UNIT - I

Time Series:

Set of data recorded at regular times.

Forecasting:

Process of predict future values of data based on what happened before.

NATURE OF TIME SERIES DATA:

⇒ Johnson & Johnson Quarterly Earnings

* 84 quarters (21 years) measured from first quarter of 1960 to last quarter of 1980.

⇒ Global Warming

* Globalmean land-ocean temparature index from 1880 to 2015 with base period 1951 - 1980.

⇒ Speech Data

* Collect data in Milliseconds (1000 points)
* Spectral Analysis

⇒ Dow Jones Industrial Average

* Data collected every month on performance of company
* Financial data - the daily returns of Dow Jones Industrial Average (DJIA) from April 20, 2006 to April 20, 2016.

⇒ El Nino and Fish population (increase in temp of ozone)

⇒ fMRI Imaging

⇒ Earthquakes and Explosion

FMRI - Functional MRI - analysis activities of nerves specially brain every 32 seconds collects the data.

TIME SERIES STASTISTICAL MODELS :

⇒ Continuous Data to Discrete Data

1) White Noise Model
2) Moving average Model
3) Auto Regression
4) Random Walk with Drift
5) Noise and signals

WHITE NOISE :

⇒ Uncorrellated random Variable with mean = 0 and Variance = $\sigma^2 w$.

$$W_t \sim Wn(0, \sigma^2 w)$$

⇒ White Independent Noise : (iid Noise)

$$W_t \sim iid(0, \sigma^2 w)$$

[iid - Identical Independent distribution]

⇒ Gaussian White Noise :

$$W_t \sim iid\,N(0, \sigma^2 w)$$

MOVING AVERAGE MODEL :

$$V_t = \frac{1}{3}(w_{t-1} + w_t + w_{t+1})$$

AUTOREGRESSION :

$$y = B_0 + B_1 x_1 + \ldots$$

$$x_t = x_{t-1} - 0.9 x_{t-2} + w_t$$

RANDOM WALK WITH DRIFT :

$$x_t = \overset{\rightarrow drift}{\delta} + x_{t-1} + w_t$$

δ - drift

Suppose $\delta = 0 \rightarrow$ Random Walk

$$t = 1, 2 \ldots\ldots$$

$$x_1 = \delta + x_0 + w_1$$

$$x_2 = \delta + x_1 + w_2$$

$$x_2 = \delta + (\delta + x_0 + w_1) + w_2$$

$$= 2\delta + x_0 + w_1 + w_2$$

$$= 2\delta + x_0 + \sum_{i=1}^{\infty} w_i$$

$$x_t = \delta t + \sum_{j=1}^{t} w_j$$

NOISE AND SIGNALS:

Noise Analysed by sign wave form

$$x_t = 2\cos\left(2\pi \frac{t+15}{50}\right) + w_t$$

SNR → signal Noise Ratio

⇒ Used in Spectral analysis

MEASURE OF DEPENDENCE:

* Mean
* Autocovariance

Two Measures of Mean:

(i) For discrete

$$\frac{\Sigma f X}{n}$$

(ii) For continuous

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot fx \cdot dx$$

Uses Of Mean

(i) Central tendency

(ii) Measures of Dispersion

Mean for Statistical Models:

(i) White Noise:

$$\mu = E(w_t) = 0$$

(ii) Moving average Model:

$$V_t = \frac{1}{3}(w_{t-1} + w_t + w_{t+1})$$

$$\mu = E(V_t) = \frac{1}{3} E[w_{t-1} + w_t + w_{t+1}]$$

$$= \frac{1}{3}[E(w_{t-1}) + E(w_t) + E(w_{t+1})]$$

$$\mu = 0$$

(iii) Auto Regression:

$$x_t = x_{t-1} - 0.9 x_{t-2} + w_t$$

$$\mu = E(X_t) = E(X_{t-1}) - 0.9 E(X_{t-2})$$

(iv) Random Walk with Drift

$$x_t = \delta_t + \Sigma w_t$$

$$E(x_t) = \delta_t + \Sigma E(w_t)$$

$$\mu = \delta_t$$

V) Noise and Signals:

$$x_t = 2\cos\left[2\pi \frac{t+15}{50}\right] + w_t$$

$$\mu = E(X_t) = E\left[2\cos\left(2\pi \frac{t+15}{50}\right) + w_t\right]$$

$$\mu = 2\cos 2\pi \frac{t+15}{50}$$

Autocovariance:

Autocovariance is the measure of linear dependence between two points on the same series at different times.

$$\gamma_x(s,t) = \mathrm{cov}(x_s, x_t) = E((x_s - \mu_s)(x_t - \mu_t))$$

$$\mathrm{Var}(X) = E((X - \mu y^2))$$

$$X_s = X_t$$

$$E(X - \mu^2)$$

$$s = t$$

Properties :-

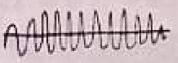
$\gamma_x (s,t) = 0$

$\gamma_x (t,t) = Var(x_t) - \delta^2$ zigma square (variance)

Nature in time series

* Very Smooth → t & s are far apart
  → γ is large

* choppy → Large Separation
  → γ is really zero

Choppy -  eg: ECG, sudden shock

Autocovariance in Statistical Models :

(i) White Noise :

$s = t$
$s \neq t$

$$\gamma_{s,t} = E\left[(w_s - \mu_s)(w_t - \mu_t) \right]$$

$$\gamma_w (s,t) = \begin{cases} \sigma^2 w , & s = t \\ 0 , & s \neq t \end{cases}$$

(ii) Moving average Model:

$$V_t = \frac{1}{3} (w_{t-1} + w_t + w_{t+1})$$

$$\gamma_{V_t} (s,t) = Cov (V_s , V_t)$$

$$= cov \left(\tfrac{1}{3} \left(w_{s-1} + w_s + w_{s+1} \right) , \tfrac{1}{3} (w_{t-1} + w_t + w_{t+1}) \right)$$

$$= \frac{1}{9} cov \left[(w_{s-1} + w_s + w_{s+1}) , (w_{t-1} + w_t + w_{t+1}) \right]$$

3 Variables Three models :

$s = t$, $s = t+1$, $s = t+2$

$s = t+3 \rightarrow 0$

(i) $s = t$

$$\gamma_v(s,t) = \frac{1}{9} \operatorname{cov}\left[(w_{t-1} + w_t + w_{t+1}), (w_{t-1} + w_t + w_{t+1})\right]$$

$$= \frac{1}{9}\left[\operatorname{cov}(w_{t-1}, w_{t-1}) + \operatorname{cov}(w_t, w_t) + \operatorname{cov}(w_{t+1}, w_{t+1})\right]$$

$$= \frac{1}{9}\left[\sigma^2 w + \sigma^2 w + \sigma^2 w\right]$$

$$= \frac{1}{3}\sigma^2 w$$

(ii) $s = t+1$

$$\gamma_v(s,t) = \frac{1}{9} \operatorname{cov}\left[(w_t + w_{t+1} + w_{t+2}), (w_{t-1} + w_t + w_{t+1})\right]$$

$$= \frac{1}{9}\left[\operatorname{cov}(w_t, w_t) + \operatorname{cov}(w_{t+1}, w_{t+1}) + \operatorname{cov}(w_{t-1}, w_{t-2})\right]$$

$$= \frac{1}{9}\left[\sigma^2 w + \sigma^2 w + 0\right]$$

$$= \frac{2}{9}\sigma^2 w$$

(iii) $s = t+2$

$$\gamma_v(s,t) = \frac{1}{9} \operatorname{cov}\left[w_{t+1} + w_{t+2} + w_{t+3}, w_{t-1} + w_t + w_{t+1}\right]$$

$$= \frac{1}{9}\left[\operatorname{cov}[w_{t+1}, w_{t+1}] + \operatorname{cov}(w_{t+3}, w_{t-1}) + \operatorname{cov}(w_{t+2}, w_t)\right]$$

$$= \frac{1}{9}\sigma^2 w$$

Final Results:

$$\gamma_v(s,t) = \begin{cases} \frac{1}{3}\sigma^2 w, & s = t \\ \frac{2}{9}\sigma^2 w, & s = t+1 \text{ (or) } |s-t| = 1 \\ \frac{1}{9}\sigma^2 w, & s = t+2 \text{ (or) } |s-t| = 2 \\ 0, & |s-t| > 2 \end{cases}$$

(iii) Random Walk With Drift

autocovariance → separation / lag

$$x_t = \sum_{i=1}^{\infty} w_i$$

$$x_s = \sum_{j=1}^{\infty} w_j$$

$$\gamma_x(s,t) = \text{cov}(x_s, x_t)$$

$$= \text{cov}\left(\sum_{j=1}^{\infty} w_j, \sum_{i=1}^{\infty} w_i\right)$$

$$\sum_{n}^{m} w_i = w_1 + w_2 + w_3 + \cdots + w_m = m\sigma^2 w$$

$$\sum^{n} w_j = \underbrace{w_1 + w_2}_{\sigma^2 w} + w_3 + \cdots + w_n = n\sigma^2 w$$

$$= \text{cov}\left(\sum_{j=1}^{\infty} w_j, \sum_{i=1}^{\infty} w_i\right)$$

$$= \min(s,t)\,\sigma^2 w$$

Autocorrelation Function :

→ single time series

→ to Measure the linear Predictability

Defined by $$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s), \gamma(t,t)}}$$

⇒ Range of Correlation (-1 to 1)

⇒ correlation → relations (identity based on rank)

Cross correlation (or) cross covariance :

→ double time series

$$\gamma_{x,y}(s,t) = \text{cov}(x_s, x_t)$$

$$= E\left[(x_s - \mu x_s)(y_t - \mu y_t)\right]$$

(i) $$\rho_{x,y}(s,t) = \frac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s)\,\gamma_y(t,t)}}$$

Stationarity of time Series :-

Definition :-

Strictly stationary time series is one for which Probabilistic behaviour of every collection values.

$\{x_{t_1}, x_{t_2}, x_{t_3} \cdots x_{t_K}\}$ is Identical to

(ii) time shifted set.

$\{x_{t_1+h}, x_{t_2+h} \cdots x_{t_K+h}\}$

It may be correlated or lag

$h = s - t \rightarrow$ difference between two time samples.

$P(x_{t_1})$ and $p(x_{t_1}+h)$ both are Identical

$\Rightarrow p\{(x_{t_1} \leq c_1, x_{t_2} \leq c_2 \cdots P(x+K) \leq c_K)\}$

(i $\Rightarrow p\,(x_{t_1+h} \leq c_1, x_{t_2+h} \leq c_2 \cdots x_{t_K+h} \leq c_K)$

$h = 0, \pm 1, \pm 2 \cdots$

Weakly Stationarity :-

Weakly Stationary time series x_t is finite Variance process such that

(i) the mean value function μ_t is constant μ_t and does not depend on time t

(ii) Auto covariance function

$\gamma_x(s,t)$ depends on time s & t only through their difference $|s-t|$

07/01 Auto correlation for stationary time series

$$P(h) = \frac{\gamma(h+t, t)}{\sqrt{\gamma(h+t, h+t)\,\gamma(t,t)}}$$

$$= \frac{\gamma(h)}{\gamma(0)}$$

$$h = |s-t|$$

$$\gamma(h) = \text{cov}(x_{t+h}, x_t)$$

$$\gamma(h) = E\left[(x_{t+h} - \mu_{t+h})(x_t - \mu_t)\right]$$

Stationary Property for

(i) White Noise :

mean : $\mu_t = 0$

autocovariance are exist

$$\gamma(h) = \begin{cases} \sigma^2_w, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

Therefore white noise satisfies the conditions of stationary. Then it is stationary.

auto-correlation :

$$P(h) = \frac{\gamma(h)}{\gamma(0)}$$

condition 1 : $h = 0$

$$P(h) = \frac{\gamma(0)}{\gamma(0)} = 1$$

condition 2 : $h \neq 0$

$$P(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{0}{\gamma(0)} = 0$$

$$P(h) = \begin{cases} 1, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

(ii) Stationarity Moving average Model

mean : $\mu v_t = 0$

auto - covariance :

$$\gamma(h) = \begin{cases} \frac{1}{3}\sigma^2_w, & h = 0 \\ \frac{2}{9}\sigma^2_w, & h = 1 \\ \frac{1}{9}\sigma^2_w, & h = 2 \\ 0, & h > 2 \end{cases}$$

Therefore moving average model satisfies the conditions of stationarity.

auto correlation :

$$P(h) = \frac{\gamma(h)}{\gamma(0)}$$

$$P(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\gamma(0)}{\gamma(0)} = 1$$

$$P(h) = \frac{\gamma(h-1)}{\gamma(0)} = \frac{\frac{2}{9}\sigma^2 w}{\frac{1}{3}\sigma^2 w} = 2/3$$

$$P(h) = \frac{\gamma(h-2)}{\gamma(0)} = \frac{\frac{1}{9}\sigma^2 w}{\frac{1}{3}\sigma^2 w} = \frac{1}{3}$$

$$P(h) = \frac{\gamma(h-3)}{\gamma(0)} = \frac{0}{\frac{1}{3}\sigma^2 w} = 0$$

$$P(h) = \begin{cases} 1, & h = 0 \\ 2/3, & h = 1 \\ 1/3, & h = 2 \\ 0, & h > 2 \end{cases}$$

(iii) stationarity of random Walk

mean : $\mu_t = \delta t$ → violates the first condition

auto covariance : $\gamma(h) = \min(s, t)\, \sigma^2_w$

Random walk violates the first condition because mean function is a function of time t.

The autocovariance of random walk depends on time s or t (it does not depends through their difference s and t)

$\therefore$ Therefore random walk is not stationary.

19/01 **Trend Stationarity :-**

→ Partial behaviour of stationarity

$$x_t = \mu_t + y_t$$

$$x_t = \alpha + \beta + \underset{\text{stationary}}{y_t} \rightarrow ①$$

The mean value of time series not existing the stationary.

Two conditions :-

⇒ Mean condition not satisfied

⇒ Autocovariance condition is satisfied

Mean :

(i) $E(x_t) = E(\alpha + \beta + y_t)$

$$\mu_x = \alpha + \beta + \mu_y \rightarrow ②$$

μ_y is not independent of time

$\therefore$ The process is not a stationary (according to the first condition)

autocovariance :

$$\gamma_x(h) = cov(x_{t+h}, x_t)$$

$$= E\left((x_{t+h} - \mu_{t+h})(x_t - \mu_t)\right) \rightarrow ③$$

sub ① & ② in ③

$(\alpha + \beta + y_{t+h} - \alpha - \beta - \mu_{t+h})$

$$= E\left[(y_{t+h} - \mu_{t+h})(y_t - \mu_t)\right]$$

$$= cov(y_{t+h}, y_t)$$

$$= \gamma_y(h)$$

Therefore It adheres autocovariance property and having stationary behaviour linear trend. This behaviour is called trend stationarity.

eg: Price of chicken series

Joint stationarity:

Two time series x_t and y_t are said to be jointly stationarity, If they are stationary and thus class covariance function is a function only of lag h. (s-t)

Cross covariance:

$$\gamma_{xy}(h) = cov(x_{t+h}, y_t)$$

$$= \left[(x_{t+h} - \mu_x)(y_t - \mu_y) \right]$$

Cross correlation function of jointly stationarity:

x_t and y_t is defined as

$$\rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0)\, \gamma_y(0)}}$$

→ cross covariance

→ auto covariance

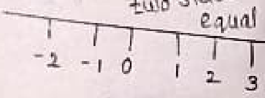
(i) cross correlation values lies between -1 to 1

$$-1 \le \rho_{xy}(h) \le 1$$

(ii) cross correlation does not symmetric about zero

$$\rho(xy)(h) \neq \rho_{xy}(-h)$$

→ symmetric → two sides equal



→ does not symmetric not equal

(iii) cross covariance function is symmetric about zero

$$\gamma_x(0) = \gamma_y(0)$$

$$\gamma_x(1) = \gamma_y(1)$$

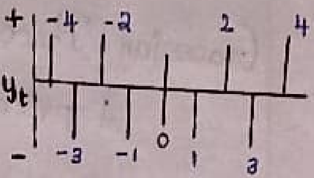
In joint stationarity auto covariance of x_t and y_t

equal at h all time points.

Prediction Using Cross Correlation

$$y_t = A x_{t-l} + w_t$$

l may be lead or lag
$(l > 0)$ $(l < 0)$



Cross Covariance :

$$\gamma_{xy}(h) = \text{cov}(y_{t+h}, x_t)$$

$$= \text{cov}(A x_{t+h-l} + w_{t+h}, x_t)$$

$$= \text{cov}(A x_{t+h-l}, x_t)$$

$$= A \,\text{cov}(x_{t+h-l}, x_t)$$

$$= A \gamma_x (h-l)$$

Cauchy - Schwartz

Largest absolute value is $\gamma_x(0)$

We have to prove $h = l$

$$\gamma(h-l) = \gamma(0)$$

$$h - l = 0$$

$$\boxed{h = l}$$

When $h = l$ it becomes autocovariance function of x_t

Linear Process

Linear process x_t is defined by combinations of white noise variates.

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$$

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \alpha$$

Auto Covariance

$$\gamma_x(h) = \sigma^2_w \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}$$

$$\text{cov}(x_{t+h}, x_t)$$

$$\text{cov}\left(\mu + \sum_{j=-\infty}^{\infty} \psi_i w_{t+h-j}, \mu + \sum_{j=-\infty}^{\infty} \psi_i w_{t-j}\right)$$

$cov\ (\psi_1 w_t, \psi_1 w_t)\ (\psi_2 w_{t+1}, \psi_2 w_{t+1})$

$$\psi \sigma^2 w + \psi_2 \sigma^2 w \quad \sigma^2 w \; A_0 x$$

$$= \sigma^2 w\ (\psi_1 + \psi_2)$$

$$= \sigma^2 w, \ \sum_{i=1}^{2} \psi_i$$

Gaussian Process:

A process $\{x_t\}$ is said to be Gaussian if the n dimensional Vectors $x = \{x_{t_1}, x_{t_2} \dots x_{t_n}\}$

For every collection of distinct time points $t_1, t_2 \dots t_n$ and every positive Integer n, have multi variated normal distribution.

Standard Normal Variate $\dfrac{e^{-(x-\mu)}}{2\sigma^2}$

Multivariate Normal Distributive function:

$$f(x) = (2\pi)^{-n/2} \, |\Gamma|^{-1/2} \exp\left\{ \tfrac{1}{2} (x-\mu)' \Gamma^{-1} (x-\mu) \right\}$$

$$Var(x) = \Gamma \{ \gamma(t_i, t_j),\ i, j = 1, 2 \dots n \}$$

Γ - Gamma

Exploratory Data Analysis:

→ Make non-stationary into stationary terms

→ Removal of trends

Simple linear regression has one dependent & one Independent variable

Trend Stationarity Model

$$x_t = \underbrace{\mu_t}_{\text{trend}} + y_t \rightarrow \text{stationary}$$

⇒ Suppose strong trend violates the behaviour of stationary.

⇒ For removal of trend we use simple linear regression.

$$\mu_t = \beta_0 + \beta_1 t$$

If the trend is fixed, We use this equation

$\Rightarrow$ The trend depends on time

OLS $\rightarrow$ Ordinary Least Square method to find B_0, B_1 values

$$x_t = B_0 + B_1 t + y_t$$

Random Walk with drift:-

$$\mu_t = \mu_{t-1} + \delta_t + w_t$$

Difference Equation:-

$$x_t = \mu_t + y_t$$

$$x_{t-1} = \mu_{t-1} + y_{t-1}$$

$$x_t - x_{t-1} = (\mu_t - \mu_{t-1}) + (y_t - y_{t-1})$$

$$= \mu_{t-1} + \delta + w_t - \mu_{t-1} + y_t - y_{t-1}$$

$$x_t - x_{t-1} = \delta + w_t + (y_t - y_{t-1})$$

When trend is fixed in difference equation

$$\mu_t = B_0 + B_1 t$$

$$\mu_{t-1} = B_0 + B_1 (t-1)$$

Sub in $x_{t-1} = (\mu_t - \mu_{t-1}) + (y_t - y_{t-1})$

$$= (B_0 + B_1 t - B_0 - B_1 (t-1)) + (y_t - y_{t-1})$$

$$= B_1 + y_t - y_{t-1}$$

Shift Operators:

$\Rightarrow$ Back shift, Forward shift

Back shift Operator:

Back shift Operator is defined by $Bx_t = x_{t-1}$

It is extended to $B^2 x_t = x_{t-2}$

$$B x_{t-1} = x_{t-2}$$

$$B^2 x_t = B \cdot B x_t \Rightarrow B x_{t-1} = x_{t-2}$$

$$B^3 x_t = B(B^2 x_t) = B(x_{t-2}) = x_{t-3}$$

$$\boxed{B^K x_t = x_{t-K}}$$

Forward shift Operator :. [Inverse of back shift]

$$x_t = B^{-1} B x_t = B^{-1} x_{t-1}$$

$$\nabla x_t = (1 - B) x_t \qquad \boxed{\because B^{-1} B = 1}$$

Finding Second order difference :.

$$\nabla^2 x_t = (1 - B)^2 x_t$$

$$= (1 - 2B + B^2) x_t$$

$$= x_t - 2B x_t + B^2 x_t$$

$$= x_t - 2x_{t-1} + x_{t-1}$$

$$\boxed{\nabla^2 x_t = (x_t - x_{t-1}) - (x_{t-1} - x_{t-2})}$$

1 st order difference → Elimination or removes the linear trend

2 nd order difference → Quadratic trend

n th order difference → n th order trend

Smoothing :.

⇒ Long term trend

⇒ Seasonal Componente

Moving average :.

$$m_t = \sum_{j=-K}^{K} a_j x_{t-j}$$

Types of smoother :.

* Kernel Smoother
* Lowess smoother
* spline smoother

Simple Kernel Smoother :-

$$m_t = \sum_{i=1}^{n} w_i(t)\, x_i$$

$$w_i(t) = K\left(\frac{t-i}{b}\right) \Big/ \sum_{j=1}^{n} \left(\frac{t-j}{b}\right)$$

K ( ) is a kernel function

Example :-

$$K(z) = \frac{1}{\sqrt{2\pi}}\, e^{(-z^2/2)}$$

↓

Gaussian

LOWESS :-

LOWESS → Locally Weighted scatterplot smoothing

Weighted may be nearest neighbour regression coefficients

Knots → All time series elements split into small intervals

Smoothing Splines :-

↳ Polynomial regression

$$x = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \cdots \beta_n x_n$$

In terms of time,

$$m_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \cdots \beta_n t^n$$

General equation :

$$x_t = \underset{\text{cubic splimes}}{\overset{\text{Polynomial Regression}}{m_t}} + w_t$$

Using cubic polynomial.

$$m_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$$

We use OLS method
ordinary least square

K intervals → Knots

↳ $[t_0 = 1, t_1]$

$[t_{1+1}, t_2]$

⋮

$[t_{K+1}, t_K = n]$

When we apply cubic polynomials for each samples, then it is called cubic splimes.

Degrees of Smoothness :

$$\sum_{t=1}^{n} [x_t - m_t]^2 + \overset{\text{degree of smoothness}}{\lambda} \underbrace{\int (m''_t)^2 \, dt}_{\text{splime driver}}$$

$m_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$

$m'_t = 0 + \beta_1 + 2\beta_2 t + 3\beta_3 t^2$

$m''_t = 0 + 2\beta_2 + 6\beta_3 t$

$m''_t = 2\beta_2 + 6\beta_3 t$ → accelarating / decelerating

(i) $\lambda = 0$

$x_t = m_t$

⇒ No smoothing

⇒ choppy ride

} Comparing these two
no smooth to very smooth

(ii) $\lambda = \infty$

check accelerating / decelarating

$m_t \to 0$

⇒ Constant velocity

⇒ Very smooth

When λ value is large, the smoothing will be good

Unit - II

Auto Regressive Model :

Auto regressive model order p is in the form of

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \phi_3 x_{t-3} \cdots \phi_p x_{t-p} + w_t$$

where x_t is an stationary white noise is mean of zero and $\phi_1 . \phi_2, \phi_3$ is an coefficients (or) constants.

$$(X_t - \mu) = \phi(x_{t-1} - \mu) + \cdots \phi(X_{t-\mu} - \mu) + w_t$$

$$X_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots \phi_p x_{t-p} + w_t \rightarrow (2)$$

$$\alpha = \mu(1 - \phi_1 + \phi_2 + \cdots \phi_p)$$

AR(P) Model in terms of Back shift operator :

$$(1 - \phi B - \phi_2 B^2 \cdots \phi_p B^p) X_t = w_t$$

Derivations :

$$BX_t = X_{t-1} \qquad B^n X_t = X_{t-n}$$

$$X_t - \phi_1 . X_{t-1} - \phi_2 X_{t-2} \cdots \phi_p X_{t-p} = w_t$$

$$X_t = \phi_1 Bx_t - \phi_2 B^2 x_t \cdots \phi_p BX_t = w_t$$

$$X_t(1 - \phi_1 B + \phi_2 B^2 + \cdots - \phi_p B^p) = w_t \rightarrow (3)$$

Auto Regressive Operator :

Auto Regressive Operator is defined by

$$\phi(B) = 1 - \phi_1 B^1 + \phi_2 B^2 \cdots \phi_p B^p$$

(3). can be Written in

$$P(B) X_t = w_t \rightarrow (4)$$

AR(1) model :- $\rightarrow$ Stationary solution model

$\rightarrow$ Explosive and casual

$\rightarrow$ get large values

$$X_t = \phi_1 X_{t-1} + w_t \rightarrow (5)$$

Iterating Backward in (5).

$$X_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t$$

$$\Rightarrow X_t = \phi^2 X_{t-2} + \phi w_{t-1} + w_t \rightarrow (2)$$

$\vdots$

K

$$\boxed{X_t = \phi^K X_{t-K} + \sum_{j=0}^{K-1} \phi^j w_{t-j}} \rightarrow (6)$$

$|\phi| < 1$ & Substitute $(Var(X_t)) < \infty$

AR (1) model can be represented in L.P

$$\boxed{X_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}} \rightarrow \text{AR(1) model} \rightarrow (7)$$

These was an stationary solution of the given model.

Let proof AR(1) is stationary :- find the

(i) mean

AR(1) process in stationary :- (ii) autocovariance

$$E(X_t) = E\left(\sum_{j=0}^{\infty} \phi \, w_{t-j}\right)$$

$$= \sum_{j=0}^{\infty} \phi_j \, E(w_{t-j}) = 0$$

Auto Covariance :-

$$\gamma(h) = cov(X_{t+h}, X_t)$$

$$= cov\left(\sum_{j=0}^{\infty} \phi_j \, w_{t+h-j}, \sum_{j=0}^{\infty} \phi_j \, w_{t-j}\right)$$

$$E\left[\left(w_{t+h} + \phi w_{t+h-1} \cdots \phi_h w_t + \phi_{h+1} w_{t-1} +\right)\left(w_t + \phi w_{t-1} + \phi w_{t-2} \cdots\right)\right]$$

$$= \frac{\delta^2 w \phi h}{1-\phi_2} \qquad \boxed{\phi_0 = 1}$$

$$\left[(1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 \ldots\right]$$

$$\frac{\delta^2 w \sum_{j=0}^{\infty} \phi^{h+j} = \phi j}{\delta^2 w \phi h \le \phi^2 j}$$

$$= \delta^2 w \phi h / 1 - \phi^2 j$$

$$= \frac{\delta^2 w \phi h}{1-\phi_2} \rightarrow 8$$

Auto Correlation for Regression Equation :-

$$p(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\delta^2 w \phi_h / (1-\phi_2)}{\delta^2 w \phi_0 (1-\phi_2)}$$

$$p(h) = \phi h$$

$\therefore$ AR (1) process is stationary

28/ Explosive Models and Casuality Models of AR (1) process :-

Explosive → Large values

Explosive Model :-

The Model is explosive when AR (1) model is stationary with magnitude of $|\phi| > 1$ in other words the model is called Explosive because the values of time series quickly become large in magnitude.

Casual Model (or) Cause :-

When an process doesn't dependent on future values, then the process is called casual in other words an causual process is AR (1) process which $|\phi| < 1$

Every Explosion has a cause :-

Consider the explosive model

$$X_t = \phi_{x-t} + w_t \quad \text{with} \quad |\phi| > 1$$

Mean : $E(X_t) = 0$

Auto covariance :- $\gamma(h) = \dfrac{\delta^2 w \, \phi^{-2} \phi^{-h}}{(1-\phi)^{-2}}$

AR(1) is stationary

Based on equation 7, we can write the model $|1/\phi| < 1$

$$y_t = \phi^{-1} y_{t-1} + U_t \quad \& \quad |\phi| < 1$$

$$U_t \sim (0, \delta^2 w \, \phi^{-2})$$

Example :-

The explosion model $X_t = 2X_{t-1} + w_t$ with $\delta^2 w = 1$. Find the casual model of given equations.

Sol:

$$y = (2)^{-1} y_{t-1} + U_t$$

$$= \frac{1}{2} y_{t-1} + U_t \quad \text{with} \quad 6_v^2 = 6\phi^{-2}v$$

$$= 1(2)^{-2}$$

$$\sigma^2 v = 1/4$$

$$\boxed{y_t = \frac{1}{2} y_{t-1} + v_t \quad \text{with} \quad \sigma^2 v = 1/4}$$

Stationary Solution for AR(1) model Using backward Operator (B) :

We have to prove that $\psi_j = \phi^j$

From, def of AR(1) model in terms of Backward shift operator

$$\phi(B) x_t = w_t$$

$$(1 - \phi B) x_t = w_t \rightarrow ①$$

From eq ⑦,

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$$

Assume that $\psi_j = \phi^j$

$$x_t = \psi(B) w_t \rightarrow ②$$

$$\psi(B) = 1 - \psi_1 B + \psi_2 B^2 \dots$$

sub ② in ①,

$$(1 - \phi B) \psi(B) w_t = w_t$$

$$\Rightarrow (1 - \phi B) \psi(B) = 1$$

$$\Rightarrow (1 - \phi B)(1 - \psi_1 B + \psi_2 B^2 \dots \psi_j B^j) = 1$$

$$\Rightarrow 1 - \psi_1 B - \phi B + \psi_1 \phi B^2 + \psi_2 B^2 + \dots (\psi_j - \psi_j \phi) B^j = 1$$

$$\Rightarrow 1 + (\psi_1 - \phi) B + (\psi_2 - \psi_1 \phi) B^2 + \dots (\psi_j - \psi_j \phi) B^j = 1$$

expansion

$$\Rightarrow 1 + \psi_1 B + \psi_2 B^2 + \dots \psi_j B^j - \phi B - \psi_1 \phi B^2 - \psi_2 \phi B^2 \dots \dots \psi_j \phi B^j + \dots = 1$$

Matching the coefficients of B.

$$\Rightarrow \psi_1 - \phi = 0 \Rightarrow \psi_1 = \phi$$

$$\Rightarrow (\psi_2 - \psi_1\phi) = 0 \Rightarrow (\psi_2 - \phi^2) = 0$$

$$\Rightarrow \boxed{\psi_2 = \phi^2}$$

We have to prove every B is complex.

$\Rightarrow$ Multiply ① with $\phi^{-1} B$

$$\phi(B)\, x_t = w_t \rightarrow ①$$

$$\phi^{-1}(B)\, \phi(B)\, x_t = \phi^{-1}(B)\, w_t$$

$$x_t = \phi^{-1}(B)\, w_t$$

This is similar to, $\phi^{-1}(B) = 1 + \phi B + \phi^2 B^2 + \cdots \phi^j B^j$

$$x_t = \psi(B)\, w_t$$

$$\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \cdots$$

Consider z be a complex number,

Consider the polynomial

$$\phi(z) = 1 - \phi z$$

$$\phi^{-1}(z) = \frac{1}{1 - \phi z}$$

$$\phi^{-1}(z) = 1 + \phi z + \phi^2 z^2 + \cdots \phi^j z^j$$

Comparing $\phi^{-1}(B)$ and $\phi^{-1}(z)$,

$$\Rightarrow B = z$$

$$\Rightarrow B^2 = z^2$$

$$\vdots$$

$$\Rightarrow B^j = z^j$$

$\therefore$ Backward shift operator is a complex number.

MA (Moving average Model):

Moving average model of order q or MA(q)

is defined $x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots \theta_q w_{t-q}$

where $w_t \sim wn(0, \sigma^2 w)$ and $\theta_1, \theta_2 \ldots \theta_q$ are Parameters

Back ward shift form of MA Model:

$$x_t = \theta(B) w_t$$

Moving average operator:

It is defined by,

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots \theta_q B^q$$

We have to prove MA(1) is stationary:

$x_t = w_t + \theta_1 w_{t-1}$ (MA1 equation) $MA_2 \rightarrow$ two coefficien

↳ one coefficient

(i) Mean:

$E(\overset{x_t}{B}) = 0$ $\Rightarrow x_{t+h} = w_{t+h} + \theta_1 w_{t+h-1}$

(ii) autocovariance:

$\gamma(h) = cov(x_{t+h}, x_t)$

$= cov[(w_{t+h} + \theta_1 w_{t+h-1}), w_t + \theta_1 w_{t-1}]$

Two conditions for two coefficients (autocovariance)

$h=0, h=1$ → first Unit

i) $h=0$

$\gamma(h) = cov(w_t + \theta w_{t-1}, w_t + \theta w_{t-1})$

$= cov(w_t, w_t) + cov(\theta w_{t-1}, \theta w_{t-1})$

$= \sigma^2 w + \theta^2 \sigma^2 w$

$= (1+\theta^2)\sigma^2 w$

(ii) $h=1$

$\gamma(h) = cov(w_{t+1} + \theta w_t, w_t + \theta w_{t-1})$

$= cov(\theta w_{t-1}, w_t) + cov(w_{t+1}, w_{t-1})$

$= \theta \sigma^2 w$

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2 w, & h = 0 \\ \theta\sigma^2 w, & h = 1 \\ 0, & h > 1 \end{cases}$$

$$\begin{cases} 1, & h = 0 \\ \dfrac{\theta}{1+\theta^2}, & h = 1 \\ 0, & h > 1 \end{cases}$$

$$\boxed{\rho(h) = \frac{\gamma(h)}{\gamma(0)}}$$

09/02

Non Uniqueness Of MA(1) Model :.

$x_t = w_t + \frac{1}{5} w_{t-1}, \quad w_t \sim \text{iid } W(0, 25)$ (θ, $\sigma^2 w$)

$y_t = v_t + 5 v_{t-1}, \quad v_t \sim \text{iid } N(0, 1)$

Autocovariance for x_t model :.

$\theta = \frac{1}{5}, \quad \sigma^2 w = 25$

$1 + \frac{1}{25} = \frac{26}{25}$

$$\gamma(h) = \begin{cases} 1 + \left(\frac{1}{5}\right)^2 \cdot 25 = 26 & h = 0 \\ \frac{1}{5} \cdot 25 = 5 & h = 1 \\ 0 & h > 1 \end{cases}$$

Autocovariance for y_t model :.

$\theta = 5, \quad \sigma^2 w = 1$

$$\gamma(h) = \begin{cases} 26, & h = 0 \\ 5, & h = 1 \\ 0, & h > 1 \end{cases}$$

For x_t and y_t, autocovariance (auto correlation) are same. It is evident of non Uniqueness of MA Model.

Invertible of MA(1) Process :-

The Model with infinite AR components is called invertible MA Model.

While writing invertible model x_t and y_t is interchanged.

$$w_t = -\theta w_{t-1} + x_t$$

$$\Rightarrow w_t = \sum_{j=0}^{\infty} (-\theta)^j x_{t-j}, \text{ if } |\theta| < 1$$

ARMA Model:

Definition :-

A time series $\{x_t : t = 0, \pm 1, \pm 2 \ldots\}$ is ARMA (p, q) (order) if it is stationary and

$$x_t = (\underbrace{\phi_1 x_{t-1} + \phi_2 x_{t-2} \ldots \phi_p x_{t-p}}_{AR}) + (\underbrace{w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \ldots \theta_q w_{t-q}}_{MA})$$

The parameters p and q are orders of AR and MA respectively.

If X has a non zero mean μ

$\phi_p \neq 0$ and $\phi_q \neq 0$

$$x_t = \alpha + \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t + \ldots \phi_q w_{t-q}$$

$$\alpha = \mu (1 - \phi_1 - \phi_2 - \ldots \phi_p)$$

ARMA in terms of Backward shift operator:-

$$\phi(B) x_t = \theta(B) w_t$$

Problems:

1) Parameter redundant (repeated) Models:
2) Stationary AR Models that depend on the future (change to casual Models)
3) MA models that are not Unique

16/02/23

Forecasting

$$x_{1:n} \rightarrow x_{n+1:m}$$

→ MSE Predictor → one-step ahead Predictor
→ Linear → m-step ahead Predictor

Forecast: Def

Forecasting is the process of predicting the feature values of time series x_{n+m}, $m = 1, 2 \ldots$ Based on the collected data

$$x_{1:n} = \{x_1, x_2 \ldots x_n\}$$

Minimum Mean Squared error:

$$x^n_{n+m} = E(x_{n+m} \mid x_{1:n})$$

Linear Predictor:

$$x^n_{n+m} = \alpha_0 + \sum_{k=1}^{n} \alpha_k x_k$$

α_0, α_k are real numbers

One Step Predictor:

x_2^1, x_3^2

$$x^n_{n+m} = \alpha_0 + \sum_{k=1}^{n} \alpha_k x_k$$

$$x_2^{1} = \alpha_0 + \sum_{k=1}^{1} \alpha_k x_k$$

$$x_2^{1} = \alpha_0 + \alpha_1 x_1$$

$$x_3^{2} = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2$$

BLP → Best Linear predictor

The Linear predictor that minimize the mean squared error is called BLP.

Given data $x_1, x_2 \dots x_n$, the BLP of $x^n_{n+m} = \alpha_0 + \sum_{k=1}^{n} \alpha_k x_k$ is formed by solving expectation of

$$E\left[\left(x_{n+m} - x^n_{n+m}\right) x_k\right] = 0 \rightarrow ①$$

Where $k = 0, 1 \dots n$

where initial condition $x_0 = 1$

① is called predictor equations

$$\varphi = E\left(x_{n+m} - \sum_{k=0}^{n} \alpha_k x_k\right)^2 \rightarrow ②$$

$$E(x_k) = \mu \qquad \frac{\partial \varphi}{\partial \alpha_k}, \; k = 0, \dots n$$

when $k = 0$

$$E\left[\left(x_{n+m} - x^n_{n+m}\right) x_k\right] = \mu$$

From eqn ①

$$\Rightarrow E\left(\left(x_{n+m} - x^n_{n+m}\right) x_0\right) = 0$$

$(x_0 = 1)$

$$\Rightarrow E\left(x_{n+m} - x^n_{n+m}\right) = 0$$

$$\Rightarrow E\left(x_{n+m}\right) - E\left(x^n_{n+m}\right) = 0$$

$$\Rightarrow E\left(x_{n+m}\right) = E\left(x^n_{n+m}\right) = \mu \rightarrow ③$$

$$x^n_{n+m} = \alpha_0 + \sum_{k=1}^{n} \alpha_k x_k$$

We can take expectation

$$E(x^n_{n+m}) = E\left(\alpha_0 + \sum_{K=1}^{n} \alpha_K x_K\right)$$

From (3), $\mu = \alpha_0 + \sum_{K=1}^{n} \alpha_K \mu$

$$\alpha_0 = \mu\left(1 - \sum_{K=1}^{n} \alpha_K\right)$$

Sub α_0 in Linear Predict equation

↳ BLP:

$$x^n_{n+m} = \mu\left(1 - \sum_{K=1}^{n} \alpha_K\right) + \sum_{K=1}^{n} \alpha_K x_K$$

BLP form:

$$x^n_{n+m} = \mu + \sum_{K=1}^{n} \alpha_K (x_K - \mu)$$

Forecasting ARMA Process:

Suppose it is Invertible and Casual

(i) Casual:

$$x_{n+m} = \sum_{j=0}^{\infty} \psi_j w_{n+m-j} \quad , \psi_0 = 1$$

(ii) Invertible:

$$w_{n+m} = \sum_{j=0}^{\infty} \pi_j x_{n+m-j} \quad , \psi_0 = 1$$

$$(i) \Rightarrow \tilde{x}_{n+m} = \sum_{j=0}^{\infty} \psi_j \tilde{w}_{n+m-j}$$

$$= \sum_{j=m}^{\infty} \psi_j w_{n+m-j}$$

$$\tilde{w}_t = E(w_t / x_n \dots x_0 \dots) = \begin{cases} 0 , & t > n \\ w_t , & t \leq n \end{cases}$$

(ii) →

$$E(w_{n+m}) = E\left(\sum_{j=0}^{\infty} \pi_j x_{n+m-j}\right)$$

$$0 = \tilde{x}_{n+m} + \sum_{j=1}^{\infty} \tilde{x}_{n+m-j}$$

$$\tilde{x}_{n+m} = - \sum_{j=1}^{\infty} \Pi_j \tilde{x}_{n+m-j}$$

$$= - \left(\sum_{j=1}^{m-1} \Pi_j \tilde{x}_{n+m-j} + \sum_{j=m}^{\infty} \Pi_j \tilde{x}_{n+m-j} \right)$$

$$\tilde{x}_t = x_t, \quad t \leq n$$

One step ahead predictor:.

$$x_{n+m} - \tilde{x}_{n+m} = \sum_{j=0}^{m-1} \psi_j \cdot w_{n+m-j}$$

$$P_{n+m} = E \left(x_{n+m} - \tilde{x}_{n+m} \right)^2$$

$$= \sigma^2_w \sum_{j=0}^{m-1} \psi_j^2$$

$E(w_{n+m-j}, w_{n+m-j})$
$= \sigma^2 w$

Estimator:.

- ⇒ yule - walker → AR
- → method of moments - MA
- → MLE - for both

first order moments
↳ mean
2nd order momements
↳ Standard deviation

Random walk with drift
↳ not Stationary

ARMA → difference operator

ARIMA → $(\overset{\nearrow AR}{p}, d, q)$ ↖MA

↙ difference operator (∇^d)

How to build ARIMA model:

Pg. 144 Book

Step 1: plotting the Data (Plot or visualize) anamalise

Step 2. transforming the data (removing anomalyes)

Step 3: Identifying dependence order of the model (p, q, d) analyse

Step 4: parameter estimation $(\phi, \theta, \nabla, \psi)$ → estimated ACF, PCF → Partial auto correlation

Step5: diagnosis →

Step6: Model choice