# String algebras

String algebras are a class of algebras defined by generators and particularly nice relations. The study of their modules goes back to [GP]; we will follow here the general framework established in [BR].

Let $k$ be an algebraically closed field, and let $Q$ be a quiver. While this is not necessary for the theory, it suffices our purposes to assume that $Q$ is finite.

**Def:** A *string algebra* is an algebra of the form $kQ/I$, where

- $kQ$ is the path algebra of $Q$ (see, for instance, [ASS, Chapter II]
- $I$ is an ideal generated by $\underbrace{\text{paths}}_{\text{of length at least two}}$ in $Q$ and such that $kQ/I$ is finite-dimensional.

The quiver $Q$ and the ideal $I$ are required to satisfy the following axioms:

(1) for any vertex $i$ of $Q$, there are at most two arrows starting in $i$ and at most two arrows ending in $i$;

(2) for any arrow $\beta$ of $Q$, there is at most one arrow $\gamma$ such that $\beta\gamma \notin I$;

(3) ———— " ———————— $\alpha$ —— " —— $\alpha\beta \notin I$.

ex:

$$Q = \begin{tikzcd} 1 \arrow[dr, "\alpha"] & & 4 \\ & 3 \arrow[ur, "\beta"] \arrow[dr, "\delta"] & \\ 2 \arrow[ur, "\gamma"] & & 5 \end{tikzcd}$$

$I = (\beta\alpha, \delta\gamma)$.

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$I = (\beta\alpha, \delta\gamma, \delta\alpha)$

$$Q = \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[loop above, "\beta"] \arrow[r, "\gamma"] & 3 \end{tikzcd}$$

$I = (\beta\alpha, \gamma\beta, \beta^5)$.

**Def:** A string algebra $kQ/I$ is a *gentle algebra* [ ] if

(4) $I$ is generated by paths of length exactly two;

(5) for any arrow $\beta$ of $Q$, there is at most one arrow $\gamma$ such that $t(\gamma) = s(\beta)$ and $\beta\gamma \in I$;

(6) —— " ———— " ———— $\alpha$ —— " —— $s(\alpha) = t(\beta)$ and $\alpha\beta \in I$.

ex: In the previous example, only the first algebra is gentle.

String algebras enjoy a particularly nice representation theory: they are tame algebras [ ] and their indecomposable representations are completely classified. We will make heavy use of this classification, which we now recall in detail.

## String and band modules

For any arrow $\beta$ of any quiver $Q$, define a formal inverse $\beta^{-1}$ with the properties that $s(\beta^{-1}) = t(\beta)$, $t(\beta^{-1}) = s(\beta)$, $\beta\beta^{-1} = e_{t(\beta)}$ and $\beta^{-1}\beta = e_{s(\beta)}$. Furthermore, let $(\beta^{-1})^{-1} = \beta$; thus $(-)^{-1}$ is an involution on the set of arrows and formal inverses of arrows of $Q$, which we extend to the set of paths in a natural way.

Def: Let $A = KQ/I$ be a string algebra.

(1) A string for $A$ is a word of the form

$$w = c_1 c_2^{-1} c_3 c_4^{-1} \cdots c_n^{\pm 1}, \quad \text{where}$$

- each $c_i$ is a path in $Q$ that is not an element of $I$;
- for each $i \in \{1, \ldots, n\}$, $t(c_i) = t(c_{i-1})$ and $s(c_i) = s(c_{i+1})$;
- $w$ is reduced, in the sense that no subword of the form $\beta\beta^{-1}$ or $\beta^{-1}\beta$ appears in $w$, where $\beta$ is an arrow of $Q$.

(2) A band is a string $b$ of length at least one such that

- $s(b) = t(b)$;
- no powers of $b$ lie in $I$;
- $b$ is not itself a power of a strictly smaller string.

ex: For the string (in fact, gentle) algebra defined by

$$Q = \begin{tikzcd} & 2 \arrow[dr, "\beta"] & & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & 3 \arrow[r, "\delta"] & 4 \end{tikzcd} \qquad I = (\delta\beta),$$

the strings are $e_1, e_2, e_3, \alpha, \beta, \gamma, \delta, \beta\alpha, \delta\gamma, \gamma^{-1}\beta, \ldots$.

the only bands are $\gamma^{-1}\beta\alpha$, $\beta\alpha\gamma^{-1}$, $\alpha\gamma^{-1}\beta$ and their inverses.

<u>Def</u>: Let $\omega$ be a string for a string algebra $A = kQ/I$. Write

$$\omega = \beta_1^{\varepsilon_1} \beta_2^{\varepsilon_2} \cdots \beta_n^{\varepsilon_n},$$

where each $\beta_i$ is an arrow in $Q$ and each $\varepsilon_i$ is in $\{\pm 1\}$.

The <u>string module</u> $M(\omega)$ is the $A$-module defined as a representation of $A$ as follows:

- Let $i_0 = s(\beta_1^{\varepsilon_1})$, and $i_m = t(\beta_m^{\varepsilon_m})$ for each $m \in \{1, \dots, n\}$.
- For each vertex $j$ of $Q$, let $M(\omega)_j$ be the vector space with basis given by $\{v_{i_m} \mid i_m = j\}$.
- For each arrow $\gamma$ of $Q$, the linear map $M(\omega)_\gamma : M(\omega)_{s(\gamma)} \longrightarrow M(\omega)_{t(\gamma)}$ is defined on the basis of $M(\omega)_{s(\gamma)}$ as follows:

$$M(\omega)_\gamma (v_{i_m}) = \begin{cases} v_{i_{m+1}} & \text{if } \beta_{m+1} = \gamma \text{ and } \varepsilon_{m+1} = 1; \\ v_{i_{m-1}} & \text{if } \beta_m = \gamma \text{ and } \varepsilon_m = -1; \\ 0 & \text{else.} \end{cases}$$

It follows from the definition that for any string $\omega$, the string modules $M(\omega)$ and $M(\omega^{-1})$ are isomorphic.

<u>ex</u>: For the previous example, let $\omega = \delta\gamma\alpha^{-1}\beta^{-1}\gamma\alpha^{-1}\beta^{-1}\gamma$.

Then $M(\omega)$ is given by

$$\begin{tikzcd} & k^2 \arrow[dr, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}"] & & \\ k^2 \arrow[ur, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[rr, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}"'] & & k^3 \arrow[r, "(0\ 0\ 1)"] & k \end{tikzcd}.$$

<u>Def</u>: Let $b$ be a band for a string algebra $A = kQ/I$. Write

$$b = \beta_1^{\varepsilon_1} \cdots \beta_n^{\varepsilon_n},$$

where each $\beta_i$ is an arrow in $Q$ and each $\varepsilon_i$ is in $\{\pm 1\}$.

Moreover, let $\lambda \in K^*$, and let $d \in \mathbb{N}_{>0}$. The <u>band module</u> $M(b,\lambda,d)$ is defined as follows:

- For each vertex $j$ of $Q$, the vector space $M(b,\lambda,d)$ is equal to $K^d$.
- For each arrow $\gamma$ different from $\beta_1$, the linear map $M(b,\lambda,d)_\gamma$ is the identity.
- The linear map $M(b,\lambda,d)_{\beta_1}$ is equal to $J_d(\lambda)^{\varepsilon_1}$, where

$$J_d(\lambda) = \begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & 1 & \ddots & \\ & & & \lambda \end{pmatrix}$$

is the $d \times d$ Jordan block of type $\lambda$.

It follows from the definition that the band modules $M(b,\lambda,d)$ and $M(b^{-1},\lambda^{-1},d)$ are isomorphic. Moreover, if $b$ and $b'$ are two bands that are <u>cyclically equivalent</u>, meaning that one is obtained from the other by cyclically permuting the arrows that constitute it, then we also have that $M(b,\lambda,d)$ and $M(b',\lambda,d)$ are isomorphic.

<u>ex:</u> In the previous example, let $b = \gamma^{-1}\beta\alpha$. Then

$$M(b,\lambda,1) = \begin{tikzcd} & k \arrow[dr, "1"] & & \\ K \arrow[ur, "1"] \arrow[rr, "\lambda^{-1}"] & & k \arrow[r, "0"] & 0 \end{tikzcd}$$

and

$$M(b,\lambda,3) = \begin{tikzcd} & K^3 \arrow[dr, "\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}"] & & \\ K^3 \arrow[ur, "\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}"] \arrow[rr, "\begin{pmatrix} \lambda^{-1}&0&0\\1&\lambda^{-1}&0\\0&1&\lambda^{-1} \end{pmatrix}"'] & & K^3 \arrow[r, "0"] & 0 \end{tikzcd}$$

<u>Theorem</u> [BR, p.161] (Let $A$ be a string algebra.) The string and band modules (over $A$) form a complete list of indecomposable $A$-modules (up to isomorphism).

Moreover,

- A string module is never isomorphic to a band module;
- two string modules $M(w)$ and $M(w')$ are isomorphic iff $w' = w^{\pm 1}$;
- two band modules $M(b,d,\lambda)$ and $M(b',d',\lambda')$ are isomorphic iff $d = d'$ and either
  - $b$ is cyclically equivalent to $b'$, and $\lambda = \lambda'$, or
  - $b^{-1}$ is cyclically equivalent to $b'$, and $\lambda^{-1} = \lambda'$.

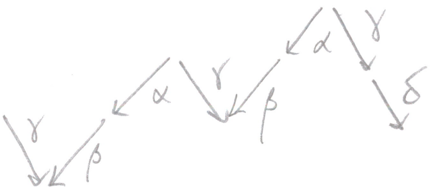
## Auslander-Reiten translation of string modules

The following notation, still following [BR], will be useful for dealing with string modules.

For a given string $w=\beta_n^{\varepsilon_n}\cdots\beta_1^{\varepsilon_1}$, we draw $w$ as follows:

- draw arrows $\beta_1,\dots,\beta_n$ from left to right;
- all arrows are pointing downwards.

ex: In the previous example, the string $w=\delta\gamma\alpha^{-1}\beta^{-1}\gamma\alpha^{-1}\beta^{-1}\gamma$ is drawn as




If $w=e_i$ is a string of length zero, then we depict it simply as "$i$".
There are some immediate advantages for this depiction of strings: For instance, the socle of $M(w)$ is the direct sum of the simple modules corresponding to the local minima in the picture of $w$; similarly, the top of $M(w)$ is the direct sum of the simple modules corresponding to the local maxima.

Def: • A string $w$ starts (resp. ends) on a peak if there is no arrow $\beta$ such that $w\beta$ (resp. $\beta^{-1}w$) is a string.

• A string $w$ starts (resp. ends) in a deep if there is no arrow $\beta$ such that $w\beta^{-1}$ (resp. $\beta w$) is a string.

Remark: The previous definition is best understood using the above depiction of strings. Starting (or ending) on a peak means that one cannot add an arrow at the start (or end) of $w$ such that the starting point (or ending point) of the new string would be higher in the picture than that of $w$. The same applies when replacing "on a peak" by "in a deep" and "higher" by "lower".

ex: The string in the previous example starts and ends on a peak, and also ends in a deep.

<u>Def</u>: • Let $w$ be a string that does not start on a peak. Let $\beta_0, \beta_1, \dots, \beta_r$ be arrows such that $w_h = w_0 \beta_0 \beta_1^{-1} \beta_2^{-1} \cdots \beta_r^{-1}$ starts in a deep. We say that $w_h$ is obtained from $w$ by <u>adding a hook</u> at the start of $w$.

• Let $w$ be a string that does not end on a peak. Let $\beta_0, \dots, \beta_r$ be arrows such that ${}_h w = \beta_r \cdots \beta_1 \beta_0^{-1} w$ ends in a deep. We say that ${}_h w$ is obtained from $w$ by <u>adding a hook</u> at the start of $w$.

<u>Remark</u>: Unless $w$ has length zero, the strings ${}_h w$ and $w_h$, when they exist, are uniquely determined. If $w$ has length zero, then there may be two possible choices for ${}_h w$ and $w_h$.

The dual of a hook is a cohook:

<u>Def</u>: • Let $w$ be a string that does not start in a deep. Let $\beta_0, \beta_1, \dots, \beta_r$ be arrows such that $w_c = w \beta_0^{-1} \beta_1 \cdots \beta_r$ starts on a peak. We say that $w_c$ is obtained from $w$ by <u>adding a cohook</u> at the start of $w$.

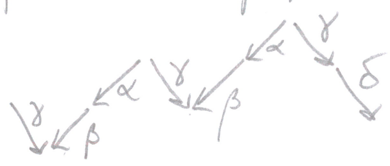
• Let $w$ be a string that does not end in a deep. Let $\beta_0, \dots, \beta_r$ be arrows such that ${}_c w = \beta_r^{-1} \cdots \beta_1^{-1} \beta_0 w$ ends on a peak. We say that ${}_c w$ is obtained from $w$ by adding a cohook at the end of $w$.

<u>Remark</u>: As for hooks, ${}_c w$ and $w_c$ are unique (if they exist) only when $w$ does not have length zero; otherwise, there may be two choices for ${}_c w$ and $w_c$.

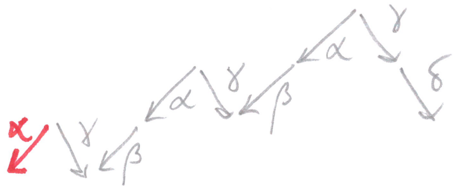
<u>ex</u>: In pictures, hooks will always look like

and cohooks like

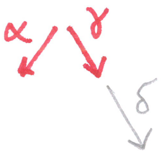
ex: In the previous example, the string

$w =$ 

does not start on a deep. Thus

$w_c =$  .

ex: In the same algebra, consider the string $w = \delta$. It does not start on a peak, and we have

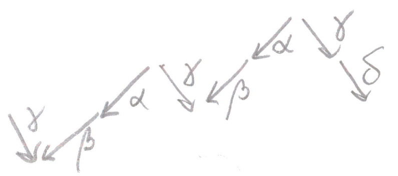
$w_h =$ 

Def: A string is a <u>straight line</u> if it is made only of arrows in $Q$ or only of formal inverses of arrows. By convention, strings of length zero are straight lines.

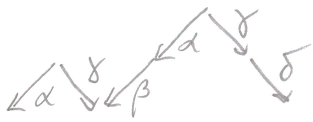
Lemma: Let $w$ be a string.

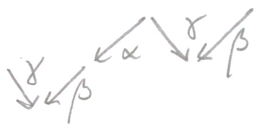
- If $w$ starts on a peak and is not a path in $Q$, then there exists a unique string $w'$ such that $w'$ does not start in a deep, and $w = w'_c$. Denote $w'$ by $w_{c^{-1}}$.
  If $w$ is a path, then put $w_{c^{-1}} = 0$.
- If $w$ ends on a peak and is not the inverse of a path in $Q$, then there exists a unique string $w'$ such that $w'$ does not end in a deep, and $w = {}_c w'$. Denote $w'$ by ${}_{c^{-1}}w$. If $w$ is the inverse of a path, put ${}_{c^{-1}}w = 0$.
- If $w$ starts in a deep and is not the inverse of a path in $Q$, then there exists a unique string $w'$ that does not start on a peak and such that $w = w'_h$. Denote $w'$ by $w_{h^{-1}}$. If $w$ is the inverse of a path, put $w_{h^{-1}} = 0$.
- If $w$ ends in a deep and is not a path in $Q$, then there exists a unique string $w'$ that does not end on a peak and such that $w = {}_h w'$. Denote $w'$ by ${}_{h^{-1}}w$. If $w$ is a path, put ${}_{h^{-1}}w = 0$.

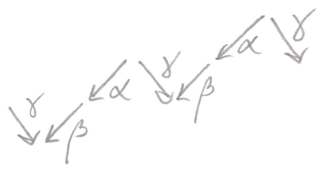
ex: In the previous example, with

$w =$ 

we have

$w_{c^{-1}} =$ 

${}_{h^{-1}}w =$ 

${}_{c^{-1}}w =$ 

Theorem [BR]: Let $A = kQ/I$ be a string algebra, and let $w$ be a string. Then $\tau(M(w)) = M(w')$, where $w'$ is obtained by following the table below.

| | $w$ does not start in a deep | $w$ starts in a deep |
|---|---|---|
| $w$ does not end in a deep | ${}_c w_c$ | $({}_c w)_{h^{-1}}$ |
| $w$ ends in a deep | ${}_{h^{-1}}(w_c)$ | ${}_{h^{-1}}w_{h^{-1}}$ |

Dually, $\tau^{-1}(M(w)) = M(w'')$, where $w''$ is given in the table below.

| | $w$ does not start on a peak | $w$ starts on a peak |
|---|---|---|
| $w$ does not end on a peak | ${}_h w_h$ | $({}_h w)_{c^{-1}}$ |
| $w$ ends on a peak | ${}_{c^{-1}}(w_h)$ | ${}_{c^{-1}}w_{c^{-1}}$ |

# Morphisms between string modules

**Def:** Let $A$ be a string algebra, and let $w = \beta_1^{\varepsilon_1} \cdots \beta_n^{\varepsilon_n}$ be a string for $A$.

(1) A substring of $w$ is a string of the form $\beta_i^{\varepsilon_i} \beta_{i+1}^{\varepsilon_{i+1}} \cdots \beta_j^{\varepsilon_j}$, where $1 \leq i \leq j \leq n$.

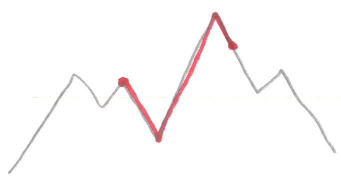
(2) A substring $w' = \beta_i^{\varepsilon_i} \cdots \beta_j^{\varepsilon_j}$ is said to be on the top of $w$ if

- $i = 1$ or $\varepsilon_{i-1} = 1$, and
- $j = n$ or $\varepsilon_{j+1} = -1$.

(3) A substring $w' = \beta_i^{\varepsilon_i} \cdots \beta_j^{\varepsilon_j}$ is said to be at the bottom of $w$ if

- $i = 1$ or $\varepsilon_{i-1} = -1$, and
- $j = n$ or $\varepsilon_{j+1} = 1$.

ex:



A substring at the top.

A substring on the bottom.

**Proposition:** Let $w$ and $w'$ be two strings for a string algebra $A$. Then the dimension of the vector space $\mathrm{Hom}_A(M(w), M(w'))$ is equal to

$$\sum_{\xi \text{ substring of } w} \left(\begin{array}{c}\#\text{ of substrings on top of } w \\ \text{equal to } \xi \text{ or } \xi^{-1}\end{array}\right) \cdot \left(\begin{array}{c}\#\text{ of substrings at the bottom of } w' \\ \text{equal to } \xi \text{ or } \xi^{-1}\end{array}\right).$$

**Proof:** Reference?

## $\tau$-orthogonality and $\tau$-rigidity for string modules

Def: Let $A$ be any finite-dimensional $k$-algebra, and let $M$ and $N$ be finitely generated $A$-modules. We say that <u>$M$ is left $\tau$-orthogonal to $N$</u> if the space $\mathrm{Hom}_A(M, \tau N)$ vanishes. A module is <u>$\tau$-rigid</u> if it is left $\tau$-orthogonal to itself.

We are interested in $\tau$-orthogonality between string modules. The reason why we restrict ourselves to string modules is the following.

Lemma: Let $A$ be a string algebra, and let $M$ be an $A$-module. If $M$ is $\tau$-rigid, then $M$ is a direct sum of string modules.

Proof: By [BR, p.175], all band modules lie in homogeneous tubes. In other words, if $N$ is a band module, then $N = \tau N$, so $\mathrm{Hom}_A(N, \tau N)$ is non-zero. $\square$

Now, let $w$ and $w'$ be two strings for a string algebra $A$. The results of the two previous sections allow us, first, to compute $\tau M(w')$, and then to check whether $\mathrm{Hom}_A(M(w), \tau M(w'))$ vanishes.

[GP] Gelfand, I.M & V.A. Ponomarev, Indecomposable representations of the Lorentz group, Russian Math. Surveys 23 (1968), p. 1-58.

[BR] M.C.R. Butler and C.M. Ringel, Auslander-Reiten sequences with few middle terms and applications to string algebras, Comm. in Algebra, 15 (1 & 2), 145-179 (1987).