4 UNCERTAINTY AND ROBUSTNESS FOR SISO SYSTEMS

4.1 Introduction to robustness

A control system is robust if it is insensitive to differences between the actual system and the model of the system which was used to design the controller. These differences are referred to as *model/plant mismatch* or simply *model uncertainty*.

$G(s)$ is a nominal model:

- the closed-loop system satisfies **nominal stability (NS)** iff

$$S = (1 + GK)^{-1} \quad \text{and} \quad KS \tag{4.1}$$

  have poles in the complex LHP.

- the closed-loop system satisfies **nominal performance (NP)** if

$$\left\| \begin{array}{c} w_p S \\ w_T T \\ w_u KS \end{array} \right\|_\infty < 1 \tag{4.2}$$

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Our approach is:

1. Determine the uncertainty set: find a mathematical representation of the model uncertainty ("clarify what we know about what we don't know").
2. Check Robust stability (RS): determine whether the system remains stable for all plants in the uncertainty set.
3. Check Robust performance (RP): if RS is satisfied, determine whether the performance specifications are met for all plants in the uncertainty set.

Notation:

Π – a set of possible perturbed plant models ("uncertainty set").

$G(s) \in \Pi$ – nominal plant model (with no uncertainty).

$G_p(s) \in \Pi$ and $G'(s) \in \Pi$ – particular perturbed plant models.

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4.2 Classes of uncertainty

1. **Parametric uncertainty.** Here the structure of the model (including the order) is known, but some of the parameters are uncertain.
2. **Neglected and unmodelled dynamics uncertainty.** Here the model is in error because of missing dynamics, usually at high frequencies, either through deliberate neglect or because of a lack of understanding of the physical process. Any model of a real system will contain this source of uncertainty.
3. **Lumped uncertainty.** Here the uncertainty description represents one or several sources of parametric and/or unmodelled dynamics uncertainty combined into a single lumped perturbation of a chosen structure.

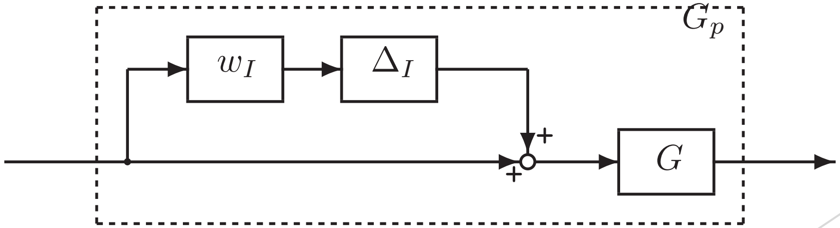

Figure 1: Plant with multiplicative uncertainty

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Multiplicative uncertainty of the form

$$\Pi_I: \quad G_p(s) = G(s)(1 + w_I(s)\Delta_I(s));$$

where

$$\underbrace{|\Delta_I(j\omega)| \leq 1 \ \forall\omega}_{\|\Delta_I\|_\infty \leq 1} \tag{4.3}$$

Here $\Delta_I(s)$ is *any* stable transfer function which at each frequency is less than or equal to one in magnitude. Some allowable $\Delta_I(s)$'s

$$\frac{s-z}{s+z}, \quad \frac{1}{\tau s + 1}, \quad \frac{1}{(5s+1)^3}, \quad \frac{0.1}{s^2 + 0.1s + 1}$$

Inverse multiplicative uncertainty

$$\Pi_{iI}: \quad G_p(s) = G(s)(1 + w_{iI}(s)\Delta_{iI}(s))^{-1};$$

$$|\Delta_{iI}(j\omega)| \leq 1 \ \forall\omega \tag{4.4}$$

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4.3 Representing uncertainty in the frequency domain

4.3.1 Uncertainty regions

Example: parametric uncertainty

$$G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad 2 \le k, \theta, \tau \le 3 \tag{4.5}$$

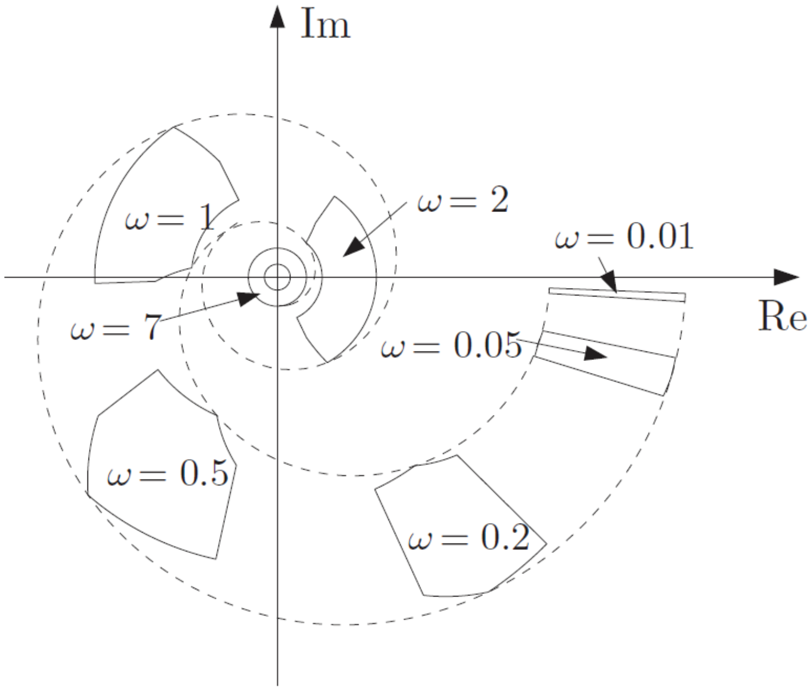

Figure 2: Uncertainty regions of the Nyquist plot at given frequencies. Data from (4.5)

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Approximate the uncertainty region by a circular disc at each frequency with center value of the nominal model.

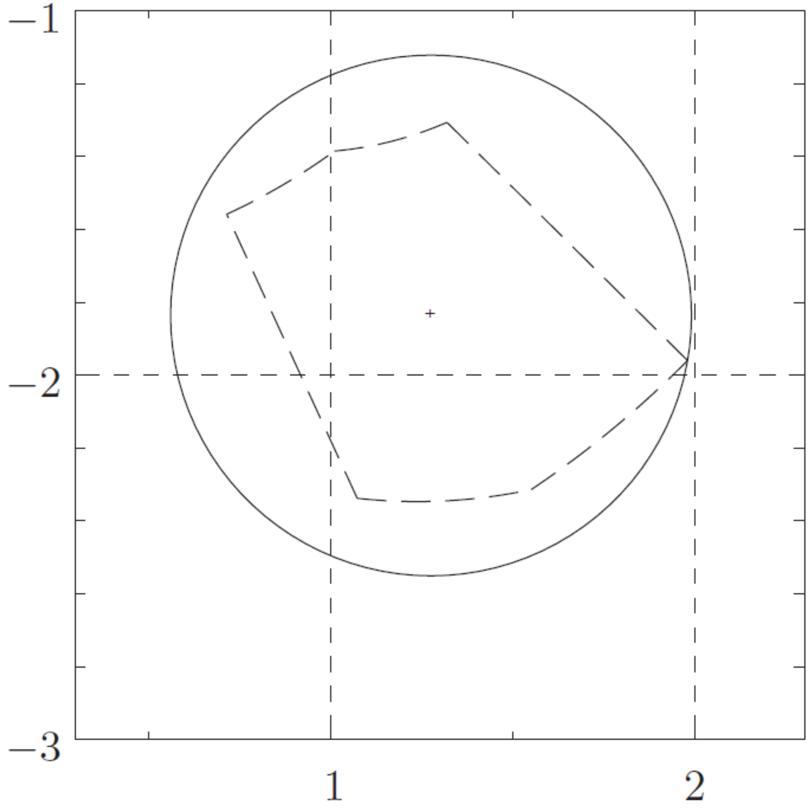

Figure 3: Disc approximation (solid line) of the original uncertainty region (dashed line). Plot corresponds to $\omega = 0.2$ in Figure 2

Introduces conservatism, i.e., include models not in the actual set

4.3.2 Representing uncertainty regions by complex perturbations

Discs with radius $|w_A(j\omega)|$ are generated from

$$\Pi_A: \ G_p(s) = G(s) + w_A(s)\Delta_A(s) \qquad (4.6)$$

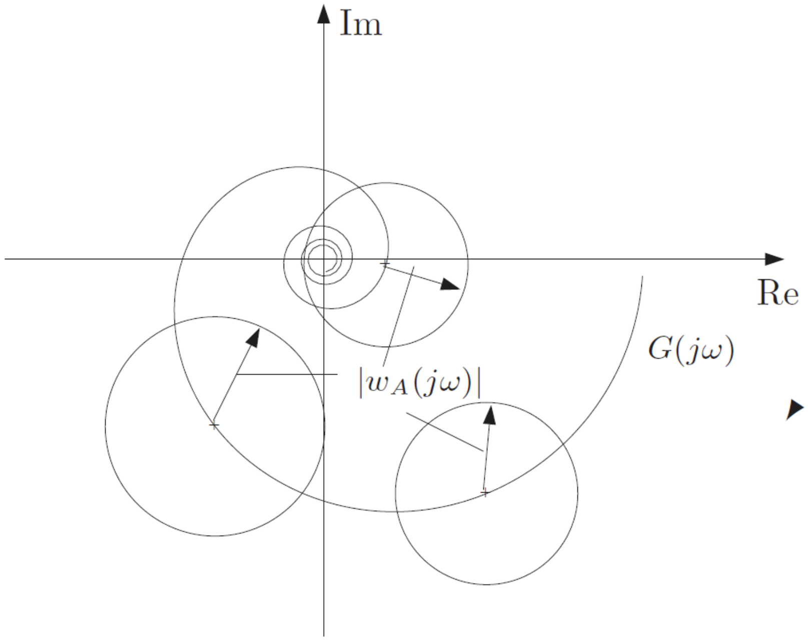

Figure 4: Disc-shaped uncertainty regions generated by complex additive uncertainty, $G_p = G + w_A\Delta_A$

Π_I can be obtained from Π_A through:

$$|w_I(j\omega)| = \frac{|w_A(j\omega)|}{|G(j\omega)|} \qquad (4.7)$$

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We use disc-shaped regions to represent uncertainty regions (Figures 3 and 4) generated by

$$\Pi_A : \quad G_p(s) = G(s) + w_A(s)\Delta_A(s); \quad |\Delta_A(j\omega)| \le 1 \; \forall\omega \tag{4.8}$$

where $\Delta_A(s)$ is *any* stable transfer function which at each frequency is no larger than one in magnitude.

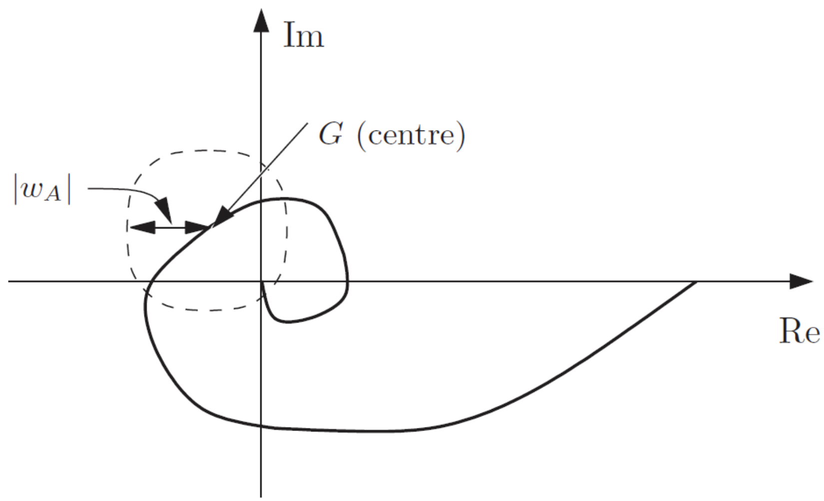

Figure 5: The set of possible plants includes the origin at frequencies where $|w_A(j\omega)| \ge |G(j\omega)|$, or equivalently $|w_I(j\omega)| \ge 1$

When $|w_A(j\omega)| > |G(j\omega)|$ or equivalently, $|w_I(j\omega)| > 1$, we have no knowledge about the phase of the system, thus we require bandwidth to be less than the frequency where $|w_I(j\omega)| = 1$.

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Alternative: *multiplicative uncertainty* description as in (4.3),

$$\Pi_I: \quad G_p(s) = G(s)(1+w_I(s)\Delta_I(s)); \quad |\Delta_I(j\omega)| \leq 1, \forall\omega \tag{4.9}$$

(4.8) and (4.9) are equivalent if at each frequency

$$|w_I(j\omega)| = |w_A(j\omega)|/|G(j\omega)| \tag{4.10}$$

4.3.3 Obtaining the weight for complex uncertainty

1. Select a nominal model $G(s)$.
2. *Additive uncertainty.* At each frequency find the smallest radius $l_A(\omega)$ which includes all the possible plants Π:

$$|w_A(jw)| \geq l_A(\omega) = \max_{G_P \in \Pi} |G_p(j\omega) - G(j\omega)| \tag{4.11}$$

3. *Multiplicative (relative) uncertainty.* (preferred uncertainty form)

$$|w_I(jw)| \geq l_I(\omega) = \max_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G(j\omega)}{G(j\omega)} \right| \tag{4.12}$$

Example 1 Multiplicative weight for parametric uncertainty. *Consider again the set of plants with parametric uncertainty given in (4.5)*

$$\Pi: \quad G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3 \tag{4.13}$$

We want to represent this set using multiplicative uncertainty with a rational weight $w_I(s)$. We select a delay-free nominal model

$$G(s) = \frac{\bar{k}}{\bar{\tau} s + 1} = \frac{2.5}{2.5s + 1} \tag{4.14}$$

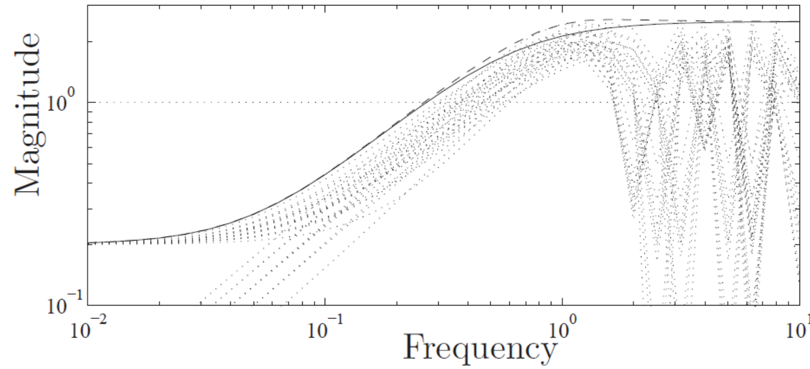

Figure 6: Relative errors $|G_p - G|/|G|$ for 27 combinations of k, τ and θ with delay-free nominal plant (dotted lines). Fit upper bound with solid line (first-order weight $|w_{I1}|$ in (4.15)) or dashed line (third-order weight $|w_I|$ in (4.16))

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$$w_{I1}(s) = \frac{Ts + 0.2}{(T/2.5)s + 1}, \quad T = 4 \tag{4.15}$$

$$w_I(s) = \omega_{I1}(s)\ \frac{s^2 + 1.6s + 1}{s^2 + 1.4s + 1} \tag{4.16}$$

Choice of nominal model

Three options for choice of nominal model $G(s)$:

- Simple model: low-order and delay-free (+) simplifies control design, (-) potentially large uncertainty.
- Mean parameter model: use average parameter values (+) simple choice, smaller uncertainty region, (-) not optimal.
- Central frequency response: use model that yields the smallest uncertainty disc at each frequency (+) smallest uncertainty, (-) complex procedure and high order model.

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Neglected dynamics as uncertainty

Assume full model:

$$G_p(s) = G_1(s)G_2(s) \tag{4.17}$$

We want to neglect $G_2(s)$ in our model. Then:

$$l_I(j\omega) = \max_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G_1(j\omega)}{G_1(j\omega)} \right| = \max_{G_2(s) \in \Pi_2} |G_2(j\omega) - 1| \tag{4.18}$$

where Π_2 denotes the uncertainty set.

Example: Neglected delay $G_2(s) = e^{-\theta s}$ with $\theta \in [0, \theta_{\max}]$

$$l_I(\omega) = \begin{cases} |1 - e^{-j\omega\theta_{\max}}| & \omega \leq \pi/\theta_{\max} \\ 2 & \omega > \pi/\theta_{\max} \end{cases} \tag{4.19}$$

$$w_I(s) = \frac{\theta_{\max} s}{0.5\theta_{\max} s + 1} \tag{4.20}$$

Unmodeled dynamics as uncertainty

Represent by some simple multiplicative weight:

$$w_I(s) = \frac{\tau s + r_0}{(\tau/r_\infty)s + 1} \tag{4.21}$$

where r_0 and r_∞ denote relative uncertainty at low and high frequencies and at $\omega = 1/\tau$ the relative uncertainty is 100%.

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4.4 SISO robust stability

4.4.1 RS with multiplicative uncertainty

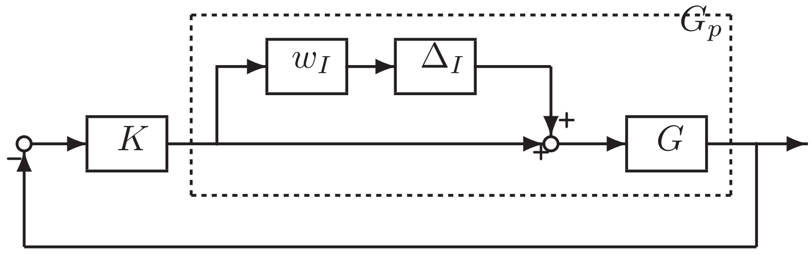

Figure 7: Feedback system with multiplicative uncertainty

Graphical derivation of RS-condition.
In Figure 8 $|-1-L| = |1+L|$ is the distance from the point -1 to the centre of the disc representing L_p, $|w_I L|$ is the radius of the disc. Encirclements are avoided if none of the discs covers -1, and we get from Figure 8

$$\begin{aligned} \text{RS} \quad &\Leftrightarrow \quad |w_I L| < |1+L|, \quad \forall\omega && (4.22) \\ &\Leftrightarrow \quad \left|\frac{w_I L}{1+L}\right| < 1, \forall\omega \Leftrightarrow |w_I T| < 1, \forall\omega && (4.23) \\ &\overset{\text{def}}{\Leftrightarrow} \quad \|w_I T\|_\infty < 1 && (4.24) \end{aligned}$$

$$\boxed{\text{RS} \Leftrightarrow |T| < 1/|w_I|, \quad \forall\omega} \qquad (4.25)$$

Consider a SISO system with multiplicative uncertainty:

$$\Pi_l : \; L_p = GK(1 + w_l \Delta_l) = L + w_l L \Delta_l, \; \|\Delta_l\|_\infty \leq 1 \tag{4.26}$$

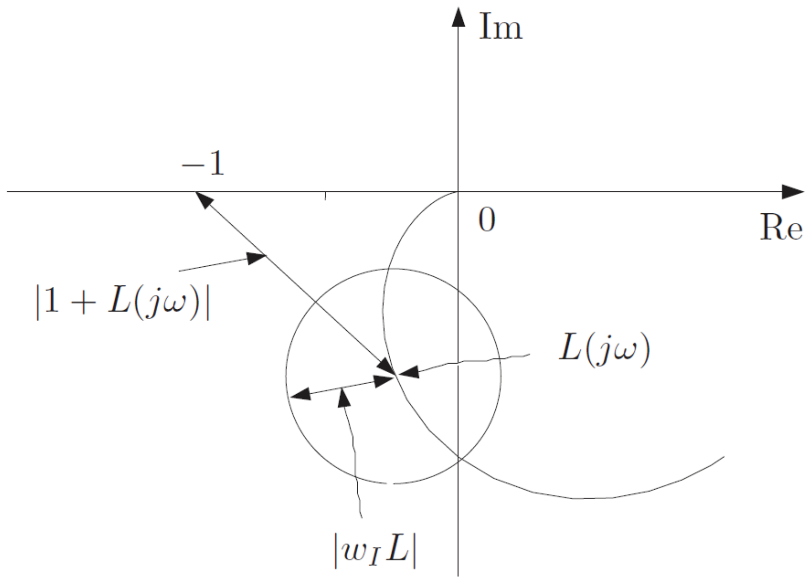

Figure 8: Nyquist plot of L_p for robust stability

Example 2 *Consider the following nominal plant and PI-controller*

$$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)} \quad K(s) = K_c \frac{12.7s+1}{12.7s}$$

$K_c = K_{c1} = 1.13$ *(Ziegler-Nichols). One "extreme" uncertain plant is* $G'(s) = 4(-3s+1)/(4s+1)^2$. *For this plant the relative error* $|(G'-G)/G|$ *is* 0.33 *at low frequencies; it is* 1 *at about* 0.1 *rad/s, and it is* 5.25 *at high frequencies* $\Rightarrow$ *uncertainty weight*

$$w_I(s) = \frac{10s+0.33}{(10/5.25)s+1}$$

which closely matches this relative error.

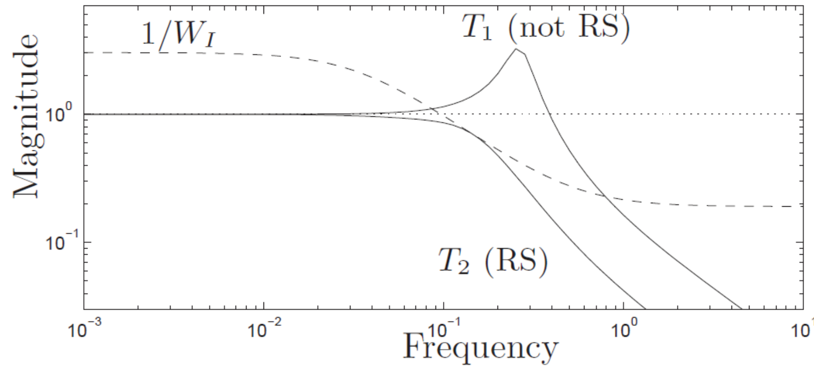

Figure 9: Checking robust stability with multiplicative uncertainty

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By trial and error we find that reducing the gain to $K_{c2} = 0.31$ just achieves RS as seen from T_2 in Fig. 9.

Remark:

The procedure is *conservative*. For K_{c2} the system with the "extreme" plant is not at the limit of instability; we can increase the gain to $k_{c2} = 0.58$ before we get instability.

$M\Delta$-structure derivation of RS-condition.

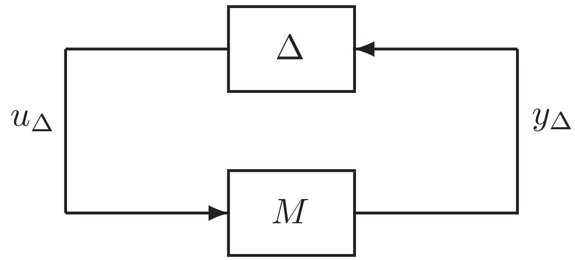

Figure 10: $M\Delta$-structure

The stability of the system in Figure 7 is equivalent to stability of the system in Figure 10, where $\Delta = \Delta_I$ and

$$M = w_I K(1+GK)^{-1}G = w_I T \qquad (4.27)$$

The Nyquist stability condition then determines RS if and only if the "loop transfer function" $M\Delta$ does not encircle -1 for all Δ. Thus,

$$\text{RS} \quad \Leftrightarrow \quad |1 + M\Delta| > 0, \quad \forall\omega, \forall|\Delta| \leq 1 \qquad (4.28)$$

$$\text{RS} \quad \Leftrightarrow \quad 1 - |M(j\omega)| > 0, \quad \forall\omega \qquad (4.29)$$

$$\Leftrightarrow \quad |M(j\omega)| < 1, \quad \forall\omega \qquad (4.30)$$

4.4.2 RS with inverse multiplicative uncertainty

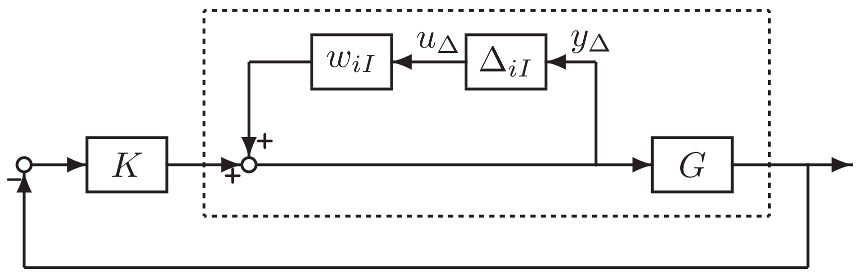

Figure 11: Feedback system with inverse multiplicative uncertainty

$$\boxed{\text{RS} \quad \Leftrightarrow \quad |S| < 1/|w_{iI}|, \quad \forall\omega} \tag{4.31}$$

4.5 SISO robust performance

4.5.1 SISO nominal performance in the Nyquist plot

$$\text{NP} \Leftrightarrow |w_P S|_\infty < 1,\ \forall\omega \Leftrightarrow |w_P| < |1+L|,\ \forall\omega \quad (4.32)$$

"Avoid" -1 with some margin $|w_P(j\omega)|$. See Figure:

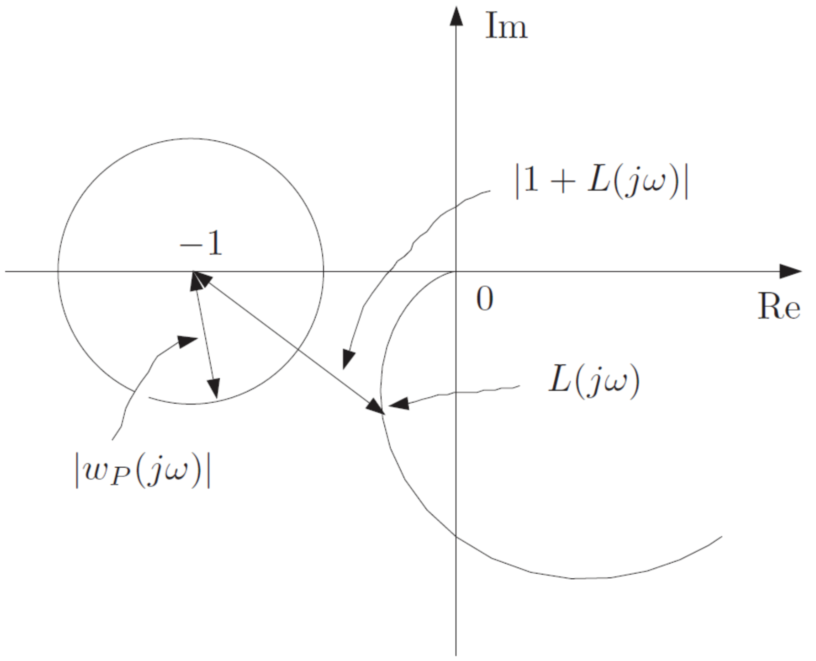

Figure 12: Nyquist plot illustration of nominal performance condition $|w_P| < |1+L|$

4.5.2 Robust performance

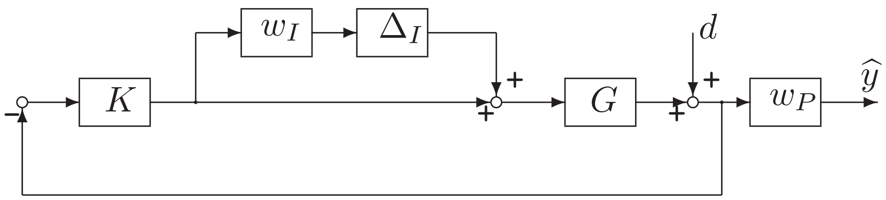

Figure 13: Diagram for robust performance with multiplicative uncertainty

For robust performance we require the performance condition (4.32) to be satisfied for *all* possible plants, that is, including the worst-case uncertainty.

$$\text{RP} \quad \stackrel{\text{def}}{\Leftrightarrow} \quad |w_P S_p| < 1 \quad \forall S_p, \forall\omega \tag{4.33}$$

$$\Leftrightarrow \quad |w_P| < |1 + L_p| \quad \forall L_p, \forall\omega \tag{4.34}$$

This corresponds to requiring $|\widehat{y}/d| < 1 \ \forall\Delta_I$ in Figure 13, where we consider multiplicative uncertainty, and the set of possible loop transfer functions is

$$L_p = G_p K = L(1 + w_I\Delta_I) = L + w_I L\Delta_I \tag{4.35}$$

Graphical derivation of RP-condition.

(Figure 14)

$$\text{RP} \quad \Leftrightarrow \quad |w_P| + |w_I L| < |1+L|, \quad \forall\omega \tag{4.36}$$

$$\Leftrightarrow \quad |w_P(1+L)^{-1}| + |w_I L(1+L)^{-1}| < 1, \forall\omega \tag{4.37}$$

$$\boxed{\text{RP} \;\Leftrightarrow\; \max_\omega \left(|w_P S| + |w_I T|\right) < 1} \tag{4.38}$$

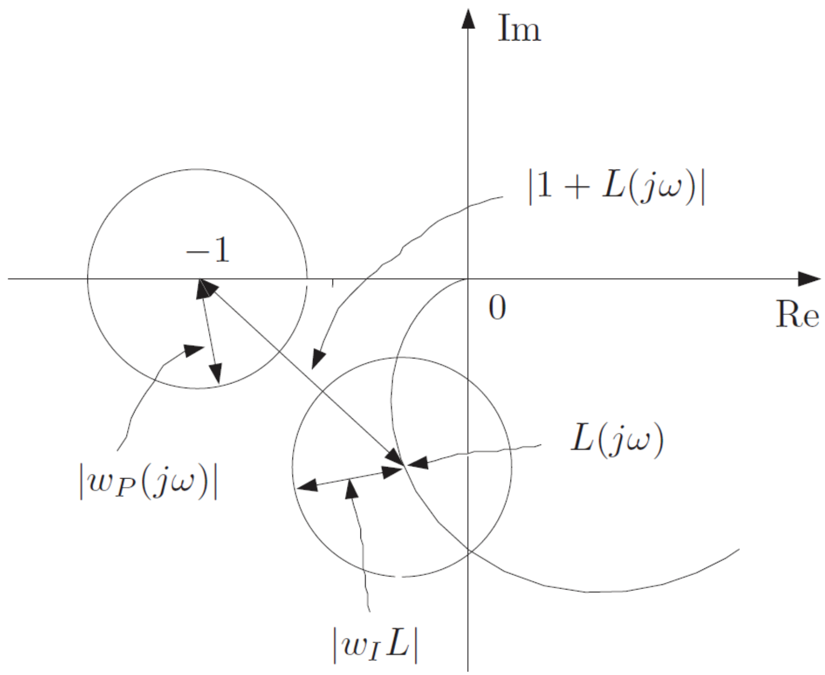

Figure 14: Nyquist plot illustration of robust performance condition $|w_P| < |1 + L_p|$

The relationship between NP, RS and RP

$$\text{NP} \quad \Leftrightarrow \quad |w_P S| < 1, \forall\omega \tag{4.39}$$

$$\text{RS} \quad \Leftrightarrow \quad |w_I T| < 1, \forall\omega \tag{4.40}$$

$$\text{RP} \quad \Leftrightarrow \quad |w_P S| + |w_I T| < 1, \forall\omega \tag{4.41}$$

- Prerequisite for RP is that we satisfy NP and RS.
- If we satisfy both RS and NP, then we have at each frequency

$$|w_P S| + |w_I T| \leq 2\max\{|w_P S|, |w_I T|\} < 2 \tag{4.42}$$

  Therefore, within a factor of at most 2, we will automatically ("for free") get RP when NP and RS are satisfied.

- We *cannot* have both $|w_P| > 1$ (i.e. good performance) and $|w_I| > 1$ (i.e. more than 100% uncertainty) at the same frequency:

$$|w_P S| + |w_I T| \geq \min\{|w_P|, |w_I|\} \tag{4.43}$$

- The RP condition can essentially be formulated as an H_∞-problem:

$$\left\| \begin{pmatrix} w_P S \\ w_I T \end{pmatrix} \right\|_\infty = \max_\omega \sqrt{|w_P S|^2 + |w_I T|^2} < 1 \tag{4.44}$$

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