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Combinatorics, Algebra, and Geometry from Cambrian to Permutree lattices

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1 Presentation

This thesis is within the range of algebraic and geometric combinatorics. We study classical structures of computer science such as *partial orders* and *lattices* over combinatorial objects. These are related to algebraic structures such as Hopf algebras and to geometrical objects such as polytopes and reflection groups. Thus we are interfacing between combinatorics, algorithmic, geometry and algebraic computation.

More specifically, we are interested in the Tamari lattice on binary trees [Tam51, HT72, MHPS12], in the weak order on permutations, and in some of their generalizations. The Cambrian lattices [Rea06] have been introduced recently by Reading as generalization of the Tamari lattice. In particular, their definition is related to finite Coxeter groups [Rea16]. On the other hand, Piland and Pons have described the Permutree lattices [PP18], which interpolate between the weak order on permutations, the Tamari lattice, and the boolean lattice. In particular, the Cambrian lattices attached to the finite Coxeter group of type A are a special case of permutree lattice. The goal of this thesis is to understand the relation between the Permutree lattices and the finite Coxeter groups, *i.e.* how to define the Permutree lattices for finite Coxeter groups other than type A? More generally, we want to survey all existing results on Cambrian lattices and see which ones can be extended to Permutree lattices. On the combinatorial side, it will give us a good comprehension of the lattice quotient and sublattices of the weak order in all types. On the geometrical side, it might lead to the definition of new polytopes. And on the algebraic side, we might obtain new Hopf algebras or related structures.

2 Thematic and context

2.1 Coxeter groups and Cambrian lattices

2.2 Permutree lattices

The permutree lattices [PP18] are an interpolation between three famous structures (see Fig. 1):

- the *weak order* on permutations of $\mathfrak{S}_n$, defined as the inclusion of inversion sets,
- the *Tamari lattice* on binary trees with n nodes, defined as the transitive closure of right rotations,

- the *boolean lattice* on binary sequences with $n-1$ letters.

The Tamari lattice is a fundamental combinatorial tool introduced by D. Tamari in [Tam51], and has motivated a wide variety of research directions since then. An overview of these research topics is available in the survey book [MHPS12]. One way to see that it is a lattice is to see it as a lattice quotient of the weak order [Rea06, **?**].

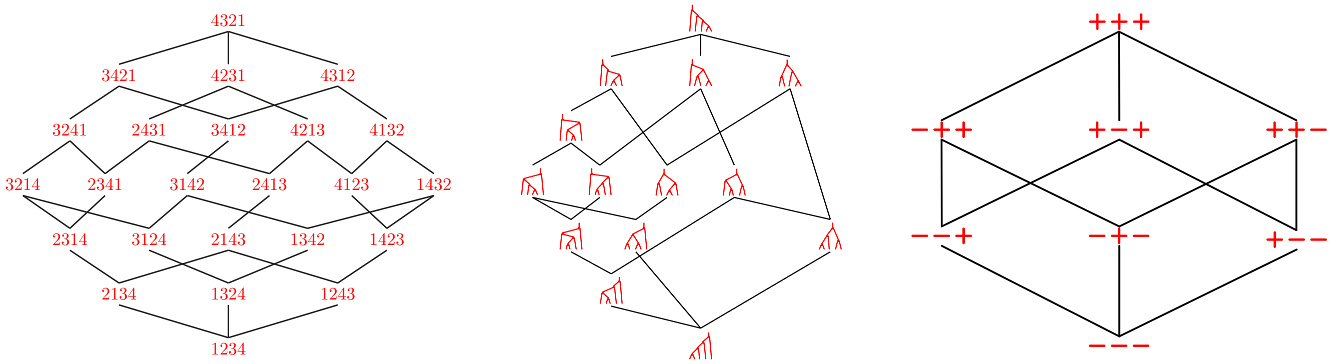

Figure 1: The weak order (left), the Tamari lattice (middle) and the boolean lattice (right).

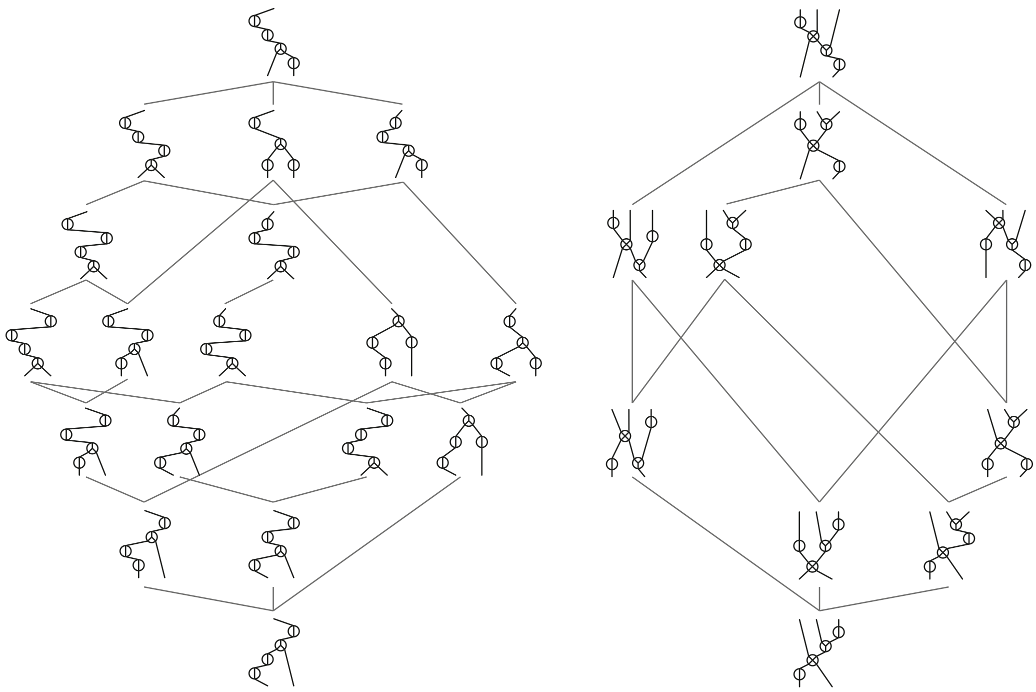

Figure 2: Some permutree lattices (right).

Given a decoration $\delta \in \{⦶, ⍦, ⍡, ⊗\}^n$, a *$\delta$-permutree* [PP18] is an oriented and labeled tree such that the node labeled i has one or two parents and one or two children depending on δ_i, and with local conditions on the labeling. Permutrees should be consider as hybrid combinatorial objects between permutations (obtained when each node has one parent and one child), binary trees (obtained when each node has one parent and two children), and binary sequences (obtained when each node has two parents and two children). There is also a natural rotation operation on δ-permutrees, with a natural orientation, and the transitive closure of this operation defines a lattice on δ-permutrees called *permutree lattice* and denoted $\mathcal{L}(\delta)$. See Fig. 2. For example, the weak order on permutations is the permutree lattice $\mathcal{L}(⦶^n)$, the Tamari lattice is the permutree lattice $\mathcal{L}(☮^n)$, and the boolean lattice is the permutree lattice $\mathcal{L}(\otimes^n)$. The permutree lattice is always a lattice quotient of the weak order on permutations.

Add something about polytopes and Hopf algebras

2.3 Specific context of the thesis

As previously described, we have two families of interesting structures, the Cambrian lattices and the Permutree lattices, both generalizing the Tamari lattice. They overlap in the sense that the Cambrian lattices of type A are a specific case of Permutree lattices. They correspond to a decoration $\delta \in \{Ⓨ, ☮\}^n$. The problem of defining Permutree lattices for other types has never been studied and raises many interesting questions:

- Each permutree can be injectively sent to a couple of permutations. In type A, these permutations can be characterized through pattern avoidance. Can we also characterize them by conditions on their reduced words as elements of the type A Coxeter group?
- Can this characterization be generalized to other types? Do they also give sublattices of the weak order?
- What would be the combinatorial description of permutrees in other types?
- Do permutree in other types lead to the definition of new polytopes?
- Can we define Hopf algebras in other types?

We have good reasons to believe that some of these questions have positive answers and lead to very interesting research and results.

Add a word about it being a trending topic, cite other stuff on the Tamari lattice

3 Objectives, method and expected results

3.1 Objectives

3.2 Method

The project can be done in 3 phases.

In phase one, the candidate must familiarize themselves with the notions at play: the different lattice structures, the finite Coxeter groups, the Permutree lattices, and the Cambrian lattices. They must gain a fine understanding of their interactions and be able to easily experiment and explore.

In phase two, they will try to better understand the relation between Permutrees and the type A Coxeter group in order to generalize some results to other classical types such as B and D.

In phase three, they will try to obtain some generic results and proof for all types and understand which properties of Cambrian lattices can be extended to all types Permutrees and why.

3.3 Computer exploration and SageMath

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3.4 Expected results

Bibliography

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