THE STRUCTURE OF ALGEBRAS

An algebra is characterized by specifying the following three components:

1. a set, called the carrier of the algebra
2. operations defined on the carrier, and
3. distinguished elements of the carrier, called the constants of the algebra.

Let $A = \langle S, \circ, \Delta, k \rangle$ and $A' = \langle S', \circ', \Delta', k' \rangle$ be algebras. Then A' is a sub algebra of A if

(i) $S' \subset S$;

(ii) $a \circ' b = a \circ b$, for all $a, b \in S'$;

(iii) $\Delta' a = \Delta a$ for all $a \in S'$;

(iv) $k' = k$.

If A' is a subalgebra of A, then A' has the same signature as A and obeys the same axioms.

SOME VARIETIES OF ALGEBRAS:

SEMIGROUPS:

A semigroup is an algebra with signature $\langle S, \circ \rangle$, where $\circ$ is a binary associative operation.

The preceding definition establishes that the variety of semigroups consists of all algebras with a single binary operation which satisfies the axiom of associativity

$$a \circ (b \circ c) = (a \circ b) \circ c$$

MONOIDS:

A monoid is an algebra with signature $\langle S, \circ, 1 \rangle$, where $\circ$ is a binary associative operation on S and 1 is a two – sided identity for the operation $\circ$, i.e., the following axioms hold for all elements a, b, c $\in S$:

$$a \circ (b \circ c) = (a \circ b) \circ c,$$

$$a \circ 1 = a,$$

$$1 \circ a = a.$$

If $\langle S, \circ, 1 \rangle$ is a monoid ant $T \subset S$, $1 \in T$, and $T \circ T \subset T$, then $\langle T, \circ, 1 \rangle$ is a subalgebra of $\langle S, \circ, 1 \rangle$; a subalgebra of a monoid is called a submonoid.

GROUPS:

A group is an algebra with signature $\langle S, \circ, ^{-}, 1\rangle$ such that $\circ$ is an associative binary operation S, the constant 1 is a two – sided identity for the operation $\circ$, and $^{-}$ is a unary operation defined over the carrier such that for all $x \in S$, $\bar{x}$ is an inverse for x with respect to $\circ$.

If $A = \langle S, \circ, ^{-}, 1\rangle$ is a group and $A' = \langle T, \circ, ^{-}, 1\rangle$ is a subalgebra of A, then A' is called a subgroup of A. A subalgebra of a group is a group.

BOOLEAN ALGEBRA:

A Boolean algebra is an algebra with signature

<S, +, . , ¯, 0, 1>

(where + and . are binary operations and ¯ is a unary operation called complementation) and the following axioms hold.

$$\left.\begin{array}{l}\text{(i) } a + b = b + a \\ \text{(ii) } ab = ba\end{array}\right\} \text{Commutative laws}$$

$$\left.\begin{array}{l}\text{(iii) } (a + b) + c = a + (b + c) \\ \text{(iv) } (ab)c = a(bc)\end{array}\right\} \text{Associative laws}$$

$$\left.\begin{array}{l}\text{(v) } a(b + c) = ab + ac \\ \text{(vi) } a + (bc) = (a + b)(a + c)\end{array}\right\} \text{Distributive laws}$$

$$\text{(vii) } a + 0 = a \qquad \text{0 is an identity for +}$$

$$\text{(viii) } a1 = a \qquad \text{1 is an identity for .}$$

$$\left.\begin{array}{l}\text{(ix) } a + \bar{a} = 1 \\ \text{(x) } a\bar{a} = 0\end{array}\right\} \text{Properties of the complement}$$