Real Analysis Exercise Sheet

1. Let X be an infinite set. For $p, q \in X$, define
$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q. \end{cases}$$
   (a) Prove that this is a metric.
   (b) Which subsets of the resulting metric space are open?
   (c) Which are closed?
   (d) Which are compact? (Definition. A metric space (X, d) is called compact if every sequence in X has a convergent subsequence whose limit lies in X; a subset $Y \subset X$ is called compact if the subspace $(Y, d|_{Y\times Y})$ is compact).
   (e) Prove that a sequence $\{x_n\} \subset X$ converges to $x \in X$ in the discrete metric if and only if there exists $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$.

2. Let (X, d) be a metric space. Show that $d'(x, y) = \sqrt{d(x, y)}$ is also a metric on X, and that the open sets for d' are the same as the open sets for d.

3. Consider $\mathbb{R}$ with the standard metric. Let $E \subset \mathbb{R}$ be a subset which has no limit points. Show that E is at most countable.

4. Let (X, d) be a compact metric space, and $f : X \to X$ a map such that $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$. Prove that there exists a point x such that $f(x) = x$. Hint: how small can $d(x, f(x))$ get? Comment: this is an example of a fixed point theorem, very popular in various sciences for showing the existence of equilibria and such.

5. Let (X, d) be a metric space.
   (a) Prove the following:
      (i) For all $x, y, z \in X$
$$|d(y, z) - d(z, x)| \leq d(x, y).$$
      (ii) For any $n \in \mathbb{N}$ and any $x_1, x_2, \ldots, x_n \in X$
$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$
   (b) For $x \in \mathbb{R}$ and $y \in \mathbb{R}$ define
      (i) $d_1(x, y) = (x - y)^2$
      (ii) $d_2(x, y) = \sqrt{|x - y|}$
      (iii) $d_3(x, y) = |x^2 - y^2|$
      (iv) $d_4(x, y) = |x - 2y|$

      Determine, for each of these, whether it is a metric or not.

6. (a) In a metric space (X, d), prove that if a sequence (x_n) converges, then its limit is unique. That is, show that if $x_n \to x$ and $x_n \to y$, then $x = y$.
   (b) Prove that a subset E of a metric space (X, d) is closed if and only if
$$E = \overline{E},$$
   where $\overline{E}$ denotes the closure of E.

7. Let $\mathbb{R}^2$ be equipped with the Taxicab metric/Manhattan metric defined by
$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$
   (a) Prove that d is a metric.
   (b) Draw the open balls $B_1((0, 0))$, $B_2((0, 0))$, and $B_{\frac{1}{2}}((0, 0))$ in $\mathbb{R}^2$.

(c) On the same axes, sketch the corresponding Euclidean balls (under the usual metric) of the same radii.

(d) What geometric shape do the taxicab balls form?

(e) Are these taxicab balls open sets in the usual metric on Euclidean space? Prove that these two metrics are equivalent.

(f) Let
$$A = \{(x, y) \in \mathbb{R}^2 : d((x, y), (1, 1)) \geq 2\}.$$
Sketch the set A. Is it closed in the taxicab metric?

8. (a) Let us consider the set $X = \mathbb{R} \setminus \{\sqrt{2}\}$ equipped with the usual metric $d(x, y) = |x - y|$. Is (X, d) a complete metric space? Justify your answer with appropriate reasoning.

(b) Let R be the set of real numbers with the usual metric $d(x, y) = |x - y|$.

(i) Show that the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence in R.

(ii) Show that the sequence $x_n = n$ is not a Cauchy sequence in R.

9. Let $C[0, 1]$ be the set of all real-valued continuous functions on [0, 1]. Define:
$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx.$$
Define a sequence $\{f_n\} \subset C[0, 1]$ by:
$$f_n(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ n(x - (\frac{1}{2} - \frac{1}{n})), & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

(a) Draw the graph of each function f_n and verify that $f_n \in C[0, 1]$ by observing that each f_n is piecewise linear and continuous.

(b) Prove that f_n pointwise converge to f where
$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

(c) Verify that $\{f_n\}$ is a Cauchy sequence in $(C[0, 1], d_1)$.

(d) Is $(C[0, 1], d_1)$ a complete metric space?

10. Let l^2 be the set of all sequences $x = \{x_n\}$ over the real numbers R such that $\sum_{n \geq 1} x_n^2$ converges. Prove that $d(x, y) = \sqrt{\sum_{n \geq 1}(x_n - y_n)^2}$ gives a metric on l^2.

11. Let (X, d) be a metric space and A be a non-empty subset of X. For $x \in X$, we define
$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

(a) Show that $d(x, A) = 0$ if and only if x lies in the closure of A.

(b) Show that if A is compact, then $d(x, A) = d(x, a)$ for some $a \in A$.

(c) The ϵ-neighborhood of A is defined to be
$$N(A; \epsilon) = \{x \in X : d(x, A) < \epsilon\}.$$
Prove that $N(A; \epsilon)$ is the union of the open balls $B_\epsilon(a)$ for $a \in A$.

(d) If A is compact and U is an open set containing A, prove that there exists $\epsilon > 0$ such that
$$A \subset N(A; \epsilon) \subset U.$$
Is this fact still true if A is just a closed set?

12. Let (X, d) be a metric space. Then verify the following properties:

(a) The union of any collection $\{G_\alpha\}$ of open sets is open; that is, $\bigcup_\alpha G_\alpha$ is open.

(b) The intersection of any collection $\{F_\alpha\}$ of closed sets is closed; that is, $\bigcap_\alpha F_\alpha$ is closed.

(c) The intersection of any finite collection $G_1, \dots, G_n$ of open sets is open; that is, $\bigcap_{i=1}^n G_i$ is open.

(d) The union of any finite collection $F_1, \dots, F_n$ of closed sets is closed; that is, $\bigcup_{i=1}^n F_i$ is closed.

13. In vector spaces, metrics are usually defined in terms of norms which measure the length of a vector. If V is a vector space defined over $\mathbb{R}$, then a norm is a function from vectors to real numbers, denoted by $||\cdot||$, satisfying:

(i) $||x|| \geq 0$ and $||x|| = 0 \iff x = 0$,

(ii) For any $\lambda \in \mathbb{R}$, $||\lambda x|| = |\lambda| ||x||$,

(iii) $||x + y|| \leq ||x|| + ||y||$.

Prove that every norm defines a metric.

14. Let M be a metric space with metric d. Show that d_1 defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on M. Observe that M itself is bounded in this metric.

15. Let A and B be two subsets of a metric space M. Recall that A°, the interior of A, is the set of interior points of A. Prove the following:

$$\text{a)} A^\circ \cup B^\circ \subseteq (A \cup B)^\circ, \quad \text{b)} A^\circ \cap B^\circ = (A \cap B)^\circ$$

Give an example of two subsets A and B of the real line such that $A^\circ \cup B^\circ \neq (A \cup B)^\circ$.

16. Let A be a subset of a metric space M. Recall that $\overline{A}$, the closure of A, is the union of A and its limit points. Recall that a point belongs to the boundary of A, ∂A, if every open ball centered at the point contains points of A and points of A^c, the complement of A. Prove that:

(a) $\partial A = \overline{A} \cap \overline{A^c}$

(b) $p \in \partial A \iff p$ is in $\overline{A}$, but not in A° (symbolically, $\partial A = \overline{A} \backslash A^\circ$)

(c) ∂A is a closed set

(d) A is closed $\iff \partial A \subseteq A$

17. Show that, in $\mathbb{R}^n$ with the usual (Euclidean) metric, the closure of the open ball $B_R(p)$, $R > 0$, is the closed ball

$$\{q \in \mathbb{R}^n : d(p, q) \leq R\}.$$

Give an example of a metric space for which the corresponding statement is false.

18. Prove directly from the definition that the set $K \subseteq \mathbb{R}$ given by

$$K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots\}$$

is compact.

19. Let K be a compact subset of a metric space M, and let $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ be an open cover of K. Show that there is a positive real number δ with the property that for every $x \in K$ there is some $\alpha \in I$ with

$$B_\delta(x) \subseteq \mathcal{U}_\alpha$$

20. Let M be a non-empty set, and let d be a real-valued function of ordered pairs of elements of M satisfying

(a) $d(x, y) = 0 \iff x = y$

(b) $d(x, y) \leq d(x, z) + d(y, z)$.

Show that d is a metric on M.

21. Determine the boundaries of the following sets, $A \subseteq X$:

    (i) $A = \mathbb{Q}$, $X = \mathbb{R}$

    (ii) $A = \mathbb{R} \backslash \mathbb{Q}$ $X = \mathbb{R}$

    (iii) $A = (\mathbb{Q} \times \mathbb{Q}) \cap B_R(0)$ $X = \mathbb{R}^2$

22. Describe the interior of the Cantor set.

23. Let M be a metric space with metric d, and let d_1 be the metric defined above (in problem 2). Show that the two metric spaces (M, d), (M, d_1) have the same open sets.

24. Let A be a proper dense subset of $\mathbb{R}$, and let U be a non-empty open subset of $\mathbb{R}$. For each of the following statements, prove or disprove:

    (a) $U \subseteq \overline{A \cap U}$.

    (b) $\overline{A \cap U} = \varnothing$.

    (c) $\overline{A \cap U} \subseteq U$.

    (d) $\overline{A \cap U} = A \cap \overline{U}$.

25. Let $\{A_i : i \in I\}$ be a family of subsets of $\mathbb{R}$, where I is an index set. Prove or disprove the following statements:

    (a) If I is finite, then
    $$\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}.$$

    (b) If I is an arbitrary (possibly infinite) index set, then
    $$\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}.$$

    (c) For any index set I,
    $$\overline{\bigcap_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}.$$

26. Recall that a subset $S \subseteq \mathbb{R}^n$ is said to be *compact* if every open cover of S admits a finite subcover. With this in mind, answer the following:

    (a) Let $S \subseteq \mathbb{R}^n$ be unbounded. Construct an open cover of S that admits no finite subcover, and prove your claim.

    (b) Let $S \subseteq \mathbb{R}^n$ be not closed. Construct an open cover of S that admits no finite subcover, and prove your claim.

27. **(Discrete metric and compactness).** Let (X, d) be a metric space with the *discrete metric*
    $$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$
    Prove the following statements.

    (a) Every subset of X is both open and closed.

    (b) The space (X, d) is bounded (i.e., has finite diameter), regardless of the underlying set X.

    (c) (X, d) is compact if and only if X is finite. *Hint:* If X is infinite, consider the open cover by singleton sets.

    (d) Conclude that in (X, d) the implication "closed and bounded $\Rightarrow$ compact" fails whenever X is infinite (since X is closed and bounded but not compact).

(e) Provide a concrete example with $X = \mathbb{N}$ under the discrete metric to illustrate the above points.

28. **Topology of the Cantor Set**

Let $C \subseteq [0, 1]$ denote the standard middle-third Cantor set.

Definitions:

- The *interior* of a set $A \subseteq \mathbb{R}$, denoted $\mathrm{int}(A)$, is the largest open set contained in A.
- A space (or subset of a space) is called *totally disconnected* if its only connected subsets are singletons (sets consisting of a single point).
- A set A is called *self-similar* if it can be written as a union of scaled and translated copies of itself. For the Cantor set, this means

$$C = \tfrac{1}{3}C \cup \left(\tfrac{2}{3} + \tfrac{1}{3}C\right).$$

Questions:

(a) (**Closedness**) Show that C is a closed subset of $[0, 1]$.

(b) (**Interior**) Show that $\mathrm{int}(C) = \varnothing$.

(c) (**Perfectness**) Show that every point of C is a limit point of C. (That is, C has no isolated points.)

(d) (**Total Disconnectedness**) Prove that the only connected subsets of C are singletons.

(e) (**Compactness**) Show that C is compact. (Hint: C is a closed subset of $[0, 1]$.)

(f) (**Product Topology** — do this only after product topology is covered in class) Show that C is homeomorphic to the product space $\{0, 1\}^{\mathbb{N}}$ with the product topology, where each copy of $\{0, 1\}$ is given the discrete topology.

(g) (**Self-similarity**) Show that C is self-similar, in the sense that

$$C = \tfrac{1}{3}C \cup \left(\tfrac{2}{3} + \tfrac{1}{3}C\right).$$

29. **Connected Subsets under Different Metrics**

Let (X, d) be a metric space. Recall that a subset $A \subseteq X$ is *connected* if it cannot be written as the union of two disjoint nonempty open subsets in the subspace topology.

Questions:

(a) (**Euclidean metric on** $\mathbb{R}$) Let $d(x, y) = |x - y|$.

i. Show that if $A \subseteq \mathbb{R}$ is connected, then for any $x, y \in A$ with $x < y$, the entire interval $[x, y]$ is contained in A.

ii. Deduce that the connected subsets of $(\mathbb{R}, d)$ are precisely the intervals (including singletons).

(b) (**Boundedness vs connectedness**) Give one example of a bounded connected subset of $\mathbb{R}$, and one example of an unbounded connected subset.

(c) (**Discrete metric on** $\mathbb{R}$) Consider the discrete metric

$$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Show that the only connected subsets of $(\mathbb{R}, d)$ are singletons.

(d) (**Truncated metric on** $\mathbb{R}$) Define $d(x, y) = \min\{1, |x - y|\}$ on $\mathbb{R}$.

i. Show that this is indeed a metric.

ii. Determine whether the connected subsets of $(\mathbb{R}, d)$ are the same as those under the Euclidean metric.

(e) (**Connectedness in** $\mathbb{R}^n$) Let $\mathbb{R}^n$ be equipped with the Euclidean metric.

i. Show that convex subsets of $\mathbb{R}^n$ are connected.

ii. Give an example of a connected subset of $\mathbb{R}^2$ that is not convex.

(f) (**Thought exercise**) Can you find (or prove impossible) a metric on $\mathbb{R}$ in which every infinite set is connected?

30. **Cardinality of Subsets of** $\mathbb{R}^n$

Let $n \geq 1$ and consider subsets of $\mathbb{R}^n$. For each statement, prove or provide a counterexample.

(a) (**Finite sets**)

i. Every finite subset of $\mathbb{R}^n$ is closed.

ii. Every finite subset of $\mathbb{R}^n$ is bounded.

(b) (**Countable sets**)

i. The set $\mathbb{Q}^n$ (vectors with rational coordinates) is countable.

ii. A countable subset of $\mathbb{R}^n$ can be dense.

(c) (**Uncountable sets**)

i. The interval $[0, 1]^n \subset \mathbb{R}^n$ is uncountable.

ii. Every uncountable subset of $\mathbb{R}^n$ contains a limit point. *(Hint: use Bolzano–Weierstrass theorem.)*

(d) (**Mixed cardinality statements — Prove or Disprove**)

i. The union of two countable sets is countable.

ii. The union of a countable set and an uncountable set is uncountable.

iii. Every subset of $\mathbb{R}^n$ is either countable or has the same cardinality as $\mathbb{R}$.

iv. Every infinite subset of $\mathbb{R}^n$ is uncountable.

v. The Cartesian product $\mathbb{Q} \times \mathbb{Q}$ is countable.

(e) (**Bonus**)

i. Show that $[0, 1] \subset \mathbb{R}$ and $[0, 1]^n \subset \mathbb{R}^n$ have the same cardinality.

ii. Show that $\mathbb{R}^n$ is uncountable for any $n \geq 1$.