SCIENCE CHINA
Information Sciences

• REVIEW •
Special Focus on Multi-source Information Fusion

November 2020, Vol. 63 210201:1–210201:19
https://doi.org/10.1007/s11432-020-3006-9

Uncertainty measure in evidence theory

Yong DENG[1,2]

[1]*The Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu 610054, China;*
[2]*School of Education, Shannxi Normal University, Xi'an 710062, China*

Received 27 May 2020/Accepted 1 July 2020/Published online 20 October 2020

Abstract As an extension of probability theory, evidence theory is able to better handle unknown and imprecise information. Owing to its advantages, evidence theory has more flexibility and effectiveness for modeling and processing uncertain information. Uncertainty measure plays an essential role both in evidence theory and probability theory. In probability theory, Shannon entropy provides a novel perspective for measuring uncertainty. Various entropies exist for measuring the uncertainty of basic probability assignment (BPA) in evidence theory. However, from the standpoint of the requirements of uncertainty measurement and physics, these entropies are controversial. Therefore, the process for measuring BPA uncertainty currently remains an open issue in the literature. Firstly, this paper reviews the measures of uncertainty in evidence theory followed by an analysis of some related controversies. Secondly, we discuss the development of Deng entropy as an effective way to measure uncertainty, including introducing its definition, analyzing its properties, and comparing it to other measures. We also examine the concept of maximum Deng entropy, the pseudo-Pascal triangle of maximum Deng entropy, generalized belief entropy, and measures of divergence. In addition, we conduct an analysis of the application of Deng entropy and further examine the challenges for future studies on uncertainty measurement in evidence theory. Finally, a conclusion is provided to summarize this study.

Keywords evidence theory, uncertainty measure, Deng entropy, Shannon entropy

Citation Deng Y. Uncertainty measure in evidence theory. Sci China Inf Sci, 2020, 63(11): 210201, https://doi.org/10.1007/s11432-020-3006-9

1 Introduction

Uncertainty plays a fundamental role in real life, such as decision-making [1, 2], information fusion [3–5], reliability analysis [6, 7], and other applications [8, 9]. How to better handle uncertainty has attracted the attention of many researchers over the past few years [10]. Currently, many methodologies exist for describing uncertain information, such as fuzzy sets [11], rough sets [12], Dempster-Shafer evidence theory (D-S evidence theory) [13, 14], and so on [15–17]. Many of these tools can be transformed into a framework on evidence theory. D-S evidence theory has attracted much attention because it can better represent uncertain information by using basic probability assignment (BPA) and implementing uncertainty reasoning [18]. D-S evidence theory not only provides an elegant mathematical framework for modeling uncertainty, but also offers a combination method to fuse information obtained from different sensors. Owing to its effectiveness in modeling and handling uncertainty, D-S evidence theory has been generalized in many aspects, such as complex evidence theory [19, 20], which provides a new perspective for uncertainty modeling. Conversely, Yang and Xu [21, 22] extended evidence theory to multiple attribute decision analysis and developed systematic methods of uncertainty reasoning [23].

Email: dengentropy@uestc.edu.cn, prof.deng@hotmail.com

As a form of informational expression, BPAs are inevitably subject to uncertainty, possibly owing to multiple sources, so quantifying this uncertainty precisely is a crucial issue in the application of evidence theory because it determines the accuracy of subsequent fusion and reasoning. Shannon entropy [24] provides a measure of uncertainty for probability theory that has been used in many fields. Similarity, fuzzy sets and rough sets also feature their own uncertainty measures. Although a number of methods exist for measuring BPA uncertainty, this is a challenging subject and scholars have not yet agreed upon a measurement standard, making this an open research issue. Therefore, this paper reviews various measurements for uncertainty in evidence theory, which can help us understand uncertainty and apply evidence theory to additional fields.

Some scholars measure the uncertainty of BPA using entropy. Deng entropy [25] has a strong ability to measure the uncertainty of basic BPAs and can be effectively applied in many fields, such as fuzzy multi-criteria decision-making [26]. Some scholars measure the uncertainty of evidence from the perspective of generalized informational quality [27]. Other scholars measure the uncertainty of BPAs from the perspective of divergence. For example, the work of [28] generalized classical Jenson-Shannon divergence to belief functions; its successful application to medical diagnostics verifies its validity [29]. Owing to the lack of consideration for the relationship between focal elements in [28], a new divergence measurement [30] for belief functions was proposed to measure the uncertainty of BPAs. The work of [31] measured the uncertainty of BPAs from the perspective of complex evidence distance.

To date, no overview has investigated the studies related to Deng entropy after reviewing various uncertainty information measures. To fill this gap and increase scholars' in-depth understanding of the uncertainty measures represented by Deng entropy, this paper has performed an in-depth study. The contributions of this paper are as follows.

(1) We review the classical uncertainty measures used in evidence theory. We first introduce the basic concepts in evidence theory, describe 17 classical uncertainty measures, and then analyze the properties that belief entropy should have. We conclude this section by discussing some current controversies regarding uncertainty measures.

(2) This paper focuses on an excellent uncertainty measure, Deng entropy, and analyzes its development process, including its proposal, its properties, its superiority through comparison, its maximum properties, its properties on the pseudo-Pascal triangle, its generalized form, and its divergence.

(3) We present the application of Deng entropy in different fields, including multi-sensor information fusion, fault diagnosis, failure mode, and effects analysis, and analyze the developmental status and future trends in the application of Deng entropy.

(4) We analyze the challenges for future studies on uncertainty measures in evidence theory, including the development of an uncertainty measurement theory, challenges in the theoretical development of Deng entropy, and challenges in the exploration of Deng entropy's physical significance.

The remaining contents are organized as follows. Section 2 briefly reviews some typical measures and open issues in the literature. In Section 3, the development of Deng entropy is introduced. An analysis of the application of Deng entropy is provided in Section 4. Section 5 analyzes the challenges faced by uncertainty measurement research in evidence theory. Section 6 provides a conclusion to the study.

2 Review of typical uncertainty measures

In this section, some basic concepts in evidence theory are introduced first, and then an overview of entropy functions for BPA is conducted, followed by some property analysis. In addition, a number of issues with the current entropies are illustrated by employing several practical examples.

2.1 Basic concepts in evidence theory

BPA. Let X be a finite set, also known as the frame of discernment.

A mass function on X is defined as a function $m : 2^X \to [0,1]$ such that $m(\phi) = 0$ and

$$\sum_{A \subseteq X} m(A) = 1. \tag{1}$$

The subsets $A \in 2^X$ such that $m(A) > 0$ are called focal elements. Note that a mass function is also called a BPA.

Belief function. Let any subset of X be A; the sum of the basic belief corresponding to all subsets of A is referred to as belief function, defined as

$$\mathrm{Bel}(A) = \sum_{B \subseteq A} m(B), \quad \forall A \subseteq X. \tag{2}$$

Plausibility function. Let any subset of X be A; the plausibility function $\mathrm{Pl}(A)$ represents the non-false trust of A, and is defined as

$$\mathrm{Pl}(A) = \sum_{B \cap A \neq \phi} m(B) = 1 - \mathrm{Bel}(\overline{A}), \quad \forall A \subseteq X. \tag{3}$$

Commonality function. The commonality function corresponding to BPA m is defined as

$$Q(A) = \sum_{B \subseteq X, B \supseteq A} m(B), \quad \forall A \subseteq X. \tag{4}$$

This concludes our brief review of evidence theory, and we discuss some existing entropy functions.

2.2 The previous entropies of BPAs

Since the evidence theory was proposed, researchers have proposed a variety of entropy functions to measure the uncertainty of BPA, which are summarized as follows.

Definition 1 (Höhle entropy). Höhle [32] first defined the entropy of BPAs in evidence theory as

$$C_{\mathrm{H}}(m) = -\sum_{A \subseteq X} m(A) \log_2 \mathrm{Bel}(A), \tag{5}$$

where $m(A)$ is a BPA in the frame of discernment X, and $\mathrm{Bel}(A)$ is the belief function in X.

Definition 2 (Smets's entropy). Smets [33] defined the entropy of BPAs by commonality function $Q(A)$ in X:

$$H_{\mathrm{s}}(m) = -\sum_{A \subseteq X} m(A) \log_2 Q(A), \tag{6}$$

where Q is the commonality function corresponding to BPAs.

Definition 3 (Yager's dissonance measure). Yager [34] proposed dissonance measure of BPAs by plausibility function $\mathrm{Pl}(A)$ in X:

$$E_{\mathrm{Y}}(m) = -\sum_{A \subseteq X} m(A) \log_2 \mathrm{Pl}(A). \tag{7}$$

Definition 4 (Weighted Hartley entropy). Dubois and Prade [35] defined weighted Hartley entropy as

$$l_{\mathrm{DP}}(m) = \sum_{A \subseteq X} m(A) \log_2 |A|, \tag{8}$$

where $|A|$ is the cardinality of A.

Definition 5 (Lamata & Moral's entropy). Based on $E_{\mathrm{Y}}(m)$ and $l_{\mathrm{DP}}(m)$, Lamata and Moral [36] proposed an entropy of BPAs as

$$H_{\mathrm{l}}(m) = E_{\mathrm{Y}}(m) + l_{\mathrm{DP}}(m). \tag{9}$$

Definition 6 (Klir & Ramer's discord). Klir and Ramer [37] proposed the discord measure of BPAs in X as

$$D_{\mathrm{KR}}(m) = -\sum_{A \subseteq X} m(A) \log_2 \sum_{B \subseteq X} m(B) \frac{|A \cap B|}{|B|}. \tag{10}$$

Definition 7 (Klir & Parviz's strife). Klir and Parviz [38] defined the strife measure in X as

$$D_{\mathrm{KP}}(m) = -\sum_{A \subseteq X} m(A) \log_2 \sum_{B \subseteq X} m(B) \frac{|A \cap B|}{|A|}. \tag{11}$$

Definition 8 (The entropy of Pal et al.). Pal et al. [39, 40] defined an entropy function based on the cardinality of A as

$$H_{\mathrm{P}}(m) = -\sum_{A \subseteq X} m(A) \log_2 \frac{m(A)}{|A|}. \tag{12}$$

Definition 9 (George & Pal's conflict measure). George and Pal [41] proposed the conflict measure of BPAs in X:

$$D_{\mathrm{GP}}(m) = -\sum_{A \subseteq X} m(A) \log_2 \sum_{B \subseteq X} m(B) \left[1 - \frac{|A \cap B|}{|A \cup B|}\right]. \tag{13}$$

Definition 10 (The entropy of Jousselme et al.). Jousselme et al. [42] defined an entropy function based on the idea of pignistic transformation in X:

$$H_{\mathrm{J}}(m) = -\sum_{A \subseteq X} \mathrm{Bet}_m(A) \log_2 \mathrm{Bet}_m(A), \tag{14}$$

where $\mathrm{Bet}_m(A) = \sum_{B \subseteq X} m(B) \frac{|A \cap B|}{|B|}, \forall A \in X$.

Definition 11 (The entropy of Jiroušek & Shenoy). Jiroušek and Shenoy [43] defined an entropy function of BPAs by the idea of plausibility transformation in X as

$$H_{\mathrm{JS}}(m) = -\sum_{A \in X} \mathrm{Pl}\_P_m(A) \log_2 \mathrm{Pl}\_P_m(A) + \sum_{B \subseteq 2^X} m(B) \log_2 |B|, \tag{15}$$

where $\mathrm{Pl}\_P_m(A) = K^{-1} \times \mathrm{Pl}(A)$ for all $A \in X$, and $K = \sum_{x \in X} \mathrm{Pl}(x)$.

Definition 12 (The entropy of Pan et al.). Pan et al. [44] defined an entropy function of BPAs by plausibility transformation and weighted Hartley entropy:

$$H_{\mathrm{PQ}}(m) = -\sum_{A \subseteq X} m(A) \log_2 \mathrm{Pm}(A) + \sum_{A \subseteq X} m(A) \log_2 |A|, \tag{16}$$

where $\mathrm{Pm}(A) = \sum_{x \in A} \mathrm{Pt}(x)$, and $\mathrm{Pt}(x) = \frac{\mathrm{Pl}(x)}{\sum_{x \in X} \mathrm{Pl}(x)}$.

Definition 13 (The uncertainty measure of Wen et al.). Wen et al. [45] proposed a novel uncertainty measure of BPAs by exponential form, which is different from the existing uncertainty measure:

$$U_{\exp}(m) = \frac{\mathrm{e} - \sum_{A \subseteq X, |A|=1} m(A) \mathrm{e}^{m(A)} - \sum_{A \subseteq X, |A| \neq 1} \frac{m(A)}{|A| \times |C|} \mathrm{e}^{\frac{m(A)}{|A| \times |C|}}}{\mathrm{e} - \frac{1}{|X|^2} \mathrm{e}^{\frac{1}{|X|^2}}}, \tag{17}$$

where C is called Core that is calculated as $C = \cup_{i=1}^{q} \{A_i | m(A_i) > 0\}$. Note that q indicates the number of focal elements.

Definition 14 (The uncertainty measure of Song & Wang). Song and Wang [46] proposed an uncertainty measure based on interval probabilities as follows:

$$\mathrm{SU}_{\mathrm{SW}}(m)=\sum_{A\in X}\left[-\frac{\mathrm{Bel}(A)+\mathrm{Pl}(A)}{2}\log_2\frac{\mathrm{Bel}(A)+\mathrm{Pl}(A)}{2}+\frac{\mathrm{Pl}(A)-\mathrm{Bel}(A)}{2}\right]. \tag{18}$$

Definition 15 (The distance-based total uncertainty measures of Yang & Han). Yang and Han [47] defined a distance-based total uncertainty measure in terms of belief interval of each singleton:

$$\mathrm{TU}^I(m)=1-\frac{1}{|X|}\cdot\sqrt{3}\cdot\sum_{A\in X}d^I\left([\mathrm{Bel}(A),\mathrm{Pl}(A)],[0,1]\right) \tag{19}$$

with

$$d^I\left([a_1,b_1],[a_2,b_2]\right)=\sqrt{\left[\frac{a_1+b_1}{2}-\frac{a_2+b_2}{2}\right]^2+\frac{1}{3}\left[\frac{b_1-a_1}{2}-\frac{b_2-a_2}{2}\right]^2}. \tag{20}$$

Definition 16 (The total uncertainty measure of Deng et al.). Deng et al. [48,49] proposed an improved belief interval-based total uncertainty measure on the basis of Yang and Han's work to get a better performance in monotonicity:

$$\mathrm{TU}_E^I(m)=\sum_{A\in X}\left[1-d_E^I\left([\mathrm{Bel}(A),Pl(A)],[0,1]\right)\right] \tag{21}$$

with

$$d_E^I\left([a_1,b_1],[a_2,b_2]\right)=\sqrt{(a_1-a_2)^2+(b_1-b_2)^2}. \tag{22}$$

Moreover, the total uncertainty measure TU_E^I has been further migrated to measure the uncertainty of D numbers [50, 51].

In addition, many of other researchers have also proposed the uncertainty measures for interval-valued belief structures. For instance, Yager [52] proposed interval-valued measures for belief structures.

Definition 17 (The total uncertainty measure of interval-valued belief structures). The measures of entropy introduced by Yager [34] and Höhle [32] are used to provide the lower and upper bounding values on BPAs as

$$S(m)=\left[S_{\mathrm{Y}}(m),S_{\mathrm{H}}(m)\right], \tag{23}$$

where $S_{\mathrm{Y}}(m)=-\sum_{A\in X}m(A)\log_2\mathrm{Pl}(A)$ and $S_{\mathrm{H}}(m)=-\sum_{A\in X}m(A)\log_2\mathrm{Bel}(A)$.

2.3 Analysis of the properties of existing entropy functions

In 1999, Klir and Wierman [53] defined five properties of uncertainty measures in evidence theory from the perspective of generalized information theory. They are probabilistic consistency, set consistency, maximum entropy, sub-additivity, and additivity. We analyze the above entropies according to these five properties and list the results in Table 1.

As can be seen in Table 1, almost all uncertainty measures satisfy the property of probabilistic consistency, when BPA is a Bayesian distribution, both $H_{\mathrm{s}}(m)$ and $l_{\mathrm{DP}}(m)$ are different from the Shannon entropy of the Bayesian distribution. More than 2/3 of the uncertainty measures satisfy the property of set consistency. Obviously, there are only three uncertainty measures that satisfy the property of maximum entropy. All of the uncertainty measures satisfy the property of additivity but most do not satisfy the property of sub-additivity. Additional details can be found in [43]. Here, we introduce some controversies regarding uncertainty measures in evidence theory, which leads to a discussion on the concept of Deng entropy.

Table 1 A summary of the properties of entropies in evidence theory

Proposer	Probability consistency	Set consistency	Maximum entropy	Additivity	Subadditivity
Höhle [32]	Yes	No	No	Yes	No
Smets [33]	No	No	No	Yes	No
Yager [34]	Yes	No	No	Yes	No
Dubois and Prade [35]	No	Yes	Yes	Yes	Yes
Lamata and Moral [36]	Yes	Yes	No	Yes	No
Klir and Ramer [37]	Yes	Yes	No	Yes	No
Klir and Parviz [38]	Yes	Yes	No	Yes	No
Pal et al. [39, 40]	Yes	Yes	No	Yes	No
George and Pal [41]	Yes	Yes	No	Yes	No
Jousselme et al. [42]	Yes	Yes	No	Yes	No
Jiroušek and Shenoy [43]	Yes	Yes	Yes	Yes	No
Pan et al. [44]	Yes	Yes	Yes	Yes	No

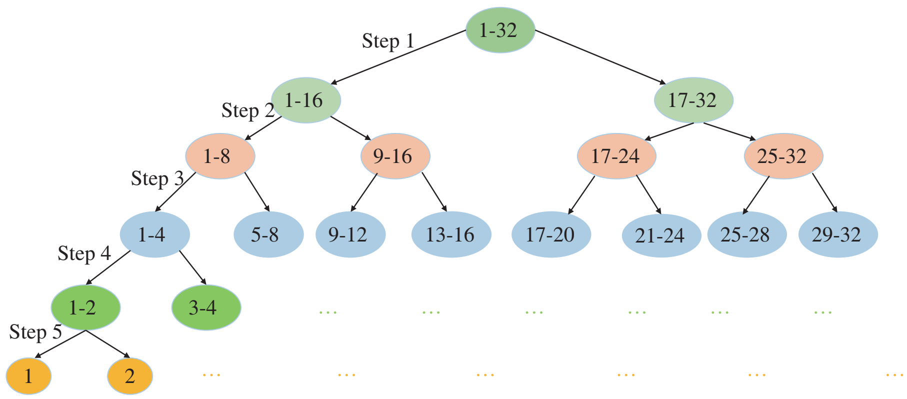

Figure 1 (Color online) The question of the team's championship.

2.4 Some controversies regarding uncertainty measures

In Subsection 2.2, various uncertainty measures are listed, which are analyzed using the five requirements proposed by Klir et al. [53]. The results demonstrate that the property of maximum entropy is violated by most of the measures. We provide a detailed analysis of this issue as follows.

Shannon entropy is often applied as informational capacity or an information boundary. A classic binary search problem is presented to demonstrate the role of Shannon entropy. 32 football teams compete and a champion will eventually emerge. A person guesses without knowing who the winner is. He/She can ask an informed person: "Is T the winner?" The informed person can only answer "yes" or "no" until the correct answer is provided. Now, the question is what is the maximum number of times the person can guess the championship. The answer is five. The process is shown in Figure 1. Since 32 teams are likely to win the championship, the result is 5, according to a Shannon entropy calculation based on the logarithmic function of 2. So, when encoded in binary form, it is $00000, 00001, 00010, \ldots$. That is, a 5-bit binary code can represent the information regarding the 32 teams and a computer can store the information in 5 bits. This shows that Shannon entropy reflects informational capacity or the information boundary and also exemplifies Shannon entropy as the basis of information theory. Klir et al. [54] believed that for all uncertain information, the information boundary (or, the maximum entropy) is the maximum Shannon entropy $\log_2 |X|$.

However, what happens when the application scenario changes; e.g., if 32 students participate in a 100-point test instead of 32 teams, how many times would it take to guess who wins the first place? 5 times is definitely not correct, because there may exist a situation such as more than one student takes first place at the same time. This kind of simultaneous situation, corresponding to mathematical multi-value mapping, is simply the suitable description form of a multi-subset proposition in evidence theory.

Therefore, in this case, the information boundary of a reasonable belief entropy should be more than 5. Because BPAs are more uncertain than the probability distribution, they require more informational capacity to describe them. The result is like this: Is student A in the first place? Is student B in the first place? If you repeat the question 32 times, you will obtain every possible first place student. Obviously, existing uncertainty information research indicates that it is necessary to break through the informational capacity boundary of Shannon entropy.

In addition, entropy is an extension of thermodynamics and earlier studies have found that entropy is an extended quantity, i.e., the value of the total system is equal to the sum of the values of each sub-system. It is often said that entropy is additive. However, existing experimental results indicate that there exists a real system whose entropy function is nonextended [55]. When Tsallis entropy [56] was proposed, researchers gradually realized that the nonadditivity of entropy is general and universal, while the additivity can only be satisfied under certain special conditions. Therefore, the definition of a reasonable entropy is still an open issue. The proposal of Deng entropy [25] provides a more satisfactory answer to this question.

3 An analysis of the developments of Deng entropy

Inspired by Shannnon entropy, Deng [25] proposed the concept of Deng entropy by taking the effect of cardinality of focal elements in BPA into account in the measurement of uncertainty.

3.1 The definition of Deng entropy

Definition 18 (Deng entropy). Deng [25] proposed Deng entropy, defined as

$$E_{\mathrm{D}} = -\sum_{A\subseteq X} m\left(A\right) \log_2 \frac{m(A)}{2^{|A|}-1}, \tag{24}$$

where m is a BPA defined on frame of discernment X and $|A|$ is the cardinality of A.

As above definition, when BPA degenerates into probability, i.e., $m(A) = 0$ if $|A| > 1$, Deng entropy also degenerates into classical Shannon entropy. Deng entropy can be divided into a measure of total nonspecificity, $m(A)\log_2(2^{|A|-1})$, and the discord, $-m(A)\log_2(m(A))$, of BPA among various focal elements. Through a simple transformation, it is found that Deng entropy is actually a type of composite measures, defined as

$$E_{\mathrm{D}} = \sum_{A\subseteq X} m(A)\log_2(2^{|A|}-1) - \sum_{A\subseteq X} m(A)\log_2(m(A)). \tag{25}$$

Deng entropy has attracted more and more attention [57–61] and has been improved from multiple perspectives [62–66] and applied in different fields [67–71]. For more details on Deng entropy, please refer to [25].

3.2 The properties of Deng entropy

Abellán [72] discussed some properties of Deng entropy, which are described below. For more information, please refer to [72].

Proposition 1 (Probabilistic consistency). Deng entropy satisfies the probabilistic consistency. Deng entropy can degenerate into classical Shannon entropy when BPA is a particular form, known as a probability distribution.

Proposition 2 (Set consistency). Deng entropy cannot satisfy the set consistency. The set consistency can be broken owing to the existing of $\log_2(2^{|A|}-1)$.

Proposition 3 (Range). Deng entropy can expand the range of Shannon entropy $\log_2(n)$. The range was broken mainly considering the number of focal elements in BPA.

Proposition 4 (Subadditivity)**.** Deng entropy does not verify the subadditivity property. There are some examples to explain this property.

Proposition 5 (Additivity)**.** Deng entropy is not also additivity. Deng entropy is non-additivity owing to the existing of $2^{|A|} - 1$.

Proposition 6 (Monotonicity)**.** Deng entropy also does not verify the monotonicity property.

3.3 Comparison of Deng entropy with other entropies

In this subsection, the advantages of Deng entropy in measuring uncertainty can be revealed by comparing it with other existing entropy functions based on several numerical examples.

Case 1. Let the frame of discernment be $X = \{1, 2, \ldots, 15\}$, and a BPA is given as $m(\{3, 4, 5\}) = 0.05, m(\{6\}) = 0.05, m(A) = 0.8, m(X) = 0.1$, where proposition A is a variable that varies from $A = \{1\}$ to $A = \{1, 2, \ldots, 14\}$. Common sense tells us that as the cardinality of A increases, the uncertainty of m increases; i.e., the entropy increases. Now, we examine whether the uncertainty measures of various entropy functions for m conform to this rule. Based on Figure 2, the results of the four entropies meet the reality, and they are weighted Hartley entropy, Lamata & Moral's entropy, the entropy of Pal et al. and Deng entropy, while the other entropy functions cannot accurately capture the change of uncertainty.

Case 2. Let the frame of discernment be $X = \{x_1, x_2, \ldots x_n\}$, and a BPA is given as $m(x_i) = 1/n$. It can be inferred that the uncertainty of m increases as n increases. Similarly, we explore the performance of various uncertainty measures in such cases. In Figure 3, the curves that lead to unreasonable results are marked as dashed lines. It is worth noting that the weighted Hartley entropy, which worked in Case 1, gets the wrong result in this case, showing that it cannot adapt to uncertainty measures in all cases. And Deng entropy, which has been doing well in Case 1, still works.

Case 3. Let the frame of discernment be $X = \{x_1, x_2\}$, and a BPA is given as $m(x_1) = a, m(x_2) = b, m(X) = 1 - a - b,\ a, b \in [0, 0.5]$. We take all the values of a and b in a given range to explore the performance of various uncertainty measures, and the results are manifested in Figure 4. Several subgraphs clearly do not fit the facts, such as Figures 4(a)–(c), (f)–(i) and (l). For other results, we mainly study the case where maximum entropy is obtained, i.e., when the maximum uncertainty is reached. Four categories can be obtained through analysis: (1) maximum entropy is obtained when $m(X) = 1$, including Figures 4(d), (k), (n), (o), (p); (2) maximum entropy is obtained when $m(x_1) = 0.5, m(x_2) = 0.5$, including Figure 4(m); (3) maximum entropy is obtained when $m(X) = 1$ and $m(x_1) = 0.5, m(x_2) = 0.5$, including Figures 4(e) and (j); (4) Deng entropy. How belief is distributed to maximize entropy is a very important issue in evidence theory, which is considered to be the meaning of information entropy. Based on Deng entropy, it can be obtained that when $m(x_1) = 0.2, m(x_2) = 0.2, m(X) = 0.6$, it has the greatest uncertainty. The rationality of this conclusion would be introduced in Subsection 3.4.

3.4 The maximum Deng entropy

Based on Subsection 3.1, we discuss the concept of maximum Deng entropy, which plays an essential role in information theory and has been used in many fields. Kang et al. [73] discussed the condition of Deng entropy maximization based on Lagrange multiplier method. The specific distribution is as follows.

Definition 19 (The maximum Deng entropy)**.** When a BPA satisfies the following condition, the Deng entropy has the maximum value [73].

$$m(A_i) = \frac{2^{|A_i|} - 1}{\sum_i 2^{|A_i|} - 1}. \tag{26}$$

The Lagrange function can be defined as

$$D_0 = -\sum_i m(A_i) \log_2 \left(\frac{m(A_i)}{2^{|A_i|} - 1} \right) + \lambda(m(A_i) - 1).$$

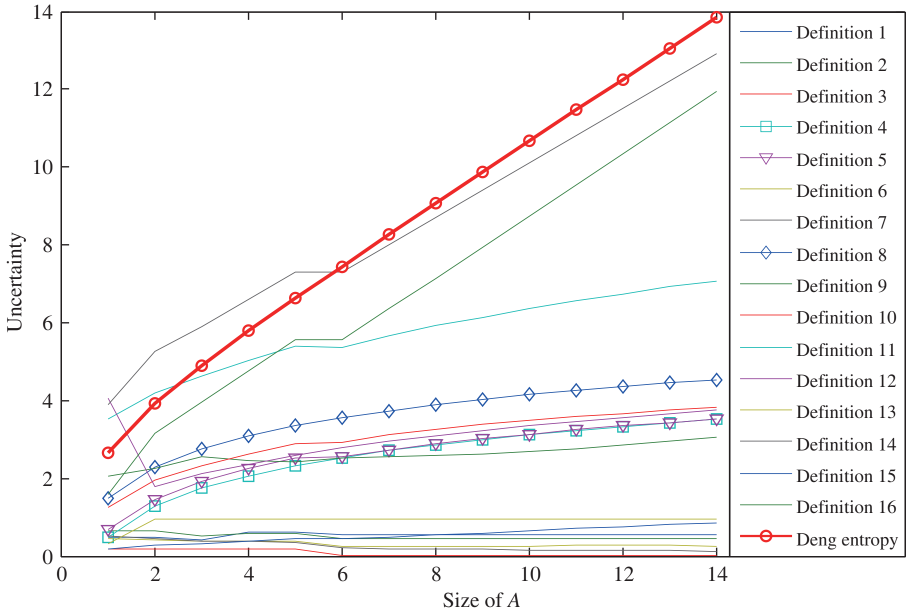

Figure 2 (Color online) Comparison results of Case 1.

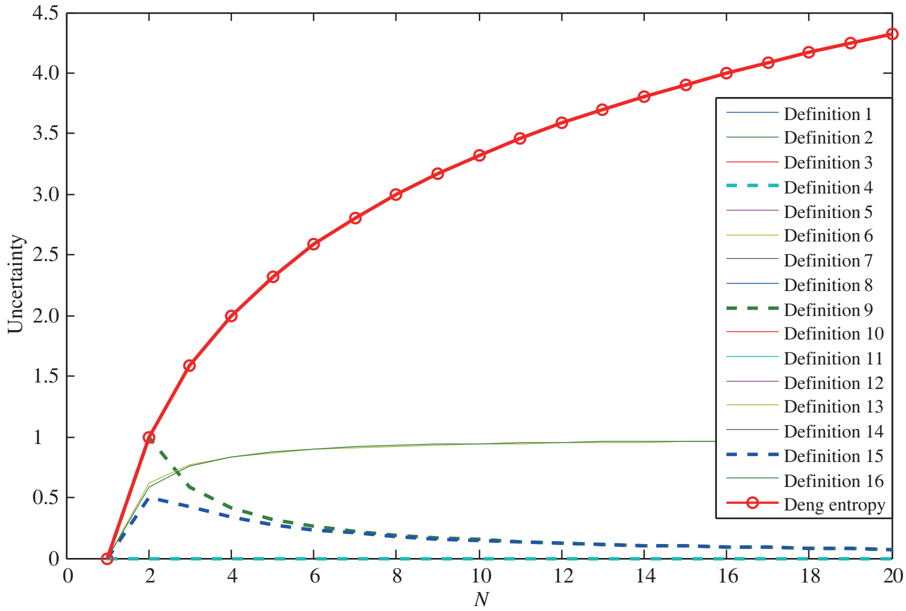

Figure 3 (Color online) Comparison results of Case 2.

Now by calculating the gradient of Lagrange function to obtain the BPA and the maximum Deng entropy,

$$\frac{\partial D_0}{\partial m(A_i)} = -\log\frac{m(A_i)}{2^{|A_i|}-1} - m(A_i)\frac{1}{\frac{m(A_i)}{2^{|A_i|-1}}\ln a}\cdot\frac{1}{2^{|A_i|}-1} + \lambda.$$

Besides, Kang et al. discussed the change of the maximum Deng entropy with the different scales of frame of discernment. It can be seen that the maximum Deng entropy shows a positive proportion trend with the change of N that represents the number of elements in the frame of discernment. Therefore, in Case 3, the belief distribution conforming to the maximum Deng entropy is obtained as $\{(x_1), 0.2; (x_2), 0.2; (x_1x_2), 0.6\}$, and the maximum Deng entropy can be calculated based on (29) as 2.3219. Going back to the example in Subsection 2.4 of guessing the first place in the exam. For a test with 32 students, one needs to guess the first place at most 50.7188 times. In addition, Deng [74] pointed

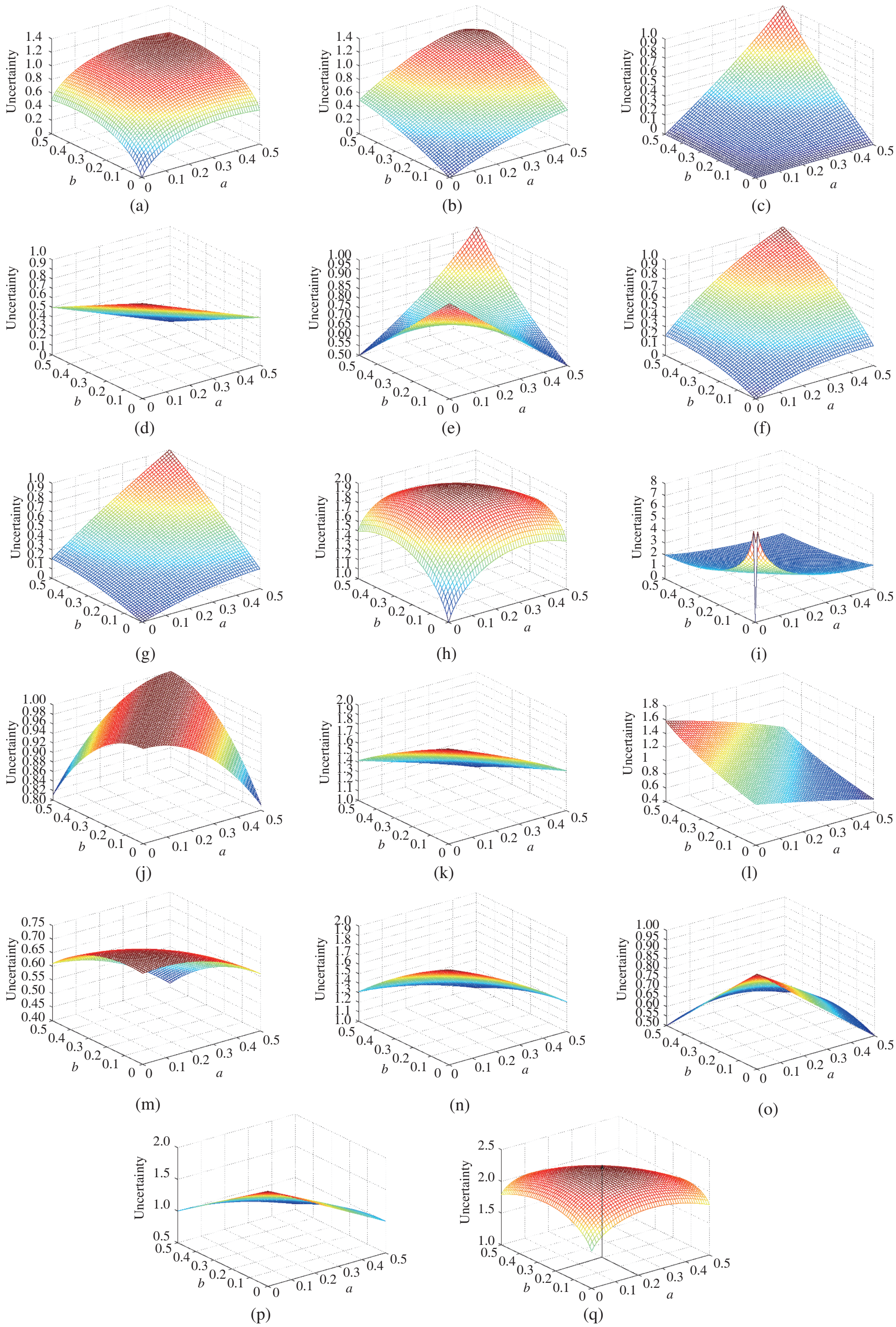

Figure 4 (Color online) Comparison results of Case 3. (a) Definition 1; (b) definition 2; (c) definition 3; (d) definition 4; (e) definition 5; (f) definition 6; (g) definition 7; (h) definition 8; (i) definition 9; (j) definition 10; (k) definition 11; (l) definition 12; (m) definition 13; (n) definition 14; (o) definition 15; (p) definition 16; (q) Deng entropy.

out that the information volume of mass function is constructed with high ordered Deng entropy, which satisfies multi-fractal.

Table 2 The belief distribution of maximum Deng entropy [76]

Frame of discrement	$\|X\|=1$	$\|X\|=2$	$\|X\|=3$	$\|X\|=4$
$\Theta=\{A\}$	1	–	–	–
$\Theta=\{A,B\}$	$(\frac{1}{5},\frac{1}{5})$	$\frac{3}{5}$	–	–
$\Theta=\{A,B,C\}$	$(\frac{1}{19},\frac{1}{19},\frac{1}{19})$	$(\frac{3}{19},\frac{3}{19},\frac{3}{19})$	$\frac{7}{19}$	–
$\Theta=\{A,B,C,D\}$	$(\frac{1}{65},\frac{1}{65},\frac{1}{65},\frac{1}{65})$	$(\frac{3}{65},\frac{3}{65},\frac{3}{65},\frac{3}{65},\frac{3}{65},\frac{3}{65})$	$(\frac{7}{65},\frac{7}{65},\frac{7}{65},\frac{7}{65})$	$\frac{15}{65}$

Table 3 The Pseudo-Pascal triangle of the maximum Deng entropy [76]

$2^{\|m\|-1}$	$2^{\|1\|-1}$	$2^{\|2\|-1}$	$2^{\|3\|-1}$	$2^{\|4\|-1}$	$2^{\|5\|-1}$	$2^{\|6\|-1}$	$2^{\|7\|-1}$	$2^{\|8\|-1}$	$2^{\|9\|-1}$	$2^{\|10\|-1}$	$2^{\|11\|-1}$	$2^{\|12\|-1}$
$N=1$	1	–	–	–	–	–	–	–	–	–	–	–
$N=2$	2	1	–	–	–	–	–	–	–	–	–	–
$N=3$	3	3	1	–	–	–	–	–	–	–	–	–
$N=4$	4	6	4	1	–	–	–	–	–	–	–	–
$N=5$	5	10	10	5	1	–	–	–	–	–	–	–
$N=6$	6	15	20	15	6	1	–	–	–	–	–	–
$N=7$	7	21	35	35	21	7	1	–	–	–	–	–
$N=8$	8	28	56	70	56	28	8	1	–	–	–	–
$N=9$	9	36	84	126	126	84	36	9	1	–	–	–
$N=10$	10	45	120	210	252	210	120	45	10	1	–	–
$N=11$	11	55	165	330	462	462	330	165	55	11	1	–
$N=12$	12	66	220	495	792	924	792	495	220	66	12	1

3.5 The Pseudo-Pascal triangle of maximum Deng entropy

As we all known, Shannon entropy has the maximum value when the probability satisfies uniform distribution. Tasllis entropy has been modelled with Pascal triangle [75]. What about the maximum Deng entropy? For this issue, Gao et al. [76] conducted the relationship between Pseudo-Pascal triangle and the maximum Deng entropy by discussing the distribution of BPAs. First, the distribution of the maximum Deng entropy is shown in Table 2.

Next, Gao et al. established the connection between Pseudo-Pascal triangle of maximum Deng entropy, as shown in Table 3. It can be seen that the BPAs have the connection with the number of frame of discernment, which can help us analyze the physical meaning of Deng entropy. From the maximum Deng entropy, it can be seen that the value of BPA is equal with the same number of focal elements in the same frame of discernment, which presents a phenomenon of stratification.

3.6 Generalized belief entropy

To study the relationship between Deng entropy, Renyi entropy and Tsallis entropy, Liu et al. [77] proposed different entropies, such as Renyi-Deng (R-D) entropy, Tsallis-Deng (T-D) entropy, and Renyi-Tsallis-Deng (R-T-D) entropy. The following is a brief introduction, with specific details referred to [77].

Definition 20 (Renyi-Deng entropy [77]). The R-D entropy is defined as

$$E_\alpha(m(A_i)) = \frac{1}{1-\alpha}\ln\left[\sum_i\left(\frac{m(A_i)}{2^{|A_i|}-1}\right)^\alpha(2^{|A_i|}-1)\right]. \tag{27}$$

Definition 21 (Tsallis-Deng entropy [77]). The T-D entropy is defined as

$$E_q(m(A_i)) = \frac{1}{q-1}\left[1-\sum_i\left(\frac{m(A_i)}{2^{|A_i|}-1}\right)^q(2^{|A_i|}-1)\right]. \tag{28}$$

Definition 22 (Renyi-Tsallis-Deng entropy [77]). The R-T-D entropy is defined as

$$E_{t,r}(m(A_i)) = \frac{1}{1-r}\left[\left[\sum_i\left(\frac{m(A_i)}{2^{|A_i|}-1}\right)^t(2^{|A_i|}-1)\right]^{\frac{1-r}{1-t}}-1\right]. \tag{29}$$

Table 4 Distribution of the applications of Deng entropy

Application subsector	Frequency	Percentage (%)	Reference
Multi-sensor information fusion	16	36.17	[28, 77, 78, 81–94]
Fault diagnosis	5	10.64	[95–99]
Pattern recognition	5	10.64	[100–104]
Failure mode and effects analysis	4	8.51	[105–108]
Risk assessment	3	6.38	[109–111]
Multi-criteria decision-making	3	6.38	[26, 112, 113]
Decision-making	2	4.26	[114, 115]
Emergency management	2	4.26	[116, 117]
Quantum decision	2	4.26	[118, 119]
Organizational management	1	2.13	[120]
IoT applications	1	2.13	[121]
Language model	1	2.13	[122]
Medical diagnosis	1	2.13	[98]

Besides, Ref. [77] also discussed the maximum entropy and the applications of proposed entropies. Liu et al. discussed the influence of the change of parameters α, r and t, and analyzed the relationship between different entropies.

3.7 Divergence based on Deng entropy

Divergence plays an essential role to measure the difference between two BPAs. Song et al. proposed a divergence based on Deng entropy. Next, it can be briefly introduced, more detail refer to [78].

Definition 23 (Divergence based on Deng entropy [78])**.** Given two BPAs m_1 and m_2, the divergence between m_1 and m_2 is defined as

$$\mathrm{Div}(m_1, m_2) = \sum_i \frac{1}{2^{|F_i|}-1} m_1(F_i) \log\left(\frac{m_1(F_i)}{m_2(F_i)}\right). \tag{30}$$

Besides, to consider symmetry, Song et al. [78] proposed a symmetrical divergence as

$$D(m_1, m_2) = D(m_2, m_1) = \frac{\mathrm{Div}(m_1, m_2) + \mathrm{Div}(m_2, m_1)}{2}. \tag{31}$$

In addition, Song et al. [78] discussed the properties of proposed divergence. Furthermore, the proposed divergence can better handle more uncertainty and high conflict conditions by comparing with existing belief divergence measures.

Deng entropy has attracted many researchers attentions. Gao et al. [79] proposed a new uncertainty measure by considering the q in Tsallis entropy changes with the number of fractal elements in BPAs and using Deng entropy to consider the influence of the number of fractal elements on uncertainty measure. Besides, they revealed the increase trend of power set with the increase of the framework of discernment. Pan et al. [80] proposed a new entropy based on Deng entropy by considering the interval probability. Owing to the benefits of Deng entropy, it has been used in many fields. Next, the applications of Deng entropy will be simply introduced.

4 An analysis of the applications of Deng entropy

In this section, the applications of Deng entropy are analyzed, including the following fields: multi-sensor information fusion, fault diagnosis, pattern recognition, decision-making, risk assessment and other applications. Table 4 [26,28,77,78,81–122] lists the frequency of publications in different application areas, as detailed below.

4.1 Applications in multi-sensor information fusion

A lot of studies on the multi-source information fusion based on Deng entropy have been conducted. For instance, for the processing of conflicting evidence, the similarity measure [83], evidence weighting idea [85], divergence measure [28, 91] and IOWA operator [93] were combined with Deng entropy respectively. And in [90], an evidential aggregation method of intuitionistic fuzzy sets based on Deng entropy was proposed. Deng et al. further defined generalized Deng entropy [77, 81] and divergence measures [78, 82] and applied these results to data fusion. What's more, several improved fusion rules were also presented, such as [86, 87, 89]. Recently, combined with Deng entropy, a multisensor fusion method was provided in [92], and a maximum of entropy for belief intervals was given in [94].

4.2 Applications in fault diagnosis

In this field, Refs. [97–99] applied Deng entropy in fault diagnosis by solving evidential conflict management problems. In addition, the method for combining evidential sensor reports was proposed based on distance function and Deng entropy in [95]. A novel multi-sensor data fusion technique was proposed for fault diagnosis in [96] by using Deng entropy and fuzzy preference approach.

4.3 Applications in failure mode and effects analysis (FMEA)

By employing Deng entropy, two methods for failure mode and effects analysis under uncertain environment were proposed [105, 106]. In addition, Liu et al. [107] and Zheng et al. [108] focused on determining the weight of uncertain information.

4.4 Applications in pattern recognition

The classical Deng entropy was improved [100] and applied in pattern recognition. An evidential reliability indicator-based fusion rule was defined [101] and used in classification. The collaborative fusion rule was presented [107] for distributed target classification in IoT environment. The concept of Deng extropy based on Deng entropy was defined [103] and used to pattern recognition. An association coefficient of BPAs was provided [104] and applied in a target recognition system.

4.5 Applications in decision-making

The applications of Deng entropy in the field of decision-making can be classified into three parts, including quantum decision, multi-attribute decision making and decision-making in artificial intelligence. For the first aspect, a Deng entropy-based uncertainty measurement was proposed [118] to quantify the interference effect that was used to quantum decision. An evidential Markov quantum decision making model was also proposed [119]. For the second aspect, a novel multi-criteria decision-making method for supplier selection was presented [112] based on Deng entropy and VIKOR method. Xiao [26, 113] proposed two multi-attribute decision making methods in uncertain environment based on Deng entropy. For the third aspect, an evidential decision tree was provided [114] based on Deng entropy. And Deng entropy was improved in [115] to achieve a better decision-making level.

4.6 Applications in risk assessment

A new weapon-target assignment model considering uncertainty and its decomposition coevolution algorithm was proposed in [109]. A fire control system operation status assessment method based on information fusion was proposed in [110] and demonstrated through a case study. In addition, an air target threat assessment method based on Deng entropy has been developed in [111].

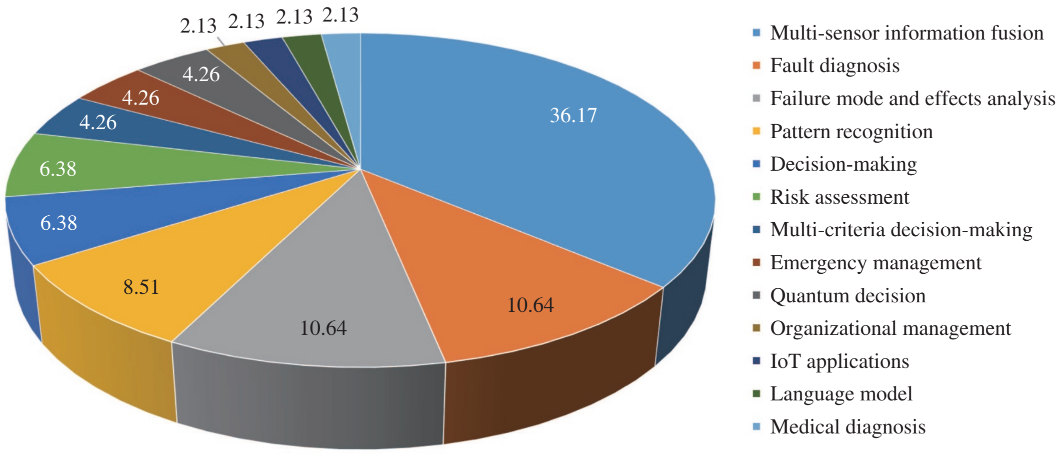

Figure 5 (Color online) The statistical distribution (%) of the application fields of Deng entropy.

4.7 Applications in other fields

Besides the mainstream applications listed above, Deng entropy also has several other applications. For example, Deng entropy was combined with DEMATEL method and applied to emergency management [116, 117]. Kang [120] constructed the stable hierarchy organization from the perspective of the maximum Deng entropy. A Deng entropy-based approach for conflict resolution in IoT applications was developed in [121]. And a few other applications, such as language model construction [122] and medical diagnosis [98] were also conducted by using Deng entropy.

4.8 Statistical analysis of Deng entropy applications

In Figure 5, we demonstrate the proportion of Deng entropy in various application fields in the form of pie chart. It can be seen from the figure that Deng entropy is most commonly used in multi-sensor information fusion owing to its advantages in uncertainty measurement. Several other crucial applications, such as fault diagnosis, FMEA, pattern recognition and decision making, also account for a large proportion. Applications such as IoT and language model are rare, and it will take time to test whether Deng entropy has a future in these fields.

In addition, Figure 6 manifests the timeline for the Deng entropy applications. It can be seen from the figure that from 2016 when Deng entropy was proposed, its application began in 2017 and increased significantly in 2019. The statistical time of this paper is July 2020, so the applications of Deng entropy in 2020 cannot be clearly defined at present, but from the development trend, we can boldly predict that its applications will continue to increase. In addition, it can be seen from the applications of each year that multi-sensor information fusion is always in the hottest position. The proportion of other applications fluctuates from year to year. As Deng entropy was proposed about only four years ago, the current statistics may not be able to explain much more, but we expect Deng entropy to reflect its higher value in the passage of time.

5 Challenges to future studies on uncertainty measures in evidence theory

In this section, we present the challenges of uncertainty measures in evidence theory for future research, including its theory and application, which are discussed in Subsections 5.1–5.3.

5.1 Challenges for the development of uncertainty measures in evidence theory

Nearly half a century has passed since evidence theory was proposed in 1967 and perfected as an uncertainty reasoning tool in 1976. Since this time, research on the measurement of uncertainty has continued interrupted. As introduced in Subsection 2.2, scholars have defined uncertainty measurement methods

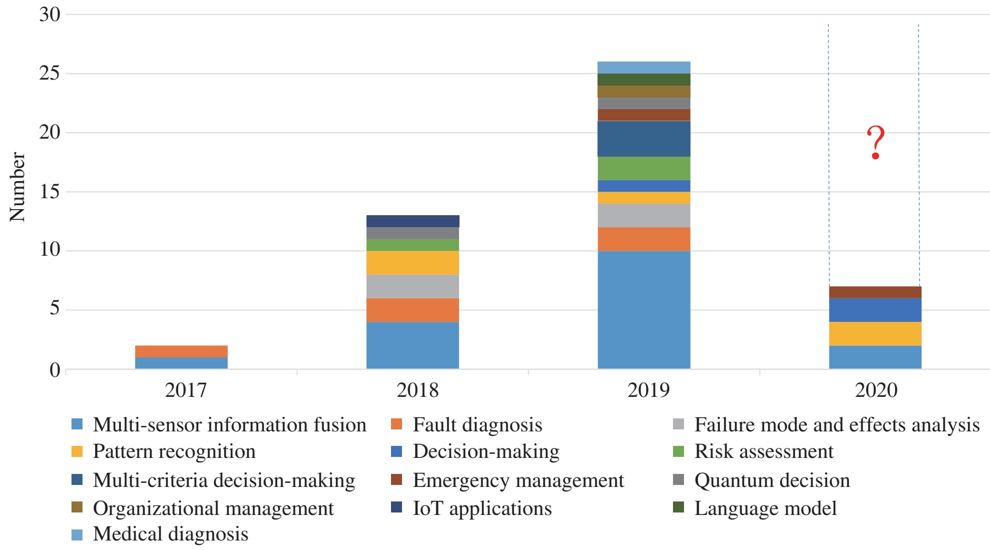

Figure 6 (Color online) Development statistics of the applications of Deng entropy.

in evidence theory from a variety of perspectives. Thus far, there is no final conclusion; i.e., the measurement of uncertainty has always been a developmental problem. From the comparison experiment in Subsection 3.3, it can be inferred that different uncertainty measures might be applicable to different scenarios, which also prompts researchers to select uncertainty measures according to their actual needs. It is not easy to answer whether there exists a measure that can reasonably solve every uncertainty measure. But it is certain that Deng entropy is a very good choice, both in terms of its theoretical significance and its practical value.

5.2 Challenges for theoretical development of Deng entropy

Since it was put forth, Deng entropy has been developed from a theoretical perspective, such as some improved forms of Deng entropy, maximum Deng entropy, the relationship between the distribution of maximum Deng entropy, and the pseudo-Pascal triangle. From current research results, the development of Deng theory seems to be close to perfect, but from the perspective of development, unfinished business remains. Some of the questions that persist are: Can Deng entropy be further extended to a more generalized form to convey more meaningful information and also be backward compatible with classical theoretical results? Is there a more factual maximum Deng entropy distribution? Is there a secret relationship between the maximum Deng entropy distribution and the pseudo-Pascal triangle? On the other hand, as Abellán [72] pointed out, Deng entropy cannot satisfy some properties that are considered to be properties of belief entropy such as consistency and sub-additivity. However, if an entropy function does not satisfy the so-called theorem, whether it is bad is another question to ponder.

5.3 Challenges for exploring the physical meaning of Deng entropy

Currently, the reason why Deng entropy attracts an increasing amount of attention is mainly due to its unique advantages in the measurement of uncertainty and its important role in the application of information fusion. However, the significance of Deng entropy is obviously greater than just that, and it has seen numerous theoretical extensions such as maximum Deng entropy and the relationship between the distribution of maximum Deng entropy and the pseudo-Pascal triangle. According to Deng entropy, we know that for a test with 32 participants, it takes at most 50.7188 times to guess who comes in the first place. However, what is the physical meaning behind this number? The existence of these doubts inspires us to explore its deeper meaning and beckons us with infinite appeal to a place closer to the truth.

6 Conclusion

In multi-source information fusion, owing to the diversity and uncontrollability of information sources, the process of expressing and processing information using evidence theory faces potential uncertainty risks. Therefore, it is very important to effectively capture and deal with uncertainty to enhance the accuracy of fusion results. With the development of evidence theory, belief entropy has become the main tool of uncertainty measurement. In recent years, a few novel uncertainty measures represented by Deng entropy have enriched the theory's development, but have also brought some controversy. In the current literature, few research reviews exist on uncertainty measures in evidence theory, especially on Deng entropy. Therefore, based on the review of uncertainty measures in evidence theory, this paper presents a comprehensive introduction and commentary on an uncertainty measure called Deng entropy that has become unusually popular in recent years. Below, we summarize the theoretical and practical contributions and significance of this paper. First, we review the main uncertainty measurement methods in existing evidence theory, analyze the properties of belief entropy, and introduce the main controversies in the study of uncertainty measures. This comprehensive review is of great reference significance for scholars in understanding uncertainty measures in evidence theory. Next, we introduce the definition of Deng entropy, analyze its properties, highlight its advantages through comparative experiments, introduce the definition of maximum Deng entropy, and discuss the relationship between the distribution of maximum Deng entropy and the pseudo-Pascal triangle. We have introduced and commented on the theory of Deng entropy in detail, which is of great significance in fully understanding Deng entropy. Finally, we investigate the application of Deng entropy in different fields and obtain a ranking according to application frequency: multi-sensor information fusion, fault diagnosis, failure mode and effects analysis, pattern recognition, decision-making, and risk assessment. This is very helpful in understanding the application status and prospects of Deng entropy. In the end, we put forth the difficulties and challenges in the future development of uncertainty measures in evidence theory and summarize these challenges from the development of various theories on the entropy function to the challenges of exploring the physical significance of Deng entropy. Challenges and the unknown are the driving force that propels us forward. We are ready to push ahead in the direction of uncertainty measurement theory with enthusiasm and an indomitable spirit.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 61973332).

References

1 Fu C, Chang W, Yang S. Multiple criteria group decision making based on group satisfaction. Inf Sci, 2020, 518: 309–329
2 Fu C, Chang W, Xue M, et al. Multiple criteria group decision making with belief distributions and distributed preference relations. Eur J Operational Res, 2019, 273: 623–633
3 He Y, Hu L F, Guan X, et al. New method for measuring the degree of conflict among general basic probability assignments. Sci China Inf Sci, 2012, 55: 312–321
4 Fei L, Feng Y, Liu L. Evidence combination using OWA-based soft likelihood functions. Int J Intell Syst, 2019, 34: 2269–2290
5 Liu Z, Pan Q, Dezert J, et al. Classifier fusion with contextual reliability evaluation. IEEE Trans Cybern, 2018, 48: 1605–1618
6 Wu B, Yan X, Wang Y, et al. An evidential reasoning-based CREAM to human reliability analysis in maritime accident process. Risk Anal, 2017, 37: 1936–1957
7 Wang Z, Gao J M, Wang R X, et al. Failure mode and effects analysis using Dempster-Shafer theory and TOPSIS method: application to the gas insulated metal enclosed transmission line (GIL). Appl Soft Comput, 2018, 70: 633–647
8 Liu Z G, Liu Y, Dezert J, et al. Evidence combination based on credal belief redistribution for pattern classification. IEEE Trans Fuzzy Syst, 2020, 28: 618–631
9 Pan Y, Zhang L, Wu X, et al. Multi-classifier information fusion in risk analysis. Inf Fusion, 2020, 60: 121–136
10 He Y, Jian T, Su F, et al. Two adaptive detectors for range-spread targets in non-Gaussian clutter. Sci China Inf Sci, 2011, 54: 386–395
11 Zadeh L A. Fuzzy sets. Inf Control, 1965, 8: 338–353
12 Pawlak Z. Rough sets. Int J Comput Inf Sci, 1982, 11: 341–356
13 Dempster A P. Upper and lower probabilities generated by a random closed interval. Ann Math Statist, 1968, 39: 957–966
14 Shafer G. A Mathematical Theory of Evidence. Princeton: Princeton University Press, 1976

15 Atanassov K T. Intuitionistic fuzzy sets. Fuzzy Sets Syst, 1986, 20: 87–96
16 Zadeh L A. A note on Z-numbers. Inf Sci, 2011, 181: 2923–2932
17 Yager R R. Pythagorean fuzzy subsets. In: Proceedings of 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), 2013. 57–61
18 Fu C, Chang W, Xu D, et al. An evidential reasoning approach based on criterion reliability and solution reliability. Comput Industrial Eng, 2019, 128: 401–417
19 Xiao F. Generalization of Dempster-Shafer theory: a complex mass function. Appl Intell, 2020, 50: 3266–3275
20 Xiao F. Generalized belief function in complex evidence theory. J Intell Fuzzy Syst, 2020, 38: 3665–3673
21 Yang J B. Rule and utility based evidential reasoning approach for multiattribute decision analysis under uncertainties. Eur J Operational Res, 2001, 131: 31–61
22 Yang J B, Xu D L. On the evidential reasoning algorithm for multiple attribute decision analysis under uncertainty. IEEE Trans Syst Man Cybern A, 2002, 32: 289–304
23 Yang J B, Xu D L. Evidential reasoning rule for evidence combination. Artif Intell, 2013, 205: 1–29
24 Shannon C E. A mathematical theory of communication. Bell Syst Technical J, 1948, 27: 379–423
25 Deng Y. Deng entropy. Chaos Solitons Fractals, 2016, 91: 549–553
26 Xiao F. EFMCDM: evidential fuzzy multicriteria decision making based on belief entropy. IEEE Trans Fuzzy Syst, 2020, 28: 1477–1491
27 Xiao F. GIQ: a generalized intelligent quality-based approach for fusing multi-source information. IEEE Trans Fuzzy Syst, 2020. doi: 10.1109/TFUZZ.2020.2991296
28 Xiao F. Multi-sensor data fusion based on the belief divergence measure of evidences and the belief entropy. Inf Fusion, 2019, 46: 23–32
29 Xiao F. A distance measure for intuitionistic fuzzy sets and its application to pattern classification problems. IEEE Trans Syst Man Cybern Syst, 2020. doi: 10.1109/TSMC.2019.2958635
30 Xiao F. A new divergence measure for belief functions in D-S evidence theory for multisensor data fusion. Inf Sci, 2020, 514: 462–483
31 Xiao F. CED: a distance for complex mass functions. IEEE Trans Neural Netw Learning Syst, 2020. doi: 10.1109/TNNLS.2020.2984918
32 Höhle U. Entropy with respect to plausibility measures. In: Proceedings of the 12th IEEE International Symposium on Multiple Valued Logic, Paris, 1982
33 Smets P. Information content of an evidence. Int J Man-Machine Studies, 1983, 19: 33–43
34 Yager R R. Entropy and specificity in a mathematical theory of evidence. Int J General Syst, 1983, 9: 249–260
35 Dubois D, Prade H. Properties of measures of information in evidence and possibility theories. Fuzzy Sets Syst, 1987, 24: 161–182
36 Lamata M T, Moral S. Measures of entropy in the theory of evidence. Int J General Syst, 1988, 14: 297–305
37 Klir G J, Ramer A. Uncertainty in the Dempster-Shafer theory: a critical re-examination. Int J General Syst, 1990, 18: 155–166
38 Klir G J, Parviz B. A note on the measure of discord. In: Proceedings of the 8th Conference on Uncertainty in Artificial Intelligence, 1992. 138–141
39 Pal N R, Bezdek J C, Hemasinha R. Uncertainty measures for evidential reasoning I: a review. Int J Approximate Reasoning, 1992, 7: 165–183
40 Pal N R, Bezdek J C, Hemasinha R. Uncertainty measures for evidential reasoning II: a new measure of total uncertainty. Int J Approximate Reasoning, 1993, 8: 1–16
41 George T, Pal N R. Quantification of conflict in Dempster-Shafer framework: a new approach. Int J General Syst, 1996, 24: 407–423
42 Jousselme A L, Liu C S, Grenier D, et al. Measuring ambiguity in the evidence theory. IEEE Trans Syst Man Cybern A, 2006, 36: 890–903
43 Jiroušek R, Shenoy P P. A new definition of entropy of belief functions in the Dempster-Shafer theory. Int J Approximate Reasoning, 2018, 92: 49–65
44 Pan Q, Zhou D, Tang Y, et al. A novel belief entropy for measuring uncertainty in Dempster-Shafer evidence theory framework based on plausibility transformation and weighted hartley entropy. Entropy, 2019, 21: 163
45 Wen K, Song Y, Wu C, et al. A novel measure of uncertainty in the Dempster-Shafer theory. IEEE Access, 2020, 8: 51550–51559
46 Wang X, Song Y. Uncertainty measure in evidence theory with its applications. Appl Intell, 2018, 48: 1672–1688
47 Yang Y, Han D. A new distance-based total uncertainty measure in the theory of belief functions. Knowledge-Based Syst, 2016, 94: 114–123
48 Deng X, Xiao F, Deng Y. An improved distance-based total uncertainty measure in belief function theory. Appl Intell, 2017, 46: 898–915
49 Deng X. Analyzing the monotonicity of belief interval based uncertainty measures in belief function theory. Int J Intell Syst, 2018, 33: 1869–1879
50 Deng X, Jiang W. A total uncertainty measure for D numbers based on belief intervals. Int J Intell Syst, 2019, 34: 3302–3316
51 Xia J, Feng Y, Liu L, et al. On entropy function and reliability indicator for D numbers. Appl Intell, 2019, 49: 3248–3266
52 Yager R R. Interval valued entropies for Dempster-Shafer structures. Knowledge-Based Syst, 2018, 161: 390–397

53 Klir G J, Wierman M J. Uncertainty-based Information: Elements of Generalized Information Theory. Berlin: Springer, 1999
54 Klir G J. Uncertainty and Information: Foundations of Generalized Information Theory. Piscataway: Wiley-IEEE Press, 2006
55 Abe S, Okamoto Y. Nonextensive Statistical Mechanics and Its Applications. Berlin: Springer, 2001
56 Tsallis C. Possible generalization of Boltzmann-Gibbs statistics. J Stat Phys, 1988, 52: 479–487
57 Wang D, Gao J, Wei D. A new belief entropy based on Deng entropy. Entropy, 2019, 21: 987
58 Ozkan K. Comparing Shannon entropy with Deng entropy and improved Deng entropy for measuring biodiversity when a priori data is not clear. J Faculty Forestry-Istanbul Univ, 2018, 68: 136–140
59 Li J, Pan Q. A new belief entropy in Dempster-Shafer theory based on basic probability assignment and the frame of discernment. Entropy, 2020, 22: 691
60 Zhou Q, Mo H, Deng Y. A new divergence measure of pythagorean fuzzy sets based on belief function and its application in medical diagnosis. Mathematics, 2020, 8: 142
61 Kuzemsky A. Temporal evolution, directionality of time and irreversibility. La Rivista del Nuovo Cimento, 2018, 41: 513–574
62 Jiang W, Wang S. An uncertainty measure for interval-valued evidences. Int J Comput Commun, 2017, 12: 631–644
63 Mambe M D, N'Takp'e T, Georges N, et al. A new uncertainty measure in belief entropy framework. Int J Adv Comput Sci Appl, 2018, 9: 600–606
64 Xie K, Xiao F. Negation of belief function based on the total uncertainty measure. Entropy, 2019, 21: 73
65 Zhao Y, Ji D, Yang X, et al. An improved belief entropy to measure uncertainty of basic probability assignments based on Deng entropy and belief interval. Entropy, 2019, 21: 1122
66 Luo C K, Chen Y X, Xiang H C, et al. Evidence combination method in time domain based on reliability and importance. J Syst Eng Electron, 2018, 29: 1308–1316
67 Vandoni J, Aldea E, Le Hégarat-Mascle S. Evidential query-by-committee active learning for pedestrian detection in high-density crowds. Int J Approx Reason, 2019, 104: 166–184
68 Khan M N, Anwar S. Time-domain data fusion using weighted evidence and Dempster-Shafer combination rule: application in object classification. Sensors, 2019, 19: 5187
69 Pan L, Deng Y. Probability transform based on the ordered weighted averaging and entropy difference. Int J Comput Commun, 2020, 15: 4
70 Wang Y, Liu F, Zhu A. Bearing fault diagnosis based on a hybrid classifier ensemble approach and the improved Dempster-Shafer theory. Sensors, 2019, 19: 2097
71 Zhang Y, Liu Y, Zhang Z, et al. A weighted evidence combination approach for target identification in wireless sensor networks. IEEE Access, 2017, 5: 21585–21596
72 Abellán J. Analyzing properties of Deng entropy in the theory of evidence. Chaos Solitons Fractals, 2017, 95: 195–199
73 Kang B, Deng Y. The maximum Deng entropy. IEEE Access, 2019, 7: 120758–120765
74 Deng Y. The information volume of uncertain informaion: (1) mass function. 2020. viXra:2006.0028
75 Tsallis C, Gell-Mann M, Sato Y. Asymptotically scale-invariant occupancy of phase space makes the entropy S_q extensive. Proc Natl Acad Sci USA, 2005, 102: 15377–15382
76 Gao X, Deng Y. The Pseudo-Pascal triangle of maximum Deng entropy. Int J Comput Commun, 2020, 15: 1–10
77 Liu F, Gao X, Zhao J, et al. Generalized belief entropy and its application in identifying conflict evidence. IEEE Access, 2019, 7: 126625–126633
78 Song Y, Deng Y. Divergence measure of belief function and its application in data fusion. IEEE Access, 2019, 7: 107465–107472
79 Gao X, Liu F, Pan L, et al. Uncertainty measure based on Tsallis entropy in evidence theory. Int J Intell Syst, 2019, 34: 3105–3120
80 Pan L, Deng Y. A new belief entropy to measure uncertainty of basic probability assignments based on belief function and plausibility function. Entropy, 2018, 20: 842
81 Li Y, Deng Y. Generalized ordered propositions fusion based on belief entropy. Int J Comput Commun, 2018, 13: 792–807
82 Song Y, Deng Y. A new method to measure the divergence in evidential sensor data fusion. Int J Distributed Sens Networks, 2019, 15: 1–8
83 Xiao F. An improved method for combining conflicting evidences based on the similarity measure and belief function entropy. Int J Fuzzy Syst, 2018, 20: 1256–1266
84 Boulkaboul S, Djenouri D. DFIOT: data fusion for Internet of Things. J Netw Syst Manage, 2020, 54: 1–25
85 Xiao F, Qin B. A weighted combination method for conflicting evidence in multi-sensor data fusion. Sensors, 2018, 18: 1487
86 An J, Hu M, Fu L, et al. A novel fuzzy approach for combining uncertain conflict evidences in the Dempster-Shafer theory. IEEE Access, 2019, 7: 7481–7501
87 Wang J, Qiao K, Zhang Z. An improvement for combination rule in evidence theory. Future Generation Comput Syst, 2019, 91: 1–9
88 Tang Y, Zhou D, Chan F. An extension to Deng's entropy in the open world assumption with an application in sensor data fusion. Sensors, 2018, 18: 1902
89 Hurley J, Johnson C, Dunham J, et al. Nonlinear algorithms for combining conflicting identification information in multisensor fusion. In: Proceedings of 2019 IEEE Aerospace Conference, 2019. 1–7

90 Liu Z, Xiao F. An evidential aggregation method of intuitionistic fuzzy sets based on belief entropy. IEEE Access, 2019, 7: 68905–68916
91 Wang Z, Xiao F. An improved multi-source data fusion method based on the belief entropy and divergence measure. Entropy, 2019, 21: 611
92 Fan X, Guo Y, Ju Y, et al. Multisensor fusion method based on the belief entropy and DS evidence theory. J Sens, 2020, 2020: 1–16
93 Tao R, Xiao F. Combine conflicting evidence based on the belief entropy and IOWA operator. IEEE Access, 2019, 7: 120724
94 Moral-Garcia S, Abellan J. Maximum of entropy for belief intervals under evidence theory. IEEE Access, 2020, 8: 118017
95 Dong Y, Zhang J, Li Z, et al. Combination of evidential sensor reports with distance function and belief entropy in fault diagnosis. Int J Comput Commun, 2019, 14: 329–343
96 Xiao F. A novel evidence theory and fuzzy preference approach-based multi-sensor data fusion technique for fault diagnosis. Sensors, 2017, 17: 2504
97 Wang Z, Xiao F. An improved multisensor data fusion method and its application in fault diagnosis. IEEE Access, 2019, 7: 3928–3937
98 Chen L, Diao L, Sang J. A novel weighted evidence combination rule based on improved entropy function with a diagnosis application. Int J Distributed Sens Netw, 2019, 15: 1–13
99 Liu F, Wang Y. A novel method of ds evidence theory for multi-sensor conflicting information. In: Proceedings of the 4th International Conference on Machinery, Materials and Computer (MACMC 2017). Paris: Atlantis Press, 2018. 343–349
100 Cui H, Liu Q, Zhang J, et al. An improved deng entropy and its application in pattern recognition. IEEE Access, 2019, 7: 18284–18292
101 Xia J, Feng Y, Liu L, et al. An evidential reliability indicator-based fusion rule for Dempster-Shafer theory and its applications in classification. IEEE Access, 2018, 6: 24912–24924
102 Zhang Y, Liu Y, Zhang Z, et al. Collaborative fusion for distributed target classification using evidence theory in IOT environment. IEEE Access, 2018, 6: 62314–62323
103 Buono F, Longobardi M. A dual measure of uncertainty: the Deng extropy. Entropy, 2020, 22: 1–10
104 Pan L, Deng Y. An association coefficient of a belief function and its application in a target recognition system. Int J Intell Syst, 2020, 35: 85–104
105 Huang Z, Jiang W, Tang Y. A new method to evaluate risk in failure mode and effects analysis under fuzzy information. 2018, 22: 4779–4787
106 Wang H, Deng X, Zhang Z, et al. A new failure mode and effects analysis method based on Dempster-Shafer theory by integrating evidential network. IEEE Access, 2019, 7: 79579–79591
107 Liu Z, Xiao F. An intuitionistic evidential method for weight determination in FMEA based on belief entropy. Entropy, 2019, 21: 211
108 Zheng H, Tang Y. Deng entropy weighted risk priority number model for failure mode and effects analysis. Entropy, 2020, 22: 280
109 Pan Q, Zhou D, Tang Y, et al. A novel antagonistic Weapon-Target assignment model considering uncertainty and its solution using decomposition co-evolution algorithm. IEEE Access, 2019, 7: 37498–37517
110 Li Y, Wang A, Yi X. Fire control system operation status assessment based on information fusion: case study. Sensors, 2019, 19: 2222
111 Liu H, Ma Z, Deng X, et al. A new method to air target threat evaluation based on Dempster-Shafer evidence theory. In: Proceedings of 2018 Chinese Control and Decision Conference (CCDC), 2018. 2504–2508
112 Fei L, Deng Y, Hu Y. DS-VIKOR: a new multi-criteria decision-making method for supplier selection. Int J Fuzzy Syst, 2019, 21: 157–175
113 Xiao F. A multiple-criteria decision-making method based on D numbers and belief entropy. Int J Fuzzy Syst, 2019, 21: 1144–1153
114 Li M, Xu H, Deng Y. Evidential decision tree based on belief entropy. Entropy, 2019, 21: 897
115 Yan H, Deng Y. An improved belief entropy in evidence theory. IEEE Access, 2020, 8: 57505–57516
116 Chen L, Li Z, Deng X. Emergency alternative evaluation under group decision makers: a new method based on entropy weight and DEMATEL. Int J Syst Sci, 2020, 51: 570–583
117 Shang X, Song M, Huang K, et al. An improved evidential DEMATEL identify critical success factors under uncertain environment. J Ambient Intell Humanized Comput, 2019
118 Huang Z, Yang L, Jiang W. Uncertainty measurement with belief entropy on the interference effect in the quantum-like Bayesian networks. Appl Math Comput, 2019, 347: 417–428
119 He Z, Jiang W. An evidential Markov decision making model. Inf Sci, 2018, 467: 357–372
120 Kang B. Construction of stable hierarchy organization from the perspective of the maximum deng entropy. In: Integrated Uncertainty in Knowledge Modelling and Decision Making. Berlin: Springer, 2019. 421–431
121 Mambe M D, Oumtanaga S, Anoh G N. A belief entropy-based approach for conflict resolution in IOT applications. In: Proceedings of 2018 1st International Conference on Smart Cities and Communities (SCCIC), 2018. 1–5
122 Prajapati G L, Saha R. Reeds: relevance and enhanced entropy based Dempster Shafer approach for next word prediction using language model. J Comput Sci, 2019, 35: 1–11