Coalgebraic Modal Logic in the Category of Nominal Sets

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Introduction

Name binding in systems

By systems, we mean state transition systems, like process calculi and software systems.
Defined individually, but 'Coalgebra' captures the commonalities.

Name binding: explicit variable scope
Used in many systems: Programming Langauges, λ-Calculus, π-calculus, etc.
Nominal logic captures the key principles. (system specific)

Proposed Approach

Coalgebraic Modal Logic in Nominal Sets
Coalgebra captures transition structure
Nominal sets captures naming structure and binding

We show generic results for soundness, completeness, and some results relevant to decidability.

Modal Logic

Modal Logics: formal, symbolic logics for reasoning about modalities. What kind of modalities?

Modal K, $S4$, Intuitionistic Logic, tense logics, temporal logics, spatial logics, obligation logics, knowledge logics, dynamic logics, multi-agent logics etc.

Coalgebraic framework organises and proves generic theorems.

Coalgebra

Functor T maps sets X to sets TX, functions f to functions Tf.
T-Coalgebra: a function $\gamma : X \to TX$

We envisage X a set of states, TX a set of successors. Each state has one successor object (some structure involving many states).

For example, if $T = \mathcal{P}$, then each state has a set of successors, modelling nondeterminism.
$\mathcal{P}$-coalgebras are Kripke frames - models for Modal K.

Coalgebraic modal logic

Logic syntax: $\mathcal{F}(\Lambda) \ni \varphi, \psi ::= p \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \heartsuit(\varphi)$
Interpretation: Subset of X
Nonmodal e.g. $[\![\neg\phi]\!] = X \smallsetminus [\![\phi]\!]$

Modality lifts a predicate on states to a predicate on successors.
$[\![\heartsuit]\!] : \mathcal{P}(X) \to \mathcal{P}(TX)$
Modal Interpretation: $[\![\heartsuit(\varphi)]\!] = \gamma^{-1}([\![\heartsuit]\!]([\![\varphi]\!]))$
This view allows categorical treatment of interpretations.
Convenient for metalogic proofs, esp. about proof theory.

Nominal Sets

First, an example: Set of process expressions ($a \in \mathbb{A}$)

$$P, Q \ ::= \ \overline{a}.P \mid a.P \mid P \| Q \mid P; Q \mid 0$$

A name permutation $(a\ b)$ swaps a and b

$$(a\ b) \cdot a.0 = b.0 \quad \textit{while} \quad (a\ b) \cdot c.0 = c.0$$

We can use $\pi \cdot x$ to create a generic test for the free names of x in any nominal set. We call this the support of x: $supp(x)$.

Nominal Sets

A set X, with a permutation action $\cdot : \mathbb{P}_{\mathbb{A}} \times X \to X$
$id \cdot x = x, \quad \pi_1 \cdot (\pi_2 \cdot x) = (\pi_1 \circ \pi_2) \cdot x$
$supp(x)$ is finite for all $x \in X$

Equivariant function is $f : X \to Y$ such that $\pi \cdot f(x) = f(\pi \cdot x)$
Nominal sets and equivariant functions form the category **NOM**

The exponent Y^X is not the set of equivariant functions $X \to Y$,
but rather the nominal set of finitely supported functions $X \to Y$.

Nominal Logic

Freshness: $a\#x$ means $a \notin supp(x)$
Name Quantifier: $\text{И} n.\phi$ means $\exists n' \# \phi \,.\, \phi[n'/n]$
Note that this is the same as $\forall n' \# \phi \,.\, \phi[n'/n]$

Unlike quantifiers such as $\exists$ and $\forall$, the fresh name quantifier И is decidable as an extension of propositional logic.

Proof theory for CML

Methodology: Framework exists in category of sets - modify it

Coalgebraic Semantics: Already exists outside **Set**
Bisimulation and Expressivity: Already exists outside **Set**
Soundness and Completeness: New outside **Set**
Decidability: Nonexistent outside **Set**
Compositionality: Already exists outside **Set**
Global Caching: Nonexistent outside **Set**

Nominalisation

Which parts will become equivariant/finitely supported?

Modalities form a nominal set: finitely supported predicate liftings.
Substitutions: $p \mapsto \phi$ arbitrary finite formula. Finitely supported.
Valuations: Interchangeable with substitutions. Finitely supported.

Carrier set, coalgebra and functor all in the category of nominal sets and equivariant functions.
Interpretation equivariant except for valuation, specifically

$$\pi \cdot [\![\phi]\!]_{X,v} = [\![\pi \cdot \phi]\!]_{X,\pi \cdot v}$$

Orbit-finite model property

CML in **Set** has a finite model property.
Finite nominal sets are trivial. In fact in **NOM** the finitely presentable objects are orbit-finite nominal sets.

CML in **NOM** has an orbit-finite model property (maybe).

$\reflectbox{N}$ quantifier

The semantics of И are natural in **NOM**
This means we can add И naturally to CML in **NOM**.

The addition is sound and complete w.r.t. the appropriate semantics and proof rules.

Recall that И is decidable. We strongly suspect that CML will remain decidable with the addition of И.

Global Caching

Tableau proof method: Never repeat analysis
Widespread applicability
Good optimality bounds

Generic decision theory for CML (restrictions on the functor)
Algorithm can be adapted to any Coalgebraic Modal Logic

Conjecture: A modification will allow $\reflectbox{N}$ to be added
In the case of $\reflectbox{N}$HML we are confident that a slight modification to the original ALC global caching algorithm will work.

Future Work in **NOM**

- More complex and relevant examples: Process calculi (π-calculus) Note: bisimilarity is undecidable in general
- Bisimilarity and Expressivity results for CML in **NOM**.
- Compositionality of modalities in CML for **NOM**
- Generic Global Caching algorithm for ИCML.

More Future Work outside **NOM**

- Sheaf Topoi
- Internal logic of the category dictates the propositional connectives
- Topoi internal logic is intuitionistic in general: makes proof theory more complicated.

Questions

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