

Chapter 7

Numerical Methods for Partial Differential Equations

1 Introduction

Many problems encountered in applications cannot be solved using any of the analytical methods previously described. It is then necessary to resort to approximate solution methods. The most commonly used approximation methods for the solution of partial differential equations are the finite difference method, the finite volume method and the finite element method. In these methods one represents the solution of the problem in a finite dimensional space of points called nodes. Appropriate discrete forms of the governing equations are used to obtain computational algorithms that allow calculation of the approximate values of the solution at the nodal locations.

Many problems in engineering and science are formulated as partial differential equations. As previously noted, the following types of partial differential equations are of frequent occurrence in applications where one considers a material occupying a domain Ω with boundary Γ for time $t > 0$.

The elliptic equation known as Laplace's equation

$$\nabla^2 u = 0$$

satisfied by the function $u(\mathbf{r})$, where $\mathbf{r} \in \Omega$ is the position vector, provides a quantitative description of many steady state problems. Laplace's equation arises in the study of steady state temperature distribution in an isotropic medium, the analysis of steady state current distribution in homogeneous conductors and in the analysis of the velocity potential due to an irrotational incompressible fluid flow, among many others.

Poisson's equation

$$\nabla^2 u = f(\mathbf{r}, u)$$

satisfied by the function $u(\mathbf{r})$, represents the problem of conduction of heat inside a planar domain undergoing internal heat generation. Poisson equation is encountered in the theory

of torsion, the theory of slow-moving incompressible viscous flow and the inverse-square law theories of electricity, magnetism and gravitation in the presence of charge densities, magnetic poles and mass densities.

The parabolic equation known as the heat or diffusion equation

$$\frac{1}{\alpha} \frac{\partial u}{\partial t} = \nabla^2 u$$

satisfied by the function $u(\mathbf{r}, t)$, where t is time and α a characteristic physical property of the material (e.g. thermal diffusivity), describes the transport of energy or diffusant through a conducting or diffusing medium and also the diffusion of magnetic field strength vector components in linear magnetic or non-magnetic materials of constant electrical conductivity

The hyperbolic equation called the wave equation

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

satisfied by the function $u(\mathbf{r}, t)$, where c is the wave velocity, describing shape perturbations from equilibrium in a vibrating medium.

Another important equation encountered in applications is the convection diffusion equation

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = \alpha \nabla^2 u$$

satisfied by the function $u(\mathbf{r}, t)$, where \mathbf{v} is the velocity field and α is the diffusivity of the fluid. This is used to describe the transport of substance, energy or magnetic field strength due to the combined effects of the velocity field and diffusion.

As a final example, one can mention the Navier-Stokes equations,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \nu \nabla^2 \mathbf{v} - \frac{1}{\rho} \nabla p$$

satisfied by the velocity vector function $\mathbf{v}(\mathbf{r}, t)$, where ν and ρ are the viscosity and the density of the fluid, respectively and p is the pressure. This system describe the laminar flow of a viscous incompressible Newtonian fluid.

The above equations must be solved subject to specified boundary and initial conditions. Boundary conditions can consist of specifying the value of the function somewhere on the boundary of the domain (boundary condition of Dirichlet type). I.e. on the boundary portion Γ_1

$$u|_{\Gamma_1} = u_1$$

Alternatively, the value of the derivative can be specified (boundary condition of Neumann type), i.e.

$$\frac{\partial u}{\partial n}|_{\Gamma_1} = u'_1$$

Finally, a linear combination of the above is also possible (boundary condition of Robin type)

$$au|_{\Gamma_1} + b\frac{\partial u}{\partial n}|_{\Gamma_1} = au_1 + bu'_1$$

Initial conditions specify either the value of the function everywhere inside the domain at $t = 0$, i.e.

$$u(\mathbf{r}, 0) = u_0$$

and perhaps the value of the time derivative, i.e.

$$\frac{\partial u}{\partial t}|_{(\mathbf{r},0)} = u'_0$$

In each case, the above equations must be solved for $u(\mathbf{r})$, $u(\mathbf{r}, t)$ or $\mathbf{v}(\mathbf{r}, t)$ subject to the specified boundary and initial conditions.

All the above partial differential equations can be written down in general operator form as follows

$$Lu = S$$

where L is a partial differential operator and S is a source term. For instance, the partial differential operators for elliptic, parabolic and hyperbolic equations, respectively

$$L = \nabla^2$$

$$L = \nabla^2 - \frac{1}{\alpha} \frac{\partial}{\partial t}$$

$$L = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

2 Errors in Numerical Computation

Besides the floating point errors associated with the finite representation of numbers in computers, other errors occur when solving differential equations numerically. In all cases one is clearly interested in the determination of the global error of the approximation e^G , representing some measure of the difference between the approximate solution obtained and the exact solution of the problem, i.e.

$$e^G = \| u_{exact} - u_h \|$$

where u_h is the approximate solution and $\| \cdot \|$ denotes some appropriate norm. One of the most important tasks in the numerical analysis of partial differential equations is the estimation of the global error involved in computation.

Moreover, if the numerical approximation is constructed directly from the differential formulation using finite differences, there is a truncation error at each mesh point e_i^T given by

$$e_i^T = Lu_i - S_i$$

where u_i and S_i are, respectively, the value of the approximate solution and that of the source term at node i . Although the global error may be difficult to estimate, the truncation error can be readily assessed from the derivatives of the approximate solution. In practice, the calculated truncation error is then used to estimate the global error.

The order of accuracy of a numerical method is a measure of the sensitivity of the truncation error to the discretization used. Specifically, if an approximate solution to a given problem is obtained using a mesh of size h , and an upper bound for the truncation error can be constructed such that

$$|e_i^T| = Ch^p$$

where C is some constant and p is an integer, then one says that the obtained approximation is p -th order accurate.

3 Numerical Methods for Elliptic Problems

3.1 Finite Difference Method

Consider the case of Poisson's equation for the unknown function $u(x, y)$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

on (x, y) in the interior $R = \{(x, y) : a < x < b, c < y < d\}$ of a planar region and subject to

$$u(x, y) = g(x, y)$$

along the boundary S of the region. As long as f and g are continuous, one can show that a unique solution exists.

A simple approach to the numerical solution of the above problem consists in partitioning the intervals $[a, b]$ and $[c, d]$ respectively into n and m subintervals with step sizes $\Delta x = h = (b-a)/n$ and $\Delta y = k = (d-c)/m$ so that the node or mesh point (x_i, y_j) is at $x_i = a + (i-1)h$ for $i = 1, \dots, N$ and $y_j = c + (j-1)k$ for $j = 1, \dots, M$. Here, $N = n + 1$ and $M = m + 1$.

In practice, a distinction must be made between interior nodes (i.e. (x_i, y_j) for $i = 2, \dots, N-1$ and $j = 2, \dots, M-1$) and the boundary nodes (represented by nodes with $i = 1$, $i = N$, $j = 1$, or $j = M$). In the finite difference method, all interior nodes are handled by the same equation. Special equations are typically required for the boundary nodes depending of the boundary conditions applied in each case.

Using now centered finite difference approximations for the partial derivatives, the following second order accurate system of simultaneous algebraic equations is obtained for all interior nodes

$$2\left[\left(\frac{h}{k}\right)^2 + 1\right]w_{i,j} - (w_{i+1,j} - w_{i-1,j}) - (w_{i,j+1} - w_{i,j-1})\left(\frac{h}{k}\right)^2 = -h^2 f(x, y)$$

This expression is often called the five point formula.

If the non-homogeneous term is $f(x, y) = 0$ (Laplace's equation) and if the mesh spacings are chosen to be the same (i.e. $h = k$) the following particularly simple system of equations is obtained for the interior nodes:

$$w_{i,j} = \frac{1}{4}[w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}]$$

Exercise. Consider steady state heat conduction in a thin square plate of edge = 100 cm. The edges $x = 0$, $x = l$ and $y = 0$ are maintained at $u = 0$ while at the edge $y = d$ $u(x, d) = 100$. No heat flow along the z direction perpendicular to the plate. The required temperature $u(x, y)$ satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This problem is readily solved in closed form by separation of variables yielding

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{100}x\right) \frac{\sinh\left(\frac{n\pi}{100}y\right)}{\sinh(n\pi)}$$

with

$$c_n = \begin{cases} 0 & n \text{ even} \\ \frac{400}{n\pi} & n \text{ odd} \end{cases}$$

Solve the above problem numerically using the finite difference method and compare your results against the analytical solution.

The five point formulae above are also applicable at the boundary nodes but they must be supplemented by specially formulated discrete equations representing the boundary conditions of the problem in order to obtain the specific finite difference formulae for the boundary nodes.

If Dirichlet boundary conditions are imposed on the boundary nodes then one has

$$\begin{aligned} w_{1,j} &= g(x_1, y_j) & j &= 1, \dots, M \\ w_{N,j} &= g(x_N, y_j) & j &= 1, \dots, M \\ w_{i,1} &= g(x_i, y_1) & i &= 1, 2, \dots, N \\ w_{i,M} &= g(x_i, y_M) & i &= 1, 2, \dots, N \end{aligned}$$

To obtain a *banded* matrix the mesh points must be relabeled sequentially from left to right and from top to bottom. The resulting system can be solved by Gaussian elimination if n and m are small and by iterative methods (e.g. SOR iteration) when they are large.

One is clearly interested in determining the quality of the approximation computed by the finite difference method. This is measured by the error. By using Taylor series expansions, the following error bound for the numerical solution is obtained for the case when $\Delta x = h = \Delta y = k = h$

$$|w_{i,j} - u(x_i, y_j)| \leq \frac{h^2}{96} (M_{xxxx} + M_{yyyy})$$

where M_{xxxx} and M_{yyyy} are bounds for u_{xxxx} and u_{yyyy} .

3.2 Finite Volume Method

An alternative discretization method is based on the idea of regarding the computation domain as subdivided into a collection of finite volumes. In this view, each finite volume is represented by an area in 2D and a volume in 3D. A node, located inside each finite volume is the locus of computational values. In rectangular Cartesian coordinates in 2D the simplest finite volumes are rectangles. For each node, the rectangle faces are formed by drawing perpendiculars through the midpoints between contiguous nodes. Discretization equations are obtained by integrating the original partial differential equation over the span of each finite volume. The method is easily extended to nonlinear problems.

Consider for instance the following problem consisting of determining $u(x, y)$ in $x \in [a, b]$, $y \in [c, d]$ such that

$$\frac{\partial}{\partial x} \left(a \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(a \frac{\partial u}{\partial y} \right) = f$$

subject to specified conditions at the boundary. The problem represents steady state heat conduction in a solid with position dependent conductivity a where energy is being internally generated at the rate f , per unit volume.

The typical control volume has dimensions $\Delta x \times \Delta y \times 1$ and the discrete equation is obtained by performing an energy balance on the finite volume. The net energy input is obtained by integrating the fluxes $a \partial u / \partial x$ and $a \partial u / \partial y$ into and out from the finite volume

in the x and y directions, respectively. Since energy is conserved, the result must equal the rate of energy generation inside the volume which is given by the volume integral of f .

A notation from cartography is used to simplify the discrete formulation. In this notation, the node representing the finite volume is called P and the neighboring nodes along the four coordinate directions are called N , S , E and W , respectively. The following discrete equation is obtained for node P ,

$$\frac{a_e(w_E - w_P) + a_w(w_P - w_W)}{\Delta x^2} + \frac{a_n(w_N - w_P) + a_s(w_P - w_S)}{\Delta y^2} = f_P$$

where the subscripts e, w, n, s are used to indicate that the values of a are to be evaluated at the corresponding finite volume faces. Note that if $a_e = a_w = a_n = a_s = a = 1$, $\Delta x = \Delta y = h$ and $f_P = 0$ the above reduces to the simple form

$$w_P = \frac{1}{4}[w_E + w_W + w_N + w_S]$$

which is identical to the expression obtained before using the method of finite differences.

As in the case of finite differences, care is required to properly deal with the boundary conditions at the boundary nodes, specially when derivative conditions are involved.

When the boundaries of the computational domain do not coincide with coordinate lines additional care must be used for the implementation of boundary conditions. Finite difference or finite volume formulae can be derived by ad-hoc methods, by conventional formulae if the irregular boundaries are represented by staircases approximating the actual boundaries or by coordinate transformation methods.

3.3 Iterative Solution of the Algebraic System

The system of algebraic equations is readily solved using iterative methods. Specifically, in the Jacobi method for the numerical solution of Laplace's equation in a uniform mesh one assumes a solution for the interior nodes and then computes an improved approximation using the five point formula, i.e.

$$w_{i,j} = \frac{1}{4}[w_{i+1,j}^o + w_{i-1,j}^o + w_{i,j+1}^o + w_{i,j-1}^o]$$

where $w_{i,j}^o$ are the initial guesses values and $w_{i,j}$ is the improved guess generated by iteration.

Since nodes $[i-1, j]$ and $[i, j-1]$ would have been already visited when computing the improved guess for node $[i, j]$, it is wise to use the improved values as soon as they become available in the memory of the computer. This is Gauss-Seidel method and is expressed as

$$w_{i,j} = \frac{1}{4}[w_{i+1,j}^o + w_{i-1,j} + w_{i,j+1}^o + w_{i,j-1}]$$

Finally, if the size of the corrections introduced by iteration is artificially enhanced by means of an overrelaxation factor one obtains the successive overrelaxation (SOR) method, i.e.

$$w_{i,j} = w_{i,j}^o + \frac{\omega}{4}[w_{i+1,j}^o + w_{i-1,j}^o + w_{i,j+1}^o + w_{i,j-1}^o + 4w_{i,j}^o]$$

where $\omega > 1$ is the overrelaxation factor. Note that this formula reduces to G-S when $\omega = 1$.

The iteration process stops whenever the maximum difference found between two contiguous iterates falls below a specified limit.

3.4 Finite Element Method

The finite element method is an alternative discrete representation obtained from the variational formulation of the original problem. Specifically, for the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = f$$

inside the domain $\Omega = \{x|x \in [a, b]\}$ and $\{y|y \in [c, d]\}$, and subject to $u = 0$ at the boundary, the associated functional I is given by

$$I(v) = \int_{\Omega} [\frac{1}{2}|\nabla v|^2 + fv] dx dy$$

The desired solution is the function $u \in v$ which minimizes the functional, i.e.

$$\delta I(u) = 0$$

and this function comes from the set of functions which satisfy the stated boundary conditions.

Discretization in the finite element method is also obtained by subdividing the computational domain into subdomains (finite elements), commonly triangles, rectangles, tetrahedra or rectangular parallelepipeds (bricks). The vertices in each of these geometries are called *nodes*. The method seeks an approximation inside each element of the form

$$w(x, y) = \sum_{i=1}^m c_i \phi_i$$

where the ϕ_i are linearly independent piecewise polynomials and the c_i are constants with respect to which the functional is minimized (i.e. $\frac{\partial I}{\partial c_i} = 0$). The constants c_i actually represent the values of u at the nodal locations.

Minimization of I with respect to the c_i 's produces a set of simultaneous linear algebraic equations of the form

$$A\mathbf{c} = \mathbf{b}$$

Solution of the above system using standard techniques yields the desired approximation.

Consider as a second example the following partial differential equation in a plane region D

$$\frac{\partial}{\partial x}(p(x, y)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(q(x, y)\frac{\partial u}{\partial y}) + r(x, y)u(x, y) = f(x, y)$$

subject to the boundary conditions

$$u(x, y) = g(x, y)$$

along portion S_1 of the *boundary* S of D and along the remaining portion S_2 of the boundary

$$p\frac{\partial u}{\partial x}\cos\theta + q\frac{\partial u}{\partial y}\cos(\pi/2 - \theta) + g_1(x, y)u = g_2(x, y)$$

where θ is the angle of the outward normal.

Now, in D let p and q be C^1 , r, f, g_1 and g_2 be all C^0 , let also $p > 0$, $q > 0$ and $r \leq 0$. Then the solution of the PDE above is the twice differentiable function $w(x, y)$ satisfying $w = g$ which *minimizes* the functional

$$I[w] = \int \int_D \left\{ \frac{1}{2} [p(x, y)(\frac{\partial w}{\partial x})^2 + q(x, y)(\frac{\partial w}{\partial y})^2 - r(x, y)w^2] + f(x, y)w \right\} dx dy + \int_{S_2} \left\{ -g_2w + \frac{1}{2}g_1w^2 \right\} dS$$

The search for w is performed in the FEM from among the (smaller) set of functions which are *linear combinations* of appropriately chosen *basis function* just as in the Rayleigh-Ritz-Galerkin method.

As before, the region D is first subdivided into a finite number of regions with simple shapes (e.g. triangles or rectangles). The vertices of the resulting network are called *nodes*. Piecewise polynomials basis functions for triangle shaped elements are

$$\phi_i(x, y) = a + bx + cy$$

The method then seeks to determine constants γ_i such that

$$w(x, y) \approx \phi = \sum_{i=1}^m \gamma_i \phi_i(x, y)$$

minimizes $I[w] \approx I[\sum_{i=1}^m \gamma_i \phi_i(x, y)]$. The condition for a minimum is

$$\frac{\partial I}{\partial \gamma_j} = 0$$

for each $j = 1, 2, \dots, n$. Note that constants $\gamma_{n+1}, \gamma_{n+2}, \dots, \gamma_m$ are determined by the boundary condition $\phi = g$.

Applying the condition for a minimum and taking advantage of the simple form of ϕ_i leads to the system

$$A\mathbf{c} = \mathbf{b}$$

where the entries of A and \mathbf{b} involve double integrals over D and path integrals over S_2 . The resulting system can then be solved using standard methods.

Exercise. A cantilever beam of length L has rectangular cross section with height $h = 2a$ and width $w = 2b$. A point load of magnitude P is applied on the free end of the beam and perpendicular to its length. Recall that the moment of inertia of the beam cross section with respect to a centroidal axis is given by

$$I = \frac{1}{12}wh^3$$

Introducing rectangular Cartesian coordinates with the $x - y$ plane parallel to the cross section of the beam and the z axis along its length, and in terms of the *stress function* $\phi(x, y)$ given by

$$\tau_{xz} = \frac{\partial \phi}{\partial y}$$

and

$$\tau_{yz} = -\frac{\partial \phi}{\partial x}$$

the problem becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1 + \nu} \frac{Py}{I}$$

which is Poisson's equation and it also represents the deflection of a uniformly stretched rectangular membrane under a continuous load.

The above problem can be solved in closed form to give

$$\phi(x, y) = -\frac{\nu}{1 + \nu} \frac{P 8b^3}{I \pi^4} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n-1} \cos[(2m+1)\pi x/2a] \sin(n\pi y/b)}{(2m+1)n[(2m+1)^2(b^2/4a^2) + n^2]}$$

Solve the above problem numerically using the finite element method and compare your results against the analytical solution.

Exercise. A thin rectangular plate of edges a and b and thickness h is constrained along all four edges and is loaded by hydrostatic pressure q acting uniformly on one side. Compute the elastic deflection of a plate of sides $a = 1$ m, $b = 0.75$ m, and thickness $h = 0.025$

m, uniformly loaded by pressure $q = 5 \times 10^5$ (N/m^2), using the finite element method and compare against the analytical solution given by

$$w(x, y) = \frac{4qa^4}{\pi^5 D} \sum_{m=1,3,5,\dots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^5} \cos\left(\frac{m\pi x}{a}\right) \left[1 - \frac{\alpha_m \tanh(\alpha_m) + 2}{2 \cosh(\alpha_m)} \cosh\left(\frac{m\pi y}{a}\right) + \frac{1}{2 \cosh(\alpha_m)} \frac{m\pi y}{a} \sinh\left(\frac{m\pi y}{a}\right) \right]$$

where

$$\alpha_m = \frac{m\pi b}{2a}$$

and

$$D = \frac{Eh^3}{12(1 - \nu^2)}$$

is the *flexural rigidity* of the plate. Assume $E = 2 \times 10^{11}$ (N/m^2), $\nu = 0.33$.

4 Parabolic Problems: Finite Difference Methods

The finite difference method is readily applied to the solution of parabolic problems. In this case a mesh of nodes along the time coordinate is introduced in addition to the spatial mesh. The various derivatives in the parabolic equation are then approximated by finite difference formulae. The resulting system of algebraic equations is then solved at each time level in order to obtain the approximate values of the solution at the spatial nodes. Depending on how one approaches the problem along the time coordinate, three different schemes are obtained: the explicit scheme, the implicit scheme and the semi-implicit scheme. These are described in detail below.

As described before, the simplest parabolic problem consists of determining the function $u(x, t)$ satisfying the parabolic equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

subject to suitable initial and boundary conditions.

4.1 Explicit Scheme

As mentioned above, in the finite difference method the space domain is subdivided into equal size subintervals to give the set of mesh points (in space) $x_i = (i - 1)\Delta x = (i - 1)h$ with $i = 1, 2, \dots, N$ while a *mesh of points in time* $t_j = (j - 1)\Delta t = (j - 1)k$ with $j = 1, 2, \dots, M$ is also created.

If a forward difference approximation is now used for the time derivative and a centered difference formula is used for the spatial derivative the following first order accurate in time, second order accurate in space, explicit scheme is obtained

$$\frac{w_{i,j+1} - w_{i,j}}{k} = \alpha \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2}$$

As the name implies, this formulae can be solved explicitly to give the values of w for all points on the space mesh at a subsequent time level given the values of w at all mesh points at the immediately preceeding time level, i.e.

$$w_{i,j+1} = (1 - \frac{2\alpha k}{h^2})w_{i,j} + \alpha \frac{k}{h^2}(w_{i+1,j} + w_{i-1,j}) = (1 - 2\alpha\nu)w_{i,j} + \alpha\nu(w_{i+1,j} + w_{i-1,j})$$

where $\nu = k/h^2$. The above formula can also be written using matrix notation as $\mathbf{w}^{(j)} = A\mathbf{w}^{(j-1)}$.

Note that each equation in the above set uses the values of w at three contiguous nodes $(i-1, i, i+1)$ for time level j to compute the value of $w_{i,j+1}$.

The explicit method, however, is only conditionally stable. This means that only for certain predetermined values of the mesh spacings h and k is the truncation error damped and thus unable to increase as the calculation progresses. For stability, it is required that the spectral radius $\rho(A) \leq 1$. As a result, for stability of the explicit scheme the mesh spacing must satisfy the Courant-Friedrichs-Lewy (CFL) restriction

$$\frac{k}{h^2} = \nu \leq \frac{1}{2\alpha}$$

Stability is frequently investigated by expanding the numerical solution in *Fourier modes*, i.e.

$$w_{i,j} = (\lambda)^j \exp[\imath k_x i \Delta x]$$

where $\imath = \sqrt{-1}$, λ is the error amplification factor and k_x is the error wave number, and substituting into the finite difference formula to obtain a relationship for the amplification factor which in turn yields the CFL restriction.

The truncation error of the scheme, $T(x, t)$ is given by

$$\begin{aligned} T(x, y) &= \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} - \alpha \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} = \\ &= \frac{1}{2}u_{tt}(x, \eta)k - \frac{1}{12}u_{xxxx}(\xi, t)h^2 \end{aligned}$$

where $x \leq \xi \leq x + h$ and $t \leq \eta \leq t + k$. An upper bound for the truncation error is

$$T(x, y) \leq \frac{1}{2}k[M_{tt} + \frac{1}{6\nu}M_{xxxx}]$$

where the M s are bounds for the corresponding derivatives. The explicit scheme is *consistent* with the original PDE since

$$T \rightarrow 0$$

as $k \rightarrow 0$.

The explicit finite difference scheme is called convergent if the computed solution at any point approaches the exact value as the mesh is refined, i.e. if

$$w_{i,j} \rightarrow u(x_i, t_j)$$

as $h \rightarrow 0$, $k \rightarrow 0$. It can be shown that the explicit scheme is convergent as long as $\nu \leq \frac{1}{2\alpha}$.

Exercise As mentioned before, the transport of diffusing species and of energy through and out of a solid can be described by a parabolic partial differential equation (often called the heat equation or the diffusion equation). Consider the following specific example consisting of determining $u(x, t)$ such that

$$\frac{\partial u}{\partial t}(x, t) = \alpha \frac{\partial^2 u}{\partial x^2}(x, t)$$

on $0 < x < l$, $0 < t$ and subject to the conditions

$$\begin{aligned} u(0, t) &= u(l, t) = 0, & t > 0 \\ u(x, 0) &= f(x), & 0 \leq x \leq l \end{aligned}$$

Here $u(x, t)$ is the concentration of diffusing substance (in mass transfer problems) or the temperature (in energy transfer problems), at position x and time t , $f(x)$ is the initial distribution and α is the diffusivity. This problem is readily solved using separation of variables. Assuming $\alpha = 1$, the solution is

$$u(x, t) = \sum_{m=1}^{\infty} a_m \sin(m\pi x) \exp[-(m\pi)^2 t]$$

where

$$a_m = 2 \int_0^1 f(x) \sin(m\pi x) dx$$

Solve the above problem using the method of finite differences with an explicit scheme. Assume $\alpha = 1$ and $f(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$ and $f(x) = 2 - 2x$ for $\frac{1}{2} \leq x \leq 1$. Use $m = 20$ and time steps just above and just below that given by the CFL condition. Compare against the analytical solution.

4.2 Implicit Scheme

An unconditionally stable scheme results for linear problems if backward differences in time are used instead, i.e.

$$\frac{w_{i,j} - w_{i,j-1}}{k} = \alpha \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2}$$

which is equivalent to

$$\frac{w_{i,j+1} - w_{i,j}}{k} = \alpha \frac{w_{i+1,j+1} - 2w_{i,j+1} + w_{i-1,j+1}}{h^2}$$

This practice produces after slight rearrangement the implicit method given by the formula

$$-\frac{\alpha k}{h^2} w_{i-1,j+1} + [1 + 2(\frac{\alpha k}{h^2})] w_{i,j+1} - \frac{\alpha k}{h^2} w_{i+1,j+1} = w_{i,j}$$

For a given time level $j + 1$, the above can be represented as

$$-a_i w_{i-1,j+1} + b_i w_{i,j+1} - c_i w_{i+1,j+1} = d_i$$

for each of the x_i nodes in the spatial mesh. Note that each equation in the above set involves the values of w at three contiguous nodes ($i - 1, i, i + 1$) for time level $j + 1$ and $w_{i,j}$ to compute the value of $w_{i,j+1}$, i.e. this is a system of simultaneous, interlinked algebraic equations. The system of equations is clearly associated with a tridiagonal matrix and is written in matrix notation as $A\mathbf{w}^{(j)} = \mathbf{w}^{(j-1)}$.

The implicit scheme is unconditionally stable as can be shown by representing the numerical solution in terms of its Fourier modes. However, although the scheme is consistent, the truncation error is only $O(k)$. In any case, since consistency and stability imply convergence (Lax equivalence theorem), the implicit scheme is convergent.

Implicit Finite Difference Heat Equation Algorithm.

- Give initial condition $f(x, y)$, boundary conditions, number of subintervals in x and y and time step Δt .
- Create coefficients for the tridiagonal matrix.
- Solve tridiagonal matrix.
- Move to the next time step.

Exercise. Same as exercise above but using the implicit scheme.

4.3 The Semi-Implicit (Crank-Nicolson) Scheme

The explicit and implicit schemes can be combined to produce a semi-implicit scheme with improved features. The schemes can be combined by simple weighing

$$\frac{w_{i,j+1} - w_{i,j}}{k} - \frac{\alpha}{2} \left[(1 - \theta) \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} + \theta \frac{w_{i+1,j+1} - 2w_{i,j+1} + w_{i-1,j+1}}{h^2} \right] = 0$$

where $0 \leq \theta \leq 1$ is the weighing factor. Note that $\theta = 0$ gives the explicit scheme and $\theta = 1$ yields the implicit scheme.

A stability analysis of this generalized scheme can be performed using Fourier modes and the result is that an unconditionally stable scheme is obtained as long as $\frac{1}{2} \leq \theta \leq 1$.

A second order accurate in time and space scheme can be obtained by averaging the forward and backward differences approximations (i.e. $\theta = \frac{1}{2}$ to produce the *Crank-Nicolson method*

$$\frac{w_{i,j+1} - w_{i,j}}{k} - \frac{\alpha}{2} \left[\frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} + \frac{w_{i+1,j+1} - 2w_{i,j+1} + w_{i-1,j+1}}{h^2} \right] = 0$$

where the associated matrix A is positive definite, diagonally dominant and tridiagonal. Either the Crout scheme or SOR iteration can be used to solve this system. Note that each equation in the above set uses the values of w at three contiguous nodes ($i - 1, i, i + 1$) for time level j and the same three nodes at time level $j + 1$ to compute the value of $w_{i,j+1}$.

Crank-Nicolson Finite Difference Algorithm.

- Give initial condition $f(x, y)$, boundary conditions, number of subintervals in x and y and time step Δt .
- Create coefficients for the tridiagonal matrix.
- Solve tridiagonal matrix.
- Move to the next time step.

Exercise. Same as exercise above but using the C-N scheme.

4.4 More General Boundary Conditions

Special care is require to handle more general boundary conditions. As an example consider the situation where the following condition applies at $x = 0$

$$\frac{\partial u}{\partial x} = \beta u + g$$

where β and g can be functions of time. To derive a discrete equation for the node located at $x = 0$ one introduces a fictitious node located at $x = -h$ and uses centered differences to write

$$\frac{w_{+h,j} - w_{-h,j}}{2h} = \frac{w_{2,j} - w_{-2,j}}{2h} = \beta w_{1,j} + g$$

The fictitious node value $w_{-h,j}$ is then eliminated by combining the above with the finite difference formula and scheme being used for the interior nodes (explicit, implicit or semi-implicit) which is assumed valid up to $x = -h$. The result is an algebraic equation for $w_{1,j}$.

4.5 One-dimensional Problems in Polar Coordinates and Boundary Conditions

When a multidimensional problem exhibits cylindrical or spherical symmetry it can be represented as a one dimensional problem in polar coordinates, i.e. for a cylindrical coordinate system

$$\frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = u_t = \frac{1}{r} (r u_r)_r = u_{rr} + \frac{1}{r} u_r$$

and for a spherical coordinate system

$$\frac{\partial u}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} = u_t = \frac{1}{r^2} (r^2 u_r)_r = u_{rr} + \frac{2}{r} u_r$$

The above two equations can be represented by the single formula

$$u_t = u_{rr} + \frac{\gamma}{r} u_r$$

where $\gamma = 1$ for cylindrical and $\gamma = 2$ for spherical coordinates.

The corresponding second order accurate finite difference formula using an explicit scheme and a uniform mesh in space are then readily obtained. Specifically, in cylindrical coordinates

$$\frac{w_{i,j+1} - w_{i,j}}{\Delta t} = \frac{1}{r_i \Delta r^2} (r_{i+\frac{1}{2}} w_{i+1,j} - 2r_i w_{i,j} + r_{i-\frac{1}{2}} w_{i-1,j})$$

and in spherical coordinates

$$\frac{w_{i,j+1} - w_{i,j}}{\Delta t} = \frac{1}{r_i^2 \Delta r^2} (r_{i+\frac{1}{2}}^2 w_{i+1,j} - 2r_i^2 w_{i,j} + r_{i-\frac{1}{2}}^2 w_{i-1,j})$$

where $r_i; i = 1, 2, \dots, N$ are the nodal positions along the radial direction, $r_{i \pm \frac{1}{2}}$ are positions of points located half-way between neighboring nodes, Δr is the mesh spacing and j is the time step index. Note $r_1 = 0$ is the node located at the center of the cylinder (or sphere).

All ideas presented before can be directly applied here with little modification. However, special care is required to handle the symmetry condition at the origin $r = 0$. For symmetry, it is required that

$$\frac{\partial u}{\partial r} = 0$$

A second order accurate finite difference formula at the origin can be obtained using a fictitious node located at $r = -\Delta r$ giving

$$\frac{w_{2,j} - w_{-2,j}}{2\Delta r} = 0$$

i.e. $w_{2,j} = w_{-2,j}$, the condition of radial symmetry around the origin. The fictitious node value $w_{-2,j}$ is then eliminated by combining the above with the finite difference formula being used for the interior nodes giving the result

$$\frac{w_{1,j+1} - w_{1,j}}{\Delta t} = \frac{2}{\Delta r^2}(w_{2,j} - w_{1,j})$$

Alternatively and equivalently, by means of a Maclaurin expansion one can show that at the origin, the following form of the diffusion equation is valid (when one has symmetry at the origin),

$$u_t = 2u_{rr}$$

for two-dimensional problems in cylindrical coordinates and

$$u_t = 3u_{rr}$$

for three-dimensional problems in spherical coordinates. These expressions can then be readily discretized to obtain finite difference formulae for the point at the origin.

If no symmetry can be assumed the following expressions can be used to approximate the Laplacian,

$$\nabla^2 u \approx \frac{4(w_{M,j} - w_{0,j})}{\Delta r^2}$$

for cylindrical systems and

$$\nabla^2 u \approx \frac{6(w_{M,j} - w_{0,j})}{\Delta r^2}$$

for spherical systems. Here $w_{M,j}$ is the nearest-neighbor mean value of w obtained by averaging over all nearest neighbor nodes to the node at the origin. The above approximations can then be used in the original PDE to obtain finite difference formulae for $w_{0,j}$.

4.6 Multidimensional Problems

Multidimensional problems for parabolic PDEs result if more than one spatial variable is involved. For instance, for the two-dimensional plate $a \leq x \leq b$, $c \leq y \leq d$, the linear diffusion equation is

$$\frac{\partial u}{\partial t} = \alpha \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

this must be solved for $u(x, y, t)$ subject to suitable boundary and initial conditions. A particularly simple example would require

$$u = 0$$

for $x = a$ and $x = b$ for all y and for $y = a$, $y = b$, for all x . This starting from

$$u(x, y, 0) = f(x, y)$$

This problem can also be solved easily by separation of variables.

The finite difference method proceeds as before by first creating a mesh in space subdividing the system into M subdomains in x and N in y yielding area elements of size $\Delta x \times \Delta y$ where $\Delta x = (b - a)/M$ and $\Delta y = (d - c)/N$. Next, a mesh of spacing Δt is also created along the time axis. The method then computes approximations to the values of u at the nodal locations, i.e. $w_{i,j,k} \approx u(x_i, y_j, t_k)$.

The simplest scheme is the explicit one given by

$$\frac{w_{i,j,k+1} - w_{i,j,k}}{\Delta t} - \alpha \left[\frac{w_{i+1,j,k} - 2w_{i,j,k} + w_{i-1,j,k}}{\Delta x^2} \frac{w_{i,j+1,k} - 2w_{i,j,k} + w_{i,j-1,k}}{\Delta y^2} \right] = 0$$

Note that each equation in the above set uses the values of w at five contiguous nodes $i, j; i - 1, j; i + 1, j; i, j + 1; i, j - 1$ for time level k to compute the value of $w_{i,j,k+1}$. As in the one-dimensional case, the explicit scheme is conditionally stable and the CFL restriction in this case requires that

$$\Delta t \leq \frac{1}{2\alpha} \frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2}$$

Several implicit schemes are possible. For instance, if the two spatial dimensions are handled implicitly one obtains the scheme

$$\begin{aligned} \frac{w_{i,j,k+1} - w_{i,j,k}}{\Delta t} - \alpha \left[\frac{w_{i+1,j,k+1} - 2w_{i,j,k+1} + w_{i-1,j,k+1}}{\Delta x^2} + \right. \\ \left. + \frac{w_{i,j+1,k+1} - 2w_{i,j,k+1} + w_{i,j-1,k+1}}{\Delta y^2} \right] = 0 \end{aligned}$$

Note that each equation in the above set uses the values of w at the five contiguous nodes $i, j; i - 1, j; i + 1, j; i, j + 1; i, j - 1$ for time level $k + 1$ to compute the value of $w_{i,j,k+1}$. Although the matrix associated with the resulting system is still sparse, at its best (depending on clever ordering of the nodes) it is pentadiagonal.

Alternatively, one may decide to handle only one dimension implicitly (say x) during the first part of the time step and switch the choice for the second part. Since only three contiguous nodes at the same time level are involved at each implicit step, a much easier to

solve tridiagonal system is obtained. This is called the *Alternating Direction Implicit (ADI) scheme*. Therefore, for the first half of the time step Δt one uses

$$\frac{w_{i,j,k+1/2} - w_{i,j,k}}{\frac{1}{2}\Delta t} - \alpha \left[\frac{w_{i+1,j,k+1/2} - 2w_{i,j,k+1/2} + w_{i-1,j,k+1/2}}{\Delta x^2} + \frac{w_{i,j+1,k} - 2w_{i,j,k} + w_{i,j-1,k}}{\Delta y^2} \right] = 0$$

to calculate $w_{i,j,k+1/2}$ and this is followed by

$$\frac{w_{i,j,k+1} - w_{i,j,k+1/2}}{\frac{1}{2}\Delta t} - \alpha \left[\frac{w_{i+1,j,k+1/2} - 2w_{i,j,k+1/2} + w_{i-1,j,k+1/2}}{\Delta x^2} + \frac{w_{i,j+1,k+1} - 2w_{i,j,k+1} + w_{i,j-1,k+1}}{\Delta y^2} \right] = 0$$

to calculate $w_{i,j,k+1}$.

Exercise. Consider the two dimensional formed by the unit square. Let $f(x, y) = 1$ and assume $u = 0$ at all four boundaries of the square. Solve the problem using the explicit method and the alternating direction implicit method and estimate the accuracy of the calculation in each case.

5 Hyperbolic Problems: Finite Difference Methods

Hyperbolic equations occur in many engineering and scientific applications. They are frequently associated with conservation principles, propagation problems and vibrations.

5.1 First Order Hyperbolic Partial Differential Equations

The most common example of first order hyperbolic PDE is the so-called conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

which must be solved for a given $f(u)$, subject to the initial condition

$$u(x, 0) = u^0(x)$$

and a sometimes a suitable boundary condition in order to obtain $u(x, t)$. An example of the above is what happens when motorists driving at steady speed on a highway approach an area with a traffic jam. As autos approach the jammed area they are forced to stop and join the jam. The size of the jam increases as the jammed area propagates in the opposite direction of the traffic.

By making

$$a = a(u) = \frac{\partial f}{\partial u}$$

one obtains the so-called advection equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$

A key feature of the advection equation is that u has constant values along certain sets of points on the $x - t$ plane called characteristics. I.e. along such characteristic directions

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \left(\frac{dx}{dt}\right) \frac{\partial u}{\partial x}$$

The characteristics are solutions of the following ODE

$$\frac{dx}{dt} = a(x, t)$$

Therefore, given a collection of points x_j at time $t = 0$, characteristic directions can be determined by numerical solution of the resulting initial value problem for $x(t)$. For instance, if a is simply a constant the solution of the advection equation becomes

$$u(x, t) = u^0(x - at)$$

a forward moving wave.

5.2 Finite Difference Methods for Advection Problems

For numerical solution of the advection equation using finite difference methods one starts by subdividing the space domain into N subdomains represented by nodes x_i with $i = 1, 2, \dots, N$ and also creating a mesh in time with nodes t_j , with $j = 1, 2, \dots, M$

A numerical approximation $w_{i,j} \approx u(x_i, t_j)$ to the solution of the advection equation can be obtained using an explicit finite difference scheme as follows (assuming $a = \text{constant}$)

$$\frac{w_{i,j+1} - w_{i,j}}{\Delta t} + a \frac{w_{i,j} - w_{i-1,j}}{\Delta x} = 0$$

Note that forward differences have been used for both time and space. Introducing $\nu = a\Delta t/\Delta x$, the scheme becomes

$$w_{i,j+1} = (1 - \nu)w_{i,j} + \nu w_{i-1,j}$$

This shows that the computed value of $w_{i,j+1}$ at time level $j+1$ depends on the values of $w_{i,j}$ and $w_{i-1,j}$. In turn, the value of $w_{i,j}$ depends on the values of $w_{i,j-1}$ and $w_{i-1,j-1}$ and that of

$w_{i-1,j}$ on those of $w_{i-1,j-1}$ and $w_{i-1,j+1}$ and so on forming a triangular *domain of dependence*. For convergence of the numerical scheme it is necessary that its domain of dependence contains the domain of dependence of the original PDE. This is the Courant-Friedrichs-Lewy condition for the convergence of the explicit scheme for the advection equation. Specifically, for $a > 0$, the CFL condition requires that

$$\nu = a \frac{\Delta t}{\Delta x} \leq 1$$

If seeking greater accuracy, one uses instead of a forward difference formula for the space derivative a central difference formula, unexpectedly, an *absolutely unstable* scheme is obtained!

A three point scheme with improved convergence characteristics is the upwind scheme in which the approximation is determined by the sign of a , i.e.

$$w_{i,j+1} = \begin{cases} (1 + \nu)w_{i,j} - \nu w_{i+1,j} & \text{if } a < 0 \\ (1 + \nu)w_{i,j} + \nu w_{i-1,j} & \text{if } a > 0 \end{cases}$$

Exercise. Use the upwind scheme to solve the hyperbolic equation

$$u_t + a(x, t)u_x = 0$$

with

$$a(x, t) = \frac{1 + x^2}{1 + 2xt + 2x^2 + x^4}$$

and subject to

$$u(x, 0) = 1$$

for $0.2 \leq x \leq 0.4$ and 0 otherwise, and

$$u(0, t) = 0$$

A more accurate method is the Lax-Wendroff scheme given by

$$w_{i,j+1} = \frac{1}{2}\nu(1 + \nu)w_{i-1,j} + (1 - \nu^2)w_{i,j} - \frac{1}{2}\nu(1 - \nu)w_{i+1,j}$$

As can be shown by Fourier mode analysis, this scheme is stable for $|\nu| \leq 1$.

Exercise. Use the Law-Wendroff scheme to solve the problem in the previous exercise and compare your results.

The box method is an implicit scheme used to solve the advection equation. The scheme is given as follows

$$w_{i+1,j+1} = w_{i,j} + (1 + \nu_{i+1/2}^{j+1/2})^{-1}(1 - \nu_{i+1/2}^{j+1/2})(w_{i+1,j} - w_{i,j+1})$$

When applied in practice the scheme is used under conditions that guarantee unconditional stability (i.e. $\nu > 0$).

Another useful method is the leapfrog scheme given by

$$w_{i,j+1} = w_{i,j-1} - (a \frac{\Delta t}{\Delta x})(w_{i+1,j} - w_{i-1,j})$$

Note that this is an explicit scheme requiring special handling of the starting conditions (e.g. Lax-Wendroff). Stability of this scheme also requires $|\nu| \leq 1$.

5.3 Finite Difference Methods for Second Order Hyperbolic Partial Differential Equations

The wave equation is the most common example of second order hyperbolic partial differential equations. The goal is to determine the function $u(x, t)$ satisfying

$$\frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t) =$$

on $0 < x < l$, $0 < t$ and subject to the conditions

$$\begin{aligned} u(0, t) &= u(l, t) = 0, & t > 0 \\ u(x, 0) &= f(x), & 0 \leq x \leq l \\ \frac{\partial u}{\partial t}(x, 0) &= g(x) & 0 \leq x \leq l \end{aligned}$$

The implementation of the finite difference method in this case proceeds as follows. First a mesh is set up by subdividing time and space in subintervals bounded by mesh points $x_i = (i - 1)h$; $i = 1, 2, \dots, N$ and $t_j = (j - 1)k$; $j = 1, 2, \dots, M$. Second order accurate finite difference formulae for all the interior points give

$$\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{k^2} - c^2 \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} = 0$$

which can be solved for $w_{i,j+1}$ to give

$$w_{i,j+1} = 2[1 - (\frac{ck}{h})^2]w_{i,j} + (\frac{ck}{h})^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}$$

for $i = 1, \dots, N$ and $j = 1, 2, \dots$

A second order Taylor polynomial can now be constructed to satisfy the derivative initial condition. The result is

$$w_{i,2} = w_{i,1} + kg(x_i) + \frac{c^2 k^2}{2} [\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{h^2}]$$

As before, the explicit method is conditionally stable. The CFL restriction in this case is $ck/h \leq 1$. Unconditionally stable implicit methods for the wave equation are also available.

6 The Convection-Diffusion Equation

The convection-diffusion equation describes the dispersion of energy or diffusant due to the combined effects of convective transport and molecular diffusion. In vector form, the equation is

$$\frac{\partial u}{\partial t} + \mathbf{v} \nabla u = \alpha \nabla^2 u$$

where \mathbf{v} is the velocity vector, and α is the molecular diffusivity. Moreover, u is the dispersed entity (i.e. energy or diffusant). The second term on the left hand side of the above equation is known as the convective term that accounts for the dispersion due to the fluid flow while the term on the right hand side is the diffusive term and accounts for dispersion due to molecular diffusion. In most prior sections, second order accurate formulae were generally recommended and used when constructing finite difference schemes for partial differential equations. Even when using apparently first order accurate Euler forward time stepping in the construction of explicit schemes for parabolic equations, the CFL stability condition makes the scheme actually second order accurate. This practice may lead to failure when constructing finite difference approximations to the convection term in the convection-diffusion equation. A successful approach in this case is based on the idea of "upwinding" introduced earlier in connection with first order hyperbolic equations. Upwinding is the practice of constructing finite difference formulae for the convective term to enforce the physically-based notion that the value of a transported quantity at a given point is essentially determined by values on the upstream side of the domain and little or not at all affected by what is happening downstream.

As a simple illustration consider the steady state, one dimensional convection-diffusion equation without source term, i.e.

$$-\alpha \frac{d^2 u}{dx^2} + v \frac{du}{dx} = 0$$

where v is the fluid velocity (assumed constant). This must be solved subject to the boundary conditions

$$u(0) = 0$$

$$u(1) = 1$$

The exact solution of this problem u_e is

$$u_e(x) = \frac{\exp(vx/\alpha) - 1}{\exp(v/\alpha) - 1}$$

Construct now a finite difference mesh consisting of uniformly spaced nodes $x_1, x_2, \dots, x_i, \dots, x_N$ and use central differences to construct the following finite difference approximation for each of the interior nodes

$$-\alpha \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} + v \frac{w_{j+1} - w_{j+1}}{2h} = 0$$

rearranging yields

$$-(1 + \frac{P_h}{2})w_{i-1} + 2w_i - (1 - \frac{P_h}{2})w_{i+1} = 0$$

where P_h is a most important parameter in this case; it is called the mesh Peclet number and is defined as

$$P_h = \frac{vh}{\alpha}$$

With the given boundary conditions, the above FD formula yields a tridiagonal system of N simultaneous linear algebraic equations that can be readily solved using standard methods and the results compared against the exact solution.

However, while the scheme above produces quite satisfactory results for $P_h < 2$, it leads to rather incorrect (and physically meaningless) values when $P_h > 2$.

Physically meaningful approximations can be obtained for $P_h > 2$ by use of the upwind scheme. To describe the upwind scheme, assume that $v > 0$ (i.e. the upwind side of the flow is towards the origin $x = 0$). Instead of the central difference formula for the convective term, use first order accurate backward differences and obtain

$$-\alpha \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} + v \frac{w_i - w_{i-1}}{h} = 0$$

now rearrange to get

$$-(1 + P_h)w_{i-1} + (2 + P_h)w_i - w_{i+1} = 0$$

This is another tridiagonal system and although the FD formula used for the convective term is less accurate than central differences, it can produce physically meaningful and accurate results.

7 The Navier-Stokes Equations

The laminar flow of an incompressible Newtonian fluid with constant properties is described by the principle of momentum conservation expressed by the equations of Navier-Stokes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \nu \nabla^2 \mathbf{v} - \frac{1}{\rho} \nabla p$$

where $\mathbf{v}(\mathbf{r}, t)$ is the unknown velocity vector, ν and ρ are the viscosity and the density of the fluid, respectively and p is the pressure. Since the momentum is a vector, the above equation actually represents three balance equations; one for each coordinate direction.

Moreover, the velocity field must also satisfy the principle of mass conservation expressed by the following equation of continuity in the case of an incompressible fluid

$$\nabla \cdot \mathbf{v} = 0$$

The combined momentum and mass conservation equations must be solved simultaneously subject to the specific boundary and initial conditions imposed in each particular case.

The study of approximate solution methods for the above system is known as computational fluid dynamics (CFD) and is an active area of investigation where many researchers have made important contributions. The field of CFD has also broad applicability in engineering and science as it allows exploration of complex flow phenomena using only mathematical models and computer simulation.

The presentation below is necessarily brief and focuses on the description of particular algorithms that have been found useful in the solution of many problems. The first case has elliptic-type features while the second case has more of a parabolic nature.

Consider the steady, two-dimensional flow of a Newtonian fluid inside a rectangular domain. Extensions to other geometries and to three dimensional situations can be readily produced. For this case, the Navier-Stokes equations become

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

representing the balance of fluid momentum along the x -direction and

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial y}$$

representing the balance of fluid momentum along the y -direction. Here u and v are, respectively the x - and y - velocity components so that $\mathbf{v} = (u, v)$ is the velocity vector.

The equation of continuity in this case becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The following finite volume method can be used to obtain approximate solutions in this case. Introduce a mesh of points (x_i, y_j) , with $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$, uniformly spaced along the x and y directions, and where Δx and Δy are, respectively, the mesh spacings along these directions. Any individual interior mesh point $(x_i, y_j) = P$ is surrounded by pairs of mesh points along the coordinate directions i.e. (x_i, y_{j+1}) , (x_i, y_{j-1}) , (x_{i+1}, y_j) , and (x_{i-1}, y_j) .

Introduce now the idea of a staggered mesh that requires computation of the unknowns u, v, p on different mesh points. Specifically, pressure is computed at P -type nodes, u -velocity is computed at midpoint nodes located a half-mesh spacing shift along the x -direction $(i + \frac{1}{2}, j)$ while v -velocity is computed at midpoint nodes located a half-mesh spacing shift along the y -direction $(i, j + \frac{1}{2})$.

If central differences are used to approximate the diffusive term and upwinding for the convective term, the following discrete formula is obtained for the u velocity at all interior

mid-points $(i + \frac{1}{2}, j)$

$$u_{i+\frac{1}{2},j} = \frac{1}{a_{i+\frac{1}{2},j}}(a_{i+\frac{3}{2},j}u_{i+\frac{3}{2},j} + a_{i-\frac{1}{2},j}u_{i-\frac{1}{2},j} + a_{i+\frac{1}{2},j+1}u_{i+\frac{1}{2},j+1} + a_{i+\frac{1}{2},j-1}u_{i+\frac{1}{2},j-1}) - (p_{i+1,j} - p_{i,j})\Delta y$$

where

$$\begin{aligned} a_{i+\frac{3}{2},j} &= \alpha \frac{\Delta y}{\Delta x} + \begin{cases} -u_{i+1,j}\Delta y & \text{if } -u_{i+1,j}\Delta y > 0 \\ 0 & \text{otherwise} \end{cases} \\ a_{i-\frac{1}{2},j} &= \alpha \frac{\Delta y}{\Delta x} + \begin{cases} u_{i,j}\Delta y & \text{if } u_{i,j}\Delta y > 0 \\ 0 & \text{otherwise} \end{cases} \\ a_{i+\frac{1}{2},j+1} &= \alpha \frac{\Delta x}{\Delta y} + \begin{cases} -v_{i+\frac{1}{2},j+1}\Delta x & \text{if } -v_{i+\frac{1}{2},j+1}\Delta x > 0 \\ 0 & \text{otherwise} \end{cases} \\ a_{i+\frac{1}{2},j-1} &= \alpha \frac{\Delta x}{\Delta y} + \begin{cases} v_{i+\frac{1}{2},j-1}\Delta x & \text{if } v_{i+\frac{1}{2},j-1}\Delta x > 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$a_{i+\frac{1}{2},j} = a_{i+\frac{3}{2},j} + a_{i-\frac{1}{2},j} + a_{i+\frac{1}{2},j+1} + a_{i+\frac{1}{2},j-1}$$

The corresponding expression for the v velocity at all interior mid-points $(i, j + \frac{1}{2})$

$$v_{i,j+\frac{1}{2}} = \frac{1}{a_{i,j+\frac{1}{2}}}(a_{i,j+\frac{3}{2}}v_{i,j+\frac{3}{2}} + a_{i,j-\frac{1}{2}}v_{i,j-\frac{1}{2}} + a_{i+1,j+\frac{1}{2}}v_{i+1,j+\frac{1}{2}} + a_{i-1,j+\frac{1}{2}}v_{i-1,j+\frac{1}{2}}) - (p_{i,j+1} - p_{i,j})\Delta x$$

where

$$\begin{aligned} a_{i,j+\frac{3}{2}} &= \alpha \frac{\Delta y}{\Delta x} + \begin{cases} -u_{i,j+1}\Delta y & \text{if } -u_{i,j+1}\Delta y > 0 \\ 0 & \text{otherwise} \end{cases} \\ a_{i,j-\frac{1}{2}} &= \alpha \frac{\Delta y}{\Delta x} + \begin{cases} u_{i,j}\Delta y & \text{if } u_{i,j}\Delta y > 0 \\ 0 & \text{otherwise} \end{cases} \\ a_{i+1,j+\frac{1}{2}} &= \alpha \frac{\Delta x}{\Delta y} + \begin{cases} -v_{i+1,j+\frac{1}{2}}\Delta x & \text{if } -v_{i+1,j+\frac{1}{2}}\Delta x > 0 \\ 0 & \text{otherwise} \end{cases} \\ a_{i-1,j+\frac{1}{2}} &= \alpha \frac{\Delta x}{\Delta y} + \begin{cases} v_{i-1,j+\frac{1}{2}}\Delta x & \text{if } v_{i-1,j+\frac{1}{2}}\Delta x > 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$a_{i,j+\frac{1}{2}} = a_{i,j+\frac{3}{2}} + a_{i,j-\frac{1}{2}} + a_{i+1,j+\frac{1}{2}} + a_{i-1,j+\frac{1}{2}}$$

The above equations are more simply written as

$$u_{i+\frac{1}{2},j} = \frac{1}{a_{i+\frac{1}{2},j}} \sum_{nb} a_{nb} u_{nb} + (p_{i,j} - p_{i+1,j}) \Delta y$$

$$v_{i,j+\frac{1}{2}} = \frac{1}{a_{i,j+\frac{1}{2}}} \sum_{nb} a_{nb} v_{nb} + (p_{i,j} - p_{i,j+1}) \Delta x$$

Note that since the various a_{nb} contain the (unknown) velocity components, the equations are algebraic but non-linear. Also, note that the pressure field $p_{i,j}$ must be known before attempting to solve them. However, the pressure field must be such that the equation of continuity is satisfied. Let an initial guess for the velocity field be $(u_{i+\frac{1}{2},j}^o, v_{i,j+\frac{1}{2}}^o)$. This velocity field would be associated with an (incorrect) pressure field p^o . An improved guess for the pressure field can be written as

$$p^n = p^o + p'$$

where p' is called the pressure correction. This correction must be such that the (corrected) velocities are given by

$$u_{i+\frac{1}{2},j}^n = u_{i+\frac{1}{2},j}^o + \frac{\Delta y}{a_{i+\frac{1}{2},j}} (p'_{i,j} - p'_{i+1,j})$$

$$v_{i,j+\frac{1}{2}}^n = v_{i,j+\frac{1}{2}}^o + \frac{\Delta x}{a_{i,j+\frac{1}{2}}} (p'_{i,j} - p'_{i,j+1})$$

Substituting these two expressions into the equation of continuity yields the following discrete equation for the pressure correction at each node

$$p'_{i,j} = \frac{1}{a_{i,j}} (a_{i-1,j} p'_{i-1,j} + a_{i+1,j} p'_{i+1,j} + a_{i,j-1} p'_{i,j-1} + a_{i,j+1} p'_{i,j+1} + b)$$

where

$$a_{i-1,j} = \frac{\Delta y^2}{a_{i-\frac{1}{2},j}}$$

$$a_{i+1,j} = \frac{\Delta y^2}{a_{i+\frac{1}{2},j}}$$

$$a_{i,j-1} = \frac{\Delta x^2}{a_{i,j-\frac{1}{2}}}$$

$$a_{i,j+1} = \frac{\Delta x^2}{a_{i,j+\frac{1}{2}}}$$

$$b = (u_{i-\frac{1}{2},j}^o - u_{i+\frac{1}{2},j}^o)\Delta y + (v_{i,j-\frac{1}{2}}^o - v_{i,j+\frac{1}{2}}^o)\Delta x$$

and

$$a_{i,j} = a_{i-1,j} + a_{i+1,j} + a_{i,j-1} + a_{i,j+1}$$

The SIMPLE algorithm proceeds by first solving the (linearized) momentum equations for an assumed (incorrect) pressure field p^o in order to compute approximations u^o, v^o . These equations can be solved by standard methods such as point by point Gauss-Seidel or SOR iteration, or even better by line by line Gauss-Seidel or SOR iteration with the TDMA used for each line.

Once the updated values u^o, v^o are available, the pressure correction equation is solved using iterative Gauss-Seidel or SOR methods in order to obtain values of the pressure correction p' . The corrected pressure is then given by

$$p_{i,j}^n = p_{i,j}^o + p'_{i,j}$$

The velocities are updated to new, corrected values u^n, v^n using the formulae

$$u_{i+\frac{1}{2},j}^n = u_{i+\frac{1}{2},j}^o + \frac{\Delta y}{a_{i+\frac{1}{2},j}}(p'_{i,j} - p'_{i+1,j})$$

$$v_{i,j+\frac{1}{2}}^n = v_{i,j+\frac{1}{2}}^o + \frac{\Delta x}{a_{i,j+\frac{1}{2}}}(p'_{i,j} - p'_{i,j+1})$$

The corrected pressure $p_{i,j}^n$ is then used as a new guess and the computation process repeated. Iterations are continued in this manner until a suitable convergence criterion is satisfied.

As a second example, consider the case of transient, two-dimensional flow in a rectangular domain. The Navier-Stokes equations become

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

and

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial y}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

To solve this problem for specified boundary and initial conditions, use again a staggered grid with mesh spacings $\Delta x, \Delta y$ and select a time step Δt .

The calculation then moves forward using an explicit method. Specifically, given values of the velocity and pressure at time level k , u^k, v^k, p^k , the pressure field is advanced to time $k+1$ by solving the discrete analog of the equation of continuity, i.e.

$$\begin{aligned} & \frac{p_{i+1,j}^{k+1} - 2p_{i,j}^{k+1} + p_{i-1,j}^{k+1}}{\Delta x^2} + \frac{p_{i,j+1}^{k+1} - 2p_{i,j}^{k+1} + p_{i,j-1}^{k+1}}{\Delta y^2} = \\ & = \frac{\rho}{\Delta t} \left(\frac{F_{i+\frac{1}{2},j}^k - F_{i-\frac{1}{2},j}^k}{\Delta x} + \frac{G_{i,j+\frac{1}{2}}^k - G_{i,j-\frac{1}{2}}^k}{\Delta y} \right) \end{aligned}$$

where

$$\begin{aligned} F_{i+\frac{1}{2},j}^k = & -\Delta t \left(\frac{u_{i+\frac{1}{2},j}^k}{\Delta x} \begin{cases} (u_{i+\frac{1}{2},j}^k - u_{i-\frac{1}{2},j}^k) & \text{if } u_{i+\frac{1}{2},j} \geq 0 \\ (u_{i+\frac{3}{2},j}^k - u_{i+\frac{1}{2},j}^k) & \text{if } u_{i+\frac{1}{2},j} < 0 \\ 0 & \text{otherwise} \end{cases} \right. \\ & \left. + \frac{v_{i+\frac{1}{2},j}^k}{\Delta y} \begin{cases} (u_{i+\frac{1}{2},j}^k - u_{i+\frac{1}{2},j-1}^k) & \text{if } v_{i+\frac{1}{2},j} \geq 0 \\ (u_{i+\frac{1}{2},j+1}^k - u_{i+\frac{1}{2},j}^k) & \text{if } v_{i+\frac{1}{2},j} < 0 \\ 0 & \text{otherwise} \end{cases} \right) \\ & - \nu \left(\frac{u_{i+\frac{3}{2},j}^k - 2u_{i+\frac{1}{2},j}^k + u_{i-\frac{1}{2},j}^k}{\Delta x^2} + \frac{u_{i+\frac{1}{2},j+1}^k - 2u_{i+\frac{1}{2},j}^k + u_{i+\frac{1}{2},j-1}^k}{\Delta y^2} \right) \end{aligned}$$

and

$$\begin{aligned} G_{i,j+\frac{1}{2}}^k = & -\Delta t \left(\frac{u_{i,j+\frac{1}{2}}^k}{\Delta x} \begin{cases} (v_{i,j+\frac{1}{2}}^k - v_{i-1,j-\frac{1}{2}}^k) & \text{if } u_{i,j+\frac{1}{2}} \geq 0 \\ (v_{i+1,j+\frac{1}{2}}^k - v_{i,j+\frac{1}{2}}^k) & \text{if } u_{i,j+\frac{1}{2}} < 0 \\ 0 & \text{otherwise} \end{cases} \right) \\ & + \frac{v_{i,j+\frac{1}{2}}^k}{\Delta y} \begin{cases} (v_{i,j+\frac{1}{2}}^k - v_{i,j-\frac{1}{2}}^k) & \text{if } v_{i,j+\frac{1}{2}} \geq 0 \\ (v_{i,j+\frac{3}{2}}^k - v_{i,j+\frac{1}{2}}^k) & \text{if } v_{i,j+\frac{1}{2}} < 0 \\ 0 & \text{otherwise} \end{cases} \\ & - \nu \left(\frac{v_{i+1,j+\frac{1}{2}}^k - 2v_{i,j+\frac{1}{2}}^k + v_{i-1,j+\frac{1}{2}}^k}{\Delta x^2} + \frac{v_{i,j+\frac{3}{2}}^k - 2v_{i,j+\frac{1}{2}}^k + v_{i,j-\frac{1}{2}}^k}{\Delta y^2} \right) \end{aligned}$$

As soon as p has been determined for the new time level, the updated velocity components are computed from the formulae

$$u_{i+\frac{1}{2},j}^{k+1} = u_{i+\frac{1}{2},j}^k + F_{i+\frac{1}{2},j}^k - \frac{\Delta t}{\rho} \left(\frac{p_{i+1,j}^{k+1} - p_{i,j}^{k+1}}{\Delta x} \right)$$

and

$$v_{i,j+\frac{1}{2}}^{k+1} = v_{i,j+\frac{1}{2}}^k + G_{i,j+\frac{1}{2}}^k - \frac{\Delta t}{\rho} \left(\frac{p_{i,j+1}^{k+1} - p_{i,j}^{k+1}}{\Delta y} \right)$$

The explicit method above is conditionally stable. Stability requires selecting the time step such that the following stability condition is satisfied

$$\Delta t \leq \frac{1}{\max_{i,j} \left(\frac{|u_{i,j}^k|}{\Delta x} + \frac{|v_{i,j}^k|}{\Delta y} + \frac{2\nu}{\Delta x^2} + \frac{2\nu}{\Delta y^2} \right)}$$

8 Exercises

1.- Find approximate solutions $u(x, y)$ to the following problem in $x \in [0, 1]$, $y \in [0, 1]$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to

$$\begin{aligned} u(0, y) &= u(1, y) = u(x, 0) = 0 \\ u(x, 1) &= \sin(\pi x) \end{aligned}$$

using the finite difference method, the finite volume method and the finite element method. Assess the accuracy of the solution obtained in each case by comparing the approximations obtained using various levels of mesh refinement against the exact solution of the problem obtained by separation of variables.

2.- Find approximate solutions $u(x, t)$ to the following problem in $x \in [0, 1]$, $t > 0$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

subject to

$$\begin{aligned} u(0, t) &= u(1, t) = 0 \\ u(x, 0) &= 1 \end{aligned}$$

using the finite difference method, the finite volume method and the finite element method. Assess the accuracy of the solution obtained in each case by comparing the approximations obtained using various levels of mesh refinement against the exact solution of the problem obtained by separation of variables.

3.- Find approximate solutions $u(x, t)$ to the following problem in $x \in [0, 1]$, $t > 0$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

subject to

$$\begin{aligned} u(0, t) &= u(x, 0) = 0 \\ u(0, t) &= t \end{aligned}$$

using the finite difference method, the finite volume method and the finite element method. Assess the accuracy of the solution obtained in each case by comparing the approximations obtained using various levels of mesh refinement against the exact solution of the problem obtained by Duhamel's method.

4.- Find approximate solutions $u(x, t)$ to the following problem in $x \in [0, 1]$, $t > 0$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

subject to

$$\begin{aligned} u(0, t) &= \\ u(1, t) &= 0 \end{aligned}$$

$$u(x, 0) = \begin{cases} 0.1x & \text{for } x \in [0, 1/2] \\ 0.1(1 - x) & \text{for } x \in [1/2, 1] \end{cases}$$

and

$$\frac{\partial u}{\partial t}|_{(x,t)} = 0$$

using the finite difference method, the finite volume method and the finite element method. Assess the accuracy of the solution obtained in each case by comparing the approximations obtained using various levels of mesh refinement against the exact solution of the problem.

5.- Find an approximate solution $u(r, z)$ to the following problem in $r \in [0, 1]$, $z \in [0, 1]$,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} = 0$$

subject to

$$\begin{aligned} u(1, z) &= u(r, 0) = 0 \\ u(r, 1) &= \cos\left(\frac{\pi r}{2}\right) \end{aligned}$$

and

$$\frac{\partial u}{\partial r}|_{(0,z)} = 0$$

using the finite difference method, the finite volume method and the finite element method. Assess the accuracy of the solution obtained in each case by comparing the approximations obtained using various levels of mesh refinement against the exact solution of the problem.

6.- Find an approximate solution $u(r, \theta)$ to the following problem in $r \in [0, 1]$, $\theta \in [0, 2\pi]$,

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} (\sin(\theta) \frac{\partial u}{\partial \theta}) = 0$$

where θ is the poloidal angle and subject to

$$u(x, 0) = \begin{cases} 1 & \text{for } \theta \in [0, \pi] \\ 0 & \text{for } \theta \in [\pi, 2\pi] \end{cases}$$

using the finite difference method, the finite volume method and the finite element method. Assess the accuracy of the solution obtained in each case by comparing the approximations obtained using various levels of mesh refinement against the exact solution of the problem.

7.- Find an approximate solution $u(r, t)$ to the following problem in $r \in [0, 1]$, $t > 0$,

$$\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) = \frac{1}{\alpha} \frac{\partial u}{\partial t} = 0$$

subject to

$$u(r, 0) = 100$$

$$\frac{\partial u}{\partial r}|_{(0,t)} = 0$$

and

$$u(1, t) = 0$$

where $\alpha = 10^{-5}$, using the finite difference method, the finite volume method and the finite element method. Assess the accuracy of the solution obtained in each case by comparing the approximations obtained using various levels of mesh refinement against the exact solution of the problem.

8.- Find approximate solutions $u(r, \theta, t)$ to the following problem in $r \in [0, 1]$, $\theta \in [-\pi, \pi]$ and $t > 0$,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial t^2}$$

subject to

$$u(1, \theta, t) = 0$$

$$u(r, \theta, 0) = 0.1(1 - r)$$

and

$$\frac{\partial u}{\partial t}|_{(r, \theta, 0)} = 0$$

using the finite difference method, the finite volume method and the finite element method. Assess the accuracy of the solution obtained in each case by comparing the approximations obtained using various levels of mesh refinement against the exact solution of the problem.

9.- Find approximate solutions $u(x, y, z, t)$ to the following problem in $x \in [0, 0.01]$, $y \in [-0.002, 0.002]$, $z \in [0, -0.003]$, $t > 0$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\alpha} \frac{\partial u}{\partial t}$$

where $\alpha = k/(\rho C_p)$ with $k = 40$, $\rho = 8700$ and $C_p = 385$ and subject to

$$u(x, y, z, 0) = 0$$

and

$$\frac{\partial u}{\partial n} = 0$$

in all boundaries except the $x - y$ plane for $z = 0$ where the condition is

$$-\frac{\partial u}{\partial z} = q_0(x, y, t)$$

where

$$q_0 = 6 \times 10^7 \exp(-10^6((x - 0.001 - 0.006t)^2 + y^2))$$

Assess the accuracy of the solution obtained by mesh extensions.

10.- Consider the problem of developing laminar flow of a Newtonian fluid entering a pipe (radius = 0.001 length 0.01 with a uniform velocity $u_0 = 0.01$. The fluid density is $\rho = 1000$ and the viscosity $\nu = 10^{-6}$.

a) Obtain an approximate solution to the problem.

b) Assume that the fluid enters the pipe with a temperature of $T_0 = 25$ degrees while the pipe wall is maintained at $T_w = 50$ degrees. Assume $k = 0.6$ and $C_p = 4000$ and obtain an approximation for the temperature of the water as a function of the distance from the pipe inlet.