Relations

Discrete Structures

Relations and Their Properties

A binary relation from set A to B
is a subset of Cartesian product $A \times B$

Example: $A = \{0,1,2\}$ $B = \{a,b\}$

A relation: $R = \{(0,a),(0,b),(1,a),(2,b)\}$

A relation on set A is a subset of $A \times A$

Example:

A relation on set $A = \{1,2,3,4\}$:

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$

Reflexive relation R on set A :

$$\forall a \in A, \quad (a,a) \in R$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(3,3),(4,3),(4,4)\}$$

Symmetric relation R :

$$(a,b) \in R \rightarrow (b,a) \in R$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(4,4)\}$$

Antisymmetric relation R :

$$(a,b) \in R \wedge (b,a) \in R \rightarrow a = b$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,2),(3,4),(4,4)\}$$

Transitive relation R :

$$(a,b)\in R \wedge (b,c)\in R \rightarrow (a,c)\in R$$

Example: $A=\{1,2,3,4\}$

$$R=\{(1,1),(1,2),(2,3),(3,4)(1,3),(1,4),(2,4)\}$$

Combining Relations

$$R_1 = \{(1,1),(2,2),(3,3)\}$$

$$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$$

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2),(3,3))$$

Composite relation: $S \circ R$

$$(a,b) \in S \circ R \leftrightarrow \exists x : (a,x) \in R \wedge (x,b) \in S$$

Note: $(a,b) \in R \wedge (b,c) \in S \rightarrow (a,c) \in S \circ R$

Example:

$$R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$$

$$S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$$

$$S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$$

Power of relation: R^n

$$R^1 = R \qquad\qquad R^{n+1} = R^n \circ R$$

Example: $R = \{(1,1),(2,1),(3,2),(4,3)\}$

$$R^2 = R \circ R = \{(1,1),(2,1),(3,1)(4,2)\}$$

$$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1)(4,1)\}$$

$$R^4 = R^3 \circ R = R^3$$

n-ary relations

An n-ary relation on sets $A_1, A_2, \ldots, A_n$
is a subset of Cartesian product $A_1 \times A_2 \times \cdots \times A_n$

Example: A relation on $N \times N \times N$

All triples of numbers (a, b, c) with $a < b < c$

$$R = \{(1,2,3), (1,2,4), (1,2,5), \ldots\}$$

Relational data model

n-ary relation R is represented with table

fields

records

R: Teaching assignments

Professor	Department	Course-number
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Rosen	Comp. Science	518
Rosen	Mathematics	575

primary key
(all entries are different)

Representing Relations with Matrices

$$A = \{a_1, a_2, a_3\} \qquad B = \{b_1, b_2, b_3, b_4, b_5\}$$

$$R = \{(a_1, a_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$$

Relation Matrix

$$M_R \qquad \begin{array}{cc} & B \\ A & \begin{array}{c|ccccc|} & b_1 & b_2 & b_3 & b_4 & b_5 \\ a_1 & 0 & 1 & 0 & 0 & 0 \\ a_2 & 1 & 0 & 1 & 1 & 0 \\ a_3 & 1 & 0 & 1 & 0 & 1 \end{array} \end{array}$$

Reflexive relation R on set A :

$$\forall a \in A, \quad (a,a) \in R$$

Diagonal elements must be 1

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(3,3),(4,3),(4,4)\}$$

$$\begin{array}{c|cccc} & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & 1 & 1 & & \\ a_2 & 1 & 1 & & \\ a_3 & & & 1 & 1 \\ a_4 & & & 1 & 1 \end{array}$$

Symmetric relation $R: \quad (a,b) \in R \rightarrow (b,a) \in R$

Matrix is equal to its transpose: $M_R = M_R^T$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(4,4)\}$$

For all i, j

$$M_R[i,j] = M_R[j,i]$$

$$\begin{array}{c|cccc|} & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & 1 & 1 & & \\ a_2 & 1 & 1 & & \\ a_3 & & & & 1 \\ a_4 & & & 1 & 1 \\ \hline \end{array}$$

Antisymmetric relation R :

$$(a,b) \in R \wedge (b,a) \in R \rightarrow a = b$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(2,2),(2,1),(3,4),(4,1),(4,4)\}$$

For all $i \neq j$

$$M_R[i,j] \neq M_R[j,i]$$

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & & 1 \\ 1 & & & 1 \end{bmatrix} \end{array}$$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Union $R \cup S$:

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Intersection $R \cap S$:

$$M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Composition $S \circ R$: Boolean matrix product

$$M_{S \circ R} = M_R \circ M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

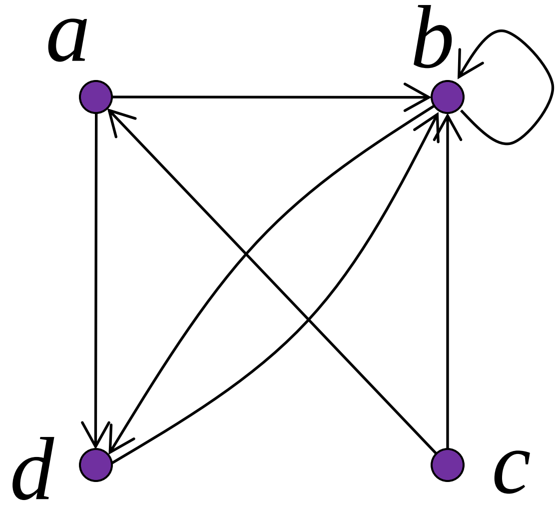
$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Power $R^2 = R \circ R$: Boolean matrix product

$$M_{R^2} = M_R \circ M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Digraphs (Directed Graphs)

$$R = \{(a,b),(a,d),(b,b),(b,d),(c,a),(c,b),(d,b)\}$$



Closures and Relations

Reflexive closure of R :

Smallest size relation that contains R and is reflexive

Easy to find

Symmetric closure of R :

Smallest size relation that contains R and is symmetric

Easy to find

Transitive closure of R:

Smallest size relation that contains R and is transitive

More difficult to find

Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

Find the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: By Theorem 3, it follows that the zero–one matrix of R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}.$$

Because

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

it follows that

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

ALGORITHM 1 A Procedure for Computing the Transitive Closure.

procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)

$\mathbf{A} := \mathbf{M}_R$

$\mathbf{B} := \mathbf{A}$

for $i := 2$ **to** n

  $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$

  $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$

return $\mathbf{B}$ {$\mathbf{B}$ is the zero–one matrix for R^*}

Find Transitive closure using Warshall's Algo

Example

$R=\{(1,1),(1,2),(2,3),(3,1),(3,2)\}$

Warshall's Algorithm

Input: *An* $(n \times n)$ 0-1 matrix M_R representing a relation R on A, $|A|=n$

Output: *An* $(n \times n)$ 0-1 matrix W representing the transitive closure of R on A

1. $W \leftarrow M_R$
2. FOR k=1,…, n
3. FOR i=1,…,n
4. FOR j=1,…,n
5. $w_{i,j} \leftarrow w_{i,j} \vee (w_{i,k} \wedge w_{k,j})$
6. END
7. END
8. END
9. RETURN W