Séminaire Lotharingien de Combinatoire XX (2020)
Article #YY, 12 pp.

Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

On type cones of $\boldsymbol{g}$-vector fans

Arnau Padrol*[1], Yann Palu‡[2],
Vincent Pilaud§[3] and Pierre-Guy Plamondon¶[4],

[1] *Institut de Mathématiques de Jussieu - Paris Rive Gauche, Sorbonne Université, Paris, France*
[2] *LAMFA, Université Picardie Jules Verne, Amiens, France*
[3] *CNRS & LIX, École Polytechnique, Palaiseau, France*
[4] *Laboratoire de Mathématiques d'Orsay, CNRS, Université Paris-Saclay, France*

Abstract. We study the type cone (*i.e.* the space of all polytopal realizations) of $\boldsymbol{g}$-vector fans of finite type cluster-like complexes (finite type cluster complexes, non-kissing complexes of gentle algebra, and graphical nested complexes). We show that this cone is often simplicial, which explains an elegant "kinematic" construction of the associahedron as a section of a high dimensional positive orthant by certain affine subspaces parametrized by a low dimensional positive orthant.

Résumé. Nous étudions le type cone (*i.e.* l'espace de toutes les réalisations polytopales) des éventails de $\boldsymbol{g}$-vecteurs de complexes de type amassés (complexes amassés de type fini, complexes platoniques d'algèbres aimables, et complexes emboités graphiques). Nous montrons que ce cone est souvent simplicial, ce qui explique une élégante construction "cinématique" de l'associaèdre comme section d'un orthant positif de haute dimension par certains sous-espaces affines paramètrés par un orthant positif de basse dimension.

Keywords: Fans · cluster algebras · non-kissing complexes · graph associahedra

1 Introduction

This paper focuses on a surprising construction of the associahedron that recently appeared in mathematical physics. Motivated by the prediction of the behavior of scattering particles, N. Arkani-Hamed, Y. Bai, S. He, and G. Yan recently described in [1, Sect. 3.2] the kinematic associahedron. It is a class of polytopal realizations of the classical associahedron obtained as sections of a high-dimensional positive orthant with well-chosen affine subspaces. This construction provides a large degree of freedom in the choice of the parameters defining these affine subspaces, and actually produces all

*arnau.padrol@imj-prg.fr. Supp. by ANR grant CAPPS (17 CE40 0018).
‡yann.palu@u-picardie.fr. Supp. by ANR grant SC3A (15 CE40 0004 01).
§vincent.pilaud@lix.polytechnique.fr. Supp. by ANR SC3A 15 CE40 0004 01 and CAPPS 17 CE40 0018.
¶pierre-guy.plamondon@math.u-psud.fr. Supp. by ANR grant SC3A (15 CE40 0004 01).

polytopes whose normal fan is affinely equivalent to that of J.-L. Loday's associahedron [12] (see Section 3.1). These realizations were then extended by V. Bazier-Matte, G. Douville, K. Mousavand, H. Thomas and E. Yıldırım [2] in the context of finite type cluster algebras (see Section 3.2) using tools from quiver representation theory.

We revisit, extend and explore further this construction using a reversed approach. Given a complete simplicial fan $\mathcal{F}$, we consider the space $\mathbb{TC}(\mathcal{F})$ of all its polytopal realizations. This space was called type cone in [13] and deformation cone in [16], who studied the case when $\mathcal{F}$ is the braid arrangement leading to the rich theory of generalized permutahedra. The type cone is known to be a polyhedral cone defined by a collection of inequalities corresponding to the linear dependences among the rays of $\mathcal{F}$ contained in pairs of adjacent maximal cones of $\mathcal{F}$ (see Definition 2). Our approach is based on an elementary but powerful observation: for any fan $\mathcal{F}$, all polytopal realizations of $\mathcal{F}$ can be described as sections of a high dimensional positive orthant with a collection of affine subspaces parametrized by the type cone $\mathbb{TC}(\mathcal{F})$ (see Proposition 6); if moreover the type cone $\mathbb{TC}(\mathcal{F})$ is a simplicial cone, it leads to a simple parametrization of all polytopal realizations of $\mathcal{F}$ by a positive orthant corresponding to the facets of the type cone $\mathbb{TC}(\mathcal{F})$ (see Corollary 7). To prove that the type cone $\mathbb{TC}(\mathcal{F})$ is simplicial, we just need to identify which pairs of adjacent maximal cones of $\mathcal{F}$ correspond to the facets of $\mathbb{TC}(\mathcal{F})$ and to show that the corresponding linear dependences among their rays positively span the linear dependence among the rays of any pair of adjacent maximal cones of $\mathcal{F}$. When applied to cluster algebras (see Section 3.2), this yields all polytopal realizations of the $\boldsymbol{g}$-vector fans, revisiting and extending results of [2].

This new perspective has several advantages, as our proof uniformly applies to *any initial seed* (acyclic or not), *any finite type cluster algebra* (simply-laced or not), and *any positive real-valued parameters* (rational or not). In contrast, note that the result of [2] is first proved for acyclic initial seeds in simply-laced cluster algebras for integer parameters, and then extended by a technical folding argument to all finite types and by an approximation argument to arbitrary real-valued parameters. These advantages of the type cone approach all follow from one essential feature: it enables to completely separate the algebraic aspects from the geometric aspects of the problem.

Besides revisiting the construction of [1, 2] and extending it to any initial seed, our type cone approach is also successful when applied to the $\boldsymbol{g}$-vector fans of other families of generalizations of the associahedron. In the present paper, we explore specifically the non-kissing complexes of gentle algebras introduced in [15] (see Section 3.3) and the graphical nested complexes studied in [3, 5, 16, 17] (see Section 3.4).

Many details and all proofs are omitted in this extended abstract for space reason, but a complete treatment can be found in [14]. In particular, the quiver representation theory is an iceberg in this abstract: while useful for intuition and essential for certain proofs (*e.g.* Theorem 15), it is mostly hidden and only appears in the Auslander-Reiten quivers used to represent the inequalities of the type cones (in Figures 3, 4 and 6).

2 Type cone of a fan

Fix an essential complete simplicial fan[1] $\mathcal{F}$ in $\mathbb{R}^n$. Let $\boldsymbol{G}$ be the $N \times n$-matrix whose rows are (representative vectors of) the rays of $\mathcal{F}$. For any height vector $\boldsymbol{h} \in \mathbb{R}^N$, we define the polytope[2] $P_{\boldsymbol{h}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h}\}$. Unfortunately, the normal fan[3] of this polytope $P_{\boldsymbol{h}}$ does not always coincide with the fan $\mathcal{F}$ we started from. The following statement gives a simple characterization of the height vectors h for which it is the case. It is a reformulation of regularity of triangulations of vector configurations, introduced in the theory of secondary polytopes [9]. We present here a convenient formulation from [4].

Proposition 1. *Let $\mathcal{F}$ be an essential complete simplicial fan in $\mathbb{R}^n$. Then the following are equivalent for any height vector $\boldsymbol{h} \in \mathbb{R}^N$:*

1. *The fan $\mathcal{F}$ is the normal fan of the polytope $P_{\boldsymbol{h}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h}\}$.*
2. *For any two adjacent maximal cones $\mathbb{R}_{\geq 0}\boldsymbol{R}$ and $\mathbb{R}_{\geq 0}\boldsymbol{R}'$ of $\mathcal{F}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$, we have $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, h_{\boldsymbol{s}} > 0$, where $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{s} = 0$ is the unique linear dependence among the rays of $\boldsymbol{R} \cup \boldsymbol{R}'$ such that $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}) + \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}') = 2$.*

Definition 2 ([13])**.** *The type cone of $\mathcal{F}$ is the cone $\mathbb{TC}(\mathcal{F})$ of all polytopal realizations of $\mathcal{F}$:*

$$\begin{aligned}\mathbb{TC}(\mathcal{F}) &:= \left\{ \boldsymbol{h} \in \mathbb{R}^N \;\middle|\; \mathcal{F} \text{ is the normal fan of } P_{\boldsymbol{h}} \right\} \\ &= \left\{ \boldsymbol{h} \in \mathbb{R}^N \;\middle|\; \sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} > 0 \; \begin{array}{l} \text{for any adjacent maximal} \\ \text{cones } \mathbb{R}_{\geq 0}\boldsymbol{R} \text{ and } \mathbb{R}_{\geq 0}\boldsymbol{R}' \text{ of } \mathcal{F} \end{array} \right\}.\end{aligned}$$

Note that the type cone is an open cone and contains a lineality subspace of dimension n (it is invariant by translation in $\boldsymbol{G}\mathbb{R}^n$). Therefore, we say that the type cone is simplicial when it has precisely $N - n$ facets.

Definition 3. *An extremal adjacent pair of $\mathcal{F}$ is a pair of adjacent maximal cones $\mathbb{R}_{\geq 0}\boldsymbol{R}, \mathbb{R}_{\geq 0}\boldsymbol{R}'$ of $\mathcal{F}$ such that the corresponding inequality $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} > 0$ in the definition of the type cone $\mathbb{TC}(\mathcal{F})$ actually defines a facet of $\mathbb{TC}(\mathcal{F})$. In other words, extremal adjacent pairs define the extremal rays of the polar of the closed type cone $\overline{\mathbb{TC}}(\mathcal{F})$.*

---

[1] A *polyhedral cone* is the positive span of finitely many vectors or equivalently, the intersection of finitely many closed linear half-spaces. The *faces* of a cone are its intersections with its supporting hyperplanes. A *fan* $\mathcal{F}$ is a set of polyhedral cones such that any face of a cone of $\mathcal{F}$ belongs to $\mathcal{F}$, and any two cones of $\mathcal{F}$ intersect along a face of both. A fan is *essential* if the intersection of its cones is the origin, *complete* if the union of its cones covers $\mathbb{R}^n$, and *simplicial* if all its cones are generated by dimension many rays.

[2] A *polytope* is the convex hull of finitely many points or equivalently, a bounded intersection of finitely many closed affine half-spaces.

[3] The *normal cone* of a face F of a polytope P is the cone generated by the normal vectors of the facets of P containing F. The *normal fan* of P is the set of normal cones of all its faces.

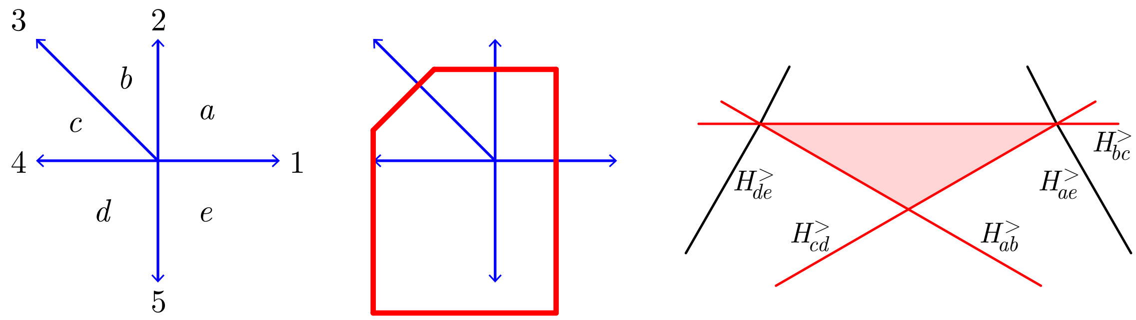


Figure 1: A 2-dimensional fan $\mathcal{F}$ with five rays $1, \dots, 5$ and five maximal cones $a, \dots, e$ (left), its polytopal realization corresponding to the height vector $(1/2, 3/4, 1, 1, 5/4)$ (middle), and a 2-dimensional slice of the type cone $\mathbb{TC}(\mathcal{F})$ (right).

Example 4. *Consider the* 2*-dimensional fan* $\mathcal{F}$ *depicted in Figure 1 (left). It has five rays labeled* $1, \dots, 5$ *and five maximal cones labeled* $a, \dots, e$*. Its type cone* $\mathbb{TC}(\mathcal{F})$ *lies in* $\mathbb{R}^5$*, but has a* 2*-dimensional lineality space, and is defined by the following five inequalities:*

$$H_{ab}^{>}: \ \boldsymbol{h}_1 + \boldsymbol{h}_3 - \boldsymbol{h}_2 > 0 \qquad H_{bc}^{>}: \ \boldsymbol{h}_2 + \boldsymbol{h}_4 - \boldsymbol{h}_3 > 0 \qquad H_{cd}^{>}: \ \boldsymbol{h}_3 + \boldsymbol{h}_5 - \boldsymbol{h}_4 > 0$$
$$H_{de}^{>}: \ \boldsymbol{h}_1 + \boldsymbol{h}_4 > 0 \qquad H_{ae}^{>}: \ \boldsymbol{h}_2 + \boldsymbol{h}_5 > 0,$$

where $H_{xy}^{>}$ *denotes the halfspace defined by the inequality corresponding to the two adjacent maximal cones* x *and* y*. For example, the height vector* $(1/2, 3/4, 5/4, 1, 5/4)$ *belongs to* $\mathbb{TC}(\mathcal{F})$*, and the corresponding polytope is represented in Figure 1 (middle). To represent* $\mathbb{TC}(\mathcal{F})$*, we slice it with a transversal* 2*-dimensional affine space, and obtain the red triangle in Figure 1 (right).*

Remark 5. *All fans considered in Section 3 have the unique exchange relation property: the linear dependence* $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{s} = 0$ *only depends on the exchanged rays* $\boldsymbol{r}$ *and* $\boldsymbol{r}'$*, not on the maximal adjacent cones* $\boldsymbol{R}$ *and* $\boldsymbol{R}'$*. We then write* $\alpha_{\boldsymbol{r},\boldsymbol{r}'}(\boldsymbol{s})$ *instead of* $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})$ *and we call extremal exchangeable pairs of* $\mathcal{F}$ *the pairs of exchangeable rays* $\boldsymbol{r}, \boldsymbol{r}'$ *of* $\mathcal{F}$ *such that the corresponding inequality* $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{r},\boldsymbol{r}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} > 0$ *defines a facet of* $\mathbb{TC}(\mathcal{F})$*.*

To conclude this section, we consider alternative polytopal realizations of the fan $\mathcal{F}$ and discuss their behavior in the situation when the type cone $\mathbb{TC}(\mathcal{F})$ is simplicial. Fix a $(N-n) \times N$-matrix $\boldsymbol{K}$ that spans the left kernel of the $N \times n$-matrix $\boldsymbol{G}$ (*i.e.* $\boldsymbol{KG} = 0$).

Proposition 6. *The map* $\boldsymbol{x} \in \mathbb{R}^n \mapsto \boldsymbol{h} - G\boldsymbol{x} \in \mathbb{R}^N$ *is an affine transformation of the polytope* $P_{\boldsymbol{h}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h}\}$ *to the polytope* $Q_{\boldsymbol{h}} := \{\boldsymbol{z} \in \mathbb{R}^N \mid \boldsymbol{K}\boldsymbol{z} = \boldsymbol{K}\boldsymbol{h} \text{ and } \boldsymbol{z} \ge 0\}$*.*

Corollary 7. *Assume that the type cone* $\mathbb{TC}(\mathcal{F})$ *is simplicial and consider inner normal vectors to its* $N-n$ *facets. Since all these vectors belong to the left kernel of* $\boldsymbol{G}$ *by Proposition 1, we can assume that they are the rows of the* $(N-n) \times N$*-matrix* $\boldsymbol{K}$*. Then, for any positive vector* $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{N-n}$*, the polytope* $R_{\boldsymbol{\ell}} := \{\boldsymbol{z} \in \mathbb{R}^N \mid K\boldsymbol{z} = \boldsymbol{\ell} \text{ and } \boldsymbol{z} \ge 0\}$ *is a realization of the fan* $\mathcal{F}$*. Moreover, the polytopes* $R_{\boldsymbol{\ell}}$ *for* $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{N-n}$ *describe all polytopal realizations of* $\mathcal{F}$*.*

3 Applications to g-vector fans

We now study the type cones of the normal fans of three families generalizing the associahedron constructed in [12]: the generalized associahedra of finite type cluster algebras [6, 7, 8, 10, 11], the gentle associahedra [15], and the graph associahedra [3, 5, 16, 17].

3.1 Classical associahedra

An n-dimensional *associahedron* is a polytope whose face lattice is the reverse inclusion lattice of dissections (*i.e.* sets of pairwise non-crossing diagonals) of a convex $(n+3)$-gon. In particular, its vertices correspond to triangulations of the $(n+3)$-gon and its facets correspond to internal diagonals of the $(n+3)$-gon. Let $X(n) := \{(a,b) \mid 0 \le a < b \le n+2\}$ denote all diagonals of the $(n+3)$-gon and $Y(n) := \{(a,b) \mid 0 \le a < b-2 \le n\} \subset X(n)$. The *$g$-vector* of a diagonal $(a,b) \in X(n)$ is $\boldsymbol{g}(a,b) := \sum_{a<\ell<b} \boldsymbol{e}_\ell - \frac{b-a-1}{n+1} \sum_{1 \le \ell \le n+1} \boldsymbol{e}_\ell$. We set $\boldsymbol{g}(D) := \{\boldsymbol{g}(a,b) \mid (a,b) \in D\}$ for a dissection D. Recall that:

- the set of cones $\mathcal{F}(n) := \{\mathbb{R}_{\ge 0}\, \boldsymbol{g}(D) \mid D \text{ dissection of the } (n+3)\text{-gon}\}$ forms a complete simplicial fan. See Figure 2 (left and middle).
- the fan $\mathcal{F}(n)$ is the normal fan of the associahedron $\mathsf{Asso}(n)$ constructed *e.g.* in [12].
- for any two adjacent triangulations T and T' with $T \smallsetminus \{(a,b)\} = T' \smallsetminus \{(a',b')\}$ such that $0 \le a < a' < b < b' \le n+2$, the linear dependence among the $\boldsymbol{g}$-vectors of $T \cup T'$ is given by $\boldsymbol{g}(a,b) + \boldsymbol{g}(a',b') = \boldsymbol{g}(a,b') + \boldsymbol{g}(a',b)$.

This provides a redundant description of the type cone of the fan $\mathcal{F}(n)$.

Corollary 8. *For any $n \in \mathbb{N}$, the type cone of the fan $\mathcal{F}(n)$ is given by*

$$\mathbb{TC}\big(\mathcal{F}(n)\big) = \left\{ \boldsymbol{h} \in \mathbb{R}^{X(n)} \;\middle|\; \begin{array}{l} \boldsymbol{h}_{(0,n+2)} = 0, \quad \textit{and} \quad \boldsymbol{h}_{(a,a+1)} = 0 \textit{ for all } 0 \le a \le n+1 \\ \boldsymbol{h}_{(a,b)} + \boldsymbol{h}_{(a',b')} > \boldsymbol{h}_{(a,b')} + \boldsymbol{h}_{(a',b)} \textit{ for all } 0 \le a < a' < b < b' \le n+2 \end{array} \right\}.$$

The next statement, illustrated in Figure 3, gives the facets of the type cone of $\mathcal{F}(n)$.

Proposition 9. *Two internal diagonals (a,b) and (a',b') of the $(n+3)$-gon form an extremal exchangeable pair for the fan $\mathcal{F}(n)$ if and only if $a = a'+1$ and $b = b'+1$, or the opposite.*

Corollary 10. *The type cone $\mathbb{TC}\big(\mathcal{F}(n)\big)$ is simplicial.*

Combining Corollaries 7 and 10 and Proposition 9, we derive the following description of all polytopal realizations of $\mathcal{F}(n)$, recovering all associahedra of [1, Sect. 3.2].

Corollary 11 ([1, Sect. 3.2])**.** *For any $n \in \mathbb{N}$ and any $\boldsymbol{\ell} \in \mathbb{R}^{Y(n)}_{>0}$, the polytope*

$$R_{\boldsymbol{\ell}}(n) := \left\{ \boldsymbol{z} \in \mathbb{R}^{X(n)} \;\middle|\; \begin{array}{l} \boldsymbol{z} \ge 0, \quad \boldsymbol{z}_{(0,n+2)} = 0 \quad \textit{and} \quad \boldsymbol{z}_{(a,a+1)} = 0 \textit{ for all } 0 \le a \le n+1 \\ \boldsymbol{z}_{(a,b-1)} + \boldsymbol{z}_{(a+1,b)} - \boldsymbol{z}_{(a,b)} - \boldsymbol{z}_{(a+1,b-1)} = \boldsymbol{\ell}_{(a,b)} \textit{ for all } (a,b) \in Y(n) \end{array} \right\}$$

is an n-dimensional associahedron, whose normal fan is $\mathcal{F}(n)$. Moreover, the polytopes $R_{\boldsymbol{\ell}}(n)$ for $\boldsymbol{\ell} \in \mathbb{R}^{Y(n)}_{>0}$ describe all polytopal realizations of the fan $\mathcal{F}(n)$.

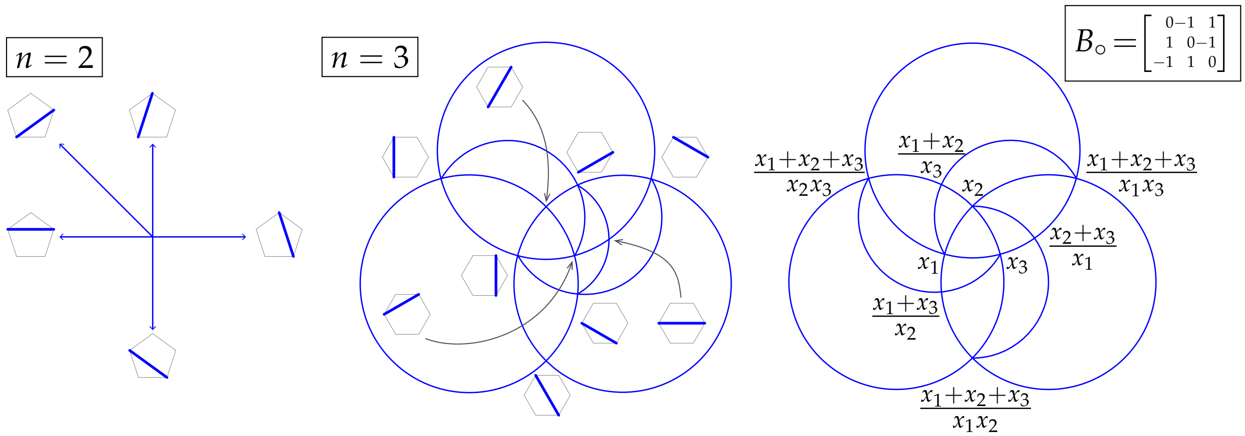


Figure 2: The fans $\mathcal{F}(2)$, $\mathcal{F}(3)$ and $\mathcal{F}(B_\circ)$. The two right 3-dimensional fans are intersected with the sphere and stereographically projected from the direction $(-1, -1, -1)$.

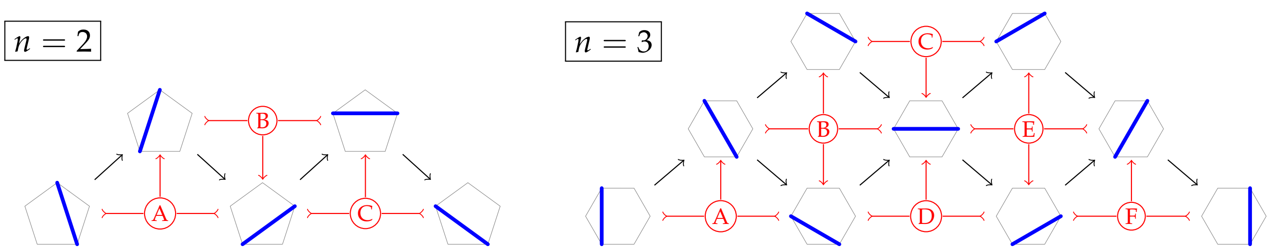


Figure 3: Facet-defining inequalities of the type cone $\mathbb{TC}(\mathcal{F}(n))$. See Proposition 9. Each circled red letter gives an inequality with left (resp. right) hand side given by the incoming (resp. outgoing) arrows. For instance, the inequality A is $\boldsymbol{h} + \boldsymbol{h} > \boldsymbol{h}$.

3.2 Cluster fan and generalized associahedra

We now consider finite type cluster algebras, defined by S. Fomin and A. Zelevinsky in [6, 7]. We skip the technical definitions and refer to the original papers for details. We fix an initial exchange matrix $B_\circ$ (acyclic or not) of finite type and denote by $\mathcal{A}(B_\circ)$ the corresponding cluster algebra with principal coefficients, and by $\mathcal{V}(B_\circ)$ its cluster variables. We denote by $\boldsymbol{g}(B_\circ, x)$ the *$\boldsymbol{g}$-vector* of a cluster variable $x \in \mathcal{V}(B_\circ)$ as defined in [8], and we set $\boldsymbol{g}(B_\circ, X) := \{\boldsymbol{g}(B_\circ, x) \mid x \in X\}$ for a cluster X of $\mathcal{A}(B_\circ)$. Recall that:

1. the set of cones $\{\mathbb{R}_{\ge 0}\, \boldsymbol{g}(B_\circ, X) \mid X \text{ cluster of } \mathcal{A}(B_\circ)\}$, together with all their faces, forms a complete simplicial fan $\mathcal{F}(B_\circ)$, called the *cluster fan* of $B_\circ$. See Figure 2.
2. the cluster fan $\mathcal{F}(B_\circ)$ is the normal fan of the *generalized associahedron* $\mathsf{Asso}(B_\circ)$, constructed for bipartite (resp. acyclic, resp. arbitrary) initial exchange matrices in [4] (resp. [10], resp. [11]).
3. for any two adjacent seeds (B, X) and (B', X') in $\mathcal{A}(B_\circ)$ with $X \smallsetminus \{x\} = X' \smallsetminus \{x'\}$, the $\boldsymbol{g}$-vectors of $X \cup X'$ with respect to $B_\circ$ satisfy one of the two linear dependences

$$\boldsymbol{g}(B_\circ, x) + \boldsymbol{g}(B_\circ, x') = \sum_{\substack{y \in X \cap X' \\ b_{xy} < 0}} -b_{xy}\, \boldsymbol{g}(B_\circ, y) \quad \text{or} \quad \boldsymbol{g}(B_\circ, x) + \boldsymbol{g}(B_\circ, x') = \sum_{\substack{y \in X \cap X' \\ b_{xy} > 0}} b_{xy}\, \boldsymbol{g}(B_\circ, y).$$

This provides a redundant description of the type cone of the cluster fan $\mathcal{F}(B_\circ)$. Denoting by $\mathbf{n}(B_\circ, x, x')$ the vector whose coefficients are given by the appropriate linear dependence above, we obtain the following statement.

Corollary 12. *For any finite type exchange matrix $B_\circ$, the type cone of $\mathcal{F}(B_\circ)$ is given by*

$$\mathbb{TC}\big(\mathcal{F}(B_\circ)\big) = \big\{\mathbf{h} \in \mathbb{R}^{\mathcal{V}(B_\circ)} \;\big|\; \langle\, \mathbf{n}(B_\circ, x, x') \mid \mathbf{h}\,\rangle > 0 \text{ for any exchangeable variables } x, x'\big\}.$$

To describe the facets of this type cone, we need the following special mutations.

Definition 13. *The mutation of a seed (B, X) in the direction of a cluster variable $x \in X$ is a <u>mesh mutation</u> that <u>starts</u> (resp. <u>ends</u>) at x if the entries b_{xy} for $y \in X$ are all non-negative (resp. all non-positive). A mesh mutation is <u>initial</u> if it ends at a cluster variable of an initial seed. We denote by $\mathcal{M}(B_\circ)$ the set of all pairs $\{x, x'\}$ where x and x' are two cluster variables of $\mathcal{A}(B_\circ)$ which are exchangeable via a non-initial mesh mutation.*

Lemma 14. *Consider two adjacent seeds (B, X) and (B', X') with $X \smallsetminus \{x\} = X' \smallsetminus \{x'\}$ connected by a non-initial mesh mutation. Then, the $\mathbf{g}$-vectors of $X \cup X'$ with respect to $B_\circ$ satisfy the linear dependence $\mathbf{g}(B_\circ, x) + \mathbf{g}(B_\circ, x') = \sum_{y \in X \cap X'} |b_{xy}|\, \mathbf{g}(B_\circ, y)$.*

For $\{x, x'\} \in \mathcal{M}(B_\circ)$ and $y \in \mathcal{V}(B_\circ)$, we denote by $\alpha_{x,x'}(y)$ the coefficient of $\mathbf{g}(B_\circ, y)$ in the linear dependence of Lemma 14. The next statement is proved in [14, Sect. 3] using techniques from quiver representation theory.

Theorem 15. *For any finite type exchange matrix $B_\circ$ (acyclic or not, simply-laced or not), the linear dependence among the $\mathbf{g}$-vectors of any mutation decomposes into positive combinations of linear dependences among $\mathbf{g}$-vectors of non-initial mesh mutations.*

Using Theorem 15 as a blackbox, we describe the facets of the type cone. See Figure 4.

Theorem 16. *For any finite type exchange matrix $B_\circ$, the type cone $\mathbb{TC}\big(\mathcal{F}(B_\circ)\big)$ is simplicial and the non-initial mesh mutations are the extremal exchangeable pairs of the cluster fan $\mathcal{F}(B_\circ)$.*

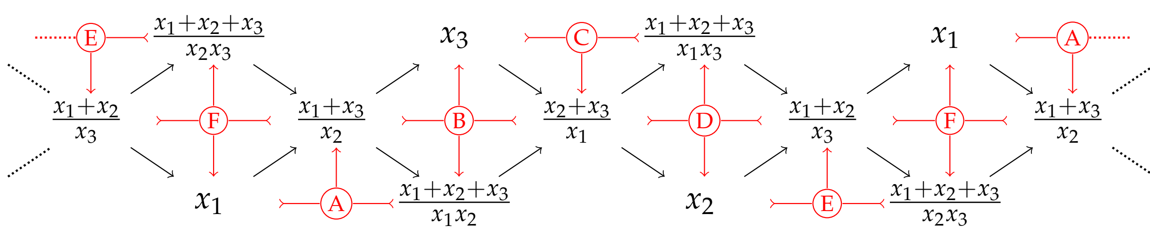


Figure 4: The facet-defining inequalities of the type cone $\mathbb{TC}\big(\mathcal{F}(B_\circ)\big)$ for the cluster fan of Figure 2 (right). See Theorem 16. The conventions are similar to Figure 3.

Combining Corollary 7 and Theorem 16, we derive the following description of all polytopal realizations of the cluster fan $\mathcal{F}(B_\circ)$. This result was stated in [2] in the special situation of acyclic seeds in simply-laced types, with a very different proof.

Theorem 17. *For any finite type exchange matrix $B_\circ$, and for any $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{\mathcal{M}(B_\circ)}$, the polytope*

$$\boldsymbol{R}_{\boldsymbol{\ell}}(B_\circ) := \Big\{ \boldsymbol{z} \in \mathbb{R}^{\mathcal{V}(B_\circ)} \;\Big|\; \boldsymbol{z} \ge 0 \text{ and } \boldsymbol{z}_x + \boldsymbol{z}_{x'} - \sum_{y \in \mathcal{V}(B_\circ)} \alpha_{x,x'}(y)\, \boldsymbol{z}_y = \boldsymbol{\ell}_{\{x,x'\}} \text{ for } \{x,x'\} \in \mathcal{M}(B_\circ) \Big\}$$

is a generalized associahedron, whose normal fan is the cluster fan $\mathcal{F}(B_\circ)$. Moreover, the polytopes $\boldsymbol{R}_{\boldsymbol{\ell}}(B_\circ)$ for $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{\mathcal{M}(B_\circ)}$ describe all polytopal realizations of $\mathcal{F}(B_\circ)$.

3.3 Non-kissing fans and gentle associahedra

We now consider non-kissing complexes of gentle algebras studied in [15]. We briefly recall all definitions needed here as they are purely combinatorial.

Fix a gentle bound quiver $\bar{Q} = (Q, I)$, *i.e.* a finite quiver Q (with vertices Q_0, arrows Q_1, and source and target maps s and t) and an ideal I of the path algebra kQ such that

(i) each vertex $a \in Q_0$ has at most two incoming and two outgoing arrows,
(ii) the ideal I is generated by paths of length exactly two,
(iii) for any $\beta \in Q_1$, there is at most one $\alpha \in Q_1$ such that $t(\alpha) = s(\beta)$ and $\alpha\beta \notin I$ (resp. $\alpha\beta \in I$) and at most one $\gamma \in Q_1$ such that $t(\beta) = s(\gamma)$ and $\beta\gamma \notin I$ (resp. $\beta\gamma \in I$).

The blossoming quiver $\bar{Q}^{❀}$ of a gentle quiver $\bar{Q}$ is the gentle quiver obtained by completing all vertices of $\bar{Q}$ with additional incoming or outgoing blossoms such that all vertices of $\bar{Q}$ become 4-valent. Blossom vertices appear in white in all pictures.

A string in $\bar{Q}$ is a word $\rho = \alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\cdots\alpha_\ell^{\varepsilon_\ell}$ with $\alpha_i \in Q_1$ and $\varepsilon_i \in \{-1, 1\}$ such that $t(\alpha_i^{\varepsilon_i}) = s(\alpha_{i+1}^{\varepsilon_{i+1}})$ and containing no factor π or π^{-1} for $\pi \in I \cup \{\alpha\alpha^{-1} \mid \alpha \in Q_1\}$. We implicitly identify the two inverse strings ρ and ρ^{-1}. A walk of $\bar{Q}$ is a maximal string of its blossoming quiver $\bar{Q}^{❀}$ (meaning that each endpoint is a blossom).

A substring of a walk $\omega = \alpha_1^{\varepsilon_1}\cdots\alpha_\ell^{\varepsilon_\ell}$ of $\bar{Q}$ is a string $\sigma = \alpha_{i+1}^{\varepsilon_{i+1}}\cdots\alpha_{j-1}^{\varepsilon_{j-1}}$ of $\bar{Q}$ for some indices $1 \le i < j \le \ell$. The substring $\sigma = \alpha_{i+1}^{\varepsilon_{i+1}}\cdots\alpha_{j-1}^{\varepsilon_{j-1}}$ is at the bottom (resp. on top) of the walk $\omega = \alpha_1^{\varepsilon_1}\cdots\alpha_\ell^{\varepsilon_\ell}$ if $\varepsilon_i = 1$ and $\varepsilon_j = -1$ (resp. if $\varepsilon_i = -1$ and $\varepsilon_j = 1$). In other words the two arrows of ω incident to the endpoints of σ point towards σ (resp. outwards from σ). We denote by $\Sigma_{\mathrm{bot}}(\omega)$ and $\Sigma_{\mathrm{top}}(\omega)$ the sets of bottom and top substrings of ω.

For a walk ω, we denote by $\mathsf{peaks}(\omega)$ (resp. $\mathsf{deeps}(\omega)$) the multiset of peaks (resp. deeps) of ω, *i.e.* vertices which are substrings on the top (resp. at the bottom) of ω. A walk is straight if it has no peak nor deep.

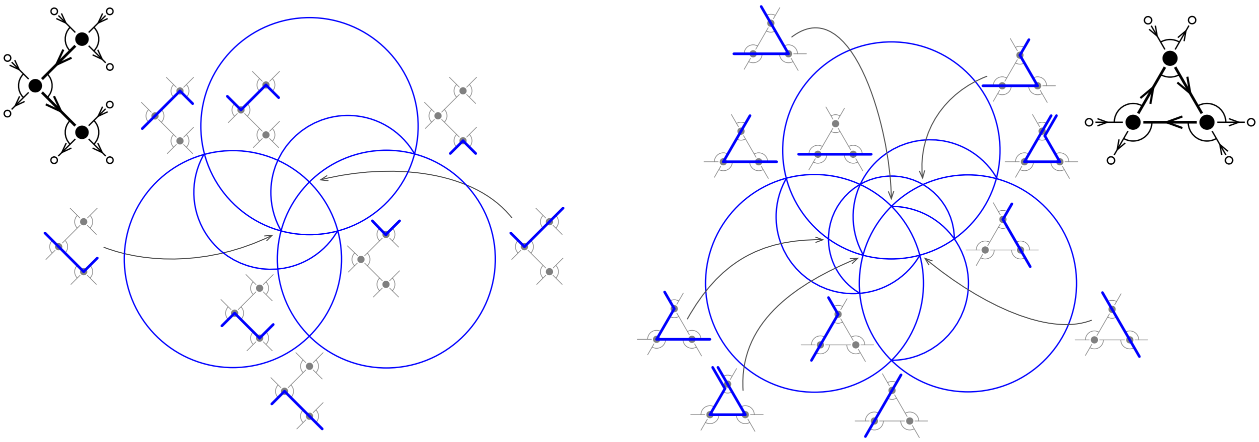
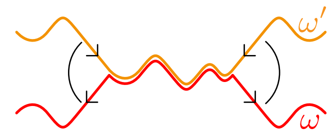


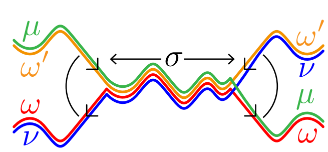
Figure 5: Two non-kissing fans. As the fans are 3-dimensional, we intersect them with the sphere and stereographically project them from the direction $(-1,-1,-1)$.



Two walks ω, ω' are <u>*kissing*</u> if $\Sigma_{\mathrm{top}}(\omega) \cap \Sigma_{\mathrm{bot}}(\omega') \neq \varnothing$ (as illustrated on the right) or the opposite. A walk is <u>*proper*</u> if it is not straight nor self-kissing. The <u>*non-kissing complex*</u> of $\bar{Q}$ is the simplicial complex $\mathcal{NK}(\bar{Q})$ whose faces are the sets of pairwise non-kissing proper walks of $\bar{Q}$.

For a multiset $V = \{\{v_1, \dots, v_k\}\}$ of Q_0, we denote by $\boldsymbol{m}_V := \sum_{i\in[k]} \boldsymbol{e}_{v_i}$, where $(\boldsymbol{e}_v)_{v\in Q_0}$ is the canonical basis of $\mathbb{R}^{Q_0}$. The <u>*g-vector*</u> of a walk ω is $\boldsymbol{g}(\omega) := \boldsymbol{m}_{\mathsf{peaks}(\omega)} - \boldsymbol{m}_{\mathsf{deeps}(\omega)}$. We set $\boldsymbol{g}(F) := \{\boldsymbol{g}(\omega) \mid \omega \in F\}$ for a non-kissing face $F \in \mathcal{NK}(\bar{Q})$. Recall that:

1. the set of cones $\mathcal{F}(\bar{Q}) := \{\mathbb{R}_{\geq 0}\, \boldsymbol{g}(F) \mid F \text{ non-kissing facet of } \mathcal{NK}(\bar{Q})\}$ forms a complete simplicial fan, called the <u>*non-kissing fan*</u> of $\bar{Q}$. See Figure 5.
2. the non-kissing fan $\mathcal{F}(\bar{Q})$ is the normal fan of the <u>*gentle associahedron*</u> $\mathsf{Asso}(\bar{Q})$, constructed in [15].
3. for any two adjacent non-kissing facets F and F' of $\mathcal{NK}(\bar{Q})$ with $F \smallsetminus \{\omega\} = F' \smallsetminus \{\omega'\}$, the linear dependence among the g-vectors of $F \cup F'$ is given by $\boldsymbol{g}(\omega) + \boldsymbol{g}(\omega') = \boldsymbol{g}(\mu) + \boldsymbol{g}(\nu)$, where $\mu := \rho'\sigma\tau$ and $\nu := \rho\sigma\tau'$ if the walks $\omega = \rho\sigma\tau$ and $\omega' = \rho'\sigma'\tau'$ kiss along σ (as illustrated on the right).



This provides a redundant description of the type cone of the non-kissing fan $\mathcal{F}(\bar{Q})$. See Figure 6 for examples of facet descriptions of these type cones.

Corollary 18. *The type cone of the non-kissing fan $\mathcal{F}(\bar{Q})$ is given by*

$$\mathbb{TC}\big(\mathcal{F}(\bar{Q})\big) = \left\{ \boldsymbol{h} \in \mathbb{R}^{\mathcal{W}(\bar{Q})} \;\middle|\; \begin{array}{l} \boldsymbol{h}_\omega = 0 \text{ for any improper walk } \omega \\ \boldsymbol{h}_\omega + \boldsymbol{h}_{\omega'} > \boldsymbol{h}_\mu + \boldsymbol{h}_\nu \text{ for any exchangeable walks } \omega, \omega' \end{array} \right\},$$

where $\mu := \rho'\sigma\tau$ and $\nu := \rho\sigma\tau'$ if the walks $\omega = \rho\sigma\tau$ and $\omega' = \rho'\sigma'\tau'$ kiss along σ.

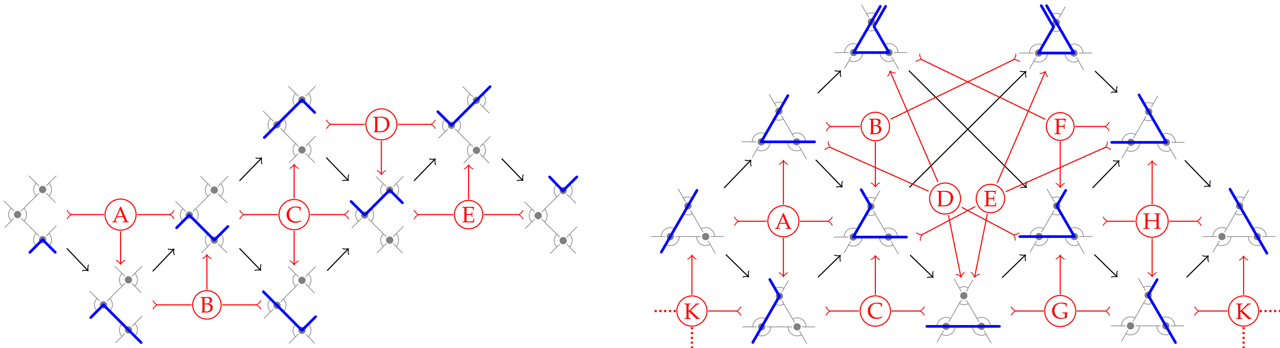


Figure 6: The facet-defining inequalities of the type cone $\mathbb{TC}(\mathcal{F}(\bar{Q}))$ for the non-kissing fan of Figure 5 (right). The conventions are similar to Figure 3.

As illustrated by the second non-kissing fan of Figure 6 (which lives in $\mathbb{R}^{11}$, has a 3-dimensional lineality space, and has 9 facets), the type cone $\mathbb{TC}(\mathcal{F}(\bar{Q}))$ is not always simplicial. However, it turns out to be for the following family of quivers.

Definition 19. *A gentle quiver $\bar{Q}$ is called:*

- *brick if any (non necessarily oriented) cycle of $\bar{Q}$ contains at least two relations in I,*
- *2-acyclic if it contains no cycle of length 2.*

For a string σ of $\bar{Q}$, we denote by $\sigma\nwarrow$ (resp. $\sigma\checkmark$) the unique string of the blossoming quiver $\bar{Q}^{❀}$ of the form $\sigma\nwarrow = \sigma\alpha_1^{-1}\alpha_2\ldots\alpha_\ell$ (resp. $\sigma\checkmark = \sigma\alpha_1\alpha_2^{-1}\ldots\alpha_\ell^{-1}$) with $\ell \geq 1$ and $\alpha_1,\ldots,\alpha_\ell \in Q_1$ and such that $t(\alpha_\ell)$ (resp. $s(\alpha_\ell)$) is a blossom of $\bar{Q}^{❀}$. The terminology usually says that $\sigma\nwarrow$ (resp. $\sigma\checkmark$) is obtained by adding a hook (resp. a cohook) to σ. We define similarly $\nearrow\sigma$ (resp. $\searrow\sigma$). The walk $\nearrow(\sigma\nwarrow) = (\nearrow\sigma)\nwarrow$ of $\bar{Q}$ is simply denoted by $\nearrow\sigma\nwarrow$, and we define similarly $\searrow\sigma\checkmark$, $\nearrow\sigma\checkmark$ and $\searrow\sigma\nwarrow$.

Proposition 20. *For any brick and 2-acyclic gentle quiver $\bar{Q}$ and any string $\sigma \in \mathcal{S}(\bar{Q})$, the walks $\searrow\sigma\checkmark$ and $\nearrow\sigma\nwarrow$ are exchangeable with distinguished substring σ.*

Proposition 21. *For any brick and 2-acyclic gentle quiver $\bar{Q}$, the extremal exchangeable pairs for the non-kissing fan of $\bar{Q}$ are precisely the pairs $\{\searrow\sigma\checkmark, \nearrow\sigma\nwarrow\}$ for all strings $\sigma \in \mathcal{S}(\bar{Q})$.*

Corollary 22. *For any brick and 2-acyclic gentle quiver $\bar{Q}$, the type cone $\mathbb{TC}(\mathcal{F}(\bar{Q}))$ of the non-kissing fan $\mathcal{F}(\bar{Q})$ is simplicial.*

Theorem 23. *For any brick and 2-acyclic gentle quiver $\bar{Q}$ and any $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{\mathcal{S}(\bar{Q})}$, the polytope*

$$R_{\boldsymbol{\ell}}(\bar{Q}) := \left\{ \boldsymbol{z} \in \mathbb{R}^{\mathcal{W}(\bar{Q})} \;\middle|\; \begin{array}{l} \boldsymbol{z} \geq 0 \qquad \text{and} \qquad \boldsymbol{z}_\omega = 0 \text{ for any improper walk } \omega \\ \boldsymbol{z}_{\searrow\sigma\checkmark} + \boldsymbol{z}_{\nearrow\sigma\nwarrow} - \boldsymbol{z}_{\nearrow\sigma\checkmark} - \boldsymbol{z}_{\searrow\sigma\nwarrow} = \boldsymbol{\ell}_\sigma \text{ for all } \sigma \in \mathcal{S}(\bar{Q}) \end{array} \right\}$$

is a realization of the non-kissing fan $\mathcal{F}(\bar{Q})$. Moreover, the polytopes $R_{\boldsymbol{\ell}}(\bar{Q})$ for $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{\mathcal{S}(\bar{Q})}$ describe all polytopal realizations of $\mathcal{F}(\bar{Q})$.

3.4 Nested fans and graph associahedra

We finally consider the graph associahedra studied in [3, 5, 16, 17]. Again, we briefly recall all definitions needed here as they are purely combinatorial.

Let G be a graph with vertex set V. A *tube* of G is a connected induced subgraph of G. Let $\mathcal{T}(G)$ denote the set of tubes of G. The inclusion maximal tubes of G are its connected components $\kappa(G)$. The tubes which are neither empty nor maximal are called *proper*. Two tubes t, t' of G are *compatible* if they are either nested (*i.e.* $t \subseteq t'$ or $t' \subseteq t$), or disjoint and non-adjacent (*i.e.* $t \cup t'$ is not a tube of G). A *tubing* on G is a set T of pairwise compatible proper tubes of G. The *nested complex* of G is the simplicial complex $\mathcal{N}(G)$ of all tubings on G.

The *$\boldsymbol{g}$-vector* of a tube t of G is the projection $\boldsymbol{g}(t)$ of the characteristic vector $\sum_{v \in t} \boldsymbol{e}_v$ of t orthogonally to $\sum_{v \in V} \boldsymbol{e}_v$. We set $\boldsymbol{g}(T) := \{\boldsymbol{g}(t) \mid t \in T\}$ for a tubing T. Recall that:

1. the set of cones $\mathcal{F}(G) := \{\mathbb{R}_{\ge 0}\, \boldsymbol{g}(T) \mid T \text{ tubing on } G\}$ forms a complete simplicial fan, called the *nested fan* of G.
2. the nested fan $\mathcal{F}(G)$ is the normal fan of the *graph associahedron* $\mathsf{Asso}(G)$, constructed in [3, 5, 16, 17]. For instance, the permutahedra, associahedra and cyclohedra are graph associahedra of complete graphs, paths, and cycles respectively.
3. for any maximal tubings T, T' on G with $T \smallsetminus \{t\} = T' \smallsetminus \{t'\}$, the linear dependence among the $\boldsymbol{g}$-vectors of $T \cup T'$ is given by $\boldsymbol{g}(t) + \boldsymbol{g}(t') = \boldsymbol{g}(t \cup t') + \sum_{s \in \kappa(t \cap t')} \boldsymbol{g}(s)$, where $\kappa(t \cap t')$ denote the set of connected components of $t \cap t'$.

This provides a redundant description of the type cone of the nested fan $\mathcal{F}(n)$.

Corollary 24. *For any graph G, the type cone of the nested fan $\mathcal{F}(G)$ is given by*

$$\mathbb{TC}\big(\mathcal{F}(G)\big) = \left\{ \boldsymbol{h} \in \mathbb{R}^{\mathcal{T}(G)} \;\middle|\; \begin{array}{l} \boldsymbol{h}_t = 0 \text{ for any improper tube } t \\ \boldsymbol{h}_t + \boldsymbol{h}_{t'} > \boldsymbol{h}_{t \cup t'} + \sum\limits_{s \in \kappa(t \cap t')} \boldsymbol{h}_s \text{ for any exchangeable tubes } t, t' \end{array} \right\}.$$

The next statement gives the facets of the type cone of the nested fan $\mathcal{F}(G)$.

Proposition 25. *Two tubes t and t' of G form an extremal exchangeable pair for the nested fan of G if and only if $t \smallsetminus \{v\} = t' \smallsetminus \{v'\}$ for some neighbor v of t' and some neighbor v' of t.*

Corollary 26. *For a graph G on V with tubes $\mathcal{T}(G)$ and a height vector $\boldsymbol{h} \in \mathbb{R}^{\mathcal{T}(G)}$, the nested fan $\mathcal{F}(G)$ is the normal fan of the polytope $\{\boldsymbol{x} \in \mathbb{R}^V \mid \langle \boldsymbol{g}(t) \mid \boldsymbol{x} \rangle \le \boldsymbol{h}_t \text{ for any tube } t \in \mathcal{T}(G)\}$ if and only if $\boldsymbol{h}_\varnothing = \boldsymbol{h}_G = 0$ and $\boldsymbol{h}_{s \smallsetminus \{v'\}} + \boldsymbol{h}_{s \smallsetminus \{v\}} > \boldsymbol{h}_s + \boldsymbol{h}_{s \smallsetminus \{v, v'\}}$ for any tube $s \in \mathcal{T}(G)$ and distinct non-disconnecting vertices v, v' of s.*

Corollary 27. *The nested fan $\mathcal{F}(G)$ has $\sum_{s \in \mathcal{T}(G)} \binom{\mathrm{nd}(s)}{2}$ extremal exchangeable pairs, where $\mathrm{nd}(s)$ is the number of non-disconnecting vertices of s.*

For instance, the number of extremal exchangeable pairs is $2^{n-2}\binom{n}{2}$ for the permutahedron, $\binom{n}{2}$ for the associahedron, and $3\binom{n}{2} - n$ for the cyclohedron. In fact, it turns out that the type cone $\mathbb{TC}\big(\mathcal{F}(G)\big)$ is simplicial if and only if G is a path.

References

[1] N. Arkani-Hamed, Y. Bai, S. He, and G. Yan. *Scattering forms and the positive geometry of kinematics, color and the worldsheet*. *J. High Energy Phys.* 5 (2018), p. 096.

[2] V. Bazier-Matte, G. Douville, K. Mousavand, H. Thomas, and E. Yıldırım. *ABHY Associahedra and Newton polytopes of F-polynomials for finite type cluster algebras*. Preprint, arXiv:1808.09986. 2018.

[3] M. P. Carr and S. L. Devadoss. *Coxeter complexes and graph-associahedra*. *Topology Appl.* 153.12 (2006), pp. 2155–2168.

[4] F. Chapoton, S. Fomin, and A. Zelevinsky. *Polytopal Realizations of Generalized Associahedra*. *Canad. Math. Bull.* 45.4 (2002), pp. 537–566.

[5] E. M. Feichtner and B. Sturmfels. *Matroid polytopes, nested sets and Bergman fans*. *Port. Math. (N.S.)* 62.4 (2005), pp. 437–468.

[6] S. Fomin and A. Zelevinsky. *Cluster algebras. I. Foundations*. *J. Amer. Math. Soc.* 15.2 (2002), pp. 497–529.

[7] S. Fomin and A. Zelevinsky. *Cluster algebras. II. Finite type classification*. *Invent. Math.* 154.1 (2003), pp. 63–121.

[8] S. Fomin and A. Zelevinsky. *Cluster algebras. IV. Coefficients*. *Compos. Math.* 143.1 (2007), pp. 112–164.

[9] I. Gelfand, M. Kapranov, and A. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Birkhäuser Boston Inc., 2008, pp. x+523.

[10] C. Hohlweg, C. Lange, and H. Thomas. *Permutahedra and generalized associahedra*. *Adv. Math.* 226.1 (2011), pp. 608–640.

[11] C. Hohlweg, V. Pilaud, and S. Stella. *Polytopal realizations of finite type* $\mathbf{g}$*-vector fans*. *Adv. Math.* 328 (2018), pp. 713–749.

[12] J.-L. Loday. *Realization of the Stasheff polytope*. *Arch. Math.* 83.3 (2004), pp. 267–278.

[13] P. McMullen. *Representations of polytopes and polyhedral sets*. *Geometriae Dedicata* 2 (1973), pp. 83–99.

[14] A. Padrol, Y. Palu, V. Pilaud, and P.-G. Plamondon. *Associahedra for finite type cluster algebras and minimal relations between* $\boldsymbol{g}$*-vectors*. Preprint, arXiv:1906.06861. 2019.

[15] Y. Palu, V. Pilaud, and P.-G. Plamondon. *Non-kissing complexes and tau-tilting for gentle algebras*. Preprint, arXiv:1707.07574. In press *Mem. Amer. Math. Soc.* 2017.

[16] A. Postnikov. *Permutohedra, associahedra, and beyond*. *Int. Math. Res. Not. IMRN* 6 (2009), pp. 1026–1106.

[17] A. Zelevinsky. *Nested complexes and their polyhedral realizations*. *Pure Appl. Math. Q.* 2.3 (2006), pp. 655–671.