0.0.1 Complex Dynamics

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Our research team spent the spring semester 2006 at the Fields Institute, Toronto, participating at the programme in Complex Dynamics, Hyperbolic Geometry and Laminations.

Most of the research in my group is in complex dynamics: the investigation of the dynamical system arising from iteration of a holomorphic function $f: \mathbb{C} \to \mathbb{C}$, and questions from geometry arising in this context. The particular strength of complex dynamics comes from the fact that complex analysis provides powerful tools for the study of dynamical systems, and the stiffness of complex analysis often allows to answer many questions successfully in terms of symbolic dynamics.

A prominent and important example is the dynamics of Newton's root-finding method: if g is a differentiable function (over the real or complex numbers), then the classical Newton method $N_g(z) = z - g(z)/g'(z)$ is often used to find zeroes of g by iterating N_g. Even though Newton's method is as old as analysis, and extremely simple, it still leaves many questions open, especially about the global structure of the basins of attraction for the various roots of g. There are a number of questions by Steve Smale (one of the fathers of the modern theory of dynamical systems); our work has helped to solve some of these questions.

Complex dynamics is an active research field that transcends Newton's method. For polynomials, there is a classical theory by Fatou and Julia (from the beginning of the 20-th century), by Douady and Hubbard (from the 1980's), by Thurston and others. The theory of transcendental functions is very different because many of the successful tools for polynomials do not apply there. Much of our research is related to bridging the gap from polynomials to transcendental functions. Even for very special transcendental functions such as exponential functions, there are still unsolved fundamental questions: one recent result of my group is the complete answer to an extension of an old question by Euler.

Highlights *Dynamics of Newton's Method and Rückert's Thesis.* One focus of the research in our group is Newton's method as a dynamical system on the complex plane. Newton's method is very easy to use and converges extremely fast near (simple) roots, but its global dynamical properties were poorly understood, which made it difficult to use in practice with guaranteed success. Figure 1 illustrates the complicated structure of the attracting basins of the various roots for a polynomial of degree 7. Our general view is on understanding Newton's method as a dynamical system, not to compete with high-performance numerical algorithms. An survey on this point of view is appearing as [17].

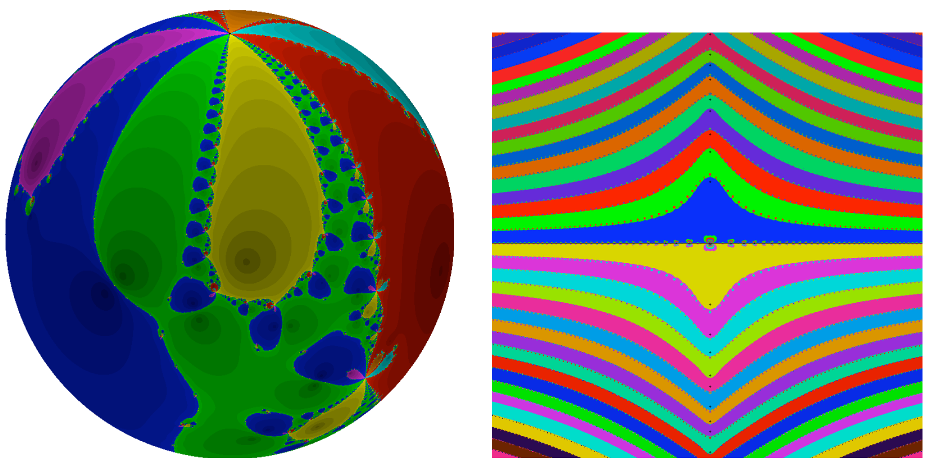

Figure 1: Left: Newton's method applied to a polynomial of degree 7 in a single complex variable. Different colors indicate domains of attraction for the different roots. Right: Newton's method applied to the Riemann ξ function: this is an entire transcendental function whose zeroes are exactly the non-trivial zeroes of the Riemann ζ function.

Recent results of our group include a good set of starting points for the Newton iteration: given only the degree d of a complex polynomial and a trivial normalization, we can specify an explicit set S_d of points so that starting Newton's method at these points is guaranteed to converge to all roots (joint with with Hubbard and Sutherland; Invent. Math. **146** (2001), 1–33); the number of points required is $O(d \log^2 d)$. It was an open question how many iterations are required to find all roots with prescribed precision. We now managed to give an efficient bound on this number: it is essentially cubic in the degree d and thus quite efficient [16]. This made it possible to turn Newton's method for polynomials into an efficient algorithm with good complexity bounds.

We have made progress on extending these results to Newton's method for transcendental entire functions. A prime candidate is the Riemann ξ function, an entire function the zeroes of which are exactly the non-trivial zeroes of the famous ζ-function: see Figure 1 for an illustration and [16] for some initial results. Three publications [1, 2, 15] study fundamental properties of Newton's method for arbitrary entire functions; a number of interesting similarities as well as differences to the polynomial case were discovered.

Finally, the difficult features of the Newton dynamics call for a classification of all possibilities, as proposed by Smale. This was achieved in [14] in an important special case. This result guides the way to a complete classification of (hyperbolic) Newton maps of polynomials; this is currently work in progress. An explicit classification of rational maps as dynamical systems is an important topic of current research; Newton maps would be the first such family (other than polynomials) that admits such a classification. The study of Newton's method was the topic of the PhD thesis of Johannes Rückert.

Symbolic Dynamics of Quadratic and Cubic Polynomials and Kaffl's thesis. Another focus of research in our group was symbolic dynamics related to iteration theory of polynomials. Polynomial dynamics is particularly successful because we have a variety of methods to describe their dynamics in terms of combinatorics and symbolic dynamics. With Bruin, we solved a number of the remaining open problems on the prototypical case of quadratic polynomials and started to collect old and new results in this area. Many of these results have been extended and generalized to "unicritical" polynomials of higher degrees by Alexandra Kaffl [10] in her PhD thesis [21], and jointly we finished this project into a monograph [18].

Alexandra Kaffl managed to extend a number of these results from the case of one-dimensional parameter space (which by now is well understood) towards parameter spaces of dimension at least 2 (a notoriously very difficult area). These results form another key part of her successful PhD thesis.

Further Research. A significant part of the research in our group is concerned with the dynamics of transcendental functions, either directly as dynamical systems [13, 5, 9, 6, 8, 11, 12], from the point of view of Newton's method [1, 2, 15], or in terms of Hausdorff dimension [4], including a survey article [7] (see also the research report of 2005).

Finally, some research was inspired by mathematicians or mathematical topics from other areas [3, 19], including a publication on distortion-reduced color representation on screens and printers [20] that resulted from discussions with people from imaging science.

Organization

1. **International Mathematical Olympiad (IMO) 2009.** I continue to chair the local organizing committee for this event that will bring excellent students from around 100 countries to Bremen.

2. **Deutsche Mathematik-Olympiaden e.V.** In May, I was elected into the German Mathematical Olympiad Council (Vorstand des Deutsche Mathematik-Olympiaden e.V.).

3. **Training to the German Team to the IMO 2006.** Since Bremen was chosen as the site of the International Mathematical Olympiad 2009, IUB has become one of the training sites for the German team to the annual IMO's. Together with Michael Stoll and Alexei Belov, we have held the annual final training sessions before the IMO.

4. **Studienstiftung des Deutschen Volkes.** Since 2002, I have served as "federführender Vertrauensdozent" for the "Studienstiftung des Deutschen Volkes"; this appointment was renewed in 2006.

Collaborations

1. *Univ. of Surrey, UK*
   Henk Bruin and Alexandra Kaffl
   Symbolic dynamics of quadratic polynomials, joint monograph

2. *Cornell University, NY/USA; Kyoto University, Japan*
   John Hubbard and Mitsuhiro Shishikura
   Thurston theory for postsingularly finite exponential maps

3. *University of Liverpool*
   Lasse Rempe
   Dynamics of transcendental functions, in particular complex exponential maps

4. *Technische Universität Berlin*
   Markus Müller
   Summation with non-integer number of terms

Grants

1. *State of Bremen*: Continued Support for Preparation of IMO 2009. More funding expected 2007–2009 by the Federal Republic of Germany and further sponsors.

2. *Konrad-Adenauer-Stiftung*: support for graduate student Alexandra Kaffl during her entire graduate education and in particular for her stay at the Fields Institute, Toronto.

Awards, Prizes, New Grants

1. *Fields Institute*, Toronto, January–May 2006: membership as senior researcher and travel support for Dierk Schleicher; full support for graduate student Nikita Selinger; partial support for graduate student Johannes Rückert

2. *Deutscher Akademischer Austauschdienst*: support for graduate Johannes Rückert during his stay at the Fields Institute, Toronto

3. *European Union, Marie Curie Research Training Network FP6*: "Conformal structures and dynamics" (CODY), network of 10 European countries, member of the German node and coordinator of IUB involvement; total funding volume: 2.7 Mio Euro.

4. *European Science Foundation*: ESF Research Networking Programmes "Harmonic and complex analysis and its applications", network of 10 European countries; member of the steering committee and coordinator of the German node. Total funding volume: 855 k Euro (funding to be confirmed).

5. *Mathematisches Forschungszentrum Oberwolfach*: co-organizer of a conference "Trends and Developments in Complex Dynamics" (with Lyubich/Stony Brook, Petersen/Roskilde and Smillie/Cornell); conference proposal accepted, expenses sponsored by Oberwolfach (approx. 10 k Euro).

6. *Centro Internazionale Matematico Estivo* in Cetraro, Cosenza, Italy, invited to offer course at summer school in 2008.

7. *Bundeswettbewerb "Jugend forscht":* high school student Simon Schmitt from Bremen participated at the German youth science fair "Jugend forscht 2006". First prize at the Bremen state level, federal award for the best interdisciplinary project, (Bundessieger-Sonderpreis für die beste interdisziplinäre Arbeit), sponsored by federal minister for education and research, Dr. Annette Schavan. Advisor and mentor of this project was D. Schleicher.

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[6] Markus Förster, Dierk Schleicher, Lasse Rempe, *Classification of Escaping Exponential Maps* (2004). ArXiv math.DS/0311427.

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[7] Dierk Schleicher, *Hausdorff dimension, its properties, and its surprises. American Mathematical Monthly*, to appear. ArXiv math.DS/0505099.

[8] Lasse Rempe, Dierk Schleicher, *Combinatorics of bifurcations in exponential parameter space*, ArXiv math.DS/0408011. Invited for a volume of Cambridge University Press in memory of Noel Baker (Phil Rippon, ed).

[9] Günter Rottenfußer, *Escaping Points of the Cosine Family*, ArXiv math.DS/0403012. To appear in a volume of Cambridge University Press in memory of Noel Baker (Phil Rippon, ed.).

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PhD Theses

[21] Alexandra Kaffl, *Hubbard trees and kneading sequences for unicritical and cubic polynomials*. PhD thesis, International University Bremen (2006). External Reviewers: Milnor (Stony Brook), Rees (Liverpool).

[22] Johannes Rückert, *Newton's method as a dynamical system*. PhD thesis, International University Bremen (2006). External Reviewer: Lyubich (Stony Brook and Toronto).