Chapter 1 Fundamental Concepts

1.1 What Is a Graph?

Definition. A *graph* G is a triple consisting a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its *endpoints*.

Definition. A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints.

A *simple graph* is a graph having no loops or multiple edges.

When u and v are endpoints of an edge, they are *adjacent* and are *neighbors*. We write $u \leftrightarrow v$ for "u is adjacent to v".

We identify a simple graph by its vertex set and edge set, treating the edge set as a set of unordered pairs of vertices ($E \subseteq V \times V$) and writing $e = uv\,(= vu)$ for an edge e with endpoints u and v.

The identification of a graph, would be insufficient for multigraphs and digraphs.

Definition. The *complement* $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E\left(\overline{G}\right)$ if and only if $uv \notin E(G)$. A *clique* in a graph is a set of pairwise adjacent vertices. An *independet set* in a graph is a set of pairwise nonadjacent vertices.

Definition. A graph G is *bipartite* if $V(G)$ is the union of two disjoint (possibly empty) independent sets called *partites sets* of G.

Definition. The *chromatic number* of a graph G, written $\chi(G)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors. A graph G is *k-partite* if $V(G)$ can be expressed as the union of k (possibly empty) independent sets.

This generalizes the idea of bipartite graphs, which are 2-partite. Vertices given the same color must form an independent set, so $\chi(G)$ is the minimum number of independent sets needet to partition $V(G)$.

Definition. A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

Definition. A graph G is *connected* if each pair of vertices G belongs to a path; otherwise is *disconnected*.

Definition. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and the assignment of endpoints to edges in H is the same as in G. We then can write $H \subseteq G$ and say that "G contains H".