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Combinatorics, Algebra, and Geometry from Cambrian to Permutree lattices

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1 Presentation

This thesis is within the range of algebraic and geometric combinatorics. We study classical structures of computer science such as partial orders and lattices on combinatorial objects. These are related to algebraic structures such as Hopf algebras and to geometrical objects such as polytopes and reflection groups. Thus we are interfacing between combinatorics, algorithmic, geometry and algebraic computation.

More specifically, we are interested in the Tamari lattice on binary trees [Tam51, HT72, MHPS12], in the weak order on permutations, and in some of their generalizations. These objects are related to a combinatorial and algebraic study of classical sorting algoritms such as the bubble-sort for the weak order and the binary search tree rotation and AVL for the Tamari lattice. The Cambrian lattices [Rea06] have been introduced recently by Reading as generalizations of the Tamari lattice. In particular, their definition is related to finite Coxeter groups [Rea16]. On the other hand, Piland and Pons have described the Permutree lattices [PP18], which interpolate between the weak order on permutations, the Tamari lattice, and the boolean lattice. In particular, the Cambrian lattices attached to the finite Coxeter group of type A are a special case of permutree lattices. The goal of this thesis is to understand the relation between the Permutree lattices and the finite Coxeter groups, *i.e.* how to define the Permutree lattices for finite Coxeter groups other than type A? More generally, we want to survey all existing results on Cambrian lattices and see which ones can be extended to Permutree lattices. On the combinatorial side, it will give us a good comprehension of the lattice quotient and sublattices of the weak order in all types. On the geometrical side, it might lead to the definition of new polytopes. And on the algebraic side, we might obtain new Hopf algebras or related structures.

2 Thematic and context

2.1 Permutree lattices

The permutree lattices [PP18] are an interpolation between three famous structures (see Fig. 1):

- the *weak order* on permutations of $\mathfrak{S}_n$, defined as the inclusion of inversion sets,
- the *Tamari lattice* on binary trees with n nodes, defined as the transitive closure of right rotations,

- the *boolean lattice* on binary sequences with $n-1$ letters.

The Tamari lattice is a fundamental combinatorial tool introduced by D. Tamari in [Tam51], and has motivated a wide variety of research directions since then. An overview of these research topics is available in the survey book [MHPS12]. One way to see that it is a lattice is to see it as a lattice quotient of the weak order [Rea06, CP17].

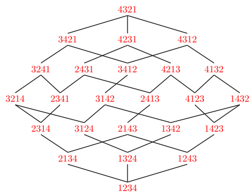
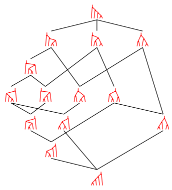
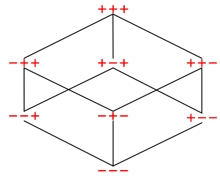
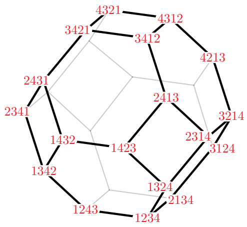
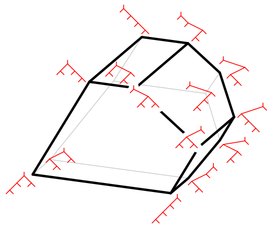
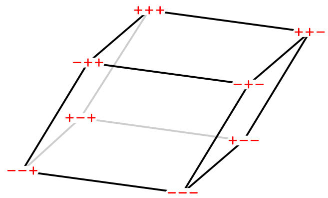
	permutations	binary trees	binary sequences
Combinatorics	Weak order  [LS96]	Tamari lattice  [MHPS12]	boolean lattice 
Geometry	Permutahedron  [Zie95, Lecture 0]	Associahedron  [Lod04, HL07]	Cube 
Algebra	Malvenuto-Reutenauer algebra [MR95, DHT02]	Loday-Ronco algebra [LR98, HNT05]	Descent Hopf algebra [GKL+95]

Figure 1: Lattice, polytopes, and Hopf algebras on 3 families of combinatorial objects

The boolean lattice is a sublattice of the Tamari lattice, itself a sublattice of the weak order. Similar patterns appear in their geometric and algebraic aspects. Indeed, the associahedron [Lod04, HL07] is a polytope whose skeleton gives the Hasse diagram of the Tamari lattice. It can be obtained from the permutahedron, whose vertices are the permutations and whose skeleton gives the weak order, by removing some facets. By removing even more facets, one obtains a cube. Algebraically, one can define structures known as *Hopf Algebras* (vector spaces with a compatible product and co-product operations) whose basis are respectively permutations [MR95, DHT02], binary trees [LR98, HNT05], and binary words [GKL+95]. The algebra on binary words is a sub Hopf algebra of the one on binary trees, itself a sub Hopf algebra of the one on permutations.

Given a decoration $\delta \in \{$⦶, ☮, ⦵, ⊗$\}^n$, a *δ-permutree* [PP18] is an oriented and labeled tree such that the node labeled i has one or two parents and one or two children depending on δ_i, and with local conditions on the labeling. Permutrees should be consider as hybrid combinatorial objects between permutations (obtained when each node has one parent and one child), binary trees (obtained when each node has one parent and two children), and binary sequences (obtained when each node has two parents and two children). There is also a natural rotation operation on δ-permutrees, with a natural orientation, and the transitive closure of this operation defines a lattice on δ-permutrees called *permutree lattice* and denoted $\mathcal{L}(\delta)$. See Fig. 2. For example, the weak order on permutations is the permutree lattice $\mathcal{L}($⦶$^n)$, the Tamari lattice

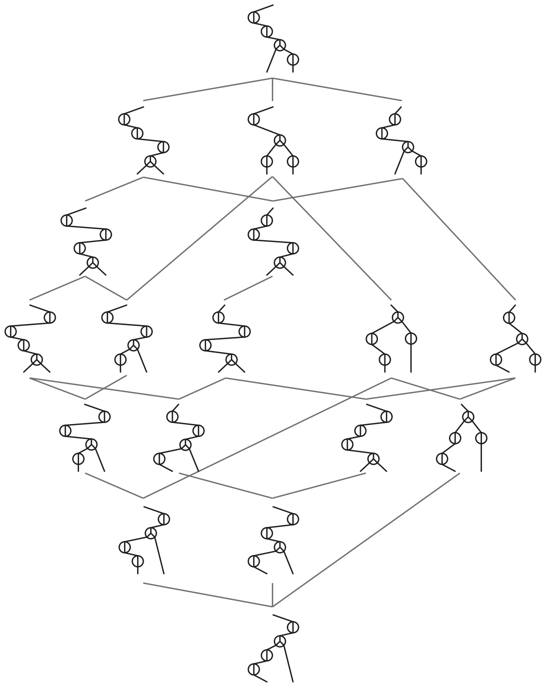


Figure 2: Some permutree lattices.

is the permutree lattice $\mathcal{L}(\text{☮}^n)$, and the boolean lattice is the permutree lattice $\mathcal{L}(\otimes^n)$. The permutree lattice is always a lattice quotient and a sublattice of the weak order on permutations. Besides, the permutrees also generalize the geometric and algebraic structures. Each permutree lattice gives a certain Permutreehedron polytope whose skeleton is the Hasse diagram of the lattice. The different permutreehedrons are contained into each other is a way that reflects the sublattices relations between permutree lattices and generalizes the permutahedron-associahedron-cube pattern. Agebraically, we can define a permutree Hopf algebra whose basis is given by the permutrees.

2.2 Coxeter groups and Cambrian lattices

The permutahedron, associahedron and cube have yet another important generalizations in the world of finite Coxeter groups. *Coxeter groups* are a family of groups generated by reflections, generalizing the classical symmetric group on permutations. See Fig. 3. Each element of a Coxeter group W has an inversion set (a subset of the root system), and the elements of the Coxeter group are naturally ordered by inclusion of their inversion sets. This order is the *weak order* on the Coxeter group W. A celebrated result of Björner states that this order is a lattice when the group is finite [Bjö84]. Moreover, it is well-known that the Hasse diagram of the weak order on W is the skeleton of a polytope called *W-permutahedron*.

The *Cambrian lattices*, defined by Reading [Rea06], are certain lattice quotients of the weak order. Their Hasse diagrams are skeleta of polytopes called *generalized associahedron*, particularly important in the theory of the finite type cluster algebras of Fomin and Zelevisnky [FZ02, FZ03]. As in type A, these generalized associahedra are obtained from the W-permutahedron by deleting some inequalities in its facet description [HLT11]. Deleting more facets leads to a parallelepiped given by the Minkowski sum of all simple roots. See Fig. 4.

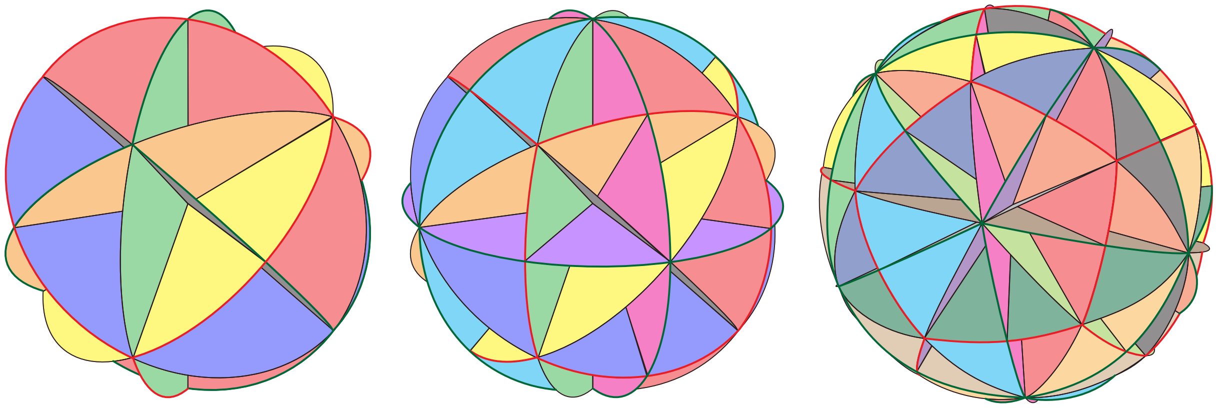


Figure 3: The Coxeter arrangements of type A_3, B_3 and H_3.

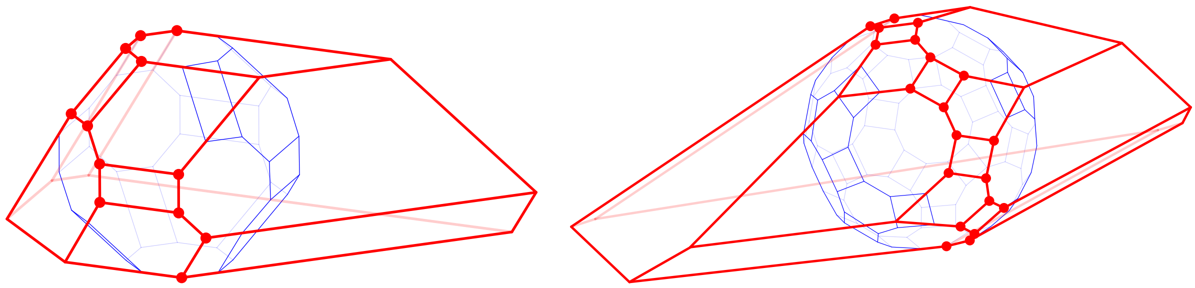


Figure 4: The permutahedra (blue) and associahedra (red) of types B_3 (left) and H_3 (right). The associahedron is obtained by deleting some inequalities in the facet description of the permutahedron.

2.3 Specific context of the thesis

When studying combinatorial objects, it is common to define a partial order sturcture on its elements. This structure is motivated by algebraic as well as algorithmic questions. A classical example is the weak order on permutations based on the bubble sort algorithm. A permutation is smaller than another one if it can be obtained by a partial application of the bubble sort. Another example is the Tamari lattice on binary trees, where the cover relations is the classical rotation of binary trees used in algorithmic in the AVL sort. These structures have long been related to difficult questions in algorithmic and complexity theory. A classical example is the diameter of the associahedron studied by Sleator, Tarjan, and Thurston [STT86] and later on by Pournin [Pou14].

As previously described, we have two families of interesting structures, the Permutree lattices and the Cambrian lattices, both generalizing the Tamari lattice. They overlap in the sense that the Cambrian lattices of type A are a specific case of Permutree lattices. They correspond to a decoration $\delta \in \{⊗, ⊗\}^n$. The problem of defining Permutree lattices for other types has never been studied.

More generally, the topic of the Tamari lattice and its generalizations has been very active in the past years, with algebraic, combinatorial and geometrical aspects on top of connections with other areas of mathematics and computer science. See for example the work of Bergeron and Préville-Ratelle on harmonic polynomials which led to the definition of the m-Tamari lattices [BPR12], the more recent generalization to ν-Tamari lattices and its geometrical realizations [PX15, CPS16], and the enumeration of intervals in relation to maps [BB09, FPR17] among many others. All three advisers of the present project have shown their expertise on the subject through their recent work [HNT05, LP18, CP17, CP15, PP18, CPP19, CP19].

3 Objectives, method and expected results

3.1 Objectives

The goal of this thesis is to understand the relation between the Permutree lattices and the finite Coxeter groups, more specifically, we will study the following questions:

- Each permutree can be injectively sent to a couple of permutations. In type A, these permutations can be characterized through pattern avoidance. Can we also characterize them by conditions on their reduced words as elements of the type A Coxeter group?
- Can this characterization be generalized to other types? Do they also give sublattices of the weak order?
- What would be the combinatorial description of permutrees in other types?
- Do permutrees in other types lead to the definition of new polytopes?
- Can we define Hopf algebras in other types?

We have good reasons to believe that some of these questions have positive answers and lead to very interesting research and results.

3.2 Method

The project can be done in 3 phases.

In phase one, the candidate must familiarize themselves with the notions at play: the different lattice structures, the finite Coxeter groups, the Permutree lattices, and the Cambrian lattices. They must gain a fine understanding of their interactions and be able to easily experiment and explore.

In phase two, they will try to better understand the relation between Permutrees and the type A Coxeter group in order to generalize some results to other classical types such as B and D.

In phase three, they will try to obtain some generic results and uniform proofs for all types and understand which properties of Cambrian lattices can be extended to all types Permutrees and why.

3.3 Computer exploration and SageMath

Florent Hivert and Viviane Pons are two active contributors to the open-source mathematical software SageMath (see `http://www.sagemath.org`). Florent Hiver, along with Nicolas Thiéry, was the initiator of a development project for experimental combinatorial research. The project was first developed on the computational system MuPAD before moving to SageMath in 2008 as Sage-Combinat. In 2015, the success of the SageMath community has been recognized and supported by the European Union through the OpenDreamKit (see `https://opendreamkit.org/`) H2020 grant: Open Digital Research Environment Toolkit for the Advancement of Mathematics. The coordinator of the project is Nicolas Thiéry (LRI, Paris-Sud). Florent Hivert and Viviane Pons are both members and Viviane Pons is the local coordinator for Paris-Sud as well as the work package leader of *Community Building, Training, Dissemination, Exploitation, and Outreach*. In particular, she has been giving many SageMath lectures, tutorials and workshops around the academic world and has organized many SageMath events.

SageMath is an open-source mathematical software under the GPL license. It combines the functionalities of many open-source software under a common interface based on the programming language python. It covers a vast range of mathematics including algebra, analysis, number theory, cryptography, numerical analysis, commutative algebra, group theory, combinatorics, graph theory, formal linear algebra, etc. The combinatorician community is particularly active, developing what is known as Sage-Combinat, whose mission is to "to improve the open-source mathematical system Sage as an extensible toolbox for computer exploration in (algebraic) combinatorics, and foster code sharing between researchers in this area". Sage-Combinat is gathering about fifty international contributors (Europe, North America, Australia, Japan, Korea, ...).

This research will strongly rely on computer exploration and mathematical experimentation. This methodology is used and promoted by all three advisers as demonstrated by the recent interventions of Viviane Pons in the python community such as her talk "Experimental pure mathematics using Sage" at PyCon 2015 and her keynote presentation "Science and open-source: what do we learn from each other?" at PyConFr 2018. The main tool will be the software SageMath. This will allow the student to join an international development team and to gain an experience in collaborative development and computer exploration. This will be very valuable for their future career.

3.4 Expected results

- publications in international journals recognized by the community (JCTA, ALCO, EJC, etc.);
- publications in international conferences of the domain (FPSAC, EuroComb);
- presentations in national and international workshops of the domain;
- contributions to the sofwatre SageMath.

4 Scientific environment

4.1 Scientific and material conditions

The thesis will take place in the GALaC team of LRI with a strong connection to the combinatorics team of LIX. The student will benefit from an active research environment with many other students working on similar subjects: 8 students (Master internships, PhD students, visiting students) working on combinatorics between GALaC and LIX this semester. A joint combinatorics seminar is organized weekly between GALaC and LIX as well as a team seminar in GALaC (at least once per month). They will also benefit from the rich combinatorial research environment of the Paris area: Flajolet seminar at IHP every two months, discrete geometry seminar at IHP, student seminar at IHP (co-organized by Justine Falque from GALaC).

4.2 International opportunities

All three advisors have strong international collaborations going on, especially in Europe and North America. Viviane Pons has been leading a PHC Amadeus project with Vienna (Austria) for the past 18 months. This will give a direct opportunity for the student to travel to Vienna and present their results at the University of Vienna and Technichal University of Vienna. Vincent Pilaud is also part of this project, which is based on a long standing collaboration between Cesar Ceballos in Vienna and both Vincent Pilaud and Viviane Pons in France.

We also have on going collaborations with researchers from Canada, especially Nantel Bergeron in Toronto (York University) and François Bergeron in Montréal (LACIM, UQAM). François Bergeron has been a visiting professor in GALaC for the past year. All three advisors are associated members of the CNRS international department LIRCO (UQAM, Montréal), which offer great travel opportunities to students.

Finally, Viviane Pons and Vincent Pilaud have a strong connection to Colombia. Vincent Pilaud has been an organizer of the ECCO (Encuentro Colombiano de Combinatoria) summer school, supported by the CIMPA program, in 2018 in Barranquilla and in 2016 in Medellin. Viviane Pons was an invited professor both times to organize the SageMath tutorials. She will be an organizer of the 2020 edition in Bogota (in charge of the CIMPA funding). The 2020 ECCO in Bogota will be the 7th edition of the school, which gathers between 100 and 200 students from all over the world (from undergrads to postdocs). Over the years, it has gained a solid reputation in the international community both from its high scientific quality and for the role it plays in connecting students from South America to the global academic world.

4.3 Possible collaboration

- University of Vienna, Austria (C. Ceballos)
- UQAM Montréal, Canada (F. Bergeron, C. Hohlweg, C. Reutenauer, F. Saliola)
- University of Los Andes, Bogota, Colombia (C. Benedetti)
- Universitad Natcional, Bogota, Colombia (R. D'Leon)
- York University, Toronto, Otario Canada (N. Bergeron, M. Zabrocki)
- University of Minnesota USA (V. Reiner)
- Université de Strasbourg (F. Chapoton)
- Université de Marne la Vallée (J.-C. Novelli, J.-Y. Thibon)

4.4 Valorization objectives

Valorization will be done through peer-reviewed publications in international journals and conferences (JCTA, ALCO, EJC, FPSAC, EuroComb), contribution to the SageMath software, and participation to national and international workshops and conferences.

5 Candidate profile

We expect the candidate to have already a good background in combinatorics with a strong understanding of the objects at play such as lattices and polytopes. The candidate must have carried a research project or internship in the domain. They must have some experience of programming.

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