

# 1 Problems set 1(Real Numbers)

1. Prove the following statements are true in every ordered field.

- (a) If  $x > 0$  then  $-x < 0$ , and vice versa.
- (b) If  $x > 0$  and  $y < z$  then  $xy < xz$ .
- (c) If  $x < 0$  and  $y < z$  then  $xy > xz$ .
- (d) If  $x \neq 0$  then  $x^2 > 0$ . In particular,  $1 > 0$ .
- (e) If  $0 < x < y$  then  $0 < 1/y < 1/x$ .

2. Let  $S \subset \mathbb{R}$  and  $a \in \mathbb{R}$ . Prove that

- (a)  $\sup(a + S) = a + \sup S$
- (b) If  $a > 0$  then  $\sup(aS) = a \sup S$
- (c) If  $a < 0$  then  $\sup(aS) = a \inf S$

3. Prove that supremum property implies infimum property.

4. Use Archimedean Property to prove the following

- (a) If  $S := \{1/n : n \in \mathbb{N}\}$ , then  $\inf S = 0$ .
- (b) If  $t > 0$ , there exists  $n_t \in \mathbb{N}$  such that  $0 < 1/n_t < t$ .
- (c) If  $y > 0$ , there exists  $n_y \in \mathbb{N}$  such that  $n_y - 1 \leq y \leq n_y$ .
- (d) If  $S := \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$ , find  $\inf S$  and  $\sup S$ .

5. Prove that between any two real numbers there is an irrational number.

6. Let  $I_n := [0, 1/n]$  for  $n \in \mathbb{N}$ . Prove that

$$\bigcap_{n=1}^{\infty} I_n = \{0\}.$$

7. Let  $J_n := (0, 1/n)$  for  $n \in \mathbb{N}$ . Prove that

$$\bigcap_{n=1}^{\infty} J_n = \emptyset.$$

8. Let  $K_n := (n, \infty)$  for  $n \in \mathbb{N}$ . Prove that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

## 2 Problem set 2(Sequences)

1. Find  $K(\epsilon)$  for the following sequences

(a)  $\frac{1}{5n+1}$  ;  $\epsilon = \frac{1}{6}$

(b)  $\frac{2}{n^3} + 5$  ;  $\epsilon = \frac{1}{8}$

(c)  $\frac{3n}{2n+1}$  ;  $\epsilon = \frac{1}{10}$

(d)  $\frac{n^2+n}{2n^2-1}$  ;  $\epsilon = \frac{1}{5}$

(e)  $(-1)^n \frac{1}{n}$  ;  $\epsilon = \frac{1}{61}$

2. Prove that limit of a sequence if it exist is unique.

3. Evaluate the following limits (also prove it using the definition of limit):

(a)  $\lim_{n \rightarrow \infty} \frac{3n}{2n+1}$

(b)  $\lim_{n \rightarrow \infty} \frac{n}{n^2+1}$

(c)  $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$

4. Let  $x_n, y_n$  be two sequences such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . Prove that:

(a)  $x_n + y_n \rightarrow x + y$

(b)  $x_n - y_n \rightarrow x - y$

(c)  $x_n y_n \rightarrow xy$

(d)  $x_n/y_n \rightarrow x/y$  if  $y \neq 0$

5. Prove that if  $x_n \rightarrow x$  then  $|x_n| \rightarrow |x|$  and  $\sqrt{|x_n|} \rightarrow \sqrt{|x|}$ .

6. Prove squeeze theroerm: If  $x_n \leq y_n \leq z_n$  for all  $n$  and  $x_n \rightarrow L$ ,  $z_n \rightarrow L$ , then  $y_n \rightarrow L$ .

7. Prove that for any real number  $x$ , there exists a sequence of rational numbers  $(x_n)$  such that  $x_n \rightarrow x$ .

### 3 Problem set 3(Sequences)

1. Is the sequence  $\log(n) - \log(n+1)$  convergent?
2. Let  $x_n$  converges to  $x$ . Prove that any subsequence of  $x_n$  is convergent to  $x$ .
3. True or False : Let  $(a_n)$  be a sequence such that  $a_{nk}$  converges for  $k \in \mathbb{N} - \{1\}$ . Then  $a_{nk}$  converges.
4. Give an example of an unbounded sequence that has a convergent subsequence.
5. Suppose that  $x_n \geq 0$  and  $(-1)^n x_n$  converges. Does it imply that  $x_n$  converges.
6. Let  $(x_n)$  be a bounded sequence and let  $s := \sup\{x_n : n \in \mathbb{N}\}$ . Show that if  $s \notin \{x_n : n \in \mathbb{N}\}$ , then there is a subsequence of  $(x_n)$  that converges to  $s$ .
7. Show directly that bounded monotone sequence is a cauchy sequence.
8. Let  $Y = (y_n)$  be defined inductively by  $y_1 := 1$ ,  $y_{n+1} := \frac{1}{4}(2y_n + 3)$  for  $n \geq 1$ . Show that  $\lim Y = 3/2$ .

## 4 Problem set 4 (Limits and Continuity)

1. Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be two functions continuous at  $x_0$ . Prove that:
  - (a)  $f + g$  is continuous at  $x_0$ .
  - (b)  $f - g$  is continuous at  $x_0$ .
  - (c)  $fg$  is continuous at  $x_0$ .
  - (d)  $f/g$  is continuous at  $x_0$  if  $g(x_0) \neq 0$ .
2. Prove that all polynomials from  $\mathbb{R} \rightarrow \mathbb{R}$  are continuous functions. Are they uniformly continuous?
3. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that  $f(x) = 0$  for all  $x \in \mathbb{Q}$ . Prove that  $f(x) = 0$  for all  $x \in \mathbb{R}$ .
4. Prove that composition of two continuous functions is continuous.
5. Prove that  $x^2$  is uniformly continuous on  $[0, 100]$
6. Prove that  $x^2$  is not uniformly continuous on  $\mathbb{R}$ .
7. Prove that  $\frac{1}{x}$  is not uniformly continuous on  $(0, 1)$ .