Real Analysis Notes - 2
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Sequences

- You are moving from sequences in the familiar metric space $(\mathbb{R}, d)$, where the distance $d(x, y) = |x-y|$, to sequences in a general **metric space** (X, d).
- **Rudin's Definition Check:** Refer to Rudin's text (or equivalent) for the formal definition of convergence in a general metric space.
- **The Formal Definition:** A sequence $\{x_n\}$ in a metric space (X, d) **converges to a limit** $x \in X$ if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \quad d(x_n, x) < \varepsilon$$

- **Key Insight:** The undergraduate absolute value $|x_n - x|$ is always replaced by the **metric (distance function)** $d(x_n, x)$.

Checking Convergence by Definition

Determine if the limit of the sequence exists. Clearly state the metric space used (e.g., $\mathbb{R}$ or $\mathbb{C}$) and justify your answer based on the definition.

1. $x_n = (-1)^n$
2. $x_n = i^n$ (Consider this sequence in the metric space $\mathbb{C}$).
3. $x_n = \frac{1}{n}$

Algebra of Limits Pitfalls

Assume all sequences are in the standard Euclidean metric space $\mathbb{R}$.

1. **Sum Rule:** Given $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n^2}$. Does $z_n = x_n + y_n$ converge? If so, state the limit.
2. **Product Rule:** Given $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n}$. Does $z_n = x_n y_n$ converge? If so, state the limit.
3. **Quotient Rule Restrictions:** If $\{x_n\}$ and $\{y_n\}$ are convergent sequences, what two conditions are required for the sequence $z_n = \frac{x_n}{y_n}$ to be convergent? *(Hint: Consider y_n terms and their limit.)*
4. **Quotient Rule Pitfall Example:** Let $x_n = 1$ and $y_n = \frac{1}{n}$. Analyze the convergence of $z_n = \frac{x_n}{y_n}$. Why does the standard Quotient Rule fail to apply, even though $y_n \neq 0$ for all n?

Subsequences and monotone subsequences

The following problems test your recall of the definitions and implications of monotone and convergent sequences in R.

1. **True or False:** Every monotone sequence of real numbers is convergent. Justify your answer.
2. **True or False:** A monotone increasing sequence of real numbers converges to its supremum. Justify your answer.
3. **True or False:** Every convergent sequence of real numbers is bounded. Justify your answer.

Monotone Subsequence Theorem

This section explores the crucial property that every sequence must contain a monotone subsequence.

Problem 4: Prove that every sequence of real numbers $\{x_n\}$ has a monotone subsequence.

1. **Case 1: The Sequence $\{x_n\}$ is Unbounded.**
   - Without loss of generality, assume $\{x_n\}$ is not **bounded above**.
   - Choose $x_{n_1} = x_1$. Since x_{n_1} is not an upper bound for the entire sequence (because the sequence is unbounded above), locate the first element after x_{n_1} which is greater than x_{n_1}. Call this x_{n_2}.
   - Since x_{n_2} is also not an upper bound, locate the first element after x_{n_2} which is greater than x_{n_2}. Call this x_{n_3}.
   - This process never ends (why? Because $\{x_n\}$ is not bounded above). This construction yields a subsequence $\{x_{n_k}\}$ that is strictly **monotone increasing**.
   - A similar argument works if the sequence is not bounded below, yielding a strictly monotone decreasing subsequence.
2. **Case 2: The Sequence $\{x_n\}$ is Bounded.**
   - By the **Bolzano-Weierstrass Theorem**, $\{x_n\}$ has a convergent subsequence, say $\{y_k\}$, and $\lim y_k = y$.
   - **Subcase 2A (Easy Case):** Suppose $\{y_k\}$ has infinitely many terms equal to its limit y. You can choose a constant subsequence from these terms, which is trivially monotone.
   - **Subcase 2B (Harder Case):** Suppose there are infinitely many terms of $\{y_k\}$ not equal to y. This means there are infinitely many terms either strictly greater than y or strictly less than y.
   - Without loss of generality, assume there are infinitely many terms of $\{y_k\}$ **greater than y**.
   - Choose the first term y_{k_1} from $\{y_k\}$ such that $y_{k_1} > y$.
   - Now, look for an element of $\{y_k\}$ in the interval (y, y_{k_1}). Since $\lim y_k = y$, and there are infinitely many terms greater than y, such an element y_{k_2} (with $k_2 > k_1$) exists such that $y < y_{k_2} < y_{k_1}$.
   - Next, look for an element y_{k_3} (with $k_3 > k_2$) in the interval (y, y_{k_2}) such that $y < y_{k_3} < y_{k_2}$.
   - Continuing this process yields a subsequence $\{y_{k_j}\}$ that is strictly **monotone decreasing**.

Sequences: Monotonicity, Boundedness, and Subsequences

This section reviews fundamental concepts regarding monotone sequences and the existence of monotone subsequences, leading into key theorems.

Conceptual True/False (Review of Definitions)

(a) **True or False:** Every monotone sequence of real numbers is convergent. Justify your answer.

(b) **True or False:** A monotone increasing sequence of real numbers converges to its supremum. Justify your answer.

(c) **True or False:** Every convergent sequence of real numbers is bounded. Justify your answer.

Sequences and Series: The Sequence of Partial Sums

This note outlines the fundamental definition of series convergence via partial sums and details the necessary condition for convergence, including the crucial counterexample.

The Fundamental Connection: Series as Sequences

- The most critical conceptual understanding is that a **series** ($\sum_{n=1}^{\infty} x_n$) is inherently a **sequence** in disguise: the **sequence of partial sums**, $\{S_N\}$.
- The sequence of partial sums $\{S_N\}$ is constructed by summing the terms of the series consecutively:

$$S_N = x_1 + x_2 + \cdots + x_N = \sum_{n=1}^{N} x_n$$

- **Convergence Definition:** The series $\sum x_n$ **converges** if and only if the sequence of partial sums $\{S_N\}$ converges.
- **Note on Terminology:** Do not confuse the **corresponding sequence** $\{x_n\}$ (the terms being added) with the **sequence of partial sums** $\{S_N\}$.

The Necessary Condition for Convergence (Test for Divergence)

If a series $\sum x_n$ converges, its sequence of partial sums $\{S_N\}$ must converge to some limit L. This requires the terms being added to approach zero.

- Since the N-th term of the series is the difference between two consecutive partial sums, $x_N = S_N - S_{N-1}$ (for $N \geq 2$), we take the limit:

$$\lim_{N\to\infty} x_N = \lim_{N\to\infty} (S_N - S_{N-1}) = L - L = 0$$

- **Conclusion:** For a series $\sum x_n$ to converge, the absolute value of the terms added, $|x_n|$, **must become smaller and smaller** as n goes to infinity. That is, $\lim_{n\to\infty} x_n = 0$.

The Problem with Intuitive Understanding: A Cautionary Pitfall

The intuitive idea that terms becoming small guarantees convergence **is flawed**. The converse of the necessary condition is false.

- The question is: If $\lim_{n\to\infty} x_n = 0$, can we be certain that the corresponding series $\sum x_n$ converges?
- The answer is a **loud NO**! The condition $\lim x_n = 0$ is **necessary** but **not sufficient** for series convergence.
- **Counterexample:** Consider the sequence $x_n = \frac{1}{n}$. The terms approach zero: $\lim_{n\to\infty} \frac{1}{n} = 0$.
- However, the corresponding series, $\sum_{n=1}^{\infty} \frac{1}{n}$ (the **Harmonic Series**), **diverges**. This demonstrates that terms may not become small *fast enough* to allow the sequence of partial sums to stabilize.

Standard Sequences and Series Reference

A summary of fundamental sequences and series, their convergence properties, and core convergence tests.

Fundamental Sequences ($\lim_{n\to\infty} x_n$)

A sequence $\{x_n\}$ converges if $\lim_{n\to\infty} x_n = L$ (a finite number).

Sequence (x_n)	**Limit (L)**	**Condition**	**Notes**
Constant	c	Always Convergent	$x_n = c$
Power	0	$p > 0$	$x_n = 1/n^p$
Geometric	0	$\|r\| < 1$	$x_n = r^n$. Diverges if $\|r\| \geq 1$.
Root of n	1	Always Convergent	$\lim \sqrt[n]{n} = 1$
Log over Power	0	$p > 0$	$\lim \frac{\ln(n)}{n^p} = 0$. Log dominates power.

Fundamental Series ($\sum_{n=1}^{\infty} a_n$)

A series converges if its sequence of partial sums converges.

Series ($\sum \mathbf{a_n}$)	Type	Convergence Condition	Notes
Harmonic	p-Series ($p = 1$)	Diverges	$\sum \frac{1}{n}$. $\lim a_n = 0$, but diverges.
p-Series	p-Series	$\mathbf{p > 1}$	$\sum \frac{1}{n^p}$.
Geometric	Geometric	$\lvert\mathbf{r}\rvert < \mathbf{1}$	$\sum r^n$. Converges to $\frac{r}{1-r}$ (if $n = 1$).
Factorial	General	Converges	$\sum \frac{1}{n!}$. Converges quickly (Ratio Test $\to 0$).

Core Convergence Tests

1. **Test for Divergence (Necessary Condition)**
   - If $\lim_{n\to\infty} a_n \neq 0$ or DNE, then $\sum a_n$ **diverges**.
   - **Caution:** If $\lim a_n = 0$, the test is inconclusive.
2. **Comparison Tests** (for non-negative terms a_n)
   - **Direct Comparison:** Use with a known convergent ($\sum b_n$ where $a_n \leq b_n$) or divergent ($\sum b_n$ where $a_n \geq b_n$) series.
   - **Limit Comparison:** If $\lim \frac{a_n}{b_n} = L$ $(0 < L < \infty)$, then $\sum a_n$ and $\sum b_n$ share convergence.
3. **Ratio Test** (Good for factorials/exponentials)
   - Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Converges if $L < 1$. Diverges if $L > 1$. Inconclusive if $L = 1$.
4. **Alternating Series Test (AST)**
   - Used for $\sum(-1)^n b_n$. Converges if $b_n \geq 0$, $\lim b_n = 0$, and $\{b_n\}$ is eventually monotone decreasing.
5. **Absolute vs. Conditional Convergence**
   - $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges. (Absolute convergence implies convergence.)
   - $\sum a_n$ **converges conditionally** if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

Rigorous Workout Problems

This section contains problems requiring detailed proofs, complex applications of convergence criteria, and deep conceptual understanding.

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1. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences of positive real numbers such that
$$\lim_{n\to\infty} \frac{x_n}{y_n} = L$$
where L is a positive real number ($L > 0$ and $L < \infty$).
Prove:
$$\lim_{n\to\infty} x_n = \infty \iff \lim_{n\to\infty} y_n = \infty$$
2. Let a_n be a sequence of positive real numbers such that
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$
a) If $L < 1$, show that $\lim_{n\to\infty} a_n = 0$. b) If $L > 1$, show that $\lim_{n\to\infty} a_n = \infty$. c) Moreover, show that
$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = L.$$
3. **Constructive Convergence of Subseries:**
Let $\{a_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$.
Constructively demonstrate the existence of a subsequence $\{a_{n_k}\}$ for which the series $\sum_{k=1}^{\infty} a_{n_k}$ converges.
Hint: It is sufficient to choose the subsequence such that $|a_{n_k}| < \frac{1}{2^k}$ for all k.

IV. Challenge Problem: The Analytical Derivation and Nature of e

This section presents the rigorous foundational proof establishing the convergence of the two principal sequences defining Euler's number (e) and demonstrates its non-rational nature.

Equivalence of Definitions (Convergence Proof)

Consider the compounded growth sequence $\{x_n\}$ and the factorial partial sum sequence $\{y_n\}$:

$$x_n := \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad y_n := \sum_{k=0}^{n} \frac{1}{k!}$$

Problem 1: Prove that $\{x_n\}$ and $\{y_n\}$ are convergent and establish the equality of their limits.

(i) **Monotonicity and Boundedness:** Demonstrate that both $\{x_n\}$ and $\{y_n\}$ are **monotone increasing** sequences. Show that $\{y_n\}$ is **bounded above** by 3, confirming its convergence (MCT).

(ii) **Upper Bound:** Derive the initial inequality $x_n \leq y_n$ by comparing the terms resulting from the **Binomial Expansion** of x_n with the terms of y_n.

(iii) **Lower Bound and Equality:** For a fixed integer m, use the truncated expansion of x_n (stopping at the m-th term) to establish the inequality:

$$\liminf_{n\to\infty} x_n \geq y_m$$

By combining this result with $x_n \leq y_n$, rigorously conclude that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. Designate this common limit as e.

The Non-Rational Nature of e

Problem 2: Prove that the number e, defined by the convergent infinite series $e = \sum_{k=0}^{\infty} \frac{1}{k!}$, is **irrational**.

(i) **Contradiction Setup:** Proceed by contradiction. Assume e is rational, so $e = \frac{p}{q}$ for $p, q \in \mathbb{N}$.

(ii) **Remainder Analysis:** Consider the remainder term $R_q = e - y_q = \sum_{k=q+1}^{\infty} \frac{1}{k!}$.

(iii) **The Conflict:** Examine the algebraic quantity $q! \cdot R_q$.

- Show that $q! \cdot R_q$ must be an integer under the initial rational assumption for e.
- Establish the bounds for the remainder:

$$0 < R_q < \frac{1}{q! \cdot q}$$

- Combine the bounds to show that $0 < q! \cdot R_q < 1$. Explain why this result contradicts the fact that $q! \cdot R_q$ must be an integer.

Proof that $\sum_{n=1}^{\infty} \cos(n)$ Diverges

The convergence of any series $\sum_{n=1}^{\infty} a_n$ requires that the limit of its terms must be zero (The Test for Divergence). That is, if the series converges, then $\lim_{n\to\infty} a_n = 0$.

Analysis of the Sequence of Terms $a_n = \cos(n)$

We first establish that the sequence of terms $\{\cos(n)\}$ diverges, using a proof by contradiction.

1. Assume, for contradiction, that $\lim_{n\to\infty} \cos(n) = \alpha$.
2. Using the identity $\cos^2(n) + \sin^2(n) = 1$, this assumption implies that $\{\sin(n)\}$ must also converge to a limit β such that $\alpha^2 + \beta^2 = 1$.
3. We utilize the trigonometric identity $\cos(1) = \cos((n+1) - n) = \cos(n+1)\cos(n) + \sin(n+1)\sin(n)$.
4. Taking the limit as $n \to \infty$ on both sides yields the result $\cos(1) = \alpha^2 + \beta^2$.
5. Combining the results leads to the contradictory statement $\cos(1) = 1$.

Since $\cos(1) \neq 1$, the initial assumption must be false.
Conclusion 1: The sequence $\{\cos(n)\}$ **diverges**.

Application of the Test for Divergence

Since the terms of the series, $a_n = \cos(n)$, do not converge to zero, the series fails the necessary condition for convergence.
Final Conclusion: By the Test for Divergence, the series $\sum_{n=1}^{\infty} \cos(n)$ **diverges**.