Real Analysis Notes - 2
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Sequences

- You are moving from sequences in the familiar metric space $(\mathbb{R}, d)$, where the distance $d(x, y) = |x-y|$, to sequences in a general **metric space** (X, d).
- **Rudin's Definition Check:** Refer to Rudin's text (or equivalent) for the formal definition of convergence in a general metric space.
- **The Formal Definition:** A sequence $\{x_n\}$ in a metric space (X, d) **converges to a limit** $x \in X$ if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \quad d(x_n, x) < \varepsilon$$

- **Key Insight:** The undergraduate absolute value $|x_n - x|$ is always replaced by the **metric (distance function)** $d(x_n, x)$.

Checking Convergence by Definition

Determine if the limit of the sequence exists. Clearly state the metric space used (e.g., $\mathbb{R}$ or $\mathbb{C}$) and justify your answer based on the definition.

1. $x_n = (-1)^n$
2. $x_n = i^n$ (Consider this sequence in the metric space $\mathbb{C}$).
3. $x_n = \frac{1}{n}$

Algebra of Limits Pitfalls

Assume all sequences are in the standard Euclidean metric space $\mathbb{R}$.

1. **Sum Rule:** Given $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n^2}$. Does $z_n = x_n + y_n$ converge? If so, state the limit.
2. **Product Rule:** Given $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n}$. Does $z_n = x_n y_n$ converge? If so, state the limit.
3. **Quotient Rule Restrictions:** If $\{x_n\}$ and $\{y_n\}$ are convergent sequences, what two conditions are required for the sequence $z_n = \frac{x_n}{y_n}$ to be convergent? *(Hint: Consider y_n terms and their limit.)*
4. **Quotient Rule Pitfall Example:** Let $x_n = 1$ and $y_n = \frac{1}{n}$. Analyze the convergence of $z_n = \frac{x_n}{y_n}$. Why does the standard Quotient Rule fail to apply, even though $y_n \neq 0$ for all n?

Subsequences and monotone subsequences

The following problems test your recall of the definitions and implications of monotone and convergent sequences in R.

1. **True or False:** Every monotone sequence of real numbers is convergent. Justify your answer.
2. **True or False:** A monotone increasing sequence of real numbers converges to its supremum. Justify your answer.
3. **True or False:** Every convergent sequence of real numbers is bounded. Justify your answer.

Monotone Subsequence Theorem

This section explores the crucial property that every sequence must contain a monotone subsequence.

Problem 4: Prove that every sequence of real numbers $\{x_n\}$ has a monotone subsequence.

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1. **Case 1: The Sequence $\{x_n\}$ is Unbounded.**
   - Without loss of generality, assume $\{x_n\}$ is not **bounded above**.
   - Choose $x_{n_1} = x_1$. Since x_{n_1} is not an upper bound for the entire sequence (because the sequence is unbounded above), locate the first element after x_{n_1} which is greater than x_{n_1}. Call this x_{n_2}.
   - Since x_{n_2} is also not an upper bound, locate the first element after x_{n_2} which is greater than x_{n_2}. Call this x_{n_3}.
   - This process never ends (why? Because $\{x_n\}$ is not bounded above). This construction yields a subsequence $\{x_{n_k}\}$ that is strictly **monotone increasing**.
   - A similar argument works if the sequence is not bounded below, yielding a strictly monotone decreasing subsequence.
2. **Case 2: The Sequence $\{x_n\}$ is Bounded.**
   - By the **Bolzano-Weierstrass Theorem**, $\{x_n\}$ has a convergent subsequence, say $\{y_k\}$, and $\lim y_k = y$.
   - **Subcase 2A (Easy Case):** Suppose $\{y_k\}$ has infinitely many terms equal to its limit y. You can choose a constant subsequence from these terms, which is trivially monotone.
   - **Subcase 2B (Harder Case):** Suppose there are infinitely many terms of $\{y_k\}$ not equal to y. This means there are infinitely many terms either strictly greater than y or strictly less than y.
   - Without loss of generality, assume there are infinitely many terms of $\{y_k\}$ **greater than y**.
   - Choose the first term y_{k_1} from $\{y_k\}$ such that $y_{k_1} > y$.
   - Now, look for an element of $\{y_k\}$ in the interval (y, y_{k_1}). Since $\lim y_k = y$, and there are infinitely many terms greater than y, such an element y_{k_2} (with $k_2 > k_1$) exists such that $y < y_{k_2} < y_{k_1}$.
   - Next, look for an element y_{k_3} (with $k_3 > k_2$) in the interval (y, y_{k_2}) such that $y < y_{k_3} < y_{k_2}$.
   - Continuing this process yields a subsequence $\{y_{k_j}\}$ that is strictly **monotone decreasing**.

Sequences: Monotonicity, Boundedness, and Subsequences

This section reviews fundamental concepts regarding monotone sequences and the existence of monotone subsequences, leading into key theorems.

Conceptual True/False (Review of Definitions)

(a) **True or False:** Every monotone sequence of real numbers is convergent. Justify your answer.

(b) **True or False:** A monotone increasing sequence of real numbers converges to its supremum. Justify your answer.

(c) **True or False:** Every convergent sequence of real numbers is bounded. Justify your answer.

Sequences and Series: The Sequence of Partial Sums

This note outlines the fundamental definition of series convergence via partial sums and details the necessary condition for convergence, including the crucial counterexample.

The Fundamental Connection: Series as Sequences

- The most critical conceptual understanding is that a **series** ($\sum_{n=1}^{\infty} x_n$) is inherently a **sequence** in disguise: the **sequence of partial sums**, $\{S_N\}$.
- The sequence of partial sums $\{S_N\}$ is constructed by summing the terms of the series consecutively:

$$S_N = x_1 + x_2 + \cdots + x_N = \sum_{n=1}^{N} x_n$$

- **Convergence Definition:** The series $\sum x_n$ **converges** if and only if the sequence of partial sums $\{S_N\}$ converges.
- **Note on Terminology:** Do not confuse the **corresponding sequence** $\{x_n\}$ (the terms being added) with the **sequence of partial sums** $\{S_N\}$.

The Necessary Condition for Convergence (Test for Divergence)

If a series $\sum x_n$ converges, its sequence of partial sums $\{S_N\}$ must converge to some limit L. This requires the terms being added to approach zero.

- Since the N-th term of the series is the difference between two consecutive partial sums, $x_N = S_N - S_{N-1}$ (for $N \geq 2$), we take the limit:

$$\lim_{N\to\infty} x_N = \lim_{N\to\infty} (S_N - S_{N-1}) = L - L = 0$$

- **Conclusion:** For a series $\sum x_n$ to converge, the absolute value of the terms added, $|x_n|$, **must become smaller and smaller** as n goes to infinity. That is, $\lim_{n\to\infty} x_n = 0$.

The Problem with Intuitive Understanding: A Cautionary Pitfall

The intuitive idea that terms becoming small guarantees convergence **is flawed**. The converse of the necessary condition is false.

- The question is: If $\lim_{n\to\infty} x_n = 0$, can we be certain that the corresponding series $\sum x_n$ converges?
- The answer is a **loud NO**! The condition $\lim x_n = 0$ is **necessary** but **not sufficient** for series convergence.
- **Counterexample:** Consider the sequence $x_n = \frac{1}{n}$. The terms approach zero: $\lim_{n\to\infty} \frac{1}{n} = 0$.
- However, the corresponding series, $\sum_{n=1}^{\infty} \frac{1}{n}$ (the **Harmonic Series**), **diverges**. This demonstrates that terms may not become small *fast enough* to allow the sequence of partial sums to stabilize.

Standard Sequences and Series Reference

A summary of fundamental sequences and series, their convergence properties, and core convergence tests.

Fundamental Sequences ($\lim_{n\to\infty} x_n$)

A sequence $\{x_n\}$ converges if $\lim_{n\to\infty} x_n = L$ (a finite number).

Sequence (x_n)	Limit (L)	Condition	Notes
Constant	c	Always Convergent	$x_n = c$
Power	0	$p > 0$	$x_n = 1/n^p$
Geometric	0	$\|r\| < 1$	$x_n = r^n$. Diverges if $\|r\| \geq 1$.
Root of n	1	Always Convergent	$\lim \sqrt[n]{n} = 1$
Log over Power	0	$p > 0$	$\lim \frac{\ln(n)}{n^p} = 0$. Log dominates power.

Fundamental Series ($\sum_{n=1}^{\infty} a_n$)

A series converges if its sequence of partial sums converges.

Series ($\sum \mathbf{a_n}$)	**Type**	**Convergence Condition**	**Notes**
Harmonic	p-Series ($p = 1$)	Diverges	$\sum \frac{1}{n}$. $\lim a_n = 0$, but diverges.
p-Series	p-Series	$\mathbf{p > 1}$	$\sum \frac{1}{n^p}$.
Geometric	Geometric	$\mathbf{\lvert r \rvert < 1}$	$\sum r^n$. Converges to $\frac{r}{1-r}$ (if $n = 1$).
Factorial	General	Converges	$\sum \frac{1}{n!}$. Converges quickly (Ratio Test $\to 0$).

Core Convergence Tests

1. **Test for Divergence (Necessary Condition)**
   - If $\lim_{n\to\infty} a_n \neq 0$ or DNE, then $\sum a_n$ **diverges**.
   - **Caution:** If $\lim a_n = 0$, the test is inconclusive.
2. **Comparison Tests** (for non-negative terms a_n)
   - **Direct Comparison:** Use with a known convergent ($\sum b_n$ where $a_n \leq b_n$) or divergent ($\sum b_n$ where $a_n \geq b_n$) series.
   - **Limit Comparison:** If $\lim \frac{a_n}{b_n} = L$ ($0 < L < \infty$), then $\sum a_n$ and $\sum b_n$ share convergence.
3. **Ratio Test** (Good for factorials/exponentials)
   - Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Converges if $L < 1$. Diverges if $L > 1$. Inconclusive if $L = 1$.
4. **Alternating Series Test (AST)**
   - Used for $\sum(-1)^n b_n$. Converges if $b_n \geq 0$, $\lim b_n = 0$, and $\{b_n\}$ is eventually monotone decreasing.
5. **Absolute vs. Conditional Convergence**
   - $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges. (Absolute convergence implies convergence.)
   - $\sum a_n$ **converges conditionally** if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

Rigorous Workout Problems

This section contains problems requiring detailed proofs, complex applications of convergence criteria, and deep conceptual understanding.

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1. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences of positive real numbers such that
$$\lim_{n\to\infty} \frac{x_n}{y_n} = L$$
where L is a positive real number ($L > 0$ and $L < \infty$).
Prove:
$$\lim_{n\to\infty} x_n = \infty \iff \lim_{n\to\infty} y_n = \infty$$
2. Let a_n be a sequence of positive real numbers such that
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$
a) If $L < 1$, show that $\lim_{n\to\infty} a_n = 0$. b) If $L > 1$, show that $\lim_{n\to\infty} a_n = \infty$. c) Moreover, show that
$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = L.$$
3. **Constructive Convergence of Subseries:**
Let $\{a_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$.
Constructively demonstrate the existence of a subsequence $\{a_{n_k}\}$ for which the series $\sum_{k=1}^{\infty} a_{n_k}$ converges.
Hint: It is sufficient to choose the subsequence such that $|a_{n_k}| < \frac{1}{2^k}$ for all k.

The Analytical Derivation and Nature of e

This section presents the rigorous foundational proof establishing the convergence of the two principal sequences defining Euler's number (e) and demonstrates its non-rational nature.

Equivalence of Definitions (Convergence Proof)

Consider the compounded growth sequence $\{x_n\}$ and the factorial partial sum sequence $\{y_n\}$:

$$x_n := \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad y_n := \sum_{k=0}^{n} \frac{1}{k!}$$

Problem 1: Prove that $\{x_n\}$ and $\{y_n\}$ are convergent and establish the equality of their limits.

(i) **Monotonicity and Boundedness:** Demonstrate that both $\{x_n\}$ and $\{y_n\}$ are **monotone increasing** sequences. Show that $\{y_n\}$ is **bounded above** by 3, confirming its convergence (MCT).

(ii) **Upper Bound:** Derive the initial inequality $x_n \leq y_n$ by comparing the terms resulting from the **Binomial Expansion** of x_n with the terms of y_n.

(iii) **Lower Bound and Equality:** For a fixed integer m, use the truncated expansion of x_n (stopping at the m-th term) to establish the inequality:

$$\liminf_{n\to\infty} x_n \geq y_m$$

By combining this result with $x_n \leq y_n$, rigorously conclude that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. Designate this common limit as e.

The Non-Rational Nature of e

Problem 2: Prove that the number e, defined by the convergent infinite series $e = \sum_{k=0}^{\infty} \frac{1}{k!}$, is **irrational**.

(i) **Contradiction Setup:** Proceed by contradiction. Assume e is rational, so $e = \frac{p}{q}$ for $p, q \in \mathbb{N}$.

(ii) **Remainder Analysis:** Consider the remainder term $R_q = e - y_q = \sum_{k=q+1}^{\infty} \frac{1}{k!}$.

(iii) **The Conflict:** Examine the algebraic quantity $q! \cdot R_q$.

- Show that $q! \cdot R_q$ must be an integer under the initial rational assumption for e.
- Establish the bounds for the remainder:

$$0 < R_q < \frac{1}{q! \cdot q}$$

- Combine the bounds to show that $0 < q! \cdot R_q < 1$. Explain why this result contradicts the fact that $q! \cdot R_q$ must be an integer.

Proof that $\sum_{n=1}^{\infty} \cos(n)$ Diverges

The convergence of any series $\sum_{n=1}^{\infty} a_n$ requires that the limit of its terms must be zero (The Test for Divergence). That is, if the series converges, then $\lim_{n\to\infty} a_n = 0$.

Analysis of the Sequence of Terms $a_n = \cos(n)$

We first establish that the sequence of terms $\{\cos(n)\}$ diverges, using a proof by contradiction.

1. Assume, for contradiction, that $\lim_{n\to\infty} \cos(n) = \alpha$.
2. Using the identity $\cos^2(n) + \sin^2(n) = 1$, this assumption implies that $\{\sin(n)\}$ must also converge to a limit β such that $\alpha^2 + \beta^2 = 1$.
3. We utilize the trigonometric identity $\cos(1) = \cos((n+1) - n) = \cos(n+1)\cos(n) + \sin(n+1)\sin(n)$.
4. Taking the limit as $n \to \infty$ on both sides yields the result $\cos(1) = \alpha^2 + \beta^2$.
5. Combining the results leads to the contradictory statement $\cos(1) = 1$.

Since $\cos(1) \neq 1$, the initial assumption must be false.
Conclusion 1: The sequence $\{\cos(n)\}$ **diverges**.

Application of the Test for Divergence

Since the terms of the series, $a_n = \cos(n)$, do not converge to zero, the series fails the necessary condition for convergence.
Final Conclusion: By the Test for Divergence, the series $\sum_{n=1}^{\infty} \cos(n)$ **diverges**.

Riemann Integration: Key Concepts and Theorems

Key Concepts & Definitions (Bartle and Sherbert)

The Riemann integral is defined using the **limit of Riemann sums** based on tagged partitions.

- **Partition (P):** A finite set of points that divides the interval $[a, b]$ into subintervals.

$$P = \{x_0, x_1, \dots, x_n\}, \quad \text{where } a = x_0 < x_1 < \cdots < x_n = b.$$

- **Norm (Mesh):** The length of the longest subinterval. Measures the "fineness" of the partition.

$$\|P\| = \max_{i=1,\dots,n} \{x_i - x_{i-1}\}.$$

- **Tagged Partition:** A partition where a "tag" point t_i is chosen from each subinterval $[x_{i-1}, x_i]$.

$$\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}.$$

- **Riemann Sum:** The approximation of the integral.

$$S(f; \dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

- **Riemann Integrability:** A function f is **Riemann integrable** on $[a, b]$, denoted $f \in \mathcal{R}[a, b]$, if the limit of the Riemann sums, as the norm $\|\dot{P}\| \to 0$, exists and equals a unique number L.

Integrability Criterion: Darboux Equivalence (Crucial!)

1. Equivalence of $\mathcal{R}$-Integrability and $\mathcal{D}$-Integrability

A function f on $[a, b]$ is **Riemann integrable ($\boldsymbol{f \in \mathcal{R}[a, b]}$)** if and only if it is **Darboux integrable ($\boldsymbol{f \in \mathcal{D}[a, b]}$)**. Both definitions define the exact same class of functions.

2. The Darboux Integrability Criterion (Squeeze Criterion)

A bounded function f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there exists a partition P_ϵ such that:

$$U(f, P) - L(f, P) < \epsilon$$

- **Upper Darboux Sum:** $U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$, where $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$.
- **Lower Darboux Sum:** $L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$, where $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$.

3. Boundedness

The set of integrable functions $\mathcal{R}[a, b]$ is a **subclass of the bounded functions** on $[a, b]$. Any function that is Riemann integrable must be bounded.

Basic Integrability Classes & Properties

Integrability Classes

- **Continuous Functions:** If f is **continuous** on $[a, b]$, then $f \in \mathcal{R}[a, b]$.
- **Monotone Functions:** If f is **monotone** (increasing or decreasing) on $[a, b]$, then $f \in \mathcal{R}[a, b]$.

Properties of the Riemann Integral

Let $f, g \in \mathcal{R}[a, b]$ and $c \in \mathbb{R}$.

- **Linearity:**

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g \quad \text{and} \quad \int_a^b (cf) = c \int_a^b f$$

- **Additivity on Intervals:** If $a < c < b$, then $\int_a^b f = \int_a^c f + \int_c^b f$.
- **Absolute Value:**

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Major Theorems

1. Fundamental Theorem of Calculus (FTC)

- **FTC Part 1 (Differentiating the Integral):** If $F(x) = \int_a^x f(t)dt$ and f is **continuous at** c, then F is differentiable at c and $\mathbf{F'(c) = f(c)}$.
- **FTC Part 2 (Evaluating the Integral):** If f is differentiable and $\mathbf{f'}$ is Riemann integrable on $[a, b]$, then:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

2. Mean Value Theorem for Integrals (MVT)

If f is **continuous** on $[a, b]$, then there exists a point $c \in [a, b]$ such that:

$$\int_a^b f(x)dx = f(c)(b - a)$$

3. Integration by Parts Formula

If f and g have continuous derivatives on $[a, b]$, then:

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)dx$$

Examples and Exercises

This section provides exercises to practice the definitions (Riemann Sums and Darboux Sums) for basic functions.

0.1 Constant Function

Example 1: The Constant Function Let $f(x) = c$ be a constant function on the interval $[a, b]$, where $c \in \mathbb{R}$.
Problem: Prove that f is Riemann integrable on $[a, b]$ and that $\int_a^b c\,dx = c(b - a)$.

Explanation/Note: We use the definition of the Riemann Sum directly. Since a constant function is both continuous and monotone, it is guaranteed to be integrable.

1. Let $P = \{x_0, x_1, \dots, x_n\}$ be **any partition** of $[a, b]$.
2. Let P be any **tagged partition** with tags $t_i \in [x_{i-1}, x_i]$.
3. Form the Riemann Sum $S(f; P)$:
$$S(f; P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$
4. Since $f(t_i) = c$ for all i, simplify the sum:
$$S(f; P) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c\sum_{i=1}^{n}(x_i - x_{i-1})$$
5. Use the telescoping property of the partition: $\sum_{i=1}^{n}(x_i - x_{i-1}) = (x_1 - x_0) + (x_2 - x_1) + \dots + (x_n - x_{n-1}) = x_n - x_0 = b - a$.
6. Conclude that $S(f; P) = c(b - a)$. Since this value is independent of the partition P and its tags, the integral is:
$$\int_a^b c\,dx = c(b - a)$$

0.2 Step Functions

Example 2: A Simple Step Function Let $f : [0, 2] \to \mathbb{R}$ be defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 2 & \text{if } x \in (1, 2] \end{cases}$$

Problem: Prove that f is Riemann integrable on $[0, 2]$ and find the value of $\int_0^2 f(x)\,dx$.

Hint (Use the Darboux Criterion or Interval Additivity):

a. Since f is bounded and has only one discontinuity at $x = 1$, it is integrable.

b. Use the **Additivity on Intervals** property with the split point $c = 1$:

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

c. **To show integrability:** Construct a partition P_ϵ that isolates the discontinuity at $x = 1$. Let the subinterval containing $x = 1$ be $[1 - \delta, 1 + \delta]$, where δ is small. Show that the difference $U(f, P_\epsilon) - L(f, P_\epsilon)$ can be made arbitrarily small by controlling the length of this single critical interval.

0.3 Piecewise Function

Example 3: A More Complex Step Function Let $g : [0, 3] \to \mathbb{R}$ be defined by:

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 4 & \text{if } x \in (0, 1] \\ 1 & \text{if } x \in (1, 3] \end{cases}$$

Problem (a): Determine the partition points $c_1, c_2, \ldots$ required to use the interval Additivity property.
Problem (b): Evaluate $\int_0^3 g(x)\,dx$.
Problem (c): Is the value of the integral affected if $g(0) = 4$ instead of $g(0) = 0$? Justify your answer using a note on measure zero sets.

0.4 IV. Oscillation and Non-Integrability

Example 4: The Dirichlet Function (A Non-Integrable Function) Let $h : [0, 1] \to \mathbb{R}$ be defined by:

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \end{cases}$$

Problem: Prove that h is **not** Riemann integrable on $[0, 1]$.

Explanation/Note: To prove non-integrability, you must show that the Darboux Criterion fails. Specifically, you must demonstrate that the oscillation $U(h, P) - L(h, P)$ cannot be made arbitrarily small for **any** partition P. Use the fact that every non-degenerate interval in $\mathbb{R}$ contains both rational and irrational numbers (density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$).

0.5 Mean Value Theorem for Integrals

Example 5: Verification of $\mathbf{MVT_{int}}$ Verify the Mean Value Theorem for Integrals for the function $f(x) = x^2$ on the interval $[0, 3]$. Find the value $c \in (0, 3)$ that satisfies the theorem.

Hints:

- **Step 1:** Calculate the definite integral $\int_0^3 x^2\,dx$.
- **Step 2:** Set the integral value equal to the MVT_{int} formula: $\int_a^b f(x)\,dx = f(c)(b - a)$.
- **Step 3:** Solve the resulting equation $3c^2 = \text{(Integral Value)}$ for c.
- **Step 4:** Ensure the resulting c value is strictly within the open interval $(0, 3)$.

0.6 Products of Integrable Functions

Example 6: $\mathcal{R}$-Integrability of the Square Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$.
Problem: Prove that the function $f^2(x) = (f(x))^2$ is also Riemann integrable on $[a, b]$.

Hint (Proof by Basic Principles):

- **Boundedness:** Since $f \in \mathcal{R}[a, b]$, f is bounded. Let $M > 0$ be a bound such that $|f(x)| \leq M$ for all $x \in [a, b]$.
- **Difference of Squares:** Use the identity $f(x)^2 - f(y)^2 = (f(x) - f(y))(f(x) + f(y))$.
- **Oscillation Control:** Use the Darboux criterion. Show that for any subinterval I_i, the oscillation of f^2 is related to the oscillation of f:

$$M_i(f^2) - m_i(f^2) \leq 2M \cdot (M_i(f) - m_i(f))$$

- Since f is integrable, the right-hand side can be made arbitrarily small by choosing a sufficiently fine partition P such that $\sum(M_i(f) - m_i(f))(x_i - x_{i-1}) < \epsilon'$.

0.7 Monotonicity of the Integral

Example 7: Non-Negativity of the Integral Let $f : [a, b] \to \mathbb{R}$ be **Riemann Integrable** on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Show that:

$$\int_a^b f(x)\, dx \geq 0$$

using basic principles of Riemann Integration.

Hint (Using the Limit of Riemann Sums):

- Start with the definition: $\int_a^b f(x)\, dx = \lim_{\|P\| \to 0} S(f; P)$, where $S(f; P) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$.
- Analyze the components: Since $f(x) \geq 0$ and $a < b$ (implying $x_i - x_{i-1} > 0$), every term in the sum, $f(t_i)(x_i - x_{i-1})$, must be non-negative.
- Conclusion: The Riemann Sum $S(f; P) \geq 0$ for all partitions P. Since the limit of a sequence of non-negative terms must be non-negative, the integral must satisfy $\int_a^b f(x)\, dx \geq 0$.

0.8 Exercise Problems

Problem:1 Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function.

a. **Monotonicity:** If f is **continuous** on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$, show that $\displaystyle\int_a^b f(x)\, dx \geq 0$.

b. **Integrand Zero Theorem:** If f is **continuous** on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$, and $\displaystyle\int_a^b f(x)\, dx = 0$, prove that $f(x) = 0$ for all $x \in [a, b]$.

c. **Necessity of Continuity:** Justify why the assumption of **continuity** cannot be dropped in parts (a) and (b). That is, provide a counterexample where $f(x) \geq 0$ and $\int_a^b f(x)\, dx = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

Hint (For Part b):

- Assume, for contradiction, that $f(c) > 0$ for some $c \in (a, b)$.
- Use the continuity of f at c to establish a δ-neighborhood around c, say $[c - \delta, c + \delta] \subset [a, b]$, where $f(x)$ remains positive, specifically $f(x) \geq \frac{1}{2} f(c)$.
- Consider the Additivity of the integral: $\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^b f$.
- Use the Monotonicity property (Part a) on the outer intervals, and directly bound the middle integral using the constant value $\frac{1}{2} f(c)$ over the interval length 2δ.

Problem:2 Consider the identity function $f(x) = x$ on $I = [0, 1]$. Let P_n be the partition that divides I into n equal parts. If $U(f, P_n)$ and $L(f, P_n)$ are the upper and lower Riemann sums, respectively, and $A_n = U(f, P_n) - L(f, P_n)$, which of the following statements is correct?

a. A_n is strictly monotonically decreasing.

b. $\displaystyle\lim_{n \to \infty} n A_n = 0$.

c. $\displaystyle\sum_{n=1}^{\infty} A_n$ is convergent.

d. $\displaystyle\sum_{n=1}^{\infty} A_n A_{n+1} = 1$.

Problem:3 Define f on $[0,1]$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ x^3 & \text{if } x \text{ is irrational} \end{cases}$$

Select all that apply.

a. f is not Riemann integrable on $[0,1]$.

b. f is Riemann integrable and $\displaystyle\int_0^1 f(x)\,dx = \frac{1}{4}$.

c. f is Riemann integrable and $\displaystyle\int_0^1 \underline{f(x)}\,dx$ and $\displaystyle\int_0^1 \overline{f(x)}\,dx$ are the lower and upper Riemann integrals of f.

d. $\displaystyle\frac{1}{4} < \int_0^1 \underline{f(x)}\,dx < \int_0^1 \overline{f(x)}\,dx < \frac{1}{3}$, where $\displaystyle\int_0^1 \underline{f(x)}\,dx$ and $\displaystyle\int_0^1 \overline{f(x)}\,dx$ are the lower and upper Riemann integrals of f.

0.9 The Thomae Function

Definition: The Thomae function (or Popcorn function) f on $[0,1]$ is defined as:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p, q \in \mathbb{N} \text{ are coprime} \end{cases}$$

A. Continuity Discussion

a. **Continuity at Irrationals ($\mathbf{x_0} \notin \mathbb{Q}$):** The function f **is continuous** at every irrational number in $[0,1]$.

> **Hint:** For any $\epsilon > 0$, choose n_0 such that $1/n_0 < \epsilon$. There are only a **finite number** of rationals p/q with $q < n_0$. Since x_0 is irrational, choose δ small enough so that the δ-neighborhood of x_0 contains none of these finite points. For any x in this neighborhood, if $x = p/q$, then $q \geq n_0$, hence $|f(x) - f(x_0)| = 1/q \leq 1/n_0 < \epsilon$.

b. **Discontinuity at Rationals ($\mathbf{x_0} \in \mathbb{Q} \setminus \{\mathbf{0}\}$):** The function f **is discontinuous** at every non-zero rational number.

> **Note:** If $x_0 = p/q$, then $f(x_0) = 1/q > 0$. Every neighborhood of x_0 contains irrationals y where $f(y) = 0$. Since $\lim_{y \to x_0} f(y) = 0 \neq f(x_0)$, f is discontinuous.

B. Integrability Discussion

The Thomae function **is Riemann integrable** on $[0,1]$, and its value is $\displaystyle\int_0^1 f(x)\,dx = 0$.

Justification:

- **Lower Sum:** Since every subinterval contains irrationals, $m_i = \inf f = 0$. Thus, the Lower Integral $\displaystyle\underline{\int_0^1} f = 0$.
- **Upper Sum:** The set of discontinuities, $\mathbb{Q} \cap [0,1]$, is a countable set, which has **measure zero**. The Lebesgue Criterion for Riemann Integrability states that a bounded function is Riemann integrable if and only if its set of discontinuities has measure zero.
- Therefore, $\displaystyle\overline{\int_0^1} f = 0$, proving $\displaystyle\int_0^1 f(x)\,dx = 0$.

C. Inverse Continuity Property and Topology

Question: Can we have a function g continuous at all rationals and discontinuous at all irrationals of $[0,1]$?
Answer: No.

a. **Definitions:**
   - $\mathbf{G_\delta}$ Set: A set that is the **countable intersection of open sets**.
   - Set of First Category (Meager): A set that is a **countable union of nowhere dense sets** ("small" sets).
   - Set of Second Category: A set that is **not** of the first category ("large" sets).

b. **Topological Impossibility:**
- The set of **points of continuity** for *any* real-valued function $g : [0, 1] \to \mathbb{R}$ must always be a $\mathbf{G}_\delta$ set.
- However, the set of rational numbers, $\mathbb{Q} \cap [0, 1]$, is an $\mathbf{F}_\sigma$ set (countable union of closed sets), and it **is not** a G_δ set.
- Since $\mathbb{Q}$ fails the necessary topological condition (G_δ property), it cannot be the entire set of continuity points for any function g.

Differentiability

Differentiability is inherently a **local property**. The derivative of a function f at a point x is defined solely by the behavior of f in an arbitrarily small neighborhood of x.

We define differentiability at a **single point** x. A function f is said to be differentiable on an entire domain, such as a closed interval $[a, b]$, if it is differentiable at **every point** in that domain.

At the endpoints a and b of the closed interval $[a, b]$, differentiability necessitates the existence and equality of the respective **right-hand and left-hand derivatives**.

The Role of Limit Points in Rudin's Definition

Rudin's definition of the derivative,

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

fundamentally involves the concept of a **limit**.

Recall our discussion from the lectures on continuity: the formal ϵ-δ definition of the **limit of a function** f at a point x *requires* x to be a **limit point** of the domain of f.

- **Differentiability:** Because the definition of $f'(x)$ is a limit, differentiability can **only be defined** at **limit points** of the function's domain.
- **Continuity vs. Differentiability:**
  - **Continuity** can be defined at **every point** in the domain (isolated or limit points). At an isolated point p, f is *automatically* continuous.
  - **Differentiability cannot be defined** at an **isolated point** because the necessary limit for the difference quotient cannot exist. Consequently, a function is **not differentiable** at an isolated point in its domain.

This distinction highlights why Theorem 5.2 (differentiability implies continuity) is crucial, as it is easy to construct functions that are continuous but fail to be differentiable in this manner (e.g., at isolated points).

Key Examples of Differentiable and Non-Differentiable Functions

Polynomials and Rational Functions

- The **constant function** $f(x) = c$ is differentiable with $f'(x) = 0$.
- The **identity function** $f(x) = x$ is differentiable with $f'(x) = 1$.
- By repeated application of the product rule, **polynomials** are differentiable on $\mathbb{R}$.
- By the quotient rule, **rational functions** f/g are differentiable at all points where the denominator $g(x) \neq 0$.

Functions Involving $\sin(1/x)$

Consider the function $f(x)$ defined as:

$$f(x) = \begin{cases} x^m \sin\left(\frac{1}{x^n}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad m, n \in \mathbb{N}$$

We analyze the differentiability at $x = 0$ using the definition:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x^{m-1} \sin\left(\frac{1}{x^n}\right)$$

- **Differentiability:** f is differentiable at $x = 0$ if and only if the limit exists. This occurs if and only if $m - n > 0$, or $m > n$.

- **Continuous Differentiability:** f is continuously differentiable at $x = 0$ if and only if $f'(x)$ is continuous at $x = 0$. For $x \neq 0$,
$$f'(x) = mx^{m-1} \sin\left(\frac{1}{x^n}\right) - nx^{m-n-1} \cos\left(\frac{1}{x^n}\right)$$
The limit $\lim_{x\to 0} f'(x)$ exists and equals $f'(0) = 0$ if and only if the exponent of x in the second term is positive: $m - n - 1 > 0$, or $m > n + 1$.

Students are requested to verify these calculations independently, as they are crucial for grasping the definitions.

The Continuity of Derivatives: Darboux's Theorem

A key fact in analysis is that the derivative of a function f defined on an interval $[a, b]$, denoted f', **need not be continuous**.

However, even if f' is discontinuous, it satisfies the **Intermediate Value Property** (IVP). This property is stated by **Darboux's Theorem** (Theorem 5.12 in Rudin's PMA):

> If f is differentiable on $[a, b]$ and if $f'(a) < \lambda < f'(b)$ (or $f'(a) > \lambda > f'(b)$), then there exists a point $x \in (a, b)$ such that $f'(x) = \lambda$.

In simpler terms, f' takes on every value between its values at the endpoints. This is a characteristic usually associated with continuous functions.

Non-existence of Simple Discontinuities

A crucial consequence of Darboux's Theorem is that the derivative function f' **cannot have simple discontinuities** (i.e., jump discontinuities). From this, it is obvious f' cannot have simple discontinuity.

Recall that a **simple discontinuity** at a point x requires the existence of both one-sided limits, $f'(x-)$ and $f'(x+)$, such that either $f'(x-) \neq f'(x)$ or $f'(x+) \neq f'(x)$.

Let us prove this by contradiction, focusing on the case where we assume a jump discontinuity at x on the right, specifically when $f'(x) < f'(x+)$.

1. Assume $f'(x) < f'(x+)$.
2. Choose an intermediate value λ such that $f'(x) < \lambda < f'(x+)$.
3. Since $f'(x+) = \lim_{z\to x^+} f'(z)$, we can find a point $y > x$ such that $f'(z) > \lambda$ for all $z \in (x, y]$.
4. Now, consider the interval $[x, y]$. We have $f'(x) < \lambda$ and $f'(z) > \lambda$ for all $z \in (x, y]$.
5. This means f' takes values on one side of λ (at x) and values on the other side of λ (on $(x, y]$), but crucially, $f'(z) \neq \lambda$ for all $z \in [x, y]$.
6. This **contradicts Darboux's Theorem** when applied to the interval $[x, y]$.

The case $f'(x) > f'(x+)$ is treated similarly. The cases involving the left-hand limit ($f'(x-) \neq f'(x)$) are also addressed using a similar contradiction argument on an interval leading up to x.

In conclusion, the discontinuities of a derivative function f' must be of the **second kind** (where at least one of the one-sided limits does not exist).

The Generalized Mean Value Theorem (Cauchy's MVT)

The Mean Value Theorems provide a fundamental link between the average rate of change of a function over an interval and its instantaneous rate of change (the derivative) at some point within that interval.

Intuition via the Speed Analogy

The following example, while having some minor technical simplifications, gives excellent intuition for the Mean Value Theorems.

In the 2012 Olympics, Usain Bolt won the 100 metres gold medal with a time of 9.63 seconds. Let $d(t)$ be the distance covered at time t. The average speed, V_a, is the total distance divided by the total time:

$$V_a = \frac{d(t_2) - d(t_1)}{t_2 - t_1} = \frac{100 \text{ m}}{9.63 \text{ s}} \approx 10.384 \text{ m/s} \approx 37.38 \text{ km/h}$$

The standard **Mean Value Theorem (MVT)** states that there must exist some time c during the race ($t_1 < c < t_2$) such that the instantaneous speed $d'(c)$ equals the average speed:

$$d'(c) = \frac{d(t_2) - d(t_1)}{t_2 - t_1}$$

This means that at some moment c, Bolt was actually running at the exact speed of 37.38 km/h.

Now, consider a second runner, Asafa Powell, who completed the race in 11.99 seconds. Since $11.99 \approx 1.245 \times 9.63$, Bolt's average speed was approximately 1.245 times the average speed of Powell.

Formal Statement of the Generalized MVT

The **Generalized Mean Value Theorem (GMVT)**, also known as **Cauchy's Mean Value Theorem**, relates the instantaneous speeds of **two** functions, f and g.

Let $f(t)$ be the distance function for Bolt and $g(t)$ be the distance function for Powell. The theorem states that there exists a point $c \in (a, b)$ such that the ratio of the instantaneous derivatives is equal to the ratio of the average rates of change:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

We can rewrite the right-hand side by dividing the numerator and denominator by $(b - a)$:

$$\frac{f'(c)}{g'(c)} = \frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}} = \frac{\text{Average Rate of } f}{\text{Average Rate of } g}$$

Applying this to our analogy, at some time c, the ratio of Bolt's instantaneous speed ($f'(c)$) to Powell's instantaneous speed ($g'(c)$) must be equal to the ratio of their average speeds (1.245). That is, at some point, Bolt was actually running at a speed exactly 1.245 times Powell's speed!

L'Hôpital's Rule

L'Hôpital's Rule provides a powerful method for evaluating limits of indeterminate forms, specifically $\frac{0}{0}$ and $\frac{\infty}{\infty}$. The underlying mechanism of the rule is rooted deeply in the Mean Value Theorem.

The Simple Case: Continuity of Derivatives

Consider the simplest case of the indeterminate form $\frac{0}{0}$. Suppose that the functions f and g are **continuously differentiable** at a, that $g'(a) \neq 0$, and that $f(a) = g(a) = 0$.

Under these specific assumptions, the limit becomes readily apparent from the definition of the derivative:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

By dividing both the numerator and the denominator by $(x - a)$, we obtain:

$$\lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}$$

Furthermore, under the assumption that f' and g' are continuous at a, the result $\frac{f'(a)}{g'(a)}$ is precisely equal to $\lim_{x \to a} \frac{f'(x)}{g'(x)}$.

This "obvious" case, while being a particular instance, is one of the most frequently used and is widely applicable, even holding for analytic functions on an open domain in the complex plane (a case not typically covered by the real-valued argument).

The Key Role of the Generalized Mean Value Theorem

The proof of the full L'Hôpital's Rule is essentially an extrapolation of this simple case, achieved by replacing the definition of the derivative with the powerful tool of the **Generalized Mean Value Theorem (GMVT)**.

It is not necessary to assume that f' and g' are continuous or even defined at a. If $f(a) = 0 = g(a)$, the GMVT states that for every x near a, there exists a point t_x between x and a such that:

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(t_x)}{g'(t_x)}$$

Since $f(a) = g(a) = 0$, we have:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f'(t_x)}{g'(t_x)}$$

As $x \to a$, the intermediate point t_x must also approach a. Therefore, if the limit $\lim_{t \to a} \frac{f'(t)}{g'(t)}$ exists, the displayed fraction must converge to the same value.

The GMVT is the true engine of the theorem. Once this relationship is established for the $\frac{0}{0}$ case, the argument can be adapted to prove L'Hôpital's Rule for the $\frac{\infty}{\infty}$ case and the limits involving $x \to \infty$ as well.

Intuitive Derivation via Linear Approximation

For an intuitive, non-rigorous understanding of why L'Hôpital's Rule works for the $\frac{0}{0}$ case, we can use the concept of **linear approximation** (or the tangent line approximation) around the point $x = a$.

If f and g are differentiable at a, they can be approximated by their tangent lines near a:

$$f(x) \approx f'(a)(x-a) + f(a)$$
$$g(x) \approx g'(a)(x-a) + g(a)$$

The quotient $\frac{f(x)}{g(x)}$ is then approximated by the ratio of these linear functions:

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a) + f(a)}{g'(a)(x-a) + g(a)}$$

Now, we impose the condition of the $\frac{0}{0}$ indeterminate form, which means that $f(a) = 0$ and $g(a) = 0$. Substituting these values simplifies the expression:

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a) + 0}{g'(a)(x-a) + 0} = \frac{f'(a)(x-a)}{g'(a)(x-a)}$$

As $x \to a$, we consider $x \neq a$, allowing us to cancel the $(x-a)$ terms:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \approx \lim_{x\to a} \frac{f'(a)}{g'(a)} = \frac{f'(a)}{g'(a)}$$

This intuitive argument, which relies on the functions being well-approximated by their tangent lines, leads directly to the core result of L'Hôpital's Rule.

It is critical for a student of analysis to be aware of scenarios where the direct use of L'Hôpital's Rule constitutes a **circular argument**. While the rule is a powerful computational tool, it is not a fundamental definition and must be proven from first principles.

A classic example of this circularity occurs when attempting to use L'Hôpital's Rule to evaluate the limit that is required to prove the derivatives of trigonometric functions:

$$\lim_{x\to 0} \frac{\sin x}{x}$$

The problem is as follows:

1. To apply L'Hôpital's Rule, one must calculate the derivatives of the numerator and denominator: $\frac{d}{dx}(\sin x)$ and $\frac{d}{dx}(x)$.
2. The derivative of the denominator is $\frac{d}{dx}(x) = 1$.
3. The derivative of the numerator is $\frac{d}{dx}(\sin x) = \cos x$.
4. **However, the proof** that $\frac{d}{dx}(\sin x) = \cos x$ itself relies on first evaluating the fundamental limit:
$$\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \cdots = \cos x \lim_{h\to 0} \frac{\sin h}{h}$$
5. Therefore, using L'Hôpital's Rule on $\lim_{x\to 0} \frac{\sin x}{x}$ requires assuming the very result that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ is needed to prove the derivative of $\sin x$.

In analysis, this limit must be proven using a geometric argument (the Squeeze Theorem) **before** L'Hôpital's Rule is even introduced or proven, thus avoiding logical circularity.

Problem: Hölder Condition of Order $\alpha > 1$

Problem: Show that if a function $f : [a, b] \to \mathbb{R}$ satisfies a **Hölder condition of order** $\alpha > 1$ then it is constant. (*For $\alpha = 2$, it is an exercise in Rudin.)*

(Partitioning Argument) *Let $f : [a, b] \to \mathbb{R}$ satisfy the Hölder condition of order $\alpha > 1$. This means there exists a constant $K \geq 0$ such that for all $x, y \in [a, b]$:*

$$|f(x) - f(y)| \leq K|x-y|^\alpha \quad (*)$$

We aim to show that $|f(x) - f(y)| = 0$ for all $x, y \in [a, b]$.

1. ***Partition the Interval:*** *Fix any two points $x, y \in [a, b]$. For an arbitrary integer $n \geq 1$, we partition the interval $[x, y]$ into n subintervals of equal length $h = \frac{|x-y|}{n}$. Let the partition points be $x_0 = x, x_1, x_2, \ldots, x_n = y$.*

2. ***Apply Triangle Inequality and Hölder Condition:*** *By the repeated application of the* ***Triangle Inequality*** *and the Hölder condition* $(*)$ *to each subinterval* $|x_i - x_{i-1}| = h$*:*

$$|f(x) - f(y)| \le \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le \sum_{i=1}^{n} K|x_i - x_{i-1}|^\alpha$$

3. ***Simplify and Bound:*** *Substituting* h *and simplifying the sum:*

$$\sum_{i=1}^{n} K \left(\frac{|x-y|}{n} \right)^\alpha = n \cdot K \frac{|x-y|^\alpha}{n^\alpha} = K|x-y|^\alpha \cdot \frac{n}{n^\alpha}$$

   This yields the following bound:

$$|f(x) - f(y)| \le K|x-y|^\alpha \cdot \frac{1}{n^{\alpha-1}}$$

4. ***Take the Limit:*** *Since* $|f(x) - f(y)|$ *is independent of* n*, and we are given* $\alpha > 1$ *(so* $\alpha - 1 > 0$*), we take the limit as* $n \to \infty$*:*

$$\lim_{n\to\infty} \left(K|x-y|^\alpha \cdot \frac{1}{n^{\alpha-1}} \right) = K|x-y|^\alpha \cdot 0 = 0$$

Since $|f(x) - f(y)|$ *is a non-negative value bounded above by a quantity that tends to zero, we conclude that* $|f(x) - f(y)| = 0$*, which implies* $f(x) = f(y)$ *for all* $x, y \in [a, b]$*. Therefore,* f *is a* ***constant function****.*