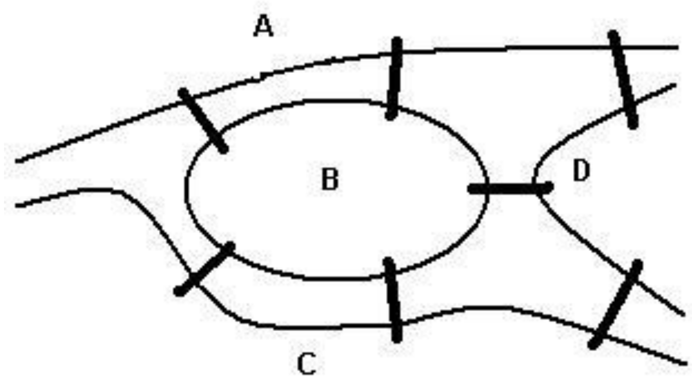
Discrete Mathematics

Graph Theory

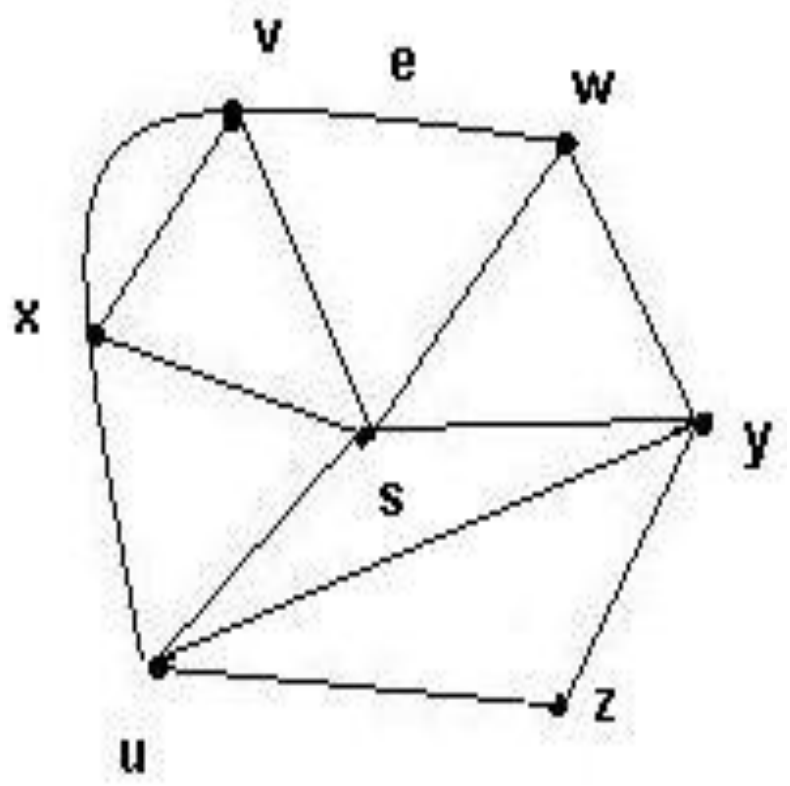
In the beginning…



- 1736: **Leonhard Euler**
  - Basel, 1707-St. Petersburg, 1786
  - He wrote *A solution to a problem concerning the geometry of a place.* First paper in graph theory.
- Problem of the Königsberg bridges:
  - Starting and ending at the same point, is it possible to cross all seven bridges just once and return to the starting point?

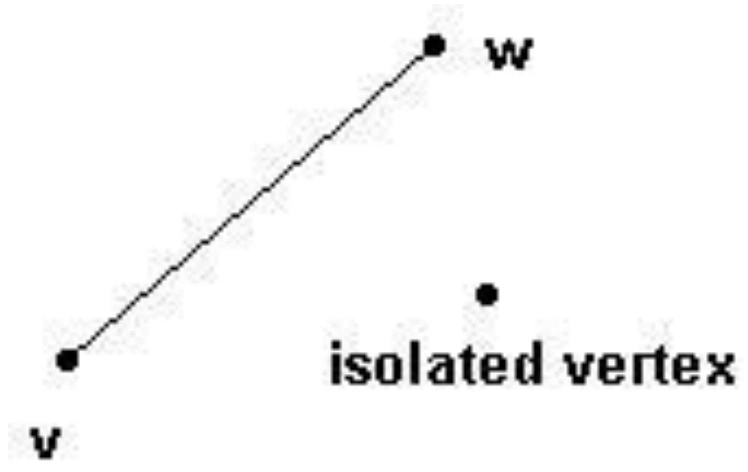
Introduction

- What is a graph G?
- It is a pair G = (V, E), where
  - V = V(G) = set of vertices
  - E = E(G) = set of edges
- **Example:**
  - V = {s, u, v, w, x, y, z}
  - E = {(x,s), $(x,v)_1$, $(x,v)_2$, (x,u), (v,w), (s,v), (s,u), (s,w), (s,y), (w,y), (u,y), (u,z),(y,z)}



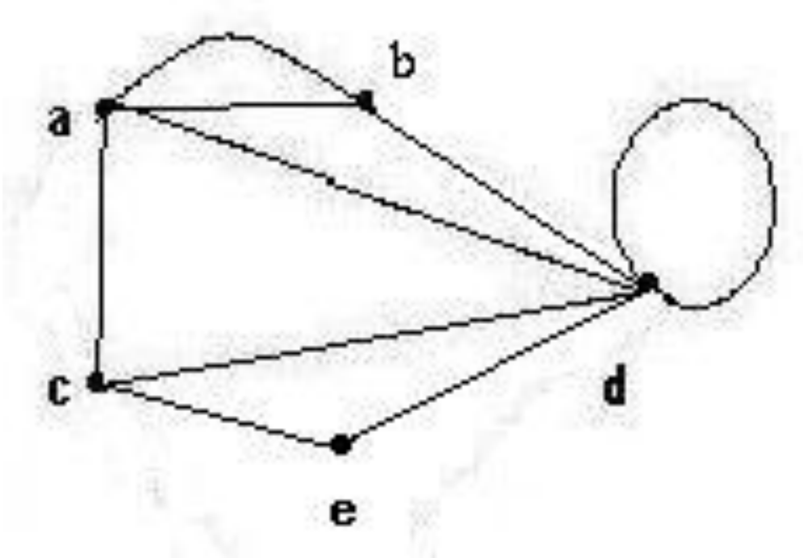
Edges

- An edge may be labeled by a pair of vertices, for instance e = (v,w).
- e is said to be *incident* on v and w.
- Isolated vertex = a vertex without incident edges.



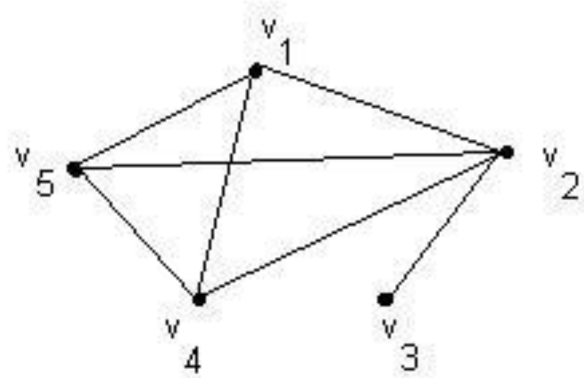
Special edges

- Parallel edges
  - Two or more edges joining a pair of vertices
    - in the example, **a** and **b** are joined by two parallel edges
- Loops
  - An edge that starts and ends at the same vertex
    - In the example, vertex **d** has a loop

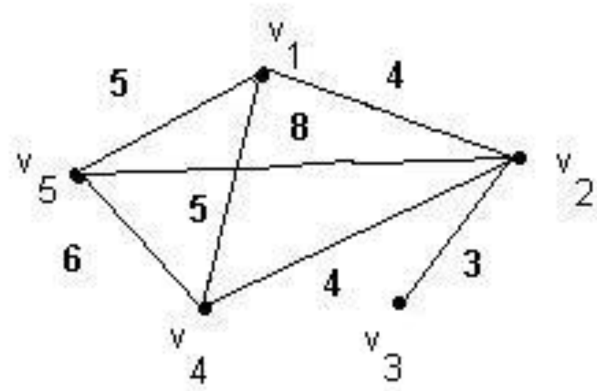


Special graphs

- Simple graph
  - A graph without loops or parallel edges.

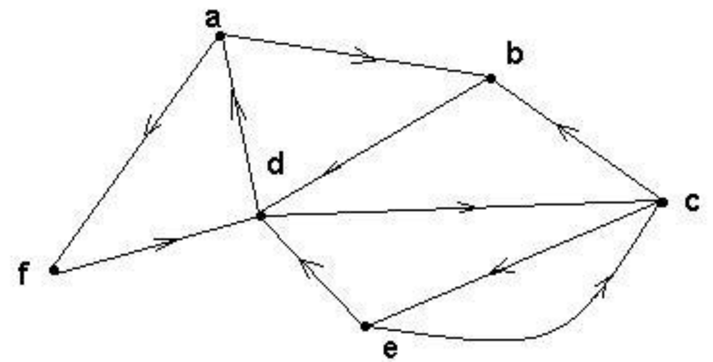


- Weighted graph
  - A graph where each edge is assigned a numerical label or “weight”.



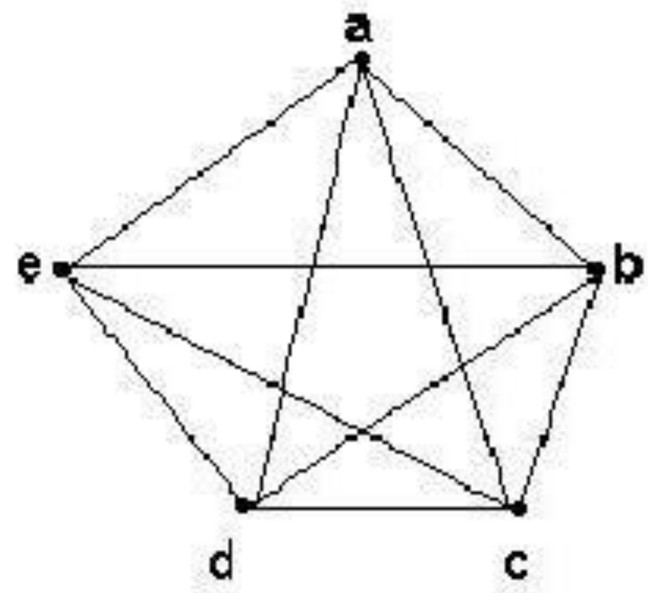
Directed graphs (digraphs)

G is a *directed graph* or *digraph* if each edge has been associated with an ordered pair of vertices, i.e. each edge has a direction



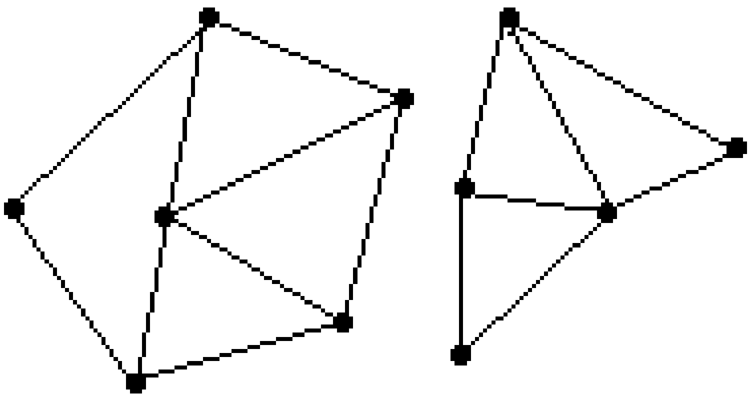
Complete graph K_n

- Let $n \geq 3$
- The *complete graph K_n* is the graph with n vertices and every pair of vertices is joined by an edge.
- The figure represents K_5



Connected graphs

- A graph is *connected* if every pair of vertices can be connected by a path
- Each connected subgraph of a non-connected graph G is connected graph G is called a *component* of G



2 connected components

Walks

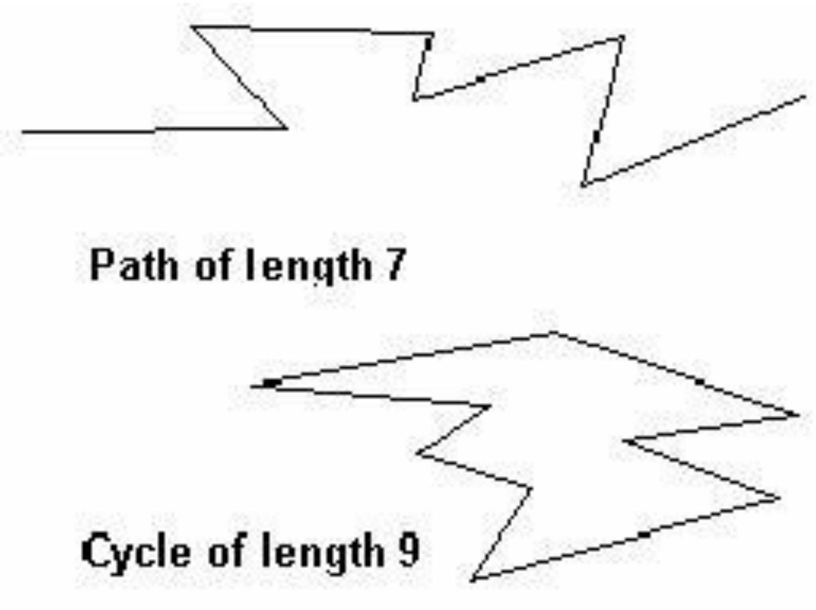
DEFINITION 10.2.1

Let u and v be two vertices in a graph G. A **walk** from u to v, in G, is an alternating sequence of $n+1$ vertices and n edges of G

$$(u = v_1, e_1, v_2, e_2, v_3, e_3, \ldots, v_{n-1}, e_{n-1}, v_n, e_n, v_{n+1} = v)$$

beginning with vertex u, called the **initial vertex**, and ending with vertex v, called the **terminal vertex**, in which v_i and v_{i+1} are endpoints of edge e_i for $i = 1, 2, \ldots, n$.

Paths and cycles



- A *path of length n* is a sequence of n + 1 vertices and n consecutive edges
- A *cycle* is a path that begins and ends at the same vertex

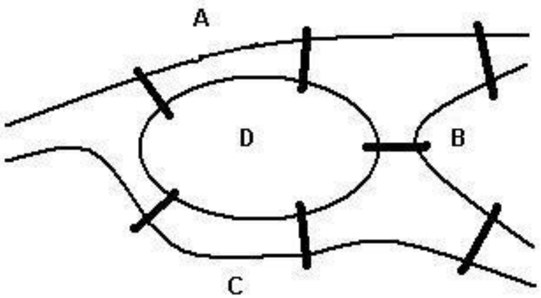
Walks, Paths, and Cycles

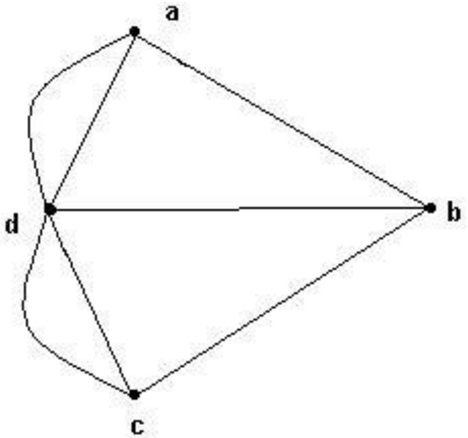
Table 10.1 Some properties of walks, trails, paths, circuits, and cycles

	Vertices	**Edges**	
Walks	Repetition allowed	Repetition allowed	
Trails	Repetition allowed	No repetition of edges	
Paths	No repetition of vertices except possibly starting and terminal vertices	No repetition of edges	
Circuits	Repetition allowed	No repetition of edges	A nontrivial closed trail
Cycles	No repetition of vertices except starting and terminal vertices	No repetition of edges	A nontrivial closed trail without repetition of vertices except starting and terminal vertices

Euler cycles

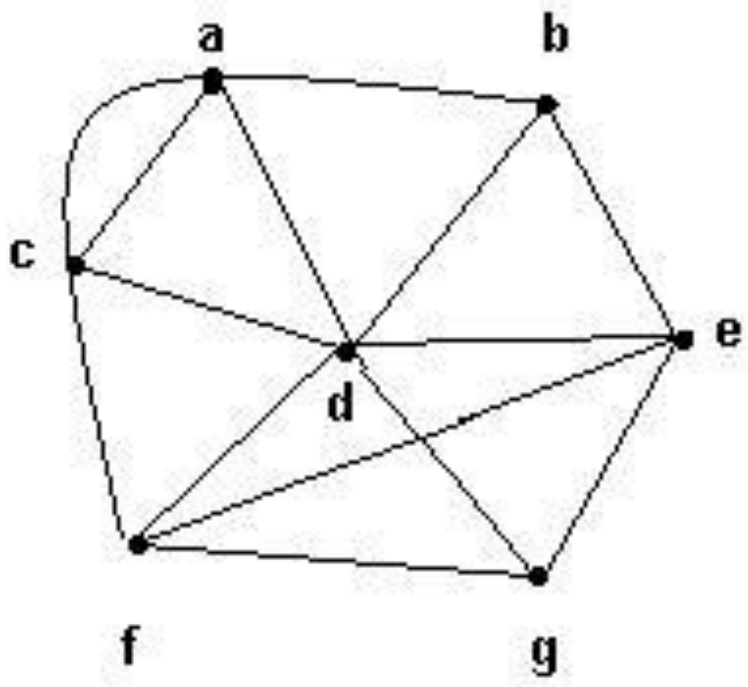
- An *Euler cycle* in a graph G is a simple cycle that passes through every edge of G only once.
- The Königsberg bridge problem:
  - Starting and ending at the same point, is it possible to cross all seven bridges just once and return to the starting point?
- This problem can be represented by a graph
- Edges represent bridges and each vertex represents a region.



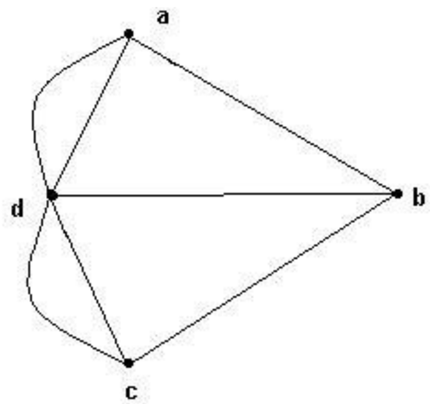


Degree of a vertex

- The *degree* of a vertex v, denoted by δ(v), is the number of edges incident on v
- Example:
  - δ(a) = 4, δ(b) = 3,
  - δ(c) = 4, δ(d) = 6,
  - δ(e) = 4, δ(f) = 4,
  - δ(g) = 3.



Euler graphs



- A graph G is an *Euler graph* if it has an Euler cycle.

Theorems 6.2.17 and 6.2.18: G is an Euler graph if and only if G is connected and all its vertices have even degree.

- The connected graph represents the Konigsberg bridge problem.
- It is not an Euler graph.
- Therefore, the Konigsberg bridge problem has *no solution*.

Sum of the degrees of a graph

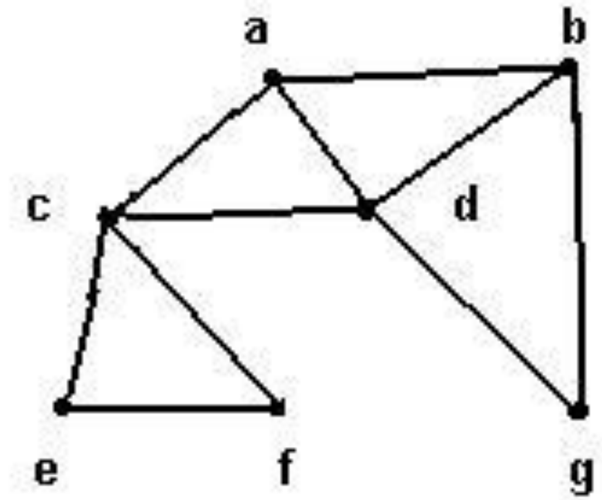
Theorem 6.2.21: If G is a graph with m edges and n vertices $v_1, v_2, \ldots, v_n$, then

$$\sum_{i=1}^{n} \delta(v_i) = 2m$$

In particular, the sum of the degrees of all the vertices of a graph is even.

Hamiltonian cycles

- *Traveling salesperson problem*
  - To visit every vertex of a graph G only once by a simple cycle.
  - Such a cycle is called a *Hamiltonian cycle*.
  - If a connected graph G has a Hamiltonian cycle, G is called a *Hamiltonian graph*.



A non-Hamiltonian graph

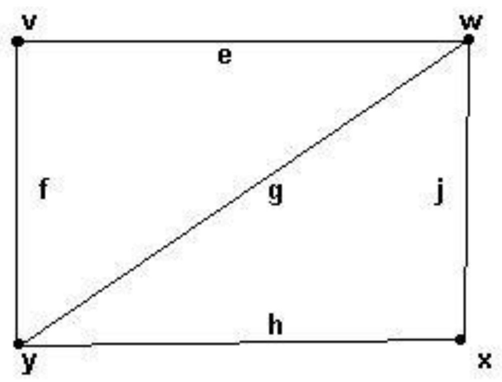
A shortest-path algorithm

- Due to Edsger W. Dijkstra, Dutch computer scientist born in 1930
- Dijkstra's algorithm finds the length of the shortest path from a single vertex to any other vertex in a connected weighted graph.
- For a simple, connected, weighted graph with n vertices, Dijkstra’s algorithms has worst-case run time $\Theta(n^2)$.

Representations of graphs

- **Adjacency matrix**
  - **Rows and columns are labeled with ordered vertices**
  - **write a 1 if there is an edge between the row vertex and the column vertex**
  - **and 0 if no edge exists between them**

	v	w	x	y
v	0	1	0	1
w	1	0	1	1
x	0	1	0	1
y	1	1	1	0



Adjacency list

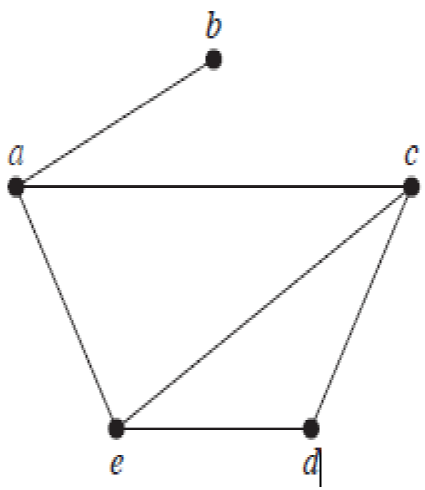


FIGURE 1 A Simple Graph.

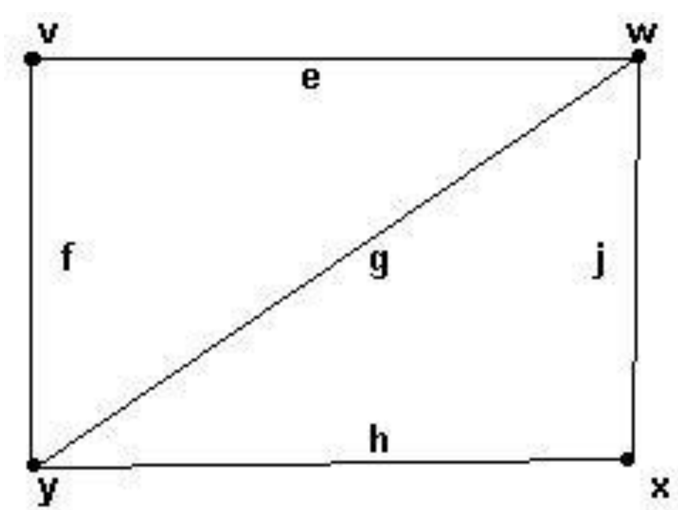
- specify the vertices that are adjacent to each vertex of the graph.

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Incidence matrix

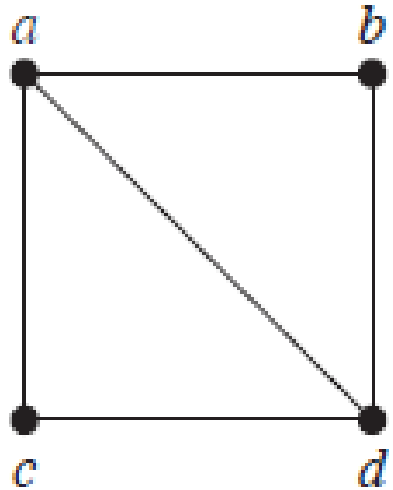
- **Incidence matrix**
  - **Label rows with vertices**
  - **Label columns with edges**
  - **1 if an edge is incident to a vertex, 0 otherwise**



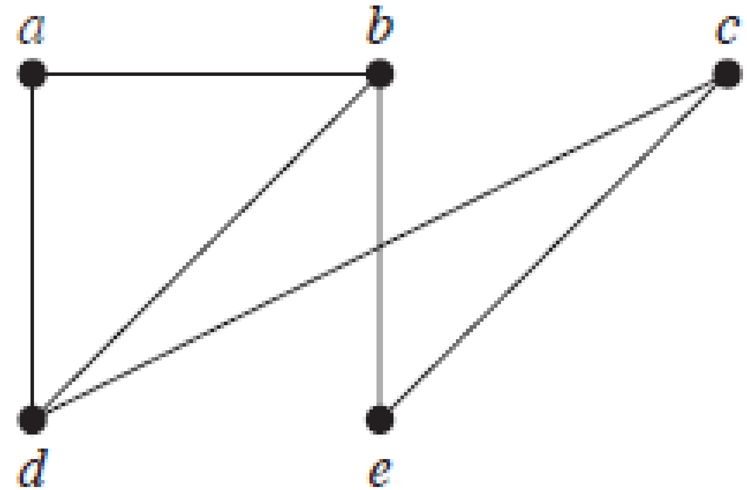
	e	f	g	h	j
v	1	1	0	0	0
w	1	0	1	0	1
x	0	0	0	1	1
y	0	1	1	1	0

In Exercises 1–4 use an adjacency list to represent the given graph.

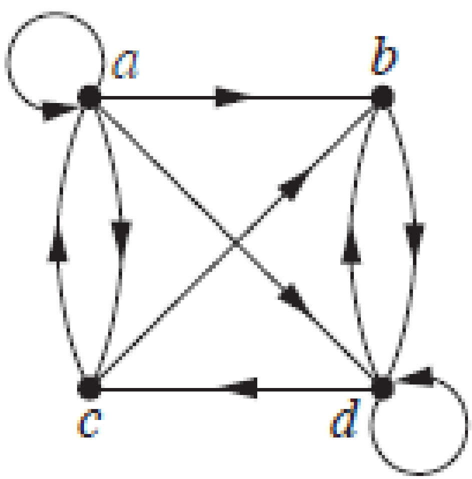
1.



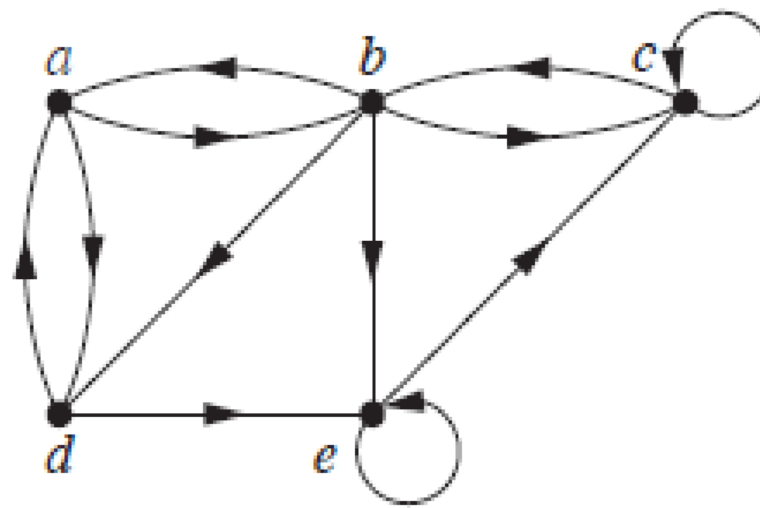
2.



3.



4.



In Exercises 22–24 draw the graph represented by the given adjacency matrix.

22. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

Isomorphic graphs

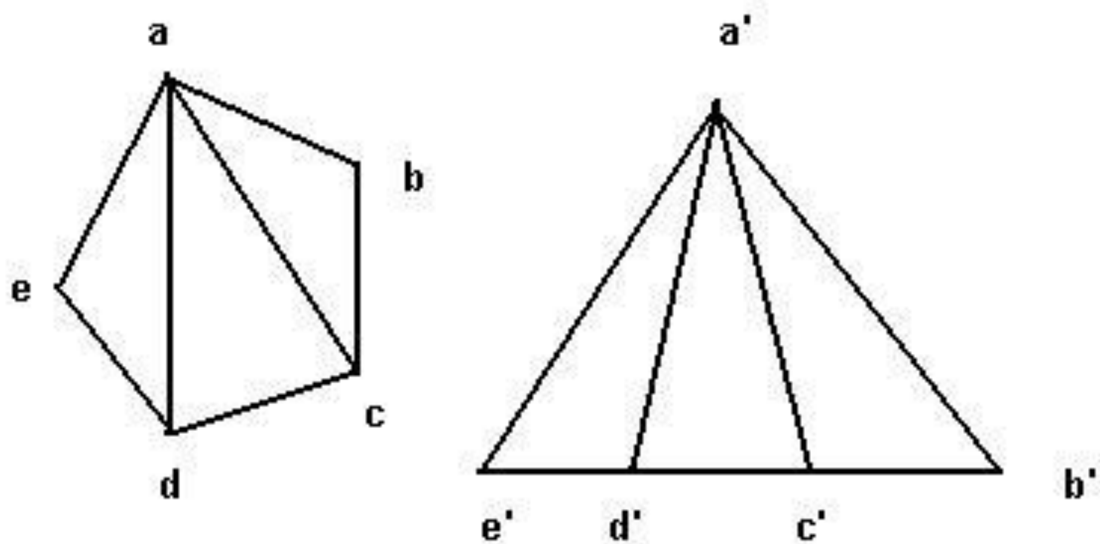
G_1 and G_2 are ***isomorphic***

If G1 and G2 have

1> Equal no. of vertices

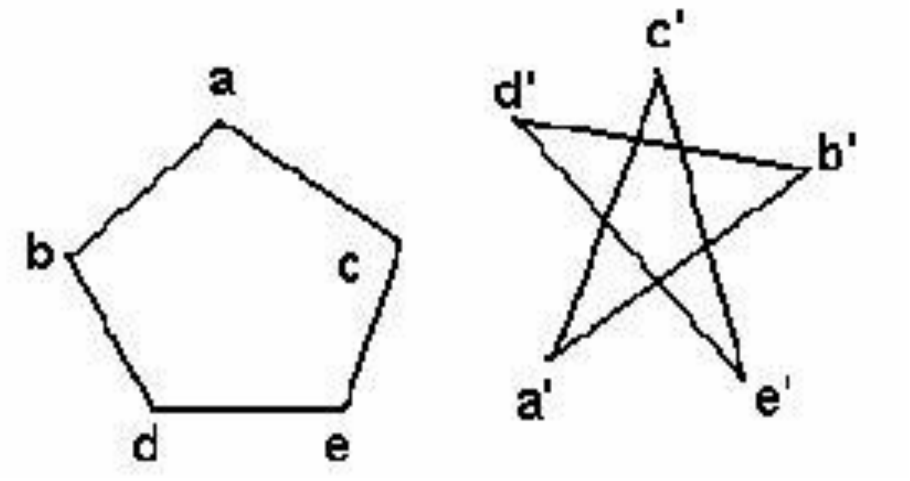
2> Equal no. of edges

3> Equal no. of vertices of particular degree



Isomorphism and adjacency matrices

- **Two graphs are isomorphic if and only if after reordering the vertices their adjacency matrices are the same**



	a	b	c	d	e
a	0	1	1	0	0
b	1	0	0	1	0
c	1	0	0	0	1
d	0	1	0	0	1
e	0	0	1	1	0

Determine whether the graphs G and H displayed in Figure 12 are isomorphic.

Solution: Both G and H have six vertices and seven edges. Both have four vertices of degree two and two vertices of degree three. It is also easy to see that the subgraphs of G and H consisting of all vertices of degree two and the edges connecting them are isomorphic (as the reader should verify). Because G and H agree with respect to these invariants, it is reasonable to try to find an isomorphism f.

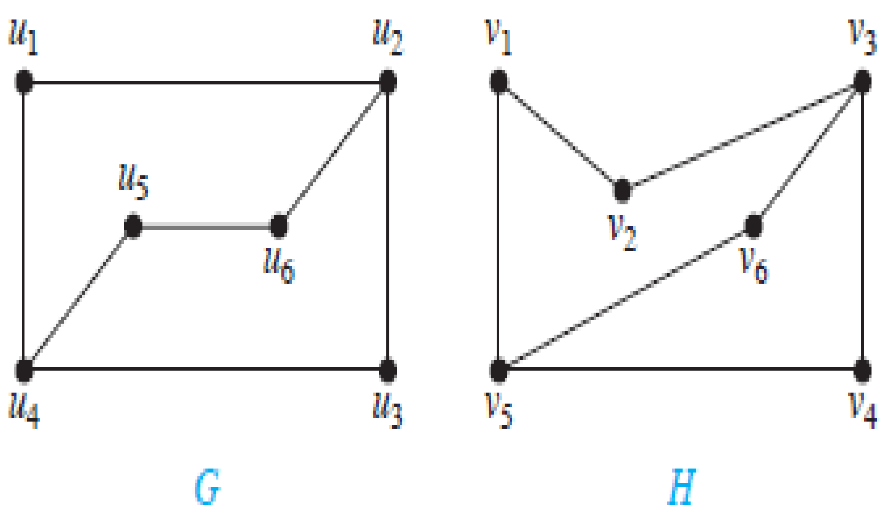


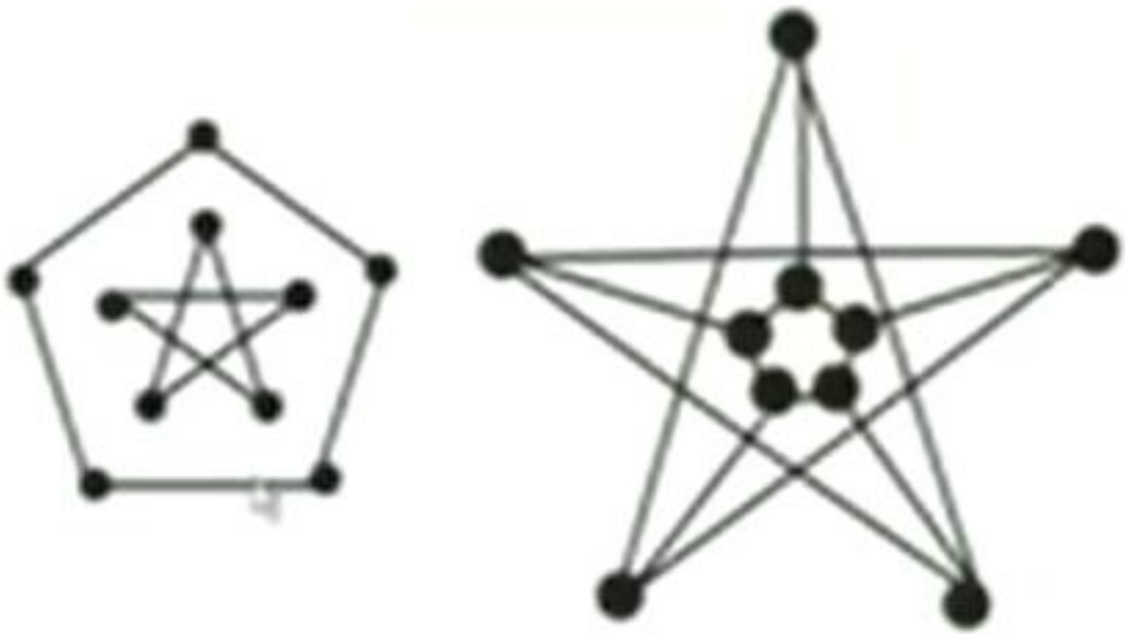
FIGURE 12 Graphs G and H.

$$
\mathbf{A}_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}
\begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\left[\begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{array}\right] \end{array},
$$

and the adjacency matrix of H with the rows and columns labeled by the images of the corresponding vertices in G,

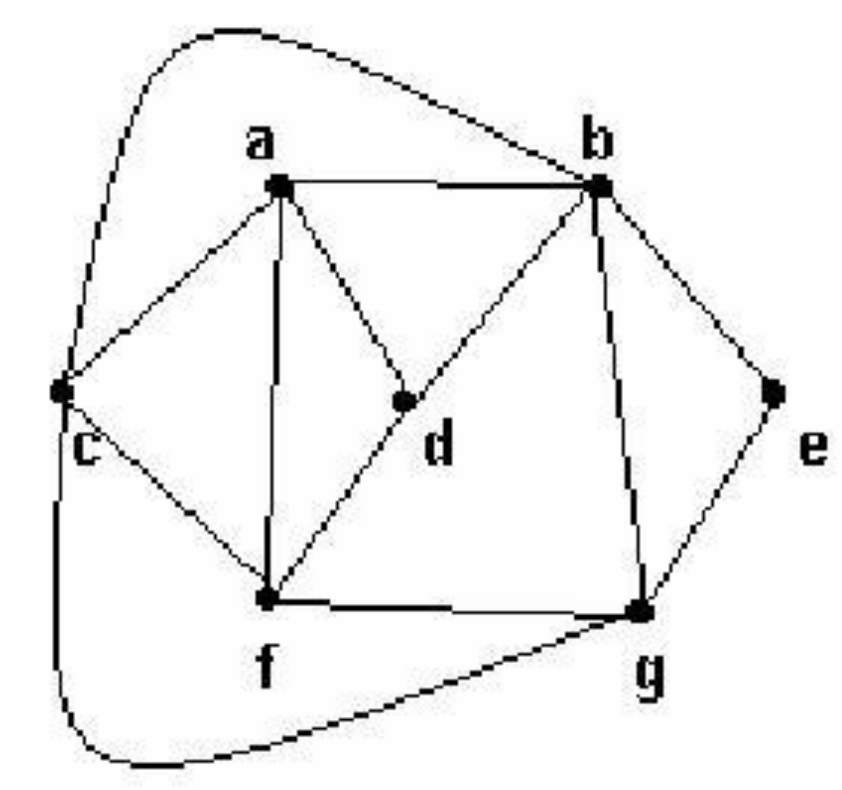
$$
\mathbf{A}_H = \begin{array}{c} \\ v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array}
\begin{array}{c} \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\
\left[\begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{array}\right] \end{array}.
$$

Because $\mathbf{A}_G = \mathbf{A}_H$, it follows that f preserves edges. We conclude that f is an isomorphism, so G and H are isomorphic. Note that if f turned out not to be an isomorphism, we would *not* have established that G and H are not isomorphic, because another correspondence of the vertices in G and H may be an isomorphism. ◀




u
x
v
w
y

Planar graphs



A graph is *planar* if it can be drawn in the plane without crossing edges

Euler's Formula

R(no. of regions)=e-v+2

Sub graph

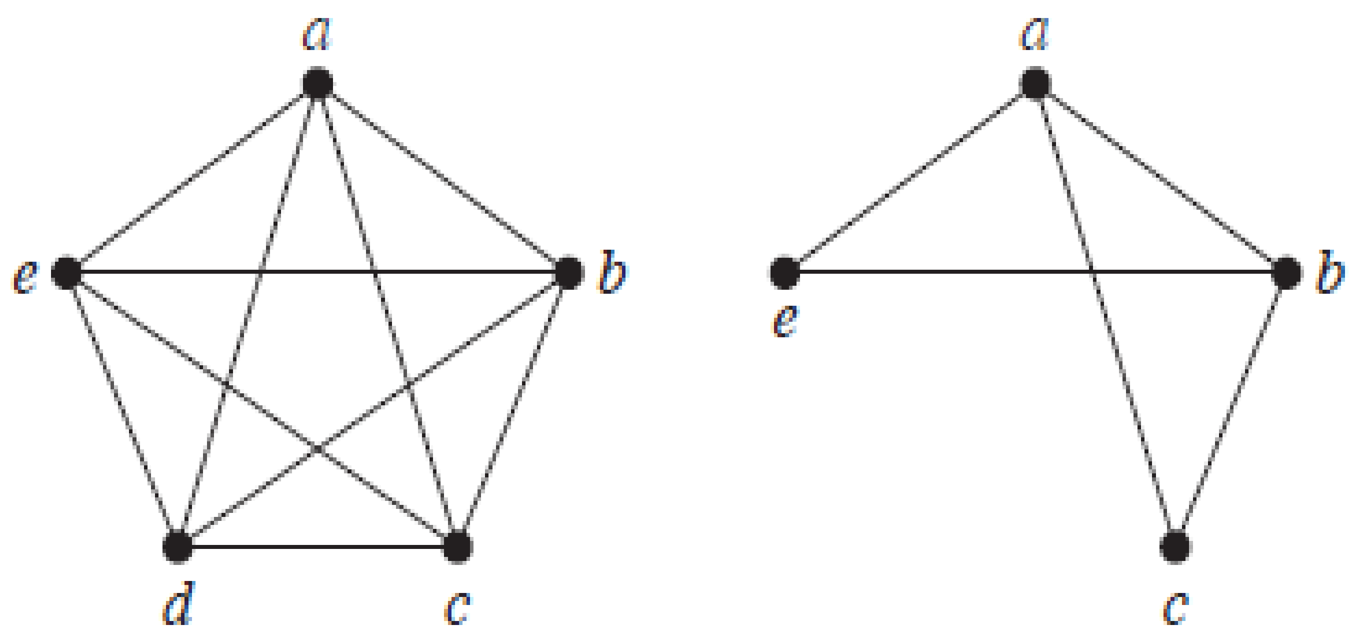


FIGURE 15 **A Subgraph of K_5.**