TYPE CONES OF $\boldsymbol{g}$-VECTOR FANS

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Abstract. We investigate the type cone of the $\boldsymbol{g}$-vector fans for different generalizations of the associahedron (hopefully generalized associahedra, gentle associahedra, graph associahedra, and maybe quotientopes and brick polytopes). The objective is to determine which are the exchangeable pairs of objects that describe the type cone. When the type cone happens to be simplicial, we derive simple descriptions of all polytopal realizations of the $\boldsymbol{g}$-vector fan.

We explain recent results of [BMDM+18]. Some advantages of our approach are:

- We manage to extend their results to other families of $\boldsymbol{g}$-vector fans: first to all $\boldsymbol{g}$-vector fans of finite type cluster algebras acyclic or not (whose polytopality was only proved recently in [HPS18]), then to all $\boldsymbol{g}$-vector fans of the τ-tilting finite gentle algebras (whose polytopality was proved recently in [PPP17]), and finally to the nested fans of graph associahedra (whose polytopality was studied in [CD06, Dev09, Pos09, FS05, Zel06]).
- We have a simpler proof, only based on the transformation between the classical descriptions of a polytope of the form $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h}\}$ and $\left\{\boldsymbol{z} \in \mathbb{R}^N \;\middle|\; \boldsymbol{K}\boldsymbol{z} = \boldsymbol{K}\boldsymbol{h} \text{ and } \boldsymbol{z} \ge 0\right\}$ (this transformation is standard in optimization). In particular, we do not need to consider only rational polytopes and pass to the limit to deal with all real descriptions.
- We observe that we obtain all realizations of the $\boldsymbol{g}$-vector fans of the considered families.

1. Polytopal realizations and type cone of a simplicial fan

1.1. Polytopes and fans.

We briefly recall basic definitions and properties of polyhedral fans and polytopes, and refer to [Zie98] for a classical textbook on this topic.

A hyperplane $H \subset \mathbb{R}^d$ is a *supporting hyperplane* of a set $X \subset \mathbb{R}^d$ if $H \cap X \neq \varnothing$ and X is contained in one of the two closed half-spaces of $\mathbb{R}^d$ defined by H.

We denote by $\mathbb{R}_{\ge 0}\boldsymbol{R} := \left\{\sum_{\boldsymbol{r}\in\boldsymbol{R}} \lambda_{\boldsymbol{r}}\, \boldsymbol{r} \;\middle|\; \lambda_{\boldsymbol{r}} \in \mathbb{R}_{\ge 0}\right\}$ the *positive span* of a set $\boldsymbol{R}$ of vectors of $\mathbb{R}^d$. A *polyhedral cone* is a subset of $\mathbb{R}^d$ defined equivalently as the positive span of finitely many vectors or as the intersection of finitely many closed linear halfspaces. The *faces* of a cone C are the intersections of C with the supporting hyperplanes of C. The 1-dimensional (resp. codimension 1) faces of C are called *rays* (resp. *facets*) of C. A cone is *simplicial* if it is generated by a set of independent vectors.

A *polyhedral fan* is a collection $\mathcal{F}$ of polyhedral cones such that

- if $C \in \mathcal{F}$ and F is a face of C, then $F \in \mathcal{F}$,
- the intersection of any two cones of $\mathcal{F}$ is a face of both.

A fan is *simplicial* if all its cones are, *complete* if the union of its cones covers the ambient space $\mathbb{R}^d$, and *essential* if it contains the cone $\{\boldsymbol{0}\}$.

A *polytope* is a subset P of $\mathbb{R}^d$ defined equivalently as the convex hull of finitely many points or as a bounded intersection of finitely many closed affine halfspaces. The *dimension* $\dim(P)$ is the dimension of the affine hull of P. The *faces* of P are the intersections of P with its supporting hyperplanes. The dimension 0 (resp. dimension 1, resp. codimension 1) faces are called *vertices* (resp. *edges*, resp. *facets*) of P. A polytope is *simple* if its supporting hyperplanes are in general position, meaning that each vertex is incident to $\dim(P)$ facets (or equivalently to $\dim(P)$ edges).

The (outer) *normal cone* of a face F of P is the cone generated by the outer normal vectors of the facets of P containing F. In other words, it is the cone of vectors $\boldsymbol{c}$ such that the linear form $\boldsymbol{x} \mapsto \langle\, \boldsymbol{c} \mid \boldsymbol{x} \,\rangle$ on P is maximized by all points of the face F. The (outer) *normal fan* of P is

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the collection of the (outer) normal cones of all its faces. We say that a complete polyhedral fan $\mathcal{F}$ in $\mathbb{R}^d$ is *polytopal* when it is the normal fan of a polytope P of $\mathbb{R}^d$, and that P is a *polytopal realization* of $\mathcal{F}$.

1.2. **Type cone.** Fix an essential complete simplicial fan $\mathcal{F}$ in $\mathbb{R}^n$. Let $\boldsymbol{G}$ the $N \times n$-matrix whose rows are the rays of $\mathcal{F}$ and let $\boldsymbol{K}$ be a $(N-n) \times N$-matrix that spans the left kernel of $\boldsymbol{G}$ (*i.e.* $\boldsymbol{KG} = 0$). For any height vector $\boldsymbol{h} \in \mathbb{R}^N$, we define the polytope

$$P_{\boldsymbol{h}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Gx} \le \boldsymbol{h}\}.$$

We say that h is *$\mathcal{F}$-admissible* if $P_{\boldsymbol{h}}$ is a polytopal realization of $\mathcal{F}$. The following classical statement characterizes the $\mathcal{F}$-admissible height vectors. It is a reformulation of regularity of triangulations of vector configurations, introduced in the theory of secondary polytopes [GKZ08], see also [DRS10]. We present here a convenient formulation from [CFZ02, Lem. 2.1].

Proposition 1.1. *Then the following are equivalent for any height vector $h \in \mathbb{R}^N$:*

(1) The fan $\mathcal{F}$ is the normal fan of the polytope $P_{\boldsymbol{h}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Gx} \le \boldsymbol{h}\}$.

(2) For any two adjacent cones $\mathbb{R}\boldsymbol{R}$ and $\mathbb{R}\boldsymbol{R}'$ of $\mathcal{F}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$, we have

$$\alpha\, h_{\boldsymbol{r}} + \alpha'\, h_{\boldsymbol{r}'} + \sum_{\boldsymbol{s} \in \boldsymbol{R} \cap \boldsymbol{R}'} \beta_{\boldsymbol{s}}\, h_{\boldsymbol{s}} > 0,$$

where

$$\alpha\, \boldsymbol{r} + \alpha'\, \boldsymbol{r}' + \sum_{\boldsymbol{s} \in \boldsymbol{R} \cap \boldsymbol{R}'} \beta_{\boldsymbol{s}}\, \boldsymbol{s} = 0$$

is the unique (up to rescaling) linear dependence with $\alpha, \alpha' > 0$ between the rays of $\boldsymbol{R} \cup \boldsymbol{R}'$.

Notation 1.2. For any two adjacent cones $\mathbb{R}\boldsymbol{R}$ and $\mathbb{R}\boldsymbol{R}'$ of $\mathcal{F}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$, we denote by $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})$ the coefficient of $\boldsymbol{s}$ in the unique linear dependence between the rays of $\boldsymbol{R} \cup \boldsymbol{R}'$, *i.e.* such that

$$\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{s} = 0.$$

These coefficients are *a priori* defined up to rescaling, but we additionally fix the rescaling so that $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}) + \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}') = 2$ (this convention is arbitrary, but will be convenient in Section 2).

In this paper, we are interested in the set of all possible realizations of $\mathcal{F}$. This was studied by P. McMullen in [McM73].

Definition 1.3. The *type cone* of $\mathcal{F}$ is the cone $\mathbb{TC}(\mathcal{F})$ of all $\mathcal{F}$-admissible height vectors $\boldsymbol{h}$:

$$\begin{aligned} \mathbb{TC}(\mathcal{F}) &:= \big\{\boldsymbol{h} \in \mathbb{R}^N \;\big|\; \mathcal{F} \text{ is the normal fan of } P_{\boldsymbol{h}}\big\} \\ &= \Big\{\boldsymbol{h} \in \mathbb{R}^N \;\Big|\; \sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} > 0 \text{ for any adjacent cones } \mathbb{R}\boldsymbol{R}, \mathbb{R}\boldsymbol{R}' \text{ of } \mathcal{F}\Big\}. \end{aligned}$$

Note that the type cone is an open cone and contains a linearity subspace of dimension n (it is invariant by translation in $\boldsymbol{G}\mathbb{R}^n$). We could thus intersect $\mathbb{TC}(\mathcal{F})$ by the kernel of $\boldsymbol{G}$, or consider the projection $\boldsymbol{K}\mathbb{TC}(\mathcal{F})$.

Definition 1.4. An *extremal adjacent pair* of $\mathcal{F}$ is a pair of adjacent cones $\{\mathbb{R}\boldsymbol{R}, \mathbb{R}\boldsymbol{R}'\}$ of $\mathcal{F}$ such that the corresponding inequality $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} > 0$ in the definition of the type cone $\mathbb{TC}(\mathcal{F})$ actually defines a facet of $\mathbb{TC}(\mathcal{F})$.

In other words, extremal adjacent pairs define the extremal rays of the polar of the type cone $\mathbb{TC}(\mathcal{F})$. Understanding the extremal adjacent pairs of $\mathcal{F}$ enables to describe its type cone $\mathbb{TC}(\mathcal{F})$ and thus all its polytopal realizations.

Remark 1.5. Since the type cone is an open N dimensional cone with a linearity subspace of dimension n, it has at least $N-n$ facets. We say that the type cone is *simplicial* when it has precisely $N-n$ facets.

1.3. **Coherent fans.** Two rays $\boldsymbol{r}$ and $\boldsymbol{r}'$ of $\mathcal{F}$ are called *compatible* if there are both contained in a cone of $\mathcal{F}$ and *exchangeable* if there are two adjacent cones $\mathbb{R}\boldsymbol{R}$ and $\mathbb{R}\boldsymbol{R}'$ of $\mathcal{F}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$. If $\boldsymbol{R}(\mathcal{F})$ denotes the set of rays of $\mathcal{F}$, a *compatibility degree* for $\mathcal{F}$ is a function $(- \parallel -) : \boldsymbol{R}(\mathcal{F}) \times \boldsymbol{R}(\mathcal{F}) \to \mathbb{R}$ such that

- $(\boldsymbol{r} \parallel \boldsymbol{r}') = -1$ if $\boldsymbol{r} = \boldsymbol{r}'$ and is non-negative otherwise,
- $(\boldsymbol{r} \parallel \boldsymbol{r}') = 0$ if $\boldsymbol{r}$ and $\boldsymbol{r}'$ are compatible,
- $(\boldsymbol{r} \parallel \boldsymbol{r}') > 0$ if $\boldsymbol{r} \neq \boldsymbol{r}'$ are incompatible,
- $(\boldsymbol{r} \parallel \boldsymbol{r}') = 1 = (\boldsymbol{r}' \parallel \boldsymbol{r})$ if and only if $\boldsymbol{r}$ and $\boldsymbol{r}'$ are exchangeable.

The definition of compatibility degree is not really useful here. — V.

Definition 1.6. We say that two exchangeable rays $\boldsymbol{r}, \boldsymbol{r}'$ of $\mathcal{F}$ admit a *unique exchange relation* when the linear dependence

$$\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{s} = 0.$$

does not depend on the pair $\{\boldsymbol{R}, \boldsymbol{R}'\}$ of adjacent cones of $\mathcal{F}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$, but only on the pair of rays $\boldsymbol{r}, \boldsymbol{r}'$. This implies in particular that the rays $\boldsymbol{s}$ for which $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s}) \neq 0$ belong to $\boldsymbol{R} \cup \boldsymbol{R}'$ for any pair $\{\boldsymbol{R}, \boldsymbol{R}'\}$ of adjacent cones of $\mathcal{F}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$. These rays are thus called *forced rays* for the exchangeable pair $\{\boldsymbol{r}, \boldsymbol{r}'\}$.

We say that the fan $\mathcal{F}$ has the *unique exchange property* if any two exchangeable rays $\boldsymbol{r}, \boldsymbol{r}'$ of $\mathcal{F}$ admit a unique exchange relation

When $\mathcal{F}$ has the unique exchange relation property, we change the notation $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})$ to $\alpha_{\boldsymbol{r},\boldsymbol{r}'}(\boldsymbol{s})$ and we obtain that the type cone of $\mathcal{F}$ is expressed as

$$\mathbb{TC}(\mathcal{F}) = \Big\{ \boldsymbol{h} \in \mathbb{R}^N \;\Big|\; \sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{r},\boldsymbol{r}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} > 0 \text{ for any exchangeable rays } \boldsymbol{r}, \boldsymbol{r}' \text{ of } \mathcal{F} \Big\}.$$

Definition 1.7. In a fan $\mathcal{F}$ with the unique exchange relation property, an *extremal exchangeable pair* is a pair of exchangeable rays $\{\boldsymbol{r}, \boldsymbol{r}'\}$ such that the corresponding inequality $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{r},\boldsymbol{r}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} > 0$ defines a facet of the type cone $\mathbb{TC}(\mathcal{F})$.

In this paper, we will only consider fans with the unique exchange relation property, and our objective will be to describe their extremal exchangeable pairs.

1.4. **Alternative polytopal realizations.** In this section, we provide alternative polytopal realizations of the fan $\mathcal{F}$. We also discuss the behavior of these realizations in the situation when the type cone $\mathbb{TC}(\mathcal{F})$ is simplicial.

We still consider an essential complete simplicial fan $\mathcal{F}$ in $\mathbb{R}^n$, the $N \times n$-matrix $\boldsymbol{G}$ whose rows are the rays of $\mathcal{F}$, and the $(N-n) \times N$-matrix $\boldsymbol{K}$ which spans the left kernel of $\boldsymbol{G}$.

Proposition 1.8. *The affine map $\Psi : \mathbb{R}^n \to \mathbb{R}^N$ defined by $\Psi(\boldsymbol{x}) = \boldsymbol{h} - G\boldsymbol{x}$ sends the polytope*

See [DRS10, Coro. 9.5.7] — V.

$$P_{\boldsymbol{h}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h}\}$$

to the polytope

$$Q_{\boldsymbol{h}} := \big\{\boldsymbol{z} \in \mathbb{R}^N \;\big|\; \boldsymbol{K}\boldsymbol{z} = \boldsymbol{K}\boldsymbol{h} \text{ and } \boldsymbol{z} \geq 0\big\}.$$

Proof. For $\boldsymbol{x}$ in $P_{\boldsymbol{h}}$, we have $\Psi(\boldsymbol{x}) \geq 0$ by definition and $\boldsymbol{K}\Psi(\boldsymbol{x}) = \boldsymbol{K}\boldsymbol{h} - \boldsymbol{K}\boldsymbol{G}\boldsymbol{x} = \boldsymbol{K}\boldsymbol{h}$ since $\boldsymbol{K}$ is the left kernel of $\boldsymbol{G}$. Therefore $\Psi(\boldsymbol{x}) \in Q_{\boldsymbol{h}}$. Moreover, the map $\Psi : P_{\boldsymbol{h}} \to Q_{\boldsymbol{h}}$ is:

- injective: Indeed, $\Psi(\boldsymbol{x}) = \Psi(\boldsymbol{x}')$ implies $\boldsymbol{G}(\boldsymbol{x} - \boldsymbol{x}') = 0$ and $\boldsymbol{G}$ has full rank since $\mathcal{F}$ is essential and complete.
- surjective: Indeed, for $\boldsymbol{z} \in Q_{\boldsymbol{h}}$, we have $\boldsymbol{K}(\boldsymbol{h} - \boldsymbol{z}) = 0$ so that $\boldsymbol{h} - \boldsymbol{z}$ belongs to the right kernel of $\boldsymbol{K}$ which is the image of $\boldsymbol{G}$. Therefore, there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{h} - \boldsymbol{z} = \boldsymbol{G}\boldsymbol{x}$. Therefore, $\boldsymbol{z} = \Psi(\boldsymbol{x})$ and $\boldsymbol{x} \in P_{\boldsymbol{h}}$ since $\boldsymbol{h} - \boldsymbol{G}\boldsymbol{x} = \boldsymbol{z} \geq 0$. ☐

Corollary 1.9. *Assume that the type cone $\mathbb{TC}(\mathcal{F})$ is simplicial and let $\boldsymbol{K}$ be the $(N-n) \times N$-matrix whose rows are the outer normal vectors of the facets of $\mathbb{TC}(\mathcal{F})$. Then the polytope*

$$R_{\boldsymbol{\ell}} := \big\{\boldsymbol{z} \in \mathbb{R}^N \;\big|\; K\boldsymbol{z} = \boldsymbol{\ell} \text{ and } \boldsymbol{z} \geq 0\big\}$$

is a realization of the fan $\mathcal{F}$ for any positive vector $\boldsymbol{\ell} \in \mathbb{R}^{N-n}_{>0}$. Moreover, the polytopes $R_{\boldsymbol{\ell}}$ for $\boldsymbol{\ell} \in \mathbb{R}^{N-n}_{>0}$ describe all polytopal realizations of $\mathcal{F}$.

Proof. Let $\boldsymbol{\ell} \in \mathbb{R}^{N-n}_{>0}$. Since $\boldsymbol{K}$ has full rank there exists $\boldsymbol{h} \in \mathbb{R}^N$ such that $\boldsymbol{K}\boldsymbol{h} = \boldsymbol{\ell}$. Since $\boldsymbol{K}\boldsymbol{h} \geq 0$ and the rows of $\boldsymbol{K}$ are precisely all outer normal vectors of the facets of the type cone $\mathbb{TC}(\mathcal{F})$, we obtain that $\boldsymbol{h}$ belongs to $\mathbb{TC}(\mathcal{F})$. Since $R_{\boldsymbol{\ell}} = Q_{\boldsymbol{h}} \sim P_{\boldsymbol{h}}$ by Proposition 1.8, we conclude that $R_{\boldsymbol{\ell}}$ is a polytopal realization of $\mathcal{F}$. Since $\mathbb{TC}(\mathcal{F})$ is simplicial, we have $\boldsymbol{K}\mathbb{TC}(\mathcal{F}) = \mathbb{R}^{N-n}_{>0}$, so that we obtain all polytopal realizations of $\mathcal{F}$ this way. $\square$

1.5. Faces of the type cone and Minkowski sums.

Lemma 1.10. *For $\boldsymbol{h}, \boldsymbol{h}' \in \mathbb{R}^N$, the polytope $P_{\boldsymbol{h}+\boldsymbol{h}'}$ is the Minkowski sum of the polytopes $P_{\boldsymbol{h}}$ and $P_{\boldsymbol{h}'}$.*

Proof. This follows from the definition of $P_{\boldsymbol{h}}$ and of Minkowski sums:

$$\begin{aligned} P_{\boldsymbol{h}} + P_{\boldsymbol{h}'} &= \{\boldsymbol{x} + \boldsymbol{x}' \mid \boldsymbol{x} \in P_{\boldsymbol{h}} \text{ and } \boldsymbol{x}' \in P_{\boldsymbol{h}'}\} = \big\{\boldsymbol{x} + \boldsymbol{x}' \bigm| \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \text{ and } \boldsymbol{G}\boldsymbol{x}' \leq \boldsymbol{h}'\big\} \\ &= \big\{\boldsymbol{y} \in \mathbb{R}^n \bigm| \boldsymbol{G}\boldsymbol{y} \leq \boldsymbol{h} + \boldsymbol{h}'\big\} = P_{\boldsymbol{h}+\boldsymbol{h}'}. \end{aligned}$$ $\square$

Lemma 1.10 ensures that convex combinations in the type cone correspond to Minkowski combinations of polytopes. This provides natural Minkowski summands for all polytopal realizations of $\mathcal{F}$.

Corollary 1.11. *Any polytope in $\mathbb{TC}(\mathcal{F})$ is the Minkowski sum of at most $N - n$ polytopes corresponding to the rays of $\boldsymbol{K}\mathbb{TC}(\mathcal{F})$.*

2. Applications to different generalizations of the associahedron

In this section, we study the type cones of complete simplicial fans arising as normal fans of three families of generalizations of the associahedron: the generalized associahedra of finite type cluster algebras [FZ02, FZ03, FZ07, HLT11], the gentle associahedra [PPP17], and the graph associahedra [CD06, Pos09, FS05, Zel06]. All these families contain the classical associahedra $\mathsf{Asso}(n)$ constructed in [SS93, Lod04]. We first describe these associahedra and their type cones as they are the prototypes of our constructions.

2.1. Classical associahedra.

We quickly recall the combinatorics and the geometric construction of [SS93, Lod04] for the associahedron $\mathsf{Asso}(n)$. The face lattice of $\mathsf{Asso}(n)$ is the reverse inclusion lattice of dissections (*i.e.* pairwise non-crossing subsets of diagonals) of a convex $(n+3)$-gon. In particular, its vertices correspond to triangulations of the $(n+3)$-gon and its facets correspond to internal diagonals of the $(n+3)$-gon. Equivalently, its vertices correspond to rooted binary trees with $(n+1)$ internal nodes, and its facets correspond to proper intervals of $[n+1]$ (*i.e.* intervals distinct from $\varnothing$ and $[n+1]$). These bijections become clear when the convex $(n+3)$-gon is drawn with its vertices on a concave curve labeled by $0, \dots, n+2$. The following statement provides three equivalent geometric constructions of $\mathsf{Asso}(n)$.

Proposition 2.1. *The associahedron $\mathsf{Asso}(n)$ can be described equivalently as:*

- *the convex hull of the points $L(T) \in \mathbb{R}^{n+1}$ for all rooted binary trees T with $n+1$ internal nodes, where the ith coordinate of $L(T)$ is the product of the number of leaves in the left subtree by the number of leaves in the right subtree of the ith node of T in inorder* [Lod04],
- *the intersection of the hyperplane $\big\{\boldsymbol{x} \in \mathbb{R}^{n+1} \bigm| \sum_{\ell \in [n+1]} x_\ell = \binom{n+2}{2}\big\}$ with the halfspaces $\big\{\boldsymbol{x} \in \mathbb{R}^{n+1} \bigm| \sum_{i \leq \ell \leq j} x_\ell \geq \binom{j-i+2}{2}\big\}$ for all proper intervals $[i,j]$ of $[n+1]$,* [SS93],
- *the Minkowski sum of the faces $\triangle_{[i,j]}$ of the standard n-dimensional simplex $\triangle$ corresponding to all proper intervals $[i,j]$ of $[n+1]$,* [Pos09].

We now focus on the normal fan $\mathcal{F}(n)$ of $\mathsf{Asso}(n)$. Since $\mathsf{Asso}(n)$ leaves in a hyperplane of $\mathbb{R}^{n+1}$, so does its normal fan. Let $\mathbb{H} := \big\{\boldsymbol{x} \in \mathbb{R}^{n+1} \bigm| \sum_{\ell \in [n+1]} x_\ell = 0\big\}$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{H}$ denote the orthogonal projection.

Proposition 2.2. *In the normal fan $\mathcal{F}(n)$ of $\mathsf{Asso}(n)$,*

- *the normal vector of the facet corresponding to an internal diagonal (a, b) of the $(n+3)$-gon is the vector* $\boldsymbol{g}(a,b) := \pi\big(\sum_{a<\ell<b} \boldsymbol{e}_\ell\big) = (n+1)\sum_{a<\ell<b} \boldsymbol{e}_\ell - (j-i+1)\sum_{1\le\ell\le n+1} \boldsymbol{e}_\ell$.
- *the normal cone of the vertex corresponding to a rooted binary tree T is the incidence cone* $\big\{\boldsymbol{x} \in \mathbb{H} \;\big|\; x_i \le x_j \text{ for all edges } i \to j \text{ in } T\big\}$.

Let us also recall the linear dependencies in this fan and observe that it has the unique exchange property. From now on, we use the convention that $\boldsymbol{g}(a, b) = 0$ when (a, b) is a boundary edge of the $(n + 3)$-gon.

Proposition 2.3. *Consider two crossing diagonals (a, b) and (a', b'), and any two triangulations T, T' such that $T \smallsetminus \{(a, b)\} = T' \smallsetminus \{(a', b')\}$. Then both triangulations T and T' contain the square $aa'bb'$, and the linear dependence between the $\boldsymbol{g}$-vectors of $T \cup T'$ is given by*

$$\boldsymbol{g}(a, b) + \boldsymbol{g}(a', b') = \boldsymbol{g}(a, b') + \boldsymbol{g}(a', b).$$

In particular, the fan $\mathcal{F}(n)$ has the unique exchange property.

From these linear dependencies, we obtain that the type cone of the fan $\mathcal{F}(n)$.

Corollary 2.4. *Let $n \in \mathbb{N}$ and $X(n) := \{(a, b) \mid 0 \le a < b \le n + 2\}$. Then the type cone of the normal fan $\mathcal{F}(n)$ of $\mathsf{Asso}(n)$ is given by*

$$\mathbb{TC}(\mathcal{F}(n)) = \left\{ \boldsymbol{h} \in \mathbb{R}^{X(n)} \;\middle|\; \begin{array}{l} \boldsymbol{h}_{(0,n+2)} = 0, \qquad \textit{and} \qquad \boldsymbol{h}_{(a,a+1)} = 0 \textit{ for all } 0 \le a \le n+1 \\ \boldsymbol{h}_{(a,b)} + \boldsymbol{h}_{(a',b')} > \boldsymbol{h}_{(a,b')} + \boldsymbol{h}_{(a',b)} \textit{ for all } 0 \le a < a' < b < b' \le n+2 \end{array} \right\}.$$

We now describe the facets of this type cone $\mathbb{TC}(\mathcal{F}(n))$.

Proposition 2.5. *Two internal diagonals (a, b) and (a', b') of the $(n + 3)$-gon form an extremal exchangeable pair for the fan $\mathcal{F}(n)$ if and only if $a = a' + 1$ and $b = b' + 1$, or the opposite.*

Proof. Let $(\boldsymbol{f}_{(a,b)})_{0\le a<b\le n+2}$ be the canonical basis of $\mathbb{R}^{\binom{n+3}{2}}$. Consider two crossing internal diagonals (a, b) and (a', b') with $0 \le a < a' < b < b' \le n + 2$. By Proposition 2.3, the linear dependence between the corresponding $\boldsymbol{g}$-vectors is given by

$$\boldsymbol{g}(a, b) + \boldsymbol{g}(a', b') = \boldsymbol{g}(a, b') + \boldsymbol{g}(a', b).$$

Therefore, the outer normal vector of the corresponding inequality of the type cone $\mathbb{TC}(\mathcal{F}(n))$ is

$$\boldsymbol{n}(a, b, a', b') := \boldsymbol{f}_{(a,b)} + \boldsymbol{f}_{(a',b')} - \boldsymbol{f}_{(a,b')} - \boldsymbol{f}_{(a',b)}.$$

Denoting

$$\boldsymbol{m}(c, d) := \boldsymbol{n}(c, d-1, c+1, d) = \boldsymbol{f}_{(c,d-1)} + \boldsymbol{f}_{(c+1,d)} - \boldsymbol{f}_{(c,d)} - \boldsymbol{f}_{(c+1,d-1)},$$

we obtain that

$$\boldsymbol{n}(a, b, a', b') = \sum_{\substack{c\in[a,a'[\\ d\in]b,b']}} \boldsymbol{m}(c, d).$$

Indeed, on the right hand side, the basis vector $\boldsymbol{f}_{(c,d)}$ appears with a positive sign in $\boldsymbol{m}(c, d+1)$ for $(c, d) \in [a, a'[\times [b, b'[$ and in $\boldsymbol{m}(c-1, d)$ for $(c, d) \in\,]a, a'] \times]b, b']$, and with a negative sign in $\boldsymbol{m}(c, d)$ for $(c, d) \in [a, a'[\times]b, b']$ and in $\boldsymbol{m}(c-1, d+1)$ for $(c, d) \in\,]a, a'] \times [b, b'[$. Therefore, these contributions all vanish except when (c, d) is one of the diagonals (a, b), (a', b'), (a, b') or (a', b). This shows that any exchange relation is a positive linear combination of the exchange relations corresponding to all pairs of diagonals (a, b) and (a', b') of the $(n + 3)$-gon such that $a = a' + 1$ and $b = b' + 1$ or the opposite.

Conversely, since $\mathcal{F}(n)$ has dimension n and $n(n + 3)/2$ rays (corresponding to the internal diagonals of the $(n+3)$-gon), we know from Remark 1.5 that there are at least $n(n+1)/2$ extremal exchangeable pairs. We thus conclude that all exchangeable pairs of diagonals $\{(a, b-1), (a+1, b)\}$ for $1 \le a < b - 2 \le n$ are extremal. □

The following statement follows from the end of the previous proof.

Corollary 2.6. *The type cone $\mathbb{TC}(\mathcal{F}(n))$ is simplicial.*

Combining Corollary 1.9, Corollary 2.6 and Proposition 2.5, we derive the following description of all polytopal realizations of the fan $\mathcal{F}(n)$, thus recovering all associahedra of [AHBHY17, Sect. 3.2]. We note that the arguments used in [AHBHY17, Sect. 3.2] were quite different from the present approach.

Corollary 2.7 ([AHBHY17, Sect. 3.2]). *For $n \in \mathbb{N}$, define $X(n) := \{(a,b) \mid 0 \le a < b \le n+2\}$ and $Y(n) := \{(a,b) \mid 0 \le a < b-2 \le n\}$. Then for any $\boldsymbol{\ell} \in \mathbb{R}^{Y(n)}_{>0}$, the polytope*

$$R_{\boldsymbol{\ell}}(n) := \left\{ \boldsymbol{z} \in \mathbb{R}^{X(n)} \;\middle|\; \begin{array}{l} \boldsymbol{z} \ge 0, \quad \boldsymbol{z}_{(0,n+2)} = 0 \quad \text{and} \quad \boldsymbol{z}_{(a,a+1)} = 0 \text{ for all } 0 \le a \le n+1 \\ \boldsymbol{z}_{(a,b-1)} + \boldsymbol{z}_{(a+1,b)} - \boldsymbol{z}_{(a,b)} - \boldsymbol{z}_{(a+1,b-1)} = \boldsymbol{\ell}_{(a,b)} \text{ for all } (a,b) \in Y(n) \end{array} \right\}$$

is an n-dimensional associahedron.

2.2. **Generalized associahedra.** The acyclic and simply-laced case was treated in [BMDM+18]. Computer experiments indicate that the type cone is always simplicial for any seed (acyclic or not) in any finite type cluster algebra. While the case of acyclic seeds can be handled by representation theory [BMDM+18], we have no proof at the moment for cyclic seeds.

One important observation is that it seems there is one extremal exchangeable pair for each positive c-vector, meaning that for each positive c-vector β, there is precisely one extremal exchangeable pair $\{x, x'\}$ of cluster variables for which the flip of x to x' (for any pair of clusters $\{X, X'\}$ with $X \smallsetminus \{x\} = X' \smallsetminus \{x'\}$) is in the direction of β. The goal is thus to determine for each positive root which exchangeable pair is extremal. This should be done using the Auslander-Reiten quiver to construct two cluster variables from a c-vector (see the next paragraph for the idea).

2.3. **Gentle associahedra.** Gentle associahedra were constructed by Y. Palu, V. Pilaud and P.-G. Plamondon [PPP17] in the context of support τ-tilting for gentle algebras. For a given τ-tilting finite gentle quiver $\bar{Q}$ (defined in the next section), the $\bar{Q}$-associahedron $\mathsf{Asso}(\bar{Q})$ is a simple polytope which encodes certain representations of $\bar{Q}$ and their τ-tilting relations. Combinatorially, the $\bar{Q}$-associahedron is a polytopal realization of the non-kissing complex of $\bar{Q}$, defined as the simplicial complex of all collections of walks on the blossoming quiver $\bar{Q}^{❀}$ which are pairwise non-kissing. The non-kissing complex encompasses two families of simplicial complexes studied independently in the literature: on the one hand the grid associahedra introduced by T. K. Petersen, P. Pylyavskyy and D. Speyer in [PPS10] for a staircase shape, studied by F. Santos, C. Stump and V. Welker [SSW17] for rectangular shapes, and extended by T. McConville in [McC17] for arbitrary grid shapes; and on the other hand the Stokes polytopes and accordion associahedra studied by Y. Baryshnikov [Bar01], F. Chapoton [Cha16], A. Garver and T. McConville [GM18] and T. Manneville and V. Pilaud [MP17b]. These two families naturally extend the classical associahedron, obtained from a line quiver. Non-kissing complexes are geometrically realized by polytopes called gentle associahedra, whose normal fan is called the non-kissing fan: its rays correspond to walks in the quiver and its cones are generated by the non-kissing walks. In this section, we describe the type cone of the non-kissing fan of a quiver $\bar{Q}$ with no self-kissing walks.

2.3.1. *Non-kissing complex and non-kissing fan of a gentle quiver.* We present the definitions and properties of the non-kissing complex of a gentle quiver, following the presentation of [PPP17].

Gentle quivers. Consider a *bound quiver* $\bar{Q} = (Q, I)$, formed by a finite quiver $Q = (Q_0, Q_1, s, t)$ and an ideal I of the path algebra kQ (the k-vector space generated by all paths in Q, including vertices as paths of length zero, with multiplication induced by concatenation of paths) such that I is generated by linear combinations of paths of length at least two, and I contains all sufficiently large paths. See [ASS06] for background.

Following M. Butler and C. Ringel [BR87], we say that $\bar{Q}$ is a *gentle bound quiver* when

(i) each vertex $a \in Q_0$ has at most two incoming and two outgoing arrows,
(ii) the ideal I is generated by paths of length exactly two,
(iii) for any arrow $\beta \in Q_1$, there is at most one arrow $\alpha \in Q_1$ such that $t(\alpha) = s(\beta)$ and $\alpha\beta \notin I$ (resp. $\alpha\beta \in I$) and at most one arrow $\gamma \in Q_1$ such that $t(\beta) = s(\gamma)$ and $\beta\gamma \notin I$ (resp. $\beta\gamma \in I$).

The algebra kQ/I is called a *gentle algebra*.

The *blossoming quiver* $\bar{Q}^{❀}$ of a gentle quiver is the gentle quiver obtained by completing all vertices of $\bar{Q}$ with additional incoming or outgoing *blossoms* such that all vertices of $\bar{Q}$ become 4-valent. We now always assume that $\bar{Q}$ is a gentle quiver with blossoming quiver $\bar{Q}^{❀}$.

Strings and walks. A *string* in $\bar{Q} = (Q, I)$ is a word of the form $\rho = \alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\cdots\alpha_\ell^{\varepsilon_\ell}$, where

(i) $\alpha_i \in Q_1$ and $\varepsilon_i \in \{-1, 1\}$ for all $i \in [\ell]$,
(ii) $t(\alpha_i^{\varepsilon_i}) = s(\alpha_{i+1}^{\varepsilon_{i+1}})$ for all $i \in [\ell - 1]$,
(iii) there is no path $\pi \in I$ such that π or π^{-1} appears as a factor of ρ, and
(iv) ρ is reduced, in the sense that no factor $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ appears in ρ, for $\alpha \in Q_1$.

The integer ℓ is called the *length* of the string ρ. For each vertex $a \in Q_0$, there is also a *string of length zero*, denoted by ε_a, that starts and ends at a. We often implicitly identify the two inverse strings ρ and ρ^{-1}, and call it an *undirected string* of $\bar{Q}$. Let $\mathcal{S}(\bar{Q})$ denote the set of strings of $\bar{Q}$.

A *walk* of $\bar{Q}$ is a maximal maximal string of its blossoming quiver $\bar{Q}^{❀}$ (meaning that each endpoint is a blossom). As for strings, we implicitly identify the two inverse walks ω and ω^{-1}, and call it an *undirected walk* of $\bar{Q}$. Let $\mathcal{W}(\bar{Q})$ denote the set of walks of $\bar{Q}$.

A *substring* of a walk $\omega = \alpha_1^{\varepsilon_1}\cdots\alpha_\ell^{\varepsilon_\ell}$ of $\bar{Q}$ is a string $\sigma = \alpha_{i+1}^{\varepsilon_{i+1}}\cdots\alpha_{j-1}^{\varepsilon_{j-1}}$ of $\bar{Q}$ for some indices $1 \le i < j \le \ell$. Note that by definition,

- the endpoints of σ are not allowed to blossom endpoints of ω,
- the position of σ as a factor of ω matters (the same string at a different position is considered a different substring).
- the string ε_a is a substring of ω for each occurence of a as a vertex of ω (take $j = i + 1$).

We denote by $\Sigma(\omega)$ the set of substrings of ω. We say that the substring $\sigma = \alpha_{i+1}^{\varepsilon_{i+1}}\cdots\alpha_{j-1}^{\varepsilon_{j-1}}$ is *at the bottom* (resp. *on top*) of the walk $\omega = \alpha_1^{\varepsilon_1}\cdots\alpha_\ell^{\varepsilon_\ell}$ if $\varepsilon_i = 1$ and $\varepsilon_j = -1$ (resp. if $\varepsilon_i = -1$ and $\varepsilon_j = 1$). In other words the two arrows of ω incident to the endpoints of σ point towards σ (resp. outwards from σ). We denote by $\Sigma_{\mathrm{bot}}(\omega)$ and $\Sigma_{\mathrm{top}}(\omega)$ the sets of bottom and top substrings of ω respectively. We use the same notation for undirected walks (of course, substrings of an undirected walk are undirected).

A *peak* (resp. *deep*) of a walk ω is a substring of ω of length zero which is on top (resp. at the bottom of ω). A walk ω is *straight* if it has no peak or deep (*i.e.* if ω or ω^{-1} is a path in $\bar{Q}^{❀}$), and *bending* otherwise. We denote by $\mathsf{peaks}(\omega)$ (resp. $\mathsf{deeps}(\omega)$) the multisets of vertices of peaks (resp. deeps) of ω.

Non-kissing complex. Let ω and ω' be two undirected walks on $\bar{Q}$. We say that ω *kisses* ω' if $\Sigma_{\mathrm{top}}(\omega) \cap \Sigma_{\mathrm{bot}}(\omega') \neq \varnothing$. In other words, ω and ω' share a common substring σ, and both arrows of ω (resp. of ω') not in σ are outgoing (resp. incoming) at the endpoints of σ. We say that ω and ω' are *kissing* if ω kisses ω' or ω' kisses ω (or both). Note that we authorize the situation where the common finite substring is reduced to a vertex a, that ω can kiss ω' several times, that ω and ω' can mutually kiss, and that ω can kiss itself. We say that a walk is *proper* if it is not straight nor self-kissing. We denote by $\mathcal{W}_{\mathrm{prop}}(\bar{Q})$ the set of all proper walks of $\bar{Q}$.

The (reduced) *non-kissing complex* of $\bar{Q}$ is the simplicial complex $\mathcal{NK}(\bar{Q})$ whose faces are the collections of pairwise non-kissing proper walks of $\bar{Q}$. As shown in [PPP17, Thm. 2.46], this simplicial complex is a combinatorial model for the support τ-tilting complex on τ-rigid modules over kQ/I. The quiver $\bar{Q}$ is called *τ-tilting finite* or *non-kissing finite* when this complex is finite (in other words, $\bar{Q}$ has finitely may non-kissing walks).

Non-kissing fan and gentle associahedron. Let $(\boldsymbol{e}_v)_{v \in Q_0}$ denote the canonical basis of $\mathbb{R}^{Q_0}$. For a multiset $V = \{\{v_1, \dots, v_k\}\}$ of Q_0, we denote by $\boldsymbol{m}_V := \sum_{i \in [k]} \boldsymbol{e}_{v_i}$. The ***g**-vector* of a walk ω is the vector $\boldsymbol{g}(\omega) \in \mathbb{R}^{Q_0}$ defined by $\boldsymbol{g}(\omega) := \boldsymbol{m}_{\mathsf{peaks}(\omega)} - \boldsymbol{m}_{\mathsf{deeps}(\omega)}$. Note that by definition, the $\boldsymbol{g}(\omega) = 0$ for a straight walk ω. We also define $\boldsymbol{g}(F) := \{\boldsymbol{g}(\omega) \mid \omega \in F\}$ for a face F of $\mathcal{NK}(\bar{Q})$. These vectors support a complete simplicial fan realization of the non-kissing complex complex. Examples are illustrated in Figure 1.

Theorem 2.8 ([PPP17, Thm. 4.17])**.** *For any non-kissing finite gentle quiver $\bar{Q}$, the set of cones*

$$\mathcal{F}(\bar{Q}) := \big\{ \mathbb{R}_{\ge 0}\, \boldsymbol{g}(F) \;\big|\; F \text{ non-kissing face of } \mathcal{NK}(\bar{Q}) \big\}$$

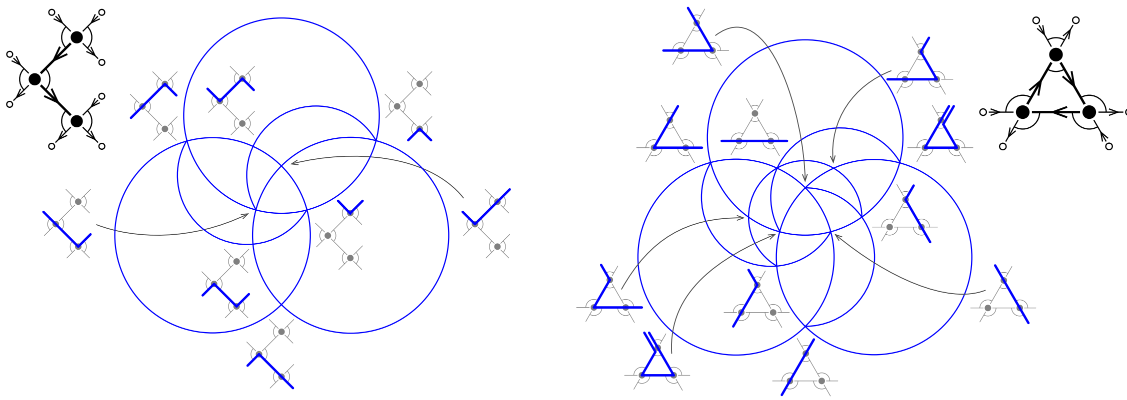

FIGURE 1. Two non-kissing fans. As the fans are 3-dimensional, we intersect them with the sphere and stereographically project them from the direction $(-1,-1,-1)$. Illustration from [PPP17].

is a complete simplicial fan of $\mathbb{R}^{Q_0}$*, called* non-kissing fan *of* $\bar{Q}$*, which realizes the non-kissing complex* $\mathcal{NK}(\bar{Q})$*.*

It is proved in [PPP17, Thm. 4.27] that the non-kissing fan comes from a polytope. For a walk ω, denote by $\textsc{kn}(\omega)$ the sum over all other walks ω' of the number of kisses between ω and ω'.

Theorem 2.9 ([PPP17, Thm. 4.27])**.** *For any non-kissing finite gentle quiver* $\bar{Q}$*, the non-kissing fan* $\mathcal{F}(\bar{Q})$ *is the normal fan of the gentle associahedron* $\mathsf{Asso}(\bar{Q})$*, defined as the intersection of the half-spaces* $\big\{\boldsymbol{x} \in \mathbb{R}^{Q_0} \bigm| \langle\, \boldsymbol{g}(\omega) \mid \boldsymbol{x}\,\rangle \le \textsc{kn}(\omega)\big\}$ *for all walks* $\omega \in \mathcal{W}_{\mathrm{prop}}(\bar{Q})$*.*

<u>*Flips in the non-kissing fan.*</u> Although we lack a characterization of the exchangeable pairs of the non-kissing complex (see Remark 2.11), we can still describe the linear dependence among the $\boldsymbol{g}$-vectors involved in a flip. The following statement is partially proved in [PPP17, Thm. 4.17].

Proposition 2.10. *Let* ω, ω' *be two exchangeable walks on* $\bar{Q}$*. Then*

(i) For any non-kissing facet F, F' *of* $\mathcal{NK}(\bar{Q})$ *with* $F \smallsetminus \{\omega\} = f' \smallsetminus \{\omega'\}$*, there exists a unique maximal common substring* σ *of* ω *and* ω' *that decomposes* $\omega = \rho\sigma\tau$ *and* $\omega' = \rho'\sigma\tau'$ *such that the facets* F *and* F' *both contain the walks* $\mu := \rho'\sigma\tau$ *and* $\nu := \rho\sigma\tau'$*.*

(ii) The substring σ *and thus the walks* μ *and* ν *actually depend on the exchangeable walks* ω *and* ω'*, not on the adjacent non-kissing facet* F *and* F'*.*

(iii) Moreover, the linear dependence between the $\boldsymbol{g}$*-vectors of* $F \cup F'$ *is given by*

$$\boldsymbol{g}(\omega) + \boldsymbol{g}(\omega') = \boldsymbol{g}(\mu) + \boldsymbol{g}(\nu).$$

In other words, the non-kissing fan of $\bar{Q}$ *has the unique exchange relation property.*

(iv) The $\boldsymbol{c}$*-vector orthogonal to all* $\boldsymbol{g}$*-vectors* $\boldsymbol{g}(\lambda)$ *for* $\lambda \in F \cap F'$ *is the multiplicity vector* $\boldsymbol{m}_\sigma$ *of the vertices of the substring* σ *of* ω *and* ω'*.*

Point (ii) to prove. — V.

Proof. Points (i), (iii) and (iv) where shown in [PPP17, Prop. 2.33, Thm. 4.17 & Prop 4.16]. □

In Proposition 2.10, the string σ is called the *distinguished* substring of ω and ω'. We say that a string σ is *distinguishable* if it is the distinguished string of an exchangeable pair. Note that an equivalent definition of distinguishable strings is given in [PPP17, Def. 2.30]. We denote by $\mathcal{S}_{\mathrm{dist}}(\bar{Q})$ the set of distinguishable strings of $\bar{Q}$.

Remark 2.11. In view of Proposition 2.10, it is tempting to look for a characterization of the exchangeable pairs ω, ω' using the kisses between ω and ω'. However, as illustrated in Figure 1 (right), note that

- two exchangeable walks may kiss along more than one string, but only one is the distinguished string,

- two non-exchangeable walks can kiss along more than one distinguishable string,
- two walks that kiss along a single distinguishable string are not always exchangeable,
- not all strings are distinguishable.

More explicit. TODO — V.

Corollary 2.12. *For any non-kissing finite gentle quiver $\bar{Q}$, the type cone of the non-kissing fan $\mathcal{F}(\bar{Q})$ is given by*

$$\mathbb{TC}(\mathcal{F}(\bar{Q})) = \left\{ \boldsymbol{h} \in \mathbb{R}^{\mathcal{W}(\bar{Q})} \;\middle|\; \begin{array}{l} \boldsymbol{h}_\omega = 0 \text{ for any improper walk } \omega \\ \boldsymbol{h}_\omega + \boldsymbol{h}_{\omega'} > \boldsymbol{h}_\mu + \boldsymbol{h}_\nu \text{ for any exchangeable walks } \omega, \omega' \end{array} \right\}.$$

Numerology. The following result will also be essential in our discussion.

Proposition 2.13 ([PPP17, 3.68])**.** *The number of distinguishable strings $\mathcal{S}_{\mathrm{dist}}(\bar{Q})$ and proper walks $\mathcal{W}_{\mathrm{prop}}(\bar{Q})$ of the quiver $\bar{Q}$ are related by*

$$|\mathcal{S}_{\mathrm{dist}}(\bar{Q})| + |Q_0| = |\mathcal{W}_{\mathrm{prop}}(\bar{Q})|.$$

Two families of examples: grid and dissection quivers. We conclude this recollections on non-kissing complexes by two families of examples.

TODO — V.

We are now ready to describe the type cone of the non-kissing fan $\mathcal{F}(\bar{Q})$. We first treat the case when $\bar{Q}$ has no self-kissing walks, which includes both grid and dissection quivers, thus in particular the case of the classical associahedron. We show that the type cone is then simplicial which gives another proof of **??** and **??**.

2.3.2. *Simplicial type cones for self-kissing free quivers.* In this section, we focus on the following family of gentle quivers, which was also considered in [, Sect. 4].

Proposition 2.14. *The following conditions are equivalent for a gentle quiver $\bar{Q}$.*

(i) any (non necessarily oriented) cycle of $\bar{Q}$ contains at least two relations in I,
(ii) any string of $\bar{Q}$ is distinguishable,
(iii) no walk on $\bar{Q}$ is self-kissing.

The quivers of this family is particularly well-behaved: they avoid all pathologies of Remark 2.11 and we will prove that the type cone of their non-kissing fan happens to be simplicial. Note that this family already contains a lot of relevant examples, including:

- classical associahedra
- grid associahedra
- dissection associahedra

More explicit. TODO — V.

For a string σ of $\bar{Q}$, we denote by $\sigma\nwarrow$ (resp. $\sigma\checkmark$) the unique string of the blossoming quiver $\bar{Q}^{❀}$ of the form $\sigma\nwarrow = \sigma\alpha_1^{-1}\alpha_2\dots\alpha_\ell$ (resp. $\sigma\checkmark = \sigma\alpha_1\alpha_2^{-1}\dots\alpha_\ell^{-1}$) with $\ell \ge 1$ and $\alpha_1, \dots, \alpha_\ell \in Q_1$ and such that $t(\alpha_\ell)$ (resp. $s(\alpha_\ell)$) is a blossom of $\bar{Q}^{❀}$. These notations are motivated by the representation of strings used in [BR87, PPP17], and the terminology usually says that $\sigma\nwarrow$ (resp. $\sigma\checkmark$) is obtained by adding a *hook* (resp. *cohook*) to σ. We define similarly $\nearrow\sigma$ (resp. $\searrow\sigma$). The walk $\nearrow(\sigma\nwarrow) = (\nearrow\sigma)\nwarrow$ of $\bar{Q}$ is simply be denoted $\nearrow\sigma\nwarrow$, and we define similarly $\searrow\sigma\checkmark$, $\nearrow\sigma\checkmark$ and $\searrow\sigma\nwarrow$.

Proposition 2.15. *For any gentle quiver $\bar{Q}$ with no self-kissing walk and any string $\sigma \in \mathcal{S}(\bar{Q})$, the walks $\searrow\sigma\checkmark$ and $\nearrow\sigma\nwarrow$ are exchangeable with distinguished substring σ.*

Proof. We just need to prove that there exists a ridge R of $\mathcal{NK}(\bar{Q})$ containing both $\nearrow\sigma\checkmark$ and $\searrow\sigma\nwarrow$ and such that $R \cup \{\searrow\sigma\checkmark\}$ and $R \cup \{\nearrow\sigma\nwarrow\}$ are adjacent facets of $\mathcal{NK}(\bar{Q})$. $\square$

TODO — V.

The following statement describes the type cone of the non-kissing fan of a gentle quiver with no self-kissing walks.

Proposition 2.16. *For any gentle quiver $\bar{Q}$ with no self-kissing walk, the extremal exchangeable pairs for the non-kissing fan of $\bar{Q}$ are precisely the pairs $\{\searrow\sigma\checkmark, \nearrow\sigma\nwarrow\}$ for all strings $\sigma \in \mathcal{S}(\bar{Q})$.*

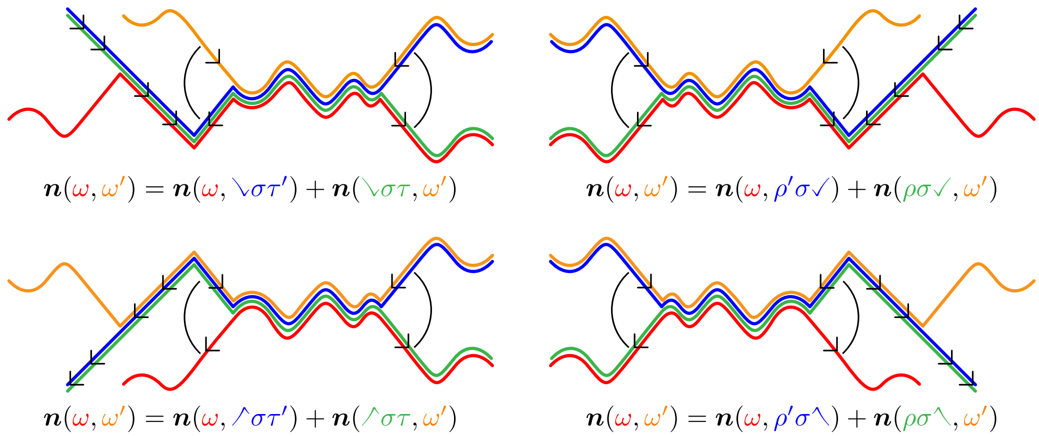

FIGURE 2. Schematic representation of te four equalities in the proof of Proposition 2.16.

Proof. Let $(\boldsymbol{f}_\omega)_{\omega\in\mathcal{W}(\bar{Q})}$ be the canonical basis of $\mathbb{R}^{\mathcal{W}(\bar{Q})}$. Consider two exchangeable walks ω and ω' with distinguished substring $\sigma \in \Sigma_{\mathrm{top}}(\omega) \cap \Sigma_{\mathrm{bot}}(\omega')$. Decompose $\omega = \rho\sigma\tau$ and $\omega' = \rho'\sigma\tau'$ and define $\mu := \rho'\sigma\tau$ and $\nu := \rho\sigma\tau'$ as in Proposition 2.10 (i). Proposition 2.10 (iii) ensures that the linear dependence between the corresponding $\boldsymbol{g}$-vectors is given by

$$\boldsymbol{g}(\omega) + \boldsymbol{g}(\omega') = \boldsymbol{g}(\mu) + \boldsymbol{g}(\nu).$$

Therefore, the outer normal vector of the corresponding inequality of the type cone $\mathbb{TC}(\mathcal{F}(\bar{Q}))$ is

$$\boldsymbol{n}(\omega, \omega') := \boldsymbol{f}_\omega + \boldsymbol{f}_{\omega'} - \boldsymbol{f}_\mu - \boldsymbol{f}_\nu.$$

We claim that this normal vector is always a positive linear combination of the normal vectors $\boldsymbol{m}(\sigma) := \boldsymbol{n}(\searrow\sigma\checkmark, \nearrow\sigma\nwarrow) = \boldsymbol{f}_{\searrow\sigma\checkmark} + \boldsymbol{f}_{\nearrow\sigma\nwarrow} - \boldsymbol{f}_{\nearrow\sigma\checkmark} - \boldsymbol{f}_{\searrow\sigma\nwarrow}$ for all string $\sigma \in \mathcal{S}(\bar{Q})$. Our proof works by descending induction on the length $\lambda(\omega, \omega') := \ell(\sigma)$ of the common substring of ω and ω'. If $\lambda(\omega, \omega')$ is big enough, then the walk ω (resp. ω') is just obtained by adding two outgoing (resp. incoming) blossoms at the end of σ, thus $\omega = \searrow\sigma\checkmark$ (resp. $\omega' = \nearrow\sigma\nwarrow$), and there is nothing to prove. Assume now that $\omega \neq \nearrow\sigma\nwarrow$ (the situations where $\omega' \neq \searrow\sigma\checkmark$ or $\omega \neq \rho\sigma\nwarrow$ are symmetric). If $\rho \neq \searrow$, observe that

- ω and $\searrow\sigma\tau'$ are exchangeable with $\boldsymbol{n}(\omega, \searrow\sigma\tau') = \boldsymbol{f}_\omega + \boldsymbol{f}_{\searrow\sigma\tau'} - \boldsymbol{f}_{\searrow\sigma\tau} - \boldsymbol{f}_\nu$, and
- $\searrow\sigma\tau$ and ω' are exchangeable with $\boldsymbol{n}(\searrow\sigma\tau, \omega') = \boldsymbol{f}_{\searrow\sigma\tau} + \boldsymbol{f}_{\omega'} - \boldsymbol{f}_{\searrow\sigma\tau'} - \boldsymbol{f}_\mu$.

We derive that

$$\boldsymbol{n}(\omega, \omega') = \boldsymbol{n}(\omega, \searrow\sigma\tau') + \boldsymbol{n}(\searrow\sigma\tau, \omega').$$

Observe moreover that since ω has outgoing arrows at the endpoints of σ, the common substring of ω and $\searrow\sigma\tau'$ strictly contains σ so that $\lambda(\omega, \searrow\sigma\tau') > \lambda(\omega, \omega')$. By induction, $\boldsymbol{n}(\omega, \searrow\sigma\tau')$ is thus a positive linear combination of $\boldsymbol{m}(\sigma)$ for $\sigma \in \mathcal{S}(\bar{Q})$. By symmetry, we obtain the four equalities

$$\boldsymbol{n}(\omega, \omega') = \begin{cases} \boldsymbol{n}(\omega, \searrow\sigma\tau') + \boldsymbol{n}(\searrow\sigma\tau, \omega') & \text{if } \rho \neq \searrow, \\ \boldsymbol{n}(\omega, \rho'\sigma\checkmark) + \boldsymbol{n}(\rho\sigma\checkmark, \omega') & \text{if } \tau \neq \checkmark, \\ \boldsymbol{n}(\omega, \nearrow\sigma\tau') + \boldsymbol{n}(\nearrow\sigma\tau, \omega') & \text{if } \rho' \neq \nearrow, \\ \boldsymbol{n}(\omega, \rho'\sigma\nwarrow) + \boldsymbol{n}(\rho\sigma\nwarrow, \omega') & \text{if } \tau' \neq \nwarrow. \end{cases}$$

These four equalities are illustrated on Figure 2. Moreover $\boldsymbol{n}(\omega, \searrow\sigma\tau')$, $\boldsymbol{n}(\searrow\sigma\tau, \nearrow\sigma\checkmark)$, $\boldsymbol{n}(\searrow\sigma\nwarrow, \nearrow\sigma\tau')$

and $\boldsymbol{n}(\diagup\sigma\tau, \omega')$ are all positive combinations of $\boldsymbol{m}(\sigma)$ for $\sigma \in \mathcal{S}(\bar{Q})$ by induction hypothesis. Applying these equalities one after the other, we obtain

$$\begin{aligned}
& \underbrace{\boldsymbol{n}(\omega, \omega')}_{\text{Equality 1}} \\
= \; & \boldsymbol{n}(\omega, \diagdown\sigma\tau') + \underbrace{\boldsymbol{n}(\diagdown\sigma\tau, \omega')}_{\text{Equality 2}} \\
= \; & \boldsymbol{n}(\omega, \diagdown\sigma\tau') + \underbrace{\boldsymbol{n}(\diagdown\sigma\tau, \diagup\sigma\tau')}_{\text{Equality 3}} + \boldsymbol{n}(\diagup\sigma\tau, \omega') \\
= \; & \boldsymbol{n}(\omega, \diagdown\sigma\tau') + \boldsymbol{n}(\diagdown\sigma\tau, \diagup\sigma\checkmark) + \underbrace{\boldsymbol{n}(\diagdown\sigma\checkmark, \diagup\sigma\tau')}_{\text{Equality 4}} + \boldsymbol{n}(\diagup\sigma\tau, \omega') \\
= \; & \boldsymbol{n}(\omega, \diagdown\sigma\tau') + \boldsymbol{n}(\diagdown\sigma\tau, \diagup\sigma\checkmark) + \boldsymbol{n}(\diagdown\sigma\checkmark, \diagup\sigma\nwarrow) + \boldsymbol{n}(\diagdown\sigma\nwarrow, \diagup\sigma\tau') + \boldsymbol{n}(\diagup\sigma\tau, \omega') \\
= \; & \boldsymbol{n}(\omega, \diagdown\sigma\tau') + \boldsymbol{n}(\diagdown\sigma\tau, \diagup\sigma\checkmark) + \quad \boldsymbol{m}(\sigma) \quad + \boldsymbol{n}(\diagdown\sigma\nwarrow, \diagup\sigma\tau') + \boldsymbol{n}(\diagup\sigma\tau, \omega')
\end{aligned}$$

where we fix the convention $\boldsymbol{n}(\lambda, \lambda) = 0$ in case $\rho = \diagup$, $\rho' = \diagdown$, $\tau = \nwarrow$ or $\tau' = \checkmark$. We conclude that $\boldsymbol{n}(\omega, \omega')$ is a positive combination of $\boldsymbol{m}(\sigma)$ for $\sigma \in \mathcal{S}(\bar{Q})$, since $\boldsymbol{n}(\omega, \diagdown\sigma\tau')$, $\boldsymbol{n}(\diagdown\sigma\tau, \diagup\sigma\checkmark)$, $\boldsymbol{n}(\diagdown\sigma\nwarrow, \diagup\sigma\tau')$ and $\boldsymbol{n}(\diagup\sigma\tau, \omega')$ are. This shows that all extremal exchangeable pairs are of the form $\{\diagdown\sigma\checkmark, \diagup\sigma\nwarrow\}$ for $\sigma \in \mathcal{S}(\bar{Q})$.

Conversely, we know from Remark 1.5 that there are at least $|\mathcal{W}_{\mathrm{prop}}(\bar{Q})| - |Q_0|$ extremal exchangeable pairs. Since this $|\mathcal{S}(\bar{Q})| = |\mathcal{S}_{\mathrm{dist}}(\bar{Q})| = |\mathcal{W}_{\mathrm{prop}}(\bar{Q})| - |Q_0|$ by Proposition 2.13, we conclude that all exchangeable pairs $\{\diagdown\sigma\checkmark, \diagup\sigma\nwarrow\}$ for $\sigma \in \mathcal{S}(\bar{Q})$ are extremal. □

The following statement now follows from Propositions 2.13 and 2.16.

Corollary 2.17. *For any gentle quiver $\bar{Q}$ with no self-kissing walk, the type cone $\mathbb{TC}(\mathcal{F}(\bar{Q}))$ of the non-kissing fan $\bar{Q}$ is simplicial.*

Combining Corollary 1.9, Corollary 2.17 and Proposition 2.16, we derive the following description of all polytopal realizations of the non-kissing fan $\mathcal{F}(\bar{Q})$ of a quiver $\bar{Q}$ with no self-kissing walk.

Corollary 2.18. *For any gentle quiver $\bar{Q}$ with no self-kissing walk and any $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{\mathcal{S}(\bar{Q})}$, the polytope*

$$R_{\boldsymbol{\ell}}(\bar{Q}) := \left\{ \boldsymbol{z} \in \mathbb{R}^{\mathcal{W}(\bar{Q})} \;\middle|\; \begin{array}{l} \boldsymbol{z} \ge 0 \qquad \text{and} \qquad \boldsymbol{z}_\omega = 0 \text{ for any improper walk } \omega \\ \boldsymbol{z}_{\diagdown\sigma\checkmark} + \boldsymbol{z}_{\diagup\sigma\nwarrow} - \boldsymbol{z}_{\diagup\sigma\checkmark} - \boldsymbol{z}_{\diagdown\sigma\nwarrow} = \boldsymbol{\ell}_\sigma \text{ for all } \sigma \in \mathcal{S}(\bar{Q}) \end{array} \right\}$$

is a realization of the non-kissing fan $\mathcal{F}(\bar{Q})$. Moreover, the polytopes $R_{\boldsymbol{\ell}}(\bar{Q})$ for $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{\mathcal{S}(\bar{Q})}$ describe all polytopal realizations of $\mathcal{F}(\bar{Q})$.

Proof. Proposition 2.16 asserts that the type cone $\mathbb{TC}(\mathcal{F}(\bar{Q}))$ has one facet for each string of $\bar{Q}$. Since all strings are distinguishable by Proposition 2.14, the number of facets of $\mathbb{TC}(\mathcal{F}(\bar{Q}))$ is $|\mathcal{S}(\bar{Q})| = |\mathcal{S}_{\mathrm{dist}}(\bar{Q})| = |\mathcal{W}_{\mathrm{prop}}(\bar{Q})| - |Q_0|$ by Proposition 2.13, so that $\mathbb{TC}(\mathcal{F}(\bar{Q}))$ is simplicial. We then conclude by a simple application of Corollary 1.9. □

We also obtain from Proposition 2.16 the following surprising property.

Corollary 2.19. *Any $\boldsymbol{c}$-vector supports exactly one extremal exchangeable pair.*

Remark 2.20. Although not needed in the proof of Proposition 2.16, we note that the extremal exchangeable pairs $\{\diagdown\sigma\checkmark, \diagup\sigma\nwarrow\}$ and their linear dependencies $\boldsymbol{g}(\diagdown\sigma\checkmark) + \boldsymbol{g}(\diagup\sigma\nwarrow) - \boldsymbol{g}(\diagup\sigma\checkmark) - \boldsymbol{g}(\diagdown\sigma\nwarrow)$ precisely correspond to the meshes of the Auslander-Reiten quiver of $\bar{Q}$.

2.3.3. *Towards the general case.* We are already missing a criterion for exchangeable pairs of walks. See [BDM+17, Sect. 9]. It seems that two exchangeable walks are always kissing along a single distinguishable string. We need to prove that to see that the non-kissing fan has the unique exchange relation property (because for a pair of exchangeable walks, I can completely reconstruct the $\boldsymbol{g}$-vector dependence of the flip if I know their distinguished substring). Note however that

(1) exchangeable pairs might kiss along additional non-distinguishable strings (example: just a loop),
(2) non-exchangeable walks might kiss along a single distinguishable string (example: see the cyclic triangle in [PPP17]).
(3) non-exchangeable walks might kiss along two or more distinguishable strings (example: see the cyclic triangle in [PPP17]).

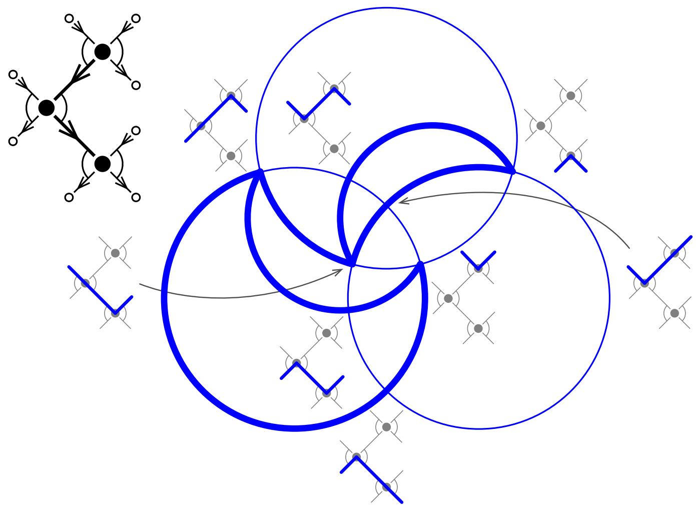

FIGURE 3. The shards corresponding to the facets of the type cone. Is there a geometric meaning?

We have some conjectures on what the extremal exchangeable pairs should be. One clear (but a bit empty) result is that extremal exchangeable pairs correspond in the Auslander-Reiten quiver to rectangles whose four vertices are non-self-kissing and that cannot be tiled with smaller such rectangles.

We checked on some (but probably not enough) small gentle quivers the following properties:

(1) Any c-vector (*i.e.* distinguishable string) is the direction of at least one extremal exchangeable pair.
(2) Consider a distinguishable string σ. Let ω (resp. ω') be the walk obtained from σ by adding two hooks (resp. two cohooks) at the endpoints of σ. If the walks ω and ω' are non-self-kissing and exchangeable, then they form the unique extremal exchangeable pair directed by σ. These extremal exchangeable pairs correspond to meshes of the Auslander-Reiten quiver.
(3) Otherwise, σ is the direction to one or more extremal exchangeable pairs obtained by moving further in the Auslander-Reiten quiver (this is really unclear at the moment).

2.4. **Graph associahedra.** Graph associahedra were defined by M. Carr and S. Devadoss [CD06] in connection to C. De Concini and C. Procesi's wonderful arrangements [DCP95]. For a given graph G, the G-associahedron $\mathsf{Asso}(G)$ is a simple polytope whose combinatorial structure encodes the connected subgraphs of G and their nested structure. More precisely, the G-associahedron is a polytopal realization of the nested complex of G, defined as the simplicial complex of all collections of tubes (connected induced subgraphs) of G which are pairwise compatible (either nested, or disjoint and non-adjacent). As illustrated in Figure 4, the graph associahedra of certain special families of graphs coincide with well-known families of polytopes: complete graph associahedra are permutahedra, path associahedra are classical associahedra, cycle associahedra are cyclohedra, and star associahedra are stellohedra. The graph associahedra were extended to the *nestohedra*, which are simple polytopes realizing the nested complex of arbitrary building sets [Pos09, FS05]. Graph associahedra and nestohedra have been geometrically realized in different ways: by successive truncations of faces of the standard simplex [CD06], as Minkowski sums of faces of the standard simplex [Pos09, FS05], or from their normal fans by exhibiting explicit inequality descriptions [Dev09, Zel06]. For a given graph G, the resulting polytopes all have the same normal fan, called nested fan of G: its rays are the characteristic vectors of the tubes, and its cones are generated by characteristic vectors of compatible tubes. In this section, we describe the type cone of the nested fan of any graph G.

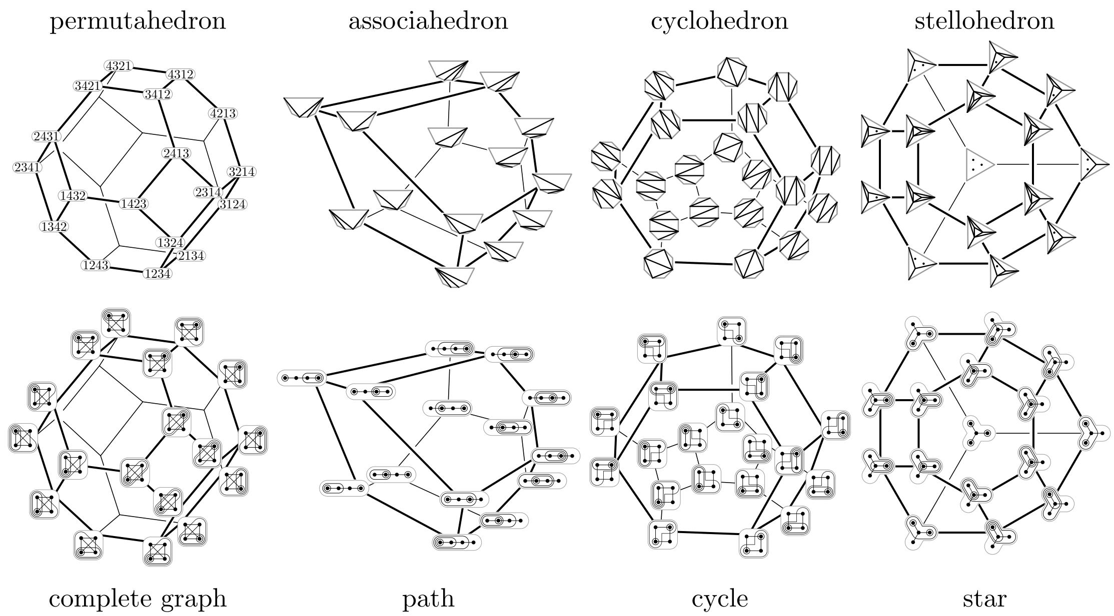

FIGURE 4. Some classical families of polytopes as graph associahedra. Illustration from [MP17a].

2.4.1. *Nested complex and nested fan of a graph.* We present the definitions and properties of the nested complex of a graph, following ideas from [CD06, Pos09, FS05, Zel06].

Nested complex. Let G be a graph with vertex set V. Let $\kappa(G)$ denote the set of connected components of G and define $n := |V| - |\kappa(G)|$. A *tube* of G is a connected induced subgraph of G. The set of tubes of G is denoted by $\mathcal{T}(G)$. The inclusion maximal tubes of G are its connected components $\kappa(G)$. The tubes which are neither empty nor maximal are called *proper*. Two tubes t, t' of G are *compatible* if they are either nested (*i.e.* $\mathsf{t} \subseteq \mathsf{t}'$ or $\mathsf{t}' \subseteq \mathsf{t}$), or disjoint and non-adjacent (*i.e.* $\mathsf{t} \cup \mathsf{t}'$ is not a tube of G). A *tubing* on G is a set T of pairwise compatible proper tubes of G. The *nested complex* of G is the simplicial complex $\mathcal{N}(G)$ of all tubings on G.

Nested fan and graph associahedron. Let $(\boldsymbol{e}_v)_{v \in V}$ be the canonical basis of $\mathbb{R}^V$. We consider the hyperplane $\mathbb{H} := \big\{ \boldsymbol{x} \in \mathbb{R}^V \;\big|\; \sum_{w \in W} x_w = 0 \text{ for all } W \in \kappa(G) \big\}$ and let $\pi : \mathbb{R}^V \to \mathbb{H}$ denote the orthogonal projection on $\mathbb{H}$. The ***g**-vector* of a tube t of G is the projection $\boldsymbol{g}(\mathsf{t}) := \pi\big(\sum_{v \in \mathsf{t}} \boldsymbol{e}_v\big)$ of the characteristic vector of t. Note that by definition, $\boldsymbol{g}(G) = 0$. We also define $\boldsymbol{g}(\mathsf{T}) := \{\boldsymbol{g}(\mathsf{t}) \mid \mathsf{t} \in \mathsf{T}\}$ for a tubing T on G. These vectors support a complete simplicial fan realization of the nested complex. Examples are illustrated in Figure 5.

Theorem 2.21 ([CD06, Pos09, FS05, Zel06])**.** *For any graph G, the set of cones*

$$\mathcal{F}(G) := \{\mathbb{R}_{\ge 0}\, \boldsymbol{g}(\mathsf{T}) \mid \mathsf{T} \text{ tubing on } G\}$$

is a complete simplicial fan of $\mathbb{H}$, called nested fan of G, which realizes the nested complex $\mathcal{N}(G)$.

It is proved in [CD06, Dev09, Pos09, FS05, Zel06] that the nested fan comes from a polytope. For any subset $U \subseteq V$, denote by $\triangle_U := \operatorname{conv}\{\boldsymbol{e}_u \mid u \in U\}$ the face of the standard simplex $\triangle_V$ corresponding to U.

Theorem 2.22 ([CD06, Dev09, Pos09, FS05, Zel06])**.** *For any graph G, the nested fan $\mathcal{F}(G)$ is the normal fan of the graph associahedron $\mathsf{Asso}(G)$, defined as the Minkowski sum $\sum_{\mathsf{t} \in \mathcal{T}(G)} \triangle_{\mathsf{t}}$ of the faces of the standard simplex corresponding to all tubes of G.*

Flips in the nested fan. The following statement follows from [MP17a, Zel06].

Proposition 2.23. *Let t, t' be two tubes of G. Then*

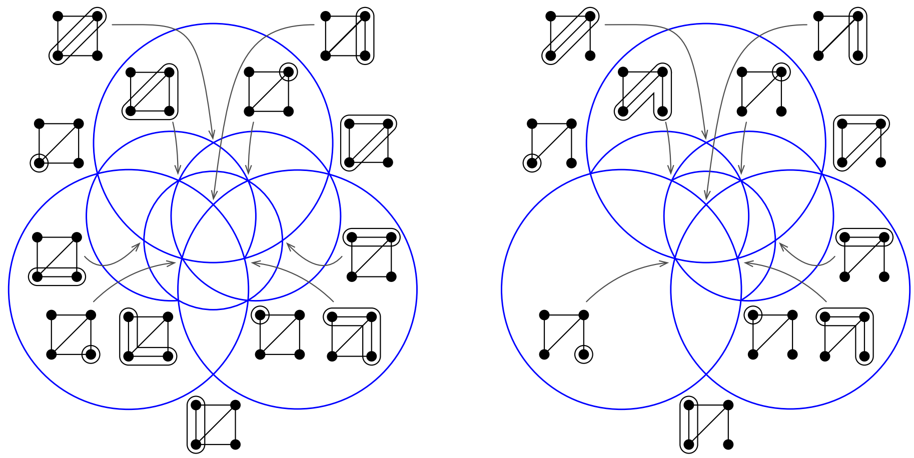

Figure 5. Two nested fans. As the fans are 3-dimensional, we intersect them with the sphere and stereographically project them from the direction $(-1,-1,-1)$.

(i) The tubes t *and* t' *are exchangeable if and only if* t' *has a unique neighbor* v *in* $\mathsf{t} \smallsetminus \mathsf{t}'$ *and* t *has a unique neighbor* v' *in* $\mathsf{t}' \smallsetminus \mathsf{t}$*.*
(ii) For any maximal tubings T, T *on* G *with* $\mathsf{T} \smallsetminus \{\mathsf{t}\} = \mathsf{T}' \smallsetminus \{\mathsf{t}'\}$*, both* $\mathsf{T} \cup \kappa(G)$ *and* $\mathsf{T}' \cup \kappa(G)$ *contain the tube* $\mathsf{t} \cup \mathsf{t}'$ *and the connected components of* $\mathsf{t} \cap \mathsf{t}'$*.*
(iii) The linear dependence between the $\boldsymbol{g}$*-vectors of* $\mathsf{T} \cup \mathsf{T}'$ *is given by*

$$\boldsymbol{g}(\mathsf{t}) + \boldsymbol{g}(\mathsf{t}') = \boldsymbol{g}(\mathsf{t} \cup \mathsf{t}') + \sum_{\mathsf{s} \in \kappa(\mathsf{t} \cap \mathsf{t}')} \boldsymbol{g}(\mathsf{s}).$$

In other words, the nested fan of G *has the unique exchange relation property.*
(iv) The $\boldsymbol{c}$*-vector orthogonal to all* $\boldsymbol{g}$*-vectors* $\boldsymbol{g}(\mathsf{s})$ *for* $\mathsf{s} \in \mathsf{T} \cap \mathsf{T}'$ *is* $\boldsymbol{e}_v - \boldsymbol{e}_{v'}$*.*

Proof. Points (i) and (ii) was proved in [MP17a]. Point (iii) follows from the fact that

$$\sum_{v \in \mathsf{t}} \boldsymbol{e}_v + \sum_{v \in \mathsf{t}'} \boldsymbol{e}_v = \sum_{v \in \mathsf{t} \cup \mathsf{t}'} \boldsymbol{e}_v + \sum_{v \in \mathsf{t} \cap \mathsf{t}'} \boldsymbol{e}_v = \sum_{v \in \mathsf{t} \cup \mathsf{t}'} \boldsymbol{e}_v + \sum_{\substack{\mathsf{s} \in \kappa(\mathsf{t} \cap \mathsf{t}') \\ v \in \mathsf{s}}} \boldsymbol{e}_v.$$

Finally for (iv), any tube $\mathsf{s} \in \mathsf{T} \cap \mathsf{T}'$ that contains v or v' actually contains both (to be compatible with t and t'). Therefore, $\boldsymbol{g}(\mathsf{s})$ is orthogonal to $\boldsymbol{e}_v - \boldsymbol{e}_{v'}$ for any tube $\mathsf{s} \in \mathsf{T} \cap \mathsf{T}'$. □

We are now ready to describe the type cone of the nested fan $\mathcal{F}(G)$. We first treat the case when G is a path so that the G-associahedron is a classical associahedron. We show that the type cone is then simplicial which gives another proof of **??** and **??**.

2.4.2. *Simplicial type cones for path associahedra.* We start with the case of the path on $n+1$ vertices. Recall that the tubes of the path are the intervals of $[n+1]$ and that two intervals $[i,j]$ and $[i',j']$ of $[n+1]$ are exchangeable if either $i < i' \le j+1 < j'+1$ or $i' < i \le j'+1 < j+1$.

Proposition 2.24. *Two intervals* $[i,j]$ *and* $[i',j']$ *of* $[n+1]$ *form an extremal exchangeable pair for the nested fan of the path if and only if* $i = i'+1$ *and* $j = j'+1$*, or the opposite.*

Proof. Let $(\boldsymbol{f}_{[i,j]})_{1 \le i \le j \le n}$ be the canonical basis of $\mathbb{R}^{\binom{n+1}{2}}$. Consider two exchangeable tubes $[i,j]$ and $[i',j']$ with $i < i' \le j+1 < j'+1$. By Proposition 2.23 (iii), the linear dependence between the corresponding $\boldsymbol{g}$-vectors is given by

$$\boldsymbol{g}([i,j]) + \boldsymbol{g}([i',j']) = \boldsymbol{g}([i,j']) + \boldsymbol{g}([i',j]).$$

Therefore, the outer normal vector of the corresponding inequality of the type cone $\mathbb{TC}(\mathcal{F}(G))$ is

$$\boldsymbol{n}(i,j,i',j') := \boldsymbol{f}_{[i,j]} + \boldsymbol{f}_{[i',j']} - \boldsymbol{f}_{[i,j']} - \boldsymbol{f}_{[i',j]}.$$

Denoting

$$\boldsymbol{m}(k,\ell) := \boldsymbol{n}(k,\ell-1,k+1,\ell) = \boldsymbol{f}_{[k,\ell-1]} + \boldsymbol{f}_{[k+1,\ell]} - \boldsymbol{f}_{[k,\ell]} - \boldsymbol{f}_{[k+1,\ell-1]},$$

we obtain that

$$\boldsymbol{n}(i,j,i',j') = \sum_{\substack{k\in[i,i'[\\ \ell\in]j,j']}} \boldsymbol{m}(k,\ell).$$

Indeed, on the right hand side, the basis vector $\boldsymbol{f}_{[k,\ell]}$ appears with a positive sign in $\boldsymbol{m}(k,\ell+1)$ for $(k,\ell) \in [i,i'[\times [j,j'[$ and in $\boldsymbol{m}(k-1,\ell)$ for $(k,\ell) \in]i,i'] \times]j,j']$, and with a negative sign in $\boldsymbol{m}(k,\ell)$ for $(k,\ell) \in [i,i'[\times]j,j']$ and in $\boldsymbol{m}(k-1,\ell+1)$ for $(k,\ell) \in]i,i'] \times [j,j'[$. Therefore, these contributions all vanish except when $[k,\ell]$ is one of the tubes $[i,j]$, $[i',j']$, $[i,j']$ or $[i',j]$. This shows that any exchange relation is a positive linear combination of the exchange relations corresponding to all pairs of tubes $[i,j]$ and $[i',j']$ of $[n+1]$ such that $i = i'+1$ and $j = j'+1$.

We now need to show that all these exchangeable pairs are extremal. Assume that $\boldsymbol{m}(k,\ell)$ can be written as the linear combination

$$\boldsymbol{m}(k,\ell) = \sum \lambda(i,j,i',j')\, \boldsymbol{n}(i,j,i',j'), \tag{1}$$

where $\lambda(i,j,i',j') \geq 0$ for all exchangeable pairs $\{[i,j],[i',j']\}$. Note that $\sum \lambda(i,j,i',j') \geq 1$ since the coefficient of $\boldsymbol{f}_{[k,\ell]}$ in $\boldsymbol{m}(k,\ell)$ is -1. Consider the linear form $\Phi : \mathbb{R}^N \to \mathbb{R}$ defined by $\Phi(\boldsymbol{f}_{[i,j]}) = -(j-i)^2$. A quick computation shows that

$$\Phi(\boldsymbol{n}(i,j,i',j')) = 2(i'-i)(j'-j).$$

Applying Φ to Equation (1), we obtain

$$1 = \sum \lambda(i,j,i',j')(i'-i)(j'-j).$$

Since $\sum \lambda(i,j,i',j') \geq 1$ we obtain that $\lambda(k,\ell-1,k+1,\ell) = 1$ and $\lambda(i,j,i',j') = 0$ for all other exchangeable pairs $\{[i,j],[i',j']\}$. □

2.4.3. *General case.* We now consider an arbitrary graph G and describe the type cone of its nested fan. Proposition 2.24 extends as follows.

Proposition 2.25. *Two tubes* t *and* t' *of* G *form an extremal exchangeable pair for the nested fan of* G *if and only if* $\mathsf{t} \smallsetminus \{v\} = \mathsf{t}' \smallsetminus \{v'\}$ *for some neighbor* v *of* t' *and some neighbor* v' *of* t.

Proof. Consider an exchangeable pair $\{\mathsf{t},\mathsf{t}'\}$ of tubes of G and let $p = |\mathsf{t} \cup \mathsf{t}'|$. By Proposition 2.23, t' has a unique neighbor v in $\mathsf{t} \smallsetminus \mathsf{t}'$ and t has a unique neighbor v' in $\mathsf{t}' \smallsetminus \mathsf{t}$. Therefore, $\mathsf{t} \smallsetminus \mathsf{t}'$ and $\mathsf{t}' \smallsetminus \mathsf{t}$ are both connected. We can thus label the elements of $\mathsf{t} \cup \mathsf{t}'$ by $\{v_1,\dots,v_p\}$ such that $\{v_i,\dots,v_j\}$ induces a tube $\mathsf{t}_{k,\ell}$ of G for any $1 \leq k \leq \ell \leq p$. The map $[k,\ell] \mapsto \mathsf{t}_{k,\ell}$ is thus an injection from the tubes of the path P_p to that of G that fulfills

- $\mathsf{t}_{k,\ell-1} \smallsetminus \{v_k\} = \mathsf{t}_{k+1,\ell} \smallsetminus \{v_\ell\}$ for any $1 \leq k \leq \ell \leq p$, and
- $\boldsymbol{g}(\mathsf{t}_{k,\ell}) = M\boldsymbol{g}([k,\ell])$ for any $1 \leq k \leq \ell \leq p$, where M is the matrix sending $\boldsymbol{e}_m$ to $\boldsymbol{e}_{v_m}$.

Therefore, this map transports the linear combinations of Proposition 2.24. We conclude that the exchange relation corresponding to the exchangeable pair $\{\mathsf{t},\mathsf{t}'\}$ is a positive linear combination of the exchange relations corresponding to the pairs $\{\mathsf{t}_{k,\ell-1},\mathsf{t}_{k+1,\ell}\}$ for all $(k,\ell) \in [1, p-|\mathsf{t}'|+1[\times]|\mathsf{t}|, p]$.

We now need to show that all these pairs are extremal. □ TODO — V.

The following statement reformulates Proposition 2.25.

Corollary 2.26. *The extremal exchangeable pairs for the nested fan of* G *are equivalently*

(i) the pairs of tubes $\mathsf{r} \cup \{v\}$ *and* $\mathsf{r} \cup \{v'\}$ *for any tube* $\mathsf{r} \in \mathcal{T}(G)$ *and distinct neighbors* v, v' *of* r,

(ii) the pairs of tubes $\mathsf{s} \smallsetminus \{v'\}$ *and* $\mathsf{s} \smallsetminus \{v\}$ *for any tube* $\mathsf{s} \in \mathcal{T}(G)$ *and distinct non-disconnecting vertices* v, v' *of* s,

We derive from Proposition 2.25 and Corollary 2.26 all polytopal realizations of the nested fan, which we can explicitly express using one of the following two equivalent conditions.

Corollary 2.27. *Consider a graph G on V with tubes $\mathcal{T}(G)$ and a height vector $\boldsymbol{h} \in \mathbb{R}^{\mathcal{T}(G)}$ such that $\boldsymbol{h}_\varnothing = \boldsymbol{h}_G = 0$ and satisfying one of the two following equivalent conditions:*

(i) $\boldsymbol{h}_{\mathsf{r}\cup\{v\}} + \boldsymbol{h}_{\mathsf{r}\cup\{v'\}} > \boldsymbol{h}_{\mathsf{r}\cup\{v,v'\}} + \boldsymbol{h}_{\mathsf{r}}$ *for any tube* $\mathsf{r} \in \mathcal{T}(G)$ *and distinct neighbors* v, v' *of* r,

(ii) $\boldsymbol{h}_{\mathsf{s}\smallsetminus\{v'\}} + \boldsymbol{h}_{\mathsf{s}\smallsetminus\{v\}} > \boldsymbol{h}_{\mathsf{s}} + \boldsymbol{h}_{\mathsf{s}\smallsetminus\{v,v'\}}$ *for any tube* $\mathsf{s} \in \mathcal{T}(G)$ *and distinct non-disconnecting vertices* v, v' *of* s.

Then the nested fan $\mathcal{F}(G)$ is the normal fan of the graph associahedron

$$\left\{ \boldsymbol{x} \in \mathbb{R}^V \;\middle|\; \langle\, \boldsymbol{g}(\mathsf{t}) \mid \boldsymbol{x} \,\rangle \le \boldsymbol{h}_{\mathsf{t}} \text{ for any tube } \mathsf{t} \in \mathcal{T}(G) \right\}.$$

We also obtain from Proposition 2.25 the following surprising property.

Corollary 2.28. *Any $\boldsymbol{c}$-vector supports at least one extremal exchangeable pair.*

Proof. Consider a $\boldsymbol{c}$-vector $\boldsymbol{e}_v - \boldsymbol{e}_{v'}$ for two distinct vertices v, v' in a common connected component of G. Let r be a path from v to v' in G and let $\mathsf{t} := \mathsf{r} \cup \{v\}$ and $\mathsf{t}' := \mathsf{r} \cup \{v'\}$. Then $\{\mathsf{t}, \mathsf{t}'\}$ is an extremal exchangeable pair with $\boldsymbol{c}$-vector $\boldsymbol{e}_v - \boldsymbol{e}_{v'}$. □

We derive from Corollary 2.26 the number of extremal exchangeable pairs of the nested fan. For a tube t of G, we denote by $\mathrm{ne}(\mathsf{t})$ the number of neighbors of t and by $\mathrm{nd}(\mathsf{t})$ the number of non-disconnecting vertices of t. In other words, it $\mathrm{ne}(\mathsf{t})$ (resp. $\mathrm{nd}(\mathsf{t})$) is the number of tubes covering t (resp. covered by t) in the inclusion poset of all tubes of G.

Corollary 2.29. *The nested fan $\mathcal{F}(G)$ has $\sum_{\mathsf{r}\in\mathcal{T}(G)} \binom{\mathrm{ne}(\mathsf{r})}{2} = \sum_{\mathsf{s}\in\mathcal{T}(G)} \binom{\mathrm{nd}(\mathsf{s})}{2}$ extremal exchangeable pairs.*

The formula of Corollary 2.29 can be made more explicit for specific families of graph associahedra discussed in the introduction and illustrated in Figure 4.

Proposition 2.30. *The number of extreme exchangeable pairs of the nested fan $\mathcal{F}(G)$ is:*

- $2^{n-2}\binom{n}{2}$ *for the permutahedron (complete graph associahedron),*
- $\binom{n}{2}$ *for the associahedron (path associahedron),*
- $3\binom{n}{2} - n$ *for the cyclohedron (cycle associahedron),*
- $n - 1 + 2^{n-3}\binom{n-1}{2}$ *for the stellohedron (star associahedron).*

Proof. For the permutahedron, choose any two vertices v, v', and complete them into a tube by selecting any subset of the $n-2$ remaining vertices. For the associahedron, choose any two vertices v, v', and complete them into a tube by taking the path between them. For the cyclohedron, choose the two vertices v, v', and complete them into a tube by taking either all the cycle, or one of the two paths between v and v' (this gives three options in general, but only two when v, v' are neighbors). For the stellohedron, choose either v as the center of the star and v' as one of the $n-1$ leaves, or v and v' as leaves of the star and complete them into a tube by taking the center and any subset of the $n-3$ remaining leaves. □

To conclude on graph associahedra, we characterize the graphs G whose nested fan has a simplicial type cone.

Proposition 2.31. *The type cone $\mathbb{TC}(\mathcal{F}(G))$ is simplicial if and only if G is a path.*

Proof. Note that any tube t with $|\mathsf{t}| \ge 2$ has two non-disconnecting vertices when it is a path, and at least three non-disconnecting vertices otherwise (the leaves of an arbitrary spanning tree of t). Therefore, each tube of $\mathcal{T}(G)$ which is not a singleton contributes to at least one extremal exchangeable pairs. We conclude that the number of extremal exchangeable pairs is at least:

$$|\mathcal{T}(G)| - |V| = |\mathcal{T}(G) \smallsetminus \{\varnothing, G\}| - (|V| - 1) = N - n,$$

with equality if and only if all tubes of G are paths, *i.e.* if and only if G is a path. □

3. Relations for $\boldsymbol{g}$-vectors in finite type cluster algebras via 2-Calabi–Yau triangulated categories

3.1. **Setting.** Let $\mathbb{K}$ be a field. Let $\mathcal{C}$ be a $\mathbb{K}$-linear triangulated category with suspension functor Σ. We fix a collection $\mathrm{ind}(\mathcal{C})$ of representatives of isomorphism classes of indecomposable objects of $\mathcal{C}$. We will assume the following:

- $\mathcal{C}$ is essentially small (in particular, $\mathrm{ind}(\mathcal{C})$ is a set);
- $\mathcal{C}$ is Hom-finite: for each pair of objects X and Y, the $\mathbb{K}$-vector space $\mathcal{C}(X, Y)$ is finite-dimensional;
- $\mathcal{C}$ is Krull–Schmidt: the endomorphism algebra of any indecomposable object is local;
- $\mathcal{C}$ is 2-Calabi–Yau: for each pair of objects X and Y, there is an isomorphism of bifunctors
$$\mathcal{C}(X, \Sigma Y) \to D\mathcal{C}(Y, \Sigma X)$$
where $D = \mathrm{Hom}_{\mathbb{K}}(-, \mathbb{K})$ is the usual duality of vector spaces;
- $\mathcal{C}$ contains a basic cluster-tilting object $T = \bigoplus_{i=1}^{n} T_i$:
$$\text{for any object } X, \quad \mathcal{C}(T, \Sigma X) = 0 \quad \text{if and only if} \quad X \in \mathrm{add}(T),$$
where $\mathrm{add}(T)$ is the smallest additive full subcategory of $\mathcal{C}$ containing the T_i's and closed under isomorphisms.

3.2. **Statement of the theorem.** We need to introduce some notations before stating the main result of this section. Let $\Lambda := \mathrm{End}_{\mathcal{C}}(T)$, and let F be the functor
$$F = \mathcal{C}(T, -) : \mathcal{C} \to \mathrm{mod}\,\Lambda.$$

Proposition 3.1 ([BMR07, KR07])**.** *The functor F induces an equivalence of $\mathbb{K}$-linear categories*
$$F : \mathcal{C}/(\Sigma T) \to \mathrm{mod}\,\Lambda,$$
where (ΣT) is the ideal of morphisms factoring through an object of $\mathrm{add}(\Sigma T)$ and $\mathrm{mod}\,\Lambda$ is the category of finite-dimensional right Λ-modules. This equivalence induces further equivalences between $\mathrm{add}(T)$ and the category of projective modules, and between $\mathrm{add}(\Sigma^2 T)$ and that of injective modules.

For categories with a cluster-tilting objects, the 2-Calabi–Yau condition implies other duality results which we shall need.

Proposition 3.2 ([Pal08])**.** *For any pair of objects X and Y in $\mathcal{C}$, there is an isomorphism of bifunctors*
$$(\Sigma T)\mathcal{C}(X, \Sigma Y) \xrightarrow{\cong} \mathcal{C}(Y, \Sigma X)/(\Sigma T),$$
where $(\Sigma T)\mathcal{C}(X, \Sigma Y)$ is the space of morphisms from X to Y factoring through (ΣT).

Remark 3.3. Although the field $\mathbb{K}$ is assumed to be algebraically closed in [Pal08], this assumption is not needed in the proof, and the result is valid over any field.

Finally, we need the existence of almost-split triangles in $\mathcal{C}$. Recall that a triangle
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$
in $\mathcal{C}$ is *almost-split* if X and Z are indecomposable, h is non-zero, and any non-section $X \to X'$ factors through f (or equivalently, any non-retraction $Z' \to Z$ factors through g). We say that a triangulated category *has almost-split triangles* if there is an almost-split triangle as above for any indecomposable object X.

Proposition 3.4. *Any triangulated category admitting a Serre functor has almost-split triangles. In particular, any 2-Calabi–Yau triangulated category has almost-split triangles.*

Cite Happel — PG.

Definition 3.5. (1) Let $K_0^{\mathrm{sp}}(\mathcal{C})$ be the *split Grothendieck group* of $\mathcal{C}$, that is, the free abelian group generated by symbols $[X]$, where $[X]$ denotes the isomorphism class of X in $\mathcal{C}$, modulo the following relations: for any objects Y and Z, we let $[Y \oplus Z] = [Y] + [Z]$.

(2) Let $K_0^{\boldsymbol{g}}(\mathcal{C})$ be the quotient of $K_0^{\mathrm{sp}}(\mathcal{C})$ by the relations $[X]+[Z]-[Y]$ for all triangles

$$X \to Y \to Z \xrightarrow{h} \Sigma X$$

with $h \in (\Sigma T)$. Denote by $\boldsymbol{g} : K_0^{\mathrm{sp}}(\mathcal{C}) \to K_0^{\boldsymbol{g}}(\mathcal{C})$ the canonical projection.

In particular, $K_0^{\mathrm{sp}}(\mathcal{C})$ is isomorphic to a free abelian group over the set $\mathrm{ind}(\mathcal{C})$. The choice of the notation for $K_0^{\boldsymbol{g}}(\mathcal{C})$ is motivated by Section 3.4, where we study relations between $\boldsymbol{g}$-vectors.

Definition 3.6. For any two objects X and Y of $\mathcal{C}$, define

$$\langle X, Y\rangle := \dim_{\mathbb{K}} \mathrm{Hom}_\Lambda(FX, FY).$$

This defines a bilinear form

$$\langle -, -\rangle : K_0^{\mathrm{sp}}(\mathcal{C}) \times K_0^{\mathrm{sp}}(\mathcal{C}) \to \mathbb{Z}.$$

Notation 3.7. (1) For any indecomposable object X of $\mathcal{C}$, let

$$X \to E \to \Sigma^{-1}X \to \Sigma X$$

be an almost split triangle (unique up to isomorphism). We let

$$\ell_X := [X] + [\Sigma^{-1}X] - [E] \in K_0^{\mathrm{sp}}(\mathcal{C}).$$

(2) For any indecomposable object Y of $\mathcal{C}$, let

$$\Sigma Y \to E' \to Y \to \Sigma^2 Y$$

be an almost split triangle (unique up to isomorphism). We let

$$r_Y := [Y] + [\Sigma Y] - [E'] \in K_0^{\mathrm{sp}}(\mathcal{C}).$$

recall Auslander — PG.

We can finally state the main theorem of this section.

Theorem 3.8. *Let $\mathcal{C}$ be a category satisying the hypotheses of Section 3.1. Then $\mathcal{C}$ has only finitely many isomorphism classes of indecomposable objects if and only if the set*

$$L := \big\{\ell_X \mid X \in \mathrm{ind}(\mathcal{C}) \setminus \mathrm{add}(\Sigma T)\big\}$$

generates the kernel of $\boldsymbol{g} : K_0^{\mathrm{sp}}(\mathcal{C}) \to K_0^{\boldsymbol{g}}(\mathcal{C})$. In this case, the set L is a basis of the kernel of $\boldsymbol{g}$, and for any $x \in \ker(\boldsymbol{g})$, we have that

$$x = \sum_{A \in \mathrm{ind}(\mathcal{C}) \setminus \mathrm{add}(\Sigma T)} \frac{\langle x, A\rangle}{\langle \ell_A, A\rangle} \ell_A.$$

Corollary 3.9. *Assume that $\mathrm{ind}(\mathcal{C})$ is finite. Let $X \to E \to Y \xrightarrow{h} \Sigma X$ be a triangle with $h \in (\Sigma T)$. Then the element $x = [X] + [Y] - [E]$ of the kernel of $\boldsymbol{g}$ is a non-negative linear combination of the ℓ_A, with $A \in \mathrm{ind}(\mathcal{C}) \setminus \mathrm{add}(\Sigma T)$.*

Proof. We know that $x = \sum_{A \in \mathrm{ind}(\mathcal{C}) \setminus \mathrm{add}(\Sigma T)} \frac{\langle x, [A]\rangle}{\langle \ell_A, [A]\rangle} \ell_A$; since $\langle \ell_A, [A]\rangle$ is positive by Lemma 3.12, we only need to show that each $\langle x, [A]\rangle$ is non-negative. The functor $F = \mathcal{C}(T, -)$ induces an exact sequence

$$FX \to FE \to FY \to 0,$$

which in turn induces an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(FY, FA) \to \mathrm{Hom}_\Lambda(FE, FA) \xrightarrow{f} \mathrm{Hom}_\Lambda(FX, FA) \to \mathrm{coker}(f) \to 0.$$

Therefore $\langle x, [A]\rangle = \dim_{\mathbb{K}} \mathrm{coker}(f) \geq 0$. $\square$

3.3. Proof of Theorem 3.8.

Lemma 3.10. *If X or Y lie in* $\operatorname{add}(\Sigma T)$*, then* $\langle X, Y\rangle = 0$.

Proof. This is because $F\Sigma T = 0$. □

Lemma 3.11. *Let* $X \to Y \to \Sigma^{-1}X \xrightarrow{h} \Sigma X$ *be an almost-split triangle. Then* $X \notin \operatorname{add}(\Sigma T)$ *if and only if* $h \in (\Sigma T)$.

Proof. If $X \in \operatorname{add}(\Sigma T)$, then h cannot be in (ΣT), otherwise it would be zero since $\mathcal{C}(T, \Sigma T) = 0$.

Assume now that $X \notin \operatorname{add}(\Sigma T)$. Let $\mathbb{K}_X$ be the residue field of the algebra $\operatorname{End}_{\mathcal{C}}(X)$. By definition of an almost-split triangle, h is in the socle of the right $\operatorname{End}_{\mathcal{C}}(X)$-module $\mathcal{C}(\Sigma^{-1}X, \Sigma X)$. Moreover, this socle is a one-dimensional $\mathbb{K}_X$-vector space; indeed, the 2-Calabi–Yau condition gives an isomorphism

$$\mathcal{C}(\Sigma^{-1}X, \Sigma X) \cong D\mathcal{C}(X, X).$$

Thus the socle of the right module $\mathcal{C}(\Sigma^{-1}X, \Sigma X)$ has the same $\mathbb{K}_X$ dimension as the top of the left module $\mathcal{C}(X, X)$. Since X is indecomposable, $\mathcal{C}(X, X)$ is local, and its top is one-dimensional over $\mathbb{K}_X$.

Now, $(\Sigma T)\mathcal{C}(\Sigma^{-1}X, \Sigma X)$ is a sub-module of $\mathcal{C}(\Sigma^{-1}X, \Sigma X)$. Therefore, if $(\Sigma T)\mathcal{C}(\Sigma^{-1}X, \Sigma X)$ is non-zero, then it contains the one-dimensional socle of $\mathcal{C}(\Sigma^{-1}X, \Sigma X)$, and thus contains h. By [Pal08],

$$(\Sigma T)\mathcal{C}(\Sigma^{-1}X, \Sigma X) \cong D\mathcal{C}(X, X)/(\Sigma T).$$

The identity morphism of X is not in (ΣT), since X is not in $\operatorname{add}(\Sigma T)$. Thus the right-hand side is non-zero, and so neither is the left-hand side. By the above, this implies that $h \in (\Sigma T)\mathcal{C}(\Sigma^{-1}X, \Sigma X)$, which finishes the proof. □

Lemma 3.12. *Let A and B be two indecomposable objects of $\mathcal{C}$.*

introduce general notation for redisue field — PG.

(1) If $A \notin \operatorname{add}(\Sigma T)$*, then*

$$\langle \ell_A, B\rangle = \begin{cases} 0 & \text{if } A \ncong B; \\ \dim_{\mathbb{K}} \mathbb{K}_A & \text{if } A \cong B. \end{cases}$$

(2) If $B \notin \operatorname{add}(\Sigma T)$*, then*

$$\langle A, r_B\rangle = \begin{cases} 0 & \text{if } A \ncong B; \\ \dim_{\mathbb{K}} \mathbb{K}_B & \text{if } A \cong B. \end{cases}$$

Proof. We only prove the first assertion; the second one is proved dually. Assume that $A \notin \operatorname{add}(\Sigma T)$. Let

$$A \xrightarrow{f} E \to \Sigma^{-1}A \xrightarrow{h} \Sigma A$$

be an almost-split triangle. By Lemma 3.11, the morphism h is in (ΣT). Applying the functor $F = \mathcal{C}(T, -)$, we get an exact sequence

$$FA \xrightarrow{Ff} FE \to F\Sigma^{-1}A \to 0.$$

Applying now the functor $\operatorname{Hom}_\Lambda(-, FB)$, we get an exact sequence

$$0 \to \operatorname{Hom}_\Lambda(F\Sigma^{-1}A, FB) \to \operatorname{Hom}_\Lambda(FE, FB) \xrightarrow{Ff^*} \operatorname{Hom}_\Lambda(FA, FB) \to \operatorname{coker}(Ff^*) \to 0$$

If $B \ncong A$, then the definition of an almost-split triangle implies that Ff^* is surjective. Thus $\operatorname{coker}(Ff^*$ 0, and by additivity of the dimension in exact sequences, we get that $\langle \ell_A, B\rangle = 0$.

add an exact sequence here; image of Ffstar is non-invertible elements — PG.

If $B \cong A$, then $\operatorname{coker}(Ff^*)$ is isomorphic to the residue field of $\operatorname{End}_\Lambda(FA)$, which is isomorphic to the residue field $\mathbb{K}_A$ of A. □

Lemma 3.13. *Let* $x \in K_0^{\mathrm{sp}}(\mathcal{C})$*, and write*

$$x = \sum_{A \in \operatorname{ind}(\mathcal{C})} \lambda_A [A].$$

Then for any $A \in \operatorname{ind}(\mathcal{C}) \setminus \operatorname{add}(\Sigma T)$, we have that

$$\lambda_A = \langle \ell_A, x\rangle = \langle x, r_A\rangle.$$

Proof. Let $B \in \operatorname{ind}(\mathcal{C}) \setminus \operatorname{add}(\Sigma T)$. Applying Lemma 3.12, we get that

$$\langle \ell_B, x\rangle = \langle \ell_B, \sum_{A\in\operatorname{ind}(\mathcal{C})} \lambda_A [A]\rangle = \sum_{A\in\operatorname{ind}\mathcal{C}} \lambda_A \langle \ell_B, [A]\rangle = \lambda_A.$$

The equality $\lambda_A = \langle x, r_A\rangle$ is proved in a similar way. □

Corollary 3.14. *Let $x \in K_0^{\mathrm{sp}}(\mathcal{C})$. Then the following are equivalent.*

(1) $x \in K_0^{\mathrm{sp}}(\operatorname{add}(\Sigma T))$;
(2) $\langle x, [A]\rangle = 0$ for all $A \in \operatorname{ind}(\mathcal{C})$;
(3) $\langle [A], x\rangle = 0$ for all $A \in \operatorname{ind}(\mathcal{C})$.

Proof. We will only proof that (1) is equivalent to (2); the proof that (1) is equivalent to (3) is similar.

Assume that (2) holds. Then, by Lemma 3.13, we have that

$$x = \sum_{A\in\operatorname{ind}(\mathcal{C})\setminus\operatorname{add}(\Sigma T)} \langle \ell_A, x\rangle + \sum_{i=1}^{n} \lambda_{\Sigma T_i}[\Sigma T_i] = \sum_{i=1}^{n} \lambda_{\Sigma T_i}[\Sigma T_i].$$

Thus $x \in K_0^{\mathrm{sp}}(\operatorname{add}(\Sigma T))$, and (1) holds.

Assume now that (1) holds. Then $\langle x, [A]\rangle = 0$ for any A by Lemma 3.10. Thus (2) holds. □

Proposition 3.15. *(1) The set $\{[\ell_A] \mid A \in \operatorname{ind}(\mathcal{C}) \setminus \operatorname{add}(\Sigma T)\} \cup \{[\Sigma T_i] \mid i = 1, \dots, n\}$ is free in $K_0^{\mathrm{sp}}(\mathcal{C})$.*
(2) The set $\{[r_A] \mid A \in \operatorname{ind}(\mathcal{C}) \setminus \operatorname{add}(\Sigma T)\} \cup \{[\Sigma T_i] \mid i = 1, \dots, n\}$ is free in $K_0^{\mathrm{sp}}(\mathcal{C})$.

Proof. We only prove (1); the proof of (2) is similar. Assume that

$$x = \sum_{A\in\operatorname{ind}(\mathcal{C})\setminus\operatorname{add}(\Sigma T)} \lambda_A \ell_A + \sum_{i=1}^{n} \lambda_i [\Sigma T_i] = 0.$$

Then $\langle x, [A]\rangle = 0$ for all $A \in \operatorname{ind}(\mathcal{C}) \setminus \operatorname{add}(\Sigma T)$. But $\langle x, [A]\rangle = \lambda_A$ by Lemma 3.12. Thus

$$x = \sum_{i=1}^{n} \lambda_i [\Sigma T_i] = 0.$$

But the $[\Sigma T_i]$ are linearly independent in $K_0^{\mathrm{sp}}(\mathcal{C})$. Thus $\lambda_i = 0$ for all $i \in \{1, \dots, n\}$. This finishes the proof. □

Proposition 3.16. *Assume that $\operatorname{ind}(\mathcal{C})$ is finite. Then the set $\{[\ell_A] \mid A \in \operatorname{ind}(\mathcal{C}) \setminus \operatorname{add}(\Sigma T)\}$ is a basis of the kernel of $\boldsymbol{g}$. Moreover, for any $x \in \ker \boldsymbol{g}$, we have that*

$$x = \sum_{A\in\operatorname{ind}(\mathcal{C})\setminus\operatorname{add}(\Sigma T)} \frac{\langle x, A\rangle}{\langle \ell_A, A\rangle} \ell_A.$$

Proof. By Proposition 3.15, the set is free. Let $x \in \ker \boldsymbol{g}$. Consider the element

$$z = x - \sum_{A\in\operatorname{ind}(\mathcal{C})\setminus\operatorname{add}(\Sigma T)} \frac{\langle x, A\rangle}{\langle \ell_A, A\rangle} \ell_A.$$

Then for any $B \in \operatorname{ind}\mathcal{C}$, we have that

$$\langle z, [B]\rangle = \left\langle x - \sum_{A\in\operatorname{ind}(\mathcal{C})\setminus\operatorname{add}(\Sigma T)} \frac{\langle x, A\rangle}{\langle \ell_A, A\rangle} \ell_A, [B] \right\rangle = \langle x, [B]\rangle - \langle x, [B]\rangle = 0,$$

where the second equality is obtained by using Lemma 3.12. By Lemma 3.14, this implies that $z \in K_0^{\mathrm{sp}}(\operatorname{add}(\Sigma T))$. Since $z \in \ker(\boldsymbol{g})$ and since $\boldsymbol{g}$ is injective on $K_0^{\mathrm{sp}}(\operatorname{add}(\Sigma T))$, we get that $z = 0$. This finishes the proof. □

We need a lemma saying that g is injec-

Corollary 3.17. *If* $\operatorname{ind}(\mathcal{C})$ *is finite, then the set*

$$\{[\ell_A] \mid A \in \operatorname{ind}(\mathcal{C}) \setminus \operatorname{add}(\Sigma T)\} \cup \{[\Sigma T_i] \mid i = 1, \dots, n\}$$

is a basis of $K_0^{\mathrm{sp}}(\mathcal{C})$.

Proof. By Proposition 3.15, the set is free. It suffices to prove that it generates $K_0^{\mathrm{sp}}(\mathcal{C})$. Let $x \in K_0^{\mathrm{sp}}(\mathcal{C})$. Consider

$$z = x - \sum_{A \in \operatorname{ind} \mathcal{C}} \frac{\langle x, A\rangle}{\langle \ell_A, A\rangle} \ell_A.$$

Then for any $B \in \operatorname{ind}(\mathcal{C})$, we have that $\langle z, [B]\rangle = 0$. By Lemma 3.14, this implies that $z \in K_0^{\mathrm{sp}}(\operatorname{add}(\Sigma T))$, and finishes the proof. $\square$

All that remains is to prove the converse in the statement of Theorem 3.8.

Proposition 3.18. *Assume that the set* $\{[\ell_A] \mid A \in \operatorname{ind}(\mathcal{C}) \setminus \operatorname{add}(\Sigma T)\}$ *is a basis of the kernel of* $\boldsymbol{g}$. *Then* $\operatorname{ind}(\mathcal{C})$ *is finite.*

Proof. Let $x \in \ker \boldsymbol{g}$, and write $x = \sum_{[A]\in\operatorname{ind}(\mathcal{C})} \lambda_A[A]$, where the sum has finite support. For any B not in the support of the sum, we have that $\langle x, [B]\rangle = 0$.

Now, $[T] + [\Sigma T]$ is in the kernel of $\boldsymbol{g}$, but $\langle [T] + [\Sigma T], [B]\rangle = \langle [T], [B]\rangle = 0$ if and only if $B \in \operatorname{add}(\Sigma T)$. Thus $\operatorname{ind}(\mathcal{C})$ has to be finite. $\square$

3.4. **Application to $\boldsymbol{g}$-vectors.** We recall the following results and definition from [DK08, Pal08].

Proposition 3.19. *Let* X *be an object of* $\mathcal{C}$. *Then there exists a triangle*

$$T_1^X \to T_X^0 \to X \to \Sigma T_1^X$$

with T_1^X *and* T_0^X *in* $\operatorname{add}(T)$.

Definition 3.20. The *index* of an object X is the element

$$\operatorname{ind}_T(X) := [T_0^X] - [T_1^X] \in K_0^{\mathrm{sp}}(\operatorname{add}(T)).$$

The notion of index is very close to the definition of the map $\boldsymbol{g} : K_0^{\mathrm{sp}}(\mathcal{C}) \to K_0^{\boldsymbol{g}}(\mathcal{C})$ of Definition 3.5. The link is given by the following result.

Proposition 3.21 ([Pal08]). *Let* $X \to Y \to Z \xrightarrow{h} \Sigma X$ *be a triangle. Then*

$$\operatorname{ind}_T(X) + \operatorname{ind}_T(Z) - \operatorname{ind}_T(Y) = 0$$

if and only if $h \in (\Sigma T)$.

Corollary 3.22. *There is an isomorphism* $\phi : K_0^{\boldsymbol{g}}(\mathcal{C}) \to K_0^{\mathrm{sp}}(\operatorname{add}(T))$ *such that* $\operatorname{ind}_T = \phi \circ \boldsymbol{g}$. *In particular,* $K_0^{\boldsymbol{g}}(\mathcal{C})$ *is a free abelian group generated by the* $[T_i]$.

Theorem 3.23. *Let* $\mathcal{C}$ *be the cluster category of a valued quiver of Dynkin type* A, B, C, D, E, F *or* G. *Let* T *be a basic cluster-tilting object in* $\mathcal{C}$, *and let* Q *be the valued Gabriel quiver of* $\operatorname{End}_{\mathcal{C}}(T)$. *Then there is a bijection*

cite???
— PG.

$$\phi : \operatorname{ind}(\mathcal{C}) \to \{\textit{cluster variables in the cluster algebra of } Q\},$$

which has the follwing properties.

- *For any* $X, Y \in \operatorname{ind}(\mathcal{C})$, $\phi(X)$ *and* $\phi(Y)$ *are compatible if and only if* $\mathcal{C}(X, \Sigma Y) = 0$.
- *For any* $X \in \operatorname{ind}(\mathcal{C})$, *the* $\boldsymbol{g}$*-vector of* $\phi(X)$ *is* $\boldsymbol{g}([X])$, *where we identify* $\mathbb{Z}^n$ *and* $K_0^{\boldsymbol{g}}(\mathcal{C})$ *via the isomorphism sending* $(a_1, \dots, a_n)$ *to* $\sum_{i=1}^n a_i[T_i]$.
- *For any* $X, Y \in \operatorname{ind}(\mathcal{C})$, $\phi(X)$ *and* $\phi(Y)$ *are exchangeable if and only if* $\dim_{\mathbb{K}_X} \mathcal{C}(X, \Sigma Y) = \dim_{\mathbb{K}_Y} \mathcal{C}(X, \Sigma Y) = 1$.

The cluster category also contains enough information to detect positive mutations.

Proposition 3.24. *Let $X, Y \in \operatorname{ind}(\mathcal{C})$ be such that $\dim_{\mathbb{K}_X} \mathcal{C}(X, \Sigma Y) = \dim_{\mathbb{K}_Y} \mathcal{C}(X, \Sigma Y) = 1$. Let*

$$X \to E \to Y \xrightarrow{h} \Sigma X \quad \text{and} \quad Y \to E' \to X \xrightarrow{h'} \Sigma Y$$

be triangles with h and h' non-zero (these triangles are unique up to isomorphism). Then $\phi(Y)$ is obtained by performing a positive mutation on $\phi(X)$ in some seed if and only if $h \in (\Sigma T)$.

TO DO, Vincent: definition of positive mutation? — PG.

Proof. □

4. Relations for *g*-vectors in brick algebras via extriangulated categories

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