Chapter 1 The Basics

1.1 Graphs

Definition. A *graph* is a pair $G = (V, E)$ of sets such that $E \subseteq V \times V$. The elements of V are the *vertices* (or *nodes*, or *points*) of the graph G, the elements of E are its *edges*.

A graph with vertex set V is said to be a graph *on* V. The vertex set of a graph G is referred to as $V(G)$, its edge set as $E(G)$.

Definition. The number of vertices of a graph G is its *order*, written as $|G|$; its number of edges is denoted by $\|G\|$. Graphs are *finite*, *infinite*, *countable* and so, according to their order.

Definition. A vertex v is *incident* with an edge e if $v \in e$, then e is an edge at v. The two vertices incident with an edge are its *ends*, and an edge *joins* its ends.

An edge $\{x, y\}$ is usually written as xy (or yx). If $x \in X$ and $y \in Y$, then xy is an X-Y *edge*. The set of all X-Y edges in a set E is denoted by $E(X, Y)$. The set of all edges in E at a vertex v is denoted by $E(v)$.

Definition. Two vertices x, y of G are *adjacent*, or *neighbours*, if xy is an edge of G. Two edges $e \neq f$ are *adjacent* if they have an end in common. If all the vertices of G are pairwise adjacent, then G is *complete*. A complete graph on n vertices is a K^n; a K^3 is called a *triangle*.

Definition. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We say G and G' are *isomorphic*, and write $G \simeq G'$, if there exists a bijection $\varphi : V \to V'$ with $xy \in E \Leftrightarrow \varphi(x)\varphi(y) \in E'$. Such a map φ is called an *isomorphism*.

We do not distinguish between isomorphic graphs, and write $G = G'$ instead of $G \simeq G'$, and talk about *the* graph.

Definition. A class of graphs that is closed under isomorphism is called a *graph property*. A map taking graphs as arguments is called a *graph invariant* if it assigns equal values to isomorphic graphs.

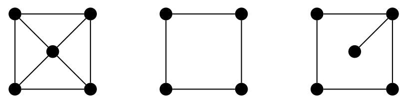
For example, "containing a triangle" is a graph property: if G contains three pairwise adjacent vertices then so does every graph isomorphic to G.

The number of vertices and the number of edges of a graph are two simple graph invariants; the greatest number of pairwise adjacent vertices is another.

Definition. We set $G \cup G' := (V \cup V', E \cup E')$, and $G \cap G' := (V \cup V', E \cap E')$. If $G \cap G' = \varnothing$, then G and G' are *disjoint*. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G (and G a *supergraph* of G'), written as $G' \subseteq G$.

If $G' \subseteq G$ and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is an *induced subgraph* of G; we say that V' *induces* or *spans* G' in G and write $G' =: G[V']$. If $U \subseteq V$ is any set of vertices, then $G[U]$ denotes the graph on U whose edges are precisely the edges of G with both ends in U. If H is a subgraph of G, not necessarily induced, we write $G[H] \equiv G[V(H)]$. Finally, $G' \subseteq G$ is a *spanning subgraph* of G is V' spans all of G, i.e. if $V' = V$.

If U is any set of vertices (usually of G), we write $G - U$ for $G[V \setminus U]$. In other words, $G - U$ is obtained from G by deleting all the vertices in $U \cap V$ and their incident edges. We write $G - G'$ instead of $G - V(G')$. For $F \subseteq V \times V$ we write $G - F := (V, E \setminus F)$ and $G + F := (V, E \cup F)$. We call G *edge-maximal* with a given graph property if G itself has the property but no graph $G + xy$ does, for non-adjacent vertices $x, y \in G$.



A graph G with subgraphs G' and G'': G' is an induced subgraph of G, but G'' is not.

More generally, when we call a graph *minimal* or *maximal* with some property but have not specified any particular ordering, we are referring to the subgraph relation. When we speak of minimal or maximal sets of vertices or edges, the reference is simply to set inclusion.

Definition. If G and G' are disjoint, we denote by $G * G'$ the graph obtained from $G \cup G'$ by joining all vertices of G to all vertices of G'. The *complement* $\overline{G}$ of G is the graph on V with edge set $V \times V \setminus E$. The *line graph* $L(G)$ of G is the graph on E in which $x, y \in E$ are adjacent vertices if and only if they are adjacent edges in G.

1.2 The degree of a vertex

Definition. Let $G = (V, E)$ a (non-empty) graph. The set of neighbours of a vertex $v \in G$ is denoted by $N_G(v)$ (or briefly $N(v)$). More generally, for $U \subseteq V$, the neighbours in $V \setminus U$ of vertices of U are called the *neighbours of* U, denoted $N(U)$.

Definition. The *degree* of a vertex $v \in G$, $d_G(v)$ (or $d(v)$ for short), is the number $|E(v)|$ of edges at v; which by definitino equals the number of neighbors of v, $|N_G(v)|$.

A vertex of degree 0 is *isolated*. The number $\delta(G) := \min\{d(v) | v \in V\}$ is the *minimum degree* of G, the number $\Delta(G) := \max\{d(v) | v \in V\}$ is the *maximum degree* of G.

If all the vertices of G have the same degree k, then G is k-*regular*.

The number $d(G) := \frac{1}{|V|} \sum_{v \in V} d_G(v)$, is the *average degree* of G.

Clearly we have that $\delta(G) \leq d(G) \leq \Delta(G)$. The average degree quantifies globally what is measured locally by the vertex degrees: the number of edges of G per vertex. Sometimes it will be convenient to express this ratio directly, as $\varepsilon(G) := |E|/|V|$. Which relates to the average degree:

$$|E| = \frac{1}{2} \sum_{v \in G} d(v) = \frac{1}{2} d(G) \cdot |V| \implies \varepsilon(G) = \frac{1}{2} d(G).$$

Proposition. *The number of vertices of odd degree in a graph is always even.*

If a graph has a large minimum degree (everywhere, locally, many edges per vertex) it also has many edges per vertex globally: $\varepsilon(G) = \frac{1}{2} d(G) \geq \frac{1}{2} \delta(G)$. Conversely its average degree may be large even with small minimum degree, in that case the vertices of large degree cannot be scattered among vertices of small degree, but concentrated in a region. Next proposition shows this region is a subgraph of G with average degree no less than G and minimum degree greater than half his average degree.

Proposition. *Every graph G with at least one edge, has a subgraph H with $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$.*

1.3 Paths and cycles

Definition. A *path* is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \dots, x_k\} \quad E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\},$$

where the x_i are all distinct. Denote this path as $P = x_0x_1 \dots x_k$, and call it a *path from x_0 to x_k*. The vertices x_0 and x_k are *linked* by P and are called its *ends*; ther vertices $x_1, \dots, x_{k-1}$ are the *inner* vertices of P. The number of edges of a path is its *length*, and the path of length k is denoted by P^k (where $P^0 = K^1$).

We denote for $0 \leq i \leq j \leq k$ the following subpaths of P

$$Px_i := x_0 \dots x_i \qquad \mathring{P} := x_1 \dots x_{k-1}$$
$$x_iP := x_i \dots x_k \qquad P\mathring{x}_i := x_0 \dots x_{i-1}$$
$$x_iPx_j := x_i \dots x_j \qquad \mathring{x}_iP := x_{i+1} \dots x_k$$
$$\mathring{x}_iP\mathring{x}_j := x_{i+1} \dots x_{j-1}$$

We use similar intuitive notation for the concatenation of paths; for example: if the union $Px \cup xQy \cup yR$ of the three paths is again a path, we may simply write $PxQyR$.

Definition. Given sets A, B of vertices, we call $P = x_0 \dots x_k$ an *A-B path* if $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$. Two or more paths are *independent* if none of them contains an inner vertex of another.

Definition. Given a graph H, we call P an *H-path* if P is non-trivial and meets H exaclty in its ends. In particular, the edge of any H-path of length 1 is never an edge of H.

Definition. If $P = x_0 \dots x_{k-1}$ is a path and $k \geq 3$, then the graph $C := P + x_{k-1}x_0$ is called a *cycle* (denoted as $C = x_0 \dots x_{k-1}x_0$). The *length* of a cycle is its number of edges (or vertices); the cycle of length k is called a k-cycle and denoted C^k.

Definition. The minimum length of a cycle (contained) in a graph G is the *girth* $g(G)$ of G; the maximum length of a cycle in G is its *circumference* (if G does not contain a cycle, we set the former to ∞, the latter to zero). An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle. Thus, an *induced cycle in G*, a cycle in G forming an induced subgraph, is one that has no chords.

If a graph has large minimum degree, it contains long paths and cycles:

Proposition. *Every graph G with $\delta(G) \geq 2$ contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$.*

Definition. The *distance* $d_G(x, y)$ in G of two vertices x, y is the length of the shortest x-y path in G; if no such path exists, we set $d(x, y) := \infty$. The greatest distance between any two vertices in G is the *diameter* of G, denoted $\operatorname{diam} G$.

Proposition. *Every graph G containing a cycle satisfies $g(G) \leq 2 \operatorname{diam} G + 1$.*

Definition. A vertex is *central* in G if its greatest distance from any other vertex is as small as possible. This distance is the *radius* of G (denoted $\operatorname{rad} G$). Formally we have $\operatorname{rad} G = \min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y)$.

It is easy to check the following relation:

$$\operatorname{rad} G \leq \operatorname{diam} G \leq 2 \operatorname{rad} G.$$

Graphs of small diameter and maximum degree must be small:

Proposition. *A graph G of radius at most k and maximum degree at most $d \geq 3$, has fewer than $\frac{d}{d-2}(d-1)^k$ vertices.*

1.4 Connectivity

Definition. A non-empty graph G is called *connected* if any two of its vertices are linked by a path in G. If $U \subseteq V(G)$ and $G[U]$ is connected, we also call U itself connecteed in G. Say a graph is *disconnected* when it is not connected.

Proposition. *Given a connected graph G, there exists an enumeration of its vertices, say $v_1, \dots, v_n$ such that $G_i := G[v_1, \dots, v_i]$ is connected for every i.*

Definition. Let $G = (V, E)$ a graph. A maximal connected subgraph f G is called a *component* of G. Note that a component, being connected, is always non-empty; the empty graph, therefore, has no components.

If $A, B \subseteq V$ and $X \subseteq V \cup E$ are such that every A-B path in G contains a vertex or an edge from X, we say that X *separates* the sets A and B in G. Note that this implies $A \cap B \subseteq X$. More generally we say that X *separates* G if $G - X$ is disconnected, that is, if X separates in G some two vertices that are not in X. A separating set of vertices is a *separator*. A vertex which separates two other vertices of the same component is a *cutvertex*, and an edge separating its ends is a *bridge*. Thus, the bridges in a graph are precisely those edges that do not lie on any cycle.

The unordered pair $\{A, B\}$ is a *separation* of G if $A \cup B = V$ and G has no edge between $A \setminus B$ and $B \setminus A$. Clearly, the latter is equivalent to saying that $A \cap B$ separates A from B. If both $A \setminus B$ and $B \setminus A$ are non-empty, the separation is *proper*. The number $|A \cap B|$ is the *order* of the separation $\{A, B\}$.

Definition. G is called *k-connected* (for $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. In other words, no two vertices of G are separated by fewer than k other vertices. Every (non-empty) graph is 0-connected, and the 1-connected graphs are precisely the non-trivial connected graphs.

The greatest integer k such that G is k-connected is the *connectivity* $\kappa(G)$ of G. Thus, $\kappa(G) = 0$ if and only if G is disconnected or a K^1, and $\kappa(K^n) = n - 1$ for all $n \geq 1$.

If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E$ of fewer than edges, then G is called *ℓ-edge-connected*. The greatest integer such that G is ℓ-edge-connected is the *edge-connectivity* $\lambda(G)$ of G. In particular, we have $\lambda(G) = 0$ if G is disconnected.

Proposition. *If G is non-trivial, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Theorem 1.1 *Let $0 \neq k \in \mathbb{N}$. Every graph G with $d(G) \geq 4k$ has a (k+1)-connected subgraph H such that $\varepsilon(G) > \varepsilon(G) - k$.*

1.5 Trees and forests