# PERMUCAMB : COMBINATOIRE, ALGÈBRE ET GÉOMÉTRIE DES TREILLIS CAMBRIENS AUX TREILLIS DE PERMUTARBRES

# PERMUCAMB : COMBINATORICS, ALGEBRA, AND GEOMETRY FROM CAMBRIAN TO PERMUTREE LATTICES

*Établissement* **Université Paris-Sud**

*École doctorale* **Sciences et Technologies de l'Information et de la Communication**

*Spécialité* **Mathématiques et Informatique**

*Unité de recherche* **LRI - Laboratoire de Recherche en Informatique**

*Directeur de la thèse* Florent HIVERT

*Co-Encadrant* Viviane PONS

*Début de la thèse le* **1 octobre 2019**

*Date limite de candidature* **20 mai 2019**

## Mots clés - Keywords

Combinatoire algébrique, Géométrie discrète, Ordre partiel, Polytope, Arbres binaires, Permutations

Algebraic combinatorics, Discrete geometry, Partial orders, Polytopes, Binary trees, Permutations

## Profil et compétences recherchées - Profile and skills required

Le candidat doit déjà avoir des connaissances de base en combinatoire avec une bonne compréhension des objets étudiés tels que les treillis et les polytopes. Il doit déjà avoir mené un projet de recherche ou un stage dans le domaine. Il doit avoir une expérience de programmation.
We expect the candidate to have already a good background in combinatorics with a strong understanding of the objects at play such as lattices and polytopes. The candidate must have carried a research project or internship in the domain. They must have some experience of programming.

## Description de la problématique de recherche - Project description

Cette thèse se situe dans le domaine de la combinatoire algébrique et géométrique. Nous étudions des structures classiques de l'informatique comme les ordres partiels et les treillis sur des objets combinatoires. Elles sont reliées à des structures algébriques telles que les algèbres de Hopf et à des notions de géométrie discrète comme les polytopes et les groupes de réflexions. Ainsi, nous sommes à l'interface entre la combinatoire, l'algorithmie, la géométrie, et le calcul algébrique.

Plus spécifiquement, nous nous intéressons au treillis de Tamari sur les arbres binaires [Tam51, HT72,MHPS12], à l'ordre faible sur les permutations et à certaines de leurs généralisations. Ces objets sont liés à une étude combinatoire et algébriques de certains algorithmes de tris classiques tels que le tri par bulle pour l'ordre faible ou les rotations d'arbres binaires et AVL pour le treillis de Tamari. Les treillis Cambriens [Rea06] sont une généralisation du treillis de Tamari récemment introduite par Reading. Leur définition repose sur les groupes de Coxeter [Rea16]. Par ailleurs, Pilaud et Pons ont décrit les treillis de Pertmutarbres [PP18] comme une interpolation entre l'ordre faible, le treillis de Tamari, et le treillis booléen. En particulier, les treillis Cambriens du groupe de Coxeter de type A sont des cas particuliers de treillis de Permutrees. Le but de cette thèse est de comprendre la relation entre les treillis de Permutrees et les groupes de Coxeter finis, c'est-à-dire comment définir les treillis de Permutrees pour les groupes de type autre que A ? Plus généralement, nous voulons parcourir l'ensemble des résultats connus sur les treillis Cambriens et voir lesquels peuvent être étendus aux treillis de Permutrees. D'un point de vue combinatoire, cela nous donnera une bonne compréhension des treillis quotients et sous-treillis de l'ordre faible dans tous les types. D'un point de vue géométrique, cela pourrait permettre de définir des nouveaux polytopes. Enfin, d'un point de vue algébrique, on peut espérer la définition de nouvelles algèbres de Hopf.

Une description détaillée du projet de thèse est disponible dans le document PDF attaché.
This thesis is within the range of algebraic and geometric combinatorics. We study classical structures of computer science such as partial orders and lattices on combinatorial objects. These are related to algebraic structures such as Hopf algebras and to geometrical objects such as polytopes and reflection groups. Thus we are interfacing between combinatorics, algorithmic, geometry and algebraic computation.

More specifically, we are interested in the Tamari lattice on binary trees [Tam51, HT72, MHPS12], in the weak order on permutations, and in some of their generalizations. These objects are related to a combinatorial and algebraic study of classical sorting algoritms such as the bubble-sort for the weak order and the binary search tree rotation and AVL for the Tamari lattice. The Cambrian lattices [Rea06] have been introduced recently by Reading as generalizations of the Tamari lattice. In particular, their definition is related to finite Coxeter groups [Rea16]. On the other hand, Piland and Pons have described the Permutree lattices [PP18], which interpolate between the weak order on permutations, the Tamari lattice, and the boolean lattice. In particular, the Cambrian lattices attached to the finite Coxeter group of type A are a special case of permutree lattices. The goal of this thesis is to understand the relation between the Permutree lattices and the finite Coxeter groups, i.e. how to define the Permutree lattices for finite Coxeter groups other than type A? More generally, we want to survey all existing results on Cambrian lattices and see which ones can be extended to Permutree lattices. On the combinatorial side, it will give us a good comprehension of the lattice quotient and sublattices of the weak order in all types. On the geometrical side, it might lead to the definition of new polytopes. And on the algebraic side, we might obtain new Hopf algebras or related structures.

Please read the attached PDF for a detailed exposition of the project.

## Thématique / Domaine / Contexte

We give a detailed description of the mathematical background (Permutree lattices, Coxeter groups and Cambrian lattices) in the attached PDF.

Algebraic and geometric combinatorics, algorithmic.

When studying combinatorial objects, it is common to define a partial order sturcture on its elements. This structure is motivated by algebraic as well as algorithmic questions. A classical example is the weak order on permutations based on the bubble sort algorithm. A permutation is smaller than another one if it can be obtained by a partial application of the bubble sort. Another example is the Tamari lattice on binary trees, where the cover relations is the classical rotation of binary trees used in algorithmic in the AVL sort. These structures have long been related to difficult questions in algorithmic and complexity theory. A classical example is the diameter of the associahedron studied by Sleator, Tarjan, and Thurston [STT86] and later on by Pournin [Pou14].

As previously described, we have two families of interesting structures, the Permutree lattices and the Cambrian lattices, both generalizing the Tamari lattice. They overlap in the sense that the Cambrian lattices of type A are a specific case of Permutree lattices. The problem of defining Permutree lattices for other types has never been studied. (See the attached PDF for more background on Permutrees and Cambrian lattices).

More generally, the topic of the Tamari lattice and its generalizations has been very active in the past years, with algebraic, combinatorial and geometrical aspects on top of connections with other areas of mathematics and computer science. See for example the work of Bergeron and Préville-Ratelle on harmonic polynomials which led to the definition of the m-Tamari lattices [BPR12], the more recent generalization to ν-Tamari lattices and its geometrical realizations [PX15, CPS16], and the enumeration of intervals in relation to maps [BB09, FPR17] among many others. All three advisers of the present project have shown their expertise on the subject through their recent work [HNT05, LP18, CP17, CP15, PP18, CPP19, CP19].

## Objectifs

The goal of this thesis is to understand the relation between the Permutree lattices and the finite Coxeter groups, more specifically, we will study the following questions:

- Each permutree can be injectively sent to a couple of permutations. In type A, these permutations can be characterized through pattern avoidance. Can we also characterize them by conditions on their reduced words as elements of the type A Coxeter group?

- Can this characterization be generalized to other types? Do they also give sublattices of the weak order?

- What would be the combinatorial description of permutrees in other types?

- Do permutrees in other types lead to the definition of new polytopes?

- Can we define Hopf algebras in other types?

We have good reasons to believe that some of these questions have positive answers and lead to very interesting research and results.

## Méthode

**Scientific approach**

The project can be done in 3 phases. In phase one, the candidate must familiarize themselves with the notions at play: the different lattice structures, the finite Coxeter groups, the Permutree lattices, and the Cambrian lattices. They must gain a fine understanding of their interactions and be able to easily experiment and explore.

In phase two, they will try to better understand the relation between Permutrees and the type A Coxeter group in order to generalize some results to other classical types such as B and D.

In phase three, they will try to obtain some generic results and uniform proofs for all types and understand which properties of Cambrian lattices can be extended to all types Permutrees and why.

**Computer exploration**

Florent Hivert and Viviane Pons are two active contributors to the open-source mathematical software SageMath (see http://www.sagemath.org). Florent Hiver, along with Nicolas Thiéry, was the initiator of a development project for experimental combinatorial research. The project was first developed on the computational system MuPAD before moving to SageMath in 2008 as Sage-Combinat. In 2015, the success of the SageMath community has been recognized and supported by the European Union through the OpenDreamKit (see https://opendreamkit.org/) H2020 grant: Open Digital Research Environment Toolkit for the Advancement of Mathematics. The coordinator of the project is Nicolas Thiéry (LRI, Paris-Sud). Florent Hivert and Viviane Pons are both members and Viviane Pons is the local coordinator for Paris-Sud as well as the work package leader of Community Building, Training, Dissemination, Exploitation, and Outreach. In particular, she has been giving many SageMath lectures, tutorials and workshops around the academic world and has organized many SageMath events.

SageMath is an open-source mathematical software under the GPL license. It combines the functionalities of many open-source software under a common interface based on the programming language python. It covers a vast range of mathematics including algebra, analysis, number theory, cryptography, numerical analysis, commutative algebra, group theory, combinatorics, graph theory, formal linear algebra, etc. The combinatorician community is particularly active, developing what is known as Sage-Combinat, whose mission is to “to improve the open-source mathematical system Sage as an extensible toolbox for computer exploration in (algebraic) combinatorics, and foster code sharing between researchers in this area”. Sage-Combinat is gathering about fifty international contributors (Europe, North America, Australia, Japan, Korea, ...).

This research will strongly rely on computer exploration and mathematical experimentation. This methodology is used and promoted by all three advisers as demonstrated by the recent interventions of Viviane Pons in the python community such as her talk “Experimental pure mathematics using Sage” at PyCon 2015 and her keynote presentation “Science and open-source: what do we learn from each other?” at PyConFr 2018. The main tool will be the software SageMath. This will allow the student to join an international development team and to gain an experience in collaborative development and computer exploration. This will be very valuable for their future career.

## Résultats attendus - Expected results

- publications in international journals recognized by the community (JCTA, ALCO, EJC,
etc.);

- publications in international conferences of the domain (FPSAC, EuroComb);

- presentations in national and international workshops of the domain;

- contributions to the sofwatre SageMath.

## Précisions sur l'encadrement - Details on the thesis supervision

La thèse sera encadrée à 50% par Viviane Pons (MCF, LRI), à 35% par Vicent Pilaud (CR, LIX, CNRS & Ecole polytechnique) et à 15% par Florent Hivert. Le candidat rencontrera au moins un.e de ses encadrant.e.s de façon hébdomadaire. Il participera au séminaire joint de combinatoire entre le LRI et le LIX qui a lieu tous les mercredi ainsi qu'aux séminaires de l'équipe GALaC qui ont lieu de façon régulière les vendredi.

## Conditions scientifiques matérielles et financières du projet de recherche

No specific security nor financial conditions.

The thesis will take place in the GALaC team of LRI with a strong connection to the combinatorics team of LIX. The student will benefit from an active research environment with many other students working on similar subjects: 8 students (Master internships, PhD students, visiting students) working on combinatorics between GALaC and LIX this semester. A joint combinatorics seminar is organized weekly between GALaC and LIX as well as a team seminar in GALaC (at least once per month). They will also benefit from the rich combinatorial research environment of the Paris area: Flajolet seminar at IHP every two months, discrete geometry seminar at IHP, student seminar at IHP (co-organized by Justine Falque from GALaC).

## Objectifs de valorisation des travaux de recherche du doctorant : diffusion, publication et confidentialité, droit à la propriété intellectuelle,...

Valorization will be done through peer-reviewed publications in international journals and conferences (JCTA, ALCO, EJC, FPSAC, EuroComb), contribution to the SageMath software, and participation to national and international workshops and conferences.

## Collaborations envisagées

University of Vienna, Austria (C. Ceballos)
UQAM Montréal, Canada (F. Bergeron, C. Hohlweg, C. Reutenauer, F. Saliola)
University of Los Andes, Bogota, Colombia (C. Benedetti)
Universitad Natcional, Bogota, Colombia (R. D'Leon)
York University, Toronto, Otario Canada (N. Bergeron, M. Zabrocki)
University of Minnesota USA (V. Reiner)
Université de Strasbourg (F. Chapoton)
Université de Marne la Vallée (J.-C. Novelli, J.-Y. Thibon)

## Ouverture Internationale

All three advisors have strong international collaborations going on, especially in Europe and North America. Viviane Pons has been leading a PHC Amadeus project with Vienna (Austria) for the past 18 months. This will give a direct opportunity for the student to travel to Vienna and present their results at the University of Vienna and Technichal University of Vienna. Vincent Pilaud is also part of this project, which is based on a long standing collaboration between Cesar Ceballos in Vienna and both Vincent Pilaud and Viviane Pons in France.

We also have on going collaborations with researchers from Canada, especially Nantel Bergeron in Toronto (York University) and François Bergeron in Montréal (LACIM, UQAM). François Bergeron has been a visiting professor in GALaC for the past year. All three advisors are associated members of the CNRS international department LIRCO (UQAM, Montréal), which offer great travel opportunities to students.

Finally, Viviane Pons and Vincent Pilaud have a strong connection to Colombia. Vincent Pilaud has been an organizer of the ECCO (Encuentro Colombiano de Combinatoria) summer school, supported by the CIMPA program, in 2018 in Barranquilla and in 2016 in Medellin. Viviane Pons was an invited professor both times to organize the SageMath tutorials. She will be an organizer of the 2020 edition in Bogota (in charge of the CIMPA funding). The 2020 ECCO in Bogota will be the 7th edition of the school, which gathers between 100 and 200 students from all over the world (from undergrads to postdocs). Over the years, it has gained a solid reputation in the international community both from its high scientific quality and for the role it plays in connecting students from South America to the global academic world.

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Dernière mise à jour le 7 mai 2019