Chapter 2 Getting Started

2.1 Insertion Sort

Sorting problem: Given a sequence of n numbers $(a_1, a_2, \ldots, a_n)$, produce a premutation $(a'_1, a'_2, \ldots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

We present a first solution to the sorting problem, the *insertion sort* algorithm, as a subroutine which takes as parameter an array $A[1..n]$ containing a sequence of length n to be sorted. When the procedure INSERTION-SORT is finished, it rearranges the elements within A so that they are sorted.

```
INSERTION-SORT($A$)
1  for $j = 2$ to $A.length$ do
2      $key = A[j]$
       // Insert $A[j]$ into the sorted sequence $A[1..j-1]$
3      $i = j - 1$
4      while $i > 0$ and $A[i] > key$ do
5          $A[i+1] = A[i]$
6          $i = i - 1$
7      end while
8      $A[i+1] = key$
9  end for
```

Loop Invariants and Correctness: In order to prove for correctness of some algorithms, we can use a *loop invariant*, which is a property of the state of the algorithm which holds during all iterations of the loop. We must show three things in order to prove that a loop invariant holds:

- **Initialization:** It is true prior to the first iteratin of the loop.
- **Maintenance:** If it is true before an iteration of the loop, it remains true before the next iteration.
- **Termination:** When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.

When the first two properties hold, we show that the loop invariant holds prior to every iteration of the loop. The third property helps us use the loop invariant to prove correctness (typically using the loop invariant along the condition which made the loop to terminate).

We can now give a loop invariant of the given INSERTION-SORT algorihtm: *At the start of each iteration of the **for** loop of lines 1 - 8 the subarray $A[1..j-1]$ consists of the elements originally in $A[1..j-1]$, but in sorted order.*

Let us see the loop invariant holds and see how it can prove correctness of the sorting algorithm.

- *Initialization:* Before the first iteration, at $j = 2$, the subarray is just $A[1]$ which trivially contains the elements of $A[1]$ in sorted order.
- *Maintenance:* Informally, the body of the **for** loop works by moving $A[j-1]$, $A[j-2]$, $A[j-3]$, and so on by one position to the right until it finds the proper position for $A[j]$. Which leaves the subarray $A[1..j]$ consisting in elements originally in $A[1..j]$ but in sorted order. Incrementing j for the next iteration of the loop preserves the loop invariant. (A more formal proof would require to prove another loop invariant for this **while** loop.)
- *Termination:* The loop terminates when $j > A.length = n$. Because each iteration increments j by one, we must have $j = n + 1$ at that time. Since the loop invariant holds at the termination time, we see that the array $A[1..n]$ consists of the elements originally in $A[1..n]$ but in sorted order. Hence the entire array A is sorted at the end, hence the algorithm is correct.

2.2 Analyzing Algorithms

Analyzing an algorithm means to predict the resource that the alorithm requires. Such resources may be memory, communication bandwidth, computer hardware, or computational time. Generally, by analyzing several algorihtms for a problem, we can identify a most efficient one (or discard inferior candidate algorithms). We will mostly study the computational time taken by algorithms in this book.

Model: Before we can analyze an algorithm, we must have a *model* of the implementation technology that we will use, including a model for the resources of that technology and their costs. Unless explicited, assume a generic one processor, random-access machine (RAM). That is, instructions executed one after another with no concurrent operations, where each instruction taking constant time (being the set of instroctions in the RAM model the commonly found in real computers: load, store, copy, arithmetic, conditional branch, subroutine call and return.

2.2.1 Analysis of Insertion Sort

The time taken by the INSERTION-SORT procedure depends on the input size, and also depending on how are arranged the item inside the array for two inputs of the same size. In general, running time will be a function of input size.

The notion of *input size* depends on the problem: it may be the number of items in the input (as in the array size in a sorting problem), or the total number of bits to represent input (like the problem of integer multiplication). Sometimes, it is more appropiate to describe the size of input with two parameters (or more), such as in a graph problem with number of edges and number of vertices.

The *running time* of an algorithm on a particular input is the time it takes to execute each primitive operation times the number of executions of those operations. We may assume that a constant amount of time is required to execute each line of pseudocode. That is, each execution of the ith line takes c_i time, where c_i is a constant.

Line Number	Cost	# of Executions
1	c_1	n
2	c_2	$n-1$
3	c_3	$n-1$
4	c_4	$\sum_{j=2}^{n} t_j$
5	c_5	$\sum_{j=2}^{n}(t_j - 1)$
6	c_6	$\sum_{j=2}^{n}(t_j - 1)$
8	c_8	$n-1$

We analyzed above the INSERTION-SORT procedure time cost of each statement and the number of times each statement is executed. For each $j = 2, 3, \ldots, n$, where $n =$

A.length, denote t_j be the number of times the **while** loop in line 5 is executed for that value of j. Then, the execution time of the algorithm is the sum of all the cost of each line times its number of executions:

$$\begin{aligned} T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j \\ + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1). \end{aligned}$$

Even for inputs of the same size, an algorithm's running time may depend on which input of that size is given. In this case, the *best case* occurs when the array is already sorted, since for each j we find that $A[i] \leq key$ in line 5 when $i = j-1$ initially, thus having $t_j = 1$. Which yields a running time:

$$\begin{aligned} T_{\text{best}}(n) &= c_1 n + c_2(n-1) + c_3(n-1) + c_4(n-1) + c_8(n-1) \\ &= (c_1 + c_2 + c_3 + c_4 + c_8)n - (c_2 + c_3 + c_4 + c_8). \end{aligned}$$

We thus can express this running time as $T_{\text{best}}(n) = an + b$ for some constants a, b; thus a linear function of n.

On the other side, if the array is in reverse order, we get the *worst case* of $t_j = j$ for each value of j. And using that $\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1$, and that $\sum_{j=2}^{n} (j-1) = \frac{n(n+1)}{2} - 1$, we can rearrange the terms as before to get $T_{\text{worst}}(n) = an^2 + bn + c$, for some constants a, b, c: a quadratic function of n.

We could also compute an *average-case* running time for a random input, and compute the expected value of the running time.

2.2.2 Order of Growth

It is the rate of growth or order of growth of the running time that interests us. We therefore consider only the most significative term as the size of input increases. Thus, the worst case running time of INSERTION-SORT has a rate of growth of n^2. We sa that it has a worst case running time of $\Theta(n^2)$. We usually consider one algorithm to be more efficient than another if its worst-case running time has a lower order of growth.

2.3 Designing algorithms

For insertion sort, we used an *incremental* approach: having sorted $A[1..j-1]$ we then insert $A[j]$ in the proper place, yielding the sorted subarray $A[1..j]$. We examine a different approach now:

2.3.1 Divide and Conquer

Many algorithms are *recursive* in sturcture: to solve a given problem, they call themselves recursively one or more times to deal with a closely related subproblem. These typically follow a *divide-and-conquer* approach:

- **Divide** the problem into a number of subproblems that are smaller instances of the same problem.
- **Conquer** he subproblems by soving them recursively. If the subproblem sizes are small enough, solve the problem in a straightforward manner.
- **Combine** the solutions to the subproblems into the solution for the original problem.

When an algorithm contains a recursive call to itelf, we can describe its running time by a *recurrence*, which describes the overall running time on a problem of size n in terms of the running time of smaller inputs.

In divide and conquer algorithms we can describe the running time as a recurrence. As said, if the problem size is small enough, say $n \leq c$ for some constant c. the straightforward solution takes constant time, write as $\Theta(1)$. Suppose that the division of the problem yields a subproblems, with each one being $1/b$ size of the original. And define $D(n)$ and $C(n)$ be the times to divide the subproblems and combine them, respectively, then we have the running time as:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq c, \\ aT(n/b) + D(n) + C(n) & \text{otherwise.} \end{cases}$$

The MERGE-SORT algorithm follows this divide-and-conquer approach:

- Divide the n-element array to be sorted into two subarrays of $n/2$ elements each.
- Sort the two subarrays recursively using merge sort.
- Merge the two sorted arrays to produce the sorted array.

The recursion "bottoms out" when the sequence to be sorted has length 1, in which case there is no work to be done, since it is an already sorted 1-element array.

The MERGE-SORT array relies on the auxiliary procedure MERGE(A, p, q, r), where A is an array, p, q and r are indices into the array such that $p \leq q < r$. The procedure assumes the subarrays $A[p..q]$ and $A[q+1..r]$ are in sorted order. It merges them to form a single sorted subarray that replaces the current subarray $A[p..r]$. This procedure can be easily implemented to have a runtime of $\Theta(n)$, where $n = r - p + 1$.

The merge sort algorithm is then:

```
MERGE-SORT$(A, p, r)$
1  if $p < r$ then
2      $q = \lfloor (p+r)/2 \rfloor$
3      MERGE-SORT$(A, p, q)$
4      MERGE-SORT$(A, q+1, r)$
5      MERGE$(A, p, q, r)$
6  end if
```

And to sort the required sequence A we call the algorithm procedure as MERGE-SORT$(A, 1, A.length)$. Assuming that n is a power of 2, we can calculate the running time of the algorithm can be calculated as.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1, \\ 2T(n/2) + \Theta(n) & \text{otherwise.} \end{cases}$$

In later chapters we will solve the recurrence using the Master Theorem.

Chapter 3 Growth Of Functions

3.1 Asymptotic notation

When studying the running-time of algorithms, we are interested in the asymptotic behaviour of a time-cost function taking values in the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$.

Definition. Let g be a function defined in $\mathbb{N}$. We denote $\Theta(g(n))$ the set of functions defined by:

$$\Theta(g(n)) = \{f(n) : \exists c_1, c_2 \in \mathbb{R}^+, \text{ and } n_0 \in \mathbb{N} \text{ such that } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n),\ \forall n \geq n_0\}.$$

If f is another function taking values in $\mathbb{N}$, we denote that f is in this set as "$f(n) = \Theta(g(n))$". We say that g is an *asymptotically tight bound* for f.

Note how the definition of $\Theta(g(n))$ requires that every member f of the set to be *asymptotically nonnegative* ($f(n) \geq 0$ for sufficiently large n). Consequently g itself must be asymptitically nonnegative, or $\Theta(g(n))$ is the empty set.

Definition. Let g be a function defined in $\mathbb{N}$. We denote $O(g(n))$ the set of functions defined by:

$$O(g(n)) = \{f(n) : \exists c \in \mathbb{R}^+, \text{ and } n_0 \in \mathbb{N} \text{ such that } 0 \leq f(n) \leq c g(n),\ \forall n \geq n_0\}.$$

If f is another function taking values in $\mathbb{N}$, we denote that f is in this set as "$f(n) = O(g(n))$". We say that g is an *asymptotic upper bound* for f.

Definition. Let g be a function defined in $\mathbb{N}$. We denote $\Omega(g(n))$ the set of functions defined by:

$$\Omega(g(n)) = \{f(n) : \exists c \in \mathbb{R}^+, \text{ and } n_0 \in \mathbb{N} \text{ such that } 0 \leq c g(n) \leq f(n),\ \forall n \geq n_0\}.$$

If f is another function taking values in $\mathbb{N}$, we denote that f is in this set as "$f(n) = \Omega(g(n))$". We say that g is an *asymptotic lower bound* for f.

Note from the above definitions that for any function g, we have:

$$\Theta(g(n)) \subseteq \Omega(g(n)), \text{ and } \Theta(g(n)) \subseteq O(g(n)).$$

Theorem 3.1 *For any two functions f and g, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.*

Asymptotic notation in equations and inequalities. When using asymptotic notation in formulas, we interpret it as standing for some anonymous function that we do not to care. For example, $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means that $2n^2 + 3n + 1 = 2n^2 + f(n)$, for some function f in the set $\Theta(n)$ (In this case $f(n) = 3n + 1$ which is indeed $\Theta(n)$).

Using asymptotic notation in this manner we can help eliminate inessential detail and clutter in an equation. For example in merge sort recurrence: $T(n) = 2T(n/2) + \Theta(n)$. If we are interested only in the asymptotic behavoir of $T(n)$, there is no point in specifying all the lower-order terms exactly; they are all understood to be included in the anonymous function denoted by the term $\Theta(n)$.

The number of anonymous functions in an expression is understood to be equal to the number of times the asymptotic notation appears. In some cases, asymptotic notation appears on the left-hand side of an equation, like: $2n^2 + \Theta(n) = \Theta(n^2)$. We interpret it using the rule: *No matter how the anonymous functions are chosen on the left hand of the equals sign, there is a way to choose the anonymous function on the right of the equal sign to make the equation valid.* Thus on our example, $\forall f \in \Theta(n)$, $\exists g \in \Theta(n^2)$ such that $2n^2 + f(n) = g(n), \forall n$.

Asymptotically tight bounds. The asymptotic upper bound provided by O-notation may or may not be asymptotically tight. For example $2n^2 + 3 = O(n^2)$ is asymptotically tight, but $n = O(n^2)$ isn't. We use o-notation to denote an upper bound that is not asymptotically tight.

Definition. We denote $o(g(n))$ the set of functions defined by:

$$o(g(n)) = \{f(n) : \forall c > 0,\ \exists n_0 \in \mathbb{N} \text{ such that } 0 \leq f(n) < c g(n),\ \forall n \geq n_0\}.$$

That is equivalent to say that if $f(n) = o(g(n))$ if, and only if, $\lim_{n\to\infty} f(n)/g(n) = 0$.

By analogy we have lower bound that are not asymptotically tight.

Definition. We denote $\omega(g(n))$ the set of functions defined by:

$$\omega(g(n)) = \{f(n) : \forall c > 0,\ \exists n_0 \in \mathbb{N} \text{ such that } 0 \leq c g(f < f(n),\ \forall n \geq n_0\}.$$

That is equivalent to say that if $f(n) = \omega(g(n))$ if, and only if, $\lim_{n\to\infty} f(n)/g(n) = \infty$.

3.1.1 Comparing Functions

Many of the relational properties of real numbers apply to asymptotic comparisons as well. For the following assume f, g be asymptotically positive functions.

Symmetry:

i) $f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$.

Reflexivity:

i) $f(n) = \Theta(f(n))$.

ii) $f(n) = O(f(n))$.

iii) $f(n) = \Omega(f(n))$.

Transitivity:

i) $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \implies f(n) = \Theta(h(n))$.

ii) $f(n) = O(g(n))$ and $g(n) = O(h(n)) \implies f(n) = O(h(n))$.

iii) $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n)) \implies f(n) = \Omega(h(n))$.

iv) $f(n) = o(g(n))$ and $g(n) = o(h(n)) \implies f(n) = o(h(n))$.

v) $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n)) \implies f(n) = \omega(h(n))$.

Transpose Symmetry:

i) $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$.

ii) $f(n) = o(g(n)) \iff g(n) = \omega(f(n))$.

We have an analogy between asymptotic comparisons of two functions and the comparisons of two real numbers. For example $f(n) = O(g(n))$ is like $a \leq b$; also $\Omega(\cdot)$ like $\geq$, $\Theta(\cdot)$ like $=$, $o(\cdot)$ like $<$, $\omega(\cdot)$ like $>$.

Remark. If we define the relation $f(n) \sim g(n)$ if, and only if, $f(n) = \Theta(g(n))$, then this relation is an equivalence relation.

Chapter 4 Divide And Conquer

In divide-and-conquer, we solve a problem recursively, applying three steps at each level of the recursion:

- **Divide** the problem into a number of subproblems that are smaller instances of the same problem.
- **Conquer** the subproblems by solveing them recursively. If the subproblem sizes are small enough, just solve them i a straightforward manner.
- **Combine** the solutions to the subproblems into the solution for the original problem.

When the subproblems are large enough to solve recursively, we call that the *recursive case*. Once the problems become small enough that we no longer recurse, we say that the recursion "bottoms out" and that we have gotten down to the *base case*. Sometimes, in addition to ubproblems that are smaller instances of the same problem, we have to solve subproblems that are NOT quite the same as the original problem, and cconsider solvim that problem part of the combine step.

4.3 The substitution method for solving recurrences

The *substitution method* for solving recurrences comprises two steps:

1. Guess the form of the solution.
2. Use the Principle of Mathematical Induction to find the constants and show that the solution works.

We can use the substitution method to establish either upper or lower bounds on a recurrence. As an example consider the recurrence given by: $T(n) = 2T(\lfloor n/2 \rfloor) + n$. We guess that the solution is $T(n) = O(n \log n)$. So, we claim that $T(n) \leq cn \log n$, for some $c > 0$. Proving the claim would then imply that $T(n) = O(n \log n)$. Using induction, we start by assuming that the bound holds for all positive $m < n$, in particular for $m = \lfloor n/2 \rfloor$, yielding $T(\lfloor n/2 \rfloor) \leq c \lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor)$. And substituting into the recurrence yields:

$$\begin{aligned} T(n) &= 2T(\lfloor n/2 \rfloor) + n \\ &\leq 2(c \lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor)) + n && \text{(H.Ind.)} \\ &\leq cn \log(n/2) + n \\ &= cn \log n + (1 - c \log 2)n \leq cn \log n, \end{aligned}$$

where the last inequality holds iff $c \geq 1/\log 2$. To complete the proof we need to check that our solution holds for a base case. That is, we need to show in our case, that we can choose c great enough so that the bound holds for the base case too.

This requeriment may lead to problems sometimes. Assume for the sake of the argument that $T(1) = 1$, then the bound $T(n) \leq cn \log n$ yields to $T(1) \leq c \cdot 1 \cdot \log 1 = 0$!! We can overcome this by taking advantage of asymptotic definition, requiring us that the bound holds for $n \geq n_0$, for some $n_0 \in \mathbb{N}$.

Subtleties In almost all cases in which the recurrence has constants or lower-order terms, it will be necessary to <u>have additional terms</u> in the upper bound to <u>cancel out</u> the constants or lower-order terms. Without the right additional terms, the inductive case of the proof will get stuck in the middle, or generate an impossible constraint; this is a signal to go back to your upper bound and determine what else needs to be added to it that will allow the proof to proceed without causing the bound to change in asymptotic terms.

Consider for example, the following recurrence:

$$\begin{cases} T(1) = 1, \\ T(n) = 2T(n-1) + c_1. \end{cases}$$

Where $c_1 > 0$ is a constant. Iterating manually on the recurrence ($T(n) = 2 \cdot (2 \cdots (2 \cdot (2 \cdot 1 + c_1) + c_1) \cdots) + c_1$), we can guess that the solution would be $O(2^n)$.

We will guess an upper bound of $k2^n - b$, where b is some constant. We include the b in anticipation of having to deal with the constant c_1 that appears in the recurrence relation, and because it does no harm. In the process of proving this bound by induction, we will generate a set of constraints on k and b, and if b turns out to be unnecessary, we will be able to set it to whatever we want at the end.

Our property, then, is $T(n) \leq k2^n - b$, for some two constants k and b. Note that this property logically <u>implies that $T(n)$ is $O(2^n)$</u>. Let's prove the claim by induction.

The base case $n = 1$ is: $T(1) = 1 \leq k2^1 - b = 2k - b$. This is true as long as $k \geq (b+1)/2$. The inductive case: We assume our property is true for $n - 1$. We now want to show that it is true for n.

$$\begin{aligned} T(n) &= 2T(n-1) + c1 \\ &\leq 2(k2n - 1 - b) + c1 && \text{(H.Ind)} \\ &= k2n - 2b + c1 \\ &\leq k2n - b \end{aligned}$$

This last inequality holds as long as $b \geq c1$. So we end up with two constraints that need to be satisfied for this proof to work, and we can satisfy them simply by letting $b = c_1$ and $k = (b+1)/2$, which is always possible, as the definition of $O(\cdot)$ allows us to choose any constant. Therefore, we have proved that our property is true, and so $T(n)$ is $O(2^n)$.

Had we not added the lower term $-b$ in the upper bound of $T(n) \leq k2^n - b$, we could have not proven the bound by induction. Indeed, in the first inequality in the hypothesis of induction we would've got $T(n) \leq 2k^n + c_1$, which is $\nleq k2^n$

Changing Variables Sometimes, a little algebraic manipulation can make an unknown recurrence similar to a known one. Consider for example:

$$T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \log n.$$

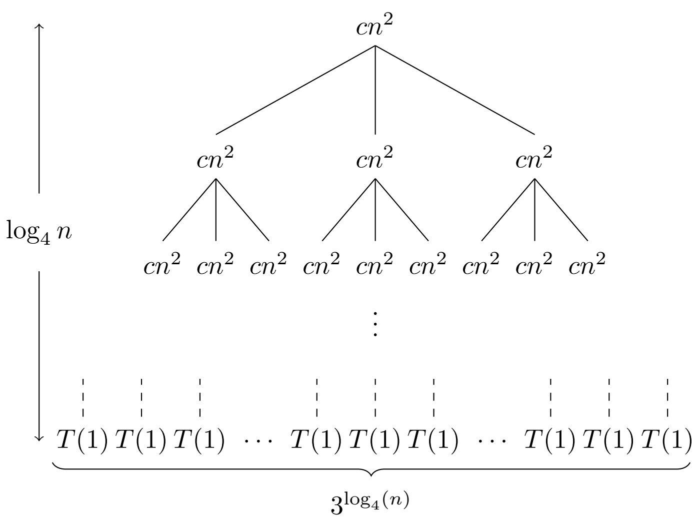
Then takign $m = \log n$, yields $T(2^m) = 2T(2^{m/2}) + m$. Renaming $S(m) = T(2^m)$, to produce the new recurrence:

$$S(m) = 2S(m/2) + m.$$

We already know $S(m) = O(m \log m)$, thus $T(n) = T(2^m) = S(m) = O(m \log m) = O(\log(n) \cdot \log(\log(n)))$.

4.4 The Recursion-Tree Method for Solving Recurrences

Drawing out a recursion tree serves as a straightforward way to devise a good guess. In a *recursion tree*, each node represents the cost of a single subproblem somewhere in the set of recursive function invocations. For example consider the recursion tree for the recurrence $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$. We start by finding an upper bound on the recurrence, and focusing only on asymptotic behavior, assume $4 \mid n$ and we write $T(n) = 3T(n/4) + cn^2$, for some $c > 0$. The recursion tree for the recurrence would be as follows:


cn^2
cn^2 cn^2 cn^2
$\log_4 n$
cn^2 cn^2 cn^2 cn^2 cn^2 cn^2 cn^2 cn^2 cn^2
$T(1)\,T(1)\,T(1)\ \cdots\ T(1)\,T(1)\,T(1)\ \cdots\ T(1)\,T(1)\,T(1)$
$3^{\log_4(n)}$

The subproblem size for a node at depth i is easily seen to be $n/4^i$. Hence, the problem hits a leaf $n = 1$ when $n/4^i = 1$, that is, $i = \log_4 n$. Thus the problem has $\log_4 n + 1$ levels.

To determine the cost at each level of the three, see that the number of nodes at level i is 3^i. Since we we reduce by 4 the size of problem each time we go down, the cost of each node at level i is $c(n/4^i)^2$. Adding for all nodes on level i, we get the total cost for level i of the tree is $3^i c(n/4^i)^2 = (3/16)^i cn^2$. The bottom level has $3^{\log_4(n)} = n^{\log_4 3}$ nodes, each contributing a cost of $T(1)$. Thus a total cost of $n^{\log_4 3}T(1) = \Theta(n^{\log_4 3})$ (assuming $T(1)$ is constant). Adding up all the costs to determine the cost of the entire tree:

$$\begin{aligned}
T(n) &= \sum_{i=0}^{\log_4(n)-1} \left(\frac{3}{16}\right)^i + \Theta(n^{\log_4 3}) \\
&< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i + \Theta(n^{\log_4 3}) \\
&= \frac{1}{1-(3/16)}cn^2 + \Theta(n^{\log_4 3}) = \frac{16}{13}cn^2 + \Theta(n^{\log_4 3}) \\
&= O(n^2).
\end{aligned}$$

Thus, we have derived a guess (not a proof: this was quite ... informal). This recurrence is indeed $\Theta(n^2)$, which can be easily seen. By the definition of the recurrence it is trivially $\Omega(n^2)$. The bound $O(n^2)$ can be proven by substitution easily.

4.5 The Master Theorem

The master method provides a way of solving recurrences of the form $T(n) = aT(n/b) + f(n)$. This recurrence describes the running time of an algorithm that divides a problem of size n into a subproblems, each of size n/b. Here $f(n)$ would represent the cost of dividing the problem and combining the results of the subproblems.

Theorem 4.1 ***(Master Theorem).*** *Let $a \geq 1$ and $b \geq 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence*

$$T(n) = aT(n/b) + f(n),$$

where we interpred n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then T(n) has the following asymptotic bounds:

1. *If $f(n) = O(n^{\log_b(a)-\epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b(a)})$.*
2. *If $f(n) = \Theta(n^{\log_b(a)}$, then $T(n) = \Theta(n^{\log_b(a)} \log n)$.*
3. *If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.*

As an example consider the recurrence given by $T(n) = 9T(n/3) + n$. For this recurrence, we have $a = 9$, $b = 3$, $f(n) = n$, thus we have that $n^{\log_b a} = n^{\log_3 9} = \Theta(n^2)$. Since $f(n) = O(n^{\log_3 9-\epsilon})$, where $\epsilon = 1$, we can apply case 1 of the theorem and conclude that $T(n) = \Theta(n^2)$.

Chapter 5 Probabilistic Analysis and Randomized Algorithms

5.1 The hiring problem

Suppose we need to hire an office assistant. You interview a person and decide to hire that person or not, and we must pay the employment agency a fee for interviewing the person (c_i). To actually hire the applicant we must pay a higher cost (c_h), since we must fire the current office assistant. So after each interview, if the new applicant is better than the current office assistant, we fire the office assistant and hire the new applicant. We wish to estimate what the price of this strategy will be.

The following HIRE-ASSISTANT, expresses this strategy for hiring among n applicants.

```
HIRE-ASSISTANT($n$)
1  $best = 0$    // Actual best candidate.
2  for $i = 1$ to $n$ do
3      interview candidate $i$
4      if candidate $i$ is better than candidate $best$ then
5          $best = i$
6          hire candidate $i$
7      end if
8  end for
```

The cost model for this problem differs from the model described in previous chapters. We focus not on the running time, but on the costs incurred by the interviewing and hiring. Interviewing has a low cost c_i, and hiring an expensive cost c_h. Let m be the number of applicants hired during the strategy, the total cost associated with this algorithm is $O(c_i n + c_h m)$. No matter how many people we hire, we always interview n candidates and thus always incur the cost $c_i n$ associated with interviewing. We therefore concentrate on analyzing $c_h m$, the hiring cost. Here the quantity m varies with each run of the algorithm.

In the worst case, candidates are in increasing order of quality, thus we hire all n candidates. However, candidates do not always come in this order, nor we know or control the order they come in. Thus it is natural to ask what we expect to happen in an average case.

Probabilistic analysis *Probabilistic analysis* is the use of probability in the analysis of problems. Most commonly we use it to analyze the running time of algorithms. In order to perform a probabilistic analysis, we must use knowledge of, or make assumptions about the sitribution of the input. When we analyze our algorithm, computing an *average-case running time*, where we take the average over the distribution of the possible inputs (we take the expected value).

For the hiring problem we can assume that the applicants come in a random order, and that between any two candidates we can decide which one is better (that is there is a *total ordering*). Thus we can rank each applicant with an integer from 1 to n, denoting $rank(i)$ the rank of applicant i (where the highest rank yields the best applicant). So we have that the list of ranks $(rank(1), rank(2), \dots, rank(n))$ is a permutation of the list $(1, 2, \dots, n)$. Assumin random ordering in the applicants, we can say that the list of ranks is equally likely to be one of the $n!$ permutations of $[n]$ (the ranks form a *uniform random permutation*).

Randomized Algorithms In the hiring problem, we assumed that the applicants come in order. Insted we can change a bit the model and say we have a list of n candidates, and on each day we choose randomly which candidate to interview. So we have gained control of the process and enforced a random order.

More generally, we call an algorithm *randomized* if its behavior is determined not only by its input but also by values produced by a *random number generator.* Assume we have a random number generator RANDOM(a, b) which models a discrete uniform random variable $U(a, b)$. When analyzing the running time of a randomized algorithm we take the expected value of the running time over the distribution of values returned by the random number generator.

5.2 Indicator random variables

Indicator rrandom variables provide a convenient method for converting between probabilities and expectations. Suppose we are given a sample space S and an event $A \subseteq S$. Then the *indicator random variable* $\mathbb{1}_A$ associated with event A is definde as

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } w \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma. *Given a sample space S and an event $A \subseteq S$, let $\mathbb{1}_A$ be the indicator variable of the event A. Then $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$.*

As an example, return to the hiring problem. Let X be te random variable whose value equals he number of times we hire a new office assistant. Now let X_i be the indicator random variable associated with the event in which candidate i is hired. Thus, we have

$$X_i = \begin{cases} 1 & \text{if candidate } i \text{ is hired,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the key step is to realize that $X = \sum_{i=1}^n X_i$. Now to calculate $\mathbb{P}(\text{candidate } i \text{ is hired})$, we have in our strategy that candidate i is hired iff it is better than all the prior $1, \dots, i-1$ candidates. Because we have assumed that the candidates arrive in a random order, the first i candidates are in random order. So any of the first i candidates could be the best-qualified so far. Thus candidate i has a probability of $1/i$ of being better qualified than the candidates 1 through $i - 1$. Hence $\mathbb{E}[X_i] = 1/i$. Now we can compute $\mathbb{E}[X]$.

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n \frac{1}{i} = \ln n + O(1).$$

Thus, although we interview n people, we actually only hire aroung $\ln n$ of them on average. So we can conclude that if the candidates are presented in random order, the HIRE-ASSISTANT algorithm has an average-case total hiring cost of $O(c_h \ln n)$.

5.3 Randomized algorithms

In the previous section we showed how knowing the distribution on the inputs can help us analyze the average-case behavior of an algorithm. Many times, we do not have such knowledge, thus precluding the average-case analysis. As mentioned, we may be able to use a randomized algorithm. In these cases we can use a randomized algorithm and force a random distribution.

In the hiring problem, we assumed a random order on inputs. We can instead, impose a distribution on the inputs. For example, before running the alorithm, randomly permute the candidates in order to enforce the property that every permutation is equally likely. Although we have modified the algorithm, we still expect to hire a new office assistant approximately $\ln n$ times. But now we expect this to be the case for any input, rather than for inputs drawn from a particular distribution. For our hiring strategy, to produced our randomized algorithm we can do the following.

```
RANDOMIZED-HIRE-ASSISTANT($n$)
1  randomly permute the list of candidates
2  $best = 0$   // Actual best candidate.
3  for $i = 1$ to $n$ do
4      interview candidate $i$
5      if candidate $i$ is better than candidate $best$ then
6          $best = i$
7          hire candidate $i$
8      end if
9  end for
```

With this simple change we have created a randomized algorithm whose performance matches the obtained by assuming that the candidates were presented ina random order. The expected hiring cost of the procedure RANDOMIZED-HIRE-ASSISTANT is $O(c_h \ln n)$.

5.3.1 Randomly permuting arrays

Many randomized algorihtms randomize the input by permuting an array. We study two algorithms for doing this. Suppose we are given an array A of n elements, without loss of generality, containing elements 1 through n. Our goal is to produce a random permutation of the array.

In the first method, we create an array P of n random elements, then sort A as in the order of P.

```
PERMUTE-BY-SORTING($A$)
1  $n = A.length$
2  let $P[1..n]$ be a new array
3  for $i = 1$ to $n$ do
4      $P[i] = $ RANDOM$(1, n^3)$
5  end for
6  sort $A$, using $P$ as sort keys
```

In line 4 we take a random number between 1 and n^3. With this range we make it likely that all the priorities in P are unique (it is easily seen that the probability that all entries are unique is at least $1 - 1/n$, and the algorithm can be implemented even if there are repetitions).Assume nevertheless, that priorities in P are unique.

The costly step comes in line 6 for sorting the array A. We will see this comparison sort can be done in $\Omega(n \log n)$ time. After this sort, if $P[i]$ is the jth smallest priority, then $A[i]$ lies in position j of the output. In this manner we obtain a permutation of A.

Proposition. *Procedure* PERMUTE-BY-SORTING *produces a uniform random permutation of the input, assuming that all the priorities are distinct.*

A better method for generating a random permutation is to permute the given array in place. The following procedure does so in $O(n)$ time.

```
PERMUTE-IN-PLACE($A$)
1  $n = A.length$
2  for $i = 1$ to $n$ do
3      swap $A[i]$ with $A[$RANDOM$(i, n)]$
4  end for
```

We shall use a loop invariant to show that the procedure produces a uniform random permutation.

Proposition. *Procedure* PERMUTE-IN-PLACE *produces a uniform random permutation of the input.*
Proof: We will use the following loop invariant: *Just prior to the ith iteration of the loop in lines 2 and 3, fo each possible $(i-1)$-permutation of the n elements, the subarray $A[1..i-1]$ contains this $(i-1)$-permutation with probability $(n-i+1)!/n!$.*

We need to see the invariant holds prior to every loop iteration. And use the loop invariant to show correctness at loop termination.

Initialization: Before the first loop, at $i = 1$, the loop invariant says for each 0-permutation, the subarray $A[1..0] = \varnothing$, contains this 0-permutation with probability $n!/n! = 1$. Which holds trivially.

Maintenance: Assume that just before the ith iteration, each possible $(i-1)$-permutation appears in the subarray $A[1..i-1]$ with probability $(n-i+1)!/n!$. We shall see, that after the ith iteration, each possible i-permutation appears in the subarray $A[1..i]$ with probability $(n-i)!/n!$.

Consider any i-permutation $(x_1, x_2, \ldots, x_i)$, which is an $(i-1)$-permutation $(x_1, x_2, \ldots, x_{i-1})$, followed by an x_i placed in $A[i]$ by the algorithm. Let E_1 denote the event in which the first $i-1$ iterations have created the particular $(i-1)$-permutation $(x_1, \ldots, x_{i-1})$ in $A[1..i-1]$; which by the loop invariant has $\mathbb{P}(E_1) = (n-i+1)!/n!$. Let E_2 be the event that the ith iteration puts x_i in $A[i]$. The i-permutation $(x1, \ldots, x_i)$ appears in $A[1..i]$, precisely when both E_1 and E_2 occur, that is the event $E_1 \cap E_2$. Now we have $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_2|E_1)\,\mathbb{P}(E_1)$.

The probability $\mathbb{P}(E_2|E_1)$ equals $1/(n-i+1)$, which is choosing randomly x_i from the array $A[i..n]$. Thus we have

$$\mathbb{P}(E_1 \cap E_2) = \frac{1}{n-i+1} \frac{(n-i+1)!}{n!} = \frac{(n-i)!}{n!}.$$

Termination: At termination, $i = n+1$, hence the subarray $A[1..n]$ is a given n-permutation with probability $(n-(n+1)+1)!/n! = 0!/n! = 1/n!$.

Thus, RANDOMIZE-IN-PLACE produces a uniform random permutation. ■

Chapter 6 Heapsort

6.1 Heaps

Definition. A *(binary) heap* data structure is an array object that we can view as a nearly complete binary tree (that is lowest level may be filled up to a point). Each node of the tree corresponds to an element of the array. An array A that represents a heap is an object with an $A.length$ attribute, which counts the number of elements in the array; and an $A.heap - size$ attribute, which represents how many elements in the heap are stored within array A; where it is satisfied $0 \leq A.heap - size \leq A.length$.

The root of the tree is $A[1]$, and given an index i of a node we can compute the indexes of the parent and left and right child nodes.

```
PARENT($i$)
1  return $\lfloor i/2 \rfloor$
```

```
LEFT($i$)
1  return $2i$
```

```
RIGHT($i$)
1  return $2i + 1$
```

Definition. There are two kinds of binary heaps: max-heaps and min-heaps. In both kinds, the values in the nodes satisfy a *heap property.* In a *max-heap*, the *max-heap property* is that for every node other than the root, $A[\textsc{Parent}(i)] \geq A[i]$. Similarly a *min-heap*, the *min-heap property* is that for every node other than the root, $A[\textsc{Parent}(i)] \leq A[i]$.

Viewing the heap as a tree, we have that the *heigth* of a node in a heap is te number of edges on the longest simple downwad path from the node to a leaf, and the *height of the heap* is the height of its root. Since a heap of n nodes is based on a complete binary tree, we have that its height is $\theta(n)$.