Set Theory Exercise Sheet
Aug-Dec 2025; Anoop V. P.

1. Prove the following:
   (a) $(\mathbb{Q} \times \mathbb{Q}^c) \cap (\mathbb{Q}^c \times \mathbb{Q}) = \phi$
   (b) $(\mathbb{Q}^c \times \mathbb{Q}^c) \subset (\mathbb{R} \times \mathbb{Q}^c)$

2. Draw the following sets :
   (a) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\} \cap \{(x,y) \in \mathbb{R} \times \mathbb{R} : (x-2)^2 + y^2 = 1\}$
   (b) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\} \cap \{(x,y) \in \mathbb{R} \times \mathbb{R} : x = y\} \cap \{(x,y) \in \mathbb{R} \times \mathbb{R} : x = 0\} \cap \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = 0\}$

3. I'd like to introduce the concept of *path connectedness* to you, not through a dry definition but with a simple story. Many of you may have heard the famous tale of *Nakula* from the *Mahabharatha*. He had the ability to ride his horse through the rain without ever getting wet. In other words, he could find a *path* in the *complement of rainfall*.

   Now, imagine the ground on which *Nakula* rode as a subset of $\mathbb{R} \times \mathbb{R}$, and the falling raindrops as points of $\mathbb{Q} \times \mathbb{Q}$. *Nakula*'s challenge was to find a *path* in the complement of $\mathbb{Q} \times \mathbb{Q}$. Put simply, the task is to show that the complement of $\mathbb{Q} \times \mathbb{Q}$ is *path connected*.

   At our level of mathematics, this means we must be able to draw straight line segments in the complement of $\mathbb{Q} \times \mathbb{Q}$ that connect any two of its points. The question is—can you do it?

4. **Barber's Paradox:** In a certain village, there is a barber who shaves all and only those men in the village who do not shave themselves.

   The question is: **Who shaves the barber?**

   - If the barber shaves himself, then by the rule he should not shave himself.
   - If the barber does not shave himself, then by the rule he must shave himself.

   Either way, we get a contradiction.

   Connection to Russell's Paradox: This story is really just a simple version of *Russell's Paradox* in set theory.

   Instead of barbers and shaving, Russell asked:

   $$R = \{\, S : S \notin S \,\}$$

   the set of all sets that do not contain themselves.

   Now ask: does R contain itself?

   - If $R \in R$, then by definition $R \notin R$.
   - If $R \notin R$, then by definition $R \in R$.

   Again, both answers lead to a contradiction—just like with the barber.

   Why It Is Important in Set Theory: The importance of this paradox is that it shows that **not every condition we can describe actually defines a valid set**.

   - In everyday language, "the barber shaves all those who do not shave themselves" sounds fine—but when analyzed carefully, it breaks down.
   - Similarly, in mathematics, if we allow *any* property to define a set, we run into contradictions like Russell's Paradox.

   This is why modern set theory (such as *Zermelo–Fraenkel set theory*) uses strict axioms to avoid such problems. The paradox teaches us that we must be **careful and precise** when building the foundations of mathematics.

   Conclusion: The Barber's Paradox is more than just a fun story—it is a gateway to understanding why set theory had to be rebuilt on solid axiomatic foundations.