6 INTRODUCTION TO MULTIVARIABLE CONTROL

6.1 Transfer functions for MIMO systems

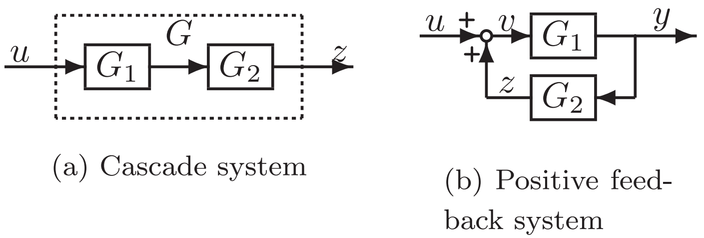

Figure 1: Block diagrams for the cascade rule and the feedback rule

1. **Cascade rule.** (Figure 1(a)) $G = G_2 G_1$
2. **Feedback rule.** (Figure 1(b)) $v = (I - L)^{-1} u$ where $L = G_2 G_1$
3. **Push-through rule.**

$$G_1(I - G_2G_1)^{-1} \quad = \quad (I - G_1G_2)^{-1}G_1$$

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MIMO Rule: *Start from the output, move backwards. If you exit from a feedback loop then include a term $(I - L)^{-1}$ where L is the transfer function around that loop (evaluated against the signal flow starting at the point of exit from the loop).*

Example

$$z = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})w \qquad (6.1)$$

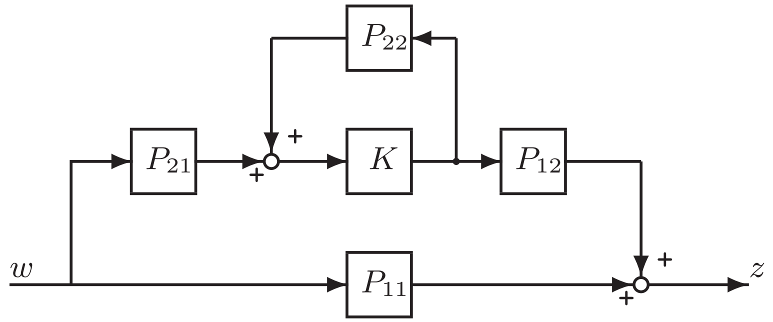

Figure 2: Block diagram corresponding to (6.1)

Negative feedback control systems

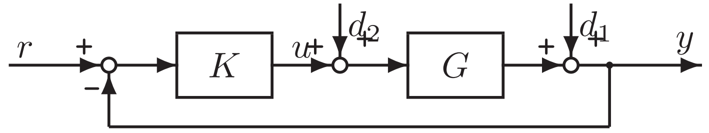

Figure 3: Conventional negative feedback control system

- L is the loop transfer function when breaking the loop at the *output* of the plant.

$$L = GK \tag{6.2}$$

Accordingly

$$S \triangleq (I+L)^{-1} \quad \text{output sensitivity} \tag{6.3}$$

$$T \triangleq I - S = (I+L)^{-1}L = L(I+L)^{-1} \quad \text{output complementary sensitivity} \tag{6.4}$$

$L_O \equiv L$, $S_O \equiv S$ and $T_O \equiv T$.

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- L_I is the loop transfer function at the *input* to the plant

$$L_I = KG \tag{6.5}$$

Input sensitivity:

$$S_I \triangleq (I + L_I)^{-1}$$

Input complementary sensitivity:

$$T_I \triangleq I - S_I = L_I(I + L_I)^{-1}$$

- Some relationships:

$$(I + L)^{-1} + (I + L)^{-1}L = S + T = I \tag{6.6}$$

$$G(I + KG)^{-1} = (I + GK)^{-1}G \tag{6.7}$$

$$GK(I+GK)^{-1} = G(I+KG)^{-1}K = (I+GK)^{-1}GK \tag{6.8}$$

$$T = L(I + L)^{-1} = (I + L^{-1})^{-1} = (I + L)^{-1}L \tag{6.9}$$

Rule to remember: "G comes first and then G and K alternate in sequence".

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6.2 Internal stability of feedback systems

Note: Checking the poles of S or T is not sufficient to determine internal stability

Example (Figure 4). In forming $L = GK$ we cancel the term $(s-1)$ (a RHP pole-zero cancellation) to obtain

$$L = GK = \frac{k}{s}, \text{ and } S = (I+L)^{-1} = \frac{s}{s+k} \qquad (6.10)$$

$S(s)$ is stable, i.e. transfer function from d_y to y is stable. However, the transfer function from d_y to u is unstable:

$$u = -K(I+GK)^{-1}d_y = -\frac{k(s+1)}{(s-1)(s+k)}d_y \qquad (6.11)$$

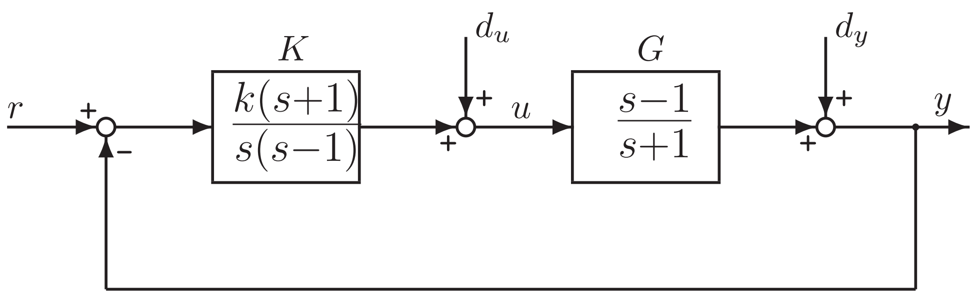

Figure 4: Internally unstable system

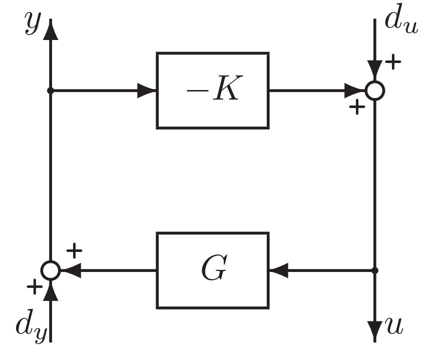

Figure 5: Block diagram used to check internal stability of feedback system

For *internal* stability consider

$$u = (I + KG)^{-1} d_u - K(I + GK)^{-1} d_y \tag{6.12}$$

$$y = G(I + KG)^{-1} d_u + (I + GK)^{-1} d_y \tag{6.13}$$

Theorem The feedback system in Figure 5 is **internally stable** if and only if all four closed-loop transfer matrices in (6.12) and (6.13) are stable.

Theorem Assume there are no RHP pole-zero cancellations between $G(s)$ and $K(s)$. Then the feedback system in Figure 5 is internally stable if and only if <u>one</u> of the four closed-loop transfer function matrices in (6.12) and (6.13) is stable.

6.3 Stabilizing controllers

6.3.1 Stable plants

Lemma For a stable plant $G(s)$ the negative feedback system in Figure 5 is internally stable if and only if $Q = K(I + GK)^{-1}$ is stable.

Proof: The four transfer functions in (6.12) and (6.13) are

$$K(I + GK)^{-1} = Q \tag{6.14}$$

$$(I + GK)^{-1} = I - GQ \tag{6.15}$$

$$(I + KG)^{-1} = I - QG \tag{6.16}$$

$$G(I + KG)^{-1} = G(I - QG) \tag{6.17}$$

which are clearly all stable if and only if Q is stable.

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Consequences: *All stabilizing negative feedback controllers for the stable plant* $G(s)$ are given by

$$K = (I - QG)^{-1}Q = Q(I - GQ)^{-1} \tag{6.18}$$

where the "parameter" Q is *any stable transfer function matrix.* (Identical to the internal model control (IMC) parameterization of stabilizing controllers.)

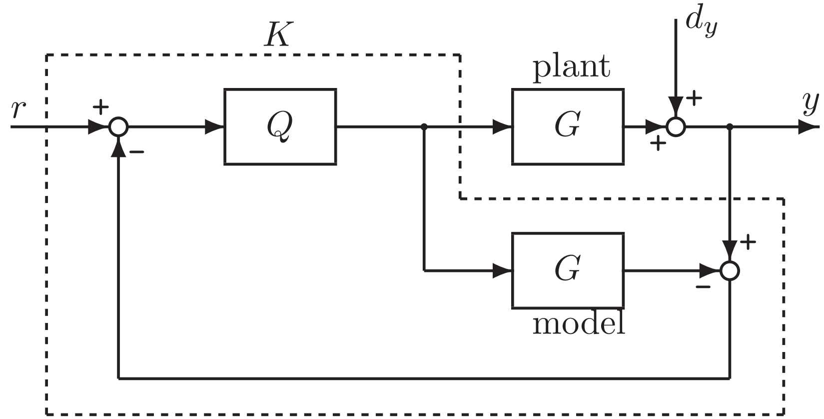

Figure 6: The internal model control (IMC) structure

6.4 Multivariable frequency response analysis

$$
\begin{aligned}
G(s) &= \text{transfer (function) matrix} \\
G(j\omega) &= \text{complex matrix representing response} \\
&\quad\ \text{to sinusoidal signal of frequency } \omega
\end{aligned}
$$

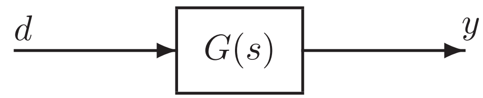

Figure 7: System $G(s)$ with input d and output y

$$y(s) = G(s)d(s) \tag{6.19}$$

Sinusoidal input to channel j

$$d_j(t) = d_{j0} \sin(\omega t + \alpha_j) \tag{6.20}$$

starting at $t = -\infty$. Output in channel i is a sinusoid with the same frequency

$$y_i(t) = y_{i0} \sin(\omega t + \beta_i) \tag{6.21}$$

Amplification (gain):

$$\frac{y_{io}}{d_{jo}} = |g_{ij}(j\omega)| \tag{6.22}$$

Phase shift:

$$\beta_i - \alpha_j = \angle g_{ij}(j\omega) \tag{6.23}$$

$g_{ij}(j\omega)$ represents the sinusoidal response from input j to output i.

Example 2×2 multivariable system, sinusoidal signals of the same frequency ω to the two input channels:

$$d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} = \begin{bmatrix} d_{10}\sin(\omega t + \alpha_1) \\ d_{20}\sin(\omega t + \alpha_2) \end{bmatrix} \tag{6.24}$$

The output signal

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_{10}\sin(\omega t + \beta_1) \\ y_{20}\sin(\omega t + \beta_2) \end{bmatrix} \tag{6.25}$$

can be computed by multiplying the complex matrix $G(j\omega)$ by the complex vector $d(\omega)$:

$$\begin{aligned} y(\omega) &= G(j\omega)d(\omega) \\ y(\omega) &= \begin{bmatrix} y_{10}e^{j\beta_1} \\ y_{20}e^{j\beta_2} \end{bmatrix}, \; d(\omega) = \begin{bmatrix} d_{10}e^{j\alpha_1} \\ d_{20}e^{j\alpha_2} \end{bmatrix} \end{aligned} \tag{6.26}$$

6.4.1 Directions in multivariable systems

SISO system ($y = Gd$): gain

$$\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|$$

The gain depends on ω, but is independent of $|d(\omega)|$.

MIMO system: input and output are vectors.

$\Rightarrow$ need to "sum up" magnitudes of elements in each vector by use of some norm

$$\|d(\omega)\|_2 = \sqrt{\sum_j |d_j(\omega)|^2} = \sqrt{d_{10}^2 + d_{20}^2 + \cdots} \quad (6.27)$$

$$\|y(\omega)\|_2 = \sqrt{\sum_i |y_i(\omega)|^2} = \sqrt{y_{10}^2 + y_{20}^2 + \cdots} \quad (6.28)$$

The *gain* of the system $G(s)$ is

$$\frac{\|y(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\sqrt{y_{10}^2 + y_{20}^2 + \cdots}}{\sqrt{d_{10}^2 + d_{20}^2 + \cdots}} \quad (6.29)$$

The gain depends on ω, and is independent of $\|d(\omega)\|_2$. However, for a MIMO system the gain depends on the *direction* of the input d.

The maximum value of the gain in (6.29) as the direction of the input is varied, is the maximum singular value of G,

$$\max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \max_{\|d\|_2=1} \|Gd\|_2 = \bar{\sigma}(G) \tag{6.30}$$

whereas the minimum gain is the minimum singular value of G,

$$\min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \min_{\|d\|_2=1} \|Gd\|_2 = \underline{\sigma}(G) \tag{6.31}$$

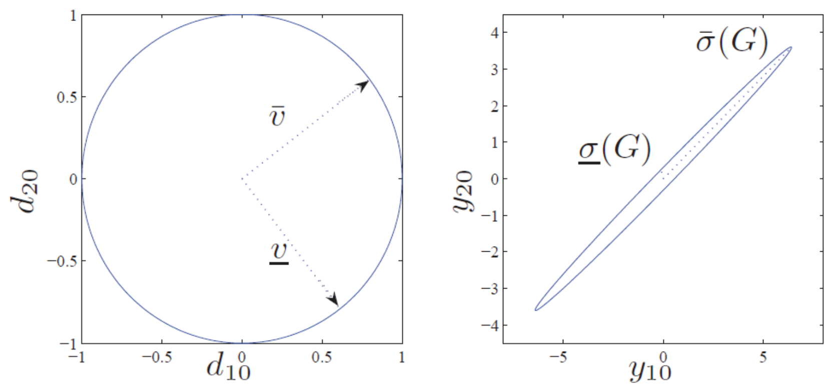

Figure 8: The maximum $\bar{\sigma}(G)$ and minimum $\underline{\sigma}(G)$ gains are obtained for $d = (\bar{v})$ and $d = (\underline{v})$ respectively.

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Example Consider the five inputs (all $\|d\|_2 = 1$)

$$\begin{aligned} d_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix},\; d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\; d_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \\ d_4 &= \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix},\; d_5 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix} \end{aligned}$$

For the 2×2 system

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \qquad (6.32)$$

The five inputs d_j lead to the following output vectors

$$y_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix},\; y_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix},\; y_3 = \begin{bmatrix} 6.36 \\ 3.54 \end{bmatrix},\; y_4 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix},\; y_5 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

with the 2-norms (i.e. the gains for the five inputs)

$$\begin{aligned} \|y_1\|_2 &= 5.83,\; \|y_2\|_2 = 4.47,\; \|y_3\|_2 = 7.30, \\ \|y_4\|_2 &= 1.00,\; \|y_5\|_2 = 0.28 \end{aligned}$$

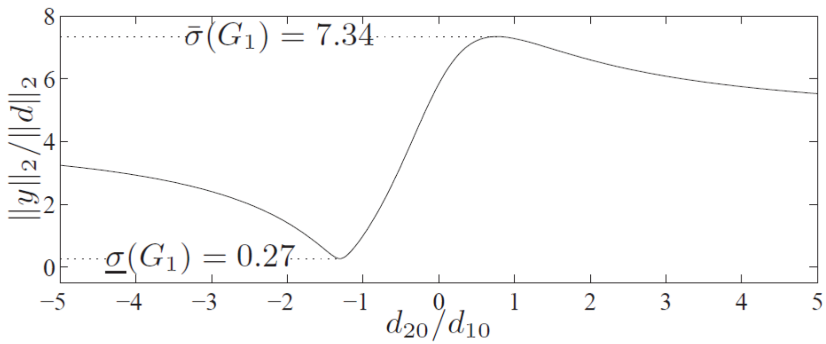

Figure 9: Gain $\|G_1 d\|_2/\|d\|_2$ as a function of d_{20}/d_{10} for G_1 in (6.32)

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6.4.2 Eigenvalues are a poor measure of gain

Example

$$G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}; \quad G\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \tag{6.33}$$

Both eigenvalues are equal to zero, but gain is equal to 100.

Problem: eigenvalues measure the gain for the special case when the inputs and the outputs are in the same direction (in the direction of the eigenvectors).

For **generalizations** of $|G|$ when G is a matrix, we need the concept of a *matrix norm*, denoted $\|G\|$. Two important properties: *triangle inequality*

$$\|G_1 + G_2\| \leq \|G_1\| + \|G_2\| \tag{6.34}$$

and the multiplicative property

$$\|G_1 G_2\| \leq \|G_1\| \cdot \|G_2\| \tag{6.35}$$

$\rho(G) \triangleq |\lambda_{max}(G)|$ (the spectral radius), does *not* satisfy the properties of a matrix norm

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6.4.3 Singular value decomposition

Any matrix G may be decomposed into its singular value decomposition,

$$G = U\Sigma V^H \tag{6.36}$$

where

Σ is an $l \times m$ matrix with $k = \min\{l, m\}$ non-negative singular values, σ_i, arranged in descending order along its main diagonal;

U is an $l \times l$ unitary matrix of output singular vectors, u_i,

V is an $m \times m$ unitary matrix of input singular vectors, v_i,

Example SVD of a real 2×2 matrix can always be written as

$$G = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cos\theta_2 & \pm\sin\theta_2 \\ -\sin\theta_2 & \pm\cos\theta_2 \end{bmatrix}^T}_{V^T} \tag{6.37}$$

U and V involve rotations and their columns are orthonormal.

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Input and output directions.

The column vectors of U, denoted u_i, represent the *output directions* of the plant. They are orthogonal and of unit length (orthonormal), that is

$$\|u_i\|_2 = \sqrt{|u_{i1}|^2 + |u_{i2}|^2 + \ldots + |u_{il}|^2} = 1 \qquad (6.38)$$

$$u_i^H u_i = 1, \quad u_i^H u_j = 0, \quad i \neq j \qquad (6.39)$$

The column vectors of V, denoted v_i, are orthogonal and of unit length, and represent the *input directions*.

$$Gv_i = \sigma_i u_i \qquad (6.40)$$

If we consider an *input* in the direction v_i, then the *output* is in the direction u_i. Since $\|v_i\|_2 = 1$ and $\|u_i\|_2 = 1$ σ_i gives the gain of the matrix G in this direction.

$$\sigma_i(G) = \|Gv_i\|_2 = \frac{\|Gv_i\|_2}{\|v_i\|_2} \qquad (6.41)$$

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Maximum and minimum singular values.

The largest gain for *any* input direction is

$$\bar{\sigma}(G) \equiv \sigma_1(G) = \max_{d\neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_1\|_2}{\|v_1\|_2} \qquad (6.42)$$

The smallest gain for any input direction is

$$\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d\neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_k\|_2}{\|v_k\|_2} \qquad (6.43)$$

where $k = \min\{l, m\}$. For any vector d we have

$$\underline{\sigma}(G) \leq \frac{\|Gd\|_2}{\|d\|_2} \leq \bar{\sigma}(G) \qquad (6.44)$$

Define $u_1 = \bar{u}, v_1 = \bar{v}, u_k = \underline{u}$ and $v_k = \underline{v}$. Then

$$G\bar{v} = \bar{\sigma}\bar{u}, \qquad G\underline{v} = \underline{\sigma}\ \underline{u} \qquad (6.45)$$

$\bar{v}$ corresponds to the input direction with largest amplification, and $\bar{u}$ is the corresponding output direction in which the inputs are most effective. The directions involving $\bar{v}$ and $\bar{u}$ are sometimes referred to as the "strongest", "high-gain" or "most important" directions.

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Example

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{6.46}$$

The singular value decomposition of G_1 is

$$G_1 = \underbrace{\begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}^H}_{V^H}$$

The largest gain of 7.343 is for an input in the direction $\bar{v} = \begin{bmatrix} 0.794 \\ 0.608 \end{bmatrix}$, the smallest gain of 0.272 is for an input in the direction $\underline{v} = \begin{bmatrix} -0.608 \\ 0.794 \end{bmatrix}$. Since in (6.46) both inputs affect both outputs, we say that the system is *interactive*. The system is *ill-conditioned*, that is, some combinations of the inputs have a strong effect on the outputs, whereas other combinations have a weak effect on the outputs. Quantified by the *condition number*;
$\bar{\sigma}/\underline{\sigma} = 7.343/0.272 = 27.0$.

Example

Shopping cart. Consider a shopping cart (supermarket trolley) with fixed wheels which we may want to move in three directions; forwards, sideways and upwards. For the shopping cart the gain depends strongly on the input direction, i.e. the plant is ill-conditioned.

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Example: Distillation process.

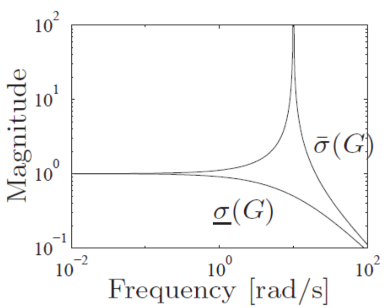
Steady-state model of a distillation column

$$G = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{6.47}$$

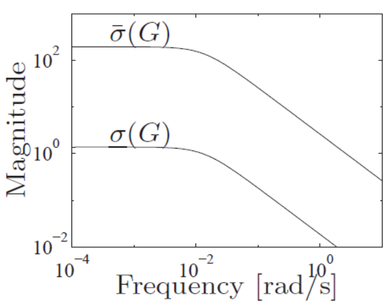
Since the elements are much larger than 1 in magnitude there should be no problems with input constraints. However, the gain in the low-gain direction is only just above 1.

$$G = \underbrace{\begin{bmatrix} 0.625 & -0.781 \\ 0.781 & 0.625 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 197.2 & 0 \\ 0 & 1.39 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.707 & -0.708 \\ -0.708 & -0.707 \end{bmatrix}^H}_{V^H} \tag{6.48}$$

The distillation process is *ill-conditioned*, and the condition number is $197.2/1.39 = 141.7$. For dynamic systems the singular values and their associated directions vary with frequency (Figure 10).



(a) Spinning satellite



(b) Distillation process

Figure 10: Typical plots of singular values

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6.4.4 Singular values for performance

Maximum singular value is very useful in terms of frequency-domain performance and robustness.

Performance measure for SISO systems:

$$|e(\omega)|/|r(\omega)| = |S(j\omega)|$$

Generalization for MIMO systems $\|e(\omega)\|_2/\|r(\omega)\|_2$

$$\underline{\sigma}(S(j\omega)) \le \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \le \bar{\sigma}(S(j\omega)) \tag{6.49}$$

For *performance* we want the gain $\|e(\omega)\|_2/\|r(\omega)\|_2$ small for any direction of $r(\omega)$

$$\begin{aligned} & \bar{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|, \ \forall\omega \\ \Leftrightarrow \quad & \bar{\sigma}(w_P S) < 1, \ \forall\omega \\ \Leftrightarrow \quad & \|w_P S\|_\infty < 1 \end{aligned} \tag{6.50}$$

where the $\mathcal{H}_\infty$ norm is defined as the peak of the maximum singular value of the frequency response

$$\|M(s)\|_\infty \triangleq \max_\omega \bar{\sigma}(M(j\omega)) \tag{6.51}$$

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Typical singular values of $S(j\omega)$ in Figure 11.

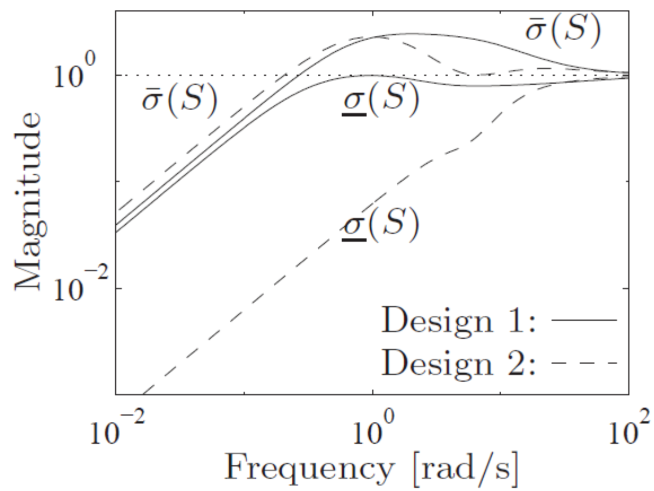

Figure 11: Singular values of S for a 2×2 plant with RHP-zero

- *Bandwidth*, ω_B: frequency where $\bar{\sigma}(S)$ crosses $\frac{1}{\sqrt{2}} = 0.7$ from below.

Since $S = (I + L)^{-1}$, the singular values inequality $\underline{\sigma}(A) - 1 \leq \frac{1}{\bar{\sigma}(I+A)^{-1}} \leq \underline{\sigma}(A) + 1$ yields

$$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1 \qquad (6.52)$$

- low ω : $\underline{\sigma}(L) \gg 1 \Rightarrow \bar{\sigma}(S) \approx \frac{1}{\underline{\sigma}(L)}$
- high ω: $\bar{\sigma}(L) \ll 1 \Rightarrow \bar{\sigma}(S) \approx 1$

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6.5 Introduction to multivariable control

- **Decentralized control.** The controller is designed as:

$$K(s) = \mathrm{diag}(k_1(s), k_2(s), \ldots, k_m(s)) \qquad (6.53)$$

where $k_i(s) = \frac{\omega_c}{s}\frac{1}{g_{ii}(s)}$.
No attempt to compensate for directionality in $G(s)$.

- **Decoupling control.** The controller is designed as:

$$K(s) = \frac{\omega_c}{s} G^{-1}(s) \qquad (6.54)$$

Compensates for directionality by employing high (low) gain in low-gain (high-gain) directions of the plant. Yields the same sensitivity in all directions:

$$S(s) = \frac{s}{s+\omega_c} I \qquad (6.55)$$

Excellent (nominal) performance.

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6.6 General control problem formulation

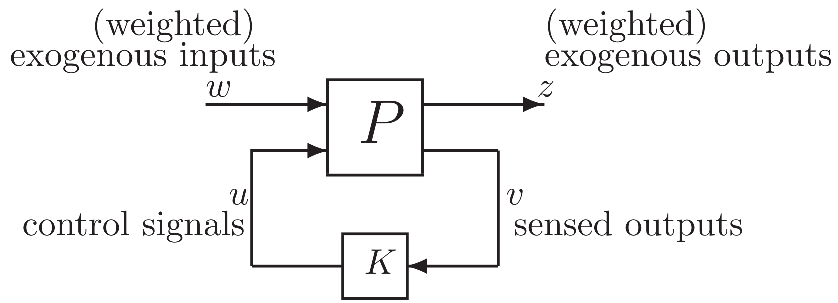

Figure 12: General control configuration for the case with no model uncertainty

The overall control objective is to minimize some norm of the transfer function from w to z, for example, the $\mathcal{H}_\infty$ norm. The controller design problem is then:

Find a controller K which based on the information in v, generates a control signal u which counteracts the influence of w on z, thereby minimizing the closed-loop norm from w to z.

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6.6.1 Obtaining the generalized plant P

The routines in MATLAB for synthesizing $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal controllers assume that the problem is in the general form of Figure 12

Example: One degree-of-freedom feedback control configuration.

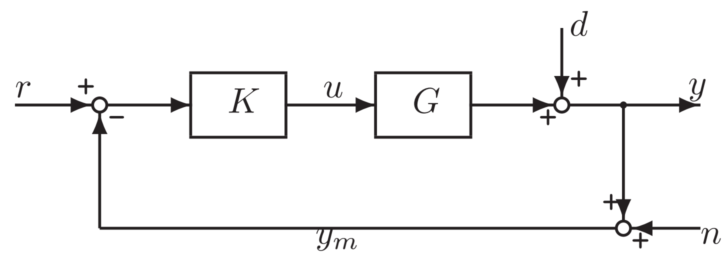

Figure 13: One degree-of-freedom control configuration

Equivalent representation of Figure 13 where the error signal to be minimized is $z = y - r$ and the input to the controller is $v = r - y_m$

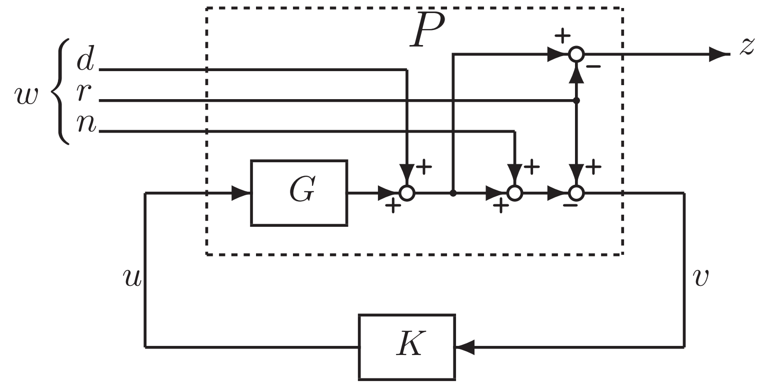

Figure 14: General control configuration equivalent to Figure 13

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix}; z = e = y - r; v = r - y_m = r - y - n \tag{6.56}$$

$$\begin{aligned} z &= y - r = Gu + d - r = Iw_1 - Iw_2 + 0w_3 + Gu \\ v &= r - y_m = r - Gu - d - n = \\ &= -Iw_1 + Iw_2 - Iw_3 - Gu \end{aligned}$$

and P which represents the transfer function matrix from $[w \quad u]^T$ to $[z \quad v]^T$ is

$$P = \begin{bmatrix} I & -I & 0 & G \\ -I & I & -I & -G \end{bmatrix} \tag{6.57}$$

Note that P does *not* contain the controller.
Alternatively, P can be obtained from Figure 14.

Remark. In MATLAB we may obtain P via `simulink`, or we may use the `sysic` program in the Robust Control toolbox. The code in Table 1 generates the generalized plant P in (6.57) for Figure 13.

Table 1: **Matlab program to generate P**

```
% Uses the Robust Control toolbox
systemnames = 'G';                              % G is the SISO plant.
inputvar = '[d(1);r(1);n(1);u(1)]';             % Consists of vectors w and u.
input_to_G = '[u]';
outputvar = '[G+d-r; r-G-d-n]';                 % Consists of vectors z and v.
sysoutname = 'P';
sysic;
```

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6.6.2 Including weights in P

To get a meaningful controller synthesis problem, for example, in terms of the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norms, we generally have to include weights W_z and W_w in the generalized plant P, see Figure 15.

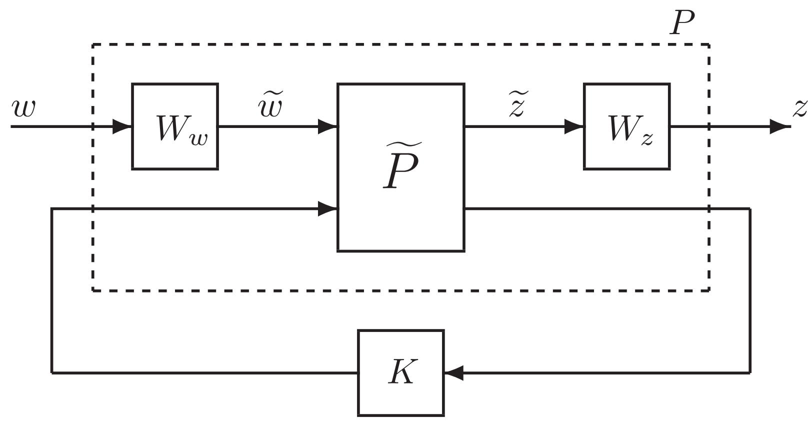

Figure 15: General control configuration for the case with no model uncertainty

That is, we consider the weighted or normalized exogenous inputs w, and the weighted or normalized controlled outputs $z = W_z\widetilde{z}$. The weighting matrices are usually frequency dependent and typically selected such that weighted signals w and z are of magnitude 1, that is, the norm from w to z should be less than 1.

Example: Stacked $S/T/KS$ problem.

Consider an $\mathcal{H}_\infty$ problem where we want to bound $\bar{\sigma}(S)$ (for performance), $\bar{\sigma}(T)$ (for robustness and to avoid sensitivity to noise) and $\bar{\sigma}(KS)$ (to penalize large inputs). These requirements may be combined into a stacked $\mathcal{H}_\infty$ problem

$$\min_K \|N(K)\|_\infty, \quad N = \begin{bmatrix} W_u KS \\ W_T T \\ W_P S \end{bmatrix} \tag{6.58}$$

where K is a stabilizing controller. In other words, we have $z = Nw$ and the objective is to minimize the $\mathcal{H}_\infty$ norm from w to z.

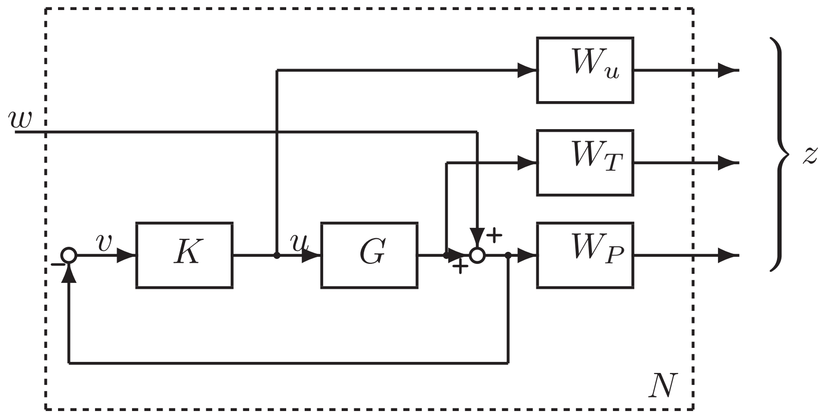

Figure 16: Block diagram corresponding to generalized plant in (6.58)

$$
\begin{aligned}
z_1 &= W_u u \\
z_2 &= W_T G u \\
z_3 &= W_P w + W_P G u \\
v &= -w - Gu
\end{aligned}
$$

so the generalized plant P from $[\,w \quad u\,]^T$ to $[\,z \quad v\,]^T$ is

$$
P = \left[\begin{array}{c|c} 0 & W_u I \\ 0 & W_T G \\ W_P I & W_P G \\ \hline -I & -G \end{array}\right] \qquad (6.59)
$$

6.6.3 Partitioning the generalized plant P

We often partition P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{6.60}$$

so that

$$\begin{aligned} z &= P_{11}w + P_{12}u && (6.61) \\ v &= P_{21}w + P_{22}u && (6.62) \end{aligned}$$

In Example "Stacked $S/T/KS$ problem" we get from (6.59)

$$P_{11} = \begin{bmatrix} 0 \\ 0 \\ W_P I \end{bmatrix}, \quad P_{12} = \begin{bmatrix} W_u I \\ W_T G \\ W_P G \end{bmatrix} \tag{6.63}$$

$$P_{21} = -I, \quad P_{22} = -G \tag{6.64}$$

Note that P_{22} has dimensions compatible with the controller K in Figure 15

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6.6.4 Analysis: Closing the loop to get N

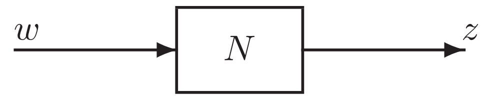

Figure 17: General block diagram for analysis with no uncertainty

For *analysis* of closed-loop performance we may absorb K into the interconnection structure and obtain the system N as shown in Figure 17 where

$$z = Nw \tag{6.65}$$

where N is a function of K. To find N, first partition the generalized plant P as given in (6.60)-(6.62), combine this with the controller equation

$$u = Kv \tag{6.66}$$

and eliminate u and v from equations (6.61), (6.62) and (6.66) to yield $z = Nw$ where N is given by

$$N = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \triangleq F_l(P, K) \tag{6.67}$$

Here $F_l(P, K)$ denotes a lower *linear fractional transformation (LFT)* of P with K as the parameter. In words, N is obtained from Figure 12 by using K to close a lower feedback loop around P. Since positive feedback is used in the general configuration in Figure 12 the term $(I - P_{22}K)^{-1}$ has a negative sign.

Example: We want to derive N for the partitioned P in (6.63) and (6.64) using the LFT-formula in (6.67). We get

$$N = \begin{bmatrix} 0 \\ 0 \\ W_P I \end{bmatrix} + \begin{bmatrix} W_u I \\ W_T G \\ W_P G \end{bmatrix} K(I+GK)^{-1}(-I) = \begin{bmatrix} -W_u KS \\ -W_T T \\ W_P S \end{bmatrix}$$

where we have made use of the identities
$S = (I + GK)^{-1}$, $T = GKS$ and $I - T = S$.

In the MATLAB Robust Control Toolbox we can evaluate $N = F_l(P, K)$ using the command `N=lft(P,K)`.

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6.6.5 Further examples

Example: Consider the control system in Figure 18, where y_1 is the output we want to control, y_2 is a secondary output (extra measurement), and we also measure the disturbance d. The control configuration includes a two degrees-of-freedom controller, a feedforward controller and a local feedback controller based on the extra measurement y_2.

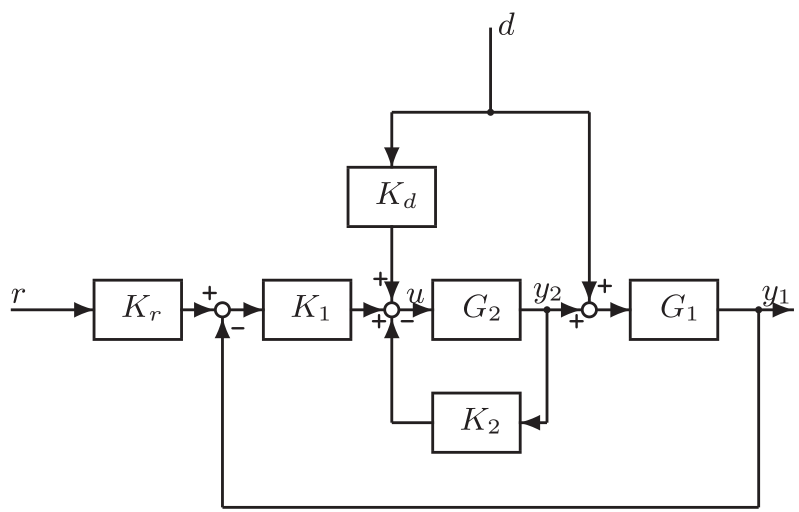

Figure 18: System with feedforward, local feedback and two degrees-of-freedom control

To recast this into our standard configuration of Figure 12 we define

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$$w = \begin{bmatrix} d \\ r \end{bmatrix}; \quad z = y_1 - r; \quad v = \begin{bmatrix} r \\ y_1 \\ y_2 \\ d \end{bmatrix} \tag{6.68}$$

$$K = [\, K_1 K_r \quad -K_1 \quad -K_2 \quad K_d \,] \tag{6.69}$$

We get

$$P = \left[\begin{array}{cc|c} G_1 & -I & G_1 G_2 \\ \hline 0 & I & 0 \\ G_1 & 0 & G_1 G_2 \\ 0 & 0 & G_2 \\ I & 0 & 0 \end{array} \right] \tag{6.70}$$

Then partitioning P as in (6.61) and (6.62) yields $P_{22} = [\, 0^T \quad (G_1 G_2)^T \quad G_2^T \quad 0^T \,]^T$.

6.6.6 * Deriving P from N

For cases where N is given and we wish to find a P such that

$$N = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

it is usually best to work from a block diagram representation. This was illustrated above for the stacked N in (6.58). Alternatively, the following procedure may be useful:

1. Set $K = 0$ in N to obtain P_{11}.

2. Define $Q = N - P_{11}$ and rewrite Q such that each term has a common factor $R = K(I - P_{22}K)^{-1}$ (this gives P_{22}).

3. Since $Q = P_{12}RP_{21}$, we can now usually obtain P_{12} and P_{21} by inspection.

Example 1 Weighted sensitivity. *We will use the above procedure to derive P when*

$N = w_P S = w_P (I + GK)^{-1}$,

where w_P is a scalar weight.

1. $P_{11} = N(K = 0) = w_P I$.

2. $Q = N - w_P I = w_P(S - I) = -w_P T = -w_P GK(I + GK)^{-1}$, *and we have* $R = K(I + GK)^{-1}$ *so* $P_{22} = -G$.

3. $Q = -w_P GR$ *so we have* $P_{12} = -w_P G$ *and* $P_{21} = I$, *and we get*

$$P = \begin{bmatrix} w_P I & -w_P G \\ I & -G \end{bmatrix} \qquad (6.71)$$

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6.6.8 A general control configuration including model uncertainty

The general control configuration in Figure 12 may be extended to include model uncertainty. Here the matrix Δ is a *block-diagonal* matrix that includes all possible perturbations (representing uncertainty) to the system. It is normalized such that $\|\Delta\|_\infty \leq 1$.

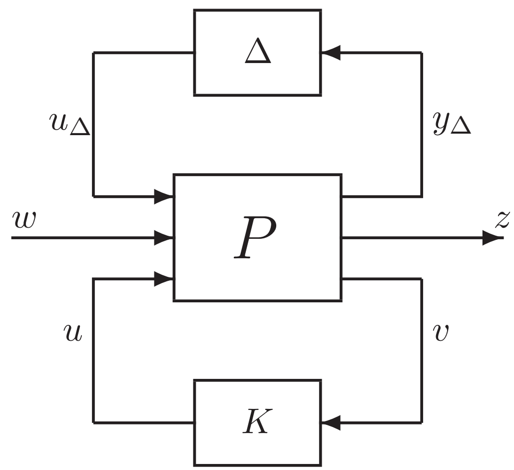

Figure 19: General control configuration for the case with model uncertainty

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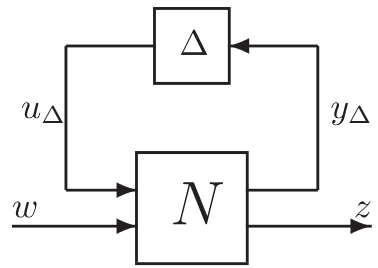

Figure 20: General block diagram for analysis with uncertainty included

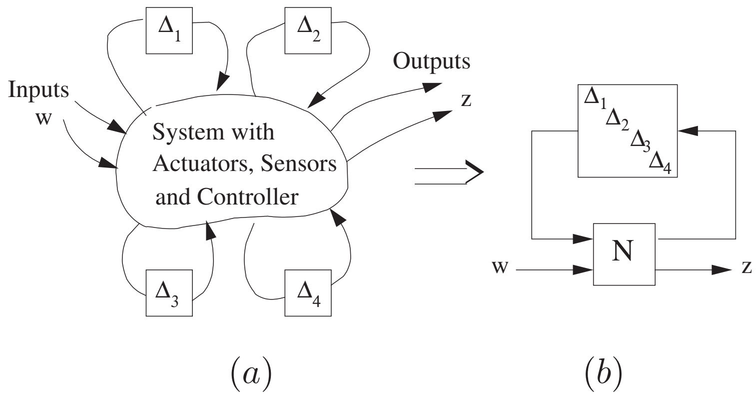

Figure 21: Rearranging a system with multiple perturbations into the $N\Delta$-structure

The block diagram in Figure 19 in terms of P (for synthesis) may be transformed into the block diagram in Figure 20 in terms of N (for analysis) by using K to close a lower loop around P. The same *lower LFT* as found in (6.67) applies, and

$$N = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \quad (6.72)$$

To evaluate the perturbed (uncertain) transfer function from external inputs w to external outputs z, we use Δ to close the upper loop around N (see Figure 20), resulting in an *upper LFT*:

$$z = F_u(N, \Delta)w; \quad (6.73)$$

$$F_u(N, \Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \quad (6.74)$$

Remark 1 Almost any control problem with uncertainty can be represented by Figure 19. First represent each source of uncertainty by a perturbation block, Δ_i, which is normalized such that $\|\Delta_i\| \leq 1$. Then "pull out" each of these blocks from the system so that an input and an output can be associated with each Δ_i as shown in Figure 21(a). Finally, collect these perturbation blocks into a large block-diagonal matrix having perturbation inputs and outputs as shown in Figure 21(b).