Relations

Ch 2 schaum's, Ch 7 Rosen

Hasse Diagram

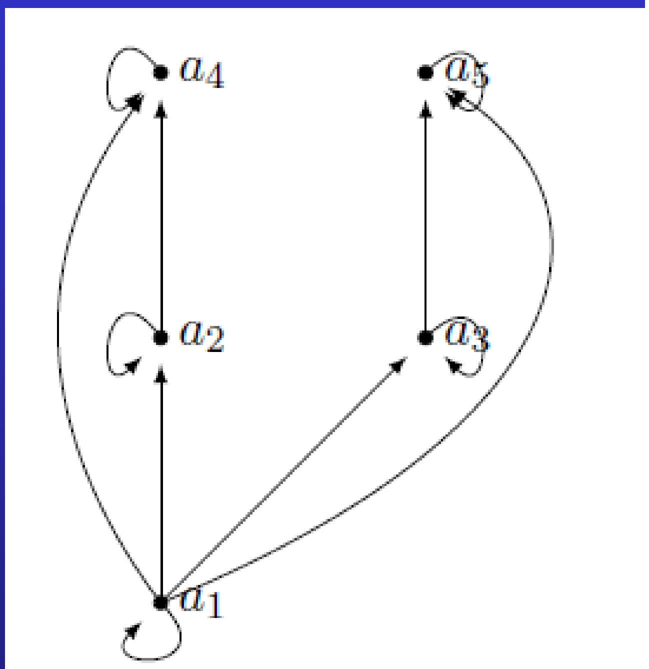
As with relations and functions, there is a convenient graphical representation for partial orders—*Hasse Diagrams*.

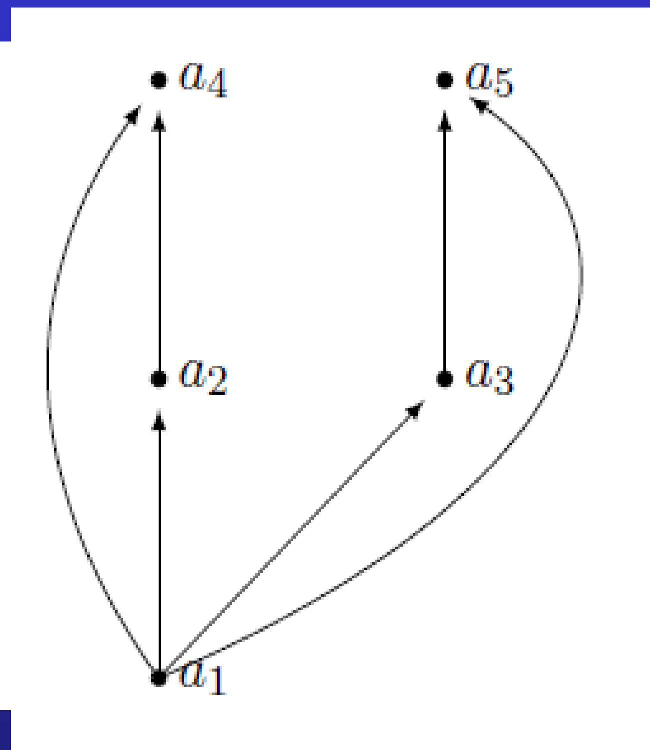
Consider the digraph representation of a partial order—since we *know* we are dealing with a partial order, we *implicitly* know that the relation must be reflexive and transitive. Thus we can simplify the graph as follows:

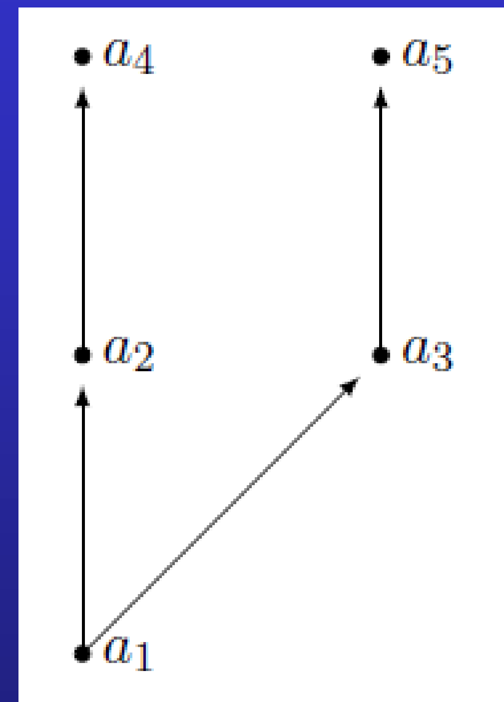
- Remove all self-loops.
- Remove all transitive edges.
- Make the graph direction-less—that is, we can assume that the orientations are *upwards*.

The resulting diagram is far simpler.

Hasse Diagram Example



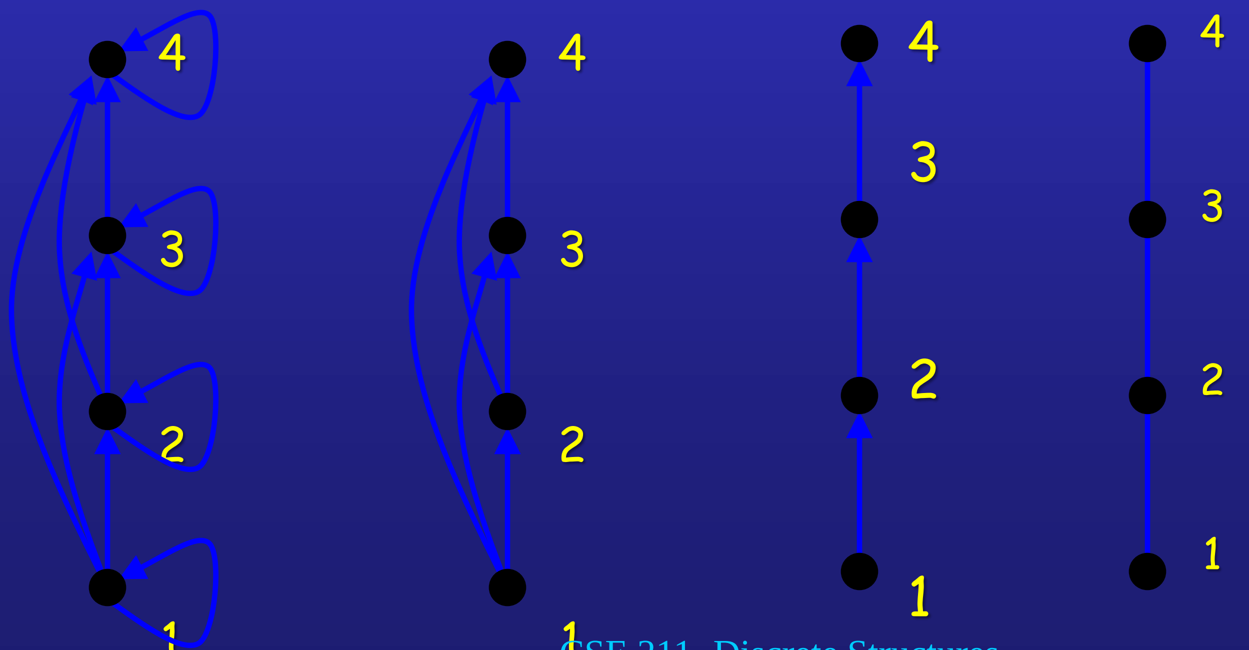




Hasse Diagrams

Consider the graph for a finite poset ({1,2,3,4},≤)

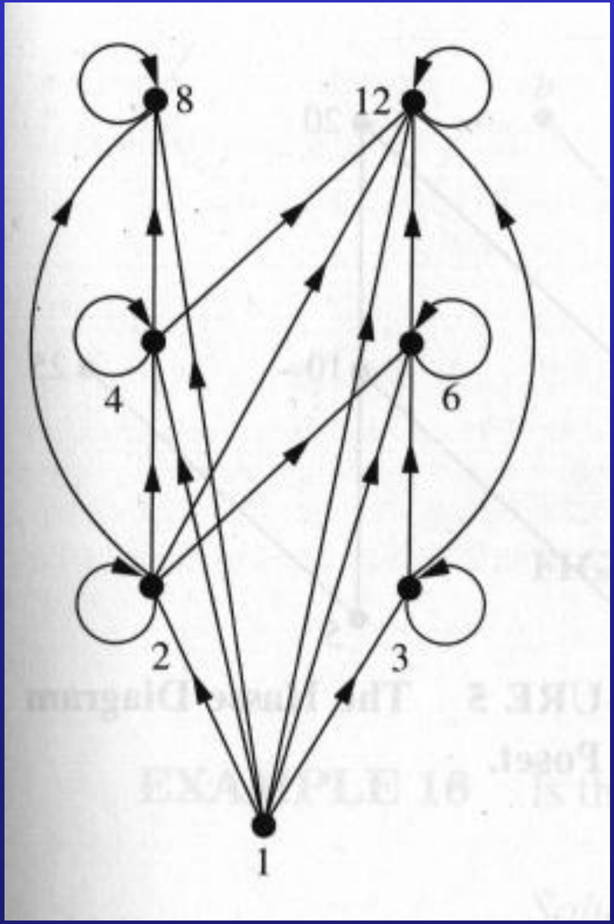
When we KNOW it's a poset, we can simplify the graph

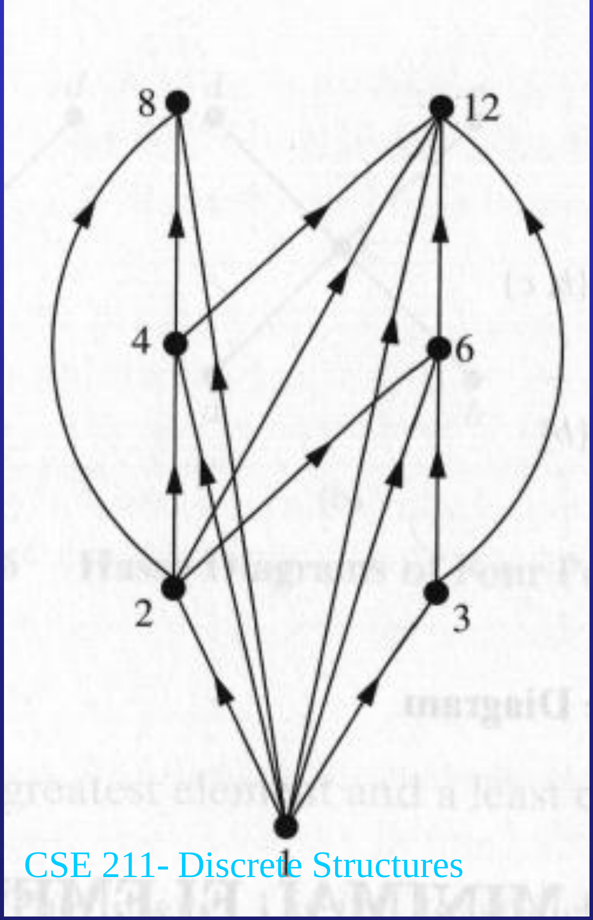


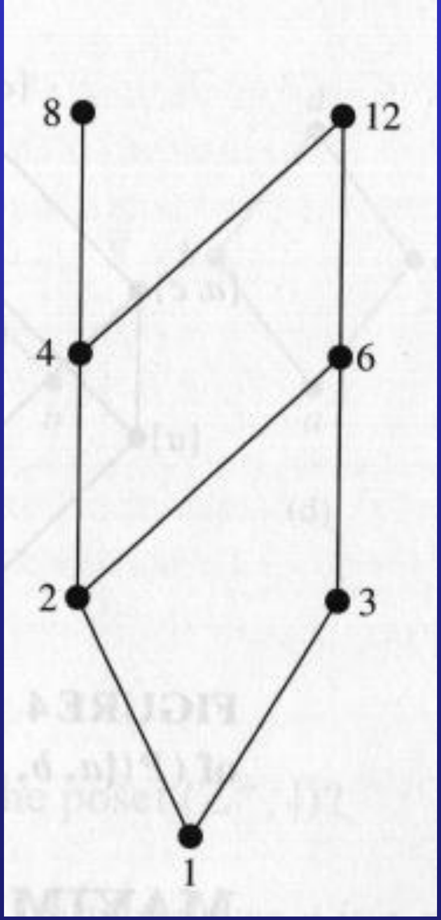
Called the Hasse diagram

Hasse Diagram

For the poset ({1,2,3,4,6,8,12}, |)







Construct the Hasse diagram of $(P(\{a, b, c\}), \subseteq)$.

The elements of $P(\{a, b, c\})$ are

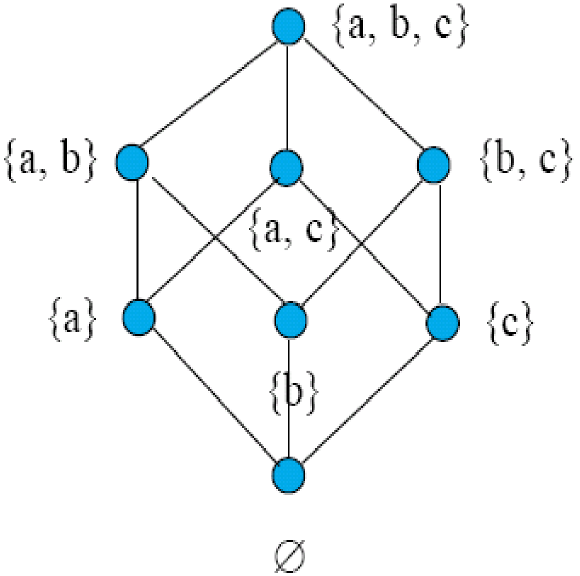
$$\varnothing$$

$$\{a\}, \{b\}, \{c\}$$

$$\{a, b\}, \{a, c\}, \{b, c\}$$

$$\{a, b, c\}$$

The digraph is



Greatest Element

Least Element

Maximal Element

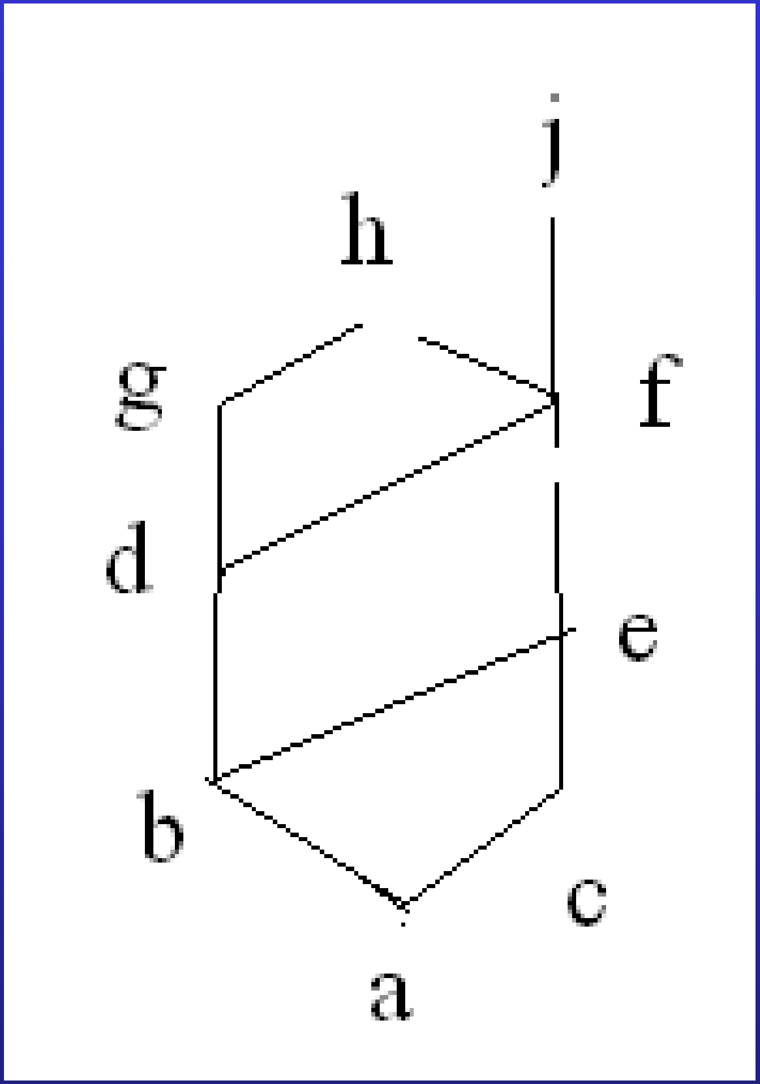
Minimal Element

Upper Bound

Lower Bound

Greatest lower bound

Least Upper bound



Find the lower and upper bounds of the subsets {a, b, c} , {j, h}, and {a, c, d, f}.

The upper bounds of {a, b, c} are e, f, j, h and its lower bound is a.

There is no upper bounds of {j, h}. And its lower bounds are a, b, c, d, e, f.

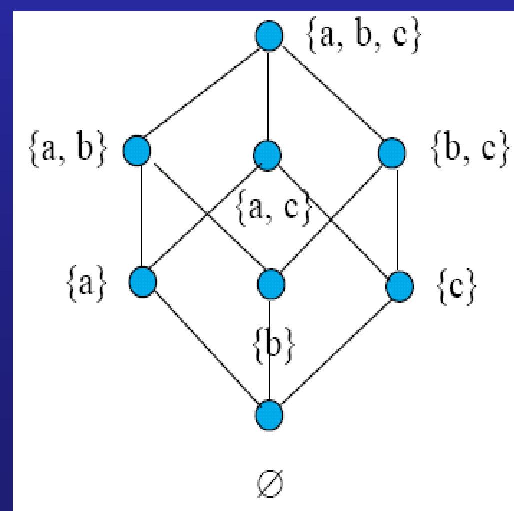
The upper bounds of {a, c, d, f} are f, h and j. and its lower bound is a.

Find the greatest lower bound and the least upper bound of {b, d, g}

Lattices

A partially ordered set in which *every pair* of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

$(P(\{a, b, c\}), \subseteq)$



Consider the elements 1 and 3.

- Upper bounds of 1 are 1, 2, 4 and 5.
- Upper bounds of 3 are 3, 2, 4 and 5.
- 2, 4 and 5 are upper bounds for the pair 1 and 3.
- There is no lub since
    - 2 is not related to 4
    - 4 is not related to 2
    - 2 and 4 are both related to 5.
- There is no glb either.

The poset is not a lattice.