Tutorial Sheet - Real Analysis

November 12, 2025

Series Convergence

Problem 1 (Condensation Test)

Check the convergence or divergence of the following infinite series:

$$\sum_{n=2}^{\infty} \frac{1}{n \log(n)}$$

Hint: Use the **Cauchy Condensation Test**. The test states that if $\{a_n\}$ is a non-increasing sequence of non-negative terms, then $\sum a_n$ converges if and only if $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Problem 2 (Alternating Series Test)

Check the convergence or divergence of the following alternating series:

$$\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$$

Hint: Use the **Alternating Series Test**. The test states that the series $\sum (-1)^n b_n$ (where $b_n \geq 0$) converges if:

1. $\lim_{n\to\infty} b_n = 0$, AND
2. $b_{n+1} \leq b_n$ for all n.

You will need to show the sequence $b_n = \frac{\log n}{n}$ is monotonically decreasing for $n > e$.

Problem 3 (FTC Counterexample)

Give the statement of the **First Fundamental Theorem of Calculus (FTC1)**. Then, check the necessity of the continuity assumption of the function f by giving a counterexample where FTC1 fails. **Hint:** The Fundamental Theorem of Calculus states:

FTC1: If f is **continuous** on $[a, b]$ and $F(x) = \int_a^x f(t)\, dt$, then $F'(x) = f(x)$ for $x \in (a, b)$.

FTC2: If f is Riemann-integrable on $[a, b]$ and G is any antiderivative of f, then $\int_a^b f(x)\, dx = G(b) - G(a)$.

To find the counterexample, think about a step function on a closed interval like $[0, 1]$.

Metric Spaces and Topology

Problem 4 (Compactness and Separability)

Prove that every **compact metric space** (X, d) is **separable**. **Hint:** Try the cover with balls of radius $1/n$ centered at each point. Find a finite subcover. Collect the center points of the subcover. Take this collection of points for all $n \in \mathbb{N}$ and show the resulting set is a countable dense subset.

Problem 5 (Homeomorphism)

Prove that any closed interval $[a, b]$, where $a < b$, is **homeomorphic** to the standard closed interval $[0, 1]$. **Remark:** The mapping function you construct and its inverse are both linear polynomial functions.

Problem 6 (Density and Open Covers)

Let (X, d) be a metric space, and let E be a **dense subset** of X. Prove that for any radius $r > 0$, the union of all open balls of radius r centered at the points of E forms a cover of X. That is, prove that $X = \bigcup_{e \in E} B_r(e)$.
Remark: This exercise is intended as writing practice to ensure the formal definition of a dense set is well-understood and can be applied correctly.

Limits of Sequences

Problem 7 (Sequence Limits)

Prove the following statements regarding limits of sequences:

1. For each $k \in \mathbb{N}$, we have:
$$\lim_{n\to\infty} (\sqrt{n+k} - \sqrt{n}) = 0$$
2. Determine the limit:
$$\lim_{n\to\infty} (\sqrt{n^2+n} - n)$$
3. Given the sequence defined by $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$, prove that:
$$\lim_{n\to\infty} x_n = 2$$

Hints for Problem 7:

1. Multiply the numerator and denominator with the conjugate, $\sqrt{n+k} + \sqrt{n}$.
2. Multiply the numerator and denominator with the conjugate, $\sqrt{n^2+n} + n$.
3. Verify that $1 \leq x_n \leq 2$ for every $n \in \mathbb{N}$ (Boundedness). Show (x_n) is an increasing sequence in $[1, 2]$ (Monotonicity). Then, let $x = \lim_{n\to\infty} x_n$ and solve the recursive formula $x = \sqrt{2 + x}$.