Algebra: Internal Exam-3 (Total Marks - 16)

PART A

1. State fundamental theorem of finitely generated Abelian group. (1 mark)
2. Define maximal ideal and give an example. (1 mark)

PART B (Answer all questions from 3 to 6 and any two questions 7, 8 and 9)

3. Find all maximal ideal of $\mathbb{Z}_8 \times \mathbb{Z}_{30}$. (1 mark)
4. Let R be a commutative ring and suppose that A is an ideal of R. Let $N(A) = \{x \in R : x^n \in A \text{ for some } n\}$. Prove

   a. $N(A)$ is an ideal of R which contains A. (3 marks)

   b. $N(N(A)) = N(A)$. (2 marks)

   The ideal $N(A)$ is often called the radical of A.
5. Is $\{(2,1),(4,1)\}$ a basis for $\mathbb{Z} \times \mathbb{Z}$? Prove your assertion. (1 mark)
6. Show that if G and G' are free abelian groups, then $G \times G'$ is free abelian. (3 marks)
7. Show that $\phi : \mathbb{C} \to \mathbb{R}$ given by $\phi(a+ib)=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for $a, b \in R$ gives an isomorphism of $\mathbb{C}$ with the subring $\phi(\mathbb{C})$ of $M_2(R)$. (2 marks)
8. Is $Q[X]/< x^2 - 6x + 6 >$ a field? Why?. (2marks)
9. Let R=$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z}_2 \right\}$ with ordinary matrix addition and multiplication modulo 2. Show that $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{r} : \ r \in R \right\}$ is not an ideal of R. (2 marks)

Algebra: Assignment Exam (Total Marks - 8)

Answer all questions.

1. Which of the following is an irreducible factor of $x^{12} - 1$ over Q.
   (a.) $x^8 + x^4 + 1$ (b.) $x^4 + 1$
   (c.) $x^4 - x^2 + 1$ (d.) $x^5 - x^4 + x^3 - x^2 + x - 1$.
   (2 marks)
2. Find all prime ideals and all maximal ideals of $\mathbb{Z}_6$ (1 mark).
3. Determine whether the polynomial $8x^3 + 6x^2 - 9x + 24$ in $\mathbb{Z}[X]$ satisfies an Eisenstein criterion for irreducibility over $\mathbb{Q}$. (1 mark)
4. Mark true/false with justification.
   a. $\mathbb{Q}$ is an ideal in $\mathbb{R}$. (1 mark)
   b. A maximal ideal of a ring R is an ideal that is not contained in any other ideal of R. (1 mark)
   c. The only ideals of a field F is $\{0\}$ and F itself. (1 mark)
   d. The intersection of two prime ideals is prime ideal. (1 mark)