8 ROBUST STABILITY AND PERFORMANCE ANALYSIS

8.1 General control configuration with uncertainty

For our robustness analysis we use a system representation in which the uncertain perturbations are "pulled out" into a block-diagonal matrix,

$$\Delta = \mathrm{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & & & \\ & \ddots & & \\ & & \Delta_i & \\ & & & \ddots \end{bmatrix} \tag{8.1}$$

where each Δ_i represents a specific source of uncertainty.

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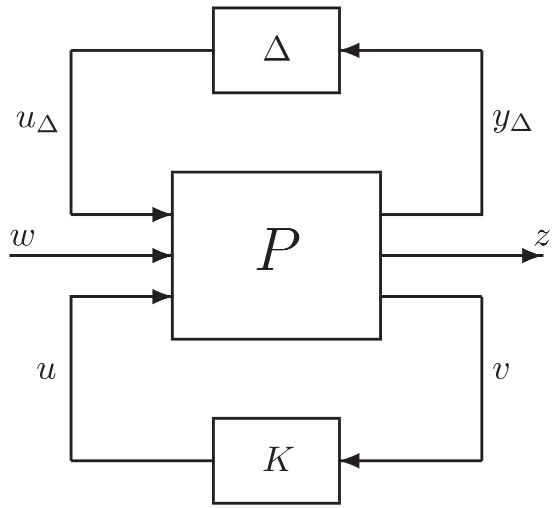


Figure 1: General control configuration (for controller synthesis)

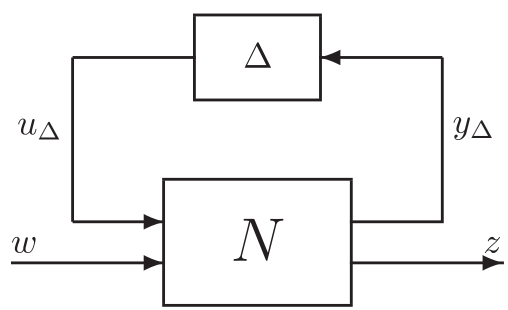


Figure 2: $N\Delta$-structure for robust performance analysis

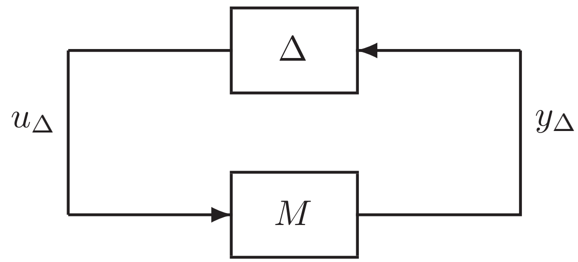
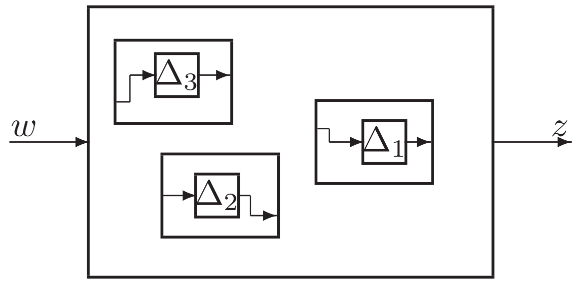


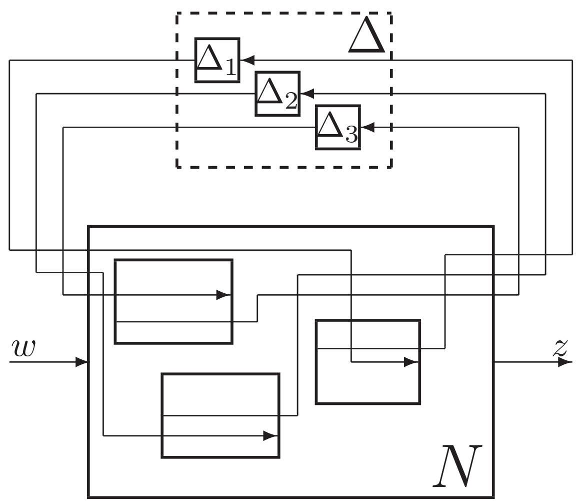
Figure 3: $M\Delta$-structure for robust stability analysis

If we also pull out the controller K, we get the generalized plant P, as shown in Figure 1. *For analysis* of the uncertain system, we use the $N\Delta$-structure in Figure 2.



(a) Original system with multiple perturbations

$\Downarrow$



(b) Pulling out the perturbations

Figure 4: Rearranging an uncertain system into the $N\Delta$-structure

Consider Figure 4 where it is shown how to pull out the perturbation blocks to form Δ and the nominal system N. As shown, N is related to P and K by a lower LFT

$$N = F_l(P, K) \triangleq P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \quad (8.2)$$

Similarly, the uncertain closed-loop transfer function from w to z, $z = Fw$, is related to N and Δ by an upper LFT,

$$F = F_u(N, \Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \quad (8.3)$$

To analyze robust stability of F, we can then rearrange the system into the $M\Delta$-structure of Figure 3 where $M = N_{11}$ is the transfer function from the output to the input of the perturbations.

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8.2 Representing uncertainty

As usual, each individual perturbation is assumed to be stable and is normalized,

$$\bar{\sigma}(\Delta_i(j\omega)) \leq 1 \ \forall\omega \tag{8.4}$$

For a complex scalar perturbation we have $|\delta_i(j\omega)| \leq 1, \ \forall\omega$, and for a real scalar perturbation $-1 \leq \delta_i \leq 1$. Since the maximum singular value of a block diagonal matrix is equal to the largest of the maximum singular values of the individual blocks, it then follows for $\Delta = \mathrm{diag}\{\Delta_i\}$ that

$$\bar{\sigma}(\Delta_i(j\omega)) \leq 1 \ \forall\omega, \ \forall i \quad \Leftrightarrow \quad \boxed{\|\Delta\|_\infty \leq 1} \tag{8.5}$$

Note that Δ has *structure*, and therefore in the robustness analysis we do *not* want to allow all Δ such that (8.5) is satisfied.

8.2.1 Unstructured uncertainty

We define *unstructured* uncertainty as the use of a "full" complex perturbation matrix Δ, usually with dimensions compatible with those of the plant, where at each frequency any $\Delta(j\omega)$ satisfying $\bar{\sigma}(\Delta(j\omega)) \leq 1$ is allowed.

Six common forms of unstructured uncertainty are shown in Figure 5. In Figure 5(a), (b) and (c) are shown three *feedforward* forms; additive uncertainty, multiplicative input uncertainty and multiplicative output uncertainty:

$$\Pi_A : \quad G_p = G + E_A; \quad E_a = w_A \Delta_a \tag{8.6}$$

$$\Pi_I : \quad G_p = G(I + E_I); \quad E_I = w_I \Delta_I \tag{8.7}$$

$$\Pi_O : \quad G_p = (I + E_O)G; \quad E_O = w_O \Delta_O \tag{8.8}$$

In Figure 5(d), (e) and (f) are shown three *feedback* or *inverse* forms; inverse additive uncertainty, inverse multiplicative input uncertainty and inverse multiplicative output uncertainty:

$$\Pi_{iA}: \qquad G_p = G(I - E_{iA}G)^{-1}; \qquad E_{iA} = w_{iA}\Delta_{iA} \tag{8.9}$$

$$\Pi_{iI}: \qquad G_p = G(I - E_{iI})^{-1}; \qquad E_{iI} = w_{iI}\Delta_{iI} \tag{8.10}$$

$$\Pi_{iO}: \qquad G_p = (I - E_{iO})^{-1}G; \qquad E_{iO} = w_{iO}\Delta_{iO} \tag{8.11}$$

The negative sign in front of the E's does not really matter here since we assume that Δ can have any sign. Δ denotes the normalized perturbation and E the "actual" perturbation. We have here used scalar weights w, so $E = w\Delta = \Delta w$, but sometimes one may want to use matrix weights, $E = W_2 \Delta W_1$ where W_1 and W_2 are given transfer function matrices.

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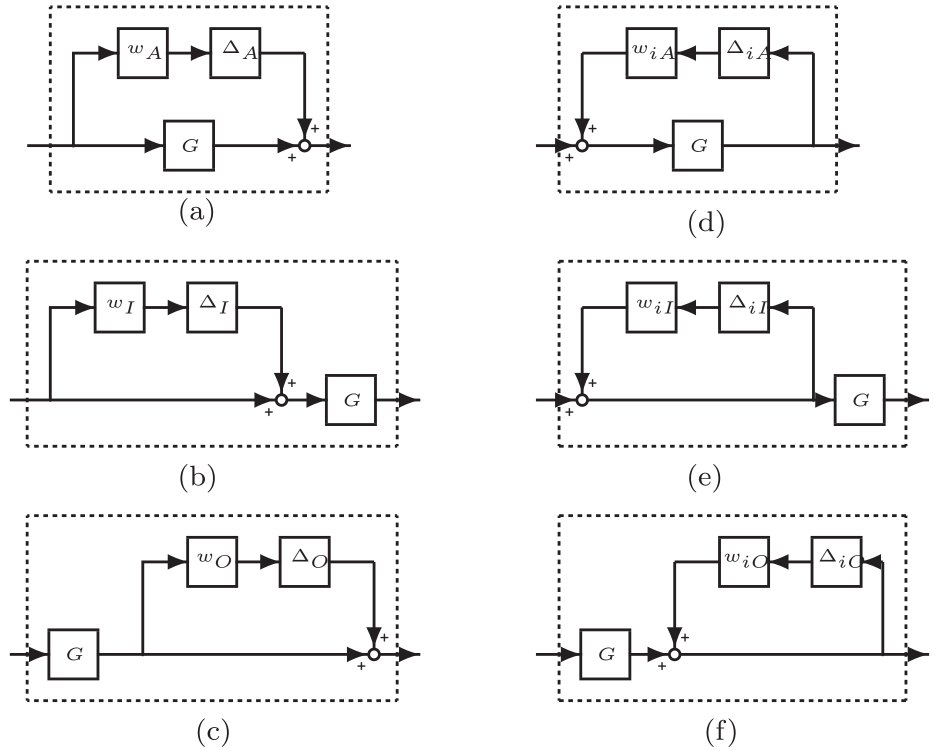

Figure 5: (a) Additive uncertainty, (b) Multiplicative input uncertainty, (c) Multiplicative output uncertainty, (d) Inverse additive uncertainty, (e) Inverse multiplicative input uncertainty, (f) Inverse multiplicative output uncertainty

8.3 Obtaining P, N and M

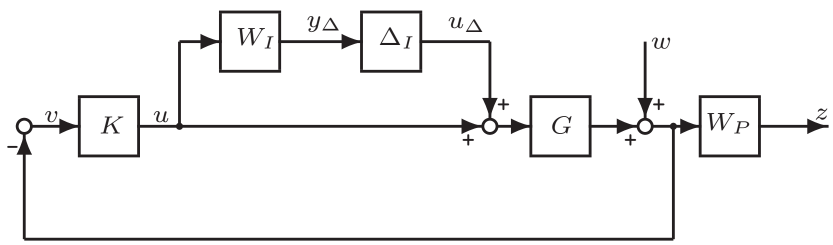

Figure 6: System with multiplicative input uncertainty and performance measured at the output

Example 1 System with input uncertainty (Figure 6). *We want to derive the generalized plant P in Figure 1 which has inputs $[u_\Delta \quad w \quad u]^T$ and outputs $[y_\Delta \quad z \quad v]^T$. By writing down the equations or simply by inspecting Figure 6 (remember to break the loop before and after K) we get*

$$P = \begin{bmatrix} 0 & 0 & W_I \\ W_P G & W_P & W_P G \\ -G & -I & -G \end{bmatrix} \tag{8.12}$$

Next, we want to derive the matrix N corresponding to Figure 2. First, partition P to be compatible with K, i.e.

$$P_{11} = \begin{bmatrix} 0 & 0 \\ W_P G & W_P \end{bmatrix}, \; P_{12} = \begin{bmatrix} W_I \\ W_P G \end{bmatrix} \tag{8.13}$$

$$P_{21} = [\,-G \quad -I\,], \;\; P_{22} = -G \tag{8.14}$$

and then find $N = F_l(P, K)$ *using (8.2). We get*

$$N = \begin{bmatrix} -W_I KG(I+KG)^{-1} & -W_I K(I+GK)^{-1} \\ W_P G(I+KG)^{-1} & W_P(I+GK)^{-1} \end{bmatrix} \tag{8.15}$$

Alternatively, we can derive N *directly from Figure 6 by evaluating the closed-loop transfer function from inputs* $\begin{bmatrix} u_\Delta \\ w \end{bmatrix}$ *to outputs* $\begin{bmatrix} y_\Delta \\ z \end{bmatrix}$ *(*without *breaking the loop before and after* K*). For example, to derive* N_{12}*, which is the transfer function from* w *to* y_Δ*, we start at the output* (y_Δ) *and move backwards to the input* (w) *using the MIMO Rule (we first meet* W_I*, then* $-K$ *and we then exit the feedback loop and get the term* $(I+GK)^{-1}$*).*

The upper left block, N_{11}*, in (8.15) is the transfer function from* u_Δ *to* y_Δ*. This is the transfer function* M *needed in Figure 3 for evaluating robust stability. Thus, we have* $M = -W_I KG(I+KG)^{-1} = -W_I T_I$*.*

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8.4 Definitions of robust stability and robust performance

1. *Robust stability (RS) analysis*: with a given controller K we determine whether the system remains stable for all plants in the uncertainty set.
2. *Robust performance (RP) analysis*: if RS is satisfied, we determine how "large" the transfer function from exogenous inputs w to outputs z may be for all plants in the uncertainty set.

In Figure 2, w represents the exogenous inputs (normalized disturbances and references), and z the exogenous outputs (normalized errors). We have $z = F(\Delta)w$, where from (8.3)

$$F = F_u(N, \Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \quad (8.16)$$

We here use the $\mathcal{H}_\infty$ norm to define performance and require for RP that $\|F(\Delta)\|_\infty \leq 1$ for all allowed Δ's. A typical choice is $F = w_P S_p$ (the weighted sensitivity function), where w_P is the performance weight (capital P for performance) and S_p represents the set of perturbed sensitivity functions (lower-case p for perturbed).

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In terms of the $N\Delta$-structure in Figure 2 our requirements for stability and performance are

$$\begin{aligned}
\text{NS} &\overset{\text{def}}{\Leftrightarrow} N \text{ is internally stable} && (8.17)\\
\text{NP} &\overset{\text{def}}{\Leftrightarrow} \|N_{22}\|_\infty < 1; \quad \text{and NS} && (8.18)\\
\text{RS} &\overset{\text{def}}{\Leftrightarrow} F = F_u(N,\Delta) \text{ is stable } \forall\Delta, \|\Delta\|_\infty \le 1; \\
&\qquad \text{and NS} && (8.19)\\
\text{RP} &\overset{\text{def}}{\Leftrightarrow} \|F\|_\infty < 1, \quad \forall\Delta, \|\Delta\|_\infty \le 1; \\
&\qquad \text{and NS} && (8.20)
\end{aligned}$$

Remark 1 Allowed perturbations. For simplicity below we will use the shorthand notation

$$\forall\Delta \quad \text{and} \quad \max_{\Delta} \tag{8.21}$$

to mean "for all Δ's in the set of allowed perturbations", and "maximizing over all Δ's in the set of allowed perturbations". By *allowed perturbations* we mean that the $\mathcal{H}_\infty$ norm of Δ is less or equal to 1, $\|\Delta\|_\infty \le 1$, and that Δ has a specified block-diagonal structure.

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8.5 Robust stability of the $M\Delta$-structure

Consider the uncertain $N\Delta$-system in Figure 2 for which the transfer function from w to z is, as in (8.16), given by

$$F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \quad (8.22)$$

Suppose that the system is nominally stable (with $\Delta = 0$), that is, N is stable. We also assume that Δ is stable. Thus, when we have nominal stability (NS), the stability of the system in Figure 2 is equivalent to the stability of the $M\Delta$-structure in Figure 3 where $M = N_{11}$.

Theorem 1 - Generalized Nyquist Criterion - Determinant stability condition (Real or complex perturbations). *Assume that the nominal system $M(s)$ and the perturbations $\Delta(s)$ are stable. Consider the convex set of perturbations Δ, such that if Δ' is an allowed perturbation then so is $c\Delta'$ where c is any* <u>*real*</u> *scalar such that $|c| \leq 1$. Then the $M\Delta$-system in Figure 3 is stable for all allowed perturbations (we have RS) if and only if*

$$\textit{Nyquist plot of } \det\left(I - M(s)\Delta(s)\right) \textit{ does not encircle the origin, } \forall\Delta \tag{8.23}$$

$$\Leftrightarrow \quad \boxed{\det\left(I - M(j\omega)\Delta(j\omega)\right) \neq 0, \quad \forall\omega, \forall\Delta} \tag{8.24}$$

$$\Leftrightarrow \quad \lambda_i(M\Delta) \neq 1, \quad \forall i, \forall\omega, \forall\Delta \tag{8.25}$$

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8.6 RS for complex unstructured uncertainty

Theorem 2 RS for unstructured ("full") perturbations. *Assume that the nominal system* $M(s)$ *is stable (NS) and that the perturbations* $\Delta(s)$ *are stable. Then the* $M\Delta$*-system in Figure 3 is stable for all perturbations* Δ *satisfying* $\|\Delta\|_\infty \le 1$ *(i.e. we have RS) if and only if*

$$\boxed{\bar{\sigma}(M(j\omega)) < 1 \quad \forall w} \qquad \Leftrightarrow \qquad \boxed{\|M\|_\infty < 1} \tag{8.26}$$

8.6.1 Application of the unstructured RS-condition

We will now present necessary and sufficient conditions for robust stability (RS) for each of the six single unstructured perturbations in Figure 5. with

$$E = W_2 \Delta W_1, \quad \|\Delta\|_\infty \leq 1 \tag{8.27}$$

To derive the matrix M we simply “isolate” the perturbation, and determine the transfer function matrix

$$M = W_1 M_0 W_2 \tag{8.28}$$

from the output to the input of the perturbation, where M_0 for each of the six cases (disregarding some negative signs which do not affect the subsequent robustness condition) is given by

$$\begin{aligned}
G_p = G + E_A : &\quad M_0 = K(I+GK)^{-1} = KS \\
G_p = G(I + E_I) : &\quad M_0 = K(I+GK)^{-1}G = T_I \\
G_p = (I + E_O)G : &\quad M_0 = GK(I+GK)^{-1} = T \\
G_p = G(I - E_{iA}G)^{-1} : &\quad M_0 = (I+GK)^{-1}G = SG \\
G_p = G(I - E_{iI})^{-1} : &\quad M_0 = (I+KG)^{-1} = S_I \\
G_p = (I - E_{iO})^{-1}G : &\quad M_0 = (I+GK)^{-1} = S
\end{aligned} \tag{8.29}$$

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Theorem 2 then yields

$$\text{RS} \quad \Leftrightarrow \quad \|W_1 M_0 W_2(j\omega)\|_\infty < 1, \forall\ w \tag{8.30}$$

For instance, from second equation in (8.29) and (8.30) we get for multiplicative input uncertainty with a scalar weight:

$$\text{RS}\ \forall G_p = G(I + w_I \Delta_I),\ \|\Delta_I\|_\infty \leq 1\ \Leftrightarrow\ \|w_I T_I\|_\infty < 1 \tag{8.31}$$

Note that the SISO-condition follows as a special case of (8.31). Similarly, it follows as a special case of the inverse multiplicative output uncertainty in (8.29):

$$\begin{aligned}&\text{RS}\ \forall G_p = (I - w_{iO}\Delta_{iO})^{-1} G,\\ &\|\Delta_{iO}\|_\infty \leq 1 \Leftrightarrow \|w_{iO} S\|_\infty < 1\end{aligned} \tag{8.32}$$

In general, the unstructured uncertainty descriptions in terms of a single perturbation are not "tight".

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8.7 RS with structured uncertainty: Motivation

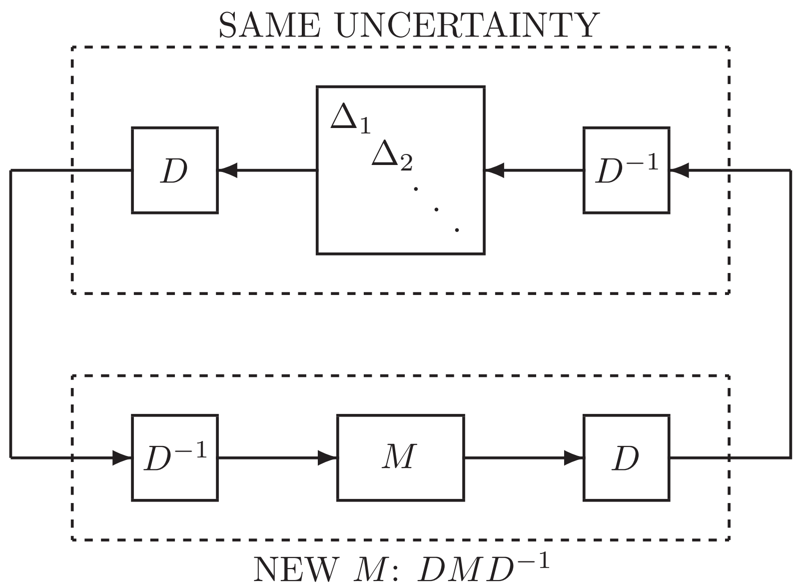

Figure 7: Use of block-diagonal scalings, $\Delta D = D\Delta$

Consider now the presence of structured uncertainty, where $\Delta = \text{diag}\{\Delta_i\}$ is block-diagonal. To test for robust stability we rearrange the system into the $M\Delta$-structure and we have from (8.26)

$$\text{RS} \quad \text{if} \quad \bar{\sigma}(M(j\omega)) < 1, \forall\omega \tag{8.33}$$

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We have here written "if" rather than "if and only if" since this condition is only necessary for RS when Δ has "no structure" (full-block uncertainty). To reduce conservatism introduce the block-diagonal scaling matrix

$$D = \mathrm{diag}\{d_i I_i\} \tag{8.34}$$

where d_i is a scalar and I_i is an identity matrix of the same dimension as the i'th perturbation block, Δ_i (Figure 7). This clearly has no effect on stability.

$$\text{RS} \quad \text{if} \quad \bar{\sigma}(DMD^{-1}) < 1, \forall\omega \tag{8.35}$$

This applies for any D in (8.34), and therefore the "most improved" (least conservative) RS-condition is obtained by minimizing at each frequency the scaled singular value, and we have

$$\boxed{\text{RS} \quad \text{if} \quad \min_{D(\omega)\in\mathcal{D}} \bar{\sigma}(D(\omega)M(j\omega)D(\omega)^{-1}) < 1, \forall\omega} \tag{8.36}$$

where $\mathcal{D}$ is the set of block-diagonal matrices whose structure is compatible to that of Δ, i.e, $\Delta D = D\Delta$.

8.8 The structured singular value

The structured singular value (denoted Mu, mu, SSV or μ) is a function which provides a generalization of the singular value, $\bar{\sigma}$, and the spectral radius, ρ. We will use μ to get necessary and sufficient conditions for robust stability and also for robust performance. How is μ defined? A simple statement is:

> *Find the smallest structured Δ (measured in terms of $\bar{\sigma}(\Delta)$) which makes $\det(I - M\Delta) = 0$; then $\mu(M) = 1/\bar{\sigma}(\Delta)$.*

Mathematically,

$$\mu(M)^{-1} \triangleq \min_{\Delta}\{\bar{\sigma}(\Delta) | \det(I - M\Delta) = 0 \text{ for structured } \Delta\} \tag{8.37}$$

Clearly, $\mu(M)$ depends not only on M but also on the allowed structure for Δ. This is sometimes shown explicitly by using the notation $\mu_{\Delta}(M)$.

Remark. For the case where Δ is "unstructured" (a full matrix), the smallest Δ which yields singularity has $\bar{\sigma}(\Delta) = 1/\bar{\sigma}(M)$, and we have $\mu(M) = \bar{\sigma}(M)$.

Example 2 Full perturbation *(Δ is unstructured).*
Consider

$$M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} =$$
$$= \begin{bmatrix} 0.894 & 0.447 \\ -0.447 & 0.894 \end{bmatrix} \begin{bmatrix} 3.162 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}^H \quad (8.38)$$

The perturbation

$$\Delta = \frac{1}{\sigma_1} v_1 u_1^H = \frac{1}{3.162} \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} \begin{bmatrix} 0.894 & -0.447 \end{bmatrix} =$$
$$= \begin{bmatrix} 0.200 & 0.200 \\ -0.100 & -0.100 \end{bmatrix} \quad (8.39)$$

with $\bar{\sigma}(\Delta) = 1/\bar{\sigma}(M) = 1/3.162 = 0.316$ *makes* $\det(I - M\Delta) = 0$. *Thus* $\mu(M) = 3.162$ *when* Δ *is a full matrix.*

Note that the perturbation Δ in (8.39) is a full matrix. If we restrict Δ to be diagonal then we need a larger perturbation to make $\det(I - M\Delta) = 0$. This is illustrated next.

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Example 2 continued. Diagonal perturbation *(Δ is structured). For the matrix M in (8.38), the smallest diagonal Δ which makes $\det(I - M\Delta) = 0$ is*

$$\Delta = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{8.40}$$

with $\bar{\sigma}(\Delta) = 0.333$. Thus $\mu(M) = 3$ when Δ is a diagonal matrix.

Definition 1 Structured Singular Value. *Let M be a given complex matrix and let $\Delta = \mathrm{diag}\{\Delta_i\}$ denote a set of complex matrices with $\bar{\sigma}(\Delta) \leq 1$ and with a given block-diagonal structure (in which some of the blocks may be repeated and some may be restricted to be real). The real non-negative function $\mu(M)$, called the structured singular value, is defined by*

$$\mu(M) \triangleq \frac{1}{\min\{k_m | \det(I - k_m M\Delta) = 0, \bar{\sigma}(\Delta) \leq 1\}} \tag{8.41}$$

If no such structured Δ exists then $\mu(M) = 0$.

A value of $\mu = 1$ means that there exists a perturbation with $\bar{\sigma}(\Delta) = 1$ which is just large enough to make $I - M\Delta$ singular. A larger value of μ is “bad” as it means that a smaller perturbation makes $I - M\Delta$ singular, whereas a smaller value of μ is “good”.

8.9 Robust stability with structured uncertainty

Consider stability of the $M\Delta$-structure in Figure 3 for the case where Δ is a set of norm-bounded block-diagonal perturbations.

Theorem 3 RS for block-diagonal perturbations (real or complex). *Assume that the nominal system M and the perturbations Δ are stable. Then the $M\Delta$-system in Figure 3 is stable for all allowed perturbations with $\bar{\sigma}(\Delta) \leq 1, \forall\omega$,* <u>*if and only if*</u>

$$\boxed{\mu(M(j\omega)) < 1, \quad \forall\omega} \tag{8.42}$$

Condition (8.42) for robust stability may be rewritten as

$$\text{RS} \;\Leftrightarrow\; \mu(M(j\omega))\,\bar{\sigma}(\Delta(j\omega)) < 1, \quad \forall\omega \tag{8.43}$$

which may be interpreted as a "generalized small gain theorem" that also takes into account the *structure* of Δ.

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Example 3 RS with diagonal input uncertainty

Consider robust stability of the feedback system in Figure 6 for the case when the multiplicative input uncertainty is diagonal. A nominal 2×2 plant and the controller (which represents PI-control of a distillation process using the decentralized configuration) is given by

$$G(s) = \frac{1}{\tau s+1}\begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix};$$

$$K(s) = \frac{1+\tau s}{s}\begin{bmatrix} -0.0015 & 0 \\ 0 & -0.075 \end{bmatrix} \tag{8.44}$$

(time in minutes). The controller results in a nominally stable system with acceptable performance. Assume there is complex multiplicative uncertainty in each *manipulated input of magnitude*

$$w_I(s) = \frac{s+0.2}{0.5s+1} \tag{8.45}$$

On rearranging the block diagram to match the $M\Delta$-structure in Figure 3 we get $M = w_I KG(I+KG)^{-1} = w_I T_I$ (recall (8.15)), and the RS-condition $\mu(M) < 1$ in Theorem 3 yields

$$\mathrm{RS} \Leftrightarrow \mu_{\Delta_I}(T_I) < \frac{1}{|w_I(j\omega)|} \quad \forall\omega, \quad \Delta_I = \begin{bmatrix} \delta_1 & \\ & \delta_2 \end{bmatrix} \tag{8.46}$$

This condition is shown graphically in Figure 8 so the *system is robustly stable. Also in Figure 8, $\bar{\sigma}(T_I)$ can be seen to be larger than $1/|w_I(j\omega)|$ over a wide frequency range. This shows that the system would be unstable for full-block input uncertainty (Δ_I full).*

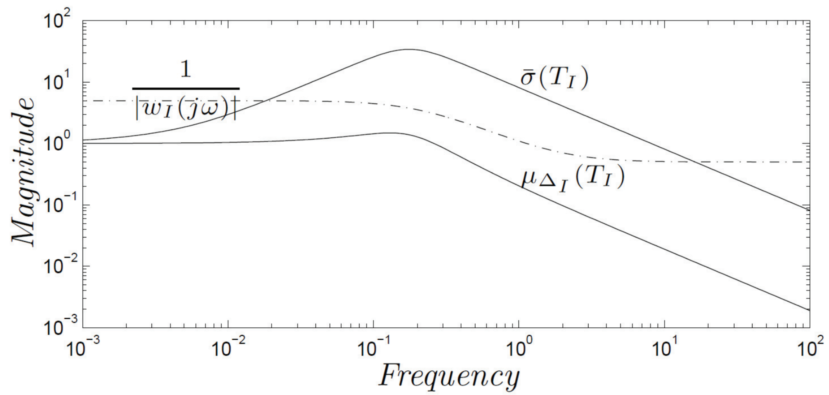

Figure 8: Robust stability for diagonal input uncertainty is guaranteed since $\mu_{\Delta_I}(T_I) < 1/|w_I|$, $\forall\omega$. The use of unstructured uncertainty and $\bar{\sigma}(T_I)$ is conservative

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8.10 Robust performance

With an $\mathcal{H}_\infty$ performance objective, the RP-condition is identical to a RS-condition with an additional perturbation block.

In Figure 9 step B is the key step.

Δ_P (where capital P denotes Performance) is always a full matrix. It is a fictitious uncertainty block representing the $\mathcal{H}_\infty$ performance specification.

8.10.1 Testing RP using μ

Theorem 4 Robust performance. *Rearrange the uncertain system into the $N\Delta$-structure of Figure 9. Assume nominal stability such that N is (internally) stable. Then*

$$\begin{aligned} \text{RP} \quad &\overset{\text{def}}{\Leftrightarrow} \quad \|F\|_\infty = \|F_u(N,\Delta)\|_\infty < 1, \quad \forall \|\Delta\|_\infty \leq 1 \\ &\Leftrightarrow \quad \boxed{\mu_{\widehat{\Delta}}(N(j\omega)) < 1, \quad \forall w} \end{aligned} \tag{8.47}$$

where μ is computed with respect to the structure

$$\widehat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix} \tag{8.48}$$

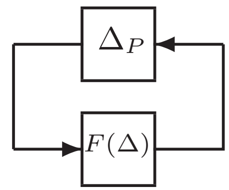
and Δ_P is a full complex perturbation with the same dimensions as F^T.

RP

STEP A $\Updownarrow$

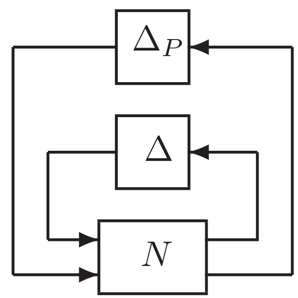
$$\| F(\Delta) \|_\infty < 1,\ \forall\ \| \Delta \|_\infty \le 1$$

STEP B $\Updownarrow$



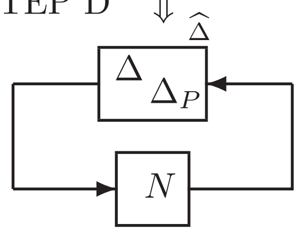
is RS, $\forall \| \Delta_P \|_\infty \le 1$
$\forall \| \Delta \|_\infty \le 1$

STEP C $\Updownarrow$



is RS, $\forall \| \Delta_P \|_\infty \le 1$
$\forall \| \Delta \|_\infty \le 1$

STEP D $\Updownarrow$



is RS, $\forall \| \widehat{\Delta} \|_\infty \le 1$

$\Updownarrow$ (RS theorem)

$$\mu_{\widehat{\Delta}}(N) < 1,\ \forall w$$

Figure 9: RP as a special case of structured RS. $F = F_u(N, \Delta)$

8.10.2 Summary of μ-conditions for NP, RS and RP

Rearrange the uncertain system into the $N\Delta$-structure, where the block- diagonal perturbations satisfy $\|\Delta\|_\infty \leq 1$.

Introduce

$$F = F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

and let the performance requirement (RP) be $\|F\|_\infty \leq 1$ for all allowable perturbations. Then we have:

$$\begin{aligned}
\text{NS} \quad &\Leftrightarrow \quad N \text{ (internally) stable} && (8.49)\\
\text{NP} \quad &\Leftrightarrow \quad \bar{\sigma}(N_{22}) = \mu_{\Delta_P} < 1,\ \forall\omega,\ \text{and NS} && (8.50)\\
\text{RS} \quad &\Leftrightarrow \quad \mu_\Delta(N_{11}) < 1,\ \forall\omega,\ \text{and NS} && (8.51)\\
\text{RP} \quad &\Leftrightarrow \quad \mu_{\widehat{\Delta}}(N) < 1,\ \forall\omega,\ \widehat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}, \\
&\qquad \text{and NS} && (8.52)
\end{aligned}$$

Here Δ is a block-diagonal matrix (its detailed structure depends on the uncertainty we are representing), whereas Δ_P always is a full complex matrix. Note that nominal stability (NS) must be tested separately in all cases.

8.11 Application: RP with input uncertainty

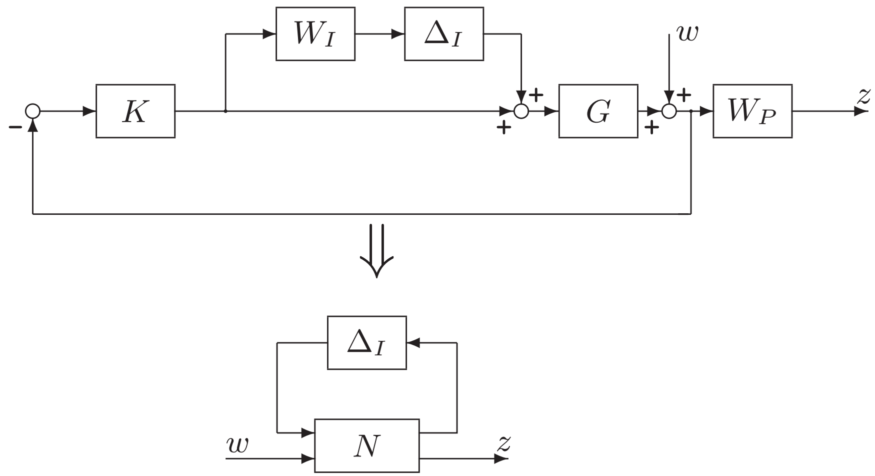


Figure 10: Robust performance of system with input uncertainty

8.11.1 Interconnection matrix

On rearranging the system into the $N\Delta$-structure, as shown in Figure 10, we get

$$N = \begin{bmatrix} w_I T_I & w_I KS \\ w_P SG & w_P S \end{bmatrix} \tag{8.53}$$

where $T_I = KG(I+KG)^{-1}$, $S = (I+GK)^{-1}$ and for simplicity we have omitted the negative signs in the 1,1 and 1,2 blocks of N, since $\mu(N) = \mu(UN)$ with unitary $U = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$.

For a given controller K we can now test for NS, NP, RS and RP using (8.49)-(8.52). Here $\Delta = \Delta_I$ may be a full or diagonal matrix (depending on the physical situation).

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8.11.2 RP with input uncertainty for SISO system

For a SISO system, conditions (8.49)-(8.52) with N as in (8.53) become

$$
\begin{aligned}
\text{NS} \quad &\Leftrightarrow \quad S,\ SG,\ KS \text{ and } T_I \text{ are stable} && (8.54)\\
\text{NP} \quad &\Leftrightarrow \quad |w_P S| < 1,\ \forall\omega && (8.55)\\
\text{RS} \quad &\Leftrightarrow \quad |w_I T_I| < 1,\ \forall\omega && (8.56)\\
\text{RP} \quad &\Leftrightarrow \quad |w_P S| + |w_I T_I| < 1,\ \forall\omega && (8.57)
\end{aligned}
$$

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8.11.3 Robust performance for 2×2 distillation process

Consider again the distillation process example from Chapter 3 (Motivating Example No. 2) and the corresponding inverse-based controller:

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}; \quad (8.58)$$

$$K(s) = \frac{0.7}{s} G(s)^{-1} \quad (8.59)$$

The controller provides a nominally decoupled system with

$$L = l\ I,\ S = \epsilon I \text{ and } T = tI \quad (8.60)$$

where

$$l = \frac{0.7}{s}, \epsilon = \frac{1}{1+l} = \frac{s}{s+0.7},$$
$$t = 1 - \epsilon = \frac{0.7}{s+0.7} = \frac{1}{1.43s+1}$$

We have used ϵ for the nominal sensitivity in each loop to distinguish it from the Laplace variable s.

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Weights for uncertainty and performance:

$$w_I(s) = \frac{s+0.2}{0.5s+1}; \quad w_P(s) = \frac{s/2+0.05}{s} \qquad (8.61)$$

The weight $w_I(s)$ may approximately represent a 20% gain error and a neglected time delay of 0.9 min. $|w_I(j\omega)|$ levels off at 2 (200% uncertainty) at high frequencies.
The performance weight $w_P(s)$ specifies integral action, a closed-loop bandwidth of about 0.05 [rad/min] (which is relatively slow in the presence of an allowed time delay of 0.9 min) and a maximum peak for $\bar{\sigma}(S)$ of $M_s = 2$.

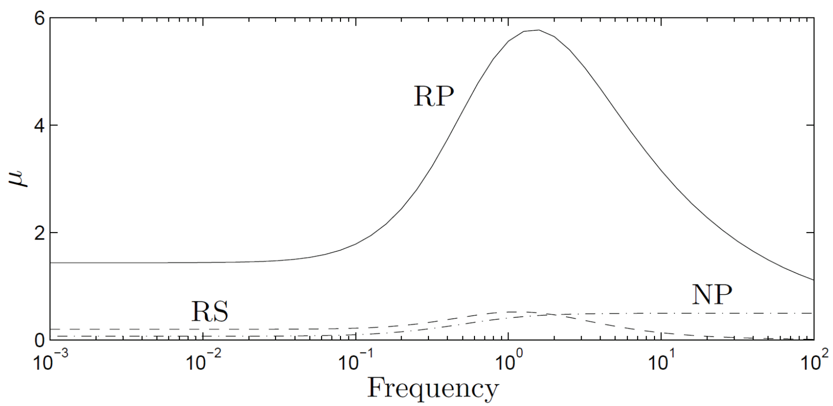

Figure 11: μ-plots for distillation process with decoupling controller

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NS Yes.

NP With the decoupling controller we have

$$\bar{\sigma}(N_{22}) = \bar{\sigma}(w_P S) = \left|\frac{s/2 + 0.05}{s + 0.7}\right|$$

(dash-dotted line in Figure 11 $\Leftarrow$ NP is OK.)

RS Since in this case $w_I T_I = w_I T$ is a scalar times the identity matrix, we have, independent of the structure of Δ_I, that

$$\mu_{\Delta_I}(w_I T_I) = |w_I t| = \left|0.2 \frac{5s + 1}{(0.5s + 1)(1.43s + 1)}\right|$$

and we see from the dashed line in Figure 11 that RS is OK.

RP Poor.

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Table 1: **Matlab program for μ-analysis**

```
% Uses the Robust Control toolbox
G0=[87.8 -86.4; 108.2 -109.6];
G=tf([1],[75 1])*G0;
G=minreal(ss(G));
%
% Inverse-based controller
%
Kinv=0.7*tf([75 1],[1 1e-5])*inv(G0);
%
% Weights
%
Wp=0.5*tf([10 1],[10 1e-5])*eye(2);
Wi=tf([1 0.2],[0.5 1])*eye(2);
%
% Generalized plant P
%
systemnames = 'G Wp Wi';
inputvar = '[ydel(2); w(2) ; u(2)]';
outputvar = '[Wi ; Wp ; -G-w]';
input_to_G = '[u+ydel]';
input_to_Wp = '[G+w]';
input_to_Wi = '[u]';
sysoutname = 'P';
cleanupsysic= 'yes'; sysic;
%
N=lft(P,Kinv);
omega = logspace(-3,3,61); Nf=frd(N,omega);
%
% mu for RP
%
blk=[1 1; 1 1; 2 2];
[mubnds,muinfo]=mussv(Nf,blk,'c');
muRP=mubnds(:,1); [muRPinf,muRPw] = norm(muRP,inf);    % (ans =5.7726)
%
%
% cont.  on next page
```

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Table 2: **Matlab program for μ-analysis (cont.)**

```
%
% Worst case weighted sensitivity
%
delta = [ultidyn('del1',[1 1]) 0;0 ultidyn('del2',[1 1])];
Np = lft(delta,N); %Perturbed model
opt = wcgopt('ABadThreshold',100);
Npw = wcgain(Np,opt);                                      % (ans = 44.98 for
%                                                             delta = 1)
% mu for RS
%
Nrs=Nf(1:2,1:2); % Picking out WiTi
[mubnds,muinfo]=mussv(Nrs,[1 1; 1 1],'c');
muRS=mubnds(:,1); [muRSinf,muRSw]=norm(muRS,inf)           % (ans = 0.5242)
%
% mu for NS (=max.  singular value of Nnp)
%
Nnp=Nf(3:4,3:4); % Picking out wP*Si
[mubnds,muinfo]=mussv(Nnp,[1 1;1 1],'c');
muNS=mubnds(:,1); [muNSinf,muNSw]=norm(muNS,inf)           % (ans = 0.500)
bodemag(muRP,'',muRS,'--',muNS,'-.',omega)
```

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8.12 μ-synthesis and DK-iteration

The structured singular value μ is a very powerful tool for the analysis of robust performance with a given controller. However, one may also seek to find the controller that minimizes a given μ-condition: this is the μ-synthesis problem.

8.12.1 DK-iteration

At present there is no direct method to synthesize a μ-optimal controller. However, for complex perturbations a method known as DK-iteration is available. It combines $\mathcal{H}_\infty$-synthesis and μ-analysis, and often yields good results. The starting point is the upper bound on μ in terms of the scaled singular value

$$\mu(N) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DND^{-1})$$

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The idea is to find the controller that minimizes the peak value over frequency of this upper bound, namely

$$\min_K(\min_{D\in\mathcal{D}} \|DN(K)D^{-1}\|_\infty) \qquad (8.62)$$

by alternating between minimizing $\|DN(K)D^{-1}\|_\infty$ with respect to either K or D (while holding the other fixed).

1. **K-step.** Synthesize an $\mathcal{H}_\infty$ controller for the scaled problem, $\min_K \|DN(K)D^{-1}\|_\infty$ with fixed $D(s)$.
2. **D-step.** Find $D(j\omega)$ to minimize at each frequency $\bar{\sigma}(DND^{-1}(j\omega))$ with fixed N.
3. Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum phase transfer function $D(s)$ and go to Step 1.

8.12.2 * Example: μ-synthesis with DK-iteration

Simplified distillation process

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{8.63}$$

The uncertainty weight $w_I I$ and performance weight $w_P I$ are given in (8.61), and are shown graphically in Figure 12. The objective is to minimize the peak value of $\mu_{\widetilde{\Delta}}(N)$, where N is given in (8.53) and $\widetilde{\Delta} = \text{diag}\{\Delta_I, \Delta_P\}$. We will consider diagonal input uncertainty (which is always present in any real problem), so Δ_I is a 2×2 diagonal matrix. Δ_P is a full 2×2 matrix representing the performance specification.

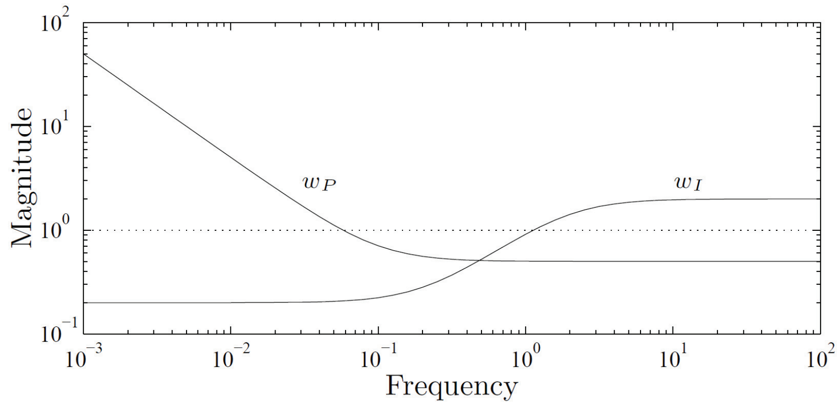

Figure 12: Uncertainty and performance weights. Notice that there is a frequency range ("window") where both weights are less than one in magnitude.

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Table 3: **Matlab program to perform *DK*-iteration**

```
% Uses the Robust Control toolbox
G0 = [87.8 -86.4; 108.2 -109.6];                                % Distillation
dyn = tf(1,[75 1]); G=dyn*eye(2)*G0;                            % process.
%
% Weights.
%
Wp = 0.5*tf([10 1],[10 1.e-5])*eye(2);                          % Approximated
Wi = tf([1 0.2],[0.5 1])*eye(2);                                % integrator.
%
% Generalized plant P. %
systemnames = 'G Wp Wi';
inputvar = '[udel(2); w(2) ; u(2)]';
outputvar = '[Wi; Wp; -G-w]';
input_to_G = '[u+udel]';
input_to_Wp = '[G+w]'; input_to_Wi = '[u]';
sysoutname = 'P'; cleanupsysic = 'yes';
sysic;
P = minreal(ss(P));
%
% Initialize.
%
omega = logspace(-3,3,61);
blk = [1 1; 1 1; 2 2];
nmeas = 2; nu = 2; d0 = 1;
D = append(d0,d0,tf(eye(2)),tf(eye(2)));                        % Initial scaling.
%
% START ITERATION.
%
% STEP 1:  Find H-infinity optimal controller
% with given scalings:
%
[K,Nsc,gamma,info] = hinfsyn(D*P*inv(D),nmeas,nu,....
                   'method','lmi','Tolgam',1e-3);
Nf = frd(lft(P,K),omega);
```

Table 4: **Matlab program to perform DK-iteration (cont.)**

```
%
% STEP 2:  Compute mu using upper bound:
%
[mubnds,Info] = mussv(Nf,blk,'c');
bodemag(mubnds(1,1),omega);
murp = norm(mubnds(1,1),inf,1e-6);
%
% STEP 3:  Fit resulting D-scales:
%
[dsysl,dsysr] = mussvunwrap(Info);
dsysl = dsysl/dsysl(3,3);
d1 = fitfrd(genphase(dsysl(1,1)),4);                        % Choose 4th order.
%
% GOTO STEP 1 (unless satisfied with murp).
%
% Alternatively use automatic software
%
% Delta = [ultidyn('D_1',[1 1]) 0;0 ultidyn('D_2',[1 1])];    % Diagonal uncertainty.
% Punc = lft(Delta,P);
% opt = dkitopt('FrequencyVector',omega);
% [K,clp,bnd,dkinfo] = dksyn(Punc,nmeas,nu,opt);
```

Iteration No. 1.

Step 1: With the initial scalings, $D^0 = I$, the $\mathcal{H}_\infty$ software produced a 6 state controller (2 states from the plant model and 2 from each of the weights) with an $\mathcal{H}_\infty$ norm of $\gamma = 1.1798$.

Step 2: The upper μ-bound gave the μ-curve shown as curve "Iter. 1" in Figure 13, corresponding to a peak value of μ=1.1798.

Step 3: The frequency-dependent $d_1(\omega)$ and $d_2(\omega)$ from Step 2 were each fitted using a 4th order transfer function. $d_1(w)$ and the fitted 4th-order transfer function (dotted line) are shown in Figure 14 and labelled "Iter. 1".

Iteration No. 2.

Step 1: With the 8 state scaling $D^1(s)$ the $\mathcal{H}_\infty$ software gave a 22 state controller and $\|D^1 N(D^1)^{-1}\|_\infty = 1.0274$.

Iteration No. 3.

Step 1: With the scalings $D^2(s)$ the $\mathcal{H}_\infty$ norm was only slightly reduced from 1.0274 to 1.0208.

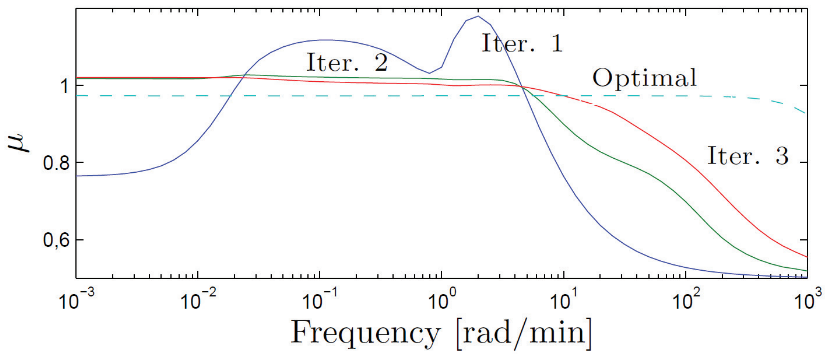

Figure 13: Change in μ during DK-iteration

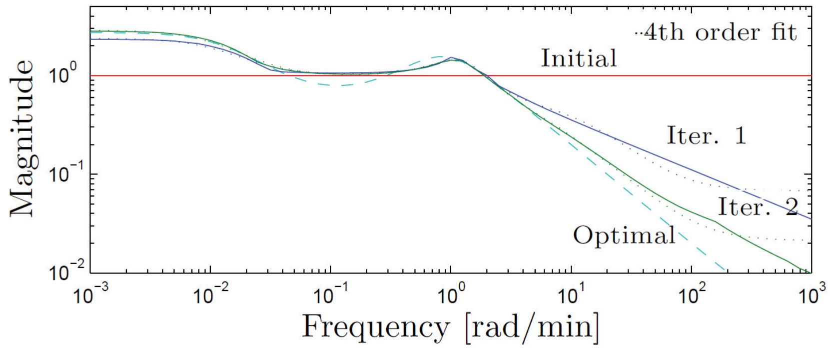

Figure 14: Change in D-scale d_1 during DK-iteration

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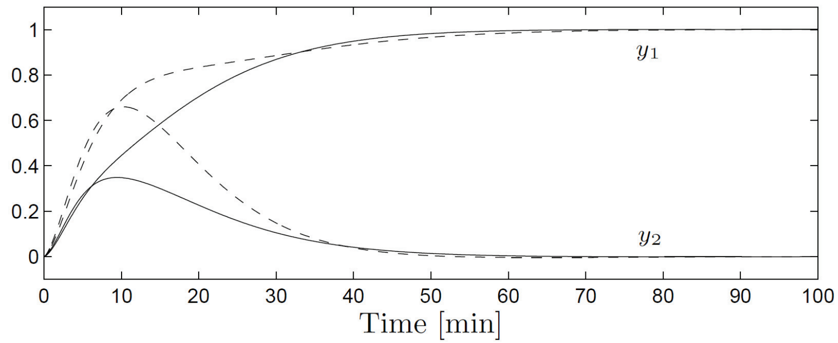

Figure 15: Setpoint response for μ-"optimal" controller K_3. Solid line: nominal plant. Dashed line: uncertain plant G_3'

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