SYLLABUS

UNIT-I

Determinants: Definition, Minors, Cofactors, Properties of Determinants, MATRICES: Definition, Types of Matrices, Addition, Subtraction, Scalar Multiplication and Multiplication of Matrices, Adjoint, Inverse, Cramers Rule, Rank of Matrix Dependence of Vectors, Eigen - Vectors of a Matrix, Caley-Hamilton Theorem (without proof). **[No. of Hrs: 12]**

UNIT-II

Limits & Continuity: Limit at a Point, Properties of Limit, Computation of Limits of Various Types of Functions, Continuity at a Point, Continuity Over an Interval, Intermediate Value Theorem, Type of Discontinuities. **[No. of Hrs: 10]**

UNIT-III

Differentiation: Derivative, Derivatives of Sum, Differences, Product & quotients, Chain Rule, Derivatives of Composite Functions, Logarithmic Differentiation, Rolle's Theorem, Mean Value Theorem, Expansion of Functions (Maclaurin's & Taylor's), Indeterminate Forms, L' Hospitals Rule, Maxima & Minima, Asymptote, Successive Differentiation & Liebnitz Theorem. **[No. of Hrs: 12]**

UNIT-IV

Integration: Integral as Limit of Sum, Riemann Sum, Fundamental Theorem of Calculus, Indefinite Integrals, Methods of Integration Substitution, By Parts, Partial Fractions, Integration of Algebraic and transcedental Functions, Reduction Formulae for Trigonometric Functions, Gamma and Beta Functions. **[No. of Hrs: 10]**

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FIRST SEMESTER
MATHEMATICS-I (BCA–101)
DECEMBER 2012

Time : 3.00 hrs. **M.M. : 75**

Note : *Attempt all questions Select one question from each unit including Q.no 1 which is compulsory.*

Q. 1.(a) Define Symmetric and Skew Symmetric Matrix. Give an example.

Ans. Symmetric Matrix : Any matric A is said to be symmetric if $A' = A$, if the transpose of the matrix is equal to the matric itself.

Then matrix A is symmetric iff $A' = A$

Example: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Skew-symmetric: Any matrix A is said to be skew-symmetric if $A' = -A$ *i.e.* if the transpose of the matrix is equal to the negative of the matric itself.

Then matrix A is skew-symmetric iff $A' = -A$

Example: $\begin{bmatrix} 0 & 3i & i-2 \\ -3i & 0 & 4-i \\ -i+2 & -4+i & 0 \end{bmatrix}$

Q. 1.(b) Find the Eigen values and Eigen vectors for the matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$.

Ans.

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$$

characteritic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -(5+\lambda).-(2+\lambda) - 4 = 0$$

$$\Rightarrow \quad (5+\lambda)(2+\lambda) - 4 = 0$$

$$\Rightarrow \quad 10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \quad \lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow \quad \lambda^2 + 6\lambda + \lambda + 6 = 0$$

$$\Rightarrow \quad \lambda(\lambda+6) + 1(\lambda+6) = 0$$

$$\Rightarrow \quad (\lambda+6)(\lambda+1) = 0$$

$\Rightarrow \quad \lambda = -1, -6$

The characteritic vector $X = \begin{bmatrix} x \\ y \end{bmatrix} \neq 0$ corresponding to characteristic root $\lambda = 1$ is given by

$$AX = \lambda X \Rightarrow (A - \lambda I)X = 0$$

$$\Rightarrow \quad (A + I)X = 0 \quad \text{[Putting } \lambda = -1]$$

$$\Rightarrow \quad \left(\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad R_2 \Rightarrow R_2 + \frac{1}{2}R_1$$

Now, coefficient matrix of there equation has rank = 1

$\therefore$ Eqns will have $2 - 1 = 1$ linerly independent solution.

$$\Rightarrow \quad -4x + 2y = 0$$

$$\Rightarrow \quad -2x + y = 0$$

$$\Rightarrow \quad y = 2x$$

Taking $x = 1, \boxed{y = 2}$

$$\Rightarrow \quad x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ is characteristic vector}$$

Now the characteristic vector $X = \begin{bmatrix} x \\ y \end{bmatrix} \neq 0$ corresponding to chharacteristic root $\lambda = -6$ is given by

$$AX = \lambda X \Rightarrow (A - \lambda I)X = 0$$

$$\Rightarrow \quad (A + 6I)X = 0 \quad \text{[Putting } \lambda = -6]$$

$$\Rightarrow \quad \left(\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0, \qquad R_2 \Rightarrow R_2 - 2R_1$$

Now, Coefficient matrix of there equation has rank = 1

$\therefore$ Eqn. will have $2 - 1 = 1$ linearly independent solution.

$$x + 2y = 0$$

$$\Rightarrow \quad x = -2y$$

Take $y = 1, \boxed{x = -2}$

$$\Rightarrow \quad x = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ is characteristic vector of } A.$$

Q. 1.(c) Evaluate the nth derivative of $\log(ax + b)$.

Ans. $y = \log(ax + b)$

$$y^1 = \frac{1}{ax+b}(a) = a \times (ax+b)^{-1}$$

$$y^2 = a \times \frac{(-1)}{(ax+b)^2}.a = \frac{-a^2}{(ax+b)^2}$$

$$y^2 = -a^2 \times (ax+b)^{-2}$$

$$y^3 = +a^2 \times 2(ax+b)^{-3} = \frac{2a^2}{(ax+b)^3}$$

$$y^n = \frac{(n-1)a^{(n-1)}(-1)^{n+1}}{(ax+b)^n} \quad \textbf{Ans.}$$

Q. 1.(d) Examine the following system of vectors for linearly dependence, if dependent, find the relation between them $X_1 = (1, 2, 3); X_2 = (2, -2, 6)$.

Ans. Given $X_1 = (1, 2, 3), X_2 = (2, -2, 6)$

Consider the matrix Eqn.

$$\lambda_1 X_1 + \lambda_2 X_2 = 0$$

i.e., $\lambda_1(1, 2, 3) + \lambda_2(2, -2, 6) = 0$

i.e.,

$$\lambda_1 + 2\lambda_2 = 0 \quad \text{...(1)}$$

$$2\lambda_1 - 2\lambda_2 = 0 \quad \text{...(2)}$$

$$3\lambda_1 - 6\lambda_2 = 0 \quad \text{...(3)}$$

From first two Eqns, we get

$$\lambda_1 = \lambda_2 \text{ and } 3\lambda_2 = 0$$

$$\Rightarrow \quad \lambda_2 = 0 \text{ and } \lambda_1 = 0$$

$\therefore X_1$ and X_2 are linearly independent

Q. 1.(e) Evaluate $\lim_{x \to \theta} \dfrac{\sin^2 x - \sin^2 \theta}{x^2 - \theta^2}$

Ans. $\lim_{x \to \theta} \dfrac{\sin^2 x - \sin^2 \theta}{x^2 - \theta^2}$

Let
$$f(x) = \frac{\sin^2 x - \sin^2 \theta}{x^2 - \theta^2}$$

$$= \frac{(\sin x - \sin\theta).(\sin x + \sin\theta)}{(x - \theta)(x + \theta)}$$

$$\lim_{x \to \theta} f(x) = \left[\frac{\cancel{\sin x} + \sin\theta}{x + \theta}\right].\lim_{x \to \theta}\left[\frac{\sin x - \sin\theta}{x - \theta}\right]$$

$$= \left[\frac{\sin\theta + \sin\theta}{\theta + \theta}\right].\lim_{x \to \theta}\left[\frac{2\cos\left(\frac{x+\theta}{2}\right).\sin\left(\frac{x-\theta}{2}\right)}{x - \theta}\right]$$

$$= \frac{\cancel{2}\sin\theta}{\cancel{2}\theta}.\lim_{x\to\theta}\left[\frac{2\cos\left(\frac{x+\theta}{2}\right).\sin\left(\frac{x-\theta}{2}\right)}{2\left(\frac{x-\theta}{2}\right)}\right]$$

$$= \frac{\sin\theta}{\theta}\lim_{x\to\theta}\frac{2\cos\left(\frac{x+\theta}{2}\right)}{2}.\lim_{x\to\theta}\frac{\sin\left(\frac{x-\theta}{2}\right)}{\left(\frac{x-\theta}{2}\right)}$$

$$= \frac{\sin\theta}{\theta}.\cos\left(\frac{\cancel{2}\theta}{\cancel{2}}\right).1$$

$$= \frac{\sin\theta.\cos\theta}{\theta} \textbf{ Ans.}$$

Q. 1.(*f*) State Leibnitz's theorem.

Ans. Statement: If u and v are any two functions of x such that all their desired differential coefficients exist then the n^{th} differential coefficients of their product is given by

$$\boxed{\begin{aligned}D^n(uv) &= D^n uv + n_{c_1}D^{n-1}uDv + n_{c_2}D^{n-2}uD^2v + n_{c_3}D^{n-3}uD^3v\\ &+ \dots + n_{c_r}D^{n-r}uD^rv + \dots + uD^nv\end{aligned}}$$

Q. 1.(*g*) Find the reduction formula for $\int \tan^2 x dx$.

Ans. Let
$$I_3 = \int \tan^3 x dx$$
$$= \int \tan x.\tan^2 x dx$$
$$I_3 = \int \tan x.(\sec^2 x - 1)dx$$
$$= \int \tan x \sec^2 x dx - \int \tan x dx$$

Let $\tan x = t$ for Ist integral so that

$dt = \sec^2 x dx$

$$I_3 = \int t dt - I_1$$
$$I_3 = \frac{t^2}{2} - I$$
$$I_3 = \frac{\tan^2 x}{2} - \int \tan x dx$$
$$I_3 = \frac{\tan^2 x}{2} - \log \sec x.$$

Q. 1. (h) Integrate $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$.

Ans.
$$I = \int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$$

Put $\frac{1}{6} = t$ i.e. $x = t^6$ so that $dx = 6t^5 dt$.

$$I = \int \frac{6t^5}{(t^6)^{\frac{1}{2}} + (t^6)^{\frac{1}{3}}}dt = \int \frac{6t^5}{t^3 + t^2}dt$$

$$= 6\int \frac{t^2.t^3}{t^2(1+t)}dt = 6\int \frac{t^3}{t+1}dt$$

Dividing t^3 by $(t + 1)$

$$= 6\int\left(t^2 - t + 1 - \frac{1}{t+1}\right)dt$$

$$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1|\right] + c$$

$$= 6\left[\frac{\left(x^{\frac{1}{6}}\right)^3}{3} - \frac{\left(x^{\frac{1}{6}}\right)^2}{2} + x^{\frac{1}{6}} - \log\left|x^{\frac{1}{6}} + 1\right|\right] + c$$

$$= \left[\frac{x^{\frac{1}{2}}}{3} - \frac{x^{\frac{1}{3}}}{2} + x^{\frac{1}{6}} - \log\left|x^{\frac{1}{6}} + 1\right|\right] + c \textbf{ Ans.}$$

Q. 1.(*i*) Show that $f(x) = |x| + |x - 1|$ is continuous at $x = 0$ and $x = 1$.

Ans.
$$f(x) = |x| + |x-1|$$
$$|x| = x \text{ if } x \geq 0$$
$$= -x \text{ if } x < 0$$
$$|x-1| = x - 1, \text{ if } x - 1 \geq 0 \text{ i.e. } x \geq 1$$
$$= -(x-1), \text{ if } x - 1 < 0 \text{ i.e. } x < 1$$
$$\therefore \quad |x| + |x-1| = -x - (x-1) = -x - x + 1$$
$$= 1 - 2x \text{ if } x < 0$$
$$|x| + |x-1| = x - (x-1)$$
$$= \cancel{x} - \cancel{x} + 1$$
$$= 1 \text{ if } 0 \leq x < 1$$
$$|x| + |x-1| = x + x - 1$$
$$= 2x - 1 \text{ if } x \geq 1$$

$$\therefore \quad |x| + |x-1| = \begin{cases} 1 - 2x & x < 0 \\ 1 & 0 \leq x < 1 \\ 2x - 1 & x \geq 1 \end{cases}$$

Now, first consider the point $x = 0$

$$\lim_{x\to 0^-} f(x)^1 = \lim_{x\to 0^-}(1 - 2x) = 1 - 2(0) = 1$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(1) = 1$$

Also $f(0) = 1$

So, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\therefore f(x)$ is continues at $x = 0$

Now Consider the point $x = 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1) = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x-1) = 2\times 1 - 1 = 1$$

Also $$f(1) = 2(1) - 1 = 1$$

So, $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$\therefore$ $f(x)$ is continues at $x = 1$

Q. 1.(j) Find the maximum and minimum values, if any of the given function, $f(x) = -|x-1| + 5$ for all $x\varepsilon R$.

Ans. $$f(x) = -|x-1| + 5 \text{ for all } x\in R$$

we know that $|x-1| \ge 0$ for all $x\in R$

$\Rightarrow -|x-1| \le 0$ for all $x\in R$

$\Rightarrow -|x-1| + 5 \le 5$ for all $x\in R$

$\therefore$ 5 is the maximum value of $f(x)$.

Also, $f(x) = 5 \Rightarrow -|x-1| + 5 = 5$

$$\Rightarrow \quad -|x-1| = 0$$

$$\Rightarrow \quad |x-1| = 0$$

$$\Rightarrow \quad x = 1$$

$\therefore$ "1" in the point of maximum value of $f(x)$.

Now, $f(x)$ can be made as small as we want, so minimum value of $f(x)$ does not exist.

UNIT-I

Q. 2. (a) If $x - 2y = 4$ and $-3x + 5y = -7$ then solve using Cramer's Rule.

Ans. $$D = \begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix}$$

$$\Rightarrow \quad 5 - 6 = -1 \ne 0 \text{ Unique solution}$$

$$D_1 = \begin{vmatrix} 4 & -2 \\ 7 & 5 \end{vmatrix} = 20 + 14 = 34$$

$$D_2 = \begin{vmatrix} 1 & 4 \\ -3 & 7 \end{vmatrix} = 7 + 12 = 19$$

$$x = \frac{+34}{-1}, y = \frac{19}{-1}$$

$$x = -34, y = -19 \text{ Ans}$$

Q. 2.(b) For what values of a and b do the equatins $-x + 2y + 3z = 6$, $x + 3y + 5z = 9$, $2x + 5y + az = b$ have (i) No solution (ii) A unique solution (iii) More than one solution.

Ans.
$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$

$$2x + 5y + 9z = b$$

Here

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & 9 \end{vmatrix}$$

$$= 1(3a-25) - 2(a-10) + 3(5-6)$$
$$= 3a - 25 - 2a + 20 + 15 - 18$$
$$= a - 8$$

$$D_1 = \begin{vmatrix} 6 & 2 & 3 \\ 9 & 3 & 5 \\ b & 5 & a \end{vmatrix}$$

$$= 6(3a-25) - 2(9a-5b) + 3(45-3b)$$
$$= \cancel{18a} - 150 - \cancel{18a} + 10b + 135 - 9b$$
$$= b - 15$$

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 1 & 9 & 5 \\ 2 & b & a \end{vmatrix}$$

$$= 1(9a-5b) - 6(a-10) + 3(b-18)$$
$$= 9a - 5b - 6a + 60 + 3b - 54$$
$$= 3a - 2b + 6$$

$$D_3 = \begin{vmatrix} 1 & 2 & 6 \\ 1 & 3 & 9 \\ 2 & 5 & b \end{vmatrix}$$

$$= 1(3b-5a) - 2(b-2a) + 6(5-6)$$
$$= 3b - 5a - 2b + 4a + 30 - 36$$
$$= -a + b - 6$$

Consistenc Either $D \ne 0$

$$a - 8 \ne 0$$
$$a \ne 8$$

or $$D = 0 = D_1 = D_2 = D_3$$

$$\Rightarrow \quad a - 8 = 0$$

$$\Rightarrow \quad \boxed{a = 8}$$

$$\Rightarrow \quad b - 15 = 0$$

$$\Rightarrow \quad \boxed{b = 15}$$

$$\Rightarrow \quad 3a - 2b + b = 0$$
$$a + b - 6 = 0$$

In this case More than one solution

OR

Q. 2.(*a*) Reduce the matrix given below into normal form and find its Rank

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}.$$

Ans.
$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

operating $R_2 \to R_2 - R_1$

$$A \sim \begin{bmatrix} 3 & -3 & 4 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

operating $C_3 \to C_3 + C_2$

$$A \sim \begin{bmatrix} 3 & -3 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

operating $C_2 \to C_2 + C_1$

$$A \sim \begin{bmatrix} 3 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

operating $R_2 \to R_2 - R_3$

$$A \sim \begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

operating $C_1 \to C_1 - 3C_3$

$$A \sim \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

operating $C_3 \Leftrightarrow C_1, C_2 \Leftrightarrow C_3, C_1 \Leftrightarrow C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Operating $R_2 \to (-1)R_2, R_3 \to (-1)R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A \sim [I_3]$

Which is the required normal from

Hence, Rank = 3

Q.2. (*b*) Verigy Cayley-Hamilton Theorem forthe matirx $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$**, hence find** A^{-1}**.**

Ans. Characterstic equation fo A is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix} = 0 \qquad \text{[Expanding by } R_1]$$

$$\Rightarrow \quad -\lambda[(1-\lambda)(4-\lambda)] + [3 + 2(1-\lambda)] = 0$$

$$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0$$

$\therefore$ Characteristic eq. of A is $\lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0$ ...(1)

By Cayley-Hamilton theorem, A satisfier characterstic equation (1)

$$A^3 - 5A^2 + 6A - 5I = 0$$

Now
$$A^3 = \begin{vmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & -19 & 51 \end{vmatrix}$$

and
$$A^2 = \begin{vmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{vmatrix}$$

$\therefore$
$$A^3 - 5A^2 + 6A - 5I$$

$$= \begin{vmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & -19 & 51 \end{vmatrix} - 5\begin{vmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{vmatrix} + 6\begin{vmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{vmatrix} - 5\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad ...(2)$$

i.e.
$$A^3 - 5A^2 + 6A - 5I = 0$$

Hence Cayley-Hamilton theorem is verified

To find A^{-1}

Pre-Multiplying (2) bg A^{-1} We have

$$A^2 - 5A + 6I - 5A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}A^2 - A + \frac{6}{5}I \qquad ...(3)$$

Putting the values of A^2 and A in (3), we have

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -2 & 5 & 14 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} + \frac{6}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix} \text{ Ans.}$$

UNIT-II

Q. 3. (a) Evaluate $\lim_{x \to 0} \frac{e^x - 1}{\sqrt{1 - \cos x}} = 1$ and show that $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$. **(5)**

Ans.

$$\underset{x \to 0}{\text{Lt}} \frac{e^x - 1}{\sqrt{2\sin^2 \frac{x}{2}}} = \underset{x \to 0}{\text{Lt}} \frac{e^x}{\sqrt{2}\sin \frac{x}{2}} = \frac{1}{\sqrt{2}} \underset{x \to 0}{\text{Lt}} \frac{e^x - 1}{\sin \frac{x}{2}}$$

$$= \frac{1}{\sqrt{2}} \underset{x \to 0}{\text{Lt}} \frac{e^x - 1}{x} . \underset{x \to 0}{\text{Lt}} \frac{2\left(\frac{1}{2}\right)x}{\sin \frac{x}{2}}$$

$$= \frac{2}{\sqrt{2}} \left[\underset{x \to 0}{\text{Lt}} \frac{e^x - 1}{x}\right]\left[\underset{x \to 0}{\text{Lt}} \frac{\frac{x}{2}}{\sin \frac{x}{2}}\right]$$

$$= \sqrt{2} \times 1 \times 1 = \sqrt{2}$$

Show $\Rightarrow$ $\underset{x \to 0}{\text{Lt}} \frac{e^x - 1}{x} = 1$

let $a = e$

$$\underset{x \to 0}{\text{lt}} \frac{e^x - 1}{x} = \log e = 1 \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{a^x - 1}{x} = \log a\right]$$

$$\underset{x \to 0}{\text{Lt}} \frac{e^x - 1}{x} = 1$$

Hence Proved.

Q. 3.(b) Determine value of a, b, c if the function **(7.5)**

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & if\ x < 0 \\ C & if\ x = 0, \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3+2}} & if\ x > 0, \end{cases}$$

is Continous at $x = 0$

Ans. L.H.S. $\lim_{x \to 0} \frac{-\sin(a+1)x + \sin x}{x}$ Putting $x = 0 - h$

$$\underset{h \to 0}{\text{Lt}} \frac{\sin(a+1)(-h) + \sin(-h)}{-h}$$

$$\underset{h \to 0}{\text{Lt}} \frac{-\sin(a+1)h - \sin h}{-h}$$

$$\Rightarrow \quad \underset{h \to 0}{\text{Lt}} \left(\frac{\sin(a+1)h}{h} + \frac{\sin h}{h}\right)$$

$$\Rightarrow \quad \underset{h \to 0}{\text{Lt}} \frac{\sin(a+1)h}{(a+1)h} \times (a+1) + \underset{h \to 0}{\text{Lt}} \frac{\sin h}{h}$$

$$\Rightarrow \quad (a+1) + 1$$

$$= (a+2) \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{\sin ax}{ax} = 1\right]$$

R.H.S: $\underset{x \to 0}{\text{Lt}}{}^{+} \frac{\sqrt{x + bx^2} - \sqrt{x}}{x^{3/2}}$

$$\Rightarrow \quad \underset{x \to 0}{\text{Lt}} \frac{\cancel{\sqrt{x}}\left(\sqrt{1 + bx} - 1\right)}{bx.\cancel{\sqrt{x}}}$$

$$\Rightarrow \quad \underset{x \to 0}{\text{Lt}} \frac{\sqrt{1 + bx} - 1}{bx} \times \frac{\sqrt{1 + bx} + 1}{\sqrt{1 + bx} + 1}$$

$$\Rightarrow \quad \underset{x \to 0}{\text{Lt}} \frac{\cancel{1} + bx - \cancel{1}}{bx\left(\sqrt{1 + bx} + 1\right)} = \underset{x \to 0}{\text{Lt}} \frac{\cancel{bx}}{\cancel{bx}(\sqrt{1 + bx} + 1)}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1}{\sqrt{1 + bx} + 1} = \underset{h \to 0}{\text{Lt}} \frac{1}{\sqrt{1 + b(0 + h)} + 1}$$

$$= \frac{1}{1 + 1} = \frac{1}{2} \text{ Ans.}$$

$$f(0) = C \Rightarrow \text{L.H.S} = \text{R.H.S}$$

$$a + 2 = \frac{1}{2} \Rightarrow a = \frac{1}{2} - 2 = \frac{-3}{2}$$

$$a = \frac{-3}{2}, b = 0, C = \frac{1}{2} \text{ Ans.}$$

OR

Q. 3.(a) Prove that $f(x) = \sin \frac{1}{x}$ is not continuous at $x = 0$. Also name the kind of discontinuity it has. **(5)**

Ans. $f(x) = \sin \frac{1}{x}$

L.H.S $\underset{x \to 0^-}{\text{Lt}} f(x) = \underset{x \to 0^-}{\sin} \frac{1}{x}$

$$\underset{h \to 0}{\text{Lt}} \sin\left(\frac{1}{0 - h}\right) = \sin\left(\frac{1}{0}\right) = +\infty$$

R.H.S $\underset{h \to 0^+}{\text{Lt}} \sin\left(\frac{1}{h}\right)$

$$\sin\left(\frac{1}{0}\right) = \infty$$

Prove $f(x) = \sin\frac{1}{x}$ is not continuous at $x = 0$

Q. 3.(b) Find the value of 'a' if $f(x) = \begin{cases} 2x-1 & ; x<2 \\ a & ; x=2, \\ x+1 & ; x>2 \end{cases}$ **is continuous at $x = 2$.** **(7.5)**

Ans. L.H.S $\underset{x\to 2^-}{\text{Lt}}\ 2x-1$

$\underset{h\to 0}{\text{Lt}}\ 2(2-h)-1$

$\Rightarrow \quad \underset{h\to 0}{\text{Lt}}\ 4-2h-1 = \underset{h\to 0}{\text{Lt}}\ 3-2h = 3-0 = 3$

R.H.S: $\quad \underset{h\to 2^+}{\text{Lt}}\ x+1$

$\Rightarrow \quad \underset{h\to 0}{\text{Lt}}\ (2+h)+1$

$\Rightarrow \quad 2+0+1 = 3$

L.H.S = R.H.S = a

$3 = 3 = a$

$a = 3$ **Ans.**

Q.4.(a) For what choice of a and b, the function $f(x) = \begin{cases} x^2 & ; x\le C \\ ax+b & ; x>C \end{cases}$ **is differentiable at $x = c$.** **(5)**

Ans. $f(x)\begin{cases} x^2 & ; x\le C \\ ax+b; & x>C \end{cases}$

If $f(x)$ is differentiable at $x = c$, then $f(x)$ is continuous at $x = c$

$$\underset{x\to c^-}{\text{Lt}}\ f(x) = \underset{x\to c^+}{\text{Lt}}\ f(x) = f(c)$$

$$\Rightarrow \quad \underset{x\to c^-}{\text{Lt}}\ x^2 = \underset{x\to c^+}{\text{Lt}}\ ax+b = c^2$$

$$\Rightarrow \quad \underset{h\to 0}{\text{Lt}}\ (c-h)^2 = \underset{h\to 0}{\text{Lt}}\left[a(c+h)+b\right] = c^2$$

$$\Rightarrow \quad ac+b = c^2 \quad ...(1)$$

At $\quad x = c$

R.H.S $\quad \underset{x\to c^+}{\text{Lt}}\ \frac{f(x)-f(c)}{x-c} = \underset{x\to c^+}{\text{Lt}}\ \frac{(ax\to b)-c^2}{x-c}$

$$\Rightarrow \quad \underset{h\to 0}{\text{Lt}}\ \frac{(a(c+h)+b)-c^2}{(c+h)-c} = \underset{h\to 0}{\text{Lt}}\ \frac{ac+ah+b-c^2}{\cancel{c}+h-\cancel{c}}$$

$$\Rightarrow \quad \underset{h\to 0}{\text{Lt}}\ \frac{(ac+b)+ah-c^2}{h} \quad \text{[using I } ac+b=c^2]$$

$$\Rightarrow \quad \underset{h\to 0}{\text{Lt}}\ \frac{\cancel{c^2}+ah-\cancel{c^2}}{h} \Rightarrow \underset{h\to 0}{\text{Lt}}\ \frac{ah}{h} = \underset{h\to 0}{\text{lt}}\ a = a$$

L.H.S $\quad \underset{h\to c^-}{\text{Lt}}\ \frac{f(x)-f(c)}{x-c} \Rightarrow \underset{x\to c^-}{\text{Lt}}\ \frac{x^2-c^2}{x-c}$ [using $(x^2-c^2) = (x-c)(x+c)$]

$$= \underset{x\to c^-}{\text{Lt}}\ \frac{(x+c)\cancel{(x-c)}}{\cancel{(x-c)}} \Rightarrow \underset{h\to c^-}{\text{Lt}}\ (x+c)$$

$$\underset{h\to 0}{\text{Lt}}\ \frac{c-h+c}{1} \Rightarrow c-0+c = 2c$$

L.H.S = R.H.S

$a = 2c$

Put this value in eq (1)

$c^2 = ac+b$

$\Rightarrow \quad c^2 = 2c\times c+b$

$c^2 = 2c^2+b$

$b = -c^2$ **Ans.**

Q.4.(b) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$**, prove that** $I_{n+1}+I_{n-1} = \frac{1}{n}$**. Deduce the value of I_5.**

Ans. $\quad I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x.\tan^2 x dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\left(\sec^2 x-1\right)dx$$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x.\sec^2 x dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x dx$$

taking $\tan x = t$ in Ist integral so that

$\Rightarrow \quad \sec^2 x\,dx = dt$

when $x\to 0$ than $t\to 0$

and $x\to\frac{\pi}{4}$ then $t\to 1$

$$\therefore \quad I_n = \int_0^1 t^{n-2}dt - I_{n-2}$$

$$\Rightarrow \quad I_n + I_{n-2} = \left[\frac{A^{n-1}}{n-1}\right]_0^1$$

$$\Rightarrow \quad I_n + I_{n-2} = \frac{1}{n-1}$$

Now put $(n+1)$ in place of n, we get

$$\boxed{I_{n+1}+I_{n-1} = \frac{1}{n}} \quad ...(1)$$

Hence proved.

Now putting $n = 4, 2$ in ...(1)

$$I_5 + I_3 = \frac{1}{4} \quad ...(2)$$

and

$$I_3 + I_1 = \frac{1}{2} \quad ...(3)$$

Now,

$$I_1 = \int_0^{\pi/4} \tan x dx \qquad \left[\log(\sec x)\right]_0^{\pi/4}$$

$$= \left[\log\left(\sec\frac{\pi}{4}\right) - \log(\sec 0)\right]$$

$$= \log\sqrt{2} - \log 1$$

$$= \log(2)^{\frac{1}{2}} - 0$$

$$I_1 = \boxed{\frac{1}{2}\log 2}$$

$\therefore$ from (3)

$$I_3 = \frac{1}{2} - I_1$$

$$I_3 = \frac{1}{2} - \frac{1}{2}\log 2$$

Putting this value in (2), we get

$$I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\log 2$$

$$\Rightarrow \quad \boxed{I_5 = \frac{1}{2}\log 2 - \frac{1}{4}}$$

OR

Q.4. (a) Given $y = x^x + (\sin x)^{\log x}$, Find $\frac{dy}{dx}$. **(5)**

Ans.

$$y = x^x + (\sin x)^{\log x} \quad ...(1)$$

Diff w.r.t x

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}\left(x^x\right)}_{a} + \underbrace{\frac{d}{dx}(\sin x)^{\log x}}_{b}$$

$$\frac{d}{dx}\left(x^x\right) = \text{Put } x^x = u \text{ taking log both sider}$$

$$\log u = x \log x$$

Diff w.r.t.x

$$\frac{1}{u}\frac{du}{dx} = \cancel{x} \times \frac{1}{\cancel{x}} + \log x.1$$

$$\frac{du}{dx} = u(1 + \log x) \qquad [\text{Here } u = x^x]$$

$$\frac{du}{dx} = x^x(1 + \log x) \quad ...(A)$$

and

$$V = (\sin x)^{\log x}$$

Taking log both sides

$$\log v = \log x \,.\, \log \sin x$$

$$\frac{1}{V}\frac{dv}{dx} = \log x \,.\, \frac{1}{\sin x} \times \cos x + \log \sin x . \frac{1}{x}$$

$$\frac{dv}{dx} = (\sin x)^{\log x}\left[\log x.\cot x + \log\frac{\sin x}{x}\right] \quad ...(B)$$

Put A and B in equation D

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= x^x(1 + \log x) + (\sin x)^{\log x}\left[\log x \cot x + \log\frac{\sin x}{x}\right] \text{ Ans.}$$

Q. 4.(b) Find the asymptotes of the curve $2y^2 - 2x^2y - 4xy^2 + 4x^3 + 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$.

Ans. (b) The Highest powers of x and y is x^3 and y^3 respectively and whose coefficients are 4 and 2 respectively and

$$4 \neq 0, 2 \neq 0 \qquad \text{[Not possible.]}$$

So, there are no asymptotes parallel to x-axis and y-axix respectively.

(b)

$$\phi_3(x, y) = 4x^3 - 2x^2y - 4xy^2 + 2y^3$$

$$\phi_2(x, y) = 4x^2 + 14xy + 6y^2$$

$$\phi_1(x, y) = 6y$$

For oblique asymptotes **Step 1:** Put $x = 1, y = m$ in above terms.

$$\phi_3(m) = 4 - 2m - 4m^2 + 2m^3$$

$$\phi_2(m) = 4 + 14m + 6m^2$$

$$\phi_1(m) = 6m$$

$$\therefore \quad \phi'(m) = -2 - 8m + 6m^2 = 6m^2 - 8m - 2$$

Step 2: The values of m are obtained by $\phi_3(m) = 0$

$$2m^3 - 4m^2 - 2m + 4 = 0$$

$$\Rightarrow \quad 2m^2(m-2) - 2(m-2) = 0$$

$$\Rightarrow \quad (m-2)(2m^2-2) = 0$$

$$\Rightarrow \quad 2(m-2)(m^2-1) = 0$$

$$\Rightarrow \quad (m-2)(m-1)(m+1) = 0$$

$$\Rightarrow \quad m = 1, 2, -1 \Rightarrow \boxed{m = -1, 1, 2}$$

Step 3: The value of c is given by

$$C = \frac{\phi_2(m)}{\phi_3(m)}$$

$$= \frac{6m^2 + 14m + 4}{6m^2 - 8m - 2}$$

For $m = 1$,

$$C = \frac{6(-1)^2 + 14(-1) + 4}{6(-1)^2 - 8(-1) - 2}$$

$$= \frac{6 - 14 + 4}{6 + 8 - 2} = \frac{-4}{12} = \frac{-1}{3}$$

for m = 1, $C = \frac{6+14+4}{6-8-2} = \frac{24}{-4} = -6$

for m = 2, $C = \frac{6\times4+14\times2+4}{6\times4-8\times2-2}$

$= \frac{24+28+4}{24-16-2} = \frac{\cancel{56}^{28}}{\cancel{6}_3} = \frac{28}{3}$

Hence, asymptotes are

$m=-1, c=-\frac{1}{3} \Rightarrow \quad y = -x-\frac{1}{3}$

$\Rightarrow \quad 3y = -3x-1 \Rightarrow 3x+3y+1=0$

$m=1, c=-6 \Rightarrow \quad y = x-6$

$\Rightarrow \quad x-y-6 = 0$

$m = 2, \quad C = \frac{28}{3} \Rightarrow y = 2x+\frac{28}{3}$

$\Rightarrow \quad 3y = 6x+28$

$\Rightarrow \quad 6x-3y+28 = 0$

UNIT- IV

Q. 5. (a) Show that $\int_0^{\pi/2}\sin^p\theta\cos^q d\theta = \frac{\Gamma(p+1)/2 . \Gamma\frac{(q+1)}{2}}{\Gamma(p+q+2)/2}$ where denote gamma function.

Ans.

$$= \int_0^{\pi/2}\sin^p x\cos^q x\,dx$$

$$= \int_0^{\pi/2}\sin^{p-1}x\cos^{q-1}x(\sin x\cos x)dx$$

$$= \int_0^{\pi/2}(\sin^2 x)^{\frac{p-1}{2}}(\cos^2 x)^{\frac{q-1}{2}}(\sin x\cos x)dx$$

$$\therefore \quad I = \frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin^2 x)^{\frac{p-1}{2}}(1-\sin^2 x)^{\frac{q-1}{2}}(2\sin x\cos x)dx \quad ...(1)$$

Put $\sin^2 x = t$ so that $2\sin\cos x\,dx = dt$

when $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1$

$\therefore$ from (1), we have

$$I = \frac{1}{2}\int_0^1 t^{\frac{p-1}{2}}(1-t)^{\frac{q-1}{2}}dt$$

$$= \frac{1}{2}B\left(\frac{p-1}{2}+1, \frac{q-1}{2}+1\right)$$

$$= \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2}\frac{T\left(\frac{p+1}{2}\right)T\left(\frac{q+1}{2}\right)}{T\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$$

Thus, We have

$$\int_0^{\frac{\pi}{2}}\sin^p x\cos^q dx = \frac{\frac{1}{2}T(p+1)T\frac{(q+1)}{2}}{T\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$$

Q. 5.(b) Integrate (i) $\int\sec^3 x dx$ (ii) $\int\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$ (iii) $\int_0^a\frac{x^7}{\sqrt{a^2-x^2}}dx$.

Ans. (*i*) $I = \int\sec^3 x\,dx \quad ...(1)$

$I = \int\sec^3 x.\sec x\,dx$

$I = \int\underset{I}{\sec x}\underset{II}{\sec^2 x\,dx}$ Integrating by part

$I = \sec x(\tan x) - \int\sec x\tan x(\tan x)dx$

$I = \sec x\tan x - \int\sec x.\tan^2 x\,dx$

$I = \sec x\tan x - \int\sec x(\sec^2 x-1)dx$

$I = \sec x\tan x - \int\sec^3 x\,dx + \int\sec x\,dx$ [using (I), $I = \int\sec^3 dx$]

$I = \sec x\tan x - I + \int\sec x\,dx \Rightarrow 2I = \sec x\tan x + \int\sec x\,dx$

$I = \frac{\sec x\tan x + \log(\sec x+\tan x)+c}{2}$ **Ans.**

Ans. (*ii*) $I = \int\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$

put $x = \tan t$ so that $dx = \sec^2 t\,dt$

$I = \int\cos^{-1}\left(\frac{1-\tan^2 t}{1+\tan^2 t}\right)\sec^2 t\,dt$

$= \int\cancel{\cos}^{-1}(\cancel{\cos}\,2t)\sec^2 t\,dt$

$= \int 2t\sec^2 t\,dt = 2\int t\sec^2 t\,dt$ [Integrating by part]

$= 2[t.\tan t] - \int 1.\tan t\,dt$

$= 2t\tan t + 2\log|\cos t| + c \quad [\therefore \int\tan t\,dt = \log(\cos t)+c]$

$= 2\tan^{-1}x.x + 2\log\left|\frac{1}{\sqrt{1+x^2}}\right| + c \quad [\because \cos t = \frac{1}{\sec t} = \frac{1}{\sqrt{1+\tan.}} = \frac{1}{\sqrt{1+x^2}}]$

$= 2x\tan^{-1}x + \cancel{2}\left(-\frac{1}{\cancel{2}}\right)\log|1+x^2| + c$

$= 2x\tan^{-1}x - \log|1+x^2| + c$ **Ans.**

Ans. (*iii*) $I = \int_0^a\frac{x^7}{\sqrt{a^2-x^2}}dx$

Put $x = a\sin\theta$ so that $dx = a\cos\theta\, d\theta$

Now, when $x = 0, \theta = 0$ and when $x = a, \theta = \pi/2$

$$I = \int_0^{\pi/2} \frac{a^7 \sin^7\theta . a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$$

$$= \int_0^{\pi/2} \frac{a^7\sin^7\theta . \cancel{a\cos\theta}}{\cancel{a\cos\theta}} d\theta$$

$$= a^7 \int_0^{\pi/2} \sin^7\theta\, d\theta$$

$$= a^7 \int_0^{\pi/2} \left(\sin^2\theta\right)^3 \sin\theta\, d\theta$$

$$= a^7 \int_0^{\pi/2} \left(1 - \cos^2\theta\right)^3 . \sin\theta\, d\theta$$

Put $\cos\theta = t$

$-\sin\theta\, d\theta = dt$

when $\theta = 0$

$\cos 0 = t \Rightarrow t = 1$

and $\theta = \dfrac{\pi}{2}$

$\cos\dfrac{\pi}{2} = t \Rightarrow t = 0$

$$= a^7 \int_1^0 -\left(1 - t^2\right)^3 dt$$

$$= a^7 \times \int_1^0 -\left(1 - 3t^2 + 3t^4 - t^6\right) dt$$

$$= a^7 \times -\left[t - 3.\frac{t^3}{3} + 3.\frac{t^5}{5} - \frac{t^7}{7}\right]_1^0 + c$$

$$= a^7 \times \left[-t + \frac{3t^3}{3} - \frac{3}{5}t^5 + \frac{1}{7}t^7\right]$$

$$= a^7\left[-1 + (1)^3 - \frac{3}{5}(1)^5 + \frac{1}{7}(1)^7\right]$$

$$= a^7\left[-\cancel{1} + \cancel{1} - \frac{3}{5} + \frac{1}{7}\right] = a^7\left[\frac{-21+5}{35}\right] = a^7 \times \frac{-16}{35}$$ **Ans.**

OR

Q. 5.(a) If $y = a\cos(\log x) + b\sin(\log x)$, then show that $x^2y_{n+2} + (2n+1)y_{n+1}x + (n^2-1)y_3 = 0$.

Ans. $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow \quad y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x) \times \frac{1}{x}$$

$$\Rightarrow \quad y_1 = \frac{1}{x}[b\cos(\log x) - a\sin(\log x)]$$

$$\Rightarrow \quad xy_1 = b\cos(\log x) - a\sin(\log x) \qquad ...(1)$$

Differentiate (1) w.r.t. x we got

$$x.\frac{d}{dx}(y_1) + y_1.1 = b.\frac{d}{dx}(\cos(\log x)) - a.\frac{d}{dx}(\sin(\log x))$$

$$\Rightarrow \quad xy_2 + y_1 = -b\sin(\log x).\frac{1}{x} - a\cos(\log x).\frac{1}{x}$$

$$= -\frac{1}{x}(a\cos(\log x) + b\sin(\log x))$$

$$= -\frac{y}{x} \qquad [\because y = a\cos(\log x) + b\sin(\log x)]$$

$$\Rightarrow \quad x^2y_2 + xy_1 + y = 0 \qquad ...(2)$$

Now, differentiating (2) 'n' times $w.r.t.x$ and applying leibritz's theorem, we get

$D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0$

$$\Rightarrow \quad \begin{aligned} &D^n(y_2)x^2 + n_{c_1}D^{n-1}(y_2)D(x^2) + {}^nC_2 D^{n-2}(y_2)D^2(x^2) + D^n(y_1).x \\ &+ {}^nC_1 D^{n-1}(y_1).D(x) + D^n(y) \end{aligned} = 0$$

$$\Rightarrow \quad y_{n+2}x^2 + ny_{n+1}(2x) + \frac{n(n-1)}{\cancel{2}}y_n.\cancel{(2)} + y_{n+1}.(x) + n.y_n.(1) + y_n = 0$$

$$\Rightarrow \quad x^2y_{n+2} + xy_{n+1}(2n+1) + y_n\left(n^2 - \cancel{n} + \cancel{n} + 1\right) = 0$$

$$\Rightarrow \quad x^2y_{n+2} + x(2n+1)y_{n+1} + (n^2+1)y_n = 0$$

Hence proved.

Q. 5. (b) Prove legendre's duplication formula $\lambda\sqrt{2m} = 2\,\boxed{m}\left(m + \frac{1}{2}\right)$ where $\boxed{n} = \int ex^{x-1}dx$ denote the gamma function.

Ans. Legendre's Duplication formula:

From the definition of beta function,

$$B(p,q) = \frac{\boxed{(p)}\boxed{(q)}}{\boxed{(p+q)}} = \int_0^1 u^{p-1}(1-u)^{q-1}du$$

Now, let $p = q = m$, them

$$\frac{\boxed{(m)}\boxed{(m)}}{\boxed{(2m)}} = \int_0^1 u^{m-1}(1-u)^{m-1}du$$

and $$u = \frac{(1+x)}{2} \text{ so } du = \frac{dx}{2}$$

and $$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \int_{-1}^{1}\left(\frac{1+x}{2}\right)^{m-1}\left(1-\frac{1+x}{2}\right)^{m-1}\left(\frac{1}{2}dx\right)$$

$$= \frac{1}{2}\int_{-1}^{1}\left(\frac{1+x}{2}\right)^{m-1}\left(\frac{1-x}{2}\right)^{m-1}dx$$

$$= \frac{1}{2^{1+2(m-1)}}\cdot\int_{-1}^{1}\left(1-x^2\right)^{m-1}dx$$

$$= 2^{1-2m}[2\int_{0}^{1}\left(1-x^2\right)^{m-1}dx$$

Now, use the beta function indentity

$$B(p,q) = 2\int_{0}^{1}x^{2p-1}.\left(1-x^2\right)^{q-1}.dx$$

to write the above as

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(2m)} = 2^{1-m}.B\left(\frac{1}{2},m\right)$$

$$= 2^{1-2m}.\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)}$$

Solving for $\Gamma(2m)$ and using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, then gives

$$\Gamma(2m) = (2\pi)^{-\frac{1}{2}}2^{2m-\frac{1}{2}}\Gamma(m).\Gamma\left(m+\frac{1}{2}\right)$$

$$\Gamma(2m) = \frac{2^{2m-1}\Gamma(m).\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}} g$$

$$\Rightarrow \quad \sqrt{\pi}.\Gamma(2m) = 2^{2m-1}\Gamma(m).\Gamma\left(m+\frac{1}{2}\right)$$ Hence proved.

END TERM EXAMINATION
FIRST SEMESTER [BCA-101]
MATHEMATICS-I [DECEMBER-2013]

Time : 3 Hours **M.M. : 75**

Note: ***Attempt Q.No. 1 which is compulsory and any two more questions from remaining.***

Q.1. (a) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ **then what are the eigen values of** A.

Ans. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ eigen value of A.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

i.e.,
$$(1-\lambda)[(2-\lambda)(3-\lambda)-0] = 0$$
$$(1-\lambda)[6-2\lambda-3\lambda+\lambda^2] = 0$$
$$(1-\lambda)(\lambda^2-5\lambda+6) = 0$$
$$(1-\lambda)(\lambda^2-2\lambda-3\lambda+6) = 0$$
$$(1-\lambda)[\lambda(\lambda-2)-3(\lambda-2)] = 0$$
$$(1-\lambda)(\lambda-2)(\lambda-3) = 0$$
$$\lambda = 1, 2, 3$$

Eigen value are 1, 2, 3

Q.1. Prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Ans. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Operate $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0$$

$\Rightarrow 0\times[(c-a)(b-c)-(a-b)^2]-0[(b-c)^2-(a-b)(c-a)]+0[(b-c)(-a-b)-(c-a)^2]$

$\Rightarrow \quad 0-0+0 = 0$

$0 = 0$ **Hence Proved**

Q.1. (c) Evaluate $\lim\limits_{x\to\infty}\dfrac{(3x^2+5x+7)}{(4x^2+4x+9)}$

Ans. $\underset{x\to\infty}{lt}\dfrac{(3x^2+5x+7)}{(4x^2+4x+9)}$

$$\Rightarrow \frac{\underset{x\to\infty}{lt}\, x^2\left[3+\frac{5x}{x^2}+\frac{7}{x^2}\right]}{\underset{x\to\infty}{lt}\, x^2\left[4+\frac{4x}{x^2}+\frac{9}{x^2}\right]} \Rightarrow \frac{\underset{x\to\infty}{lt}\left[3+\frac{5}{x}+\frac{7}{x^2}\right]}{\underset{x\to\infty}{lt}\left[4+\frac{4}{x}+\frac{9}{x^2}\right]}$$

$$\Rightarrow \frac{\left[3+\frac{5}{\infty}+\frac{7}{\infty^2}\right]}{\left[4+\frac{4}{\infty}+\frac{7}{\infty^2}\right]} \Rightarrow \left[\frac{3+0+0}{4+0+0}\right]=\frac{3}{4} \textbf{ Ans.}$$

Q.1. (d) At what points is the function $\dfrac{x}{(x-1)(x-2)}$ **is continous?**

Ans. $\dfrac{x}{(x-1)(x-2)}$

Put $x = 1$ in equation (1)

$$\frac{1}{(1-1)(1-2)} = \frac{1}{0(-1)}=\frac{1}{0}=\infty$$

Put $x = 2$ in eq. (1)

$$\frac{2}{(2-1)(2-2)} = \frac{2}{1\times 0}=\frac{2}{0}=\infty$$

Put $x = 3$ in equation (1)

$$\frac{3}{(2)(1)} = \frac{3}{2}=1.5$$

Function is continous at $x \varepsilon R$ except at 1 and 2 as the value is infinity at that point.

Q.1. (e) Using chain rule, differentiate log $\{\sin(x^2+1)\}$

Ans. Let $\quad y = \log\{\sin(x^2+1)\}$

Let $\quad \sin(x^2+1) = u$

$\therefore \quad y = \log(u)$

$$\frac{dy}{du} = \frac{1}{u}\cdot\frac{du}{dx} \qquad ...(1)$$

Let $\quad x^2+1 = v$

$\therefore \quad u = \sin v$

$$\frac{du}{dx} = \cos v.\frac{dv}{dx} \qquad ...(2)$$

and $\quad \dfrac{dv}{dx} = 2x$

Using Chain Rule $\dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}$

$$= \frac{1}{u}\times\cos V.2x$$

$$= \frac{1}{\sin(x^2+1)}.\cos(x^2+1).2x$$

$$= 2\left(\frac{\cos(x^2+1)}{\sin(x^2+1)}\right)$$

$= 2x\cot(x^2+1)$ **Ans.**

Q.1. (f) Find dy/dx if $y = \tan^{-1}\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

Ans. $\quad y = \tan^{-1}\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

$$y = \tan^{-1}\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}$$

$$\left(\begin{array}{l}\because 1-\cos 2x = 2\sin^2 x\\ 1+\cos 2x = 2\cos^2 x\end{array}\right)$$

$$y = \tan^{-1}\sqrt{\tan^2 x}$$

$$y = \tan^{-1}(\tan x)$$

$$y = x$$

Differentiate w.r.t. x.

$\dfrac{dy}{dx} = 1$ **Ans.**

Q.1. (g) Evaluate $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$

Ans. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$

$$\Rightarrow \lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right)$$

$$\Rightarrow \lim_{x\to 0}\left[\frac{1-\cos x}{x\cos x+\sin x}\right]$$

$$\Rightarrow \lim_{x\to 0}\left[\frac{\sin x}{-x\sin x+\cos x+\cos x}\right]$$

$$= \frac{\sin 0}{-0\sin 0+\cos 0+\cos 0} = \frac{0}{0+1+1} = \frac{0}{2} = 0$$

Q.1. (*h*) State Rolle's theorem.

Ans. If a function $f(x)$ is.

(*i*) Continuous in the closed Interval $[a, b]$ *i.e.*, $a \le x \le b$

(*ii*) Differenciable in the open interval (a, b) *i.e.*, $a < x < b$ and

(*iii*) $f(a) = f(b)$ then there exist at least one point 'c' in the open interval (a, b) (*i.e.*, $a < c < b$) such that $f'(c) = 0$

Q.1. (*i*) Integrate $\int \log x\,dx$

Ans. $\int \underset{I}{\log x} \cdot \underset{II}{1}\,dx$

Integrating by Part

$$\Rightarrow \log x.\int 1dx - \int\left[\frac{d}{dx}(\log x)\int 1dx\right]dx$$

$$\Rightarrow \log x.x - \int \frac{1}{x}\times x\,dx$$

$$\Rightarrow x\log x - \int 1dx$$

$\Rightarrow x\log x - x + c$ **Ans.**

Q.1. (*j*) Evaluate the gamma function $\Gamma(7/2)$

Ans. $\Gamma(7/2)$

We know

$$\Gamma\tfrac{7}{2} = \frac{7}{2}\Gamma\frac{5}{2}$$

$$= \frac{7}{2}\times\frac{5}{2}\Gamma\frac{3}{2}$$

$$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\Gamma\frac{1}{2}$$

$$= \frac{105}{8}\sqrt{\pi}$$

UNIT-I

Q.2. (*a*) Find the eigen values and eigen vectors of $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ **(6)**

Ans. Given

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

and

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(2-\lambda)(-1-\lambda)-1]-1[1(1+\lambda)-0]-2[-1-0]=0$

$\Rightarrow (1-\lambda)[-2-2\lambda+\lambda+\lambda^2-1]-1(1+\lambda)+2=0$

$\Rightarrow (1-\lambda)[-3-\lambda+\lambda^2]-1-\lambda+2=0$

$\Rightarrow -3-\lambda+\lambda^2+3\lambda+\lambda^2-\lambda^3-1-\lambda+2=0$

$$\Rightarrow -\lambda^3+2\lambda^2+\lambda-2 = 0$$

$$\Rightarrow \lambda^3-2\lambda^2-\lambda+2 = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2-\lambda-2) = 0$$

$$\Rightarrow (\lambda-1)(\lambda+1)(\lambda-2) = 0$$

so $\lambda = 1, \lambda = -1, \lambda = 2$ are the eigen values.

The characteristics vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ne 0$ corresponding to root $\lambda = 1$ is given by

$$(A-\lambda I)X = 0$$

$$\Rightarrow (A-I)X = 0$$

$$\Rightarrow \begin{bmatrix} 1-1 & 1 & -2 \\ -1 & 2-1 & 1 \\ 0 & 1 & -1-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_1 \Leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_3 \Rightarrow R_3 - R_2$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Now, coefficient of these equation has rank = 2.

$\therefore$ Equation will have $3 - 2 = 1$ linearly independent solution.

$$-x + y - z = 0$$

and $\quad y - 2z = 0 \Rightarrow y = 2z$

and $\quad -x + y - z = 0$

$\Rightarrow \quad -x + 2z - z = 0$

$\Rightarrow \quad x = z$

By taking $\quad z = 1$

We have, $\quad x = 1, y = 2$

$\therefore \quad x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the eigen vector of A.

For $\quad \lambda = -1$

$$(A - \lambda I)X = 0$$

$\Rightarrow \quad (A + I)X = 0$

$$\Rightarrow \begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_2 \Rightarrow R_2 - 3R_3$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_1 \Rightarrow R_1 - R_3$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_1 \Rightarrow R_1 + 2R_2$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_1 \Leftrightarrow R_2, R_2 \Leftrightarrow R_3$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Now, coefficient of these equation has rank = 2.

$\therefore$ Equation will have $3 - 2 = 1$ linearly independent solution.

$\therefore -x + z = 0 \Rightarrow x = z$ and $y = 0$

By taking $\quad z = 1$

We have $\quad x = 1$

$\therefore \quad x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector of A.

for $\lambda = 2$, Eigen vector is given by $(A - 2I)X = 0$

$$\Rightarrow \begin{bmatrix} 1-2 & 1 & -2 \\ -1 & 2-2 & 1 \\ 0 & 1 & -1-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_1 \Rightarrow R_1 - R_2$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_3 \Leftrightarrow R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_1 \Leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Which shows rank = 2, so equation will have 3 – 2 = 1 linearly independent solution.

$\therefore$ $-x + z = 0 \Rightarrow x = z$

and $y - 3z = 0 \Rightarrow y = 3z$

Take $z = 1$

We have $x = 1, y = 3$

$\therefore$ $x = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ is eigen vector of A.

Q.2. (b) Examine the system of vectors for linear dependence, if so, find the relation between them.

$$\mathbf{X_1 = (1, -1, 1), X_2 = (2, 1, 1), X_3 = (3, 0, 2)}$$

Sol. $X_1 = (1, -1, 1), X_2 = (2, 1, 1)$

$X_3 = (3, 0, 2)$

Consider the Relation:

$a(1, -1, 1) + b(2, 1, 1) + C(3, 0, 2) = (0, 0, 0)$

$(a + 2b + 3c, -a + b, a + b + 2c) = (0, 0, 0)$ ...(1)

$a + 2b + 3c = 0$ ...(2)

$-a + b = 0$ ...(3)

$a + b + 2c = 0$ ...(4)

from (3) $\Rightarrow a = b$

Put $a = b$ in eq. (2)

Eq (2) becomes $b + 2b + 3c = 0$

$3b + 3c = 0$

$c = -b$

Put $a = b, c = -b$ in equation (4)

$b + b - 2b = 0$

$2b - 2b = 0$

$0 = 0$

Which is true given different real value of a, we get Infinite non-zero real value of a, b.

Hence the given vector are linearly dependent.

Q.3. (a) Prove that the matrix $A \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ satisfies its characteristic equation. Hence find A^{-1}. **(6)**

Sol. We have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

Now, characteristic eqn. of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -(1+\lambda) & 4 \\ 3 & 1 & -(1+\lambda) \end{vmatrix} = 0$$

$\Rightarrow (1 - \lambda) [(1 + \lambda)^2 - 4] - 2 [-2 (1 + \lambda) - 12] + 3 [2 + 3(1 + \lambda)] = 0$

$\Rightarrow (1 - \lambda) [1 + \lambda^2 + 2\lambda - 4] - 2 [-2 - 2\lambda - 12] + 3 [2 + 3 + 3\lambda] = 0$

$\Rightarrow (1 - \lambda) [\lambda^2 + 2\lambda - 3] -2 [-14 - 2\lambda] + 3 [5 + 3\lambda] = 0$

$\Rightarrow \lambda^2 + 2\lambda - 3 - \lambda^3 - 2\lambda^2 + 3\lambda + 28 + 4\lambda + 15 + 9\lambda = 0$

$\Rightarrow -\lambda^3 - \lambda^2 + 18\lambda + 40 = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 18\lambda - 40 = 0$ ...(1)

Now, we shall show square matrix A satisfies eqn. (1)

i.e. we need to show

$$A^3 + A^2 - 18A - 40I = 0$$

Now,

We have $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} . \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}.$$

$$= \begin{bmatrix} 1+4+9 & 2-2+3 & 3+8-3 \\ 2-2+12 & 4+1+4 & 6-4-4 \\ 3+2-3 & 6-1-1 & 9+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

Now $A^3 = A^2.A$

$$= \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} . \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 14+6+24 & 28-3+8 & 42+12-8 \\ 12+18-6 & 24-9-2 & 36+36+2 \\ 2+8+42 & 4-4+14 & 6+16-14 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

Now, $A^3 + A^2 - 18A - 40I$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - \begin{bmatrix} 18 & 36 & 54 \\ 36 & -18 & 72 \\ 54 & 18 & -18 \end{bmatrix} - 40\begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 44+14-18-40 & 33+3-36-0 & 46+8-54-0 \\ 24+12-36-0 & 13+9+18-40 & 74-2-72-0 \\ 52+2-54-0 & 14+4-18-0 & 8+14+18-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, matrix A satisfies its characteristic equation.

Now, in order to find A^{-1}

Consider

$$A^3 + A^2 - 18A - 40I = 0$$

$\Rightarrow \quad A^{-1}(A^3 + A^2 - 18A - 40I) = A^{-1}.0$ [Pre Multiplying by A^{-1}]

$\Rightarrow \quad A^2 + A - 18I - 40A^{-1} = 0$

$$\Rightarrow \quad A^{-1} = \frac{1}{40}\left[A^2 + A - 18I\right]$$

Now, Calculate

$$A^2 + A - 18I$$

$$= \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

So, $$A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

Q.3. (*b*) Test the consistency, of the given system of equations, using rank. Also, find the solution, if any

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 7z = 30 \qquad (6.5)$$

Sol. Writing the given system of equation in matrix form *i.e.*, $AX = B$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

The system of equation is consistent if $\rho(A) = \rho[A : B]$ The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 14 \\ 1 & 4 & 7 & : & 30 \end{bmatrix}$$

Now we reduce the augmented matrix $[A : B]$ to echelon form by applying elementary row transformations only.

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 8 \\ 0 & 3 & 6 & : & 24 \end{bmatrix}$$

Operating $R_3 \to R_3 - 3R_2$

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 8 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

which is in echelon form in which number of non-zero rows is 2.

$\therefore \quad \rho[A : B] = 2$

Also $\quad \rho(A) = 2$

$\therefore \quad \rho(A) = \rho[A : B] = 2$

Therefore the given system of equation is consistent.

Also since $\rho(A) < 3$ (number of unknowns), therefore the system has an infinite number of solution and it will contain $3 - 2 = 1$ arbitrary constant.

The equation corresponding to the matrix are

$$x + y + z = 6$$

$$y + 2z = 8 \Rightarrow y = 8 - 2z$$

$$\therefore \quad x = 6 - y - z$$

$$= 6 - (8 - 2z) - z$$

$$= 6 - 8 + 2z - z$$

$$= z - 2$$

Putting $z = k$, we get

$$x = k - 2, y = 8 - 2k, z = k \text{ Ans.}$$

UNIT-II

Q.4. (a) Evaluate the following limits:

(a) $\lim\limits_{x \to 0}\left(\dfrac{\sqrt{1+3x} - \sqrt{1-3x}}{x}\right)$

Sol. $\lim\limits_{x \to 0}\left(\dfrac{\sqrt{1+3x} - \sqrt{1-3x}}{x}\right)$

Rationalise

$$\lim_{x \to 0}\left(\frac{\sqrt{1+3x} - \sqrt{1-3x}}{x} \times \frac{\sqrt{1+3x} + \sqrt{1-3x}}{\sqrt{1+3x} + \sqrt{1-3x}}\right)$$

$$\Rightarrow \lim_{x \to 0}\left[\frac{(\sqrt{1+3x})^2 - (\sqrt{1-3x})^2}{x(\sqrt{1+3x} + \sqrt{1-3x})}\right]$$

$$\Rightarrow \lim_{x \to 0}\left[\frac{1+3x-1+3x}{x(\sqrt{1+3x} + \sqrt{1-3x})}\right]$$

$$\Rightarrow \lim_{x \to 0}\left[\frac{6x}{x(\sqrt{1+3x} + \sqrt{1-3x})}\right]$$

$$\Rightarrow \lim_{x \to 0}\left[\frac{6}{\sqrt{1+3x} + \sqrt{1-3x}}\right] = \frac{6}{\sqrt{1+0} + \sqrt{1-0}}$$

$$= \frac{6}{1+1} = \frac{6}{2} = 3 \text{ Ans.}$$

(b) $\lim\limits_{x \to 0} \dfrac{(\sin 3x + 7x)}{(4x + \sin 2x)}$

Sol. $\lim\limits_{x \to 0} \dfrac{(\sin 3x + 7x)}{(4x + \sin 2x)}$

$$\Rightarrow \lim_{x \to 0} \frac{x\left(\frac{\sin 3x}{x} + 7\right)}{x\left(4 + \frac{\sin 2x}{x}\right)} = \lim_{x \to x}\left(\frac{3 \times \frac{\sin 3x}{3x} + 7}{4 + 2 \times \frac{\sin 2x}{2x}}\right)$$

$$\Rightarrow \frac{\left(3 \times \lim\limits_{x \to 0} \frac{\sin 3x}{3x} + 7\right)}{\left(4 + 2 \times \lim\limits_{x \to 0} \frac{\sin 2x}{2x}\right)} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\Rightarrow \frac{(3 \times 1 + 7)}{(4 + 2 \times 1)} = \frac{(3+7)}{(4+2)} = \frac{10}{6} = \frac{5}{3} \text{ Ans.}$$

Q.4. (b) Discuss the type of discontinuities, if any, of the given function $f(x)$ defined [0, 1] at the points $x = 0, 1/2, 1$:

$$f(0) = 0$$

$$f(x) = x + 1/2, 0 < x < 1/2$$

$$f(1/2) = 1/2$$

$$f(x) = 3x - 1/2, 1/2 < x < 1$$

$$f(l) = 1 \qquad \textbf{(6.5)}$$

Ans. Consider the point $x = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}\left(x + \frac{1}{2}\right) = 1/2$$

Also $\quad f(0) = 0$

$$\therefore \quad \lim_{x \to 0^+} \neq f(0)$$

So, f is discontinuous of the first kind from right at $x = 0$

Now, consider the point $x = 1/2$

$$\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} (x + 1/2) = \frac{1}{2} + \frac{1}{2} = 1$$

$$\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+}\left(3x - \frac{1}{2}\right) = \frac{3}{2} - \frac{1}{2} = 1$$

But $\quad f(1/2) = 1/2 \text{ (Given)}$

i.e., $\quad \lim\limits_{x \to \frac{1}{2}^+} f(x) = \lim\limits_{x \to \frac{1}{2}^+} f(x) \neq f(1/2)$

$\therefore f$ has a removable discontinuity at $x = \dfrac{1}{2}$

Now, consider the point $x = 1$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}\left(3x - \frac{1}{2}\right)$$

$$= 3(1) - \frac{1}{2} = \frac{5}{2}$$

Given is $f(1) = 1$

so, $\qquad \underset{x\to 1^-}{Lt} f(x) = f(1)$

$\therefore f$ has a discontinuity of first kind from left at $x = 1$.

Q.5. (a) If $\lim\limits_{x\to\infty}\left\{\dfrac{x^2+1}{x+1}-(ax+b)\right\}=2$ find a, b. (6)

Sol.

$$\underset{x\to\infty}{lt}\left[\frac{x^2+1}{x+1}-(ax+b)\right] = 2$$

$$\Rightarrow \underset{x\to\infty}{lt}\left[\frac{x^2+1+2x-2x}{x+1}-(ax+b)\right] = 2$$

$$\Rightarrow \underset{x\to\infty}{lt}\left[\frac{(x+1)^2-2x}{x+1}-(ax+b)\right] = 2$$

$$\Rightarrow \underset{x\to\infty}{lt}\left[\frac{(x+1)^2}{(x+1)}-\frac{2x}{(x+1)}-(ax+b)\right] = 2$$

$$\Rightarrow \lim_{x\to\infty}\left[(x+1)-\frac{2}{(x+1)}-(ax+b)\right] = 2$$

Dividing by x

$$\Rightarrow \underset{x\to\infty}{lt}\left[\left(\frac{x}{x}+\frac{1}{x}\right)-\frac{2}{1+1/x}-\left(\frac{ax}{x}+\frac{b}{x}\right)\right] = 2$$

$$\left[1+\frac{1}{\infty}-\frac{2}{1+\frac{1}{\infty}}-a+\frac{b}{\infty}\right] = 2 \qquad \left[\because \frac{1}{\infty}=0\right]$$

$$\Rightarrow \quad 1+0+2-a+0 = 2$$

$$\Rightarrow \quad 3-a = 2$$

$$-a = 2-3$$

$$-a = -1$$

$$a = 1$$

$$\therefore \quad \underset{x\to\infty}{lt}\frac{(1-a)x^2-(a+b)x+(1-b)}{(x+1)}$$

$$= \frac{0-(1+b)x+(1-b)}{x+1} = 2$$

$$= \underset{x\to\infty}{lt}\frac{-(1+b)+\left(\frac{1-b}{x}\right)}{1+\frac{1}{x}} = 2$$

$$= \frac{-(1+b)+\left(\frac{1-b}{x}\right)}{1+\frac{1}{\infty}} = 2$$

$$\Rightarrow \quad -(1+6) = 2$$

$$\Rightarrow \quad b = -3, a = 1 \textbf{ Ans.}$$

$\Rightarrow s$

Q.5. (b) Show that $f(x)$ is discontinuous at $x = 0$ where

$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}, & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad \textbf{(6.5)}$$

Also locate the type of discontinuity.

$$\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^-}{Lt}\left(\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right)$$

$$= \lim_{h\to 0}\left(\frac{e^{\frac{1}{0-h}}-1}{e^{\frac{1}{0-h}}+1}\right)$$

$$= \underset{h\to 0}{Lt}\left(\frac{e^{-\frac{1}{h}}-1}{e^{-1/h}+1}\right)$$

$$= \underset{h\to 0}{Lt}\left(\frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1}\right)$$

Now as $h \to 0$, $\frac{1}{h} \to \infty$

i.e., $\qquad e^{\frac{1}{h}} \to \infty$

$$= \left(\frac{\frac{1}{\infty}-1}{\frac{1}{\infty}+1}\right)$$

$$= \left(\frac{0-1}{0+1}\right)$$

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$$\boxed{\underset{x\to 0^-}{Lt} f(x) = -1}$$

$$\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^+}{Lt}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$$

$$= \underset{h\to 0}{Lt}\left(\frac{e^{\frac{1}{0+h}}-1}{e^{\frac{1}{0+h}}+1}\right)$$

$$= \underset{h\to 0}{Lt}\left(\frac{1-e^{-1/h}}{1+e^{-1/h}}\right)$$

[$\because$ Dividing numerator and denominator by $e^{1/h}$]

$$\boxed{\underset{x\to 0}{Lt} f(x) = \frac{1-0}{1+0} = 1}$$

Given is $\quad f(0) = 1$

So, we have

$$\underset{x\to 0^+}{Lt} f(x) = f(0) = 1$$

So, f is continuous from right at $x = 0$ and also

$$\lim_{x\to 0^-} f(x) \neq f(0)$$

So, f is discontinuous from left at $x = 0$.

UNIT-III

Q.6. (a) Differentiate: x^{x^2} with respect to x. **(6)**

Sol. Let $y = x^{x^2}$ ...(1)

$\Rightarrow$ $\log y = \log\left(x^{x^2}\right)$

$\Rightarrow$ $\log y = x^2 . \log x$

Differentiating w.r.t. x

$\Rightarrow$ $$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x^2.\log x)$$

$$= x^2 \times \frac{1}{x} + \log x.(2x)$$

$\Rightarrow$ $$\frac{1}{y}\frac{dy}{dx} = (x + 2x\log x)$$

$\Rightarrow$ $$\frac{dy}{dx} = y(x + 2x\log x)$$

$\Rightarrow$ $$\frac{dy}{dx} = x^{x^2}(x + 2x\log x)$$

Q.6. (b) Find the n^{th} derivative of $\sin(ax + b)$ and hence using Leibnitz theorem find the n^{th} derivative of $e^x \sin x$. **(6.5)**

Sol. n^{th} derivative of $\sin(ax + b)$

Let $\quad y = \sin(ax + b)$

Differentiate y w.r.t. x, we get

$$\frac{dy}{dx} = y_1 = a\cos(ax + b)$$

$\Rightarrow$ $$y_1 = a\sin\left(ax + b + \frac{\pi}{2}\right)$$

$$\left[\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$$

Again Differentiate w.r.t. x, we get

$$y_2 = a^2\cos\left(ax + b + \frac{\pi}{2}\right)$$

$$= a^2\sin\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^2\sin\left(ax + b + \frac{2\pi}{2}\right)$$

Again Differentiate w.r.t. x, we get

$$y_3 = a^2\cos\left(ax + b + \frac{2\pi}{2}\right)$$

$$= a^3\sin\left(ax + b + \frac{2\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^3\sin\left(ax + b + \frac{3\pi}{2}\right)$$

Similarly, In general we have,

$$\boxed{y_n = \frac{d^n y}{dx^n} = a^n.\sin\left(ax + b + \frac{n\pi}{2}\right)}$$

n^{th} derivative of $e^x \sin x$:

As we know, n^{th} derivative of

$$y_n = D^n\left(e^{ax}.\sin(bx + c)\right) = (a^2 + b^2)^{n/2}.e^{ax}.\sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$$

Put $\quad a = 1, b = 1$ and $c = 0$ in above formula.

$$\Rightarrow \quad D^n(e^x.\sin x) = (1+1)^{n/2}e^x.\sin(x+n\tan^{-1}(1))$$

$$\Rightarrow \quad y_n = (2)^{n/2}.e^x.\sin\left(x+n\tan^{-1}\left(\tan\frac{\pi}{4}\right)\right)$$

$$\Rightarrow \quad \boxed{y_n = (2)^{n/2}.e^x.\sin\left(x+n\frac{\pi}{4}\right)}$$

Q.7. (a) Evaluate: (a) $\lim\limits_{x\to 0}\dfrac{e^x-1-\log(1+x)}{x^2}$ **(b)** $\lim\limits_{x\to 0}\dfrac{\log(\sin 3x)}{\log(\sin x)}$ **(6)**

Sol. (a) $\underset{x\to 0}{lt}\dfrac{e^x-1}{x^2}-\dfrac{\log(1+x)}{x^2}$

$$\Rightarrow \underset{x\to 0}{lt}\frac{1}{x}\left[\frac{e^x-1}{x}-\frac{\log(1+x)}{x}\right]$$

$$\Rightarrow \underset{x\to 0}{lt}\frac{1}{x}\left[\underset{x\to 0}{lt}\left(\frac{e^x-1}{x}\right)-\underset{x\to 0}{lt}\log\frac{(1+x)}{x}\right]$$

$$\frac{1}{0}[1-1] = 0[0]=0$$

$$\left[\because \underset{x\to 0}{lt}\frac{e^x-1}{x}=1 \ \& \ \underset{x\to 0}{lt}\log\frac{(1+x)}{x}-1\right]$$

(b) $\underset{x\to 0}{\text{Lt}}\dfrac{\log(\sin 3x)}{\log(\sin x)}\left[\text{Indeterminate form }\dfrac{\infty}{\infty}\right]$

$$= \underset{x\to 0}{\text{Lt}}\frac{\frac{1}{\sin 3x}\times 3\cos 3x}{\frac{1}{\sin x}\times\cos x}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{3\cot 3x}{\cot x} \qquad \left[\text{Again form }\frac{\infty}{\infty}\right]$$

$$= \underset{x\to 0}{\text{Lt}}\frac{3\tan x}{\tan 3x} \qquad \left[\because \cot x=\frac{1}{\tan x}\right]\left[\text{form }\frac{0}{0}\right]$$

$$= \underset{x\to 0}{\text{Lt}}\frac{\cancel{3}\sec^2 x}{\cancel{3}\sec^2 3x}$$

$$= \frac{(1)^2}{(1)^2}=1 \ \textbf{Ans.}$$

Q.7.(b) Determine the maximum value of $f(x) = \sin x + \cos x$ in $(0, \frac{\pi}{2})$.

Ans. Let $f(x) = y = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

Put $f'(x) = 0$

$\Rightarrow \cos x - \sin x = 0$

$\Rightarrow \cos x = \sin x$

$\Rightarrow \tan x = 1$

$$\Rightarrow \quad x = \frac{\pi}{4}\left[\frac{\pi}{4}\varepsilon\left(0,\frac{\pi}{2}\right)\right]$$

Now, for $x=\frac{\pi}{4}$.

If x is sufficiently close and to the left of $\frac{\pi}{4}$,

then $\cos x > \sin x$

$\Rightarrow f'(x) < 0$

i.e. $x=\frac{\pi}{4}$ is a point of local maximum and the maximum value is

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4}+\cos\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}}=\sqrt{2}=\mathbf{1.414}.$$

UNIT-IV

Q.8 (a) Integrate (i) $\displaystyle\int\frac{3x+2}{(x-1)(2x+3)}dx$ **(ii)** $\displaystyle\int\frac{x^2+1}{x^4+1}dx$ **(6)**

Sol. (i) $\displaystyle\int\frac{3x+2}{(x-1)(2x+3)}dx = \int\frac{3x+2}{2x^2+x-3}dx$

Let $3x+2 = \lambda\left(\frac{d}{dx}(2x^2+x-3)\right)+\mu$

$3x+2 = \lambda X(4x+1)+\mu$

$3x+2 = \lambda\, 4x+\lambda+\mu$

Comparing it

$3 = 4\lambda$ $\qquad \lambda+\mu=2$

$\lambda = \frac{3}{4}$ $\qquad \mu = 2-\lambda$

$\mu = 2-\frac{3}{4}$

$\mu = \frac{8-3}{4}=\frac{5}{4}$

Now $3x+2 = \frac{3}{4}(4x+1)+\frac{5}{4}$

$$I = \int\frac{\frac{3}{4}(4x+1)+\frac{5}{4}}{2x^2+x-3}dx$$

$$\Rightarrow \frac{3}{4}\int\frac{4x+1}{2x^2+x-3}dx+\frac{5}{4}\int\frac{dx}{2x^2+x-3}$$

$$I = \frac{3}{4}I_1 - \frac{5}{4}I_2$$

$$I_1 = \int \frac{4x+1}{2x^2+x-3}$$

$$= \log|2x^2+x-3| + c_1$$

$$I_2 = \frac{5}{8}\int \frac{dx}{x^2+\frac{x}{2}-\frac{3}{2}}$$

$$= \frac{5}{8}\int \frac{dx}{x^2+\frac{x}{2}+\frac{1}{16}-\frac{1}{16}-\frac{3}{2}}$$

$$= \frac{5}{8}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2-\left(\frac{5}{4}\right)^2}$$

Let $z = x + \frac{1}{4}$

$$dz = dx$$

$$\therefore \quad \frac{5}{8}I_2 = \int \frac{dz}{z^2-\left(\frac{5}{4}\right)^2} = \frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{z-\frac{5}{4}}{z+\frac{5}{4}}\right| + C_2$$

$$\Rightarrow \quad \log\left|\frac{x+\frac{1}{4}-\frac{5}{4}}{x+\frac{1}{4}+\frac{5}{4}}\right| \Rightarrow \frac{5}{8}\log\left|\frac{x-1}{x+\frac{3}{2}}\right|$$

$$I = \frac{3}{4}I_1 - \frac{5}{8}I_2$$

$$I = \frac{3}{4}\log|2x^2+x-3| - \frac{5}{8}\log\left|\frac{x-1}{x+\frac{3}{2}}\right| \text{ Ans.}$$

(ii) $\int \frac{x^2+1}{x^4+1}dx$

$$= \int \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2+\frac{1}{x^2}\right)}dx$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx$$

Put $\quad x - \frac{1}{x} = t$

so that $\quad \left(1+\frac{1}{x^2}\right)dx = dt$

$$\int \frac{dt}{t^2+2} = \frac{1}{2}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c_1$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + c_1 \text{ Ans.}$$

Q.8 (b) State and prove the relationship between beta and gamma functions. (6.5)

Sol. To prove that $B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ where $m > 0, n > 0$.

Proof: By definition: $\quad \Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx \quad$ ...(1)

Put $\quad x = tz$ so that $dx = t.\,dz$.

When $x = 0$, $z = 0$ and when $x \to \infty$, $z \to \infty$

$\therefore$ From (1), We have

$$\Gamma(n) = \int_0^\infty (tz)^{n-1}e^{-tz}(t dz$$

$$= \int_0^\infty t^n.z^{n-1}e^{-tz}dz$$

Multiplying both sides by $e^{-t}.\,t^{m-1}$, we have

$$e^{-t}t^{m-1}.\Gamma(n) = e^{-t}t^{m-1}\int_0^\infty (t^n)z^{n-1}e^{-tz}dz$$

$$= \int_0^\infty t^{m+n-1}z^{n-1}e^{-tz}.e^{-t}dz$$

$$= \int_0^\infty t^{m+n-1}z^{n-1}e^{-t(z+1)}dz$$

Integrating both sides w.r.t t between the limit 0 to ∞

$$e^{-t}t^{m-1}\Gamma(n)dt = \int_0^\infty\int_0^\infty t^{m+n-1}z^{n-1}e^{-t(z+1)}\,dz.dt$$

$$= \int_0^\infty z^{n-1}\left[\int_0^\infty t^{m+n-1}e^{-t(z+1)}dt\right]dz \quad \text{...(2)}$$

Putting $t(z+1) = y$ so that $t = \frac{y}{z+1} \Rightarrow dt = \frac{dy}{z+1}$

when $t = 0, y = 0$ and when $t \to \infty, y \to \infty$

$\therefore$ from (2), we have

$$\Gamma(n)\int_0^\infty e^{-t}t^{m-1}dt = \left[\int_0^\infty z^{n-1}\left(\int_0^\infty \frac{y}{z+1}\right)^{m+n-1}.e^{-y}\frac{dy}{z+1}\right]dz$$

$$= \int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}}\left[\int_0^\infty y^{m+n-1}e^{-y}dy\right]dz$$

$$= \int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}}\Gamma(m+n)dz$$

$$\Gamma(n)\Gamma(m) = \Gamma(m+n).\int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}}dz$$

$$\Gamma(m)(n) = \Gamma(m+n)B(m,n)$$

Hence, we have $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Q.9. (a) Express $\int_0^1 x^5(1-x^3)^{3/2}dx$ in terms of beta function. (6)

Sol. (a) Put

$$x^n = z$$

$$x = z^{1/n}$$

$\Rightarrow$ $$dx = \frac{1}{n}z^{1/n-1}dz$$

At $x = 0, z = 0$ and at $x = 1, z = 1$

$\therefore$ $$\int_0^1 x^m(1-x^n)^p dx = \int_0^1 z^{m/n}(1-z)^p\frac{1}{n}z^{\frac{1}{n}-1}dz$$

$$= \frac{1}{n}\int_0^1 z^{m/n+\frac{1}{n}-1}.(1-z)^p dz$$

$$= \frac{1}{n}B\left(\frac{m+1}{n}, P+1\right) \quad ...(1)$$

Comparing $\int_0^1 x^m(1-x^n)^p dx$ with $\int_0^1 x^5(1-x^3)^{3/2}dx$

We get $m = 5, n = 3, P = \frac{3}{2}$

Put $m = 5, n = 3, P = 3$ in (1) we get

$$\int_0^1 x^5(1-x^3)^{3/2}dx = \frac{1}{3}B\left(\frac{5+1}{3}, \frac{3}{2}+1\right)$$

$$= \frac{1}{3}B\left(\frac{6}{3}, \frac{5}{2}\right)$$

$$= \frac{1}{3}B\left(\underset{m}{2}, \underset{n}{\frac{5}{2}}\right)$$

[$\because$ $B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$ if m, n are +ve integer]

$$= \frac{1}{3}.\frac{(2-1)!\left(\frac{5}{2}-1\right)!}{\left(2+\frac{5}{2}-1\right)!}$$

$$= \frac{1}{3}.\frac{1!.\frac{3}{2}!}{\left(\frac{7}{2}\right)!}$$

$$= \frac{1}{3}\times\frac{1\times\frac{3}{2}!}{\left(\frac{7}{2}\right)\times\left(\frac{5}{2}\right)\times\left(\frac{3}{2}\right)!}$$

$$= \frac{1}{3}\times\frac{1}{\frac{35}{4}} = \frac{1}{3}\times\frac{4}{35}$$

$$= \frac{4}{105} \text{ Ans.}$$

Q.9. (b) Obtain the reduction formula for $\int \sin^0 x dx$. Hence integrate $\int \sin^5 x dx$ (6,5)

Sol. Let

$$I_n = \int \sin^n x\, dx$$

$$= \int \sin^{n-1} x.\sin x\, dx$$

Integrating by parts by taking $\sin^{n-1}x$ as first function and $\sin x$ as second function.

$$I_n = -\sin^{n-1}x(\cos x) - \int(n-1).\sin^{n-2}x.\cos x.(-\cos x)dx$$

$$I_n = -\sin^{n-1}x.\cos x + (n-1)\int \sin^{n-2}x.\cos^2 x\, dx$$

$$I_n = -\sin^{n-1}x.\cos x + (n-1)\int \sin^{n-2}(1-\sin^2 x)\,dx$$

$$I_n = -\sin^{n-1}x.\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$\Rightarrow I_n = -\sin^{n-1}x\,.\cos x + (n-1)I_{n-2} - (n-1)I_n$$
$$\Rightarrow I_n + (n-1)I_n = -\sin^{n-1}x.\cos x + (n-1)I_{n-2}$$
$$\Rightarrow I_n(1+n-1) = -\sin^{n-1}x.\cos x + (n-1)I_{n-2}$$

$$\Rightarrow \boxed{I_n = \frac{-\sin^{n-1}x.\cos x}{n} + \frac{n-1}{n}I_{n-2}} \quad ...(1$$

which is the required reduction formula.

To calculate $\int \sin^5 x dx$

Put $n = 5$ in eq. 1

$$I_5 = \frac{-\sin^4 x.\cos x}{5} + \frac{4}{3}I_3 \quad ...(2$$

Now, put $n = 3$ in eq. 1

$$I_3 = \frac{-\sin^2 x.\cos x}{3} + \frac{2}{3}I_1 \quad ...(3$$

Now $I_1 = \int \sin x\,dx = -\cos x$

Put this value in eq. 3

$$\Rightarrow I_3 = \frac{-\sin^2 x.\cos x}{3} + \frac{2}{3}(-\cos x)$$

$$\Rightarrow I_3 = \frac{-\sin^2 x.\cos x}{3} - \frac{2}{3}\cos x$$

Now, put this value in eq. 2

$$I_5 = \frac{-\sin^4 x.\cos x}{5} + \frac{4}{5}\left(\frac{-\sin^2 x.\cos x}{3} - \frac{2}{3}\cos x\right)$$

$$\Rightarrow I_5 = \frac{-\sin^4 x\cos x}{5} - \frac{4}{15}\sin^2 x\cos x - \frac{8}{15}\cos x$$

or $$\int \sin^5 x dx = \frac{-\sin^4 x.\cos x}{5} - \frac{4}{15}\sin^2 x\cos x - \frac{8}{15}\cos x$$

FIRST SEMESTER [BCA]
END TERM EXAMINATION [2014]
MATHEMATICS-I [BCA-101]

Time : 3 hrs. **M.M. : 75**

Note: *Attempt any five questions. Question No.1 is compulsory. Select one question form each unit.*

Q.1 (a) If A = $\begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$ find the matrix X such that 3A + 5B + 2X = 0. **(3)**

Ans: Given $A = \begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$

$$3A + 5B + 2X = 0$$

$$3\begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix} + 2X = 0$$

$$\begin{bmatrix} 27 & 3 \\ 12 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 25 \\ 35 & 60 \end{bmatrix} + 2X = 0$$

$$\begin{bmatrix} 32 & 28 \\ 47 & 69 \end{bmatrix} + 2X = 0$$

$$2X = -\begin{bmatrix} 32 & 28 \\ 47 & 69 \end{bmatrix}$$

$$X = -\frac{1}{2}\begin{bmatrix} 32 & 28 \\ 47 & 69 \end{bmatrix}$$

Q.1. (b) Prove that if (verify by finding AA^{-1}) A = $\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$ then A^{-1} = $\begin{bmatrix} -1/4 & 3/8 \\ 1/2 & 1/4 \end{bmatrix}$ **(3)**

Ans: $$A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$$

Let $$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$a_{11} = (-1)^{1+1}[2] = 2$$
$$a_{12} = (-1)^{1+2}[4] = -4$$
$$a_{21} = (-1)^{2+1}[3] = -3$$
$$q_{22} = (-1)^{2+2}[2] = 2$$

Matrix formed by its cofactors

$$\begin{bmatrix}2 & -4\\ -3 & 2\end{bmatrix}$$

Transposing the above matrix

$$\text{Adjoint A}\begin{bmatrix}2 & -3\\ -4 & 2\end{bmatrix}$$

$$|A| = \begin{vmatrix}2 & 3\\ 4 & 2\end{vmatrix} = 4-12 = -8$$

$$\text{Inverse of the matrix} = \frac{\text{adjoint } A}{|A|}$$

$$-\frac{1}{8}\begin{bmatrix}2 & -3\\ -4 & 2\end{bmatrix} = \begin{bmatrix}-\frac{2}{8} & \frac{3}{8}\\ \frac{+4}{8} & \frac{-2}{8}\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{1}{4} & \frac{3}{8}\\ 1/2 & -1/4\end{bmatrix}$$

Q.1. (c) Show that $y = \dfrac{x^2-1}{x-1}$ is continous except at x=1. What is the removable discontinuity nature? **(3)**

Ans: Given that $y = \dfrac{x^2-1}{x-1}$

$$y = \frac{(x+1)(x-1)}{(x-1)}$$

$$y = (x+1)$$

L.H.L at x = 1

Put $x = 1+h$

$$\underset{h\to 0}{\text{limit}}\ (1+h+1) = 2$$

R.H.L Put $x = 1-h$

R.H.L at $x = 1$,

$$\underset{h\to 0}{\text{limit}}\ (1-h+1) = 2$$

at $x = 1, y = 2$

Hence at the point $x = 1$, L.H.L = R.H.L = $y(x)$ function is continous at $x = 1$, Otherwise function is discontinuous.

Q.1. (d) Find $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$ **(3)**

Ans: Find $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$

$\because$ form is $\frac{\infty}{\infty}$

Differentiating we have by L. Hospital Rule

$$\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \Rightarrow \lim_{x\to 0}\frac{1}{2\sqrt{1+x}} = \frac{1}{2}$$

$$= \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \frac{1}{2} \text{ Ans}$$

Q.1. (e) Using Taylor's series, find the value of $f\left(\dfrac{21}{20}\right)$ if $f(x) = x^3 - 6x^2 + 7$. **(3)**

Ans: Use Taylor series find the value of

$f\left(\dfrac{20}{21}\right)$ if $f(x)\, x^3 - 6x^2 + 7$

Since
$$f(x) = x^3 - 6x^2 + 7$$
$$f^{I}(x) = 3x^2 - 12x$$
$$f^{II}(x) = 6x - 12$$
$$f^{III}(x) = 6$$
$$f^{Iv}(x) = 0$$

Now By Taylor series

$$f(x+h) = f(x) + h f^{I}(x) + \frac{h^2}{2!} f^{II}(x) \frac{h^3}{3!} f^{III}(x) + \ldots\ \frac{h^n}{n!} f^n(x)$$

$$f(x+h) = (x^3 - 6x^2 + 7) + h(3x^2 - 12x) + \frac{h^2}{2}(6x-12) + \frac{h^3}{3!}\times 6 + 0$$

$$f\left(1+\frac{1}{20}\right) = (1-6+7) + \frac{1}{20}(3-12) + \frac{1}{2}\times\left(\frac{1}{20}\right)^2(6-12) + \frac{1}{6}\times\left(\frac{1}{20}\right)^3\times 6$$

$$= 2 + \frac{(-9)}{20} + \frac{1}{400}\times\frac{1}{2}\times(-6) + \frac{1}{8000}$$

$$= 2 - \frac{2}{20} - \frac{3}{400} + \frac{1}{8000} \Rightarrow 2 - \left(\frac{3600+60-1}{8000}\right)$$

$$= 2 - \frac{3659}{8000} \Rightarrow 2 - 0.457 \Rightarrow 1.543 \text{ Ans.}$$

Q.1. (f) Show that sin (x) (1 + cos x) is maximum when $x = \frac{\pi}{3}$ (3

Ans: Show that sin x (1+ cos x) is max at a $x = \pi/3$

Let,

$$y = \sin x(1+\cos x)$$

$$\frac{dy}{dx} = \sin x(-\sin x) + (1+\cos x)\cos x$$

$$\frac{dy}{dx} = \sin^2 x + \cos x + \cos^2 x$$

$$\frac{dy}{dx} = \cos^2 x - \sin^2 x + \cos x$$

Put $\frac{dy}{dx} = 0$

$$\cos^2 x - \sin^2 x + \cos x = 0$$

$$\cos^2 x - 1 + \cos^2 x + \cos x = 0$$

$$2\cos^2 x + \cos x - 1 = 0$$

$$2\cos^2 x + \cos x - 1 = 0$$

$$2\cos^2 x + 2\cos x - \cos - 1 = 0$$

$$2\cos x(\cos x + 1) - 1(\cos x + 1) = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0$$

$$2\cos x + 1 = 0$$

$$\cos x = -1/2$$

$$2\cos x = 1$$

$$\cos x = 1/2$$

$$= \cos\frac{\pi}{3}$$

$$x = \frac{\pi}{3}$$

Now $\frac{dy}{dx} = \cos^2 x - \sin^2 x + \cos x$

$$\frac{d^2y}{dx^2} = -2\cos\sin x - 2\sin x\cos x - \sin x$$

$$= -4\sin x\cos x - \sin x$$

$$\frac{d^2y}{dx^2} = -4\sin x\cos x - \sin x$$

$$\frac{d^2y}{dx^2\left(at\ x = \frac{\pi}{3}\right)} = -4\sin\frac{\pi}{3}\cos\frac{\pi}{3} - \sin\frac{\pi}{3}$$

$$= -4\times\frac{\sqrt{3}}{2}\times\frac{1}{2} - \frac{\sqrt{3}}{2}$$

$$= -2\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$

$$= \frac{-3\sqrt{3}}{2}.$$

since at $x = \frac{\pi}{3}$ $\frac{d^2y}{dx^2} < 0$

$\therefore$ f(x) is Maximum at $x = \pi/3$

Q.1. (g) Evaluate $\int e^x\left(\frac{x-1}{x^2}\right)dx$ **(3)**

Ans: Evaluate $\int e^x\left[\frac{x-1}{x^2}\right]dx$

$$= \int\frac{xe^x}{x^2}dx - \int\frac{e^x}{x^2}dx$$

$$= \int\frac{e^x}{x}dx - \int\frac{e^x}{x^2}dx$$

$$= \int\underset{II}{e^x}\ \underset{I}{x^{-1}}\,dx - \int\frac{e^x}{x^2}dx$$

Using integrating by parts $= x^{-1}\int e^x dx - \int\left(\frac{d}{dx}(x)^{-1} + \int e^x dx\right)dx - \int\frac{e^x}{x^2}dx$

$$x^{-1}e^x + \int\frac{1}{x^2}e^x dx - \int\frac{e^x}{x^2}dx$$

$$= x^{-1}e^x = \frac{e^x}{x} + c$$

Hence $\int e^x\left(\frac{x-1}{x^2}\right)dx = \frac{e^x}{x} + c$

Q.1. (h) Evaluate $\int e^x\cos^2 x\,dx$. ***(4)***

Ans: Evaluate $\int e^x\cos^2 x\,dx$

$$= \int e^x\frac{(1+\cos 2x)}{2}dx$$

$$= \frac{1}{2}\int(e^x + e^x\cos 2x)\,dx$$

$$= \frac{1}{2}\int e^x dx + \frac{1}{2}\int e^x \cos 2x\, dx$$

Let

$$I = \int \frac{e^x}{II} \frac{\cos 2x}{I} dx$$

$$I = \cos 2x \int e^x dx - \int\left(\frac{d}{dx}\cos 2x \int e^x dx\right)dx$$

$$I = \cos 2x \,.\, e^x - \int \frac{\sin 2x}{2} e^x\, dx$$

$$I = e^x \cos 2x - \frac{1}{2}\int \frac{e^x}{II}\frac{\sin 2x}{I} dx$$

$$I = e^x \cos 2x - \frac{1}{2}\left[\sin 2x \int e^x dx - \int(\sin 2x \int e^x dx) dx\right]$$

$$I = e^x \cos 2x - \frac{1}{2}\left[\sin 2x e^x + \int \frac{\cos 2x}{2} e^x dx\right]$$

$$I = e^x \cos x - \frac{1}{2}e^x \sin 2x - \frac{1}{4}\int e^x \cos 2x\, dx$$

$$I = e^2\left[\cos 2x - \frac{1}{2}\sin 2x\right] - \frac{1}{4}I$$

$$I + \frac{I}{4} = e^x\left[\cos 2x - \frac{1}{2}\sin 2x\right]$$

$$\frac{5I}{4} = e^x\left[\cos 2x - \frac{1}{2}\sin 2x\right]$$

$$I = \frac{4}{5}e^x\left[\cos 2x - \frac{1}{2}\sin 2x\right]$$

Now

$$\frac{1}{2}\int e^x dx + \frac{1}{2}\int e^x \cos 2x dx$$

$$\frac{1}{2}e^x + \frac{4}{5}\times\frac{1}{2}e^x\left[\cos 2x - \frac{1}{2}\sin 2x\right] + c$$

$$\frac{e^x}{2}\left[1 + \frac{4}{5}\cos 2x - \frac{1}{2}\sin 2x\right] + c$$

UNIT-I

Q.2. (a) Given A = $\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$**, find A^{-1} and A^4 using Cayley-Hamilto**

Theorem. (6

Ans: Given $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ find A^{-1} and A^4 by cayley-Hamilton theorem the characteristic equation of the matrix is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} - \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} -1 & -1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & 2-\lambda \\ 1 & -1 \end{vmatrix} = 0$$

$$(2-\lambda)[(2-\lambda)(2-\lambda)-1] - 1[-(2-\lambda)+1] + 1[1-2+\lambda] = 0$$

$$(2-\lambda)[4-2\lambda-2\lambda+\lambda^2-1] - [-2+\lambda+1] + [\lambda-1] = 0$$

$$(2-\lambda)[\lambda^2-4\lambda+3] - [\lambda-1] + [\lambda-1] = 0$$

$$\underline{2\lambda^2} - \underline{8\lambda} + \underline{6} - \lambda^3 + 4\underline{\lambda^2} - \underline{3\lambda} - \lambda + 1 + \lambda - 1 = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

According the cayley Hamilton theorem every square matrix satisfies its characteristics equation

$$A^3 - 6A^2 + 11A - 6I = 0$$

Now

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} (4-1+1) & (2+2-1) & (2-1+2) \\ (-2-2-1) & (-1+4+1) & (-1-2-2) \\ (2+1+2) & (1-2-2) & (1+1+4) \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & 3 & 3 \\ -5 & 4 & -5 \\ 5 & -3 & 6 \end{bmatrix}$$

$$\text{Now } A^3 = A^2 A = \begin{bmatrix} 4 & 3 & 3 \\ -5 & 4 & -5 \\ 5 & -3 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (8-3+3) & (4+6-3) & (4-3+6) \\ (-10-4-5) & (-5+8+5) & (-5-4-10) \\ (10+3+6) & (5-6-6) & (5+3+12) \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 8 & 7 & 7 \\ -19 & 8 & -19 \\ 19 & -7 & 20 \end{bmatrix}$$

Now

$$A^4 = \begin{bmatrix} 8 & 7 & 7 \\ -19 & 8 & -19 \\ 19 & -7 & 20 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} (16-7+7) & (8+14-7) & (8-7+14) \\ (-38-8-19) & (-19+16+19) & (-19-8-38) \\ (38+7+20) & (19+14+20) & (19+7+40) \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 16 & 15 & 15 \\ -65 & 16 & -65 \\ 65 & 43 & 66 \end{bmatrix}$$

Now using cayley-Hamilton theorem to find A^{-1}

$$A^3 - 6A^2 + 11A - 6I = 0$$

Multiplying by A^{-1} we get

$$A^{-1}\left[A^3 - 6A^2 + 11A - 6I\right] = 0$$

$$A^{-1}A^3 - 6A^2A^{-1} + 11AA^{-1} - 6IA^{-1} = 0$$

$$A^2 - 6A + 11I - 6IA^{-1} = 0$$

$$-6AI = -A^2 + 6A - 11I$$

$$A^{-1} = \frac{1}{6}\left[A^2 - 6A + 11I\right]$$

$$A^{-1} = \frac{A^2}{6} - \frac{6}{6}A + \frac{11}{6}I$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 3 & 3 \\ -5 & 4 & -5 \\ 5 & -3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \frac{11}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 3 & 3 \\ -5 & 4 & -5 \\ 5 & -3 & 6 \end{bmatrix} + \frac{11}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 4+11 & 3 & 3 \\ -5 & 4+11 & -5 \\ 5 & -3 & 6+11 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 4 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 15 & 3 & 3 \\ -5 & 15 & -5 \\ 5 & -3 & 17 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 4 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{15}{6}-2 & \frac{3}{6}-1 & \frac{3}{6}-4 \\ \frac{-5}{6}+1 & \frac{15}{6}-2 & \frac{-5}{6}+1 \\ \frac{5}{6}-1 & \frac{-3}{6}+1 & \frac{17}{6}-2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & -7/2 \\ 1/6 & 1/2 & 1/6 \\ -1/6 & 1/2 & 5/6 \end{bmatrix}$$

Hence

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -1/2 & -7/2 \\ 1/6 & 1/2 & 1/6 \\ -1/6 & 1/2 & 5/6 \end{bmatrix}$$

Q.2. (*b*) If $\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ **compute AB and BA and show that AB ≠ BA.** **(6.5)**

Ans: Given

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$$

Now
$$AB = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2+3+0 & 3+6+0 & 4+9+0 \\ -2+2-1 & -3+4+1 & -4+6+2 \\ 0+0-2 & 0+0+2 & 0+0+4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}$$

Now
$$BA = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2-3+0 & 6+6+0 & 0+3+8 \\ 1-2+0 & 3+4+0 & 0+2+6 \\ -1-1+0 & -3+2+0 & 0+1+4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}$$

it is clear from above $AB \neq BA$

Q.3. (a) If the matrix $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ **is orthogonal, then find the values of a, and c.** (6

Ans: Since the matrix A is orthogonal
$$AA^T = I = A^TA$$

Now
$$A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$$

Now
$$A^T = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$$

Now
$$AA^T = I$$

$$\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0+4b^2+c^2 & 0+2b^2-c^2 & 0-2b^2+c^2 \\ 0+2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ 0-2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix}$$

$$\begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$4b^2+c^2 = 1,\ 2b^2-c^2 = 0\ c^2 = 2b^2$

$a^2+b^2+c^2 = 1\ 4b^2+c^2 = 1\ 4b^2+2b^2 = 1$

$$6b^2 = 1$$
$$b^2 = 1/6$$
$$b = \frac{1}{\sqrt{6}}$$

then
$$c^2 = 2 \times \frac{1}{6} = 1/3$$
$$c = \frac{1}{\sqrt{3}}$$

Now
$$a^2+b^2+c^2 = 1$$
$$a^2 + \frac{1}{6} + \frac{1}{3} = 1$$
$$a^2 + \frac{1+2}{6} = 1$$

$\Rightarrow$
$$a^2 + \frac{3}{6} = 1$$
$$a^2 + 1/2 = 1$$
$$a^2 = 1/2$$
$$a = \frac{1}{\sqrt{2}}$$

Hence
$$a = \frac{1}{\sqrt{2}}, b = \frac{1}{\sqrt{6}}, c = \frac{1}{\sqrt{2}}$$

Q.3. (b) Determine the rank of the following matrix using elementary row transformation: $\begin{bmatrix} 2 & 1 & -3 & 4 \\ 2 & 4 & -2 & 5 \\ 0 & 3 & 1 & 3 \\ 2 & 1 & -3 & -2 \end{bmatrix}$ **(6.5)**

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Ans:

$$\begin{bmatrix} 2 & 1 & -3 & 4 \\ 2 & 4 & -2 & 5 \\ 0 & 3 & 1 & 3 \\ 2 & 1 & -3 & -2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_4 \to R_4 - R_1 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} R_4 \to R_4 - R_3$$

$$\begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in echelon form. Hence the rank of the matrix is 3.

UNIT II

Q.4. (a) For what value of x does $y = \dfrac{x+1}{(x+2)(x+3)}$ tends to infinity? Indicate the form of the graph of the function and describe its discontinutities. **(6)**

Ans: For what value of x does

$$y = \frac{x+1}{(x+2)(x+3)} \text{ tends to infinity}$$

$$\lim_{x\to\infty} y = \frac{\frac{1}{y}+1}{\left(\frac{1}{y}+2\right)\left(\frac{1}{y}+3\right)}$$

$$\lim_{x\to\infty} y = \frac{y+1}{(1+2y)(1+3y)}$$

$$\lim_{x\to\infty} y = \frac{y+1}{(1+2y)(1+3y)}$$

$$\lim_{y\to\infty} y = \frac{\infty+1}{(1+2\times\infty)(1+3\times\infty)}$$

$$\lim_{y\to\infty} y = \frac{1}{1\times 1} = 1$$

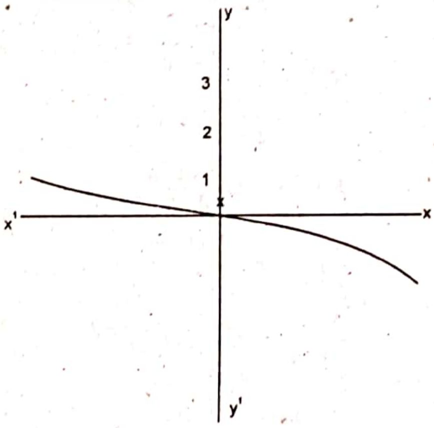
$$\lim_{y\to\infty} y = 1$$

Now to draw the graph of the function we have

$$y = \frac{x+1}{(x+2)(x+3)}$$

Put
$$x = 0, y = 1/6$$
$$x = 1, y = 1/6$$
$$x = 2, y = \frac{3}{4\times 5} = 3/20$$
$$x = 3, y = \frac{4}{5\times 6} = \frac{2}{15}$$

Now the graph of the function is



Q.4. (b) Evaluate $\lim\limits_{m\to\infty} p\left(1+\dfrac{i}{m}\right)^{mn}$. **(6.5)**

Ans: Evaluate the limit

$$\lim_{m\to\infty} p\left[1+\frac{i}{m}\right]^{mn}$$

Put
$$m = y$$
as
$$m \to \infty; y \to \infty$$

$$\lim_{y\to\infty} p\left[1+\frac{i}{y}\right]^{yn}$$

$$\lim_{y \to \infty} p\left[\left(1+\frac{i}{y}\right)^{y}\right]^{n}$$

$$= p\left[e^{y}\right]^{n} = pe^{yn}$$

Q.5. (a) A function f is defined as follows:- $f(x) = \begin{cases} \dfrac{9x}{x+2}, & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ \dfrac{x+3}{x}, & \text{if } x > 1 \end{cases}$. Examine the continuity of f in the interval (–3, 3). **(6)**

Ans: A function f is defined as follows

$$f(x) = \begin{cases} \dfrac{9x}{x+2} & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ \dfrac{x+3}{x} & \text{if } x > 1 \end{cases}$$

in the interval (–3, 3)

L.H.L. at $x = 1$

Put $x = 1 - h$

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} \frac{9(1-h)}{(1-h+2)}$$

$$= \lim_{h \to 0} \frac{9(1-h)}{3-h}$$

$$= \lim_{h \to \infty} \left(\frac{9}{3}\right) = \frac{9}{3} = 3$$

Put $x = 1 + h$

R.H.L $\lim_{h \to 0} = \dfrac{1+h+3}{(1+h)}$

$$\lim_{h \to 0} = \lim_{h \to 0} \frac{4+h}{1+h}$$

$$= \lim_{h \to 0} (4) = 4$$

at $x = 1, f(1) = 3$

Since at $x = 1$, L.H.L $\neq$ R.H.L $\neq f(1)$

$\therefore$ function is discontinuoes at $x = 1$ in the interval (–3, 3).

Q.5. (b) Find the value of a so that the function $f(x) = \begin{cases} ax+5 & if \quad x \le 2 \\ x-1 & if \quad x > 2 \end{cases}$ is continuous at x = 2. **(6.5)**

Ans: Find the value of a so that the function

$$f(x) = \begin{cases} ax+5 & if \quad x \le 2 \\ x-1 & if \quad x > 2 \end{cases} \text{ is continous at 2}$$

L.H.L at $x = 2$

Put $x = 2 - h$

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} a(2-h)+5$$

$$\lim_{h \to 0} (2a-h+5) = (2a+5)$$

Now R.H.L at $x = 2$ Put $x = 2 + h$

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} (2+h-1) = \lim_{h \to \infty} (1+h) = 1$$

at $x = 2 f(x) = ax + 5$

$f(x) = 2a + 5$

The function is continues

$\therefore$ L.H.L = R.H.L = $f(x)$

at $x = 2$

$$2a + 5 = 1 = 2a + 5$$

$$2a + 5 = 1$$

$$2a = 1 - 5$$

$$2a = -4$$

$$a = -2$$

UNIT III

Q.6. (a) Find $\dfrac{dy}{dx}$ if-

(i) $y = \sin\sqrt{x}$ (ii) $x^y.y^x = K$ is a constant. (iii) $y = \sin^3 2x$. **(6)**

Ans: (i) $y = \sin\sqrt{x}$

$$\frac{dy}{dx} = \cos\sqrt{x} \times \frac{1}{2\sqrt{x}} \times 1$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \cos\sqrt{x}$$

$$\frac{dy}{dx} = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

(ii) $2x^y.y^x = k$

Taking log on both sides

$$\log[x^y y^x] = \log k$$

$$\log x^y + \log y^x = \log k$$

$$y \log x + x \log y = \log k$$

$$y \times \frac{1}{x} + \log x \frac{dy}{dx} + \frac{xdy}{dx} + \log y1 = 0$$

$$\frac{dy}{dx}[x + \log x] + \frac{y}{x} + \log y = 0$$

$$\frac{dy}{dx}[x + \log x] = -\frac{y}{x} - \log y$$

$$\frac{dy}{dx} = \frac{-\frac{y}{x} - \log y}{[x + \log x]}$$

$$\frac{dy}{dx} = \frac{-\left[\frac{y}{x} + \log y\right]}{[x + \log x]}$$

Q.6. (b) Find all the asymptotes of the curve $y^2(x - 2a) = x^3 - a^2$. (6.5)

Ans: Let $y = mx + c$ be the oblique Asymplote

$$y^2(x - 2a) = x^3 - a^3$$

Equating to zero highest power of y we get Asymptote parallel to y axis

$$x - 20 = 0;\ x = 2a$$

Now to find oblique Asymptote we have

$$y^2(x - 2a) - x^3 - a^3 = 0$$

$$xy^2 - 2ay^2 - x^3 - a^3 = 0$$

$$-x^3 + xy^2 - 2ay^2 - a^3 = 0$$

$$x^3 - xy^2 + 2ay^2 + a^3 = 0$$

Put $x = 1$, and $y = m\ \phi_3(x, y), \phi_2(x, y)$

$$\phi_3(x, y) = 1$$

Now $$\phi_2(x, y) = -1 \times m^2 + 2a\,m^2 = 0$$

$$2am^2 - m^2 = 0$$

$$m^2[2a - 1] = 0$$

$$m = 0$$

$$C\phi'n(m) + \phi n + (m) = 0$$

$$c \times 0 + 0$$

$$c = 0$$

$$y = mx + c$$

$$y = 0$$

Hence there are two Asymptotes

$$x = 2a, \quad y = 0$$

Q.7. (a) Find the nth derivative of log (2x + 3).

Ans: Let $$y = \log(2x + 3)$$

$$y_1 = \frac{1 \times 2}{(2x + 3)}$$

$$y_1 = \frac{2}{(2x + 3)}$$

$$y_2 = 2(-(2x + 3)^{-2}) + (2x + 3) \times 0$$

$$y_2 = \frac{-2}{(2x + 3)2} + 0$$

$$y_3 = -2\left[-2(2x + 3)^{-3}\right]$$

$$y_3 = +\frac{4}{(2x + 3)^3}$$

In general $$y_n = \frac{(-1)^n 2^n}{(2x + 3)^n} \text{ Ans.}$$

Q.7. (b) Show that the function $f(x) = x^2 + \frac{250}{x}$ has a minimum value at $x = 5$. (6.5)

Ans: Show that the function

$$f(x) = x^2 + \frac{250}{x} \text{ has a minimum value at } x = 5$$

$$f^{I}(x) = 2x - \frac{250}{x^2}$$

Put $$f^{I}(x) = 0$$

$$2x - \frac{250}{x^2} = 0$$

$$2x = \frac{250}{x^2}$$

$$2x^3 = 250$$

$$x^3 = 125$$

$$\therefore x = 3\sqrt{125} = 5$$

$$x = 5$$

Now $$f^{I}(x) = 2x - \frac{250}{x^2}$$

$$f^{II}(x) = 2 + \frac{2 \times 250}{x^3}$$

$$f^{II}(x) = 2 + \frac{500}{x^3}$$

$$f^{II}(x) = 2 + \frac{500}{(5)^3}$$

at $x = 5$

$$= 2 + \frac{500}{125} = 2+4$$

$$f^{II}(x) = 6 > 1$$

at $x = 5$

since at $x = 5f(x) > 0$

$\therefore$ function has its minima at $x = 5$

UNIT IV

Q.8. (a) Find the following integrals:

(i) $\int xe^{-x}dx$ **(ii)** $\int x^n \log x dx$.

Ans: (i) $\int \underset{I}{x} \underset{II}{e^{-x}} dx$

By using, integration by parts

$$x\int e^{-x}dx - \int\left(\frac{d}{dx}(x)\int e^{-x}dx\right)dx$$

$$= -xe^{-x} + \int 1 \times e^{-x}dx$$

$$= -xe^{-x} - e^{-x} + c$$

$$= -xe^{-x} - e^{-x} + c$$

(ii) $\int \underset{II}{x^n} \underset{I}{\log x} dx$

$$\log x\int x^n dx - \int\left(\frac{d}{dx}\log x\int x^n dx\right)dx$$

$$= \log x\frac{x^{n+1}}{n+1} - \int\frac{1}{x} \times \frac{x^{n+1}}{n+1}dx$$

$$\log x\frac{x^{n+1}}{n+1} - \int\frac{x^{n+1-1}}{n+1}dx$$

$$\frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^n dx$$

$$\frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\left[\frac{x^{n+1}}{n+1}\right] + c$$

$$\frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2} + c \text{ Ans.}$$

Q.8. (b) Find out the Reduction Formulae for $\int_0^{\pi/4} \sin^n x dx$, n being a positive integer. **(6.5)**

Ans: Let $I_n = \int \sin^n x dx = \int \sin^{n-1} x.\sin x dx$ By integrating by parts we get

$$\int \sin^{n-1} x \sin x dx$$

$$= \sin^{n-1} x\int \sin x dx - \int\left(\frac{d}{dx}(\sin^{n-1} x).\int \sin x dx\right)dx$$

$$= -\sin^{n-1} x\cos x - \int\left[(n-1)\sin^{n-2} x\cos^2 x\right]dx$$

$$= -\sin^{n-1} x\cos x + \int\left[(n-1)\sin^{n-2} x(1-\sin^2 x)\right]dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$$

$$I_n = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x dx - (n-1)In$$

$$I_n(1+n-1) = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2x} x dx$$

$$I_n = \frac{1}{n}\left[-\sin^{n-1} x\cos x + (n-1)In-2\right]$$

then $$\int_0^{\pi} \sin^n x dx = \frac{1}{n}\left[-\sin^{n-1} x\cos x + (n-1)In-2\right]_0^{\pi/4}$$

$$= \frac{1}{n}\left[-\sin^{n-1}\frac{\pi}{4}\cos\frac{\pi}{4} + (n-1)In-2\right] - 0$$

$$\int_0^{\pi} \sin^n x dx = \frac{1}{\pi}(n-1)In-2$$

Q.9. (a) Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$. **(6)**

Ans: Prove that

$$\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$$

R.H.S $\beta(m+1,n) + \beta(m,n+1) = \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(x+1-x)dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= \beta(m,n)$$

$\therefore$ $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$ proved

Q.9. (b) Evaluate $\int \frac{dx}{2x^2 - 3x + 5}$.

Ans: Evaluate $\int \frac{dx}{2x^2 - 3x + 5}$

Resolving into partial fractions we have

Let $\frac{1}{(2x^2 - 3x + 5)}$

$$= \frac{1}{2x^2 - 5x - 2x + 5}$$

$$= \frac{1}{(2x-5)(x-1)}$$

Let $\frac{1}{(2x-5)(x-1)} = \frac{A}{(2x-5)} + \frac{B}{(x-1)}$

where A, & B are constants

Put $2x - 5 = 0$

$x = 5/2$

in $\frac{1}{(x-1)} = \frac{1}{\left(\frac{5}{2} - 1\right)} = \frac{1}{3/2}$

$A = 2/3$

Now put $x - 1 = 0$

$x = 1$ in $\frac{1}{(2x-5)}$

$$= \frac{1}{(2 \times 1 - 5)} = -1/3$$

$B = -1/3$

Now $\frac{1}{(2x-5)(x-1)} = \frac{A}{(2x-5)} + \frac{B}{(x-1)}$

$$\frac{2}{3(2x-5)} - \frac{1}{3(x-1)}$$

$$\int \frac{1}{(2x-5)(x-1)} dx = \frac{2}{3} \int \frac{1}{(2x-5)} dx - \frac{1}{3} \int \frac{1}{(x-1)} dx$$

$$\frac{2}{3} \times \frac{1}{2} \log (2x-5) - \frac{1}{3} \log(x-1) + \log c$$

$$= \frac{1}{3} \log(2x-5) - \frac{1}{3} Log(x-1) + \log c \text{ Ans.}$$

END TERM EXAMINATION [DEC.-2015]
FIRST SEMESTER [BCA]
MATHEMATICS-I [BCA - 101]

Time: 3 Hrs. **MM : 75**

Note: Attempt any five questions including Q.no. 1 which is compulsory. select one question from each unit.

Q.1. (a) Find Matrices A and B If **(3)**

$$2A - B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} \text{ and } 2B + A = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix}$$

Ans. Solving both Equations we have

$$2A - B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} \quad ...(i)$$

$$A + 2B = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix} \quad ...(ii)$$

Equation (1) × 2

$$4A - 2B = \begin{bmatrix} 12 & -12 & 0 \\ -8 & 4 & 2 \end{bmatrix} \quad ...(i)$$

$$A + 2B = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix} \quad ...(ii)$$

$$5A = \begin{bmatrix} 15 & 10 & 5 \\ -10 & 1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

Put the value of A in (ii)

$$A + 2B = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix}$$

$$2B = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

$$2B = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix} 2B = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}$$

$$2B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} \textbf{ Ans.}$$

Q.1. (b) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 7 & -3 \end{bmatrix}$$

Ans. Since the order of the matrix is 3×4

So rank of the matrix can not exceed by 3

Now using elementary transformations to find the rank.

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 7 & -3 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} R_3 \to R_3 - 7R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since no. of non zero rows are two

$\therefore$ Rank of matrix is 2.

Q.1. (c) Evaluate $\lim\limits_{x \to 3} \dfrac{x^5 - 243}{x^4 - 81}$

Ans. $\lim\limits_{x \to 3} \dfrac{x^5 - 243}{x^4 - 81}$ Form is $\left(\dfrac{0}{0}\right)$ Apply L Hospital rule.

Putting the value of limit we.

Get

$$\Rightarrow \qquad = \lim_{x \to 3} \frac{5}{4} \times x$$

$$\Rightarrow \qquad \frac{5}{4} \times 3 = \frac{15}{4}$$

Q.1. (d) Evaluate the $\lim\limits_{x \to 0} \left(\dfrac{\sin ax}{\tan bx}\right)^k$ where $K \in R$ **(3)**

Ans.
$$= \lim_{x \to 0} (\sin ax)^k \times \lim_{x \to 0} \left(\frac{1}{\tan bx}\right)^k$$

$$\lim_{x \to 0} \frac{(\sin ax)^k}{(ax)^k} \times (ax)^k \times \lim_{x \to 0} \frac{(bx)^k}{(\tan bx)^k} (bx)^k$$

$$\Rightarrow \qquad a^k \times 1 \times \frac{1}{b^k} \times 1$$

$$= \frac{a^k}{b^k} = \left(\frac{a}{b}\right)^k \textbf{ Ans.}$$

Q.1. (e) Use Taylor's theorem to prove that

$$\log(x + h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots\infty \qquad \textbf{(3)}$$

Ans. By Taylor Theorem we have

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \ldots + \frac{x^n}{n!} f^n(h) + \ldots$$

Let

$$f(x+h) = \log(x+h)$$

$$f(x) = \log x, \; h = h$$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f'''(x) = -\frac{2}{x^3}$$

$$\log(x+h) = \log x + h \times \frac{1}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots$$

$$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots$$ **Hence Proved.**

Q.1. (f) If $y = e^{(x+1)^3}$ find $\frac{dy}{dx}$ (3)

Ans. Taking log on both sides we have

$$\log y = \log e^{(x+1)^3}$$

$$\log y = (x+1)^3 \log e$$

$$\frac{1}{y}\frac{dy}{dx} = 3(x+1)^2 \times 1$$

$$\frac{1}{y}\frac{dy}{dx} = 3(x+1)^2$$

$$\boxed{\frac{dy}{dx} = 3y(x+1)^2}$$

$$\frac{dy}{dx} = 3e^{(x+1)^3}.(x+1)^2$$ **Ans.**

Q.1. (g) Evaluate $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

Ans. By Rationalizing the above Integral

$$\int \frac{\left[\sqrt{x+a}-\sqrt{x}\right]dx}{\left[\sqrt{x+a}+\sqrt{x}\right]\left[\sqrt{x+a}-\sqrt{x}\right]}$$

$$= \int \frac{\left[\sqrt{x+a}-\sqrt{x}\right]dx}{x+a-x}$$

$$= \frac{1}{a}\int \sqrt{x+a}\,dx - \frac{1}{a}\int \sqrt{x}\,dx$$

Put

$$x+a = t$$

$$dx = dt$$

$$\frac{1}{a}\int \sqrt{t}\,dt - \frac{1}{a}\int \sqrt{x}\,dx$$

$$\frac{1}{a}\int t^{1/2}dt - \frac{1}{a}\int x^{1/2}\,dx$$

$$\frac{\frac{1}{a}[t]^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{a}\frac{(x)^{3/2}}{\frac{3}{2}} + c$$

$$\frac{2}{3a}(x+a)^{3/2} - \frac{2}{3a}(x)^{3/2} + c$$

$$\frac{2}{3a}\left[(x+a)^{3/2} - (x)^{3/2}\right] + c$$

Q.1. (h) Obtain the reduction formula for $\int \cos^n x dx$ (4)

Ans. Let

$$I_n = \int \cos^n x dx$$

$$= \int \cos^{n-1} x \cos x dx$$

Now by Integration by Parts

$$\cos^{n-1} x \int \cos x\, dx - \int\left[\frac{d}{dx}\cos^{n-1} x \int \cos x\, dx\right]dx$$

$$= \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x \sin^2 x\, dx$$

$$= \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x(1-\cos^2 x)dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$

$$I_n = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)I_n$$

$$I_n(1 + n - 1) = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$I_n = \frac{1}{n}[\cos^{n-1} x \sin x + (n-1)I_{n-2}] \text{ Ans.}$$

UNIT – I

Q.2. (a) For what value of 'a' and 'b' does the following system of Equation $x + 2y + 3z = 1, x + 3y + 5z = 2$ and $2x + 5y + az = b$ has **(6.**

(i) no solution,

(ii) Unique solution, (iii) Infinite solution.

Ans. The given system of Equation is

$$x + 2y + 3z = 1$$
$$x + 3y + 5z = 2$$
$$2x + 5y + az = b$$

Matrix form of the Equation is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & a \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ b \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Now $$A:B = \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 1 & 3 & 5 & : & 2 \\ 2 & 5 & a & : & b \end{bmatrix} \begin{matrix} \text{using } R \to R_2 - R_1 \\ R_3 \to R_3 \to 2R_1 \end{matrix}$$

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 0 & 1 & 2 & : & 1 \\ 0 & 1 & a-6 & : & b-2 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & a-8 & : & b-3 \end{bmatrix}$$

For unique solution Rank (A) = Rank $(A : B) = 3 \Rightarrow$

$$a - 8 \neq 0, \quad b - 3 \neq 0$$

or $\quad a \neq 8, \quad$ and $b \neq 3$

For no solution: rank $(A) \neq$ rank $(A : B)$

$$\text{rank } (A) < \text{rank } (A : B)$$

$\Rightarrow \quad a - 8 = 0, \quad$ and $b - 3 \neq 0$

or $\quad a = 8$, and $b \neq 3$

For many solution rank (A) = rank $(A : B) < 3$

$\Rightarrow \quad a - 8 = 0$, and $b - 3 = 0$

or $\quad a = 8$, and $b = 3$ Ans.

Q.2. (b) If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **find A^{-1} and show that $A^2 = I$ where I is the identity matrix.** **(6)**

Ans. Let $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

Where, c_{11}, c_{12}, c_{13} ... etc are

Co-factors

$$c_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} = 0 = 0$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = 0 = 0$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 0 - (-1) = 1$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = -(0 - 0) = 0$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = (0 - 1) = -1$$

$$c_{23} = (-1)^{2+3}\begin{vmatrix}1 & -1\\ 1 & 0\end{vmatrix} = -(0+1) = -1$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix}-1 & 1\\ -1 & 0\end{vmatrix} = (0+1) = 1$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix}-1 & 1\\ -2 & 0\end{vmatrix} = -(0+2) = 2$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix}1 & -1\\ 2 & -1\end{vmatrix} = (-1+2) = 1$$

Matrix formed by Co – factors is

$$\begin{bmatrix}0 & 0 & 1\\ 0 & -1 & -1\\ 1 & 2 & 1\end{bmatrix}$$ Transposing this Matrix

$$\text{Adjoint } A = \begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{bmatrix}$$

Now the determinant of the matrix is

$$A = \begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$$

$$|A| = 1\begin{vmatrix}-1 & 0\\ 0 & 0\end{vmatrix} + 1\begin{vmatrix}2 & 0\\ 1 & 0\end{vmatrix} + 1\begin{vmatrix}2 & -1\\ 1 & 0\end{vmatrix}$$

$$|A| = 0 + (0-0) + (0+1) = 1$$

$$\boxed{|A| = 1}$$

Now the $$A^{-1} = \frac{\text{Adjoint } A}{|A|}$$

$$A^{-1} = \begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{bmatrix} \times \frac{1}{1}$$

$$A^{-1} = \begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{bmatrix}.$$

Now to Prove $A^3 = I$

$$A = \begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$$

$$A^2 = A \times A = \begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}1-2+1 & -1+1+0 & 1+0+0\\ 2-2+0 & -2+1+0 & 2-0+0\\ 1+0+0 & -1+0+0 & 1+0+0\end{bmatrix}$$

$$A^2 = \begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$$

$$A^3 = \begin{bmatrix}0+0+1 & 0+0+0 & 0+0+0\\ 0-2+2 & 0+1+0 & 0+0+0\\ 2-2+0 & -1+1+0 & 1+0+0\end{bmatrix}$$

$$A^3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$ **Hence** $\boxed{A^3 = I}$ **Ans.**

Q.3. (a) Solve the following system of Equations by Cramer's rule.

$$x-4y-z = 1$$
$$2x-5y+2z = 39$$
$$-3x+2y+z = 1$$

Ans. Let
$$A = \begin{bmatrix}1 & -4 & -1\\ 2 & -5 & 2\\ -3 & 2 & 1\end{bmatrix}\quad x = \begin{bmatrix}x\\ y\\ z\end{bmatrix}\quad B = \begin{bmatrix}1\\ 39\\ 1\end{bmatrix}$$

$$D = \begin{bmatrix}1 & -4 & -1\\ 2 & -5 & 2\\ -3 & 2 & 1\end{bmatrix}$$

$$D = 1\begin{bmatrix}-5 & 2\\ 2 & 1\end{bmatrix}+4\begin{bmatrix}2 & 2\\ -3 & 1\end{bmatrix}-1\begin{bmatrix}2 & -5\\ -3 & 2\end{bmatrix}$$

$$D = (-5-4)+4(2+6)-1(4+15)$$
$$D = -9+32-19$$
$$D = -28+32 = 4 \qquad \boxed{D=4}$$

$$D_1 = \begin{bmatrix}1 & -4 & -1\\ 39 & -5 & 2\\ 1 & 2 & 1\end{bmatrix}$$

$$D_1 = 1\begin{bmatrix}-5 & 2\\ 2 & 1\end{bmatrix}+4\begin{bmatrix}39 & 2\\ 1 & 1\end{bmatrix}-1\begin{bmatrix}39 & -5\\ 1 & 2\end{bmatrix}$$

$$D_1 = (-5-4)+4(39-2)-1(78+5)$$
$$D_1 = -9+148-83$$
$$= -92+148 = 56. \qquad \boxed{D_1 = 56}$$

$$D_2 = \begin{bmatrix}1 & 1 & -1\\ 2 & 39 & 2\\ -3 & 1 & 1\end{bmatrix}$$

$$D_2 = 1\begin{bmatrix}39 & 2\\ 1 & 1\end{bmatrix}-1\begin{bmatrix}2 & 2\\ -3 & 1\end{bmatrix}-1\begin{bmatrix}2 & 39\\ -3 & 1\end{bmatrix}$$

$$D_2 = [39-2]-[2+6]-[2-117]$$

$$D_2 = 37-8+115$$
$$D_2 = 152-8 = 144 \quad \boxed{D_2 = 144}$$

Now
$$D_3 = \begin{bmatrix}1 & -4 & 1\\ 2 & -5 & 39\\ -3 & 2 & 1\end{bmatrix}$$

$$D_3 = 1\begin{bmatrix}-5 & 39\\ 2 & 1\end{bmatrix}+4\begin{bmatrix}2 & 39\\ -3 & 1\end{bmatrix}+1\begin{bmatrix}2 & -5\\ -3 & 2\end{bmatrix}$$

$$D_3 = (-5-78)+4(2-117)+(4-15)$$
$$D_3 = -83+4(-115)+(-11)$$
$$D_3 = -83-460-11$$
$$\boxed{D_3 = -554}$$

Now
$$x = \frac{D_1}{D} = \frac{56}{4} = 14$$

$$y = \frac{D_2}{D} = \frac{144}{4} = 36$$

$$z = \frac{D_3}{D} = \frac{-54}{4} = -136 = \boxed{z=-136}$$

Q.3. (b) Find the eigen values and eigen vectors for the following matrix.

$$A = \begin{bmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{bmatrix} \qquad \textbf{(6.5)}$$

Ans. Let I be the identify matrix of same order. The characteristic equation of the atrix is $(A-\lambda I) = 0$

$$\begin{bmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix} = 0$$

$$\begin{bmatrix}2-\lambda & 0 & 1\\ 0 & 2-\lambda & 0\\ 1 & 0 & 2-\lambda\end{bmatrix} = 0$$

Solving the above equation we have

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{aligned}
(2-\lambda)(2-\lambda)(2-\lambda) - 0 + 1(0-(2-\lambda)) &= 0 \\
(2-\lambda)^3 - (2-\lambda) &= 0 \\
(2-\lambda)[(2-\lambda)^2 - 1] &= 0 \\
(2-\lambda)[\lambda^2 - 4\lambda + 4 - 1] &= 0 \\
(2-\lambda)[\lambda^2 - 4\lambda + 3] &= 0 \\
(2-\lambda)(\lambda-3)(\lambda-1) &= 0
\end{aligned}$$

eigen values are

$$\lambda = 1, 2, 3$$

Now eigen vector for eigen value $\lambda = 1$

$$(A-\lambda_1)X = 0$$

$$= (A-I)x = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Augmented matrix $\begin{bmatrix} 1 & 0 & 1 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 1 & 0 & 1 & : & 0 \end{bmatrix}$

Using $R_3 \to R_3 - R_1$

$\begin{bmatrix} 1 & 0 & 1 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$ By Back Substitution we get

$$\begin{aligned} x + z &= 0 \\ y &= 0 \end{aligned}$$

The eigen vector is $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Now eigen vector for $\quad \lambda = 2$

$$(A-\lambda I)X = 0 \Rightarrow (A-2I)X = 0 \Rightarrow \begin{bmatrix} 2-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 2.2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Augmented matrix is $\begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix}_{R_3 \leftrightarrow R_1}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x = 0, z = 0 \text{ Let } y = k$$

The eigen vector is $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

eigen vector for $\lambda = 3$

$$(A-\lambda I)X = 0 \Rightarrow (A-3I)X = 0 \Rightarrow \begin{bmatrix} 2-3 & 0 & 1 \\ 0 & 2-3 & 0 \\ 1 & 0 & 2-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Augmented matrix $\begin{bmatrix} -1 & 0 & 1 & : & 0 \\ 0 & -1 & 0 & : & 0 \\ 1 & 0 & -1 & : & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -1 & : & 0 \\ 0 & -1 & 0 & : & 0 \\ 1 & 0 & -1 & : & 0 \end{bmatrix}_{R_3 \to R_3 - R_1} \quad R_1 \to (R_1) \quad = \begin{bmatrix} 1 & 0 & -1 & : & 0 \\ 0 & -1 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x - z = 0, \quad -y = 0$$

Let $z = k, \Rightarrow x = k$

eigen vector is $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$= k\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ **Ans.**

UNIT-II

Q.4. (a) Evaluate $\lim_{x \to 1}\left([x] + \frac{|x-1|}{x-1} + 2\right)$

Ans. To evaluate Left hand limit we substitute $x = 1 - h$ and replace the li[mit] $\to 1^-$ by $h \to 0$

$$\lim_{x \to 1^-}\left([x] + \frac{|x-1|}{x-1} + 2\right) = \lim_{h \to 0}\left([1-h] + \frac{|1-h-1|}{|h|} + 2\right)$$

$$= \lim_{h \to 0}\left(0 + \frac{|-h|}{-h} + 2\right)$$

$$= \lim_{h \to 0}\left(0 - \frac{h}{h} + 2\right)$$

$$= \lim_{h \to 0}(0 - 1 + 2) = 1$$

To evaluate Right hand limit we put $x = 1 + h$ and replace the limit $x \to 1^+$ by $h \to 0$

$$\therefore \quad \lim_{x \to 1^+}\left([x] + \frac{|x-1|}{x-1} + 2\right) = \lim_{h \to 0}\left((1+h) + \frac{|1+h-1|}{1+h-1} + 2\right)$$

$$= \lim_{h \to 0}\left(1 + \frac{|h|}{h} + 2\right)$$

$$= \lim_{h \to 0}\left(1 + \frac{h}{h} + 2\right)$$

$$= \lim_{h \to 0}(1 + 1 + 2) = 4$$

Since L.H.L is not equal to R.H.L.

$\therefore$ Limit does not exist.

Q.4. (b) For what choice of 'a' and 'b' is the function continuous $\forall x \in \mathbf{R}$

$$f(x) = \begin{cases} ax^2 + b, & x < 2 \\ 2 & x = 2 \\ 2ax + b, & x > 2 \end{cases} \qquad \textbf{(6.5)}$$

Ans. Left hand limits at $x = 2$

Put $x = 2 - h$ and taking $\lim h \to 0$

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} a(2-h)^2 + b$$

$$= \lim_{h \to 0} a(2)^2 + b$$

$$= 4a + b$$

Right hand limit at $x = 2$. Put $x = (2 + h)$

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} 2a(2+h) + b$$

$$= \lim_{h \to 0} (2a \times 2 + b)$$

$$= 4a + b$$

at $x = 2$ $f(x) = 2$ is given

Since the function is continuous

$$\therefore \quad \lim_{h \to 0} f(x) = \lim_{h \to 0^+} f(x) = f(2)$$

$$4a+b = 4a+b = 2$$
$$4a+b = 2$$
$$4a = 2-b$$

Put $\quad a = 0 \quad b = 2$

Put $\quad b = 0 \quad a = 1/2$

values of 'a' and 'b' are $\frac{1}{2}, 2$

Q.5. (a) For what value of λ does the lim $x \to 1$ $f(x)$ exists, where f is defin[ed] by the rule (6)

$$f(x) = \begin{cases} 2\lambda x+3 & 2f\ x<1 \\ 1-\lambda x\times 2 & 2f\ x>1 \end{cases}$$

Ans. Right hand limit for the function

Put $x = 1$ and Taking limit $h \to 0$

$$\lim_{h\to 0} f(x) = \lim_{h\to 0} (1-\lambda(1+\lambda)2)$$
$$= \lim_{h\to 0} (1-2\lambda-2\lambda)$$
$$= (1-2\lambda)$$

Left hand limit of the function.

Put $x = 1-h$ and Taking limit $h \to 0$

$$\lim_{h\to 0} f(x) = \lim_{h\to 0} (2\lambda x+3)$$
$$= \lim_{h\to 0} 2\lambda(1-h)+3$$
$$= \lim_{h\to 0} (2\lambda-2\lambda+3)$$
$$= 2\lambda+3$$

Let $\quad 1-2\lambda = 2\lambda+3$

$$1-3 = 4\lambda$$
$$4\lambda = -2 \quad \lambda = -2/4 \quad \lambda = -1/2$$

For $\lambda = -\frac{1}{2}$ L.H.L = R.H.L

$\therefore$ For $\lambda = -\frac{-1}{2}$ Limit $x \to 1$ exists.

Q.5. (b) Discuss the nature of discontinuity at $x = 0$ of $f(x)$ (6)

$$f(x) = \begin{cases} \frac{\sin[x]}{[x^2]} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Ans. To discuss the continuity of the function we need to check that

$$\lim_{x\to 0^{-1}} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

To evaluate L.H.L. We put $x = 0-h$, Taking limit $h \to 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin[x]}{x} = \lim_{h\to 0} \frac{\sin[-h]}{-h}$$

$$\lim_{h\to 0} \frac{\sin(-1)}{-h} = \infty$$

To evaluate R.H.L Put $x = 0+h$, Taking limit $h \to 0$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} \frac{\sin[h]}{h}$$
$$= \lim_{h\to 0} \frac{\sin(0)}{h} = 0$$

$$\lim_{x\to 0^-} \neq \lim_{x\to 0^+} f(x) = f(0)$$

Function is not continuous at $= x = 0$ and function has discontinuity of second kind.

UNIT- III

Q.6. (a) Find all asymptote of $y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 - 1 = 0$ (6.5)

Ans. The given equation of curve is

$$y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 - 1 = 0$$

Put $y = m$ and $x = 1$ in 4th degree terms

$$\phi_4(m) = m^4 - 2m^3 + 2m - 1$$

Put $\quad y = m, x = 1$, in third degree term

$$\phi_3(m) = -3 + 3m - 3m^3$$

Similarly

$$\phi_2(m) = 2m^2 - 2$$
$$\phi_1(m) = 0$$
$$\phi_0(m) = 0 - 1 = -1$$

Put $\quad \phi_4(m) = 0$

$$m^4 - 2m^3 + 2m - 1 = 0$$

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$$m^3(m-1) - m^2(m-1) - m(m-1) + 1(m-1) = 0$$
$$(m-1)[m^3 - m^2 - m + 1] = 0$$
$$(m-1)[m^2(m-1) - 1(m-1)] = 0$$
$$(m-1)(m-1)(m^2-1) = 0$$
$$(m-1)(m-1)(m-1)(m+1) = 0$$
$$m = 1, 1, 1, -1$$

Now the value of c for $m = 1$

$$C\phi'_4(m) + \phi_3(m) = 0$$
$$c(4m^3 - 6m^2 + 2) - 3 + 3m - 3m^3 = 0$$

at $m = -1$, we have

$$c(-4-6+2) - 3 - 3 + 3 = 0$$
$$-8c - 3 = 0$$
$$\Rightarrow \quad c = -\frac{3}{8}$$

The Asymplote will be.

$$y = m_x + C$$
$$y = -x\frac{-3}{8}$$
$$\boxed{8x + 8y + 3 = 0}$$

Now to obtain the value of c for $m = 1$

$$\frac{c^3}{3!}\phi_4'''(m) + \frac{c^2}{2!}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0$$
$$\frac{c^3}{6}(24m - 12) + \frac{c^2}{2}(-18m) + c(4m) = 0$$
$$c(2c-1)(c-4) = 0$$
$$c = 0, 1/2, 4$$

Thus Asymptotes are

$$y = 1 \times x + 0 \Rightarrow y = x$$
$$y = 1 \times x + \frac{1}{2} \Rightarrow 2y - 2x = 1$$
$$y = 1 \times x + 4 \Rightarrow y = x + 4$$

Hence all Asymptotes are

$$\boxed{y = e, 2y - 2x = 1, y = x + 4, 8x + 8y + 3}$$

Q.6.(b) If $x^y + y^x = a^b$ find $\frac{dy}{dx}$. **(6)**

Ans.
$$x^y + y^x = a^b$$

Taking log on both sides

$$\log x^y + \log y^x = \log a^b$$
$$y \log x + x \log y = b \log a$$

Differentiating on both sides w.r.t.x

$$y \times \frac{1}{x} + \log x \frac{dy}{dx} + x \times \frac{1}{y}\frac{dy}{dx} + \log y \times 1 = 0$$
$$\frac{dy}{dx}\left[\log x + \frac{x}{y}\right] + \left(\log y + \frac{y}{x}\right) = 0$$
$$\frac{dy}{dx}\left[\frac{y \log x + x}{y}\right] = -\left[\frac{x \log y + y}{x}\right]$$
$$\frac{dy}{dx} = -\frac{y}{x}\left[\frac{x \log y + y}{y \log x + x}\right] \textbf{ Ans.}$$

Q.7. (a) If $y = \sin^{-1}x$ then show that

(i) $[1-x^2]y_2 - xy_1 = 0$

(ii) $(1-x^2)y_{n+2} = (2n+1)x\,y_{n+1} - n^2 y_n = 0$

Ans. (i) Since $y = \sin^{-1} x$

Differentiating on both sides w.r.t x

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$
$$y_1\sqrt{1-x^2} = 1$$

Squaring on both sides

$$y_1^2(1-x^2) = 1$$
$$y_1^2 \times (-2x) + (1-x^2) \times 2y_1 y_2 = 0$$
$$-2x\, y_1^2 + (1-x^2) \times 2y_1y_2 = 0$$
$$2y_1[(1-x^2)y_2 - xy_1] = 0$$
$$[(1-x^2)y_2 - xy_1] = 0 \textbf{ Proved}$$

(ii) Differentiating Result (i) n time by Lei bnitz theorem.

$$D^n(1-x^2)y_2 - D^n xy_1 = 0$$

$$n_{c_0} D^n_{y_2}(1-x^2)+n_{c_1}D^{n-1}_{y_2}D(1-x^2)+n_{c_2}D^{n-2}_{y_2}D^2(1-x^2) \quad -n_{c_0} D^n_{y_1} x + n_{c_1} D^{n-1} y_1 D(x)$$

$= 0$

$$(1-x^2)\,y_{n+2} - 2nxy_{n+1} - \frac{2n(n-1)}{2} y_n - ny_{n+1} - ny_n = 0$$

$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - n^2 y_n = 0$$

Q.7. (b) Examine the given function for maxima or minima **(6)**

$$f(x) = \frac{(x-1)(x-6)}{x-10},\; x \neq 10.$$

Ans.

$$f(x) = \frac{x^2-7x+6}{x-10}$$

$$f'(x) = \frac{2x-7)(x-10)-1\times(x^2+7x+6)}{1}$$

$$f'(x) = 2x^2-10x-7x+70-x^2-7x-6$$

$$f'(x) = x^2-34x+64=0$$

$$f'(x) = x^2-(32+2)\,x+64=0$$

$$x^2-32x-2x+64=0$$

$$x(x-32)-2(x-32)=0$$

$$(x-32)\,(x-2)=0$$

$$x = 2, 32$$

$$f'(x) = x^2-34x+64$$

$$f''(x) = 2x-34$$

Now at

$$x = 2$$

$$f''(x) = 2\times 2-34$$

$$f''(x) = 4-34$$

$$f''(x) = -30$$

since at $x = 2, f''(x) < 0$

$\therefore$ function has maxima at $x = 2$

its maximum value is $f(2) = \frac{(2-1)(2-6)}{2-10}$

$$f(2) = \frac{1\times -4}{-8}$$

$$\boxed{f(2) = 1/2}$$

Now

$$f''(x) = 2\times 32-34$$

$$\text{at } x = 32 = 64-34=30$$

since at $x = 32\; f''(x) > 0$

$\therefore$ function is minimum at $x = 32$ its minimum value is

$$f(32) = \frac{(32-1)(32-6)}{(32-10)}$$

$$f(32) = \frac{31\times \cancel{26}\,13}{\cancel{22}\,11}$$

$$f(32) = \frac{31\times 13}{11} = \frac{403}{11}$$

$\boxed{f(32) = 36.67}$ **Ans.**

UNIT-IV

Q.8.(a) Evaluate

(i) $\int \log(1+x)dx$ (ii) $\int_0^2 \frac{5x}{(x^2+1)}dx$ **(6)**

Ans. (i) $\int_{II}^{I} \frac{\log(1+x)dx}{I}$ Using Integration by parts

$$\log(1+x)\int dx - \int\left(\frac{d}{dx}\log(1+x)\int dx\right)dx$$

$$x\log(1+x) - \int\frac{1}{(1+x)}\times 1\times x\,dx$$

$$x\log(1+x) - \int\frac{x}{(x+1)}dx$$

$$x\log(1+x) - \left[\int\frac{x+1-1}{(x+1)}dx\right]$$

$$x\log(1+x) - \left[\int\frac{x+1}{x+1}dx - \int\frac{1}{(x+1)}dx\right]$$

$$x\log(1+x) - \left[\int dx - \log(x+1)\right]+c$$

$$\boxed{x\log(1+x)-x-\log(x+1)+c}$$

(ii) $\int_0^2 \frac{5x}{x^2+1}dx$

Put

$$x^2+1 = t$$

$$2x\,dx = dt$$

$$x\,dx = \frac{dt}{2}$$

$$\int_0^2 \frac{5 \times \frac{1}{2} dt}{t}$$

$$= \frac{5}{2}\int_0^2 \frac{dt}{t}$$

$$= \frac{5}{2}[\log t]_0^2 + c$$

$$= \frac{5}{2}[\log (x^2+1)]_0^2 + c$$

$$= \frac{5}{2}[\log(4+1) - \log(0+1)] + c$$

$$= \frac{5}{2}[\log 5 - \log 1] + c$$

Q.8. (b) Obtain the reduction formula for $\int \tan^n x\,dx$. Also evaluat

$\int_0^{\pi/4} \tan^n x\,dx$ (6.5)

Ans.

Let
$$I_n = \int \tan^n x\,dx = \int \tan_x^{n-2} \tan x dx$$

$$= \int \tan_n^{n-2} (\sec^2 x - 1)dx$$

$$= \int \tan_x^{n-2} \sec^2 x\,dx - \int \tan_{xdx}^{n-2}$$

$$I_n = \int \tan_x^{n-2} \sec^2 xdx - I_n - 2$$

$$= \int \tan_x^{n-2} \sec^2 xdx - I_n - 2$$

$$= \tan_n^{n-2} \int \sec^2 xdx - \int \left(\frac{d}{dx} \tan_x^{n-2} \int \sec^2 x\right) dx - I_n - 2$$

$$\ldots^{n-1} - \int (n-2) \tan_x^{n-2} (1 + \tan^2 x) dx - I_n - 2$$

$$I_n = \tan_x^{n-1} - (n-2)\int \tan_x^{n-2} dx - (n-2)\int \tan^n xdx - I_n - 2$$

$$I_n = \tan_x^{n-1} - (n-2)I_n - 2 - (n-2)I_n - 2 - I_n - 2$$

$$I_n (1 + n - 2) = \tan_x^{n-1} - (n-2)I_n - 2 - (n - 2 + 1)I_n - 2$$

$$I_n = \frac{1}{(n-1)}\left[\tan_x^{n-1} - (n-1)\, I_n - 2\right]$$

$$\boxed{I_n = \frac{\tan_x^{n-1}}{(n-1)} - I_n - 2}$$

Now
$$\int_x^{\pi/4} \tan^n xdn = \left(\frac{\tan_{\frac{\pi}{4}}^{n-1}}{n-1}\right) - \frac{\tan_0^{n-1}}{(n-1)} - I_n - 2$$

$$I_n = \boxed{\frac{1}{(n-1)} - I_n - 2} \qquad \boxed{I_n + I_n - 2 = \frac{1}{n-1}}$$

Q.9. (a) Show that

$$\int_0^{\pi/2} \sin^p \theta \cos^q \theta = \frac{\sqrt{\frac{p+1}{2}}\sqrt{\frac{\sqrt{q+1}}{2}}}{2\sqrt{\frac{p+q+2}{2}}}, p,q > -1 \quad \textbf{(6.5)}$$

Ans. We know that

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

and
$$\beta(p, q) = \frac{\sqrt{p}\,\sqrt{q}}{\sqrt{p+q}}$$

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\sqrt{\frac{p+1}{2}}\,\sqrt{\frac{q+1}{2}}}{\sqrt{\frac{p+1}{2} + \frac{q+1}{2}}}$$

$$= \frac{\sqrt{\frac{p+1}{2}}\sqrt{\frac{q+1}{2}}}{\sqrt{\frac{p+q+2}{2}}}$$

$$\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^n\theta\,d\theta = \frac{\sqrt{\frac{p+1}{2}}\cdot\sqrt{\frac{q+1}{2}}}{2\sqrt{\frac{p+q+2}{2}}} \quad \textbf{Proved}$$

Q.9. (b) Evaluate $\int_0^1 \frac{xe^x}{(x+1)^2}dx$

Ans.

$$\int_0^1 \frac{xe^x}{(x+1)^2}dx$$

$$\Rightarrow \int_0^1 \frac{(x+(-1)e^x}{(x+1)^2}dx$$

$$\Rightarrow \int_0^1 \frac{(x+1)e^x}{(x+1)^2}dx - \int\frac{e^x}{(x+1)^2}dx$$

$$\Rightarrow \int_0^1 \frac{e^x}{(x+1)}dx - \int\frac{e^x}{(x+1)^2}dx$$

$$\Rightarrow \int_0^1 \underset{II}{e^x}\underset{I}{\frac{(x+1)}{}}dx - \int\frac{e^x}{(x+1)^2}dx$$

$$\Rightarrow (x+1)\int_0^1 e^x dx - \int_0^1\left(\frac{d}{dx}(x+1)^{-1}\int e^x dx\right)dx - \int\frac{e^x}{(x+1)^2}dx$$

$$\Rightarrow (x+1)(ex)_0^1 - \int\frac{-1\times e^x dx}{(1+x)^2} - \int\frac{e^x}{(x+1)^2}dx$$

$$\Rightarrow (x+1)(e^1-e^0) + \cancel{\int\frac{e^x}{(x+1)}} - \cancel{\int\frac{e^x}{(x+1)^2}}dx$$

$$= (e-1)(x+1) + c$$

$$\boxed{\int_0^1 \frac{x\,e^x}{(x+1)^2}dx = (e-1)(x+1)+c} \quad \textbf{Ans.}$$

END TERM EXAMINATION [DEC. 2016]
FIRST SEMESTER [BCA]
MATHEMATICS–I [BCA–101]

Time : 3 hrs. **M.M. : 75**

Note: *Attempt any five questions including Q. no. 1 which is compulsory. Select one question from each unit.*

Q. 1.(a) Prove that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew symmetric matrix. **(5)**

Ans. Let A be any square matrix.

Define: $P = \frac{A+A^T}{2}$ and $Q = \frac{A-A^T}{2}$

Then, $P^T = \left(\frac{A+A^T}{2}\right)^T = \frac{1}{2}(A+A^T)^T$

$$= \frac{1}{2}(A^T+(A^T)^T) = \frac{1}{2}(A^T+A)$$

$\Rightarrow$ $P^T = \frac{A+A^T}{2}$

$\Rightarrow$ $P^T = P$ $\quad\therefore$ P is symmetric matrix

and $Q^T = \left(\frac{A-A^T}{2}\right)^T = \frac{1}{2}(A-A^T)^T$

$$= \frac{1}{2}(A^T-(A^T)^T) = \frac{1}{2}(A^T-A)$$

$\Rightarrow$ $Q^T = \frac{-1}{2}(A-A^T)$

$\Rightarrow$ $Q^T = -Q$ $\quad\therefore$ Q is skew-symmetric matrix

Now $P+Q = \frac{A+A^T}{2} + \frac{A-A^T}{2} = \frac{A+A^T+A-A^T}{2}$

$$= \frac{2A}{2}$$

$\Rightarrow$ $\boxed{P+Q=A}$

$\Rightarrow$ A is expressible as sum of a symmetric matrix and a skew symmetric matrix

Now, we show uniqueness

let $A = P_1 + Q_1$, where P_1 is symmetric and Q_1 is skew symmetric

Also, $A = P+Q$

$$P + Q = P_1 + Q_1 \quad \ldots(1)$$

$\Rightarrow$ $$(P+Q)^T = (P_1 + Q_1)^T$$

$\Rightarrow$ $$P^T + Q^T = P_1^T + Q_1^T$$

$\Rightarrow$ $$P - Q = P_1 - Q_1 \quad \ldots(2)$$

$\Rightarrow$ (Since, P, P_1 are symmetric. & Q, Q_1 are skew symmetric)

Adding (1) and (2), we get

$$2P = 2P_1 \Rightarrow \boxed{P = P_1}$$

Similarly, subtracting (2) from (1), we get

$$\boxed{Q = Q_1}$$

Hence, every square matrix is uniquely expressible as sum of a symmetric matrix and a skew symmetric matrix.

Q.1. (*b*) For what value of x, the matrix **(5)**

$$A = \begin{bmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{bmatrix}$$ **is singular.**

Ans. Given, A is singular

$$|A| = 0$$

$\Rightarrow$ $$\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$$

$\Rightarrow$ $$\begin{vmatrix} x+1 & x+1 & x+1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0 \quad (R_1 \to R_1 + R_2 + R_3)$$

$\Rightarrow$ $$(x+1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$$ (Taking common $x + 1$ from R_1)

$\Rightarrow$ $$(x+1)\begin{vmatrix} 0 & 0 & 1 \\ -x+2 & x-2 & 1 \\ 0 & -x+2 & 1 \end{vmatrix} = 0 \quad \begin{pmatrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - C_3 \end{pmatrix}$$

$\Rightarrow$ $$(x+1)[0 - 0 + 1((-x+2)^2)] = 0$$

$\Rightarrow$ $$(x+1)(x-2)^2 = 0$$

$\Rightarrow$ $$x = -1, 2$$

Alternate:

$$|A| = 0$$

$$\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$$

or $\Rightarrow (x-1)\{(x-1)^2 - 1\} - 1\{x - 1 - 1\} + 1\{1 - x + 1\} = 0$

$\Rightarrow (x-1)(x^2 + 1 - 2x - 1) - (x-2) + (2-x) = 0$

$\Rightarrow (x-1)(x^2 - 2x) - x + 2 + 2 - x = 0$

$\Rightarrow (x-1)x(x-2) - 2x + 4 = 0$

$\Rightarrow (x-1)x(x-2) - 2(x-2) = 0$

$\Rightarrow (x-2)[x(x-1) - 2] = 0$

$\Rightarrow (x-2)(x^2 - x - 2) = 0$

$\Rightarrow x = 2$ or $x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

$\therefore x = 2, -1$

Q.1. (*c*) Using properties without expanding prove that: **(5)**

$$\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Ans. Given $$\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

take common $x + y + z$ from R_1

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$= (x+y+z) \times 0$ (Since, R_1 and R_3 are identical)

$= 0$

Q.1. (*d*) Show that $f(x) = \begin{cases} 2x-1; & x < 2 \\ 3; & x = 2 \\ x+1; & x > 2 \end{cases}$ **is continuous at x = 2.** **(5)**

Ans. Given
$$f(x) = \begin{cases} 2x-1 & , x<2 \\ 3 & , x=2 \\ x+1 & , x>2 \end{cases}$$

Left hand limit at $x = 2 = \lim_{x\to 2^-} f(x)$
$$= \lim_{x\to 2^-} 2x-1$$
$$= \lim_{h\to 0} 2(2-h)-1 = 3$$

Right hand limit at $x = 2 = \lim_{x\to 2^+} f(x)$
$$= \lim_{x\to 2^+} x+1$$
$$= \lim_{h\to 0} 2+h+1 = 3$$

and,
$$f(2) = 3 \quad \text{(given)}$$

$\Rightarrow$
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$\Rightarrow f$ is continuous at $x = 2$.

Q.1. (e) Show that function $f(x) = \sin x\,(1+\cos x)$ is maximum when $x = \frac{\pi}{3}$. (5)

Ans.
$$f(x) = \sin x\,(1+\cos x)$$
$$f'(x) = \sin x(-\sin x) + (1+\cos x)\cos x$$
$$= -\sin^2 x + \cos x + \cos^2 x$$
$$= \cos^2 x - \sin^2 x + \cos x$$

$\Rightarrow$
$$f'(x) = \cos 2x + \cos x$$

and
$$f''(x) = -2\sin 2x - \sin x$$

For points of local maxima or minima,

Put
$$f'(x) = 0$$

$\Rightarrow$
$$\cos 2x + \cos x = 0$$

$\Rightarrow$
$$2\cos^2 x - 1 + \cos x = 0$$

$\Rightarrow$
$$2\cos^2 x + \cos x - 1 = 0$$

$\Rightarrow$
$$\cos x = \frac{-1\pm\sqrt{1+8}}{4} = \frac{-1\pm 3}{4}$$

$\Rightarrow$
$$\cos x = -1, \frac{1}{2}$$

$\Rightarrow$
$$x = \pi, \pi/3$$

Now,
$$f''\left(\frac{\pi}{3}\right) = -2\sin\left(2\frac{\pi}{3}\right) - \sin\frac{\pi}{3}$$

$$= -2\sin\left(\pi - \frac{\pi}{3}\right) - \sin\frac{\pi}{3}$$
$$= -2\sin\frac{\pi}{3} - \sin\frac{\pi}{3}$$
$$= -3\sin\frac{\pi}{3} = -3\frac{\sqrt{3}}{2} < 0$$

$\Rightarrow$
$$x = \frac{\pi}{3} \text{ is a point of maxima.}$$

and
$$f''(\pi) = -2\sin 2\pi - \sin\pi = 0$$

$\Rightarrow f(x)$ is maximum when $x = \pi/3$.

UNIT-I

Q.2. (a) If the matrix is orthogonal, then find the values of a, where matrix is

$$A = \begin{bmatrix} a & -2a & 2a \\ 2a & -a & -2a \\ 2a & 2a & a \end{bmatrix}$$ (6.5)

Ans. Given the matrix is orthogonal

$\Rightarrow$
$$A^TA = AA^T = I$$
$$A^T = \begin{pmatrix} a & 2a & 2a \\ -2a & -a & 2a \\ 2a & -2a & a \end{pmatrix}$$
$$A^TA = I$$

$\Rightarrow$
$$\begin{pmatrix} a & 2a & 2a \\ -2a & -a & 2a \\ 2a & -2a & a \end{pmatrix}\begin{pmatrix} a & -2a & 2a \\ 2a & -a & -2a \\ 2a & 2a & a \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$
$$\begin{pmatrix} 9a^2 & 0 & 0 \\ 0 & 9a^2 & 0 \\ 0 & 0 & 9a^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$
$$9a^2 = 1$$

$\Rightarrow$
$$a = \pm\frac{1}{3}$$

Hence proved

Q.2. (b) Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} -2 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. **Also find** A^{-1}.

Ans.

$$A = \begin{bmatrix} -2 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 2 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$R_2 \rightarrow R_2 + R_3$

$$\begin{vmatrix} -2-\lambda & 2 & 1 \\ 0 & 1-\lambda & 1-\lambda \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

taking $(1-\lambda)$ comman from R_2

$$(1-\lambda)\begin{vmatrix} -2-\lambda & 2 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$C_3 \rightarrow C_3 - C_2$

$$(1-\lambda)\begin{vmatrix} -2-\lambda & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

Expanding along R_2 we get

$\Rightarrow \quad (1-\lambda)[(-2-\lambda)(3-\lambda)-(-1)] = 0$

$\Rightarrow \quad (\lambda^2-\lambda-5)(1-\lambda) = 0$

$\Rightarrow \quad \lambda^2-\lambda-5-\lambda^3+\lambda^2+5\lambda = 0$

$\Rightarrow \quad \lambda^3-2\lambda^2-4\lambda+5 = 0$

(1) is the characterstic equation of A

Now verifying Cayley Hamilton Theorem

L.H.S

$$A^3-2A^2-4A+5I = 0$$

Now, calculate

$$A^2 = A.A = \begin{bmatrix} -2 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -5 \\ 1 & -2 & 6 \end{bmatrix}$$

$$A^3 = A^2.A$$

$$= \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -5 \\ 1 & -2 & 6 \end{bmatrix}\begin{bmatrix} -2 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 6 & 0 \\ -6 & 9 & -14 \\ 6 & -8 & 15 \end{bmatrix}$$

Putting the value of A^3, A^2 and A in (2) we get

$$\Rightarrow A^3-2A^2-4A+5I = \begin{bmatrix} 7 & 6 & 0 \\ -6 & 9 & -14 \\ 6 & -8 & 15 \end{bmatrix} - 2\begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -5 \\ 1 & -2 & 6 \end{bmatrix} - 4\begin{bmatrix} -2 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Hence proved

Q.3. (a) Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$ **(6)**

Ans. Given,

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

For eigen values, we have to solve the equation $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1-\lambda]=0$$

$$\Rightarrow \quad (-2-\lambda)(-\lambda+\lambda^2-12)+4(\lambda+3)+3(\lambda+3)=0$$

$$\Rightarrow (-2-\lambda)(\lambda+3)(\lambda-4)+7(\lambda+3)=0$$

$$\Rightarrow (\lambda+3)[-(2+\lambda)(\lambda-4)+7]=0$$

$$\Rightarrow (\lambda+3)[-\lambda^2+2\lambda+8+7)=0$$

$$\Rightarrow (\lambda+3)(-\lambda^2+2\lambda+15)=0$$

$$\Rightarrow -(\lambda+3)(\lambda^2-2\lambda+15)=0$$

$$\Rightarrow -(\lambda+3)(\lambda+3)(\lambda-5)=0$$

$$\Rightarrow \lambda = -3, -3, 5$$

$\therefore$ Eigen values are $-3, -3, 5$

For eigen value $\lambda = 5$, eigen vectors are given by non-zero solution of

$$AX = \lambda X, \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

an

$$(A-\lambda I)X = 0$$

i.e.

$$(A-5I)X = 0$$

i.e.

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we write augmented matrix, as

$$\begin{bmatrix} -7 & 2 & -3 & : & 0 \\ 2 & -4 & -6 & : & 0 \\ -1 & -2 & -5 & : & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & : & 0 \\ 2 & -4 & -6 & : & 0 \\ -7 & 2 & -3 & : & 0 \end{bmatrix} \qquad \begin{matrix} R_1 \leftrightarrow R_3 \\ R_1 \to (-1)R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & : & 0 \\ 0 & -8 & -16 & : & 0 \\ 0 & 16 & 32 & : & 0 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 7R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & : & 0 \\ 0 & 1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & : & 0 \\ 0 & 1 & 2 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \qquad R_3 \to R_3 - R_2$$

$$\therefore \quad \begin{aligned} x_1 + 2x_2 + 5x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

x_3 can take any value

take $\boxed{x_3 = 1}$

$$\therefore \quad x_2 + 2x_3 = 0 \Rightarrow \boxed{x_2 = -2}$$

and

$$x_1 + 2x_2 + 5x_3 = 0$$

$$\Rightarrow \quad x_1 - 4 + 5 = 0$$

$$\Rightarrow \quad \boxed{x_1 = -1}$$

$\therefore$ Eigen vectors corresponding to $\lambda = 5$ are span of $(-1, -2, 1)$

For eigen value $\lambda = -3$, eigen vectors are given by

$$(A-\lambda I)X = 0$$

$$\Rightarrow \quad (A+3I)X = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & : & 0 \\ 2 & 4 & -6 & : & 0 \\ -1 & -2 & 3 & : & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\Rightarrow \quad x_1 + 2x_2 - 3x_3 = 0$$

where x_2 and x_3 are free variables (i.e. they can take any value.)

$$\therefore \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 + 3x_3 \\ x_2 + 0 \\ 0 + x_3 \end{pmatrix}$$

$$\Rightarrow \quad X = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} x_3$$

$\therefore$ Eigen vector corresponding to $\lambda = -3$ are given by span of $\{(-2,1,0), (3, 0, 1)\}$

Q.3. (b) Examine the following system of vectors for linearly dependence, if dependent, find the relation between them

$$X_1 = (1,-1,1);\ X_2 = (2,1,1);\ X_3 = (3,0,2).$$

Ans. Given

$$X_1 = (1,-1,1)$$
$$X_2 = (2,1,1)$$
$$X_3 = (3,0,2)$$

let a, b, c be scalars such that

$$aX_1 + bX_2 + cX_3 = 0$$
$$\Rightarrow\ a(1,-1,1) + b(2,1,1) + c(3,0,2) = (0,0,0)$$
$$a + 2b + 3c = 0$$
$$\Rightarrow\ -a + b + 0 = 0$$
$$a + b + 2c = 0$$

Its Augmented matrix is given by

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \quad R_2 \to \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\therefore\ a + 2b + 3c = 0$$
$$b + c = 0$$

and c is free variable.

Take $\boxed{c = 1}$

then, $b + c = 0 \Rightarrow \boxed{b = -1}$

and $a + 2b + 3c = 0 \Rightarrow a - 2 + 3 = 0 \Rightarrow \boxed{a = -1}$

$\therefore$ For $a = -1, b = -1, c = 1$

we have $aX_1 + bX_2 + cX_3 = 0$

$\Rightarrow X_1, X_2, X_3$ are linearly dependent.

UNIT-II

Q.4. (a) Find the value of a so that the function $f(x) = \begin{cases} ax+5 \text{ if } x \le 2 \\ x-1 \text{ if } x > 2 \end{cases}$ is continuous at $x = 2$. **(6.5)**

Ans.
$$f(x) = \begin{cases} ax+5, & x \le 2 \\ x-1, & x > 2 \end{cases}$$

Given: f(x) is continous at x = 2

$$\Rightarrow\ \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) \quad ...(*)$$

Now
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax + 5 = 2a + 5$$

and
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x - 1 = 1$$

$$\therefore\ (*) \Rightarrow\ 2a + 5 = 1$$
$$\Rightarrow\ 2a = -4$$
$$\Rightarrow\ \boxed{a = -2}$$

Q.4. (b) Evaluate:- **(3)**

(i) $\lim_{x\to 2}\left(\frac{x^3-8}{x-2}\right)$;

Ans. (i) We have,
$$\lim_{x\to 2}\left(\frac{x^3-8}{x-2}\right)$$
$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)}$$
$$= \lim_{x\to 2} (x^2+2x+4)$$
$$= 2^2 + 2\times 2 + 4$$
$$= 12$$

Q.4. (b) (ii) $\lim_{x\to 0}\frac{|x|}{x}; x \ne 0$ **(3)**

Ans. (ii)
$$\lim_{x\to 0}\frac{|x|}{x}, x \ne 0$$

LHL at $x = 0$:
$$\lim_{x\to 0^-}\frac{|x|}{x}$$
$$= \lim_{n\to 0}\frac{|0-n|}{0-n} = \lim_{n\to 0}\frac{|-n|}{-n}$$

$$= \lim_{n\to 0}\frac{n}{-n} = \lim_{n\to 0} -1 = -1$$

RHL at $x = 0$:

$$\lim_{x\to 0^+}\frac{|x|}{x}$$

$$= \lim_{n\to 0}\frac{|0+n|}{0+n} = \lim_{n\to 0}\frac{|n|}{n} = \lim_{x\to 0}\frac{n}{n}$$

$$= \lim_{n\to 0} 1 = 1$$

Since $\lim_{x\to 0^-}\frac{|x|}{x} \neq \lim_{x\to 0^+}\frac{|x|}{x}$

∴ $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Q.5. (a) Evaluate:-

(i) $\lim_{x\to\sqrt{2}}\left(\frac{x^4-4}{x^2+3\sqrt{2}x-8}\right)$; **(ii)** $\lim_{x\to 0}\frac{\sqrt{1+3x}-\sqrt{1-3x}}{x}$ **(6)**

Ans. (i)

$$\lim_{x\to\sqrt{2}}\frac{x^4-4}{x^2+3\sqrt{2}x-8}$$

$$= \lim_{x\to\sqrt{2}}\frac{(x^2-2)(x^2+2)}{x^2+4\sqrt{2}x-\sqrt{2}x-8}$$

$$= \lim_{x\to\sqrt{2}}\frac{(x-\sqrt{2})(x+\sqrt{2})(x^2+2)}{x\left(x+4\sqrt{2}\right)-\sqrt{2}(x+4\sqrt{2})}$$

$$= \lim_{x\to\sqrt{2}}\frac{\cancel{(x-\sqrt{2})}(x+\sqrt{2})(x^2+2)}{\cancel{(x-\sqrt{2})}(x+4\sqrt{2})}$$

$$= \lim_{x\to\sqrt{2}}\frac{(x+\sqrt{2})(x^2+2)}{x+4\sqrt{2}}$$

$$= \frac{\left(\sqrt{2}+\sqrt{2}\right)(2+2)}{\sqrt{2}+4\sqrt{2}} = \frac{(2\cancel{\sqrt{2}})(4)}{5\cancel{\sqrt{2}}}$$

$$= \boxed{\frac{8}{5}}$$ **Ans.**

(ii)

$$\lim_{x\to 0}\frac{\sqrt{1+3x}-\sqrt{1-3x}}{x}$$

$$= \lim_{x\to 0}\frac{\sqrt{1+3x}-\sqrt{1-3x}}{x}\times\frac{\sqrt{1+3x}+\sqrt{1-3x}}{\sqrt{1+3x}+\sqrt{1-3x}}$$

$$= \lim_{x\to 0}\frac{(1+3x)-(1-3x)}{x\left(\sqrt{1+3x}+\sqrt{1-3x}\right)}$$

$$= \lim_{x\to 0}\frac{6\cancel{x}}{\cancel{x}\left(\sqrt{1+3x}+\sqrt{1-3x}\right)} = \lim_{x\to 0}\frac{6}{\sqrt{1+3x}+\sqrt{1-3x}}$$

$$= \frac{6}{2} = \boxed{3}$$ **Ans.**

Q.5. (*b*) If the function $f(x) = \begin{cases} 3ax+b; & \text{for } x>1 \\ 11; & \text{for } x=1 \\ 5ax-b; & \text{for } x<1 \end{cases}$ **is continuous at $x = 1$, find the values of a and b.** **(6.5)**

Ans.

$$f(x) = \begin{cases} 3ax+b, & x>1 \\ 11, & x=1 \\ 5ax-b, & x<1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(5ax-b) = 5a-b$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(3ax+b) = 3a+b$$

Given $f(1) = 11$

Given: $f(x)$ is continuous at $x = 1$

⇒ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

⇒ $5a-b = 3a+b = 11$ ...(eq.1)

⇒ $5a-b = 3a+b$

⇒ $2a = 2b$

⇒ $\boxed{a=b}$

Now, $5a-b = 11$ (using (eq. 1.)

⇒ $5a-a = 11$

⇒ $4a = 11$

⇒ $\boxed{a=\frac{11}{4}}$ ∴ $\boxed{b=\frac{11}{4}}$

UNIT-III

Q.6. (a) Find $\frac{dy}{dx}$ if:-

(i) $y = \sin\sqrt{x}$ (ii) $x^y . y^x = k$, where k is a constant (iii) $y = \sin^3 2x$ **(6)**

Ans. (i)

$$y = \sin\sqrt{x}$$

$$\frac{dy}{dx} = \cos\sqrt{x}.\frac{1}{2\sqrt{x}}$$

(ii)

$$x^y y^x = k$$

$$\log(x^y . y^x) = \log k$$

$\Rightarrow$ $$\log(x^y) + \log(y^x) = \log k$$

$\Rightarrow$ $$y\log x + x\log y = \log k$$

$\Rightarrow$ Differentiate w.r.t. x

$\Rightarrow$ $$y\frac{1}{x} + x\frac{1}{y}.\frac{dy}{dx} = 0$$

$$\frac{y}{x} + \frac{x}{y}\frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{-y}{x}\times\frac{y}{x}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{-y^2}{x^2}$$

(iii)

$$y = \sin^3 2x$$

$$\frac{dy}{dx} = 3\times\sin^2 2x\times\frac{d}{dx}(\sin 2x)$$

$$\frac{dy}{dx} = 3\sin^2 2x\times\cos 2x\times\frac{d}{dx}(2x)$$

$$\frac{dy}{dx} = 3\sin^2 2x\times\cos 2x\times 2$$

$$\frac{dy}{dx} = 6\sin 2x\times\sin 2x\cos 2x$$

$$\frac{dy}{dx} = 3\sin 2x\times 2\sin 2x\cos 2x$$

$$\frac{dy}{dx} = 3\sin 2x\sin 4x$$

Q.6. (b) Find the nth derivative of log (2x + 3) **(6.5)**

Ans.

$$y = \log(2x+3)$$

$$y' = \frac{1}{2x+3}2 = \frac{2}{2x+3}$$

$\therefore$ $$y^{(n)} = (y')^{n-1} = 2\frac{d}{dx^{n-1}}\left(\frac{1}{2x+3}\right)$$

$$= \frac{2.(-1)^{n-1}(n-1)!2^{n-1}}{(2x+3)^{n-1+1}}$$

Since, $$D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n(n!)a^n}{(ax+b)^{n+1}}$$

$\Rightarrow$ $$y^{(n)} = \frac{(-1)^{n-1}(n-1)!2^n}{(2x+3)^n}$$

Q.7. (a) Find all the asymptotes of the curve $y^2(x-2a) = x^3 - a^2$. **(6.5)**

Ans.

$$y^2(x-2a) = x^3 - a^2$$

$\Rightarrow$ $$y^2(x-2a) - x^3 + a^2 = 0$$

or $$-x^3 + a^2 + y^2(x-2a) = 0$$

Terms of highest power of x is $-x^3$ and its coefficient is -1 which cannot be zero

$\therefore$ No assymptote parelled to x axis

Terms of highest power of y is y^2 and its coefficient is $x-2a$

$\therefore$ Putting its coefficient = 0

we have $$x - 2a = 0$$

$\Rightarrow$ $$x = 2a$$

Hence $x = 2a$ is the assymptote parelled to y axis:

Q.7. (b) If $y = \sin(m\sin^{-1}x)$, then prove that

$$(1-x^2)y_{n+2} = (n^2 - m^2)y_n + (2n+1)xy_{n+1}$$ **(6)**

Ans.

$$y = (\sin(m\sin^{-1}x))$$

$$y_1 = \cos(m\sin^{-1}x)\frac{d}{dx}(m\sin^{-1}x)$$

$$y_1 = \cos(m\sin^{-1}x)\frac{m}{\sqrt{1-x^2}}$$

Squaring both the sides and cross multiplying we get

$$y_1^2(1-x^2) = m^2\cos^2(m\sin^{-1}x)$$

$$y_1^2(1-x^2) = m^2(1-\sin^2(m\sin^{-1}x)$$

$$y_1^2(1-x^2) = m^2(1-y^2)$$

$$y_1^2(1-x^2) + m^2y^2 = m^2$$

Again differentiating w.r.t x we get

$$(1-x^2)\frac{d}{dx}y_1^2 + y_1^2\frac{d}{dx}(1-x^2) + m^2\frac{d}{dx}y^2 = 0$$

$$(1-x^2)2y_1\frac{d}{dx}y_1 + (-2x)y_1^2 + m^2 2y\frac{dy}{dx} = 0$$

$$2y_1y_2(1-x^2) - 2xy_1^2 + m^2 2yy_1 = 0$$

$$2y_1[y_2(1-x^2) - xy_1 + m^2y] = 0$$

$$y_2(1-x^2) - xy_1 + m^2y = 0$$

Differentiating (iii) n times w.r.t x and using Leibnitz's theorem we get

$$\Rightarrow D^n(y_2(1-x^2) - D^n(xy_1) + m^2D^n(y_1) = 0$$

$$\Rightarrow D^n y_2(1-x^2) + {}^nC_1 D^{n-1}(y_2)D(1-x^2) + {}^nC_2 D^{n-2}y_2D^2(1-x^2)$$
$$-[D^n(y_1)x + {}^nC_1 D^{n-1}(y_1)D(x)] + m^2D^n(y)$$

$$\Rightarrow y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2) - \left[y_{n+1}x + ny_n\right] + m^2y_n\Big] = 0$$

$$(1-x^2)y_{n+2} - xy_{n+1}[2n+1] + y_n[m^2-n^2] = 0$$

i.e. $$(1-x^2)y_{n+2} = xy_{n+1}(2n+1) + y_n(n^2-m^2)$$

Hence proved

UNIT-IV

Q.8. (a) Solve the following integrals: (6)

(i) $\int xe^{-x}dx$ (ii) $\int\frac{x^4+1}{x^2+1}dx$ (iii) $\int x^n \log x\,dx$

Ans. (i)

$$I = \int x e^{-x}\,dx$$

$$I = x.\frac{e^{-x}}{(-1)} - \int\frac{d}{dx}(x).\frac{e^{-x}}{(-1)}dx$$

$$= -x.e^{-x} + \int 1.e^{-x}dx$$

$$= -x\,e^{-x} + \frac{e^{-x}}{(-1)} + c$$

$$= -e^{-x}(x+1) + c$$

(ii)

$$I = \int\frac{x^4+1}{x^2+1}dx$$

$$= \int\frac{x^4+1+2x^2-2x^2}{x^2+1}dx$$

$$I = \int\frac{(x^2+1)^2-2x^2}{x^2+1}dx$$

$$\Rightarrow \quad I = \int\left[\frac{(x^2+1)^2}{x^2+1} - \frac{2x^2}{x^2+1}\right]dx$$

$$= \int\left[(x^2+1) - 2\left(\frac{x^2+1-1}{x^2+1}\right)\right]dx$$

$$= \int\left[x^2+1-2\left(1-\frac{1}{x^2+1}\right)\right]dx$$

$$= \int\left[x^2+1-2+\frac{2}{x^2+1}\right]dx$$

$$= \int\left[x^2-1+\frac{2}{x^2+1}\right]dx$$

$$I = \frac{x^3}{3} - x + 2\tan^{-1}x + c$$

(iii)

$$I = \int x^n \log x\,dx$$

Integrating By parts

$$= \log x.\frac{x^{n+1}}{n+1} - \int\frac{1}{x}.\frac{x^{n+1}}{n+1}dx$$

$$= \frac{x^{n+1}.\log x}{n+1} - \frac{1}{n+1}\int x^n dx$$

$$= \frac{x^{n+1}\log x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c$$

Q.8. (b) Prove that $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$ **(6.5)**

Ans. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \begin{pmatrix} m>0 \\ n>0 \end{pmatrix}$

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx \quad (n>0)$$

$$\Gamma(m) = \int_0^\infty e^{-t} t^{n-1} dt$$

∴ Now

put $x^2 = t \Rightarrow 2x\,dx = dt$

$\Rightarrow$ $$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1} dx$$

Similarly $$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1} dy$$

Consider, $$\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1} dx \int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} . y^{2n-1} dx dy$$

take $x = r\cos\theta, y = r\sin\theta \; \{(0 \le r \le \infty\} \; \{0 \le \theta \le \pi/2\}$

$dxdy = r d\theta dr$

∴ $$\Gamma(m)\Gamma(n) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta dr$$

$$= \left(2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta\right) \times \left(2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} dr\right)$$

$$= \left(2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta\right) \times \Gamma(m+n) \quad \text{...(eq. 1)}$$

Now $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

∴ $$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} 2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta . \cos^{2n-1}\theta . d\theta$$

∴ (from equation 1) $\Rightarrow$ $\Gamma(m)\Gamma(n) = \beta(m,n)\,\Gamma(m+n)$

$\Rightarrow$ $$\beta(m,n) = \frac{\Gamma(m)+\Gamma(n)}{\Gamma(m+n)}$$ Hence proved.

Q.9. (a) Find out the reduction formulae for $\int_0^{\pi/4} \sin^n x dx$, n being a positive integer. **(6.5)**

Ans. $$I_n = \int_0^{\pi/4} \sin^n x \, dx$$

$$I_n = \int_0^{\pi/4} \sin^{n-1} x \sin x \, dx$$

$$I_n = \sin^{n-1} x.(-\cos x)\Big|_0^{\pi/4} \int (n-1)\sin^{n-2} x.\cos x(-\cos x) dx$$

$$I_n = -\cos x \sin^{n-1} x\Big|_0^{\pi/4} + (n-1)\int_0^{\pi/4} \sin^{n-2} x.\cos^2 x \, dx$$

$$I_n = -\cos x \sin^{n-1} x\Big|_0^{\pi/4} + (n-1)\int_0^{\pi/4} \sin^{n-2} x(1-\sin^2 x) dx$$

$$I_n = -\cos x \sin^{n-1} x\Big|_0^{\pi/4} + (n-1)\int_0^{\pi/4} \sin^{n-2} x - \sin^n x \, dx$$

$\Rightarrow$ $$I_n = -\cos x \sin^{n-1} x\Big|_0^{\pi/4} + (n-1)I_{n-2} - (n-1)I_n$$

$\Rightarrow$ $$I_n(1+n-1) = -\cos x \sin^{n-1} x\Big|_0^{\pi/4} + (n-1)I_{n-2}$$

$\Rightarrow$ $$I_n = \frac{-\cos x.\sin^{n-1} x}{n}\Big|_0^{\pi/4} + \left(\frac{n-1}{n}\right) I_{n-2}$$

Q.9. (b) If $\int_0^{\pi/4} \tan^n x dx$, then prove that $I_n - I_{n-1} = \frac{1}{n-1}$; n being a positive integer > 1. Hence, evaluate I_5. **(6)**

Ans. $$I_n = \int_0^{\pi/4} \tan^n x \, dx$$

$$I_n = \int_0^{\pi/4} \tan^{n-1} x . \tan x \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x . \tan^2 x \, dx$$

$$I_n = \int_0^{\pi/4} \tan^{n-2} x (\sec^2 x - 1) dx$$

$$I_p = \int \tan^{n-2} \sec^2 x dx - \int \tan^{n-2} x dx$$

taking $\tan x = t$ in I[st] integral so that

$$\sec^2 x dx = dt$$

When $x \to 0$ then $t \to 0$ and $x \to \frac{\pi}{4}$ then $t \to 1$

$$\therefore \qquad I_n = \int_0^1 t^{n-2} dt - I_{n-2}$$

or

$$I_n + I_{n-2} = \left[\frac{t^{n-1}}{n-1}\right]_0^1$$

$$I_n + I_{n-2} = \frac{1}{n-1}$$

Hence first part is proved.

Putting $n = 5,3$ in (eq 1)

$$I_5 + I_3 = \frac{1}{4}$$

$$I_3 + I_1 = \frac{1}{2}$$

$$I_1 = \int_0^{\pi/4} \tan x dx = [\log \sec x]_0^{\pi/4} = \left[\log \sec \frac{\pi}{4} - \log \sec 0\right]$$

$$I_1 = \log \sqrt{2} - \log 1 = \frac{1}{2} \log 2$$

Now, $$I_3 = \frac{1}{2} - I_1 = \frac{1}{2} - \frac{1}{2} \log 2$$

Similarly, $$I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \frac{1}{2} + \frac{1}{2} \log 2$$

$$I_5 = \boxed{\frac{1}{2} \log 2 - \frac{1}{4}} \text{ Ans.}$$

END TERM EXAMINATION [DEC. 2017]
FIRST SEMESTER [BCA]
MATHEMATICS–I [BCA–101]

Time : 3 hrs. **M.M. : 75**

Note: *Attempt any five questions including Q.no.1 which is compulsory, Select one question from each unit.*

Q.1. (a) Solve the following system of equations by Cramer's rule. **(5)**

$2y - 3z = 0,\ x + 3y = -4,\ 3x + 4y = 3.$

Ans. Cramer's Rule:

$$2y - 3z = 0, \quad x + 3y = -4, \quad 3x + 4y = 3$$

from above equations

$$A = \begin{bmatrix} 0 & 2 & -3 \\ 1 & 3 & 0 \\ 3 & 4 & 0 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}.$$

$$\Delta = \det A = \begin{vmatrix} 0 & 2 & -3 \\ 1 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix}$$

$$= -2(0) - 3(4 - 9) = 15$$

$$\Delta_1 = \begin{vmatrix} 0 & 2 & -3 \\ -4 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix} \text{(replace the 1st column of A by B)}$$

$$= 15$$

$$\Delta_2 = \begin{vmatrix} 0 & 0 & -3 \\ 1 & -4 & 0 \\ 3 & 3 & 0 \end{vmatrix} = \text{(Replace the 2nd column of A by B)}$$

$$= -3\begin{vmatrix} 1 & -4 \\ 3 & 3 \end{vmatrix} = -3(3 + 12) = -45.$$

$$\Delta_3 = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & -4 \\ 3 & 4 & 3 \end{vmatrix} = -2\begin{vmatrix} 1 & -4 \\ 3 & 3 \end{vmatrix} = -2(3 + 12) = -30.$$

$$x = \frac{\Delta_1}{\Delta} = \frac{15}{15} = 1, y = \frac{\Delta_2}{\Delta} = \frac{-45}{15} = -3, z = \frac{\Delta_3}{\Delta} = -2$$

Q. 1. (b) Solve $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$ (5)

Ans.
$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$$

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 0$$

$$R_2 \to R_2 - \frac{1}{3} R_3$$

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 0 & 2 & 8 \\ -6 & -24 & -60 \end{vmatrix} = 0$$

$$C_3 \to C_3 - 4C_2$$

$$\begin{vmatrix} x-2 & 2x-3 & -5x+8 \\ 0 & 2 & 0 \\ -6 & -24 & 36 \end{vmatrix} = 0$$

Expand determinant along second row

$$2\begin{vmatrix} x-2 & -5x+8 \\ -6 & 36 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2[36x - 72 - 30x + 48] = 0$$

$$\Rightarrow 2[6x - 24) = 0 \Rightarrow 12x = 48 \Rightarrow \boxed{x = 4}$$

Q. 1. (c) Find the maximum and minimum values of $f(x) = x + \sin 2x$ in $[0, 2\pi]$ (5)

Ans. Find the maximum value of $f(x) = x + \sin 2x$ in $[0, 2\pi]$

Step 1: $f'(x) = 1 + 2\cos 2x$

Step 2: put $f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2} \Rightarrow 2x = \frac{4\pi}{3}, \frac{2\pi}{3} \Rightarrow x = \frac{2\pi}{3}, \frac{\pi}{3}$

Step 3: $f''(x) = -4\sin 2x$

Step 4: $(f''(x))_{x=\frac{2\pi}{3}} = -4\frac{\sin 4\pi}{3} = -4\left(\frac{-\sqrt{3}}{2}\right) = 2\sqrt{3} > 0$

$f(x)$ has minimum at $x = \frac{4\pi}{6}$

$$(f''(x))_{x=\frac{\pi}{3}} = -4\sin\left(\frac{2\pi}{3}\right) = -2\sqrt{3} < 0$$

Maximum value $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + 2\cos\frac{2\pi}{3} = \frac{2\pi + 3\sqrt{3}}{6}$

Q. 1. (d) Evaluate $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$. (5)

Ans. $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx = \int \frac{\sqrt{\sin x}}{\sqrt{\cos x}\sin x \cos x} dx = \int \frac{1}{\sqrt{\sin x}\sqrt{\cos x}\cos x} dx$

$$= \int \frac{1}{\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\sqrt{\cos x}\sqrt{\cos x}\cos x} dx$$

$$= \int \frac{dx}{\sqrt{\tan x}\cos^2 x} = \int \frac{\sec^2 x}{\sqrt{\tan x}} dx \left(\begin{array}{c}\text{since} \\ \sec x = \frac{1}{\cos x}\end{array}\right)$$

put $\tan x = t \Rightarrow \frac{d}{dx}\tan x = \frac{dt}{dx} \Rightarrow \boxed{\sec^2 x\, dx = dt}$

$$\int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}} dt = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = 2\sqrt{t} + c$$

$$= 2\sqrt{\tan x} + c.$$

Q. 1. (e) Show that $\lim_{x \to 0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$ **does not exist.** (5)

Ans. R.H.L.

$$\lim_{x \to 0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \lim_{\substack{x \to 0 \\ x > 0}} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

Now, as $x > 0 \Rightarrow \frac{1}{x}$ $\Rightarrow$ as $x \to 0$, $\frac{1}{x} \to \infty$

$$\Rightarrow e^{\frac{1}{x}} \to$$

$$\lim_{x\to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \lim_{\substack{x\to 0\\ x>0}} \frac{\cancel{e^{\frac{1}{x}}}\left[1-\frac{1}{e^{\frac{1}{x}}}\right]}{\cancel{e^{\frac{1}{x}}}\left[1+\frac{1}{e^{\frac{1}{x}}}\right]}$$

$$= 1 \qquad \left(\text{Since } e^{\frac{1}{x}} \to \infty \text{ as } x \to 0\right).$$

L.H.L

$$\lim_{x\to 0^-} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \lim_{h\to 0} \frac{e^{\frac{1}{-h}}-1}{e^{\frac{1}{-h}}+1}$$

$$= \lim_{\substack{h\to 0\\ h>0}} \frac{e^{-\frac{1}{h}}-1}{e^{\frac{1}{h}}+1} = \lim_{h\to 0} \frac{\frac{1}{e^{\frac{1}{h}}}-1}{\frac{1}{e^{\frac{1}{h}}}+1} \frac{\frac{1}{\infty}-1}{\frac{1}{\infty}+1} = -1$$

$$\boxed{\text{as } h \to 0, e^{\frac{1}{h}} \to \infty}$$

Since R.H.L ≠ L.H.L.

$$\Rightarrow \lim_{x\to c} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} \text{ does not exist.}$$

UNIT-I

Q. 2. (a) Find eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$. **(6.5)**

Ans. The characteristic equation of the given matrix A

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)(6-\lambda)(2-\lambda) + 4(-4(6-\lambda)) = 0$

$\Rightarrow (2-\lambda)(6-\lambda)(2-\lambda) - 16(6-\lambda) = 0$

$\Rightarrow (6-\lambda)[(2-\lambda)^2 - 16] = 0$

$\Rightarrow (6-\lambda)(4+\lambda^2-4\lambda-16) = 0$

$\Rightarrow (6-\lambda)(\lambda^2-4\lambda-12) = 0$

$\Rightarrow (6-\lambda)(\lambda^2-6\lambda+2\lambda-12) = 0$

$\Rightarrow (6-\lambda)[\lambda(\lambda-6)+2(\lambda-6)]$

$\Rightarrow (6-\lambda)(\lambda+2)(\lambda-6) = 0$

$\Rightarrow \boxed{\lambda = 6, 6, -2}$ .... Eigen values

Eigen vector for $\lambda = 6$

Consider $[A-6I]X = 0$

$$\Rightarrow \begin{bmatrix} 2-6 & 0 & 4 \\ 0 & 6-6 & 0 \\ 4 & 0 & 2-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -4x_1 + 4x_3 &= 0 \\ 4x_1 - 4x_3 &= 0 \end{aligned}\right\} \Rightarrow \boxed{x_1 = x_3}$$

$$\text{Eigen vector } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, x_1, x_2 \neq 0$$

Eigen vector for $\lambda = -2$

Consider $[A+2I]X = 0$

$$\Rightarrow \begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \left.\begin{aligned} 4x_1 + 4x_3 &= 0 \\ 8x_2 &= 0 \\ 4x_1 + 4x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= -x_3 \\ &\boxed{x_2 = 0} \end{aligned}$$

$$\text{Eigen vector } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, x_1 \neq 0$$

Q. 2. (b) Find whether or not the following set of vectors are linearly dependent or independent.

$X_1 = (1,1,0)$ $X_2 = (1,0,1)$, $X_3 = (0,1,1)$. **(6)**

Ans. Consider

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 1(0-1) - 1(1-0) + 0(1-0)$$
$$= 1(-1) - 1(1) + 0$$
$$= -1 - 1$$
$$= -2 \neq 0$$

Hence $x_1 = (1, 1, 0), x_2 = (1, 0, 1), x_3 = (0, 1, 1)$ are linearly independent.

Q. 3. (a) Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ **and hence find** A^{-1}. **(6.5)**

Ans. The characteristic equation is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(2-\lambda)^2 - 1)] + 1[(-1)(2-\lambda) + 1] + 1[1 - 2 + \lambda] = 0$$
$$\Rightarrow (2-\lambda)^3 - (2-\lambda) - (2-\lambda) + \cancel{1} + \cancel{1} - \cancel{2} + \lambda = 0$$
$$\Rightarrow (2-\lambda)^3 - 2 + \lambda - 2 + \lambda + \lambda = 0$$
$$\Rightarrow 8 - \lambda^3 - 6\lambda(2-\lambda) - 4 + 3\lambda = 0$$
$$\Rightarrow 8 - \lambda^3 - 12\lambda + 6\lambda^2 - 4 + 3\lambda = 0$$
$$\Rightarrow -\lambda^3 - 9\lambda + 6\lambda^2 + 4 = 0 \Rightarrow \boxed{\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0}$$

To verify Cayley's Hamilton theorem:

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9 & 21-30+9 \\ -21+30-9 & 22-36+18-4 & -21+30-9 \\ 21-30+9 & -21+30-9 & 22-30+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayle–Hamilton theorem varified.

To find A^{-1}

$$A^3 - 6A^2 + 9A - 4I = 0$$
$$4I = A^3 - 6A^2 + 9A$$

Multiply both side by A^{-1}

$$4A^{-1} = A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} = A^2 - 6A + 9I$$

$$A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

$$A^{-1} = \frac{1}{4}\left\{\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}.$$

$$= \frac{1}{4}\left\{\begin{bmatrix} 6-12+9 & -5+6 & 5-6 \\ -5+6 & 6-12+9 & -5+6 \\ 5-6 & -5+6 & 6-12+9 \end{bmatrix}\right\}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Q. 3. (b) Find the rank of matrix $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix}$. **(6)**

Ans. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 5R_1$, $R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$ Hence $\boxed{\text{rank } A = 4}$

UNIT-II

Q. 4. (a) Discuss the continuity of the function $f(x) = \begin{cases} -x, & x \le 0 \\ x, & 0 < x \le 1 \\ 2-x, & 1 < x \le 2 \\ 1, & x > 2 \end{cases}$ **at each point** $x = 0, 1, 2.$ **(6.5)**

Ans.
$$f(x) = \begin{cases} -x, & x \le 0 \\ x, & 0 < x \le 1 \\ 2-x, & 1 < x \le 2 \\ 1, & x > 2 \end{cases}$$

At $x = 0$:
$$f(0) = 0$$

R.H.L. $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{\substack{x \to 0 \\ x > 0}} f(x) = \lim\limits_{x \to 0} x = 0$

L.H.L. $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{\substack{x \to 0 \\ x < 0}} f(x) = \lim\limits_{x \to 0} (-x) = 0$

At $x = 0$ R.H.L. = L.H.L = $0 = f(0)$

Hence $f(x)$ is continuous at $x = 0$

At $x = 1$:
$$f(1) = 1.$$

R.H.L. $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{\substack{x \to 1 \\ x > 1}} f(x) = \lim\limits_{x \to 1} (2 - x) = 2 - 1 = 1$

L.H.L. $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{\substack{x \to 1 \\ x < 1}} f(x) = \lim\limits_{x \to 1} x = 1.$

At $x = 1$: R.H.L. = L.H.L. = $f(1) = 1$

Hence at $x = 1$ $f(x)$ is continuous.

At $x = 2$:
$$f(2) = 0$$

R.H.L $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{\substack{x \to 2^+ \\ x > 2}} f(x) = \lim\limits_{x \to 2} 1 = 1$

L.H.L. $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{\substack{x \to 2 \\ x < 2}} f(x) = \lim\limits_{x \to 2} (2 - x) = 0$

At $x = 2$: R.H.L. $\ne$ L.H.L.

Hence $f(x)$ is not continuous at $x = 2$.

Q. 4. (b) $\lim\limits_{x \to 0} \left(\dfrac{\cos mx - \cos nx}{x^2} \right).$ **(6)**

Ans. $\lim\limits_{x \to 0} \dfrac{\cos mx - \cos nx}{x^2} \left(\dfrac{0}{0} \right)$

$$= \lim_{x \to 0} \frac{\frac{d}{dx}[\cos mx - \cos nx]}{\frac{d}{dx}(x^2)}$$

$$= \lim_{x \to 0} \frac{-m \sin mx + n \sin nx}{2x} \left(\frac{0}{0} \right)$$

$$= \lim_{x \to 0} \frac{-m^2 \cos mx + n^2 \cos nx}{2}$$

$$= \frac{-m^2 + n^2}{2} = \frac{(n - m)(m + n)}{2}.$$

Q. 5. (a) Let $f(x) = \begin{cases} 1 & x \le 3 \\ ax + b, & 3 < x < 5 \\ 7, & 5 \le x \end{cases}$. **Find the values of a and b so that $f(x)$ is continuous.** **(6.5)**

Ans. Since $f(x)$ is continuous function, So $f(x)$ is continuous at $x = 3, x = 5.$

Continuous at $x = 3$:
$$\lim_{x \to 3^-} f(x) = f(3) = \lim_{x \to 3^+} f(x).$$

$$\lim_{\substack{x \to 3 \\ x < 3}} 1 = f(3) \Rightarrow \boxed{1 = 1}$$

$$\lim_{x \to 3^+} f(x) = f(3) \Rightarrow \lim_{\substack{x \to 3 \\ x > 3}} ax + b = 1 \Rightarrow 3a + b = 1 \quad \text{...(1)}$$

Continuous at $x = 5$.
$$\lim_{x \to 5} f(x) = f(5) = 7$$

$$\Rightarrow \quad \lim_{\substack{x \to 5 \\ x < 5}} (ax + b) = 7 \Rightarrow 5a + b = 7 \quad \text{...(2)}$$

from (1) and (2)
$$\boxed{a = 3, b = -8}$$

Q. 5. (b) Evaluate: (i) $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{4x^2 + 1}}$ **(ii)** $\lim\limits_{x \to 0} \dfrac{x^3 \cot x}{1 - \cos x}$. **(6)**

Ans. (i) $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{4x^2 + 1}} = \lim\limits_{x \to \infty} \dfrac{\cancel{x}}{\cancel{x}\sqrt{4 + \frac{1}{x^2}}} = \lim\limits_{x \to \infty} \dfrac{1}{\sqrt{4 + \frac{1}{x^2}}}$

$$= \frac{1}{\sqrt{4+0}} = \frac{1}{2}.$$

(ii) $\lim_{x\to 0}\frac{x^3\cot x}{(1-\cos x)} = \lim_{x\to 0}\frac{x^3\cos x}{\sin x(1-\cos x)}$

$$= \lim_{x\to 0}\frac{x^3\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+.....\right)}{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+....\right)\left(\cancel{1}-\cancel{1}+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-....\right)}$$

$$= \lim_{x\to 0}\frac{\cancel{x^3}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+.....\right)}{\cancel{x^3}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}....\right]\left[\frac{1}{2!}-\frac{x^2}{4!}+.....\right]}$$

$$= \frac{1-0+0-0}{(1-0+0-)\left(\frac{1}{2}-0+0.....\right)}$$

$$= \frac{1}{\frac{1}{2}} = 2$$

UNIT-III

Q. 6. (a) Verify the hypothesis and conclusion of Lagrange's mean value theorem for the function $f(x) = \frac{1}{4x-1}, 1 \le x \le 4$. **(6.5)**

Ans. Given $f(x) = \frac{1}{4x-1}$ $\quad 1 \le x \le 4 \Rightarrow f'(x) = \frac{-4}{(4x-1)^2}$

Given $f(x)$ is continuous and differentiable on $[1, 4]$. i.e. $f(x)$ is continuous and differentiable on $[1, 4]$ and $(1, 4)$ respectively.

Thus both the condition of LMVT are satisfied.

$= \exists\ C \in [1, 4]$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{16-1}-\frac{1}{4-1}}{3}$$

$$\Rightarrow \quad f'(c) = \frac{\frac{1}{15}-\frac{1}{3}}{3} = \frac{\frac{1-5}{15}}{3} = \frac{-4}{45}$$

$$\Rightarrow \quad \frac{-\cancel{4}}{(4c-1)^2} = \frac{-\cancel{4}}{45} \Rightarrow 45 = (4c-1)^2 \Rightarrow 8c^2+1-8c = 45$$

$$\Rightarrow \quad 8c^2 - 8c - 44 = 0$$

$$\Rightarrow \quad c = \frac{8 \pm \sqrt{64+32\times 44}}{16} = \frac{8 \pm \sqrt{1408}}{16}$$

$$\Rightarrow \quad c = 2.8452 \in (1, 4).$$

Q. 6. (b) Expand log sin x in powers of $(x-2)$ by Taylor's series. **(6)**

Ans. $f(x) = \log \sin x$

Step 1: let $f(x) = \log \sin x$

Step 2: write $f(x) = f(2+x-2)$

Step 3: Let $x-2 = h$ So that $f(2+x-2) = f(2+h)$

Step 4: By Taylor's series $f(2+h) = f(2) + hf'(2) + \frac{h^2}{2!} + f''(2) + ...\infty$ ....(1)

Now $f(x) = \log \sin x$

$$f'(x) = \frac{1}{\sin x}\cos x = \cot x$$

$$f''(x) = \frac{(\sin x)(-\sin x)-\cos x\cos x}{\sin^2 x} = \frac{-(\sin^2 x+\cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = -\sin^{-2} x = \frac{-1}{\sin^2 x}$$

$$f'''(x) = (-1)(-2)\sin^{-3} x = 2\sin^{-3} x = \frac{2}{\sin^3 x}$$

$f(2) = \log \sin 2 \quad f'(2) = \cot 2 \quad f''(2) = \frac{-1}{\sin^2 2}, f'''(2) = \frac{2}{\sin^3 2}$

put these values in (1)

$$f(2+h) = \log \sin 2 + h\cot 2 + \frac{h^2}{2!}\left(\frac{-1}{\sin^2 2}\right) + \frac{h^3}{3!}\left(\frac{2}{\sin^3 2}\right) + ...\infty$$

Replace h by $x-2$

$$f(x) = \log \sin x = \log \sin 2 + \cot 2\,(x-2)\,\frac{-1}{\sin^2 2\ 2!}(x-2)^2 + ...\infty.$$

Q. 7. (a) If $y = \left[x+\sqrt{1+x^2}\right]^m$, show that $(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0$. Also find $y_n(0)$. **(6.5)**

Ans. If $\quad y = \left[x+\sqrt{1+x^2}\right]^m$ ...(1)

$$y_1 = m\left[x+\sqrt{1+x^2}\right]^{m-1}\left(1+\frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}\right)$$

$$= m\left[x+\sqrt{1+x^2}\right]^{m-1}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right) = \frac{m\left[x+\sqrt{1+x^2}\right]^m}{\sqrt{1+x^2}}$$

$$= \frac{my}{\sqrt{1+x^2}}. \quad \text{...(2)}$$

Squaring both side and cross multiplying

$$y_1^2(1+x^2) = m^2y^2$$

Differentiate w.r.t x both side

$$2y_1(1+x^2)y_2 + 2xy_1^2 = m^2 2yy_1$$

Divide both side by $2y_1$

$$(1+x^2)y_2 + xy_1 = m^2y$$

$$\Rightarrow \quad (1+x^2)y_2 + xy_1 - m^2y = 0 \quad \text{...(3)}$$

Differentiating n times w.r.t. x and applying Leibnitz's rule

$$\left[(1+x^2)y_{n+2} + ny_{n+1}(2x) + \frac{n(n-1)}{\cancel{2}!}y_n(\cancel{2})\right] + \left[xy_{n+1} + ny_n\right] - m^2y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n - \cancel{ny_n} + \cancel{ny_n} - m^2y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0 \quad \text{...(4)}$$

put $x = 0$ in (1), (2), (3)

$$y(0) = 1$$

$$y_1(0) = \frac{my(0)}{1} = m$$

$$y_2(0) = m^2y(0) = m^2$$

put $x = 0$ in (4)

$$y_{n+2}(0) + (n^2-m^2)y_n(0) = 0 \Rightarrow y_{n+2}(0) = (m^2-n^2)y_n(0) \quad \text{...(5)}$$

put $\quad n = 1, 2, 3....$ in (5)

$$y_3(0) = (m^2-1)\,m = m(m^2-1)$$

$$y_4(0) = (m^2-2^2)\,m^2 = m^2(m^2-2^2)$$

$$y_5(0) = (m^2-3^2)\,y_3(0) = m(m^2-1^2)(m^2-3^2)$$

$$y_6(0) = (m^2-4^2)\,y_4(0) = m^2(m^2-2^2)(m^2-4^2) \quad n \text{ is even}$$

$$y_n(0) = \begin{cases} m^2(m^2-2^2)(m^2-4^2)....(m^2-(n-2)^2) & n \text{ even} \\ m(m^2-1)(m^2-3^2)....(m^2-(n-2)^2) & n \text{ odd} \end{cases}$$

Q. 7. (b) Find asymptotes of the curve **(6)**

$$x^3 + 2x^2y - xy^2 - 2y^3 + x^2 - y^2 - 2x - 3y = 0.$$

Ans. $x^3 + 2x^2y - xy^2 - 2y^3 + x^2 - y^2 - 2x - 3y = 0.$

Since the coefficients of highest degree terms in x and y are constant therefore there will be no asymptote parallel to x–axis or y–axis.

Now, $(x^3 - 2y^3 + 2x^2y - xy^2) - (y^2 - x^2) - (2x + 3y) = 0$

$$\therefore \quad \phi_3(x,y) = x^3 - 2y^3 + 2x^2y - xy^2$$

$$\phi_2(x,y) = -(y^2 - x^2)$$

$$\phi_1(x,y) = -2x - 3y$$

put $\quad x = 1$ and $y = m$.

$$\phi_3(m) = 1 - 2m^3 + 2m - m^2$$

$$\phi_2(m) = -m^2 + 1$$

Now put $\quad \phi_3(m) = 0$

$$-2m^3 + 2m - m^2 + 1 = 0$$

$$\Rightarrow \quad 2m^3 + m^2 - 2m - 1 = 0$$

$$\Rightarrow \quad m^2(2m+1) - 1(2m+1) = 0$$

$$\Rightarrow \quad (m^2-1)(2m+1) = 0$$

$$\Rightarrow \quad m = 1, -1, -1/2.$$

Now

$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{m^2-1}{-6m^2+2-2m}$$

when $\quad m = 1, \; c = 0$

$\therefore$ Asymptote is $\quad y = mx + c \Rightarrow \boxed{y = x}$

When $\quad m = -1, \; c = 0$

$\therefore$ Asymptote is $\quad y - x = 0$

when $\quad m = -1/2$

$$c = \frac{\frac{1}{4} - 1}{-\cancel{6}^{3} \times \frac{1}{\cancel{4}_2} + 2 - \cancel{2}\left(-\frac{1}{\cancel{2}}\right)} = \frac{\frac{-3}{4}}{\frac{-3}{2} + 2 + 1}$$

$$= \frac{-\frac{3}{4}}{3 - \frac{3}{2}} = \frac{\frac{-3}{4}}{\frac{3}{2}} = -\frac{\cancel{3}}{\cancel{4}_2} \times \frac{\cancel{2}}{\cancel{3}} = -\frac{1}{2}$$

$\therefore$ Asymptote is $\quad y = \frac{-1}{2}x - \frac{1}{2} \Rightarrow 2y + x + 1 = 0.$

UNIT-IV

Q. 8. (a) Evaluate (i) $\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx$ **(ii)** $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx.$ **(6)**

Ans. (i) $\int \frac{x+\tan^{-1}x}{(1+x^2)^{3/2}}dx$

put $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$$\int \frac{\theta\tan\theta\sec^2\theta \, d\theta}{(1+\tan^2\theta)^{3/2}} = \int \frac{\theta\tan\theta\sec^2\theta \, d\theta}{(\sec^2\theta)^{3/2}}$$

$$= \int \frac{\theta\tan\theta\sec^2\theta \, d\theta}{\sec^3\theta} = \int \frac{\theta\tan\theta}{\sec\theta}d\theta = \int \theta\sin\theta \, d\theta$$

$$= -\theta\cos\theta + \sin\theta \text{ (By Integration by Parts)}$$
$$= -\tan^{-1}x\cos\tan^{-1}x + \sin\tan^{-1}x.$$

(ii) $$\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

$$= \int e^x\left(\frac{\sin x-1}{\cos x-1}\right)dx$$

Use following Result

1. $\cos x = 1-2\sin^2\frac{x}{2}$
$\Rightarrow \cos x - 1 = -2\sin^2\frac{x}{2}$
2. $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$= \int\frac{e^x\sin x}{\cos x-1}dx - \int\frac{e^x}{\cos x-1}dx$$

$$= \int e^x\frac{\cancel{2}\,\cancel{\sin}\frac{x}{2}\cos\frac{x}{2}}{-\cancel{2}\sin^{\cancel{2}}\frac{x}{2}} - \int\frac{e^x}{-2\sin^2\frac{x}{2}}dx$$

$$= -\int e^x\cot\frac{x}{2}dx + \frac{1}{2}\int e^x\operatorname{cosec}^2\frac{x}{2}dx \quad ...(1)$$

Now, $$\int\cot\frac{x}{2}e^x = \cot\frac{x}{2}e^x - \int\frac{-\operatorname{cosec}^2\frac{x}{2}}{2}e^xdx$$

$$= \cot\left(\frac{x}{2}\right)e^x + \frac{1}{2}\int\operatorname{cosec}^2\frac{x}{2}e^xdx \quad ...(2)$$

from (1) and (2)

$$\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx = -\left[\cot\frac{x}{2}e^x\right] - \frac{1}{2}\int\cancel{\operatorname{cosec}}^2\frac{x}{2}e^xdx + \frac{1}{2}\int\cancel{\operatorname{cosec}}^2\frac{x}{2}e^xdx$$

$$= -\left(\cot\frac{x}{2}\right)e^x$$

Q. 8. (b) Show that $\beta(p,q) = \int_0^1\frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx$. (6.5)

Ans. $$\beta(p,q) = \int_0^\infty\frac{x^{q-1}}{(1+x)^{p+q}}dx$$

$$\Rightarrow \beta(p,q) = \int_0^1\frac{x^{q-1}\,dx}{(1+x)^{p+q}} + \int_1^\infty\frac{x^{q-1}dx}{(1+x)^{p+q}} \quad ...(1)$$

Now consider, $$\int_1^\infty\frac{x^{q-1}}{(1+x)^{p+q}}dx$$

put $$x = \frac{1}{z} \Rightarrow dx = \frac{-1}{z^2}dz$$

$$\therefore \int_1^\infty\frac{x^{q-1}\,dx}{(1+x)^{p+q}} = \int_1^0\frac{\left(\frac{1}{z}\right)^{q-1}}{\left(1+\frac{1}{z}\right)^{p+q}}\left(\frac{-1}{z^2}\right)dz$$

$$= -\int_1^0\frac{\frac{1}{z^{q-1}}}{\frac{(z+1)^{p+q}}{z^{p+q}}}\times\frac{1}{z^2}dz = \int_0^1\frac{z^{p-1}}{(1+z)^{p+q}}dz$$

$$= -\int_0^1\frac{x^{p-1}dx}{(1+x)^{p+q}} \text{ (By change of variable)}$$

so $$\beta(p,q) = \int_0^1\frac{x^{q-1}dx}{(1+x)^{p+q}} + \int_0^1\frac{x^{p-1}dx}{(1+x)^{p+q}} = \int_0^1\frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx.$$

Q. 9. (a) If $I_{m,n} = \int\cos^m x\sin nx\,dx$, prove that $(m+n)I_{m,n} = -\cos^m x\cos nx + mI_{m-1,n-1}$.

Hence evaluate $\int_0^{\pi/2}\cos^5 x\sin 3x\,dx$. (7.5)

Ans. If $I_{m,n} = \int\cos^m x\sin(nx)dx$ T.P. $(m+n)I_{m,n} = -\cos^m x\cos nx + mI_{m-1,n-1}$

$$I_{m,n} = \int\cos^m x\sin(nx)dx$$

Apply Integration by parts:

$$I_{m,n} = -\cos^m x\frac{\cos nx}{n} - \int m\cos^{m-1}(x)(-\sin x)\left(\frac{-\cos nx}{n}\right)dx$$

$$= \frac{-1}{n}\cos^m x\cos nx - \frac{m}{n}\int\cos^{m-1}(x)\sin x\cos nx\,dx$$

Now $\sin(n-1)x = \sin(nx-x) = \sin(nx)\cos x - \cos nx\sin x$
$\Rightarrow \cos nx\sin x = \sin(nx)\cos x - \sin(n-1)x$.

$$I_{m,n} = \frac{-\cos^m x\cos nx}{n} - \frac{m}{n}\int\cos^{m-1}(x)[\sin nx\cos x - \sin(n-1)x]dx$$

$$= \frac{-\cos^m x\cos nx}{n} - \frac{m}{n}\int\cos^{m-1}(x)\sin(nx)\cos x + \frac{m}{n}\int\cos^{m-1}(x)\sin(n-1)ndx$$

$$= \frac{-\cos^m x\cos nx}{n} - \frac{m}{n}I_{m,n} + \frac{m}{n}I_{m-1,n-1}$$

$$\Rightarrow I_{m,n}\left(1+\frac{m}{n}\right) = -\frac{\cos^m x\cos nx + mI_{m-1,n-1}}{n}$$

$$\Rightarrow \boxed{(m+n)I_{m,n} = -\cos^m x \cos nx + m\,I_{m-1,n-1}}$$

To Evaluate:

$$I_{5,3} = \int_0^{\pi/2} \cos^5 x \sin 3x\,dx$$

$$= \left(\frac{-\cos^5 x\cos 3x}{8}\right)_0^{\pi/2} + \frac{5}{8}I_{4,2}$$

$$= \frac{1}{m+n} + \frac{m}{m+n}I_{m-1,n-1}$$

$$= \frac{1}{8} + \frac{5}{8}I_{4,2}$$

$$= \frac{1}{8} + \frac{5}{8}\left[\frac{1}{6} + \frac{4}{6}I_{3,1}\right]$$

$$= \frac{1}{8} + \frac{5}{8}\left[\frac{1}{6} + \frac{4}{6}\left\{\frac{1}{4} + \frac{3}{4}I_{2,0}\right\}\right] \quad \left\{I_{2,0} = \int_0^{\pi/2}\cos^2 x \sin 0 dx = 0\right\}$$

$$= \frac{1}{8} + \frac{5}{8}\left[\frac{1}{6} + \frac{\cancel{4}}{6} \times \frac{1}{\cancel{4}}\right]$$

$$= \frac{1}{8} + \frac{5}{8} \times \frac{\cancel{2}}{\cancel{6}_3} = \frac{1}{8} + \frac{5}{24} = \frac{3+5}{24} = \frac{1}{3}$$

Q. 9. (b) Evaluate $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx$. **(5)**

Ans.

$$\int_0^1 x^{3/2}(1-x)^{3/2}dx$$

$$= \int_0^1 x^{\frac{5}{2}-1}(1-x)^{\frac{5}{2}-1}dx$$

$$= \beta\left(\frac{5}{2},\frac{5}{2}\right)$$

$$= \frac{\sqrt{\frac{5}{2}}\,\sqrt{\frac{5}{2}}}{\sqrt{5}}$$

$$= \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\pi\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\pi}{4!} = \frac{9\pi^2}{16\times 4!}$$

Use:
1. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}$
2. $\beta(m,n) = \dfrac{\sqrt{(m)}\cdot\sqrt{(n)}}{\sqrt{(m+n)}}$
3. $\sqrt{n} = (n-1)\sqrt{n-1}$ 4. $\sqrt{\tfrac{1}{2}} = \sqrt{\pi}$

END TERM EXAMINATION [DEC. 2018]
FIRST SEMESTER (BCA)
MATHEMATICS-I [BCA-101]

Time : 3 hrs. **M.M. : 75**

Note: *Attempt any five questions including Q.no. 1 which is compulsory.*

Q.1. (a) Evaluate the determinant of the matrix $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$. **(3)**

Ans.

$$\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix} = \frac{1}{a}\left(b^3a - c^3a\right) - a^2(a-a) + bc\left(\frac{c^2}{b} - \frac{b^2}{c}\right)$$

$$= b^3 - c^3 + c^3 - b^3 = 0$$

Q. 1. (b) Use Cramer's rule to sole the system of equations. **(3)**

$x + y + z + 1 = 0;\ ax + by + cz + d = 0;\ a^2x + b^2y + c^2z + d^2 = 0$

Ans. Given

$$x + y + z = -1$$
$$ax + by + cz = -d$$
$$a^2x + b^2y + c^2z = -d^2$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -1 \\ -d \\ -d^2 \end{bmatrix}$$

$$\Delta = \det A = 1(bc^2 - b^2c) - 1(ac^2 - a^2c) + 1(ab^2 - a^2b)$$
$$= bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b$$
$$= c^2(b-a) + a^2(c-b) + b^2(a-c)$$

$$\Delta_1 = \begin{vmatrix} -1 & 1 & 1 \\ -d & b & c \\ -d^2 & b^2 & c^2 \end{vmatrix} = -1(bc^2 - b^2c) - 1(-dc^2 + d^2c) + 1(-db^2 + d^2b)$$
$$= b^2c - bc^2 + dc^2 - d^2c + d^2b - db^2$$
$$= b^2(c-d) + c^2(d-b) + d^2(b-c)$$

$$\Delta_2 = \begin{vmatrix} 1 & -1 & 1 \\ a & -d & c \\ a^2 & -d^2 & c^2 \end{vmatrix} = 1(-dc^2 + d^2c) + 1(ac^2 + a^2c) + 1(-d^2a + a^2d)$$

$$= d^2(c-a) + c^2(a-d) + a^2(d-c)$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & -1 \\ a & b & -d \\ a^2 & b^2 & -d^2 \end{vmatrix}.$$

$$= 1(-bd^2 + b^2d) - 1(-ad^2 + a^2d) - 1(ab^2 - a^2b)$$

$$= -bd^2 + b^2d + ad^2 - a^2d - ab^2 + a^2b$$

$$= a^2(b-d) + b^2(d-a) + d^2(a-b)$$

$\therefore$ $\quad x = \dfrac{\Delta_1}{\Delta}, \; y = \dfrac{\Delta_2}{\Delta}, \; z = \dfrac{\Delta_3}{\Delta}$ provide $a \neq b \neq c$.

Q. 1. (c) Find the maximum value of $y = \left(\frac{1}{x}\right)^x$ **(3)**

Ans. Maximum value of $y = \left(\frac{1}{x}\right)^x$.

Take log of both side

$$\log y = \log\left(\frac{1}{x}\right)^x = x\log\left(\frac{1}{x}\right)$$

$$= x[\log 1 - \log x] = x(-\log x)$$

Differentiate both side w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{-1}{x}\right) - \log x = -1 - \log x$$

$$\frac{dy}{dx} = y(-1 - \log x)$$

but $\quad \dfrac{dy}{dx} = 0$

$\Rightarrow \quad y(-1 - \log x) = 0$

$\Rightarrow \quad -1 - \log x = 0$

$\Rightarrow \quad \log x = -1$

$\Rightarrow \quad \boxed{x = e^{-1} = \frac{1}{e}}$

Now

$$y''(x) = y'(-1 - \log x) + y\left(-\frac{1}{x}\right)$$

$$= y'(-1 - \log x) - \frac{y}{x}$$

Now $\quad y''(x)$ at $x = \dfrac{1}{e}$

$$y''\left(\frac{1}{e}\right) = -(e)^{1/e} \qquad e < 0$$

$\therefore \quad y = \left(\dfrac{1}{x}\right)^x$ is maximum when $x = \dfrac{1}{e}$.

Maximum value is $(e)^{1/e}$

Q. 1. (d) Evaluate $\int \cos mx . \cos nx \, dx$, when (i) $m \neq n$ (ii) $m = n$. **(3)**

Ans. $\quad \int \cos mx \cos nx \, dx$, when $m \neq n$

$$= \frac{1}{2}\int \cos(mx - nx) + \cos(mx + nx)\,dx$$

$$= \frac{1}{2}\int \cos(m-n)x\,dx + \frac{1}{2}\int \cos(m+n)x\,dx$$

$$= \frac{1}{2}\frac{\sin(m-n)x}{m-n} + \frac{1}{2}\frac{\sin(m+n)x}{m+n}$$

When $\quad m = n$

$$\int \cos mx \cos mx dx = \frac{1}{2}\int \cos 2mx\,dx + \frac{1}{2}\int dx$$

$$= \frac{1}{2}\frac{\sin 2mx}{2m} + \frac{1}{2}x$$

Q. 1. (e) Evaluate $\lim_{x\to 0}\left(e x^{1/x} + 1\right)$, if it exists. **(3)**

Ans. $\quad \lim_{x\to 0}\left(e x^{1/x} + 1\right) = \lim_{x\to 0} e x^{1/x} + 1$

$$= e \lim_{x\to 0}(x)^{1/x} + 1$$

$$= e \cdot \lim_{e\; x\to 0} \frac{\ln x}{x} + 1$$

$$= e \times 0 + 1 = 1.$$

UNIT-I

Q. 2. (a) Show that the vectors

$x_1 = (1, 2, 4), x_2 = (2, -1, 3), x_3 = (0, 1, 2)$ and $x_4 = (-3, 7, 2)$ are linearly dependent and find the relation between them. **(8)**

Ans. Let $\alpha_1, \alpha_2, \alpha_3$ and α_4 be scalars such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0$$

$$\therefore \alpha_1(1, 2, 4) + \alpha_2(2, -1, 3) + \alpha_3(0, 1, 2) + \alpha_4(-3, 7, 2) = 0$$

$$\therefore \quad \alpha_1 + 2\alpha_2 - 3\alpha_4 = 0$$
$$2\alpha_1 - \alpha_2 + \alpha_3 + 7\alpha_4 = 0$$
$$4\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4 = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \alpha_1 + 2\alpha_2 + (-3)\alpha_4 = 0$$
$$-5\alpha_2 + \alpha_3 + 13\alpha_4 = 0$$
$$\alpha_3 + \alpha_4 = 0$$

Let $\alpha_4 = k \qquad [\therefore \alpha^3 = -k]$

$$\alpha_2 = \frac{12k}{5}, \; \alpha_1 = \frac{-9k}{5}$$

$\alpha_1, \alpha_2, \alpha_3$ and α_4 (not all zero), so the vectors x_1, x_2, x_3 and x_4 and linear dependent Relationship is given by

$$\alpha_1 = -\frac{9k}{5}, \; \alpha_2 = \frac{12k}{5}, \; \alpha_3 = -k, \; \alpha_4 = k$$

$$9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$$

Q. 2. (b) Find the eigen values and eigen vectors of $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **(7)**

Ans. The characteristic equation of the given matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)[(3-\lambda)^2 - 1] + 2[(-2)(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$$

$$\Rightarrow (6-\lambda)[(3-\lambda)^2 - 6 + \lambda - 12 + 4\lambda + 4 - 6 + 2\lambda = 0$$
$$\Rightarrow (6-\lambda)(3-\lambda)^2 + 7\lambda) + 20 = 0$$
$$\Rightarrow (6-\lambda)(9 + \lambda^2 - 6\lambda) + 7\lambda + 20 = 0$$
$$\Rightarrow 54 + 6\lambda^2 - 36\lambda - 9\lambda - \lambda^3 + 6\lambda^2 + 7\lambda + 20 = 0$$
$$\Rightarrow -\lambda^3 + 12\lambda^2 - 38\lambda + 74 = 0$$
$$\Rightarrow \lambda^3 - 12\lambda^2 + 38\lambda - 74 = 0$$
$$\Rightarrow (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$$
$$\Rightarrow \lambda = 2, 2, 8$$

Thus, eigen values are 2, 2, 8.

Eigen vector

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Taking $\lambda = 8$, we get

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$
$$-3x_2 - 3x_3 = 0 \quad \Rightarrow \quad x_2 = -x_3$$
$$-2x_1 + 2x_3 + 2x_3 = 0$$
$$-2x_1 = -4x_3$$
$$x_1 = 2x_3$$

Eigen vector $(2x_3, -x_3, x_3) = x_3[2 \;\; -1 \;\; 1]^T$

Eigen vector for $\lambda = 2$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 + \frac{1}{2}R_1$$

$$R_3 \to R_3 - \frac{1}{2}R_1$$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $4x_1 + 2x_3 - 2x_2 = 0$

$\Rightarrow$ $2x_1 - x_2 + x_3 = 0$

$\Rightarrow$ $x_2 = 2x_1 + x_3$

Eigen vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 + x_3 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Q.3. (a) Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ **Find adj(A) by using Cayley-Hamilton theorem. (8)**

Ans.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$|A| = 1(1-1) - 2(3) - 1(-3) = -6 + 3 = -3.$$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & -1 \\ 3 & -1 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)^2 - 1] - 2[3] - 1[-3+\lambda] = 0$$

$$(1-\lambda)[(1-\lambda)^2 - 1] - 6 + 3 - \lambda = 0$$

$$(1-\lambda)^3 - 1 - \lambda - 3 - \lambda = 0$$

$$\cancel{1} - \lambda^3 - 3\lambda + 3\lambda^2 \cancel{-1} - 2\lambda - 3 = 0$$

$$-\lambda^3 - 5\lambda + 3\lambda^2 - 3 = 0$$

$$\lambda^3 - 3\lambda^2 + 5\lambda + 3 = 0$$

By Cayley Homilton Theorem

$$A^3 - 3A^2 + 5A + 3 = 0$$

Multiply by A^{-1} both side

$$3A^{-1} = 3A - 5I - A^2$$

$$\frac{\text{adj } A}{|A|} = \frac{1}{3}\left[3A - A^2 - 5I\right]$$

$$\text{adj } A = \frac{-3}{3}\begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 9 & -3 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 5 & -4 \\ -3 & 2 & -2 \\ 6 & 4 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= -\begin{bmatrix} -1 & 2 & -1 \\ 3 & -4 & -1 \\ 3 & -7 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & -4 \\ -3 & 2 & -2 \\ 6 & 4 & -1 \end{bmatrix}$$

Q. 3. (b) Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ **(7)**

Ans. Rank of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$$R_2 \to R_2 - \frac{1}{2}R_1, \quad R_3 \to R_3 - \frac{3}{2}R_1, \quad R_4 \to R_4 - 3R_1$$

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -\frac{5}{2} & -\frac{3}{2} & -\frac{7}{2} \\ 0 & -\frac{7}{2} & \frac{9}{2} & -\frac{1}{2} \\ 0 & -6 & 3 & -4 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{-5/2}, \quad R_3 \to \frac{R_3}{-7/2}$$

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 1 & -\frac{9}{7} & \frac{1}{7} \\ 0 & -6 & 3 & -4 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \quad R_4 \to R_4 + 6R_2$$

$$= \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & -66/35 & -44/5 \\ 0 & 0 & \frac{-33}{5} & \frac{-22}{7} \end{bmatrix}$$

$$R_3 \to \frac{R_3}{\frac{-66}{35}}, \quad R_4 \to R_4 / {-33/5}$$

$$= \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 1 & -14/3 \\ 0 & 0 & 1 & 10/21 \end{bmatrix}$$

$$R_4 \to R_4 - R_3$$

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 1 & -14/3 \\ 0 & 0 & 1 & \frac{108}{21} \end{bmatrix}$$

$$\boxed{\therefore \text{ rank } A = 4}$$

UNIT-II

Q. 4. (a) Discusss the continuity of the function (8)

$$f(x) = \frac{x e^{1/x}}{1 + e^{1/x}}, \text{ when } x \neq 0, f(0) = 0$$

$$f(x) = \begin{cases} \dfrac{x e^{1/x}}{1 + e^{1/x}}, & x \neq 0 \\ 0 & , \text{when } x = 0 \end{cases}$$

Ans. L.H.S. at $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} \frac{(-h)e^{-1/h}}{1 + e^{-1/h}} \left[\begin{array}{l} h \to 0 \dfrac{1}{h} \to \infty \\ -\dfrac{1}{h} \to -\infty \\ e^{-1/h} \to e^{-\infty} = 0 \end{array} \right. = 0 \quad \text{...(1)}$$

R.H.S. at $x = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} \frac{(h)e^{1/h}}{1 + e^{1/h}} = \lim_{h \to 0} \frac{h e^{1/h}}{e^{1/h}\left[\frac{1}{e^{1/h}} + 1\right]} = 0 \quad \text{..(2)}$$

from (1) and (2)

L.H.S $= f(0) = 0 = $ R.H.S.

Hence f(x) is continucy at $x = 0$.

Q. 4. (b) solve $\lim_{x \to 0} \dfrac{(1+x)^{1/x} - e + \frac{e^x}{2}}{x^2}$. (7)

Ans.

$$\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{e^x}{2}}{x^2}$$

$$= \frac{\lim_{x \to 0} (1+x)^{1/x} - e + 1/2}{\lim_{x \to 0} x^2}$$

$$= \infty \quad \left(\text{since } \lim_{x \to \infty} (1+x)^{1/x} = 0 \right)$$

Q. 5. (a) Discuss the continuity of the function (7)

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & , \text{if } x < 0 \\ (x+1) & , \text{if } x \geq 0 \end{cases}$$

Ans.

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & , \text{if } x < 0 \\ x+1 & , \text{if } x \geq 0 \end{cases}$$

continuity at $x = 0 \quad f(0) = 1.$

R.H.S. at $x = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{\substack{x \to 0^+ \\ x > 0}} f(x) = \lim_{x \to 0} (x+1) = 1.$$

L.H.S. at $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{\substack{x \to 0 \\ x < 0}} \frac{\sin x}{x} = 1$$

Since R.H.S. $= f(0) = 1 = $ L.H.S.

Therefore f is continuous at 0.

And f is continuous on Real Number.

Q. 5. (b) Evaluate (i) $\lim_{x \to 0} \dfrac{(1 + x^n - 1)}{x}$ **(ii)** $\lim_{x \to 0} \dfrac{\log(\tan^2 2x)}{\log(\tan^2 x)}$ (8)

Ans. (i) $\lim_{x \to 0} \dfrac{(1+x^n - 1)}{x} = \lim_{x \to 0} \dfrac{1 + x^n - 1}{x} = \lim_{x \to 0} n x^{n-1} = 0$

(ii)
$$\lim_{x \to 0} \frac{\log(\tan^2 2x)}{\log(\tan^2 x)} = \lim_{x \to 0} \frac{\frac{1}{\tan^2 2x} \times 2 \tan 2x \sec^2 2x\, 2}{\frac{1}{\tan^2 x} 2 \sec^2 x}$$

$$= \lim_{x \to 0} \frac{{}^2\cancel{4} \sec^2 2x}{\tan 2x} \frac{\tan x}{\cancel{2} \sec^2 x}$$

$$= \lim_{x \to 0} \frac{\sec^2 2x}{\sec^2 x} \lim_{x \to 0} \frac{\tan x}{\tan 2x}$$

$$= 1 \times \lim_{x \to 0} \frac{\sec^2 x}{\sec^2 2x\, 2} = \frac{1}{2}$$

UNIT-III

Q. 6. (a) Verify Lagrange's Mean value Theorem for (8)

$$f(x) = 2x^2 - 7x + 10, 2 \leq x \leq 5$$

Ans. $f(x) = 2x^2 - 7x + 10, 2 \leq x \leq 5$

Since $f(x)$ is a polynomial which is continuous and differentiable everywhere

$\Rightarrow f(x)$ is continuous in [2, 5] and differentiable in (2, 5)

Thus, both the conditions of LMV has been satisfied.

Therefore $\exists$ atleast one value c of x in (2, 5) for which

$$f'(x) = \frac{f(b)-f(a)}{b-a}$$

$$4c-7 = \frac{25-4}{3} = \frac{21}{3} = 7$$

$$4c = 14$$

$$c = \frac{14}{4} \in (2,5)$$

Q. 6. (b) Expand log x in powers of $(x-1)$ by Taylor's theorem and hence find the value of $\log_e(1.1)$. **(7)**

Ans.

$$f(x) = f(1+x-1)$$

$$= f(1+h) \text{ where } h = x-1$$

$$= f(1) + hf'(1) + \frac{h^2}{2!}f''(1) + \frac{h^3}{3!}f'''(1) + \ldots\ldots\infty \quad \ldots(1)$$

$$f(x) = \log x \Rightarrow f(1) = \log 1 = 0$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \Rightarrow f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$$

put these values in (1)

$$f(x) = \log x = (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!}(2)\ldots\ldots\infty$$

Q. 7. (a) If $y = e^{m\cos^{-1}x}$ show that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n = 0$ and calculate $y_n(0)$. **(8)**

Ans.

$$y = e^{m\cos^{-1}x} \quad \ldots(1)$$

$$\therefore \quad y_1 = e^{m\cos^{-1}x}\frac{-m}{\sqrt{1-x^2}} = \frac{-my}{\sqrt{1-x^2}}\left(y = e^{m\cos^{-1}x}\right) \quad \ldots(2)$$

Squaring and cross-multiplying both side

$$y_1^2(1-x^2) = m^2y^2$$

Differentiate w.r.t. x again

$$(1-x^2)2y_1y_2 - 2xy_1^2 = m^2 2yy_1$$

Dividing both sides by $2y_1$, we get

$$(1-x^2)y_2 - xy_1 = m^2y$$

$$\therefore \quad (1-x^2)y_2 - xy_1 - m^2y = 0 \quad \ldots(3)$$

Differentiating 'x' times both side w.r.t. x and applying Leibnitz's Rule, we get

$$\left[(1-x^2)y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2)\right] - \left[xy_{n+1} + ny_n\right] - m^2y_n = 0$$

$$\Rightarrow (1-x_2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

Q. 7. (b) Find all the asymptotes of the curve. **(7)**

$y^3 + 4xy^2 + 4x^2y + 5y^2 + 15xy + 10x^2 - 2x + 1 = 0$

Ans. Let

$$\phi(x,y) = (y^3 + 4xy^2 + 4x^2y) + (5y^2 + 15xy + 10x^2) - 2x + 1 = 0$$

Since the coefficient of x^3 and y^3 are constant, therefore there is no asymptotes parallel to the x, y-axis. Here degree is 3 i.e. n = 3

Here degree is 3 i.e. n = 3

put $x = 1, \ y = m$

$$\phi(m) = (m^3 + 4m^2 + 4m) + (5m^2 + 15m + 10) - 2 + 1 = 0$$

$$\phi_3(m) = m^3 + 4m^2 + 4m, \ \phi'_3(m) = 3m^2 + 8m + 4, \ \phi''_3(m) = 6m + 8$$

$$\phi_2(m) = 5m^2 + 15m + 10, \ \phi'_2(m) = 10m + 15$$

put $\phi_3(m) = 0$

$$m^3 + 4m^2 + 4m = 0$$

$$m = 0, \ m^2 + 2m + 2m + 4 = 0$$

$$m = 0, \ m(m+2) + 2(m+2) = 0$$

$$m = 0, \ (m+2)^2 = 0$$

$$m = 0, \ m = -2, -2$$

When $m = 0$

$$c = \frac{-\phi_2(m)}{\phi'_3(m)} = -\frac{5m^2+15m+10}{3m^2+8m+4} =$$

$$c = -\frac{10}{4}$$

$\therefore$ Asymptotes is

$$y = mx + c = -\frac{10}{4}$$

$$\boxed{4y + 10 = 0}$$

Now for repeated values of m

$$\frac{c^2}{2!}\phi''_n(m) + \frac{c}{1!}\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

$$\Rightarrow \quad \frac{c^2}{2}\phi''_3(m) + c\phi'_2(m) + \phi_1(m) = 0$$

$$\Rightarrow \quad \frac{c^2}{2}(6m+8) + c(10m+15) + 2 = 0$$

When
$$m = -2$$

$$\frac{c^2}{2}(-4)+c(-5)+2 = 0$$

$$-4c^2-10c+4 = 0$$

$$4c^2+10c-4 = 0$$

$$C = \frac{-10\pm\sqrt{100+64}}{8} = \frac{-10\pm\sqrt{164}}{8}$$

$\therefore$ Corresponding asymptotes are

$$y = mx+c = -2x+\left(\frac{-10+\sqrt{164}}{8}\right)$$

$$y = -2x+\left(\frac{-10-\sqrt{164}}{8}\right)$$

UNIT-IV

Q.8. Prove that $\beta(m, n) = \beta(n, m)$ (7)

Ans.
$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}dx$$

$$= \int_0^1 (1-x)^{m-1}x^{n-1}dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx$$

$$= \beta(n,m)$$

Q. 8. (b) (i) Evaluate $\int_0^{2a} x^{3/2}(2a-x)^{1/2}\,dx$ (ii) Evaluate $\int_0^2 x(8-x^3)^{1/3}dx$

Ans. (i)
$$\int_0^{2a} x^{3/2}\sqrt{2a-x}\,dx$$

$$\int_0^{2a} x^{3/2}(2a-x)^{1/2}dx = \int_0^{2a} x^{3/2}(2a)^{1/2}\left(1-\frac{x}{2x}\right)^{1/2}dx$$

$$= (2a)^{1/2}\int_0^{2a} x^{3/2}\left(1-\frac{x}{2a}\right)dx$$

put
$$\frac{x}{2a} = t, \text{ when } x = 0, t = 0$$
$$x = 2a, t = 1$$

$$= (2a)^{1/2}(2a)\int_0^1 (2a)^{3/2}\ {}^{+3/2}(1-t)^{1/2}dt$$

$$= (2a)^3\int_0^1 t^{\left(\frac{5}{2}-1\right)}(1-t)^{1/2}dt = (2a)^3\beta\left(\frac{5}{2},1\right)$$

(ii) $\int_0^2 x(8-x^3)^{1/3}dx$

Put
$$x^3 = 8z \Rightarrow x = 2z^{1/3}$$

$$dx = \frac{2}{3}z^{-2/3}dz$$

When $x = 0, z = 0$ when $x = 2, z = 1$

$$\int_0^1 2z^{1/3}(8-8z)^{1/3}\frac{2}{3}z^{-2/3}dz$$

$$= \frac{4}{3}\int_0^1 z^{-2/3+1/3}2(1-z)^{1/3}dz$$

$$= \frac{8}{3}\int_0^1 z^{-1/3}(1-z)^{1/3}dz$$

$$= \frac{8}{3}\int_0^1 z^{2/3-1}(1-z)^{4/3-1} = \frac{8}{3}\beta\left(\frac{2}{3},\frac{4}{3}\right).$$

Q. 9. (a) If $I_n = \int_0^{\pi/4}\tan^n x\,dx$, show that $I_n + I_{n-2} = \frac{1}{n-1}$ (8)

Ans.
$$I_n = \int_0^{\pi/4}\tan^n x\,dx$$

$$= \int_0^{\pi/4}\tan^{n-2}x\tan^2 x\,dx$$

$$= \int_0^{\pi/4}\tan^{n-2}x(\sec^2 x-1)dx$$

$$= \int_0^{\pi/4}\tan^{n-2}x\sec^2 x\,dx - \int_0^{\pi/4}\tan^{n-2}x\,dx$$

$$= \int_0^{\pi/4}\tan^{n-2}x\sec^2 x\,dx - I_{n-2}$$

put
$$\tan x = z \Rightarrow \sec^2 x\,dx = dz$$

$\therefore$
$$\int\tan^{n-2}x\sec^2 x\,dx = \int z^{n-2}dz = \frac{z^{n-1}}{n-1} = \frac{\tan^{n-1}x}{n-1}$$

$\therefore$
$$I_n = \frac{\tan^{n-1}x}{n-1}\Big|_0^{\pi/4} - I_{n-2}$$

$$= \left[\frac{\tan^{n-1}\frac{\pi}{4}}{n-1} - \frac{\tan^{n-1}0}{n-1}\right] - I_{n-2}$$

$$I_m = \frac{1}{n-1} - I_{n-2}$$

$$\Rightarrow \qquad \boxed{I_n + I_{n-2} = \frac{1}{n-1}}$$

Q. 9. (b) Evaluate $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$ (7)

Ans. $$\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$$

let $$x = \tan z$$
$$dx = \sec^2 z\, dz$$

$$= \int \tan^{-1}\frac{2\tan z}{1-\tan^2 z}\sec^2 z dz$$

$$= \int \tan^{-1}\tan 2z \sec^2 z\, dz$$

$$= \int 2z \sec^2 z\, dz$$

$$= 2\int z \sec^2 z\, dz$$

$$= 2\left[z\int \sec^2 z\, dz - \int\left(\int \sec^2 z\, dz\right)dz\right]$$

$$= 2\left[z.\tan z - \int \tan z\, dz\right]$$

$$= 2z\tan z - 2\int\frac{\sin z}{\cos z}dz$$

$$= 2z\tan z + 2\log\cos z$$
$$= 2\tan^{-1}x \tan\tan^{-1}x + 2\log\cos\tan^{-1}x$$
$$= 2(\tan^{-1}x)x + 2\log\cos\tan^{-1}x.$$