Chapter 1 The Basics

1.1 Graphs

Definition. A *graph* is a pair $G = (V, E)$ of sets such that $E \subseteq V \times V$. The elements of V are the *vertices* (or *nodes*, or *points*) of the graph G, the elements of E are its *edges.*

A graph with vertex set V is said to be a graph *on* V. The vertex set of a graph G is referred to as $V(G)$, its edge set as $E(G)$.

Definition. The number of vertices of a graph G is its *order*, written as $|G|$; its number of edges is denoted by $\|G\|$. Graphs are *finite*, *infinite*, *countable* and so, according to their order.

Definition. A vertex v is *incident* with an edge e if $v \in e$, then e is an edge at v. The two vertices incident with an edge are its *ends*, and an edge *joins* its ends.

An edge $\{x, y\}$ is usually written as xy (or yx). If $x \in X$ and $y \in Y$, then xy is an *X-Y edge.* The set of all X-Y edges in a set E is denoted by $E(X, Y)$. The set of all edges in E at a vertex v is denoted by $E(v)$.

Definition. Two vertices x, y of G are *adjacent*, or *neighbours*, if xy is an edge of G. Two edges $e \neq f$ are *adjacent* if they have an end in common. If all the vertices of G are pairwise adjacent, then G is *complete*. A complete graph on n vertices is a K^n; a K^3 is called a *triangle.*

Definition. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We say G and G' are *isomorphic*, and write $G \simeq G'$, if there exists a bijection $\varphi : V \to V'$ with $xy \in E \Leftrightarrow \varphi(x)\varphi(y) \in E'$. Such a map φ is called an *isomorphism.*

We do not distinguish between isomorphic graphs, and write $G = G'$ instead of $G \simeq G'$, and talk about *the* graph.

Definition. A class of graphs that is closed under isomorphism is called a *graph property.* A map taking graphs as arguments is called a *graph invariant* if it assigns equal values to isomorphic graphs.

For example, "containing a triangle" is a graph property: if G contains three pairwise adjacent vertices then so does every graph isomorphic to G.

The number of vertices and the number of edges of a graph are two simple graph invariants; the greatest number of pairwise adjacent vertices is another.

Definition. We set $G \cup G' := (V \cup V', E \cup E')$, and $G \cap G' := (V \cup V', E \cap E')$. If $G \cap G' = \varnothing$, then G and G' are *disjoint.* If $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G (and G a *supergraph* of G'), written as $G' \subseteq G$.

If $G' \subseteq G$ and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is an *induced subgraph* of G; we say that V' *induces* or *spans* G' in G and write $G' =: G[V']$. If $U \subseteq V$ is any set of vertices, then $G[U]$ denotes the graph on U whose edges are precisely the edges of G with both ends in U. If H is a subgraph of G, not necessarily induced, we write $G[H] \equiv G[V(H)]$. Finally, $G' \subseteq G$ is a *spanning subgraph* of G is V' spans all of G, i.e. if $V' = V$.

If U is any set of vertices (usually of G), we write $G - U$ for $G[V \setminus U]$. In other words, $G - U$ is obtained from G by deleting all the vertices in $U \cap V$ and their incident edges. We write $G - G'$ instead of $G - V(G')$. For $F \subseteq V \times V$ we write

$G - F := (V, E \setminus F)$ and $G + F := (V, E \cup F)$. We call G *edge-maximal* with a given graph property if G itself has the property but no graph $G + xy$ does, for non-adjacent vertices $x, y \in G$.
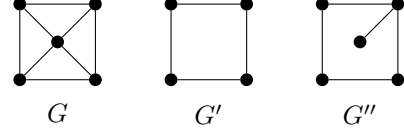
A graph G with subgraphs G' and G'': G' is an induced subgraph of G, but G'' is not.

More generally, when we call a graph *minimal* or *maximal* with some property but have not specified any particular ordering, we are referring to the subgraph relation. When we speak of minimal or maximal sets of vertices or edges, the reference is simply to set inclusion.

Definition. If G and G' are disjoint, we denote by $G * G'$ the graph obtained from $G \cup G'$ by joining all vertices of G to all vertices of G'. The *complement* $\overline{G}$ of G is the graph on V with edge set $V \times V \setminus E$. The *line graph* $L(G)$ of G is the graph on E in which $x, y \in E$ are adjacent vertices if and only if they are adjacent edges in G.

1.2 The degree of a vertex

Definition. Let $G = (V, E)$ a (non-empty) graph. The set of neighbours of a vertex $v \in G$ is denoted by $N_G(v)$ (or briefly $N(v)$). More generally, for $U \subseteq V$, the neighbours in $V \setminus U$ of vertices of U are called the *neighbours of* U, denoted $N(U)$.

Definition. The *degree* of a vertex $v \in G$, $d_G(v)$ (or $d(v)$ for short), is the number $|E(v)|$ of edges at v; which by definitino equals the number of neighbors of v, $|N_G(v)|$.

A vertex of degree 0 is *isolated.* The number $\delta(G) := \min\{d(v) | v \in V\}$ is the *minimum degree* of G, the number $\Delta(G) := \max\{d(v) | v \in V\}$ is the *maximum degree* of G.

If all the vertices of G have the same degree k, then G is *k-regular.*

The number $d(G) := \frac{1}{|V|} \sum_{v \in V} d_G(v)$, is the *average degree* of G.

Clearly we have that $\delta(G) \leq d(G) \leq \Delta(G)$. The average degree quantifies globally what is measured locally by the vertex degrees: the number of edges of G per vertex. Sometimes it will be convenient to express this ratio directly, as $\varepsilon(G) := |E|/|V|$. Which relates to the average degree:

$$|E| = \frac{1}{2} \sum_{v \in G} d(v) = \frac{1}{2} d(G) \cdot |V| \implies \varepsilon(G) = \frac{1}{2} d(G).$$

Proposition. *The number of vertices of odd degree in a graph is always even.*

If a graph has a large minimum degree (everywhere, locally, many edges per vertex) it also has many edges per vertex globally: $\varepsilon(G) = \frac{1}{2} d(G) \geq \frac{1}{2} \delta(G)$. Conversely its average degree may be large even with small minimum degree, in that case the vertices of large degree cannot be scattered among vertices of small degree, but concentrated in a region. Next proposition shows this region is a subgraph of G with average

degree no less than G and minimum degree greater than half his average degree.

Proposition. *Every graph G with at least one edge, has a subgraph H with $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$.*

1.3 Paths and cycles

Definition. A *path* is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \ldots, x_k\} \quad E = \{x_0 x_1, x_1 x_2, \ldots, x_{k-1} x_k\},$$

where the x_i are all distinct. Denote this path as $P = x_0 x_1 \ldots x_k$, and call it a *path from x_0 to x_k*. The vertices x_0 and x_k are *linked* by P and are called its *ends*; ther vertices $x_1, \ldots, x_{k-1}$ are the *inner* vertices of P. The number of edges of a path is its *length*, and the path of length k is denoted by P^k (where $P^0 = K^1$).

We denote for $0 \leq i \leq j \leq k$ the following subpaths of P

$$Px_i := x_0 \ldots x_i$$
$$x_i P := x_i \ldots x_k$$
$$x_i P x_j := x_i \ldots x_j$$

$$\mathring{P} := x_1 \ldots x_{k-1}$$
$$P\mathring{x}_i := x_0 \ldots x_{i-1}$$
$$\mathring{x}_i P := x_{i+1} \ldots x_k$$
$$\mathring{x}_i P \mathring{x}_j := x_{i+1} \ldots x_{j-1}$$

We use similar intuitive notation for the concatenation of paths; for example: if the union $Px \cup xQy \cup yR$ of the three paths is again a path, we may simply write $PxQyR$.

Definition. Given sets A, B of vertices, we call $P = x_0 \ldots x_k$ an *A-B path* if $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$. Two or more paths are *independent* if none of them contains an inner vertex of another.

Definition. Given a graph H, we call P an *H-path* if P is non-trivial and meets H exaclty in its ends. In particular, the edge of any H-path of length 1 is never an edge of H.

Definition. If $P = x_0 \ldots x_{k-1}$ is a path and $k \geq 3$, then the graph $C := P + x_{k-1} x_0$ is called a *cycle* (denoted as $C = x_0 \ldots x_{k-1} x_0$). The *length* of a cycle is its number of edges (or vertices); the cycle of length k is called a *k-cycle* and denoted C^k.

Definition. The minimum length of a cycle (contained) in a graph G is the *girth $g(G)$* of G; the maximum length of a cycle in G is its *circumference* (if G does not contain a cycle, we set the former to ∞, the latter to zero). An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle. Thus, an *induced cycle in G*, a cycle in G forming an induced subgraph, is one that has no chords.

If a graph has large minimum degree, it contains long paths and cycles:

Proposition. *Every graph G with $\delta(G) \geq 2$ contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$.*