Real Analysis Exercise Sheet

1. Let X be an infinite set. For $p, q \in X$, define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q. \end{cases}$$

   (a) Prove that this is a metric.
   (b) Which subsets of the resulting metric space are open?
   (c) Which are closed?
   (d) Which are compact? (Definition. A metric space (X, d) is called compact if every sequence in X has a convergent subsequence whose limit lies in X; a subset $Y \subset X$ is called compact if the subspace $(Y, d|_{Y\times Y})$ is compact).
   (e) Prove that a sequence $\{x_n\} \subset X$ converges to $x \in X$ in the discrete metric if and only if there exists $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$.

2. Let (X, d) be a metric space. Show that $d'(x, y) = \sqrt{d(x, y)}$ is also a metric on X, and that the open sets for d' are the same as the open sets for d.

3. Consider $\mathbb{R}$ with the standard metric. Let $E \subset \mathbb{R}$ be a subset which has no limit points. Show that E is at most countable.

4. Let (X, d) be a compact metric space, and $f : X \to X$ a map such that $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$. Prove that there exists a point x such that $f(x) = x$. Hint: how small can $d(x, f(x))$ get? Comment: this is an example of a fixed point theorem, very popular in various sciences for showing the existence of equilibria and such.

5. Let (X, d) be a metric space.

   (a) Prove the following:
      (i) For all $x, y, z \in X$
      $$|d(y, z) - d(z, x)| \leq d(x, y).$$
      (ii) For any $n \in \mathbb{N}$ and any $x_1, x_2, \ldots, x_n \in X$
      $$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$
   (b) For $x \in \mathbb{R}$ and $y \in \mathbb{R}$ define
      (i) $d_1(x, y) = (x - y)^2$
      (ii) $d_2(x, y) = \sqrt{|x - y|}$
      (iii) $d_3(x, y) = |x^2 - y^2|$
      (iv) $d_4(x, y) = |x - 2y|$

      Determine, for each of these, whether it is a metric or not.

6. (a) In a metric space (X, d), prove that if a sequence (x_n) converges, then its limit is unique. That is, show that if $x_n \to x$ and $x_n \to y$, then $x = y$.
   (b) Prove that a subset E of a metric space (X, d) is closed if and only if
   $$E = \overline{E},$$
   where $\overline{E}$ denotes the closure of E.

7. Let $\mathbb{R}^2$ be equipped with the Taxicab metric/Manhattan metric defined by
$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$
   (a) Prove that d is a metric.
   (b) Draw the open balls $B_1((0, 0))$, $B_2((0, 0))$, and $B_{\frac{1}{2}}((0, 0))$ in $\mathbb{R}^2$.

(c) On the same axes, sketch the corresponding Euclidean balls (under the usual metric) of the same radii.

(d) What geometric shape do the taxicab balls form?

(e) Are these taxicab balls open sets in the usual metric on Euclidean space? Prove that these two metrics are equivalent.

(f) Let

$$A = \{(x, y) \in \mathbb{R}^2 : d((x, y), (1, 1)) \geq 2\}.$$

Sketch the set A. Is it closed in the taxicab metric?

8. (a) Let us consider the set $X = \mathbb{R} \setminus \{\sqrt{2}\}$ equipped with the usual metric $d(x, y) = |x - y|$. Is (X, d) a complete metric space? Justify your answer with appropriate reasoning.

   (b) Let R be the set of real numbers with the usual metric $d(x, y) = |x - y|$.

      (i) Show that the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence in R.

      (ii) Show that the sequence $x_n = n$ is not a Cauchy sequence in R.

9. Let $C[0, 1]$ be the set of all real-valued continuous functions on [0, 1]. Define:

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx.$$

Define a sequence $\{f_n\} \subset C[0, 1]$ by:

$$f_n(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ n(x - (\frac{1}{2} - \frac{1}{n})), & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

   (a) Draw the graph of each function f_n and verify that $f_n \in C[0, 1]$ by observing that each f_n is piecewise linear and continuous.

   (b) Prove that f_n pointwise converge to f where

$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

   (c) Verify that $\{f_n\}$ is a Cauchy sequence in $(C[0, 1], d_1)$.

   (d) Is $(C[0, 1], d_1)$ a complete metric space?

10. Let l^2 be the set of all sequences $x = \{x_n\}$ over the real numbers R such that $\sum_{n \geq 1} x_n^2$ converges. Prove that $d(x, y) = \sqrt{\sum_{n \geq 1} (x_n - y_n)^2}$ gives a metric on l^2.

11. Let (X, d) be a metric space and A be a non-empty subset of X. For $x \in X$, we define

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

   (a) Show that $d(x, A) = 0$ if and only if x lies in the closure of A.

   (b) Show that if A is compact, then $d(x, A) = d(x, a)$ for some $a \in A$.

   (c) The ϵ-neighborhood of A is defined to be

$$N(A; \epsilon) = \{x \in X : d(x, A) < \epsilon\}.$$

   Prove that $N(A; \epsilon)$ is the union of the open balls $B_\epsilon(a)$ for $a \in A$.

   (d) If A is compact and U is an open set containing A, prove that there exists $\epsilon > 0$ such that

$$A \subset N(A; \epsilon) \subset U.$$

   Is this fact still true if A is just a closed set?

12. Let (X, d) be a metric space. Then verify the following properties:

(a) The union of any collection $\{G_\alpha\}$ of open sets is open; that is, $\bigcup_\alpha G_\alpha$ is open.
(b) The intersection of any collection $\{F_\alpha\}$ of closed sets is closed; that is, $\bigcap_\alpha F_\alpha$ is closed.
(c) The intersection of any finite collection $G_1, \ldots, G_n$ of open sets is open; that is, $\bigcap_{i=1}^n G_i$ is open.
(d) The union of any finite collection $F_1, \ldots, F_n$ of closed sets is closed; that is, $\bigcup_{i=1}^n F_i$ is closed.

13. In vector spaces, metrics are usually defined in terms of norms which measure the length of a vector. If V is a vector space defined over $\mathbb{R}$, then a norm is a function from vectors to real numbers, denoted by $||\cdot||$, satisfying:

(i) $||x|| \geq 0$ and $||x|| = 0 \iff x = 0$,
(ii) For any $\lambda \in \mathbb{R}$, $||\lambda x|| = |\lambda|||x||$,
(iii) $||x + y|| \leq ||x|| + ||y||$.

Prove that every norm defines a metric.

14. Let M be a metric space with metric d. Show that d_1 defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on M. Observe that M itself is bounded in this metric.

15. Let A and B be two subsets of a metric space M. Recall that A°, the interior of A, is the set of interior points of A. Prove the following:

$$\text{a)} A^\circ \cup B^\circ \subseteq (A \cup B)^\circ, \quad \text{b)} A^\circ \cap B^\circ = (A \cap B)^\circ$$

Give an example of two subsets A and B of the real line such that $A^\circ \cup B^\circ \neq (A \cup B)^\circ$.

16. Let A be a subset of a metric space M. Recall that $\overline{A}$, the closure of A, is the union of A and its limit points. Recall that a point belongs to the boundary of A, ∂A, if every open ball centered at the point contains points of A and points of A^c, the complement of A. Prove that:

(a) $\partial A = \overline{A} \cap \overline{A^c}$
(b) $p \in \partial A \iff p$ is in $\overline{A}$, but not in A° (symbolically, $\partial A = \overline{A} \backslash A^\circ$)
(c) ∂A is a closed set
(d) A is closed $\iff \partial A \subseteq A$

17. Show that, in $\mathbb{R}^n$ with the usual (Euclidean) metric, the closure of the open ball $B_R(p)$, $R > 0$, is the closed ball

$$\{q \in \mathbb{R}^n : d(p, q) \leq R\}.$$

Give an example of a metric space for which the corresponding statement is false.

18. Prove directly from the definition that the set $K \subseteq \mathbb{R}$ given by

$$K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\}$$

is compact.

19. Let K be a compact subset of a metric space M, and let $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ be an open cover of K. Show that there is a positive real number δ with the property that for every $x \in K$ there is some $\alpha \in I$ with

$$B_\delta(x) \subseteq \mathcal{U}_\alpha$$

20. Let M be a non-empty set, and let d be a real-valued function of ordered pairs of elements of M satisfying

(a) $d(x, y) = 0 \iff x = y$
(b) $d(x, y) \leq d(x, z) + d(y, z)$.

Show that d is a metric on M.

21. Determine the boundaries of the following sets, $A \subseteq X$:

    (i) $A = \mathbb{Q}$, $X = \mathbb{R}$

    (ii) $A = \mathbb{R} \backslash \mathbb{Q}$ $X = \mathbb{R}$

    (iii) $A = (\mathbb{Q} \times \mathbb{Q}) \cap B_R(0)$ $X = \mathbb{R}^2$

22. Describe the interior of the Cantor set.

23. Let M be a metric space with metric d, and let d_1 be the metric defined above (in problem 2). Show that the two metric spaces (M, d), (M, d_1) have the same open sets.