STRING MODULES, NON-KISSING COMPLEXES AND WHATNOT (TEMPORARY TITLE)

YANN PALU, VINCENT PILAUD, AND PIERRE-GUY PLAMONDON

Abstract. This is a work in progress.

Contents

1. Introduction

Conventions. If Q is a quiver and k is a field, we denote by kQ the path algebra of Q. Arrows of a quiver Q are composed from right to left: if $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ is a quiver, then ba is a path while ab is not. Modules over an algebra are assumed to be *left* modules. By these conventions, a module over kQ is equivalent to a representation of Q.

2. String modules

2.1. **String algebras.** String algebras are a class of algebras defined by generators and particularly nice relations. The study of their representations goes back to [4]; we will follow the general framework established in [3].

Let k be an algebraically closed field, and let Q be a quiver. While this is not necessary for the theory, it suffices our purposes to assume that Q is finite.

Definition 2.1. A *string algebra* is an algebra of the form kQ/I, where I is an ideal generated by paths of length at least two in Q and such that kQ/I is finite-dimensional, and satisfying the following properties:

(1) for any vertex i of Q, there are at most two arrows starting at i and at most two arrows ending in i;
(2) for any arrow β of Q, there is at most one arrow γ such that $\beta\gamma \notin I$;

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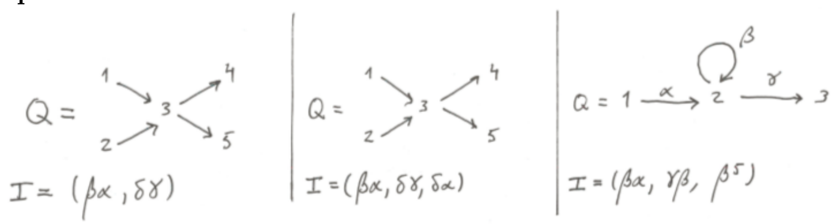
(3) for any arrow β of Q, there is at most one arrow α such that $\alpha\beta \notin I$.

Example 2.2.



Definition 2.3 ([?])**.** A string algebra kQ/I is a *gentle algebra* if

(1) I is generated by paths of length exactly two;
(2) for any arrow β of Q, there is at most one arrow γ such that $t(\gamma) = s(\beta)$ and $\beta\gamma \in I$;
(3) for any arrow β of Q, there is at most one arrow α such that $s(\alpha) = t(\beta)$ and $\alpha\beta \in I$.

Example 2.4. In Example 2.2, only the first algebra is gentle.

String algebras enjoy a particularly nice representation theory: they are tame algebras (as proved in [?]) and their indecomposable representations are completely classified. We will make heavy use of this classification, which we recall in the next section

2.2. **String and band modules.** For any arrow β of any quiver Q, define a formal inverse β^{-1} with the properties that $s(\beta^{-1}) = t(\beta), t(\beta^{-1}) = s(\beta), \beta^{-1}\beta = e_{s(\beta)}$ and $\beta\beta^{-1} = e_{t(\beta)}$, where e_i is the path of length zero starting and ending at the vertex i. Furthermore, let $(\beta^{-1})^{-1}$; thus $(-)^{-1}$ is an involution on the set of arrows and formal inverses of arrows of Q. We extend this involution to the set of all paths involving arrows or their formal inverses, by further setting $(e_i)^{-1} = e_i$ for all vertices i of Q.

Definition 2.5. Let $A = kQ/I$ be a string algebra.

(1) A *string* for A is a word of the form
$$w = \beta_1^{\varepsilon_1}\beta_2^{\varepsilon_2}\cdots\beta_n^{\varepsilon_n},$$
where
- each β_i is an arrow in Q and each ε_i is an element of $\{1, -1\}$;
- for each $i \in \{1, 2, \ldots, n-1\}$, $t(\beta_{i+1}^{\varepsilon_{i+1}}) = s(\beta_i^{\varepsilon_i})$;
- if p is a path in Q such that p or p^{-1} appears as a subword of w, then p is not an element of I; and
- w is reduced, in the sense that no subword of the form $\beta\beta^{-1}$ or $\beta^{-1}\beta$ appears in w, with β an arrow of Q.

The integer n is called the *length* of the string w.

For each vertex i of Q, there is also a *string of length zero*, denoted by e_i, that starts and ends at i.

(2) A *band* for A is a string b of length at least one such that
- $s(b) = t(b)$;
- all powers of b are strings; and
- b is not itself a power of a strictly smaller string.

Example 2.6. For the algebra

$$Q = \begin{tikzcd} & 2 \arrow[dr, "\beta"] & & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & 3 \arrow[r, "\delta"] & 4 \end{tikzcd} \qquad I = (\delta\beta)$$

some strings are $e_1, e_2, e_3, e_4, \alpha^{\pm}, \beta^{\pm}, \gamma^{\pm}, \delta^{\pm}, \delta\gamma, \gamma^{-1}\beta, \ldots$.
The only bands are $\gamma^{-1}\beta\alpha, \beta\alpha\gamma^{-1}, \alpha\gamma^{-1}\beta$ and their inverses.

Definition 2.7. Let $w = \beta_1^{\varepsilon_1}\beta_2^{\varepsilon_2}\cdots\beta_n^{\varepsilon_n}$ be a string for a string algebra $A = kQ/I$. The *string module* $M(w)$ is the A-module defined as a representation of Q as follows:

- Let $i_0 = t(\beta_1^{\varepsilon_1})$, and $i_m = s(\beta_m^{\varepsilon_m})$ for each $m \in \{1, \ldots, n\}$.
- For each vertex j of Q, let $M(w)_j$ be the vector space with basis given by $\{v_m \mid i_m = j\}$.
- For each arrow γ of Q, the linear map $M(w)_\gamma : M(w)_{s(\gamma)} \to M(w)_{t(\gamma)}$ is defined on the basis of $M(w)_{s(\gamma)}$ by

$$M(w)_\gamma(v_m) = \begin{cases} v_{m-1} & \text{if } \beta_m = \gamma \text{ and } \varepsilon_m = 1; \\ v_{m+1} & \text{if } \beta_{m+1} = \gamma \text{ and } \varepsilon_{m+1} = -1; \\ 0 & \text{else.} \end{cases}$$

It follows from the definition that for any string w, the string modules $M(w)$ and $M(w^{-1})$ are isomorphic.

Example 2.8. For the algebra given in Example 2.6, let $w = \delta\gamma\alpha^{-1}\beta^{-1}\gamma\alpha^{-1}\beta^{-1}\gamma$. Then $M(w)$ is given by

$$\begin{tikzcd} & k^2 \arrow[dr, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}"] & & \\ k^3 \arrow[ur, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"] \arrow[rr, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"'] & & k^3 \arrow[r, "\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}"] & k \end{tikzcd}$$

Notation 2.9. For convenience, even though 0 is not a string by our definition, we will define $M(0)$ to be the zero module.

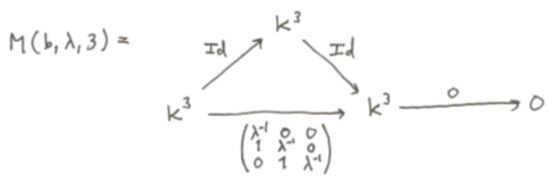
Definition 2.10. Let $b = \beta_1^{\varepsilon_1}\beta_2^{\varepsilon_2}\cdots\beta_n^{\varepsilon_n}$ be a band for a string algebra $A = kQ/I$. Moreover, let $\lambda \in k^*$, and let $d \in \mathbb{N}_{>0}$. The *band module* $M(b, \lambda, d)$ is defined as follows:

- For each vertex j of Q, the vector space $M(b, \lambda, d)_j$ is equal to k^d.
- For each arrow γ of Q different from β_1, the linear map $M(b, \lambda, d)_\gamma$ is the identity.

- The linear map $M(b,\lambda,d)_{\beta_1}$ is equal to $J_d(\lambda)^{\varepsilon_1}$, where
$$J_d(\lambda) = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$
is the $d \times d$ Jordan block of type λ.

It follows from the definition that the band modules $M(b,\lambda,d)$ and $M(b^{-1},\lambda^{-1},d)$ are isomorphic. Moreover, if b and b' are two bands that are *cyclically equivalent*, in the sense that one is obtained from the other by cyclically permuting the arrows that constitute it, then we also have that $M(b,\lambda,d)$ and $M(b',\lambda,d)$ are isomorphic.

Example 2.11. For the algebra defined in Example 2.6, let $b = \gamma^{-1}\beta\alpha$. Then b is a band, and the following are band modules:



Theorem 2.12 ([3], p.161)**.** *Let A be a string algebra. Then the string and band modules over A form a complete list of indecomposable A-modules, up to isomorphism. Moreover,*

- *A string module is never isomorphic to a band module.*
- *Two string modules $M(w)$ and $M(w')$ are isomorphic if and only if $w' = w^{\pm 1}$.*
- *Two band modules $M(b,\lambda,d)$ and $M(b',\lambda',d')$ are isomorphic if and only if $d = d'$, and either*
  - *b is cyclically equivalent to b', and $\lambda = \lambda'$, or*
  - *b^{-1} is cyclically equivalent to b', and $\lambda^{-1} = \lambda'$.*

2.3. **Auslander-Reiten translation of string modules.** The following notation, still taken from [3], will be useful for dealing with string modules.

For a given string $w = \beta_1^{\varepsilon_1}\beta_2^{\varepsilon_2}\cdots\beta_n^{\varepsilon_n}$, we draw w as follows:

- draw all arrows $\beta_1,\ldots,\beta_n$ from left to right;
- draw all arrows pointing downwards.

Moreover, the strings e_i of length zero are depicted simply as i.

This is best understood on an example.

Example 2.13. For the algebra defined in Example 2.6, the string $w = \delta\gamma\alpha^{-1}\beta^{-1}\gamma\alpha^{-1}\beta^{-1}\gamma$ is drawn as follows:

There are some immediate advantages for this depiction of strings. For instance, the socle of $M(w)$ is the direct sum of the simple modules corresponding to the local minima in the picture of w; similarly, the top of $M(w)$ is the direct sum of the simple modules corresponding to the local maxima.

Definition 2.14. (1) A string w *starts* (or *ends*) *on a peak* if there is no arrow β such that $w\beta$ (or $\beta^{-1}w$, respectively) is a string.
(2) A string w *starts* (or *ends*) *in a deep* if there is no arrow β such that $w\beta^{-1}$ (or βw, respectively) is a string.

Remark 2.15. Definition 2.14 is best understood by drawing strings. Starting (or ending) on a peak means that one cannot add an arrow at the start (or the end) of w such that the starting point (or ending point) of the new string would be higher in the picture than that of w. The same applies when replacing "on a peak" by "in a deep" and "higher" by "lower".

Example 2.16. The string in Example 2.13 starts and ends on a peak, and also ends in a deep.

Definition 2.17. (1) Let w be a string that does not start on a peak. Let $\beta_0, \beta_1, \ldots, \beta_r$ be arrows such that $w_h := w\beta_0\beta_1^{-1}\cdots\beta_r^{-1}$ starts in a deep. We say that w_h is obtained from w by *adding a hook* at the start of w.
(2) Let w be a string that does not end on a peak. Let $\beta_0, \beta_1, \ldots, \beta_r$ be arrows such that ${}_hw := \beta_r\beta_{r-1}\cdots\beta_1\beta_0^{-1}w$ ends in a deep. We say that ${}_hw$ is obtained from w by *adding a hook* at the end of w.

Remark 2.18. (1) Unless w has length zero, the strings ${}_hw$ and w_h, when they are defined, are uniquely determined. If w has length zero, then there may by two possible choices for ${}_hw$ and w_h.
(2) It is possible that $r = 0$.

The dual of a hook is a cohook.

Definition 2.19. (1) Let w be a string that does not start in a deep. Let $\beta_0, \beta_1, \ldots, \beta_r$ be arrows such that $w_c = w\beta_0^{-1}\beta_1\cdots\beta_r$ starts on a peak. We say that w_c is obtained from w by *adding a cohook* at the start of w.
(2) Let w be a string that does not end in a deep. Let $\beta_0, \beta_1, \ldots, \beta_r$ be arrows such that ${}_cw = \beta_r^{-1}\beta_{r-1}^{-1}\cdots\beta_1^{-1}\beta_0 w$ ends on a peak. We say that ${}_cw$ is obtained from w by *adding a cohook* at the end of w.

Remark 2.20. (1) As when adding hooks, if w has length zero, then the strings ${}_cw$ and w_c, when they are defined, are uniquely determined. If w has length zero, then there may by two possible choices for ${}_cw$ and w_c.

(2) It is possible that $r = 0$.

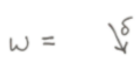
Example 2.21. In pictures, hooks always look like

while cohooks look like

Example 2.22. In Example 2.6, the string

does not start on a deep. Thus

Example 2.23. Still in Example 2.6, the string $w = \delta$ does not start on a peak, and we have



Lemma 2.24 (and Definition)**.** *Let w be a string for a string algebra $A = kQ/I$.*

(1) *If w starts on a peak and is not a path in Q, then there exists a unique string w' such that w' does not start in a deep, and $w = w'_c$. We denote w' by $w_{c^{-1}}$, and we say that is obtained from w by* removing a cohook *at the start of w. If w is a path in Q, then we put $w_{c^{-1}} = 0$.*
(2) *If w ends on a peak and is not the inverse of a path in Q, then there exists a unique string w' such that w' does not end in a deep, and $w =_c w'$. We denote w' by ${}_{c^{-1}}w$, and we say that is obtained from w by* removing a

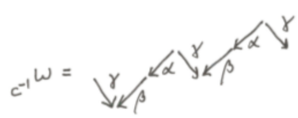
cohook *at the end of* w*. If* w *is the inverse of a path in* Q*, then we put* ${}_{c^{-1}}w = 0$.

(3) *If* w *starts in a deep and is not the inverse of a path in* Q*, then there exists a unique string* w' *such that* w' *does not start on a peak, and* $w = w'_h$*. We denote* w' *by* $w_{h^{-1}}$*, and we say that is obtained from* w *by* removing a hhook *at the start of* w*. If* w *is a the inverse of a path in* Q*, then we put* $w_{h^{-1}} = 0$.

(4) *If* w *ends in a deep and is not a path in* Q*, then there exists a unique string* w' *such that* w' *does not end on a peak, and* $w =_h w'$*. We denote* w' *by* ${}_{h^{-1}}w$*, and we say that is obtained from* w *by* removing a hook *at the end of* w*. If* w *is a path in* Q*, then we put* ${}_{h^{-1}}w = 0$.

Example 2.25. For the algebra defined in Example 2.6 and the string

we have



Theorem 2.26 ([3])**.** *Let* $A = kQ/I$ *be a string algebra, and let* w *be a string. Then* $\tau(M(w)) = M(w')$*, where* w' *is obtained by following the table below.*

	w *does not start in a deep*	w *starts in a deep*
w *does not end in a deep*	${}_cw_c$	$({}_cw)_{h^{-1}}$
w *ends in a deep*	${}_{h^{-1}}(w_c)$	${}_{h^{-1}}w_{h^{-1}}$

Dually, $\tau^{-1}(M(w)) = M(w'')$*, where* w'' *is given in the table below.*

	w *does not start on a peak*	w *starts on a peak*
w *does not end on a peak*	${}_hw_h$	$({}_hw)_{c^{-1}}$
w *ends on a peak*	${}_{c^{-1}}(w_h)$	${}_{c^{-1}}w_{c^{-1}}$

Finally, applying τ *or* τ^{-1} *to a band module gives the same band module.*

2.4. **Morphisms between string modules.**

Definition 2.27. Let $A = kQ/I$ be a string algebra, and let $w = \beta_1^{\varepsilon_1} \cdots \beta_n^{\varepsilon_n}$ be a string for A.

(1) A *substring of* w is a string of the form $\beta_i^{\varepsilon_i} \cdots \beta_j^{\varepsilon_j}$, with $1 \leq i \leq j \leq n$. A string e_i of length zero is also considered to be a substring of w is one of the β_i starts or ends in i.

(2) A substring $w' = \beta_i^{\varepsilon_i} \cdots \beta_j^{\varepsilon_j}$ is said to be *on top of* w if
   - $i = 1$ or $\varepsilon_{i-1} = 1$, and
   - $j = n$ or $\varepsilon_{j+1} = -1$.

   If w' is of length zero, we say that it is on top of w if it corresponds to a local maximum in the picture of w.

(3) A substring $w' = \beta_i^{\varepsilon_i} \cdots \beta_j^{\varepsilon_j}$ is said to be *at the bottom of* w if
   - $i = 1$ or $\varepsilon_{i-1} = -1$, and
   - $j = n$ or $\varepsilon_{j+1} = 1$.

   If w' is of length zero, we say that it is on top of w if it corresponds to a local minimum in the picture of w.

A substring on top

A substring at the bottom

Example 2.28.

Proposition 2.29. *Let w and w' be two strings for a string algebra A. Then the dimension of the k-vector space* $\mathrm{Hom}_A(M(w), M(w'))$ *is equal to*

$$\sum_{\xi \text{ substring of } w} (\#\text{of substrings equal to } \xi \text{ or } \xi^{-1} \text{ on top of } w) \cdot (\#\text{of substrings equal to } \xi \text{ or } \xi^{-1} \text{ at the bottom of } w)$$

Proof. Reference? □

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E-mail address: `pierre-guy.plamondon@math.u-psud.fr`