2)} \qquad (5.27)$$

Least common denominator:

$$\phi(s) = (s+1)(s+2)^2(s-1) \qquad (5.28)$$

The system therefore has four poles: $s=-1$, $s=1$ and two at $s=-2$.

Note MIMO-poles are essentially the poles of the elements. A procedure is needed to determine multiplicity.

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5.5 Zeros

- SISO system: zeros z_i are the solutions to $G(z_i) = 0$.

In general, zeros are values of s at which $G(s)$ loses rank.

Example

$$\left[Y = \frac{s+2}{s^2+7s+12} U\right]$$

Compute the response when

$$\begin{aligned}
u(t) &= e^{-2t},\ y(0) = 0,\ \dot{y}(0) = -1 \\
\mathcal{L}\{u(t)\} &= \frac{1}{s+2} \\
s^2Y &- sy(0) - \dot{y}(0) + 7sY - 7y(0) + 12Y = 1 \\
s^2Y &+ 7sY + 12Y + 1 = 1 \\
&\Rightarrow Y(s) = 0
\end{aligned}$$

> Assumption: $g(s)$ has a zero z, $g(z) = 0$.
> Then for input $u(t) = u_0 e^{zt}$ the output is $y(t) \equiv 0$, $t > 0$. (with appropriate initial conditions)

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5.5.1 Zeros from state-space realizations

Setup:

$$\begin{aligned} u &= u_z e^{zt},\ x(t) = x_z e^{zt},\ y(t) \equiv 0 \\ \dot{x} &= z e^{zt} x_z = A e^{zt} x_z + B u_z e^{zt} \\ & \begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} x_z \\ u_z \end{bmatrix} = 0 \end{aligned}$$

and

$$\begin{aligned} y &= Cx + Du \\ &= C e^{zt} x_z + D u_z e^{zt} \equiv 0 \end{aligned}$$

Combined

$$\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_z \\ u_z \end{bmatrix} = 0$$

The zeros are the solutions of

$$\det \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = 0$$

MATLAB

```
zero = tzero(A,B,C,D)
```

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5.5.2 Zeros from transfer functions

Definition Zeros. z_i is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$. The zero polynomial is defined as $z(s) = \prod_{i=1}^{n_z}(s - z_i)$ where n_z is the number of finite zeros of $G(s)$.

Theorem The zero polynomial $z(s)$, corresponding to a minimal realization of the system, is the greatest common divisor of all the numerators of all order-r minors of $G(s)$, where r is the normal rank of $G(s)$, provided that these minors have been adjusted in such a way as to have the pole polynomial $\phi(s)$ as their denominators.

Example

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix} \tag{5.29}$$

The normal rank of $G(s)$ is 2.

Minor of order 2: $\det G(s) = \frac{2(s-1)^2-18}{(s+2)^2} = 2\frac{s-4}{s+2}$.

Pole polynomial: $\phi(s) = s + 2$.

Zero polynomial: $z(s) = s - 4$.

Note Multivariable zeros have no relationship with the zeros of the transfer function elements.

Example

$$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix} \tag{5.30}$$

Minor of order 2 is the determinant

$$\det G(s) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)} \tag{5.31}$$

$$\phi(s) = 1.25^2(s+1)(s+2)$$

Zero polynomial = numerator of (5.31)

$\Rightarrow$ no multivariable zeros.

Example

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \end{bmatrix} \tag{5.32}$$

- The normal rank of $G(s)$ is 1
- no value of s for which $G(s) = 0$

  $\Rightarrow G(s)$ has no zeros.

5.6 Remarks on poles and zeros

5.6.1 Directions of poles and zeros

Let $G(s) = C(sI - A)^{-1}B + D$.

Zero directions. Let $G(s)$ have a zero at $s = z$. Then $G(s)$ loses rank at $s = z$, and there exist non-zero vectors u_z and y_z such that

$$G(z)u_z = 0, \quad y_z^H G(z) = 0 \tag{5.33}$$

u_z = input zero direction

y_z = output zero direction

y_z gives information about which output (or combination of outputs) may be difficult to control.

SVD:

$$G(z) = U\Sigma V^H$$

u_z = last column in V

y_z = last column of U

(corresponding to the zero singular value of $G(z)$)

Pole directions. Let $G(s)$ have a pole at $s = p$. Then $G(p)$ is infinite, and we may write

$$G(p)u_p = \infty, \quad y_p^H G(p) = \infty \tag{5.34}$$

u_p = input pole direction

y_p = output pole direction.

Example

Plant in (5.29) has a RHP-zero at $z = 4$ and a LHP-pole at $p = -2$.

$$
\begin{aligned}
G(z) \quad &= \quad G(4) = \frac{1}{6}\begin{bmatrix} 3 & 4 \\ 4.5 & 6 \end{bmatrix} \\
&= \quad \frac{1}{6}\begin{bmatrix} 0.55 & -0.83 \\ 0.83 & 0.55 \end{bmatrix}\begin{bmatrix} 9.01 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}^H
\end{aligned}
$$

$$
u_z = \begin{bmatrix} -0.80 \\ 0.60 \end{bmatrix} \quad y_z = \begin{bmatrix} -0.83 \\ 0.55 \end{bmatrix} \tag{5.35}
$$

For pole directions consider

$$
G(p+\epsilon) = G(-2+\epsilon) = \frac{1}{\epsilon^2}\begin{bmatrix} -3+\epsilon & 4 \\ 4.5 & 2(-3+\epsilon) \end{bmatrix} \tag{5.36}
$$

The SVD as $\epsilon \to 0$ yields

$$
G(-2+\epsilon) = \frac{1}{\epsilon^2}\begin{bmatrix} -0.55 & -0.83 \\ 0.83 & -0.55 \end{bmatrix}\begin{bmatrix} 9.01 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}^H
$$

$$
u_p = \begin{bmatrix} 0.60 \\ -0.80 \end{bmatrix} \quad y_p = \begin{bmatrix} -0.55 \\ 0.83 \end{bmatrix} \tag{5.37}
$$

Note Locations of poles and zeros are independent of input and output scalings, their directions are *not*.

5.6.2 Comments on poles and zeros

1. For square systems the poles and zeros of $G(s)$ are "essentially" the poles and zeros of $\det G(s)$.

   This fails when zero and pole in different parts of the system cancel when forming $\det G(s)$.

   $$G(s) = \begin{bmatrix} (s+2)/(s+1) & 0 \\ 0 & (s+1)/(s+2) \end{bmatrix} \tag{5.38}$$

   $\det G(s) = 1$, although the system obviously has poles at -1 and -2 and (multivariable) zeros at -1 and -2.

2. System (5.38) has poles and zeros at the same locations (at -1 and -2). Their directions are different. They do not cancel or otherwise interact.

3. There are no zeros if the outputs contain direct information about all the states; that is, if from y we can directly obtain x (e.g. $C = I$ and $D = 0$);

4. Zeros usually appear when there are fewer inputs or outputs than states

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5. **Moving poles.** (a) feedback control ($G(I+KG)^{-1}$) moves the poles, (b) series compensation (GK, feedforward control) can cancel poles in G by placing zeros in K (but not move them), and (c) parallel compensation ($G+K$) cannot affect the poles in G.

6. **Moving zeros.** (a) With feedback, the zeros of $G(I+KG)^{-1}$ are the zeros of G plus the poles of K. , i.e. the zeros are unaffected by feedback. (b) Series compensation can counter the effect of zeros in G by placing poles in K to cancel them, but cancellations are not possible for RHP-zeros due to internal stability. (c) The only way to move zeros is by parallel compensation, $y=(G+K)u$, which, if y is a physical output, can only be accomplished by adding an extra input (actuator).

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Example

Effect of feedback on poles and zeros.

SISO plant $G(s) = z(s)/\phi(s)$ and $K(s) = k$.

$$T(s) = \frac{L(s)}{1+L(s)} = \frac{kG(s)}{1+kG(s)} = \frac{kz(s)}{\phi(s)+kz(s)} = k\frac{z_{cl}(s)}{\phi_{cl}(s)} \tag{5.39}$$

Note the following:

1. Zero polynomial: $z_{cl}(s) = z(s)$
   $\Rightarrow$ zero locations are unchanged.
2. Pole locations are changed by feedback.
   For example,

$$k \to 0 \quad \Rightarrow \quad \phi_{cl}(s) \to \phi(s) \tag{5.40}$$

$$k \to \infty \quad \Rightarrow \quad \phi_{cl}(s) \to z(s).\widetilde{z}(s) \tag{5.41}$$

where roots of $\widetilde{z}(s)$ move with k to infinity (complex pattern)
(cf. root locus)

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5.7 Stability analysis in the frequency domain

Generalization of Nyquist's stability test for SISO systems.

5.7.1 Open- and closed-loop characteristic polynomials

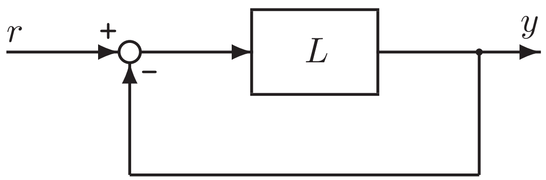

Figure 1: Negative feedback system

Open Loop:

$$L(s) = C_{ol}(sI - A_{ol})^{-1}B_{ol} + D_{ol} \tag{5.42}$$

Poles of $L(s)$ are the roots of the *open-loop* characteristic polynomial

$$\phi_{ol}(s) = \det(sI - A_{ol}) \tag{5.43}$$

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Assume no RHP pole-zero cancellations between $G(s)$ and $K(s)$. Then, internal stability of the *closed-loop* system is equivalent to the stability of $S(s) = (I + L(s))^{-1}$.

The realization of $S(s)$ can be derived as follow:

$$\begin{aligned} \dot{x} &= A_{ol}x + B_{ol}(r - y) && (5.44) \\ -e &= r - y = r - C_{ol}x - D_{ol}(r - y) && (5.45) \end{aligned}$$

or

$$r - y = (I + D_{ol})^{-1}(r - C_{ol}x) \tag{5.46}$$

and

$$\dot{x} = (A_{ol} - B_{ol}(I + D_{ol})^{-1}C_{ol})x + B_{ol}(I + D_{ol})^{-1}r \tag{5.47}$$

Therefore the state matrix of $S(s)$ is:

$$A_{cl} = A_{ol} - B_{ol}(I + D_{ol})^{-1}C_{ol} \tag{5.48}$$

And the closed-loop characteristic polynomial is

$$\phi_{cl}(s) \triangleq \det(sI - A_{cl}) = \det(sI - A_{ol} + B_{ol}(I + D_{ol})^{-1}C_{ol}) \tag{5.49}$$

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Relationship between characteristic polynomials

From (5.42) we get

$$\det(I + L(s)) = \det(I + C_{ol}(sI - A_{ol})^{-1}B_{ol} + D_{ol}) \quad (5.50)$$

Schur's formula yields (with $A_{11} = I + D_{ol}, A_{12} = -C_{ol}, A_{22} = sI - A_{ol}, A_{21} = B_{ol}$)

$$\det(I + L(s)) = \frac{\phi_{cl}(s)}{\phi_{ol}(s)} \cdot c \qquad (5.51)$$

where $c = \det(I + D_{ol})$ is a constant (cf. SISO result from RSI).

Side calculation:

$$\det \begin{bmatrix} I + D_{ol} & -C_{ol} \\ B_{ol} & sI - A_{ol} \end{bmatrix}$$

$$= \det [I + D_{ol}] \det \left[sI - A_{ol} + B_{ol} (I + D_{ol})^{-1} C_{ol}\right]$$

$$= \det [sI - A_{ol}] \det \left[I + D_{ol} + C_{ol} (sI - A_{ol})^{-1} B_{ol}\right]$$

5.7.2 MIMO Nyquist stability criteria

Assume $L(s)$ is stabilizable and detectable. Then the following Theorem holds.

Theorem: Generalized (MIMO) Nyquist theorem. Let P_{ol} denote the number of open-loop unstable poles in $L(s)$. The closed-loop system with loop transfer function $L(s)$ and negative feedback is stable if and only if the Nyquist plot of $\det(I + L(s))$

i) makes P_{ol} anti-clockwise encirclements of the origin, and

ii) does not pass through the origin.

Note
By "Nyquist plot of $\det(I + L(s))$" we mean "the image of $\det(I + L(s))$ as s goes clockwise around the Nyquist D-contour".

5.7.4 Small gain theorem

$$\rho(L(j\omega)) \stackrel{\Delta}{=} \max_i |\lambda_i(L(j\omega))| \tag{5.52}$$

Theorem: Spectral radius stability condition. Consider a system with a stable loop transfer function $L(s)$. Then the closed-loop system is stable if

$$\rho(L(j\omega)) \stackrel{\Delta}{=} \max_i |\lambda_i(L(j\omega))| < 1 \quad \forall\omega \tag{5.53}$$

Proof: Assume the system is unstable. Therefore $\det(I + L(s))$ encircles the origin, and there is an eigenvalue, $\lambda_i(L(j\omega))$ which is larger than 1 at some frequency. If $\det(I + L(s))$ does encircle the origin, then there must exists a gain $\epsilon \in (0, 1]$ and a frequency ω' such that

$$\det(I + \epsilon L(j\omega')) = 0 \tag{5.54}$$

or

$$\prod_i \lambda_i(I + \epsilon L(j\omega')) = 0 \tag{5.55}$$

$$\Leftrightarrow \quad 1 + \epsilon\lambda_i(L(j\omega')) = 0 \quad \text{for some } i \tag{5.56}$$

$$\Leftrightarrow \quad \lambda_i(L(j\omega')) = -\frac{1}{\epsilon} \quad \text{for some } i \tag{5.57}$$

$$\Rightarrow \quad |\lambda_i(L(j\omega'))| \geq 1 \quad \text{for some } i \tag{5.58}$$

$$\Leftrightarrow \quad \rho(L(j\omega')) \geq 1 \tag{5.59}$$

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Interpretation: If the system gain is less than 1 in all directions (all eigenvalues) and for all frequencies ($\forall\omega$), then all signal deviations will eventually die out, and the system is stable.

Spectral radius theorem is conservative because phase information is not considered.

Small Gain Theorem. Consider a system with a stable loop transfer function $L(s)$. Then the closed-loop system is stable if

$$\|L(j\omega)\| < 1 \quad \forall\omega \tag{5.60}$$

where $\|L\|$ denotes any matrix norm satisfying $\|AB\| \le \|A\| \cdot \|B\|$, for example the singular value $\bar{\sigma}(L)$.

Note The small gain theorem is generally more conservative than the spectral radius condition in (5.53).

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5.8 System norms

Figure 2: System G

Figure 2: System with stable transfer function matrix $G(s)$ and impulse response matrix $g(t)$.

Question: given information about the allowed input signals $w(t)$, how large can the outputs $z(t)$ become?

We use the 2-norm,

$$\|z(t)\|_2 = \sqrt{\sum_i \int_{-\infty}^{\infty} |z_i(\tau)|^2 d\tau} \tag{5.61}$$

and consider inputs:

1. $w(t)$ is a series of unit impulses.
2. $w(t)$ is any signal satisfying $\|w(t)\|_2 = 1$.

The relevant system norms in the two cases are the $\mathcal{H}_2$ and $\mathcal{H}_\infty$norms, respectively.

5.8.1 $\mathcal{H}_2$ norm

$G(s)$ strictly proper.

For the $\mathcal{H}_2$ norm we use the Frobenius norm spatially (for the matrix) and integrate over frequency, i.e.

$$\|G(s)\|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\mathrm{tr}(G(j\omega)^H G(j\omega))}_{\|G(j\omega)\|_F^2 = \sum_{ij} |G_{ij}(j\omega)|^2} \, d\omega} \tag{5.62}$$

$G(s)$ must be strictly proper, otherwise the $\mathcal{H}_2$ norm is infinite. By Parseval's theorem, (5.62) is equal to the $\mathcal{H}_2$ norm of the impulse response

$$\|G(s)\|_2 = \|g(t)\|_2 \triangleq \sqrt{\int_0^{\infty} \underbrace{\mathrm{tr}(g^T(\tau)g(\tau))}_{\|g(\tau)\|_F^2 = \sum_{ij} |g_{ij}(\tau)|^2} \, d\tau} \tag{5.63}$$

- Note that $G(s)$ and $g(t)$ are dynamic *systems* while $G(j\omega)$ and $g(\tau)$ are constant *matrices* (for a given value of ω or τ).

- We can change the order of integration and summation in (5.63) to get

$$\|G(s)\|_2 = \|g(t)\|_2 = \sqrt{\sum_{ij} \int_0^\infty |g_{ij}(\tau)|^2 d\tau} \tag{5.64}$$

where $g_{ij}(t)$ is the ij'th element of the impulse response matrix, $g(t)$. Thus $\mathcal{H}_2$ norm can be interpreted as the 2-norm output resulting from applying unit impulses $\delta_j(t)$ to each input, one after another (allowing the output to settle to zero before applying an impulse to the next input). Thus $\|G(s)\|^2 = \sqrt{\sum_{i=1}^m \|z_i(t)\|_2^2}$ where $z_i(t)$ is the output vector resulting from applying a unit impulse $\delta_i(t)$ to the i'th input.

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Numerical computations of the $\mathcal{H}_2$ norm.

Consider $G(s) = C(sI - A)^{-1}B$. Then

$$\|G(s)\|_2 = \sqrt{\operatorname{tr}(B^T Q B)} \quad \text{or} \quad \|G(s)\|_2 = \sqrt{\operatorname{tr}(C P C^T)} \tag{5.65}$$

where Q = observability Gramian

and P = controllability Gramian

5.8.2 $\mathcal{H}_\infty$ norm

$G(s)$ proper.

For the $\mathcal{H}_\infty$ norm we use the singular value (induced 2-norm) spatially (for the matrix) and pick out the peak value as a function of frequency

$$\|G(s)\|_\infty \stackrel{\Delta}{=} \max_\omega \bar{\sigma}(G(j\omega)) \tag{5.66}$$

The $\mathcal{H}_\infty$ norm is the peak of the transfer function "magnitude".

Time domain performance interpretations of the $\mathcal{H}_\infty$ norm.

- Worst-case steady-state gain for sinusoidal inputs at any frequency.
- Induced (worst-case) 2-norm in the time domain:

$$\|G(s)\|_\infty = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} = \max_{\|w(t)\|_2=1} \|z(t)\|_2 \tag{5.67}$$

(In essence, (5.67) arises because the worst input signal $w(t)$ is a sinusoid with frequency ω^* and a direction which gives $\overline{\sigma}(G(j\omega^*))$ as the maximum gain.)

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Numerical computation of the $\mathcal{H}_\infty$ norm.
Consider

$$G(s) = C(sI - A)^{-1}B + D$$

$\mathcal{H}_\infty$ norm is the smallest value of γ such that the Hamiltonian matrix H has no eigenvalues on the imaginary axis, where

$$H = \begin{bmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^TC)^T \end{bmatrix} \tag{5.68}$$

and $R = \gamma^2 I - D^T D$

5.8.3 Difference between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

Frobenius norm in terms of singular values

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_i \sigma_i^2(G(j\omega))d\omega} \tag{5.69}$$

Thus when optimizing performance in terms of the different norms:

- $\mathcal{H}_\infty$: "push down peak of largest singular value".
- $\mathcal{H}_2$: "push down whole thing" (all singular values over all frequencies).

Example

$$G(s) = \frac{1}{s+a} \tag{5.70}$$

$\mathcal{H}_2$ norm:

$$\begin{aligned}\|G(s)\|_2 &= (\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{|G(j\omega)|^2}_{\frac{1}{\omega^2+a^2}} d\omega)^{\frac{1}{2}} \\ &= (\frac{1}{2\pi a}\left[\tan^{-1}(\frac{\omega}{a})\right]_{-\infty}^{\infty})^{\frac{1}{2}} = \sqrt{\frac{1}{2a}}\end{aligned}$$

Alternatively: Consider the impulse response

$$g(t) = \mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}, t \geq 0 \tag{5.71}$$

to get

$$\|g(t)\|_2 = \sqrt{\int_0^{\infty}(e^{-at})^2 dt} = \sqrt{\frac{1}{2a}} \tag{5.72}$$

as expected from Parseval's theorem.

$\mathcal{H}_\infty$ norm:

$$||G(s)||_\infty = \max_\omega |G(j\omega)| = \max_\omega \frac{1}{(\omega^2+a^2)^{\frac{1}{2}}} = \frac{1}{a} \tag{5.73}$$

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Example

There is no general relationship between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms.

$$f_1(s) = \frac{1}{\epsilon s + 1}, \quad f_2(s) = \frac{\epsilon s}{s^2 + \epsilon s + 1} \tag{5.74}$$

$$\begin{aligned} ||f_1||_\infty = 1 \quad ||f_1||_2 = \infty \\ ||f_2||_\infty = 1 \quad ||f_2||_2 = 0 \end{aligned} \tag{5.75}$$

Why is the $\mathcal{H}_\infty$ norm so popular? In robust control it is convenient for representing unstructured model uncertainty, and because it satisfies the multiplicative property:

$$\|A(s)B(s)\|_\infty \leq \|A(s)\|_\infty \cdot \|B(s)\|_\infty \tag{5.76}$$

What is wrong with the $\mathcal{H}_2$ norm? It is *not* an induced norm and does *not* satisfy the multiplicative property.

Example

Consider again $G(s) = 1/(s+a)$ in (5.70), for which $\|G(s)\|_2 = \sqrt{1/2a}$.

$$\begin{aligned} \|G(s)G(s)\|_2 &= \sqrt{\int_0^\infty |\underbrace{\mathcal{L}^{-1}[(\frac{1}{s+a})^2]}_{te^{-at}}|^2} \\ &= \sqrt{\frac{1}{a}\frac{1}{2a}} = \sqrt{\frac{1}{a}}\ \|G(s)\|_2^2 \end{aligned} \tag{5.77}$$

for $a < 1$,

$$\|G(s)G(s)\|_2 > \|G(s)\|_2 \cdot \|G(s)\|_2 \tag{5.78}$$

which does not satisfy the multiplicative property.

$\mathcal{H}_\infty$ norm does satisfy the multiplicative property

$$\|G(s)G(s)\|_\infty = \frac{1}{a^2} = \|G(s)\|_\infty \cdot \|G(s)\|_\infty$$