Planar Graphs and Partially Ordered Sets

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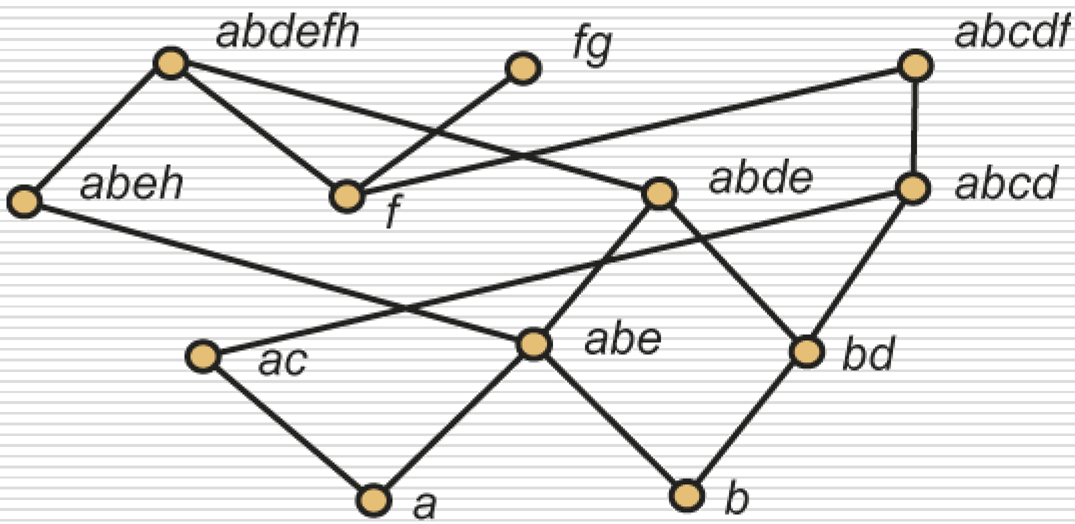
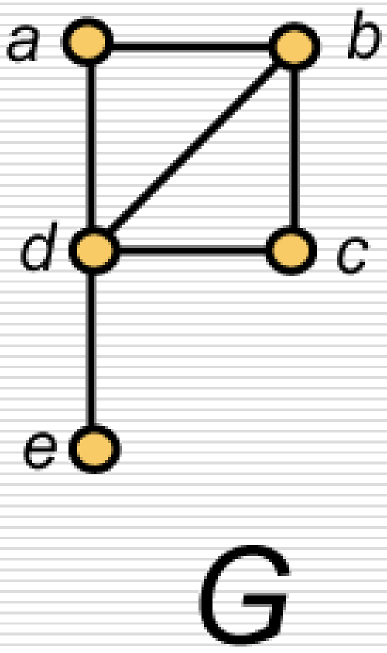
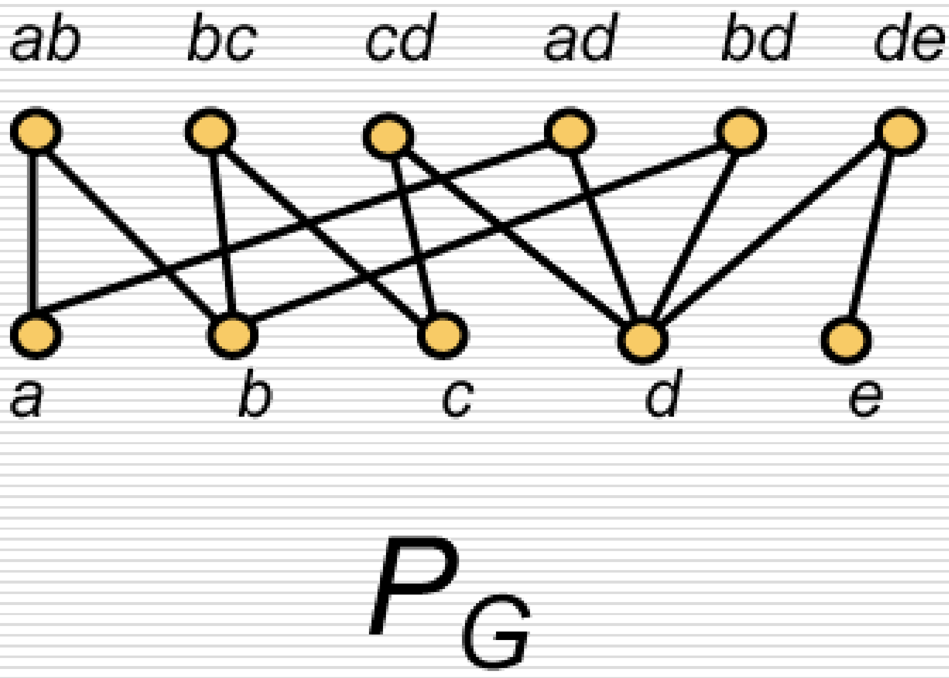
Inclusion Orders

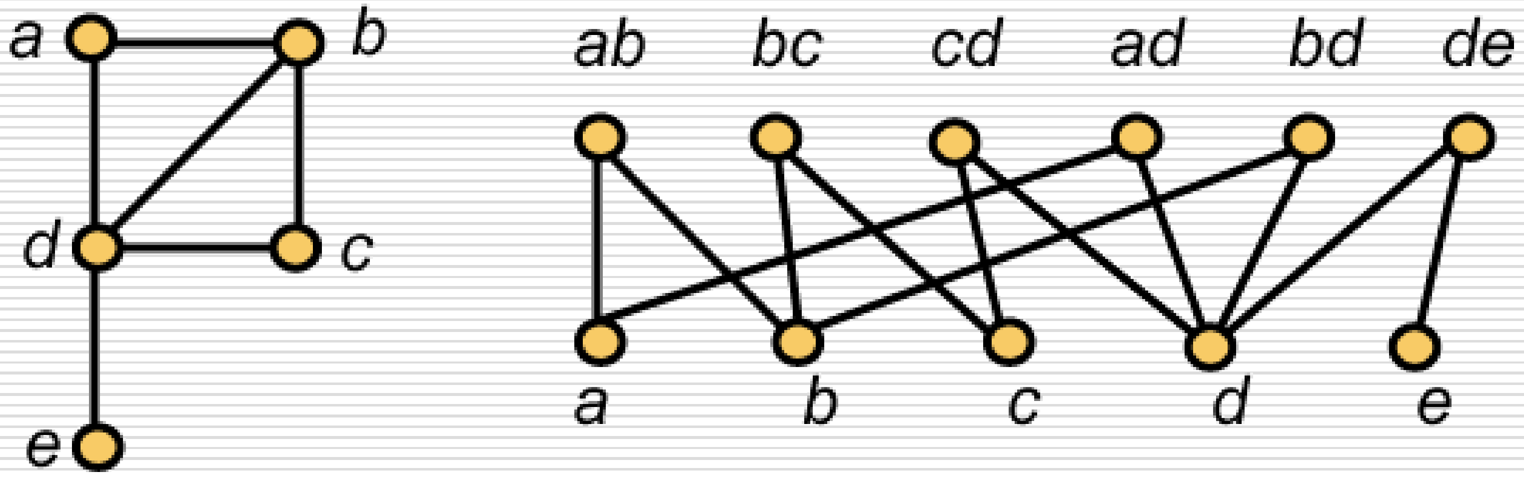


Incidence Posets



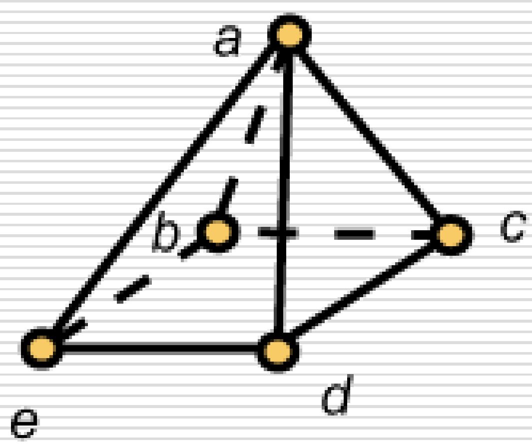


Vertex-Edge Posets

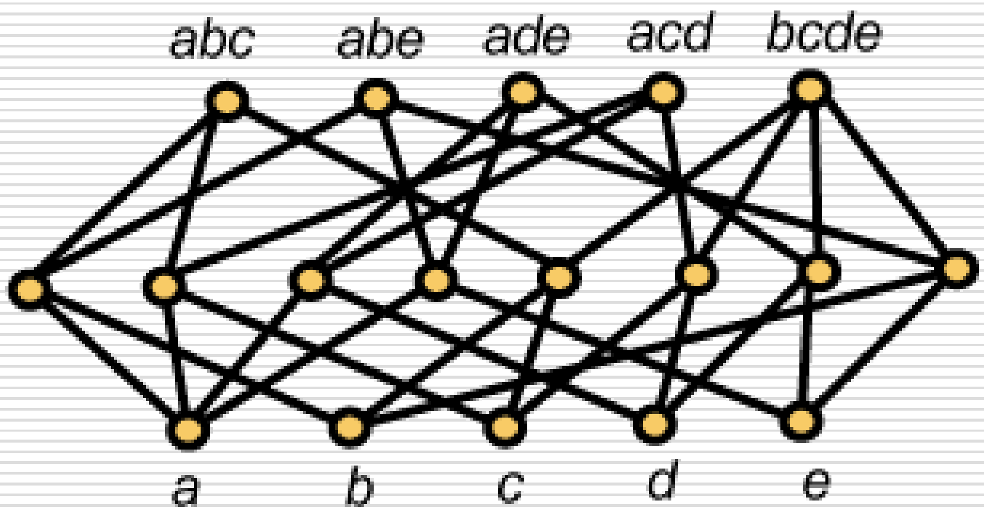


G $\quad$ P_G

Vertex-Edge-Face Posets

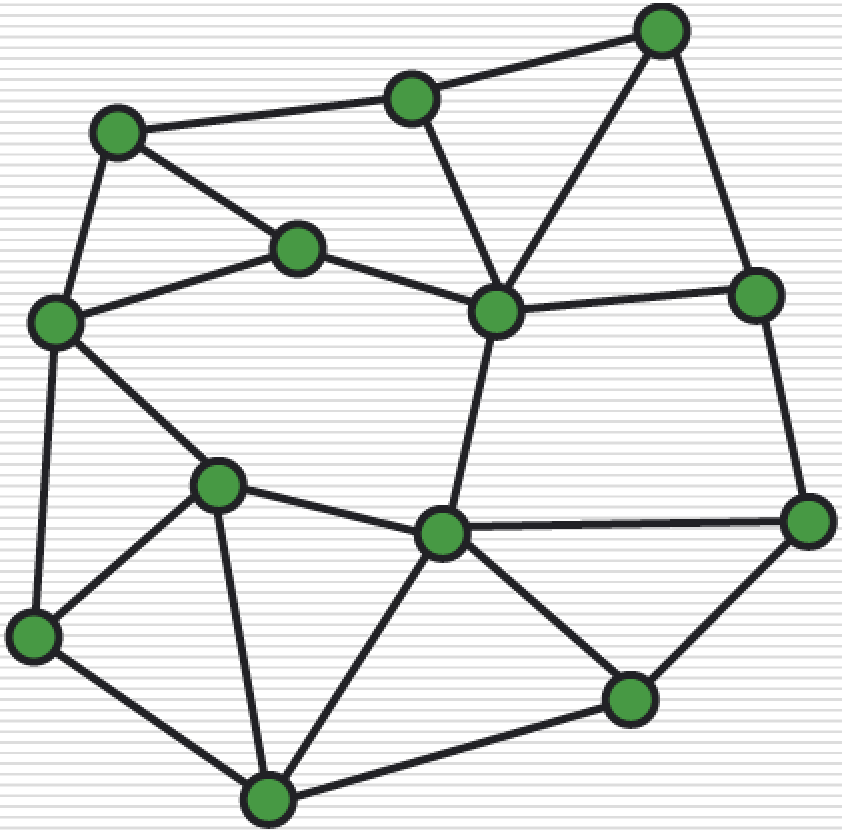


M

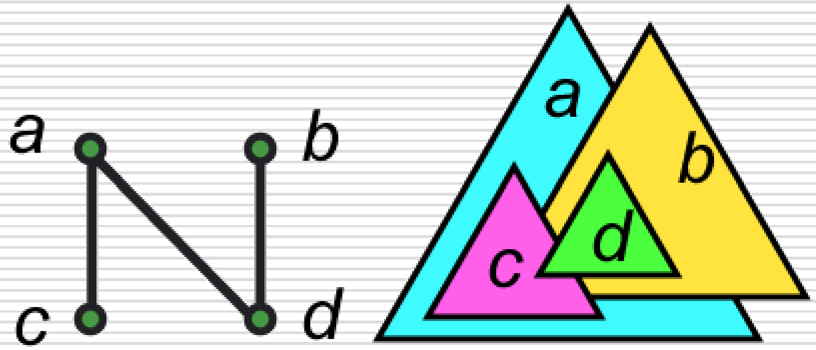


P_M

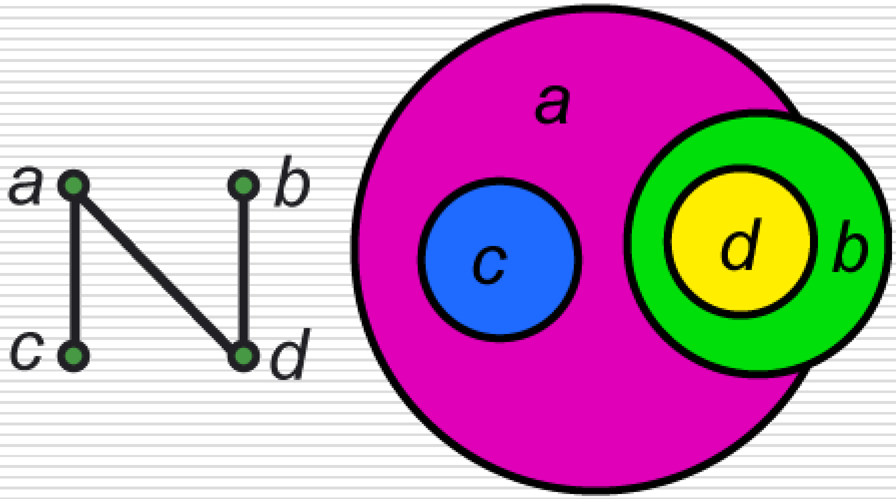
Vertex-Edge-Face Posets for Planar Graphs



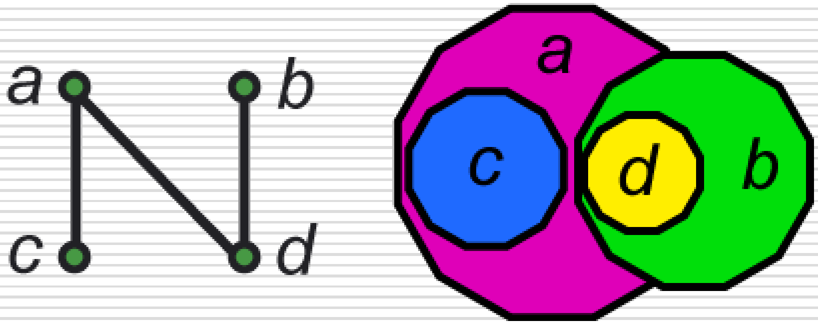
Triangle Orders



Circle Orders



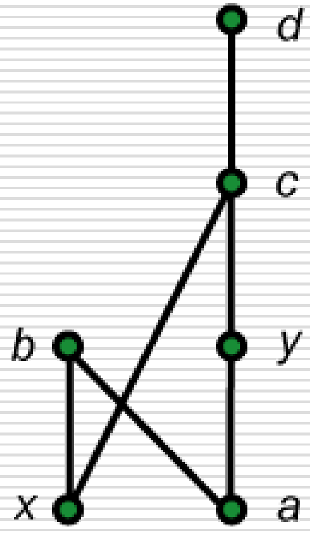
N-gon Orders



Dimension of Posets

- The dimension of a poset P is the least *t* so that P is the intersection of *t* linear orders.
- Alternately, dim(P) is the least *t* for which P is isomorphic to a subposet of R^t

A 2-dimensional poset

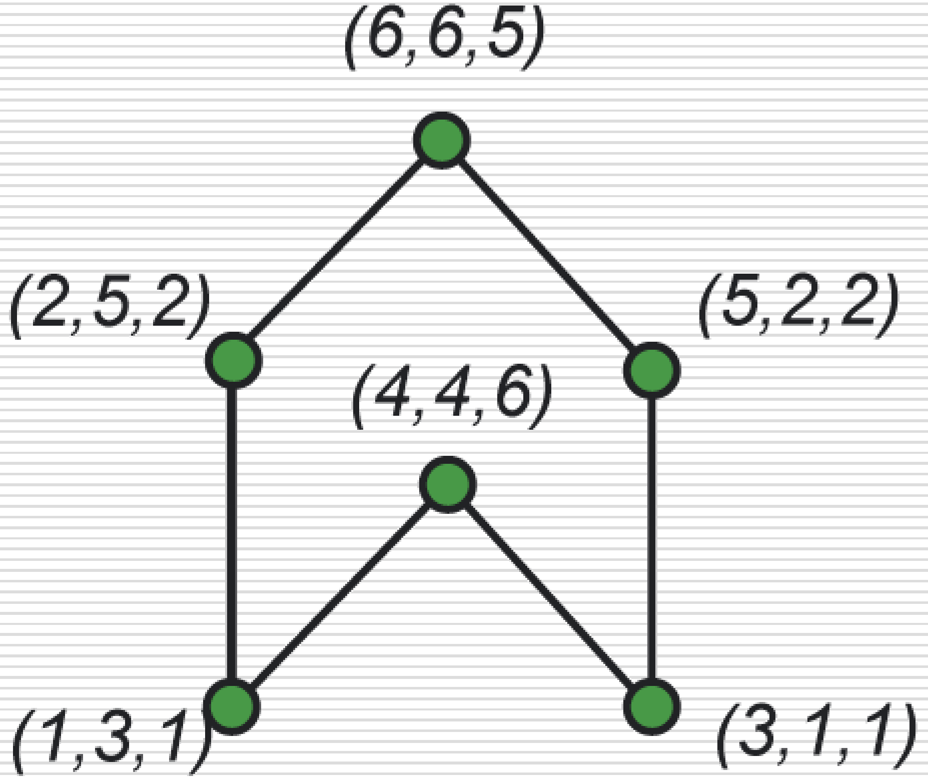


P

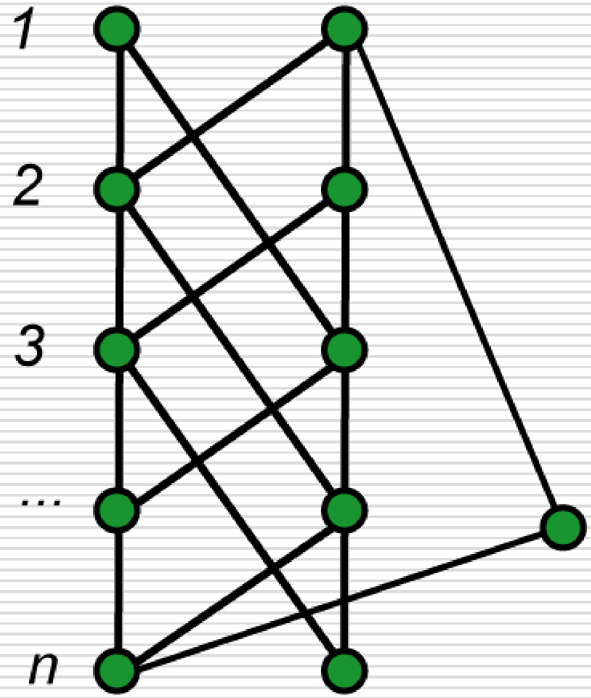
$L_1 = a < y < x < c < d < b$

$L_2 = x < a < b < y < c < d$

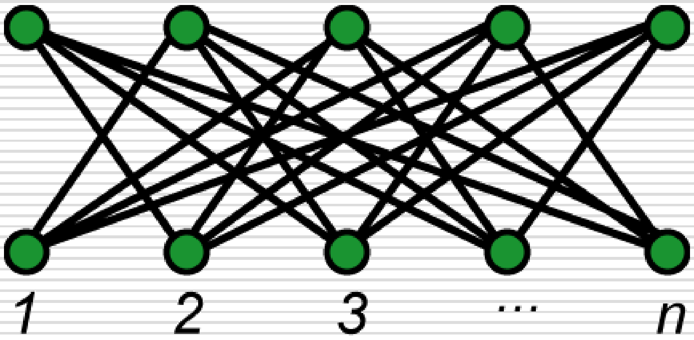
A 3-dimensional poset



A Family of 3-dimensional Posets

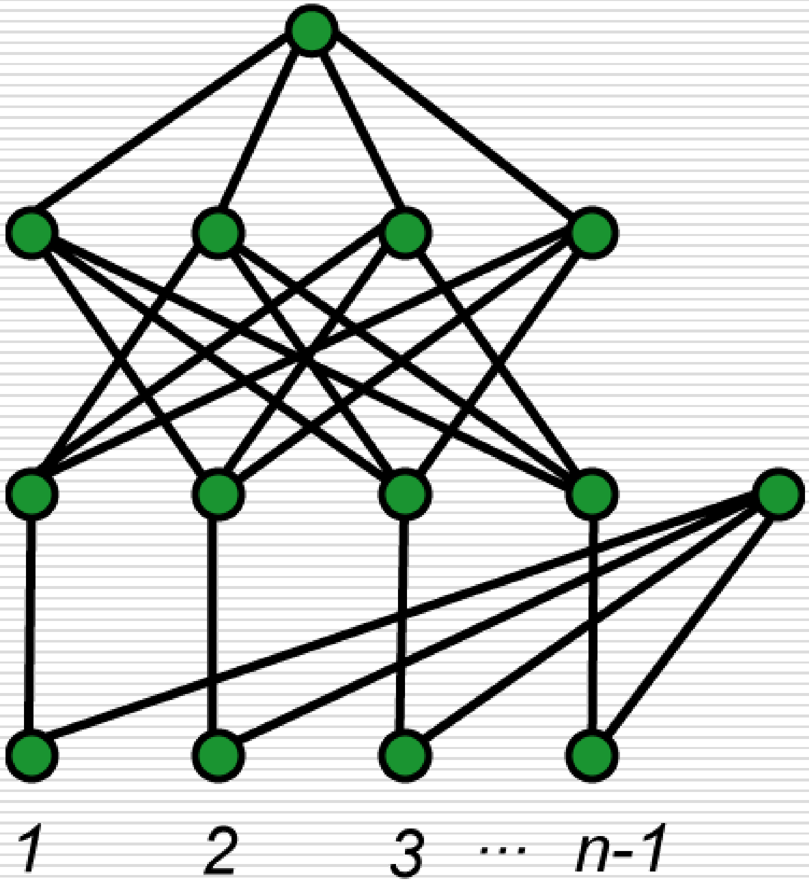


Standard Examples of n-dimensional posets



Fact: When $n \geq 2$, a poset on $2n+1$ points has dimension at most n. The standard example is the only such poset when $n \geq 4$.

Another Example of an n-dimensional Poset



Complexity Issues

- It is easy to show that the question: dim(P) $\leq$ 2? is in P.
- Yannakakis showed in 1982 that the question: dim(P) $\leq t$? is NP-complete for fixed $t \geq 3$.
- The question: dim(P) $\leq$ t? is NP-complete for height 2 posets for fixed $t \geq 4$.
- Still not known whether: dim(P) $\leq$ 3? is NP-complete for height 2 posets.

Schnyder's Theorem (1989)

A graph is planar if and only if the dimension of its incidence poset is at most 3.

Proposition

A poset has dimension at most 3 if and only if it is a triangle order.

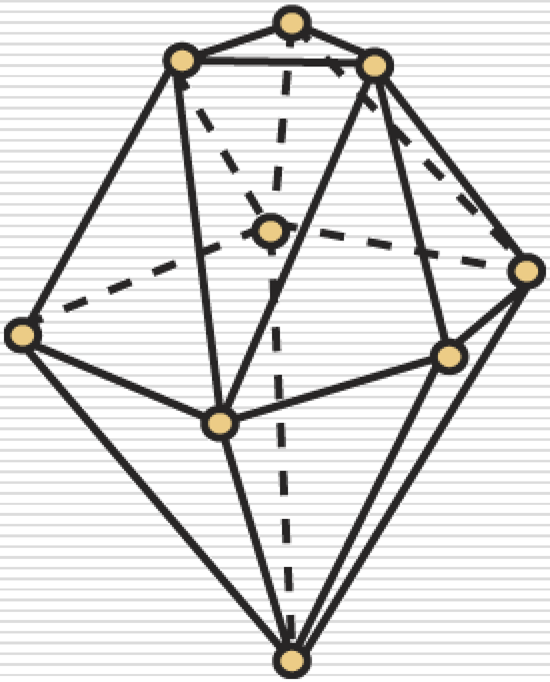
Schnyder’s Theorem (restated)

A graph is planar if and only if its incidence poset is a triangle order.

3-Connected Planar Graphs

- **Theorem** (Brightwell and Trotter, 1993): If G is a planar 3-connected graph and P is the vertex-edge-face poset of G, then $\dim(P) = 4$.
- The removal of any vertex or any face from P reduces the dimension to 3.

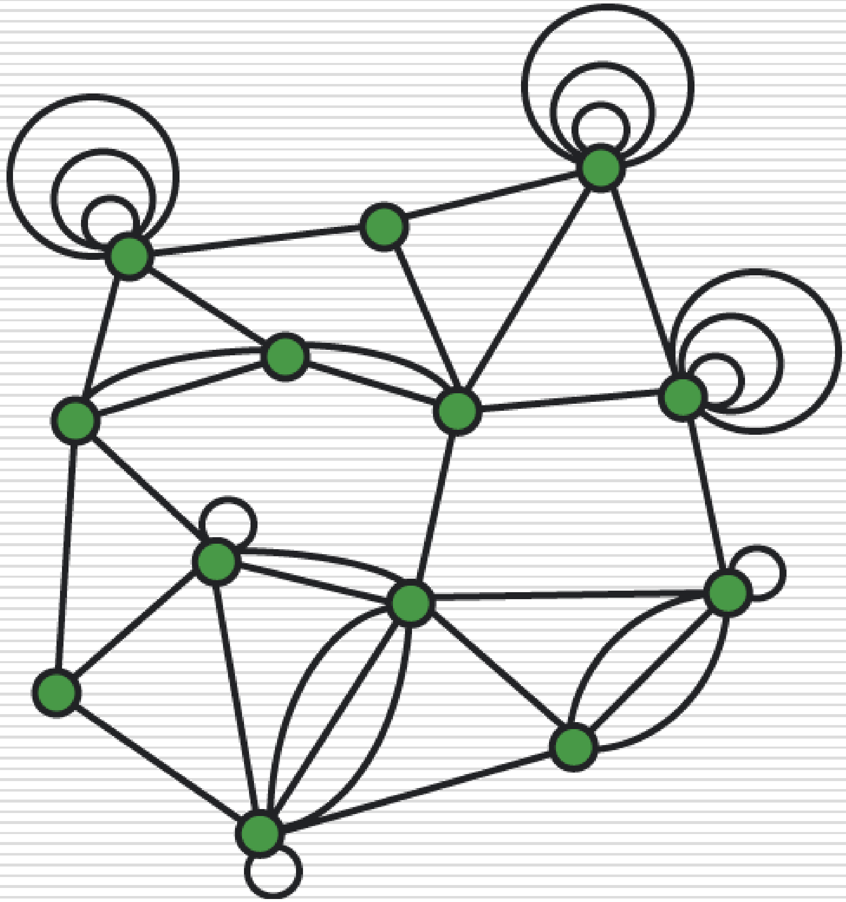
Convex Polytopes in R^3



Convex Polytopes in R^3

- **Theorem** (Brightwell and Trotter, 1993): If M is a convex polytope in R^3 and P is its vertex-edge-face poset, then $\dim(P) = 4$.
- The removal of any vertex or face from P reduces the dimension to 3.

Planar Multigraphs

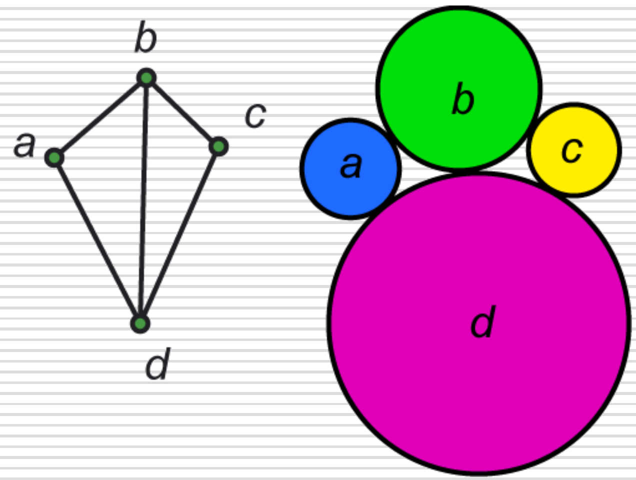


Planar Multigraphs

- **Theorem** (Brightwell and Trotter, 1997): Let D be a non-crossing drawing of a planar multigraph G, and let P be the vertex-edge-face poset determined by D. Then $\dim(P) \leq 4$.
- Different drawings may determine posets with different dimensions.

The Kissing Coins Theorem

Theorem (Koebe, 1936; Andreev, 1970; Thurston, 1985) A graph G is planar if and only if it has a representation by "kissing coins."



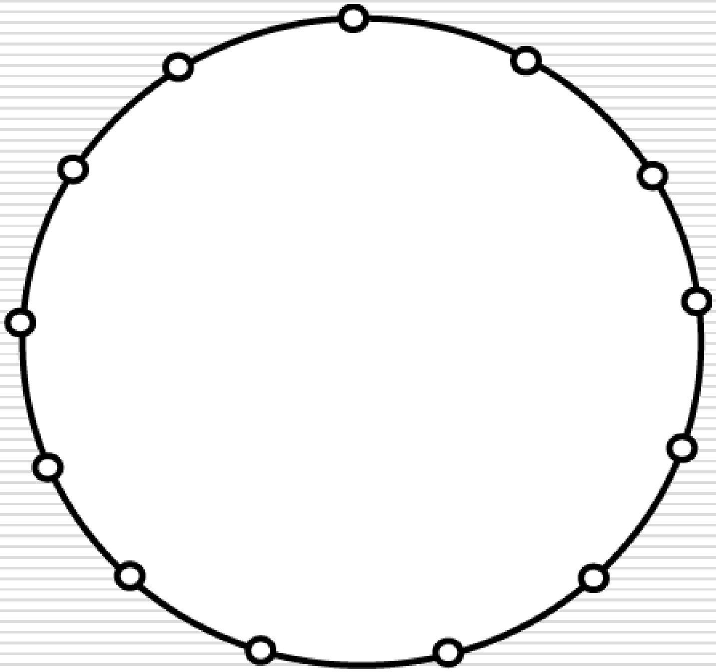
Planar Graphs and Circle Orders

Theorem (Scheinerman, 1993)
A graph is planar if and only if its incidence poset is a circle order.

Remarks on Circle Orders

- Every poset of dimension at most 2 is a circle order – in fact with circles having co-linear centers.
- Using Warren's theorem and the Alon/Scheinerman degrees of freedom technique, it follows that "almost all" 4-dimensional posets are **not** circle orders.

Standard Examples are Circle Orders



More Remarks on Circle Orders

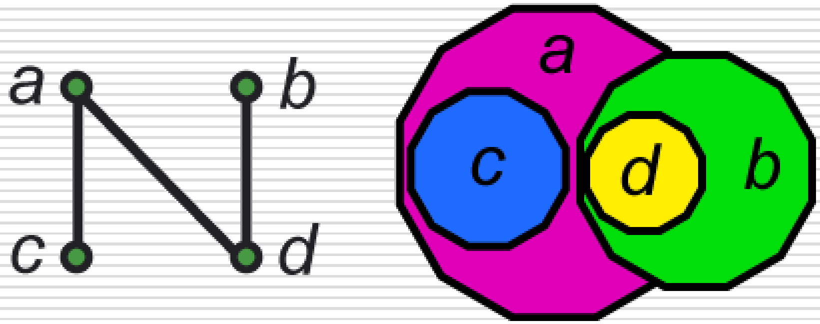
- Every 2-dimensional poset is a circle order.
- For each $t \geq 3$, some t-dimensional posets are circle orders.
- But, for each fixed $t \geq 4$, almost all t-dimensional posets are not circle orders.
- Every 3-dimensional poset is an ellipse order with parallel major axes.

Fundamental Question for Circle Orders (1984)

Is every finite 3-dimensional poset a circle order?

Support for a Yes Answer

Fact: For every $n > 2$, if P is a 3-dimensional poset, then P is an n-gon order



Support for a No Answer

Theorem (Scheinerman and Wierman, 1988): The countably infinite poset $\mathbf{Z}^3$ is not a circle order.

More Troubling News

Theorem (Fon-Der-Flaass, 1993): The countably infinite poset **N** x **2** x **3** is not a sphere order.

A Triumph for Ramsey Theory

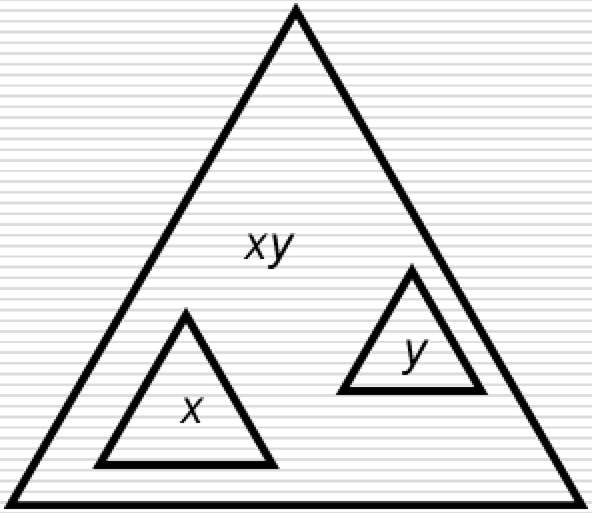
Theorem (Fishburn, Felsner, and Trotter, 1999) There exists a finite 3-dimensional poset which is not a sphere order.

Schnyder’s Theorem

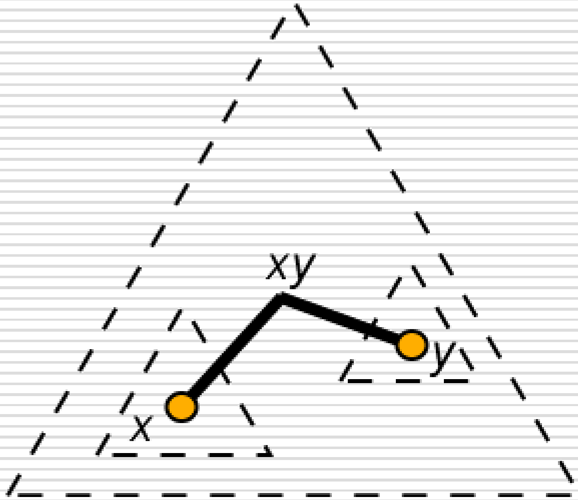
A graph is planar if and only if the dimension of its incidence poset is at most 3.

Easy Direction (Babai and Duffus, 1981)

Suppose the incidence poset has dimension at most 3.



Easy Direction

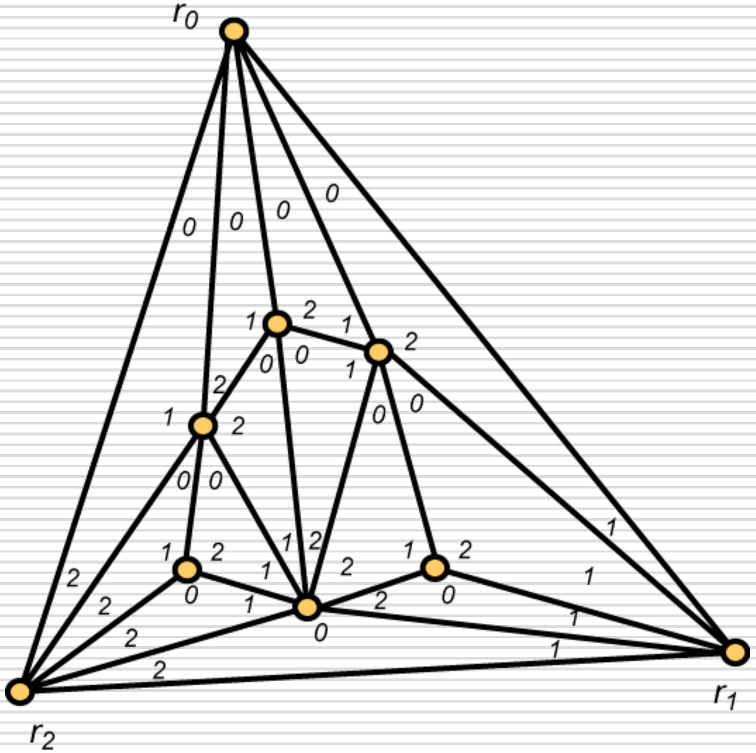


There are no non-trivial crossings. It follows that G is planar.

The Proof of Schnyder's Theorem

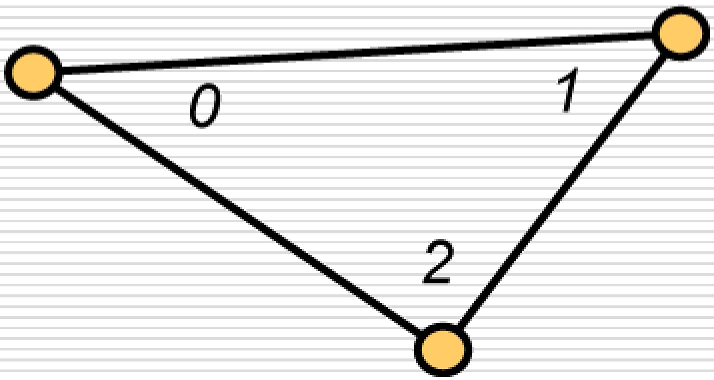
- Normal labelings of rooted planar triangulations.
- Uniform angle lemma.
- Explicit decomposition into 3 forests.
- Inclusion property
- Three auxiliary partial orders

A Normal Labeling



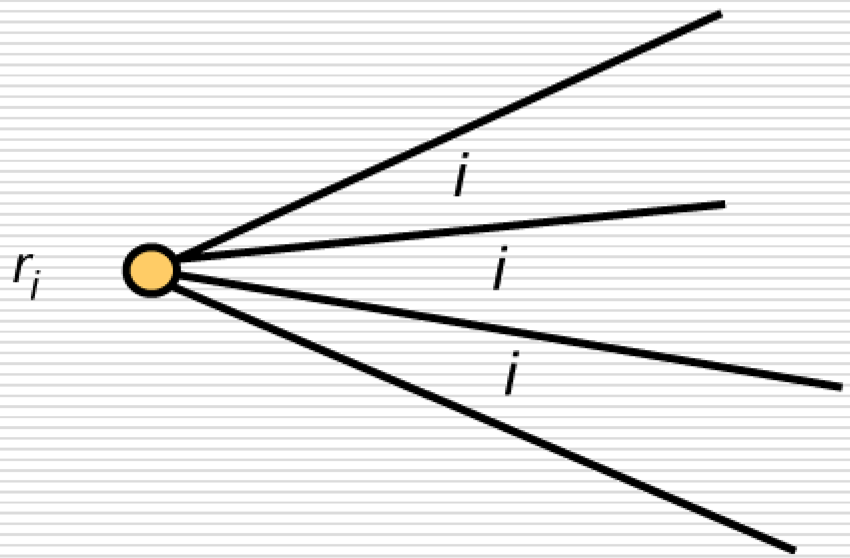
Normal Labeling - 1

1. Internal Faces



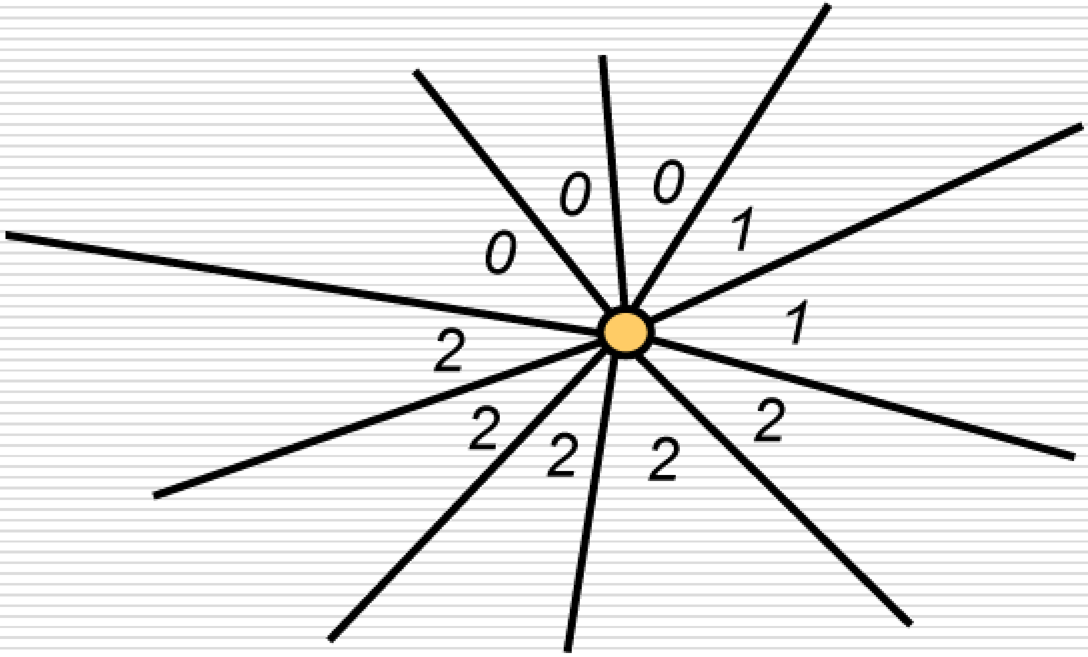
Normal Labeling - 2

2. External Vertices



Normal Labeling - 3

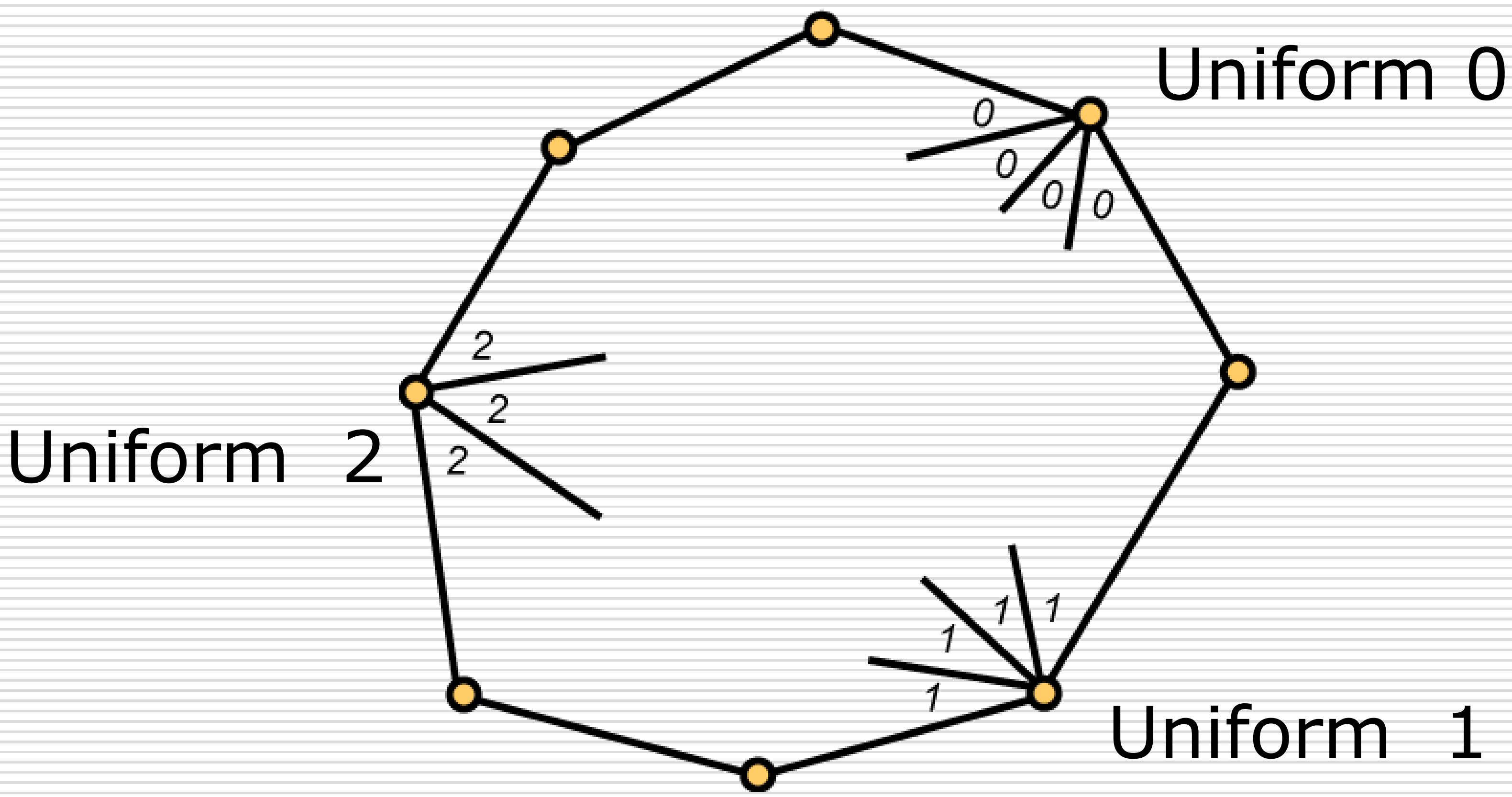
3. Internal Vertices



Lemma (Schnyder)

Every rooted planar triangulation admits a normal labeling.

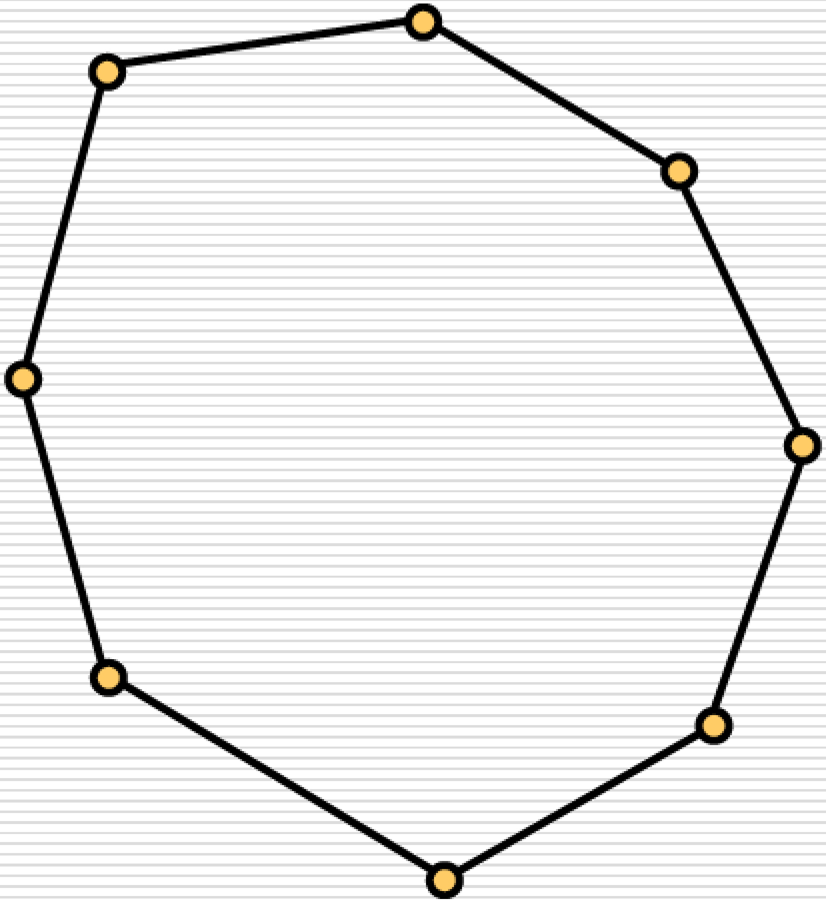
Uniform Angles on a Cycle



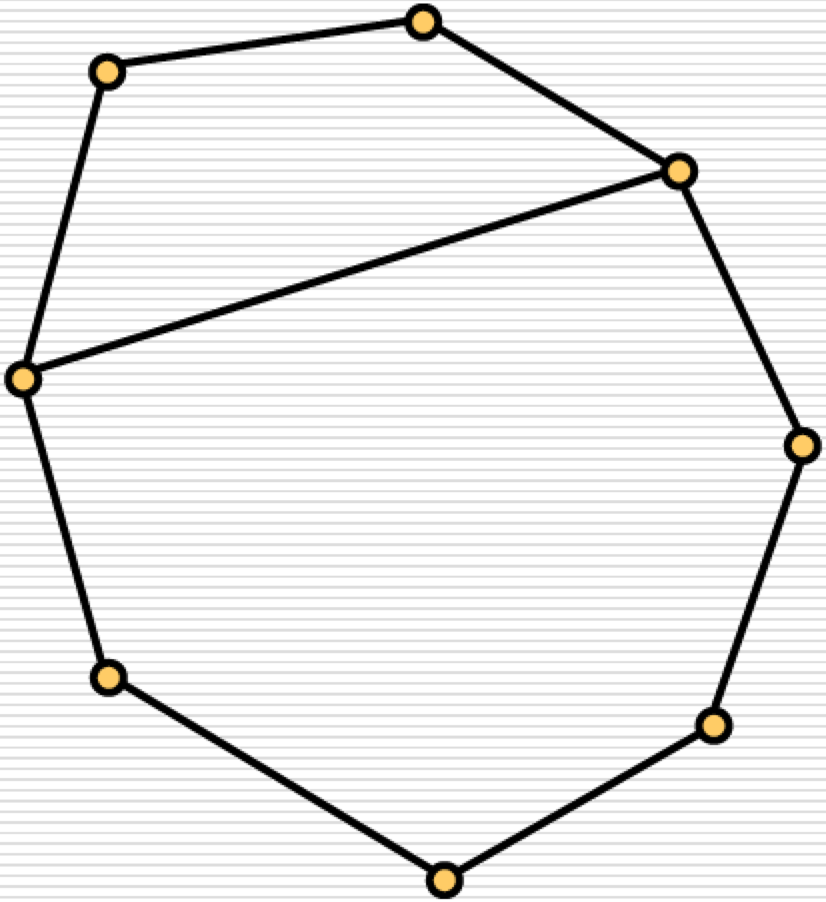
Uniform Angle Lemma (Schnyder)

If T is a rooted planar triangulation, C is a cycle in T, and L is a normal labeling of T, then for each $i = 1,2,3$, there is a uniform i on C.

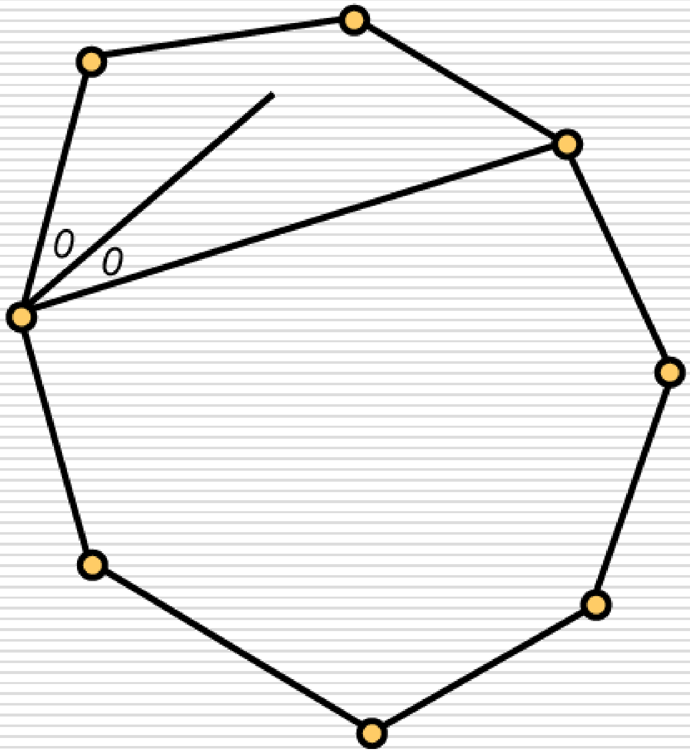
Suppose C has no Uniform 0



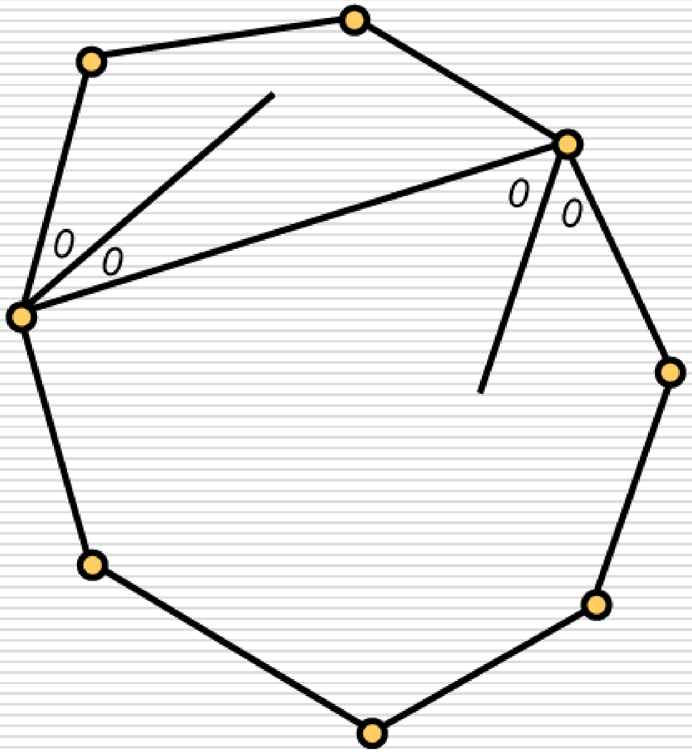
Case 1: C has a Chord



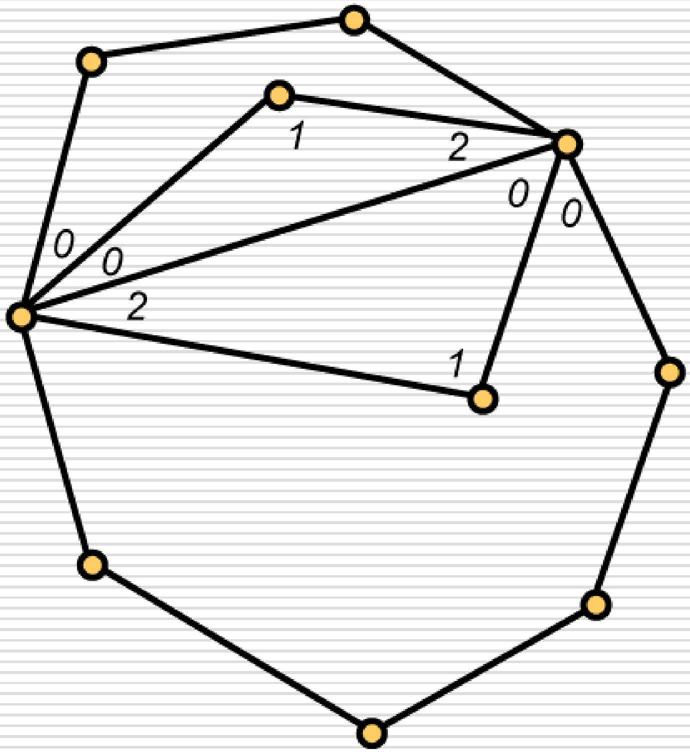
Uniform 0 on Top Part



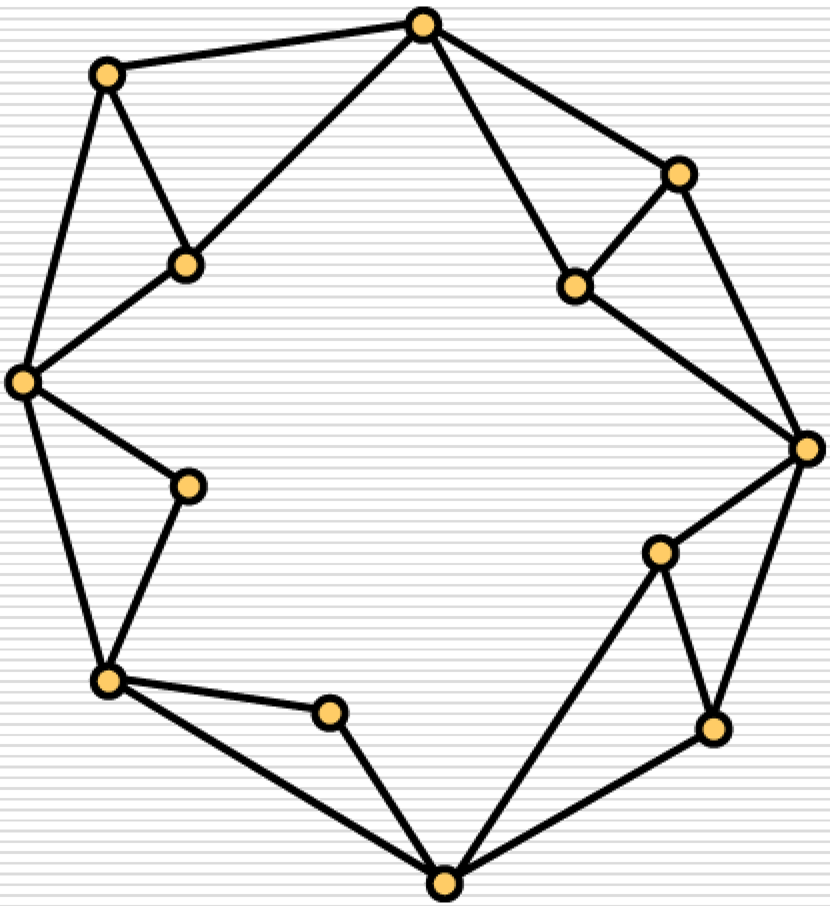
Uniform 0 on Bottom Part



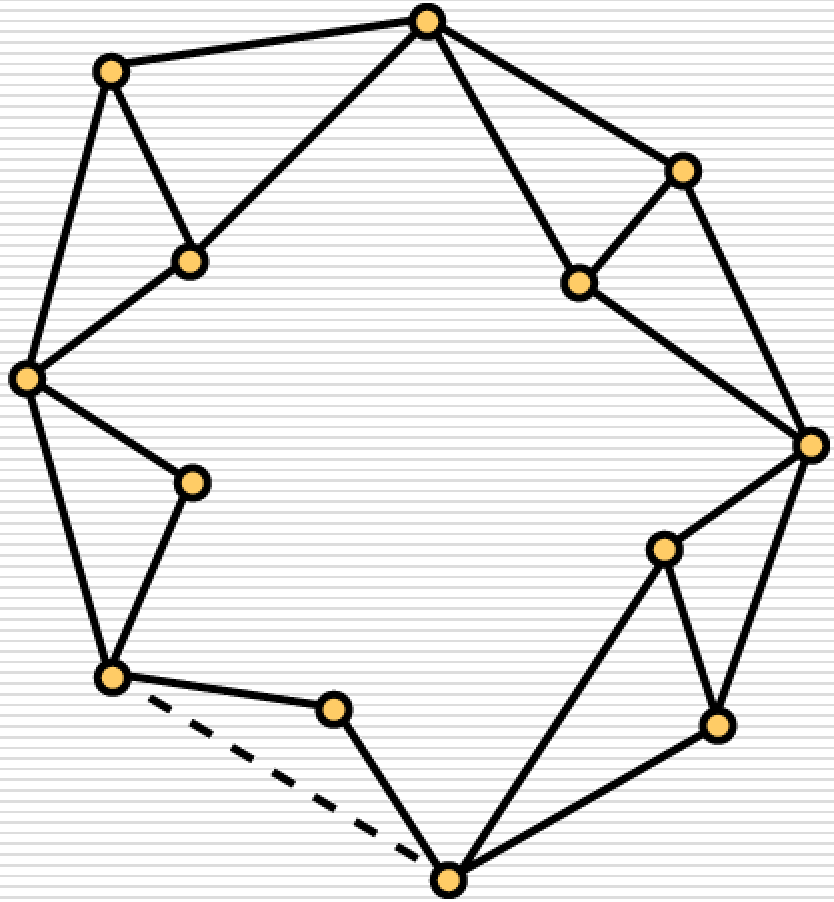
Faces Labeled Clockwise: Contradiction!!



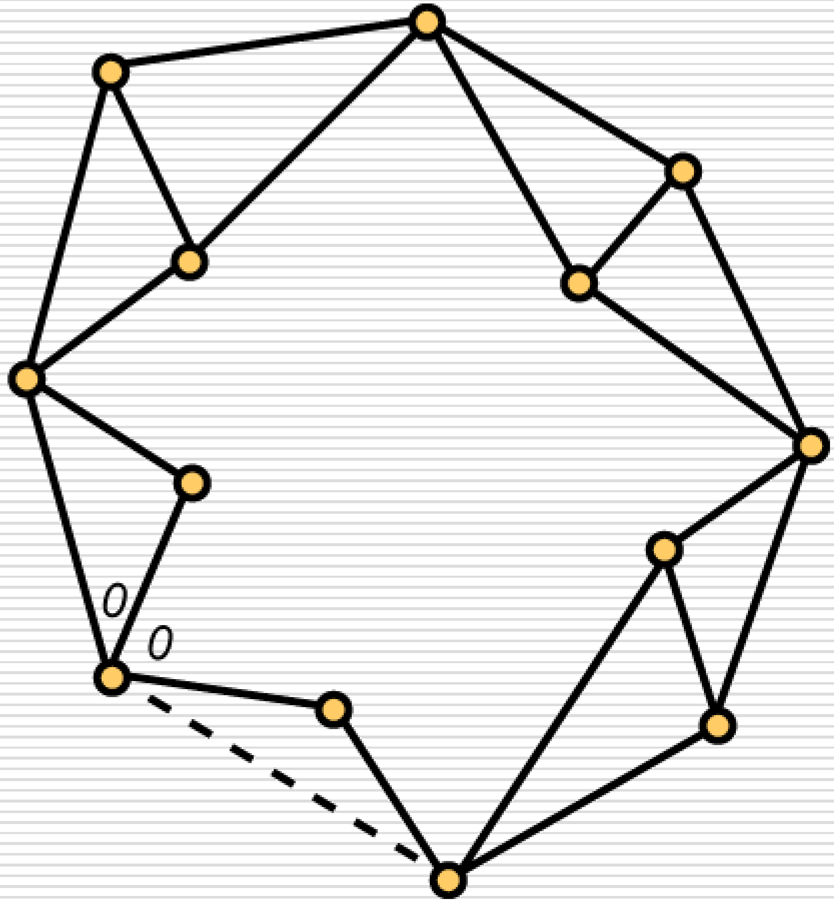
Case 2: C has No Chords



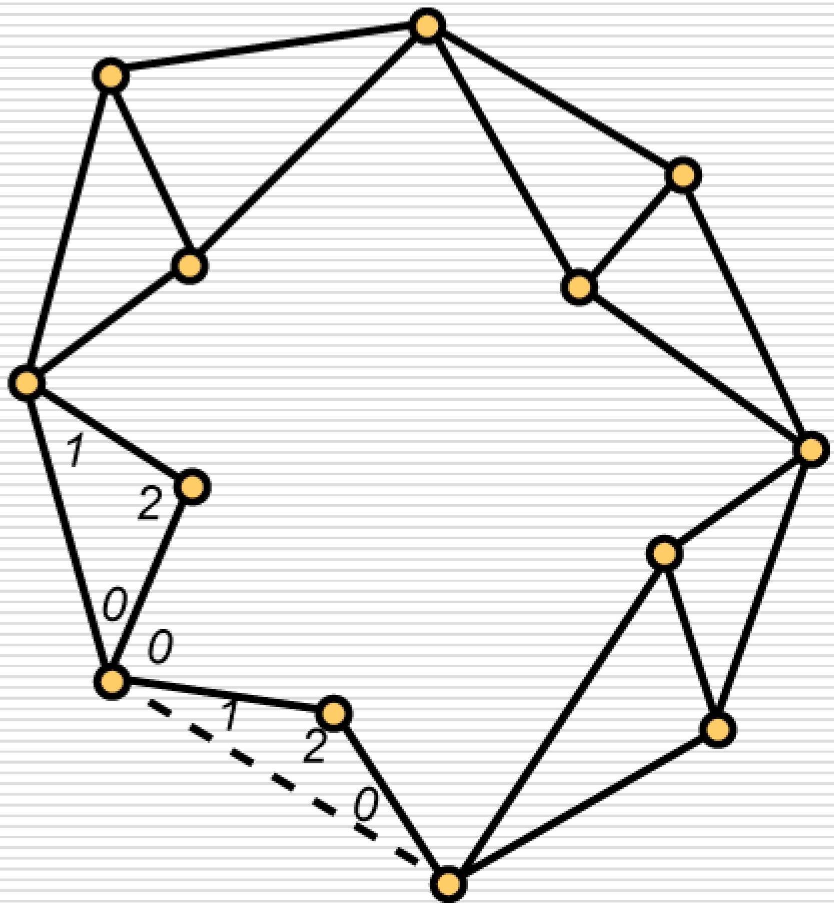
Remove a Boundary Edge



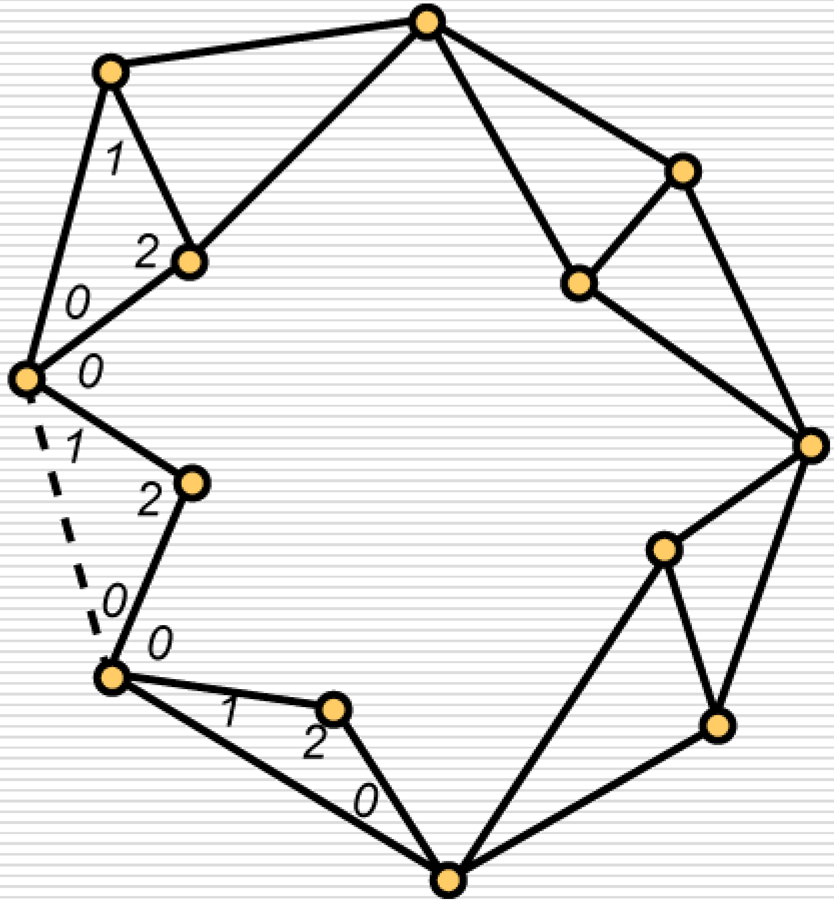
Without Loss of Generality



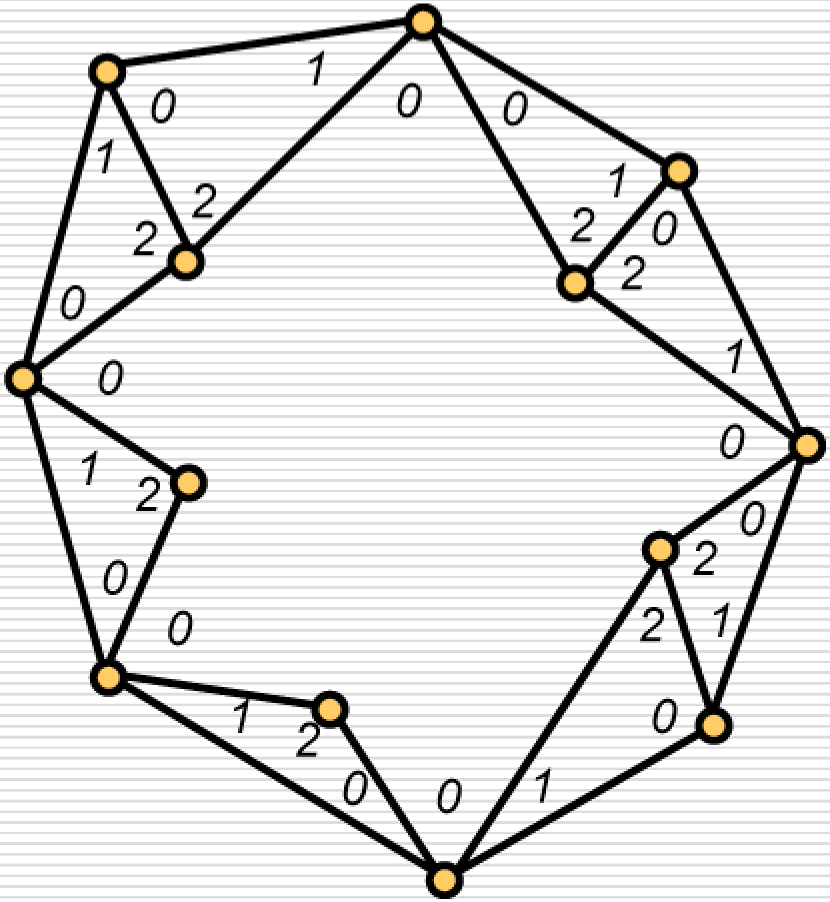
Labeling Properties Imply:



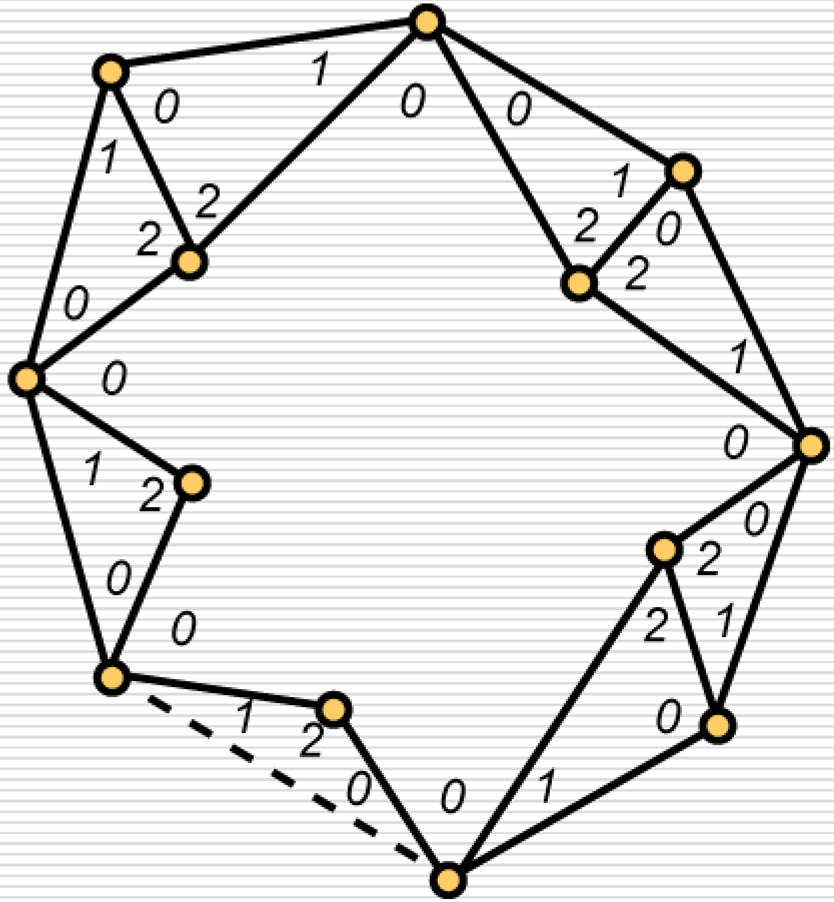
Remove Next Edge



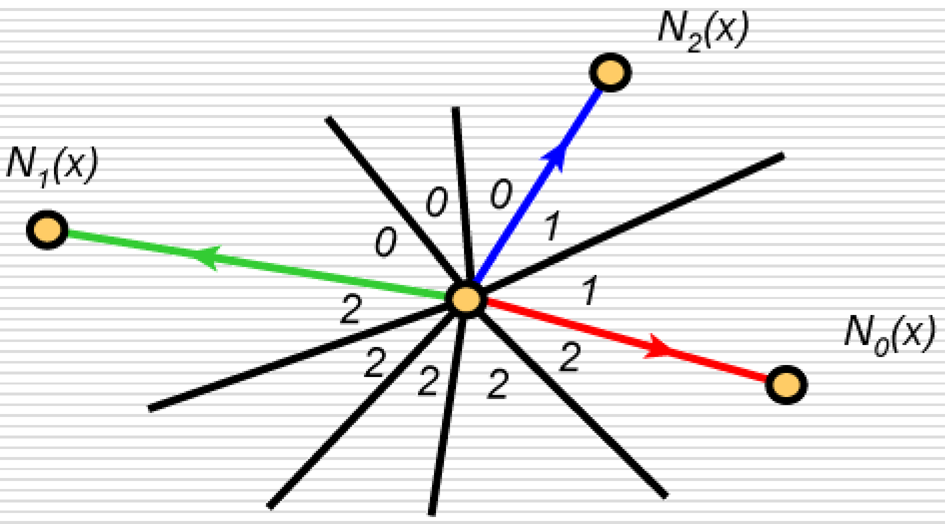
Continue Around Cycle



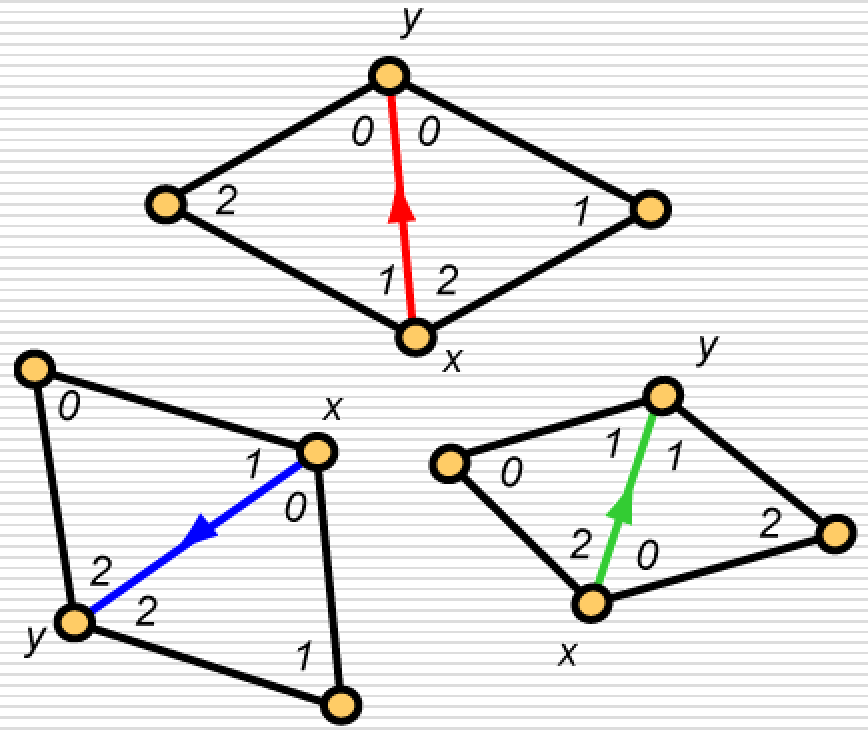
The Contradiction



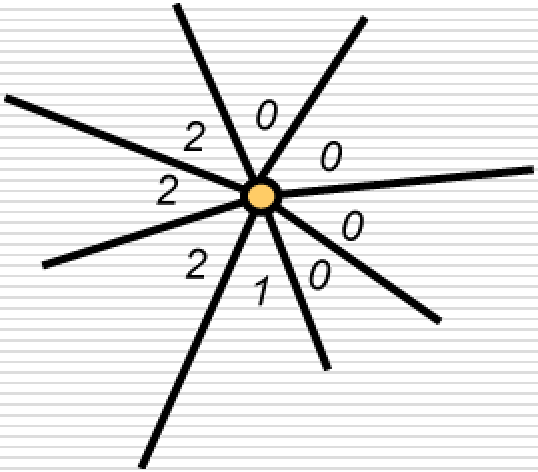
Three Special Edges



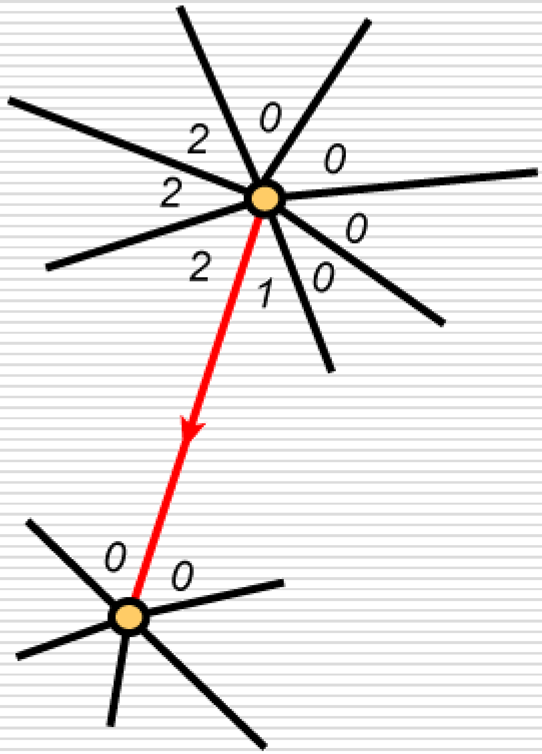
Shared Edges



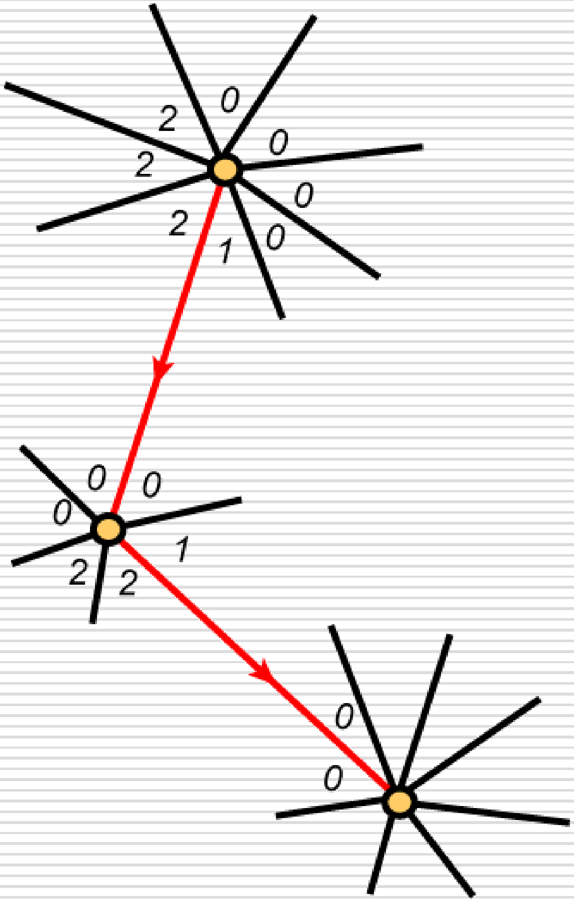
Local Definition of a Path



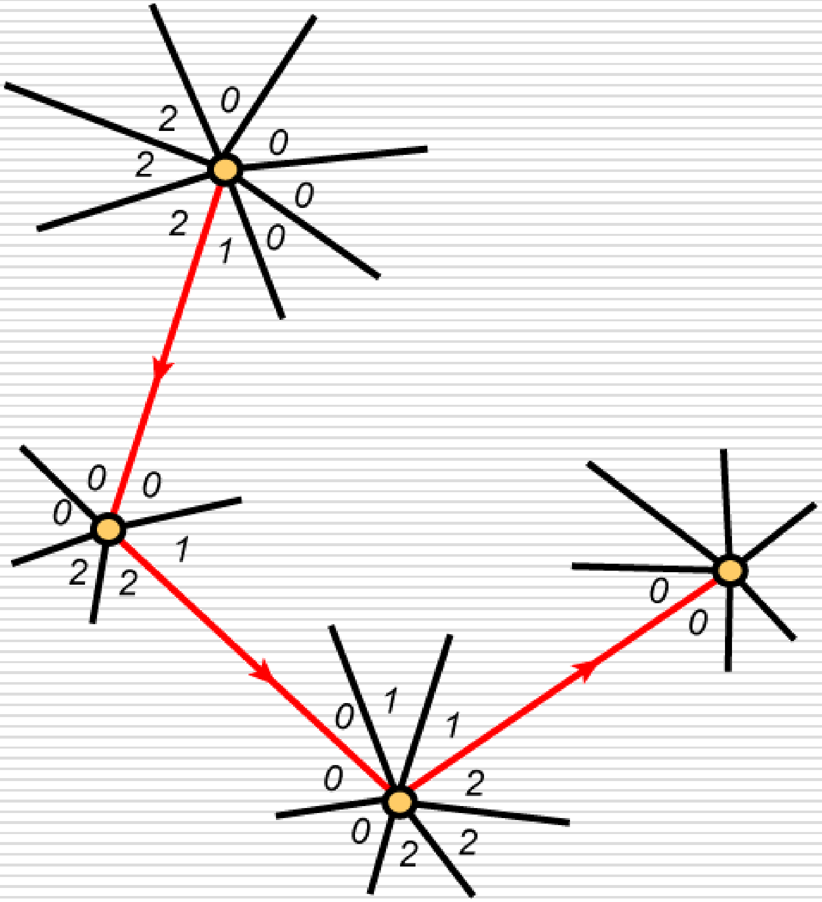
Red Path from an Interior Vertex



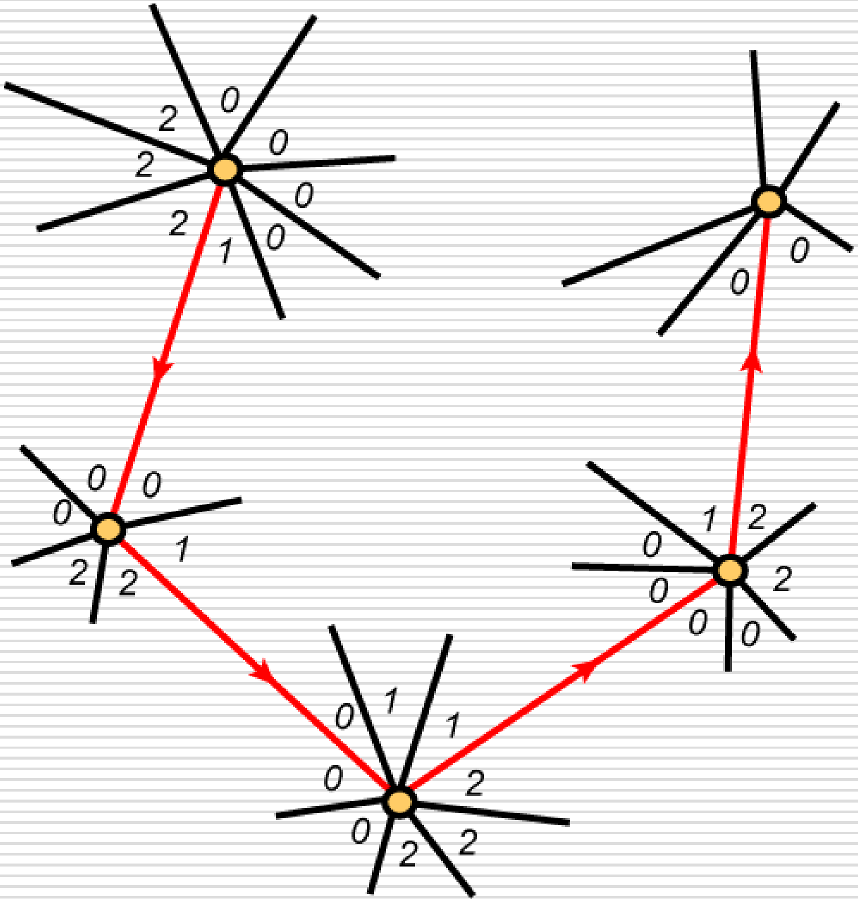
Red Path from an Interior Vertex



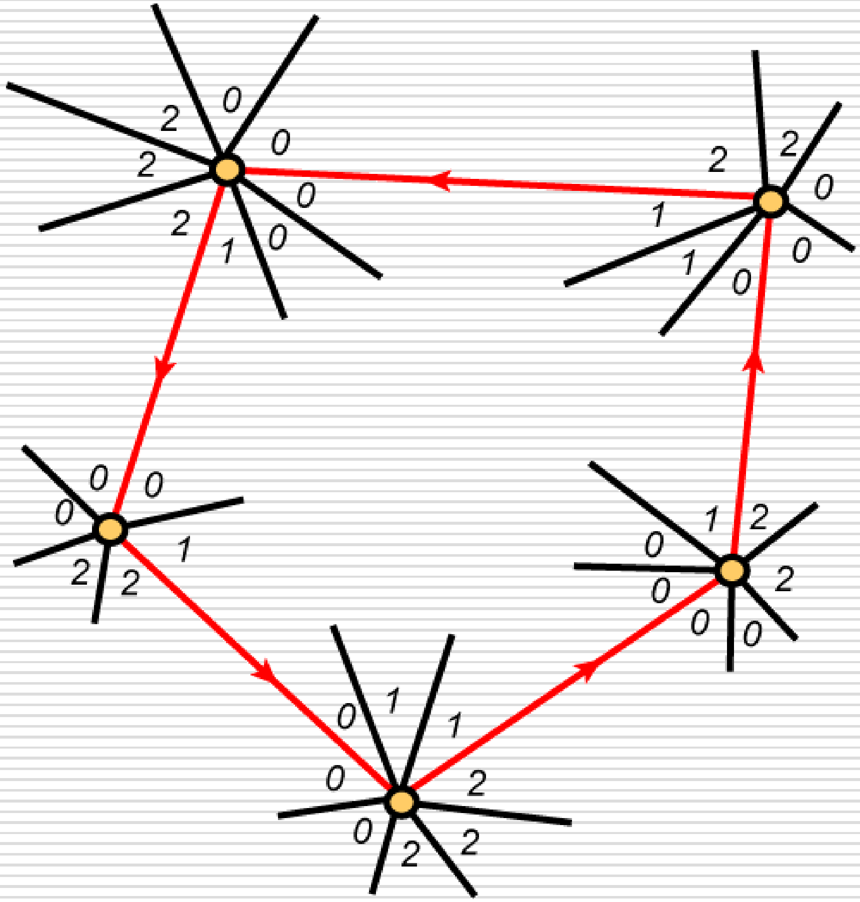
Red Path from an Interior Vertex



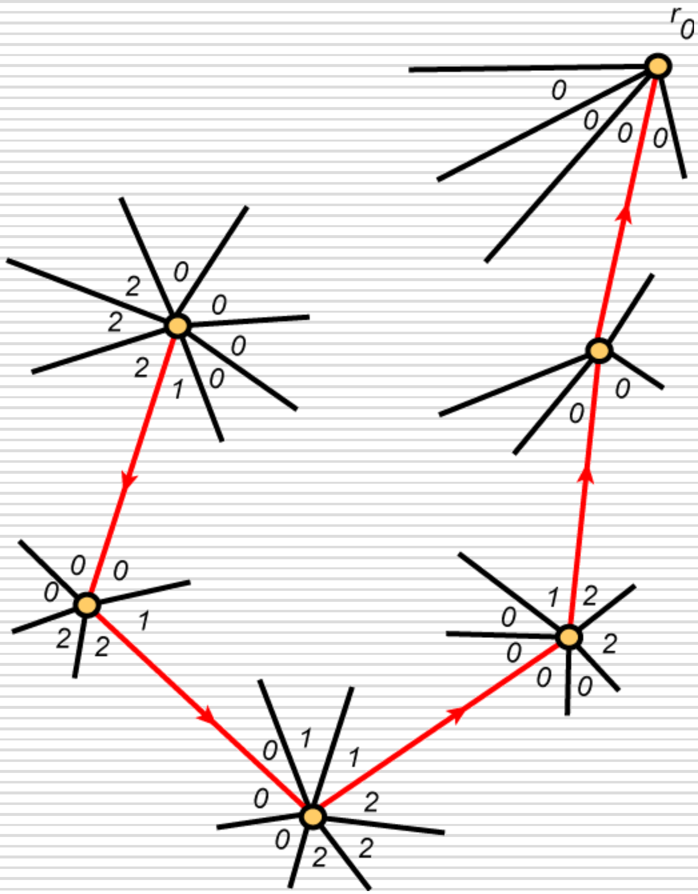
Red Path from an Interior Vertex



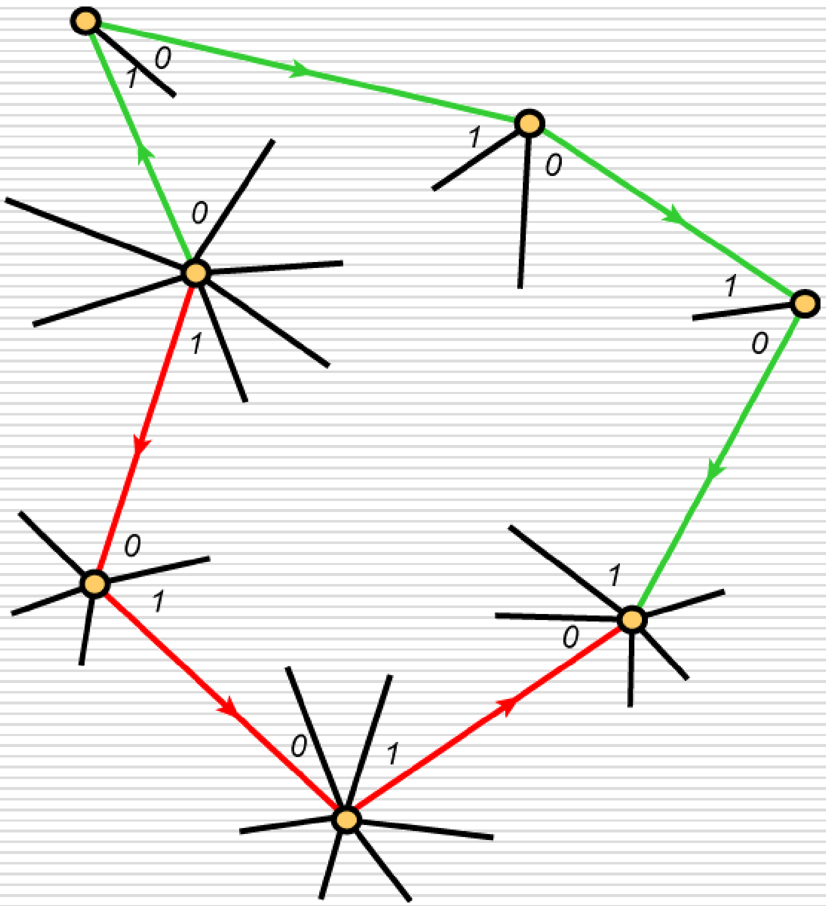
Red Cycle of Interior Vertices??



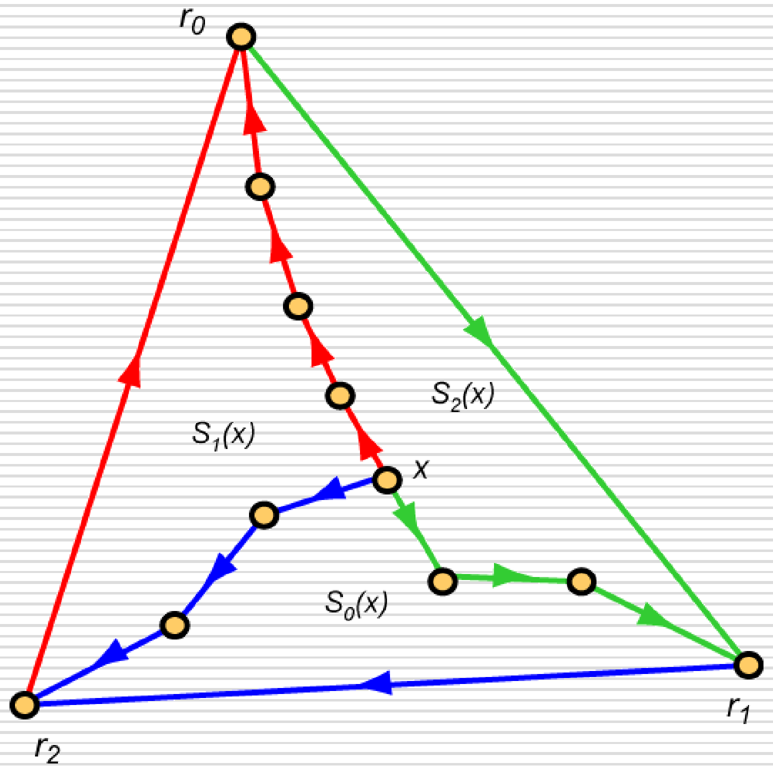
Red Path Ends at Exterior Vertex r_0



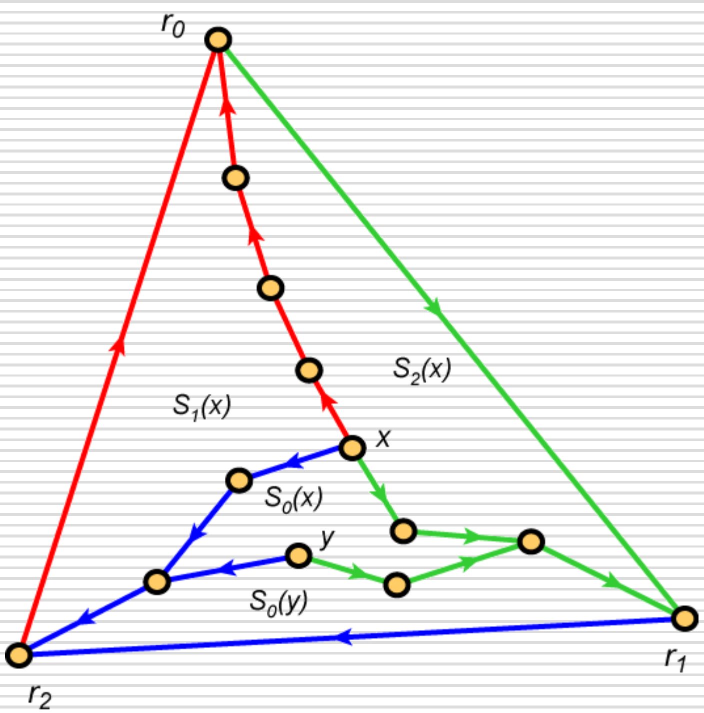
Red and Green Paths Intersect??



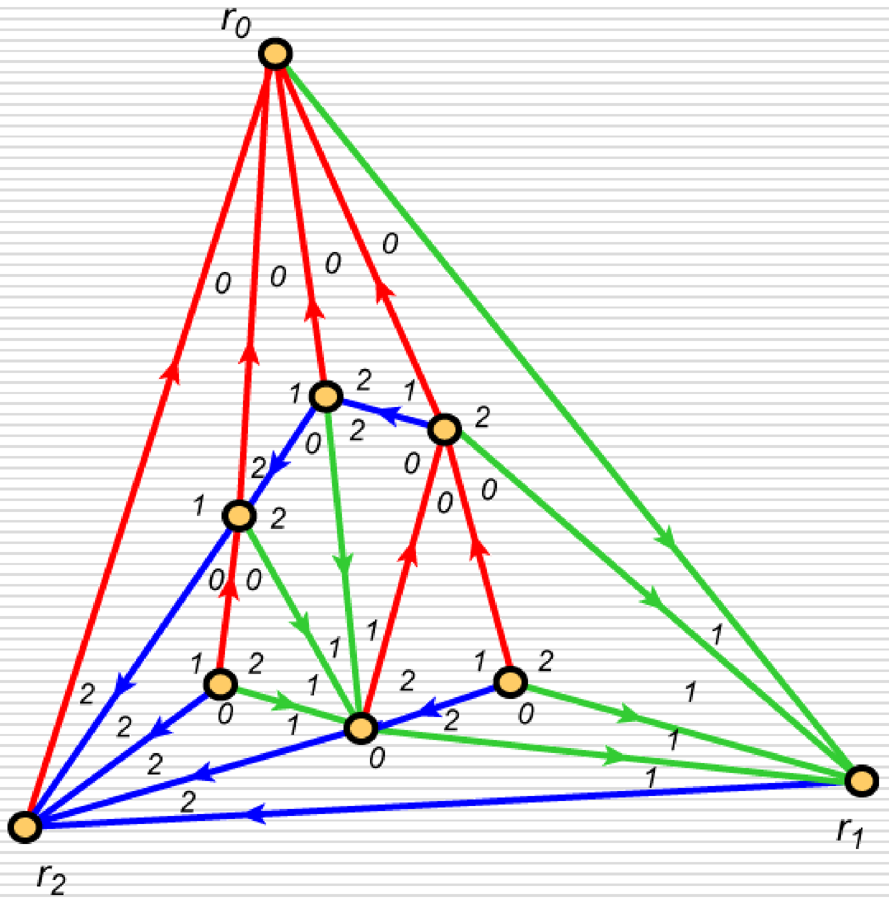
Three Vertex Disjoint Paths



Inclusion Property for Three Regions



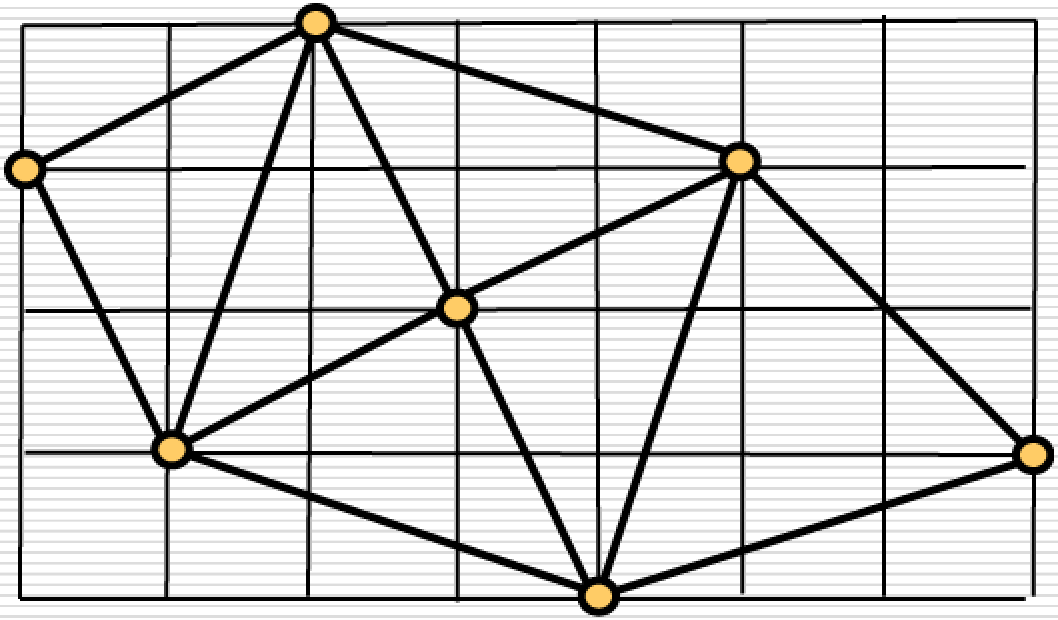
Explicit Partition into 3 Forests



Final Steps

- The regions define three inclusion orders on the vertex set.
- Take three linear extensions.
- Insert the edges as low as possible.
- The resulting three linear extensions have the incidence poset as their intersection.
- Thus, $\dim(P) \leq 3$.

Grid Layouts of Planar Graphs



Corollary (Schnyder, 1990)

For each interior vertex x and each $i = 1,2,3$, let x_i denote the number of vertices in region $S_i(x)$. Then place vertex x at the grid point (x_1, x_2) to obtain a grid embedding without edge crossings.

Algebraic Structure

Theorem (de Mendez, 2001)
The family of all normal labelings of a rooted planar triangulation forms a distributive lattice.