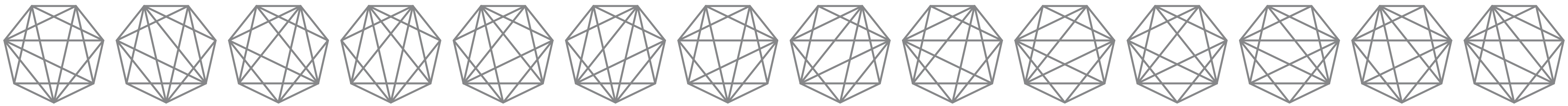
ANR EGOS
Embedded graphs and their oriented structures

MULTITRIANGULATIONS

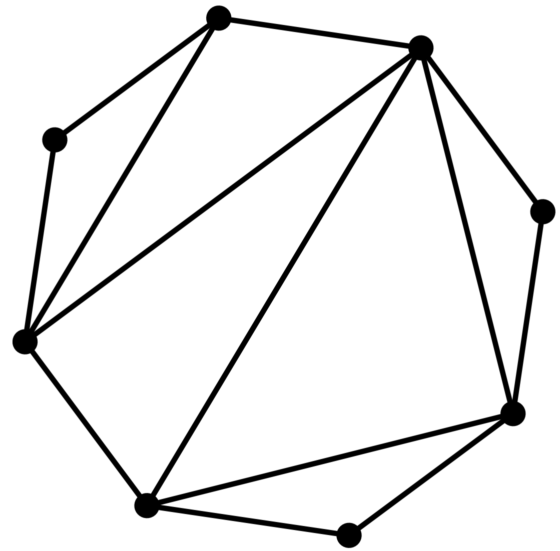


Vincent Pilaud (CNRS & École Polytechnique)

1. Triangulations of a convex polygon

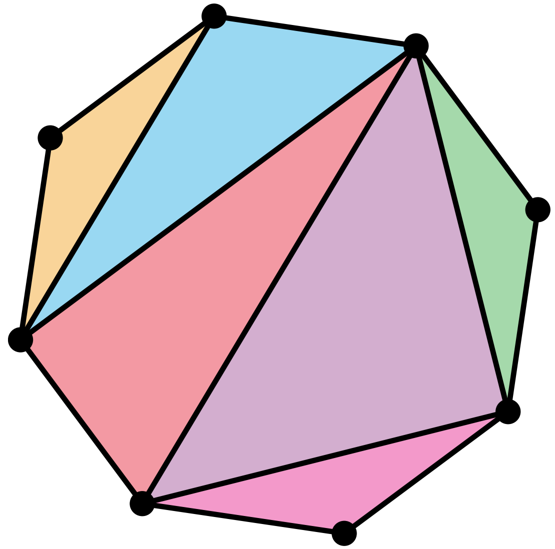
TRIANGULATIONS OF A CONVEX POLYGON

triangulation $=$ maximal crossing-free set of diagonals



TRIANGULATIONS OF A CONVEX POLYGON

triangulation = maximal crossing-free set of diagonals
= complex of triangles

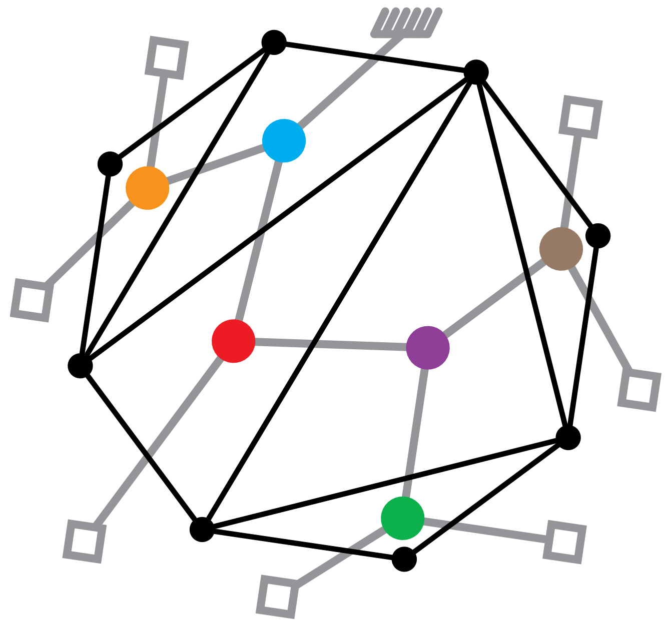


TRIANGULATIONS OF A CONVEX POLYGON

triangulation $=$ maximal crossing-free set of diagonals
$=$ complex of triangles

Counted by the Catalan number $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$

Duals are binary trees



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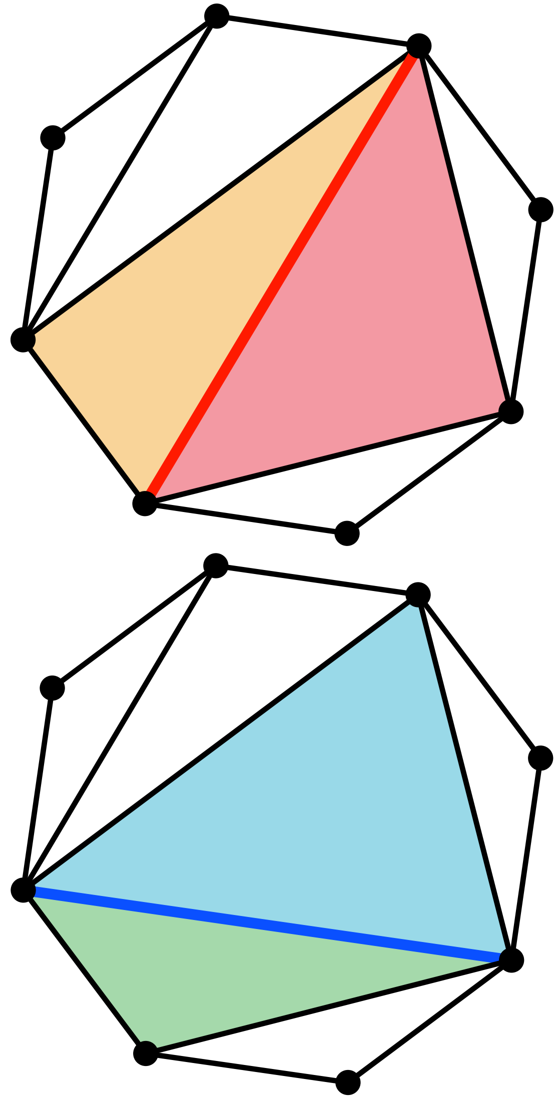
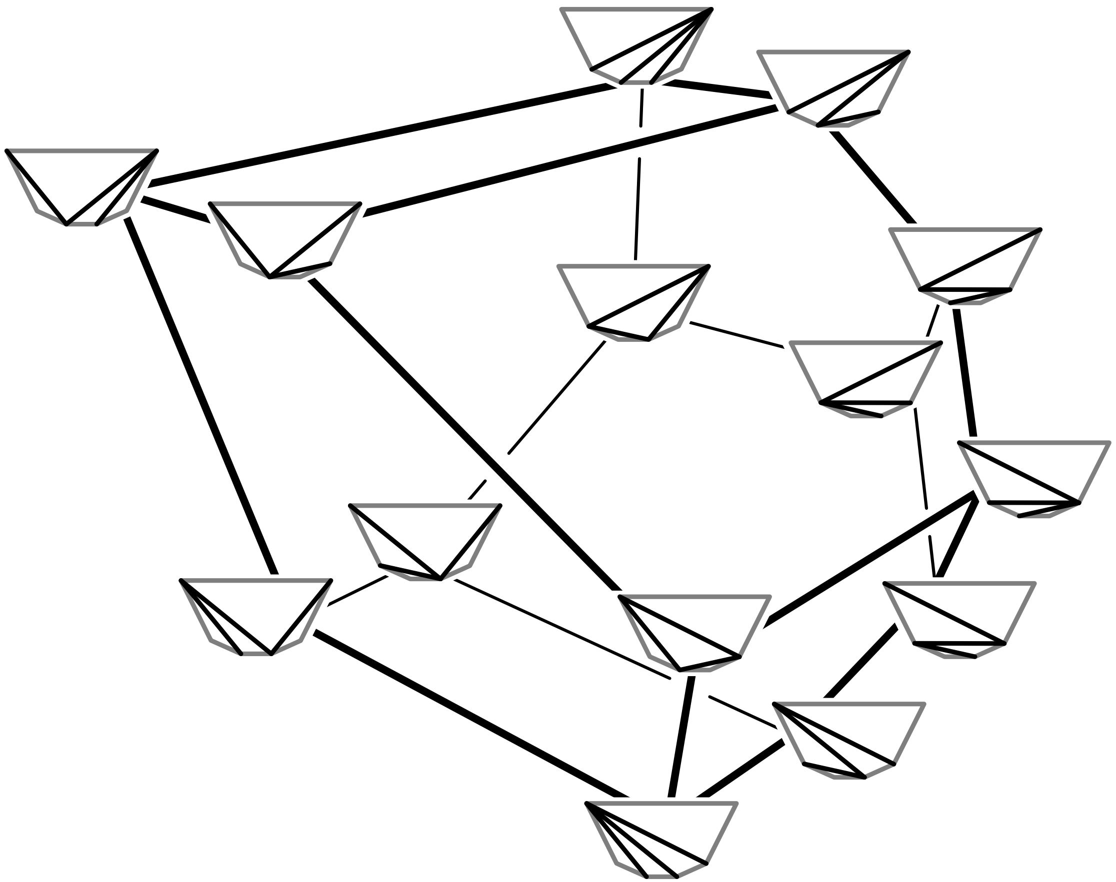


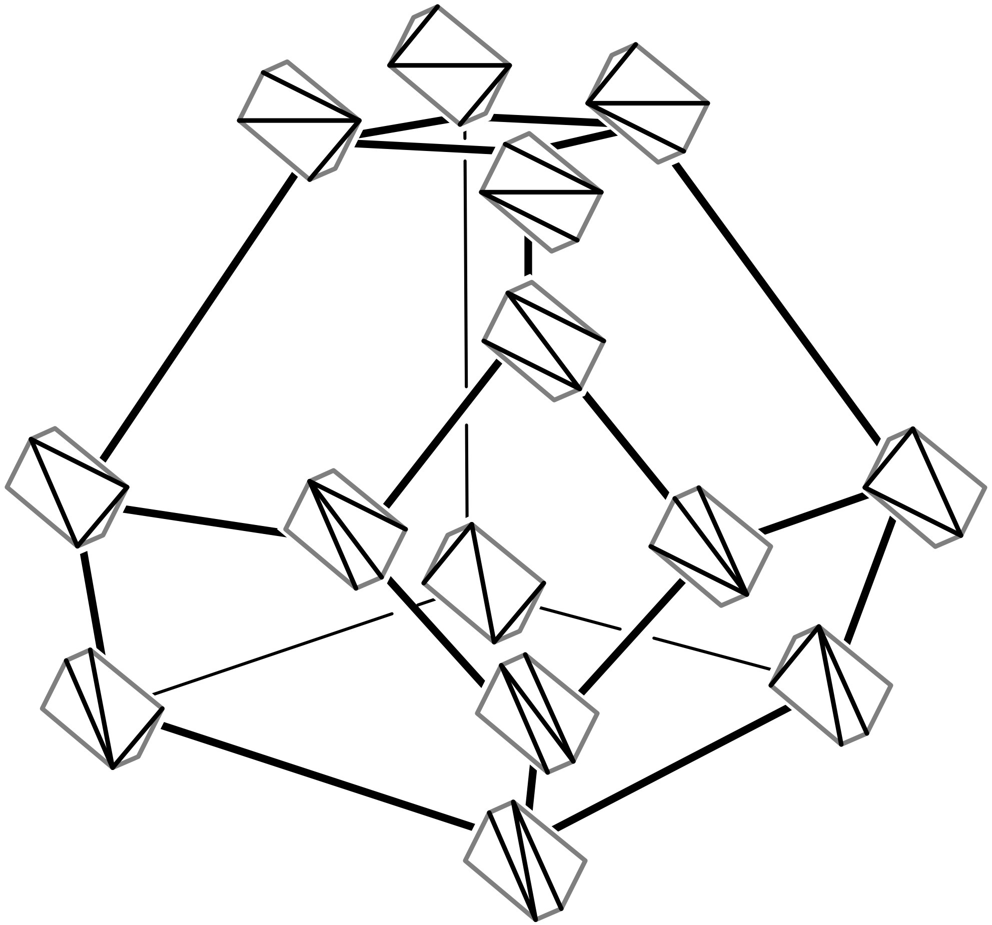
flip $=$ exchange an internal diagonal e with the unique common bisector f of the two triangles adjacent to e

Tamari lattice $=$ order induced by slope increasing flips

The graph of flips is $(n-3)$-regular and connected.
It is the graph of the $(n-3)$-dimensional associahedron

ASSOCIAHEDRON



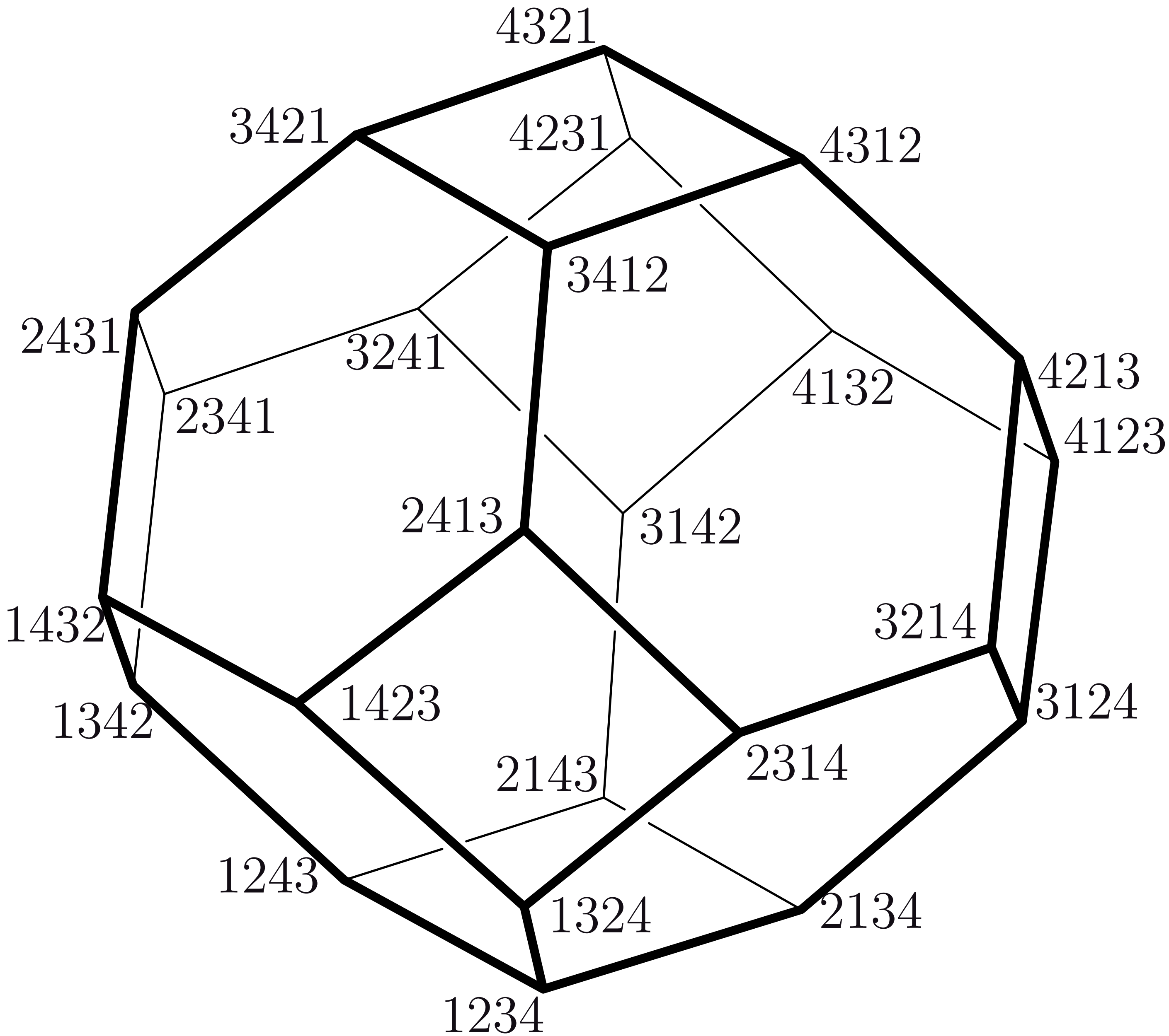


$$L(T)_j = \big(j - \max\{i \in [0, j-2] \mid ij \in T\}\big) \cdot \big(\min\{k \in [j+2, n+2] \mid jk \in T\} - j\big).$$

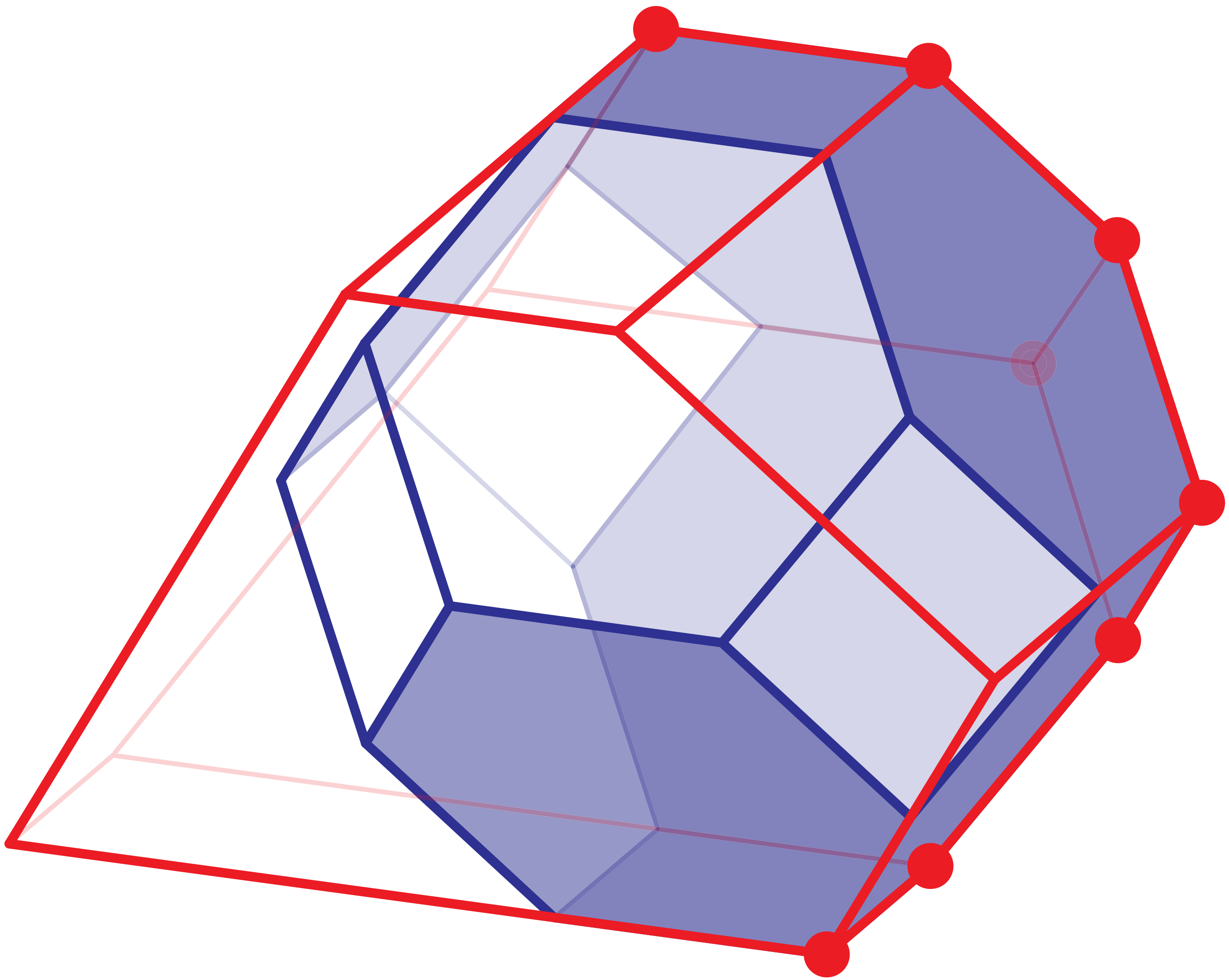
J.-L. Loday, Realization of the Stasheff polytope, 2004.

C. Hohlweg & C. Lange, Realizations of the associahedron and cyclohedron, 2007.

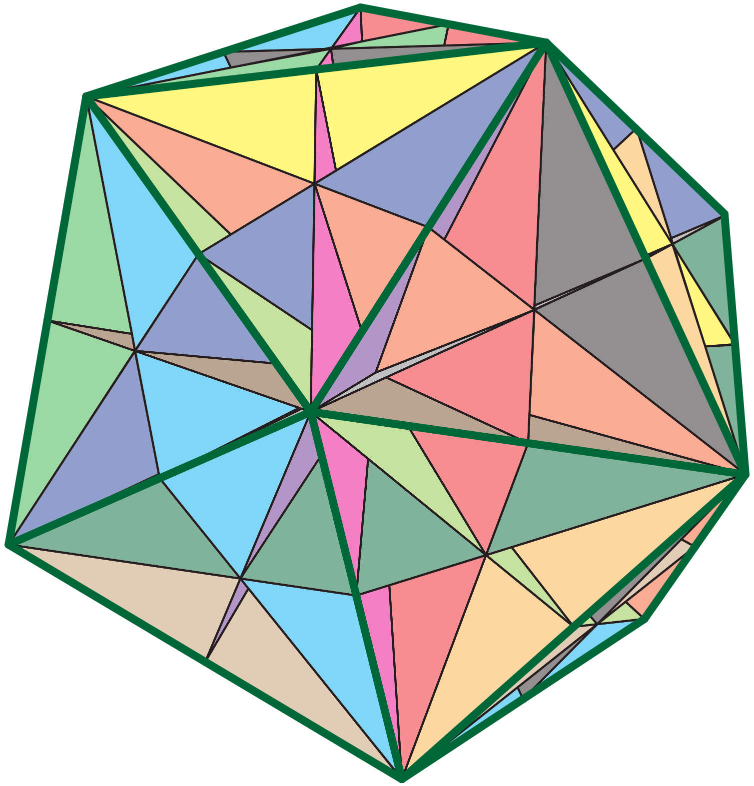
PERMUTAHEDRON

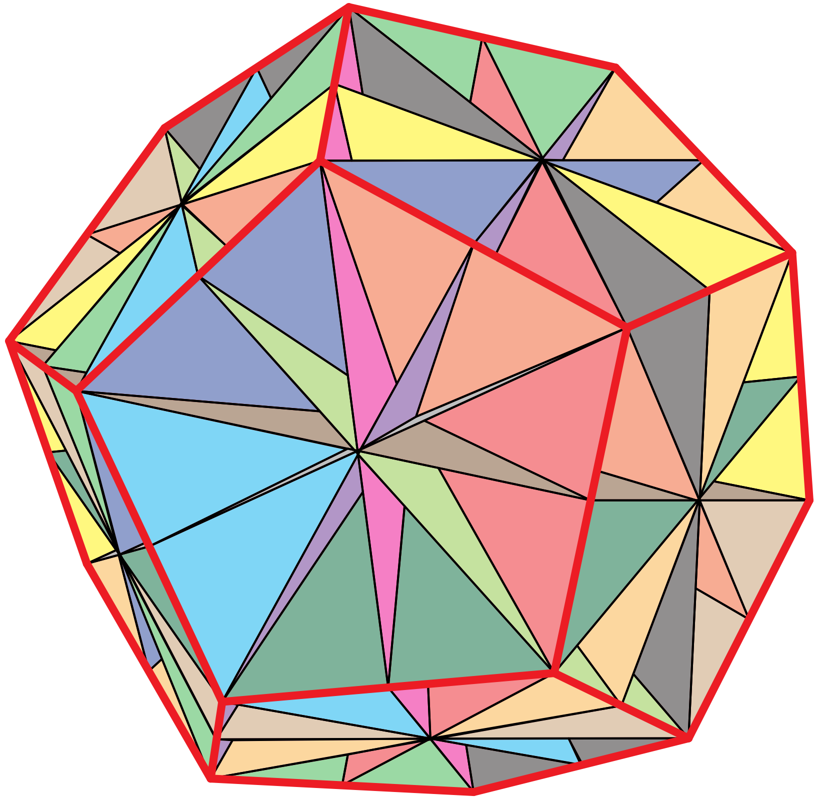


ASSOCIAHEDRON FROM PERMUTAHEDRON



DIGRESSION: COXETER GROUPS, CLUSTER ALGEBRAS, GENERALIZED ASSOCIAHEDRA





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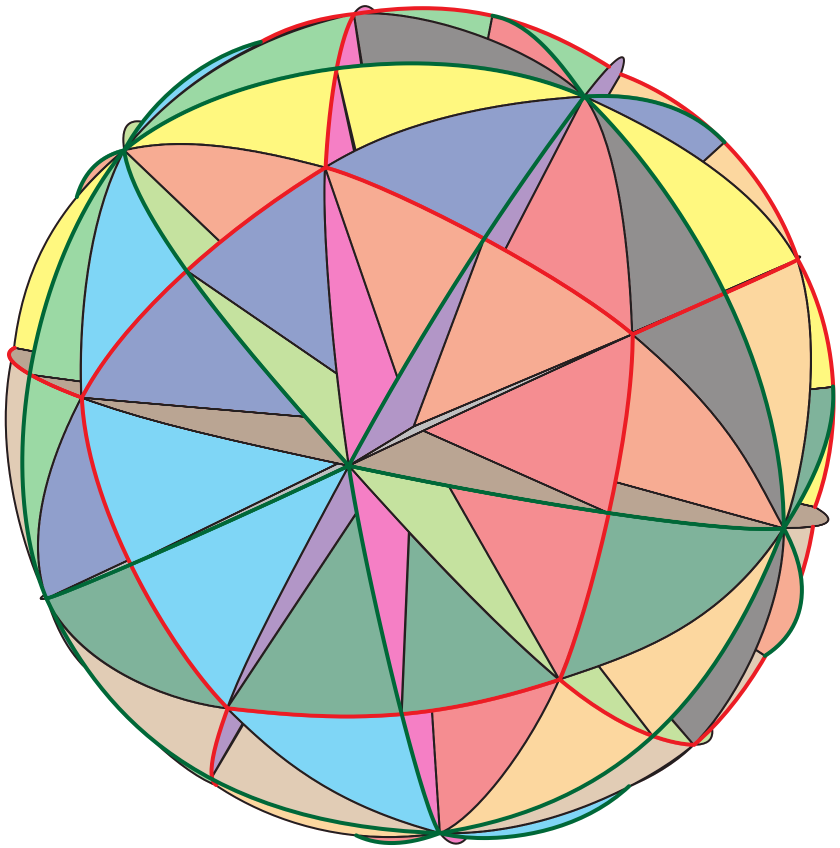 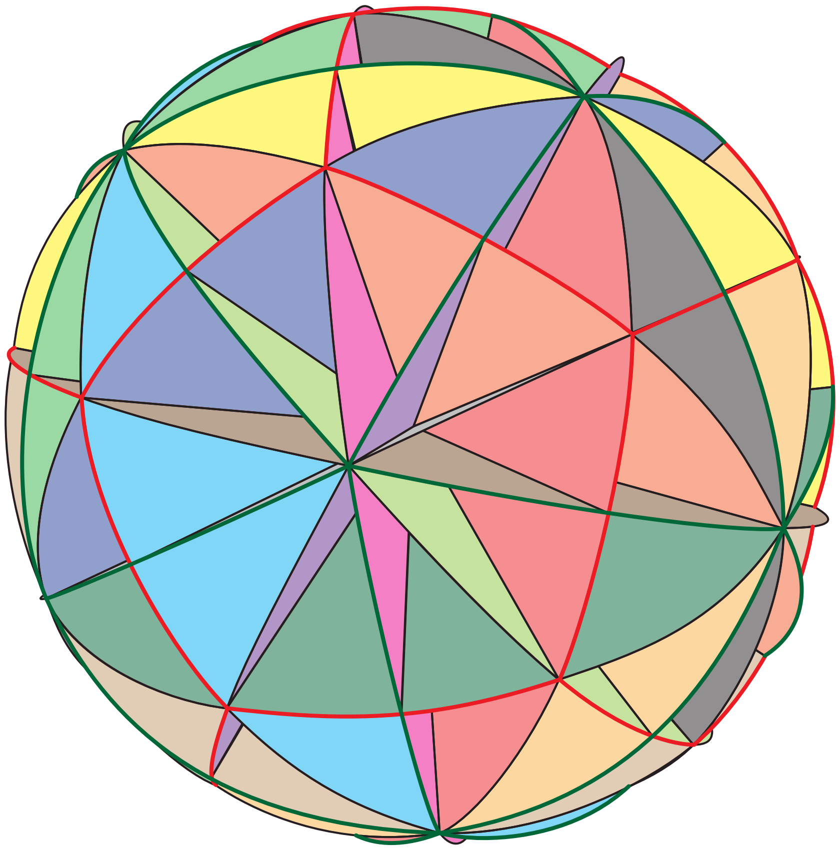

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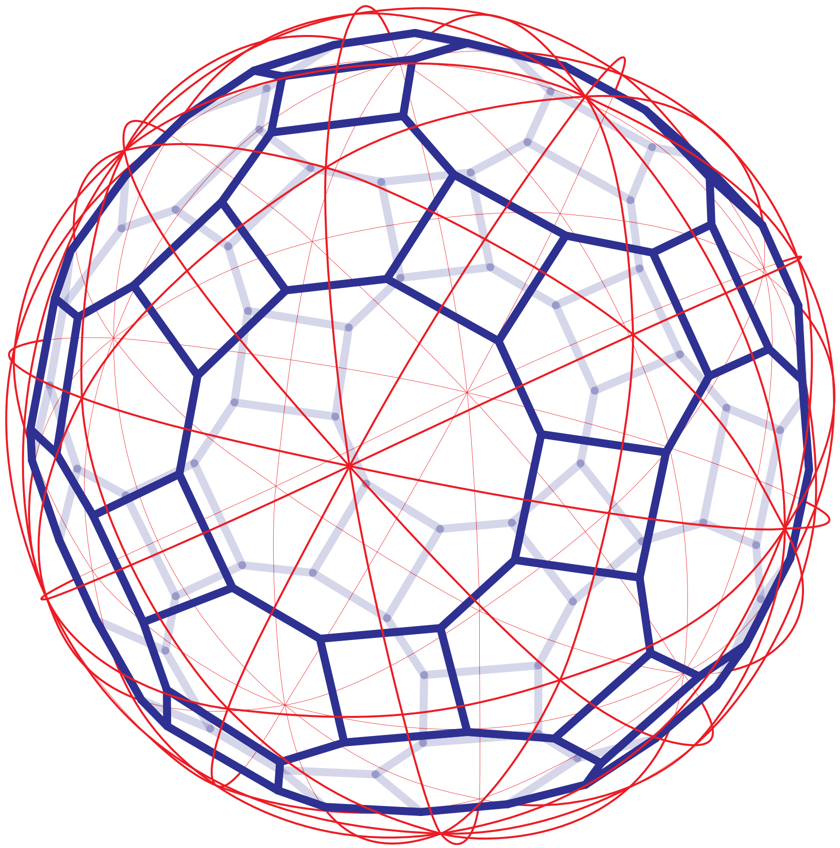

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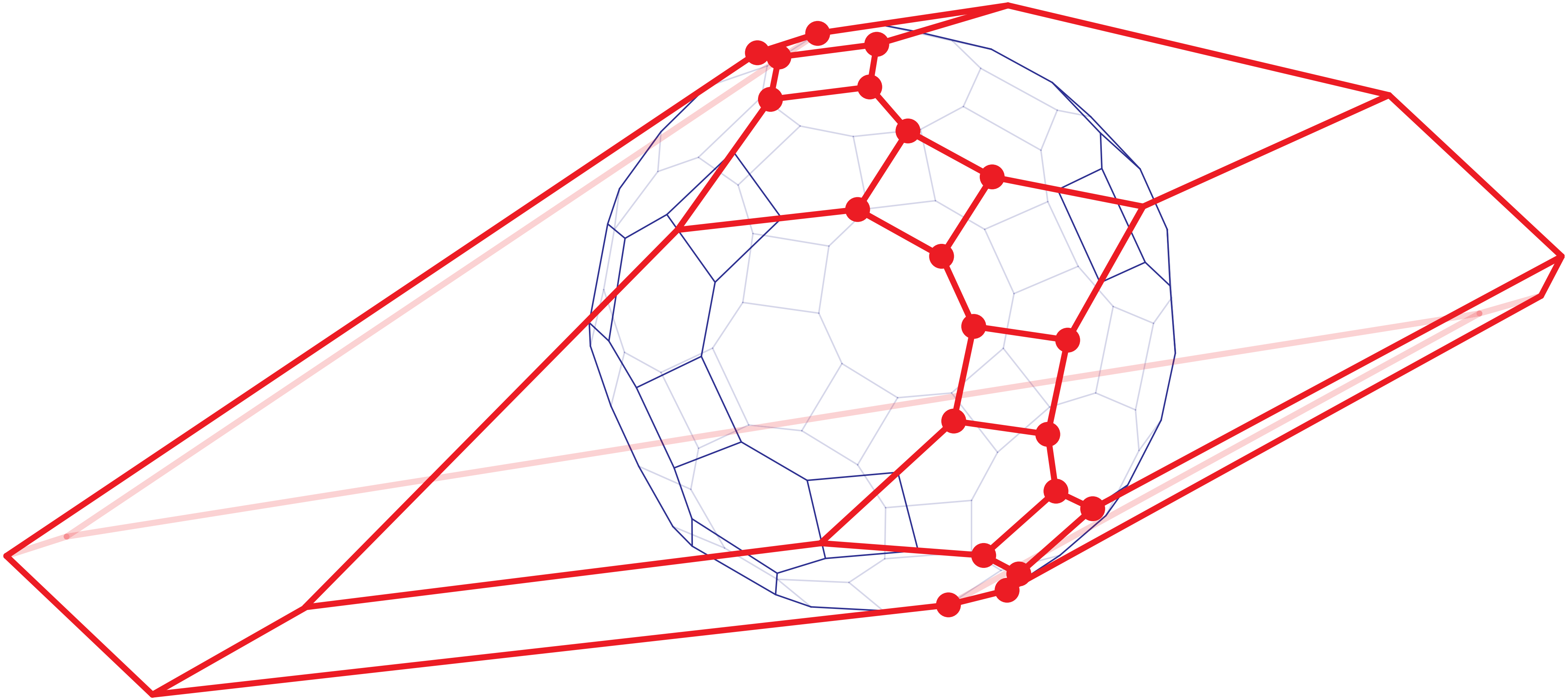


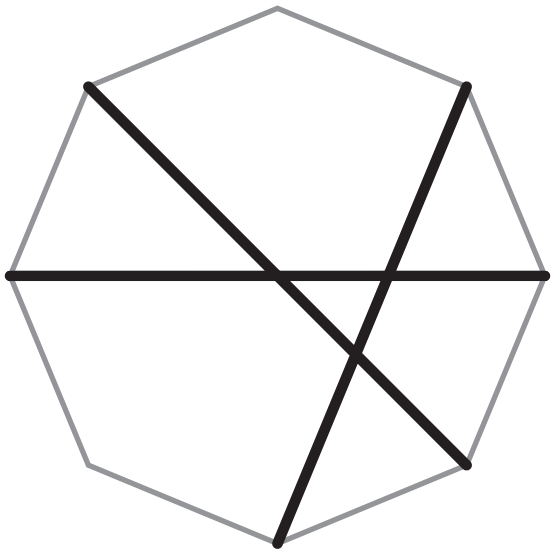
2. Multitriangulations as complexes of star polygons

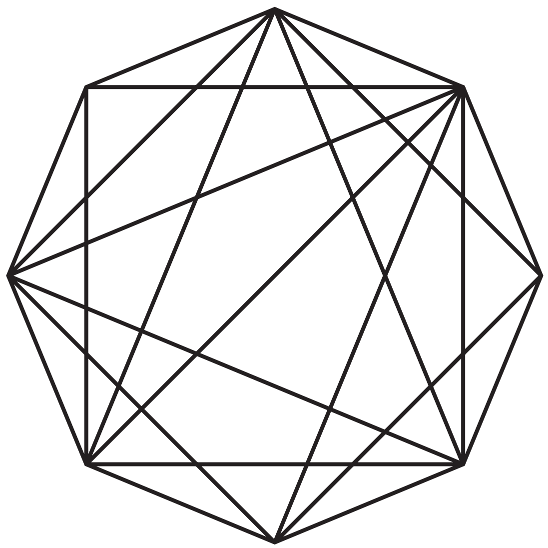
MULTITRIANGULATIONS

$k \geq 1$ and $n \geq 2k+1$ two fixed integers.

ℓ-crossing $=$ set of ℓ mutually crossing diagonals of the convex n-gon.

k-triangulation $=$ maximal $(k+1)$-crossing-free set of diagonals of the n-gon.





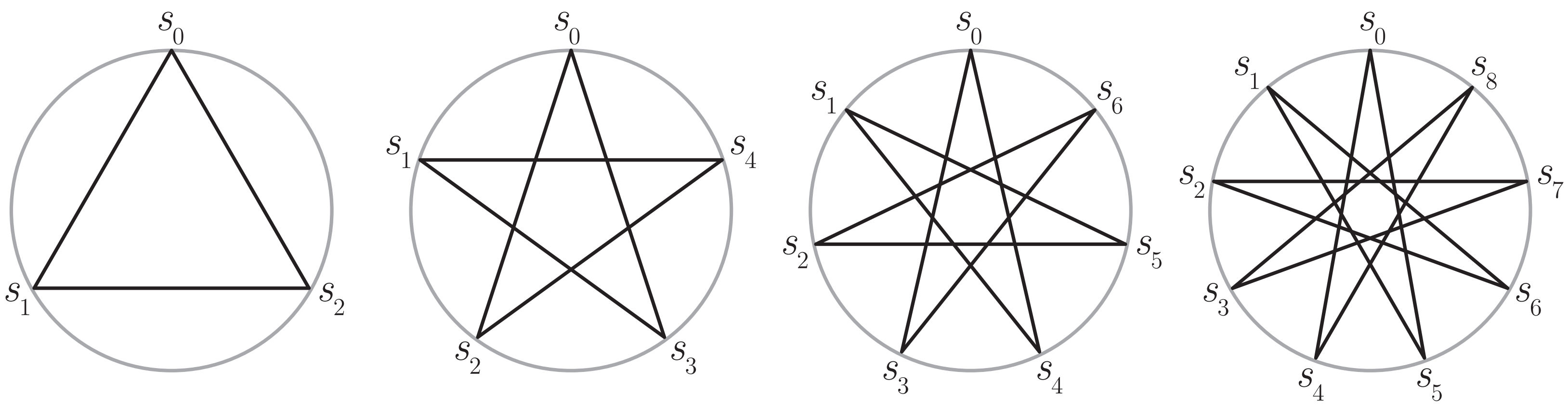
V. Capoyleas & J. Pach, A Turán-type theorem on chords of a convex polygon, 1992.

T. Nakamigawa, A generalization of diagonal flips in a convex polygon, 2000.

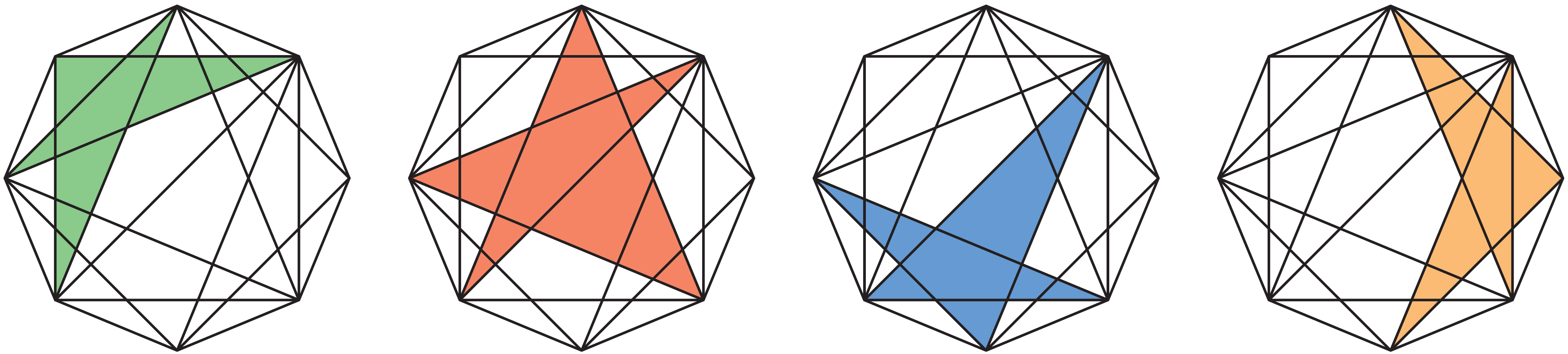
A. Dress, J. Koolen & V. Moulton, On line arrangements in the hyperbolic plane, 2002.

J. Jonsson, Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005.

STARS IN MULTITRIANGULATIONS



k-star $=$ star polygon with vertices $s_0, s_1, \dots, s_{2k}$ cyclically ordered
and edges $[s_0, s_k], [s_1, s_{1+k}], \dots, [s_k, s_{2k}], [s_{k+1}, s_0], \dots, [s_{2k}, s_{k-1}]$.



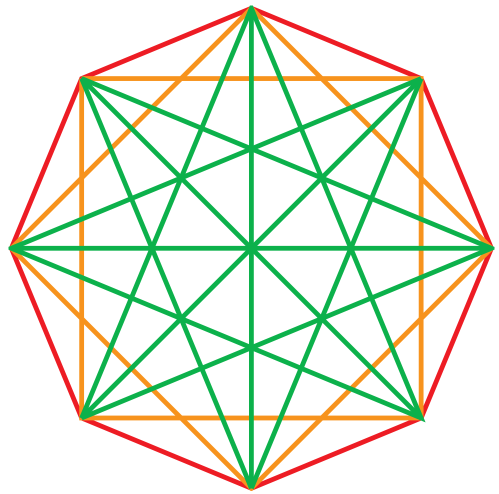
COMPLEXES OF STARS

THEOREM. In a k-triangulation T,

(i) a k-relevant diagonal belongs to exactly two k-stars of T,

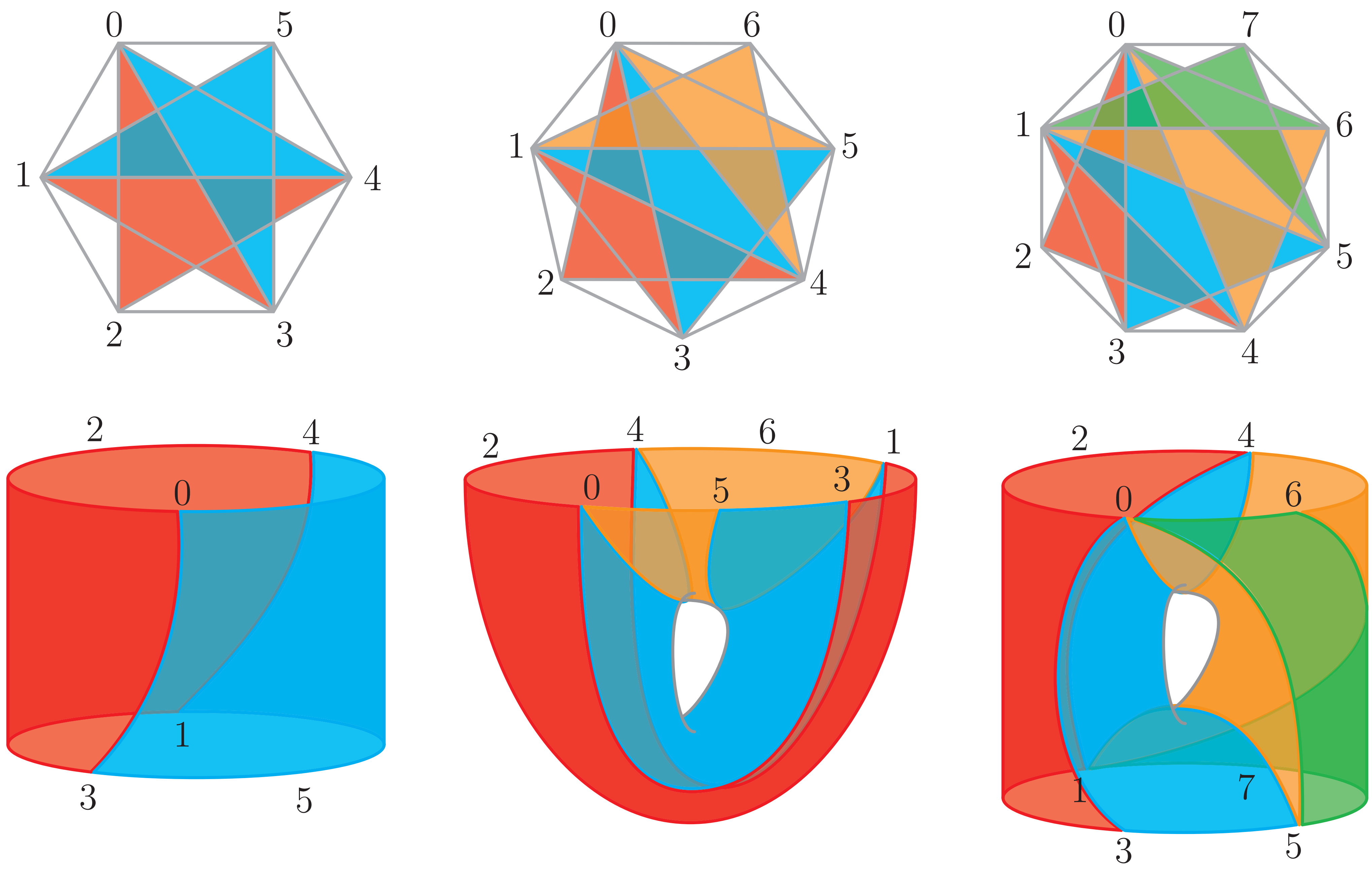
(ii) a k-boundary diagonal belongs to exactly one k-star of T,

(iii) a k-irrelevant diagonal does not belong to any k-star of T.

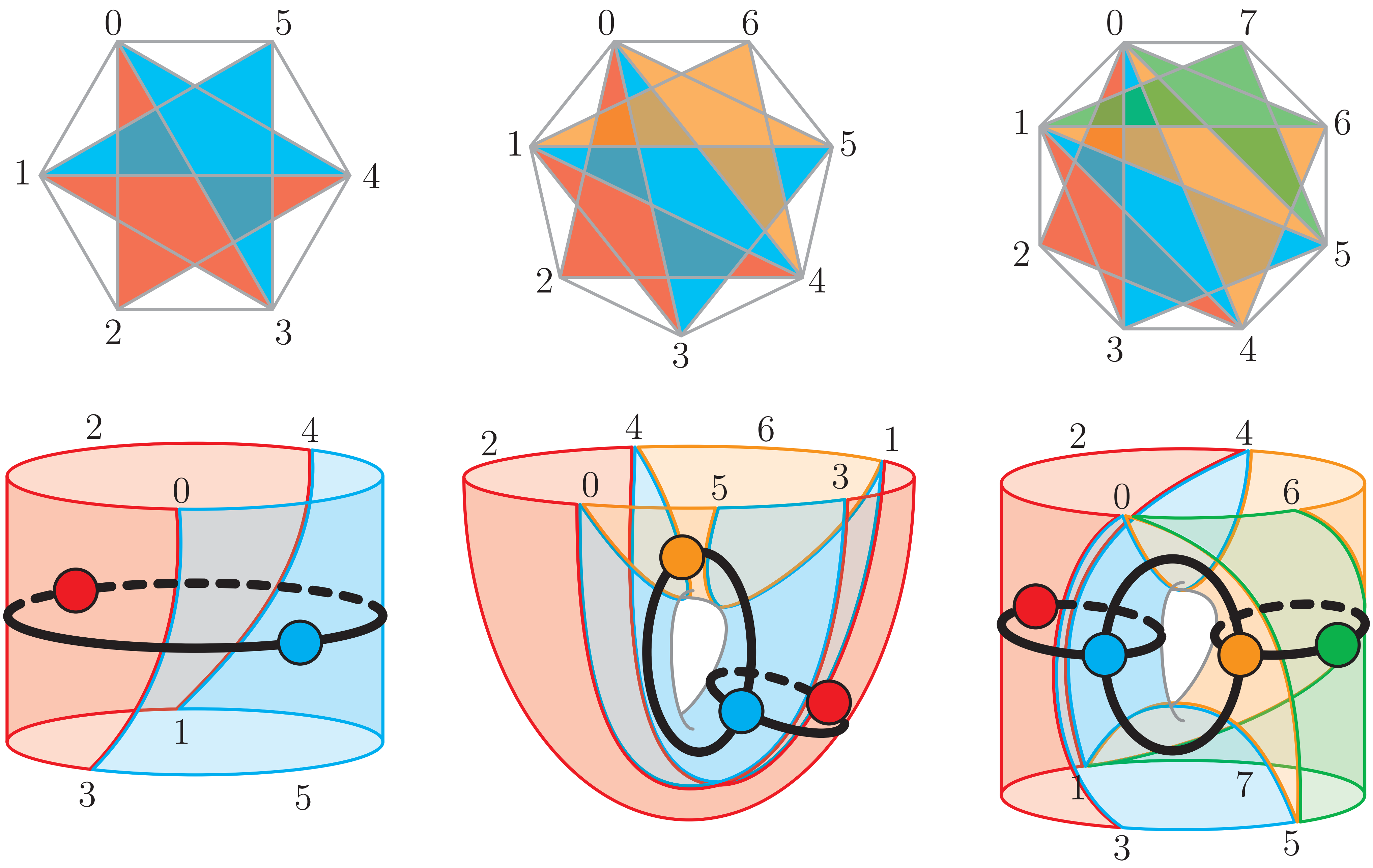


V. P. & F. Santos, Multitriangulations as complexes of star polygons, 2009.

DECOMPOSITIONS OF SURFACES

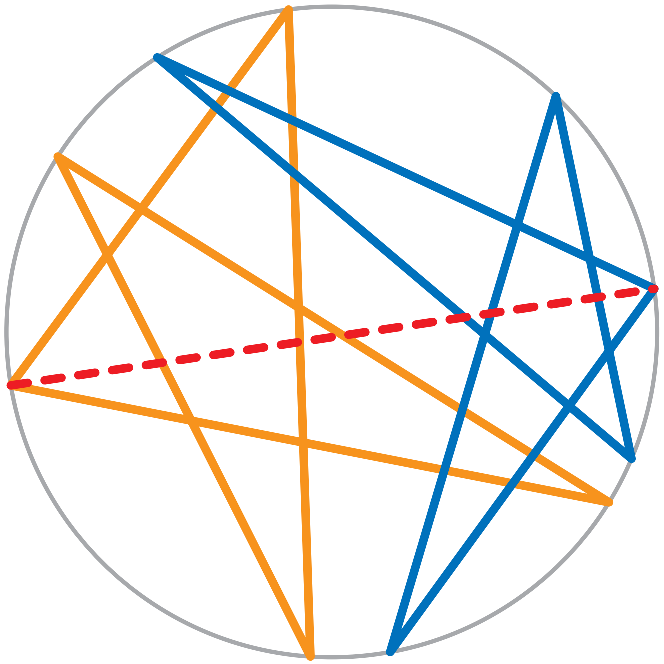


DUAL MULTIGRAPH



BISECTORS

THEOREM. T a k-triangulation of the n-gon. Every pair of k-stars of T have a unique common bisector. Reciprocally, any diagonal not in T is the common bisector of a unique pair of k-stars of T.

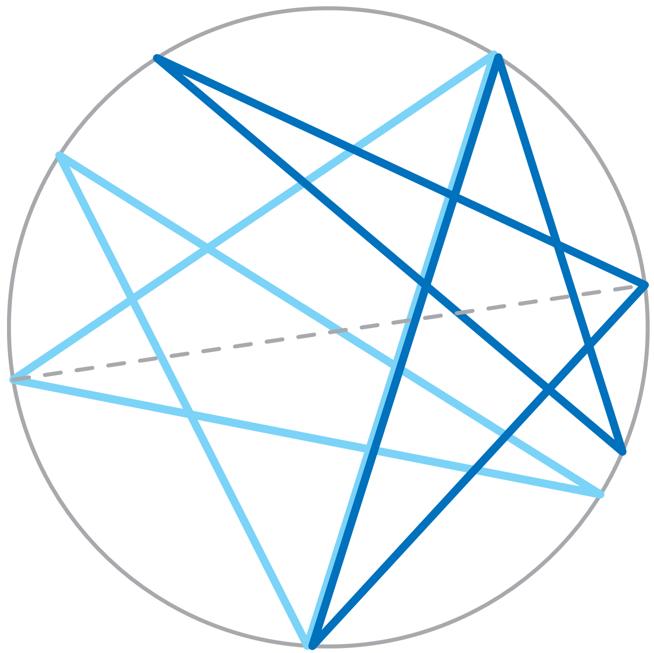


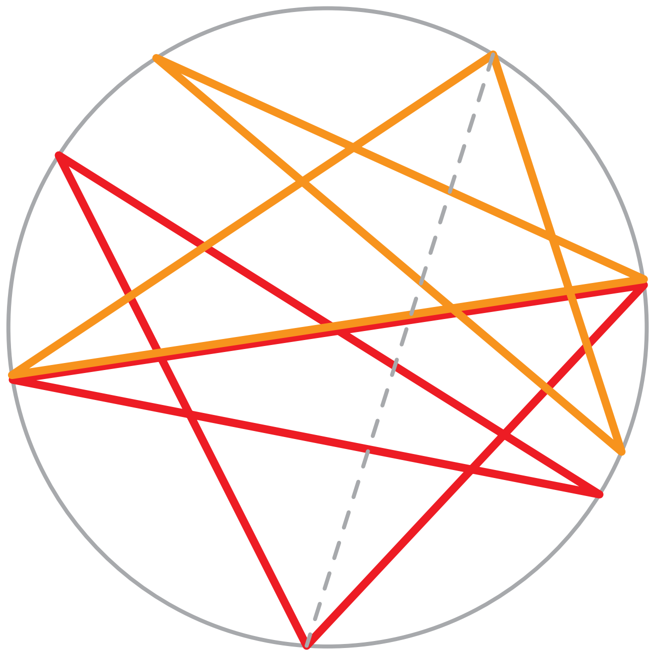
COROLLARY. Any k-triangulation of the n-gon contains exactly

$n - 2k$ k-stars and $k(2n - 2k - 1)$ diagonals.

FLIP GRAPH

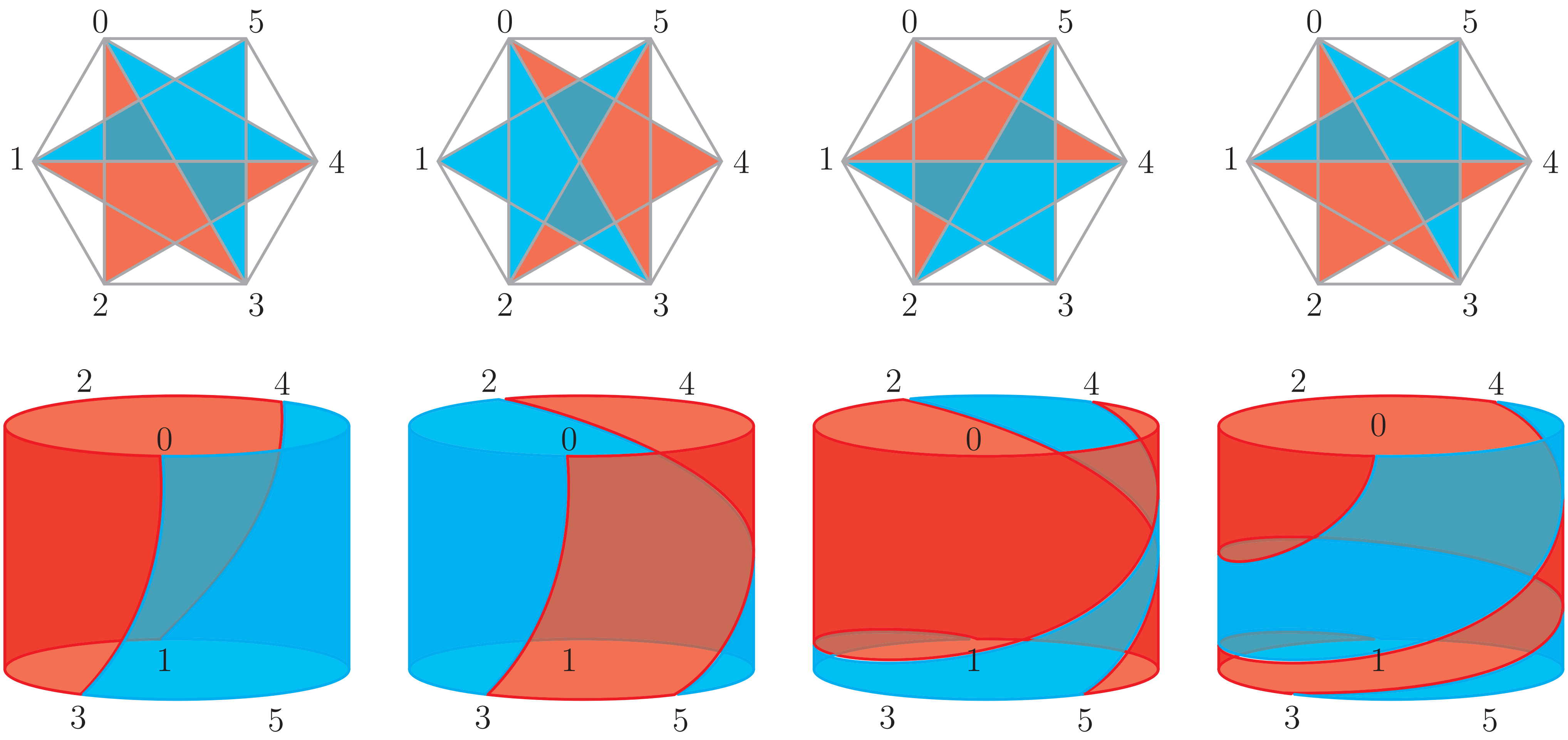
THEOREM. Let e be a k-relevant diagonal of a k-triangulation T, let R and S be the two k-stars of T containing e, and let f be the common bisector of R and S.
Then $T \triangle \{e, f\}$ is the only k-triangulation other than T containing $T \smallsetminus \{e\}$.





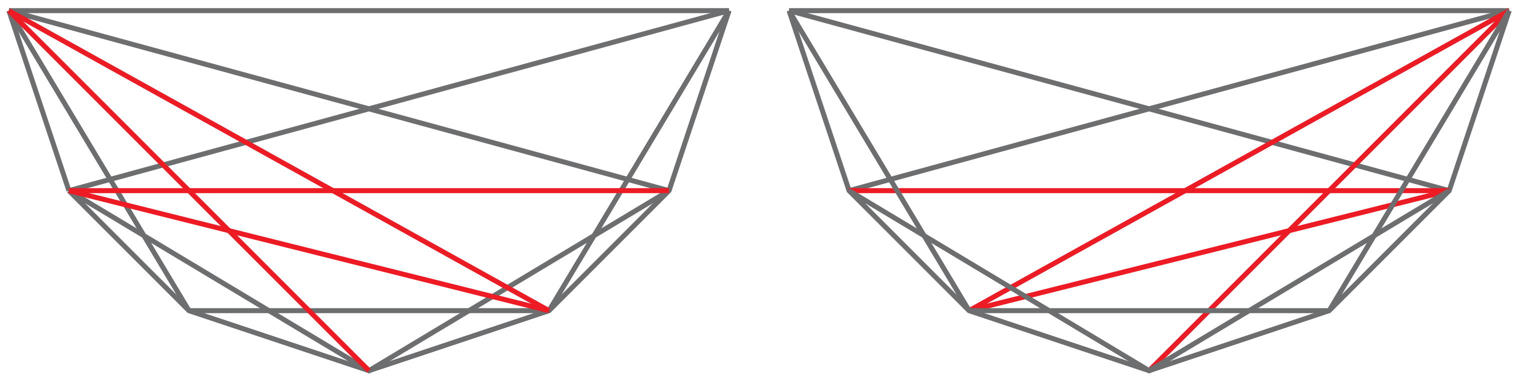
THEOREM. The graph of flips is connected, regular of degree $k(n-2k-1)$, and its diameter is at most $2k(n-2k-1)$.

FLIPS ON SURFACES



fundamental group of the flip graph $G_{n,k} \longmapsto$ mapping class group of the surface $\mathcal{S}_{n,k}$

Multi-tamari lattice



and their generalizations...

Diameter $\delta_{n,k}$ of the graph of flips on k-triangulations of the n-gon:

$$2\left\lfloor\frac{n}{2}\right\rfloor\left(k+\frac{1}{2}\right)-k(2k+3)\ \leq\ \delta_{n,k}\ \leq\ 2k(n-4k-1),$$

when $n>4k^2(2k+1)$.

3. Multiassociahedron

THE (POLYTOPAL?) SIMPLICIAL COMPLEX $\Delta_{n,k}$

$k \geq 1$ and $n \geq 2k+1$ two fixed integers.

ℓ-crossing $=$ set of ℓ mutually crossing diagonals of the convex n-gon.

k-relevant diagonal $=$ at least k vertices on each side
$=$ diagonals which may appear in a $(k+1)$-crossing.

$\Delta_{n,k}$ $=$ simplicial complex of $(k+1)$-crossing-free sets of k-relevant diagonals of the convex n-gon.

THEOREM. $\Delta_{n,k}$ is a topological sphere of dimension $k(n-2k-1)-1$.

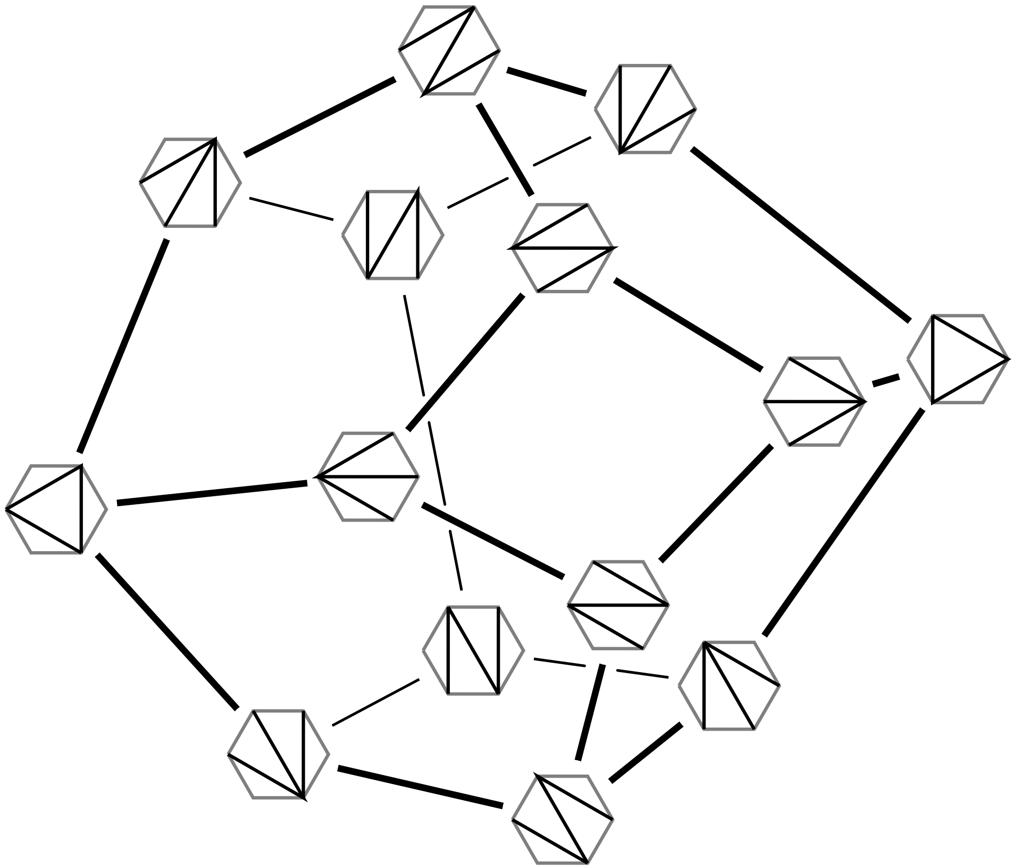
J. Jonsson, Generalized triangulations of the n-gon, 2003.

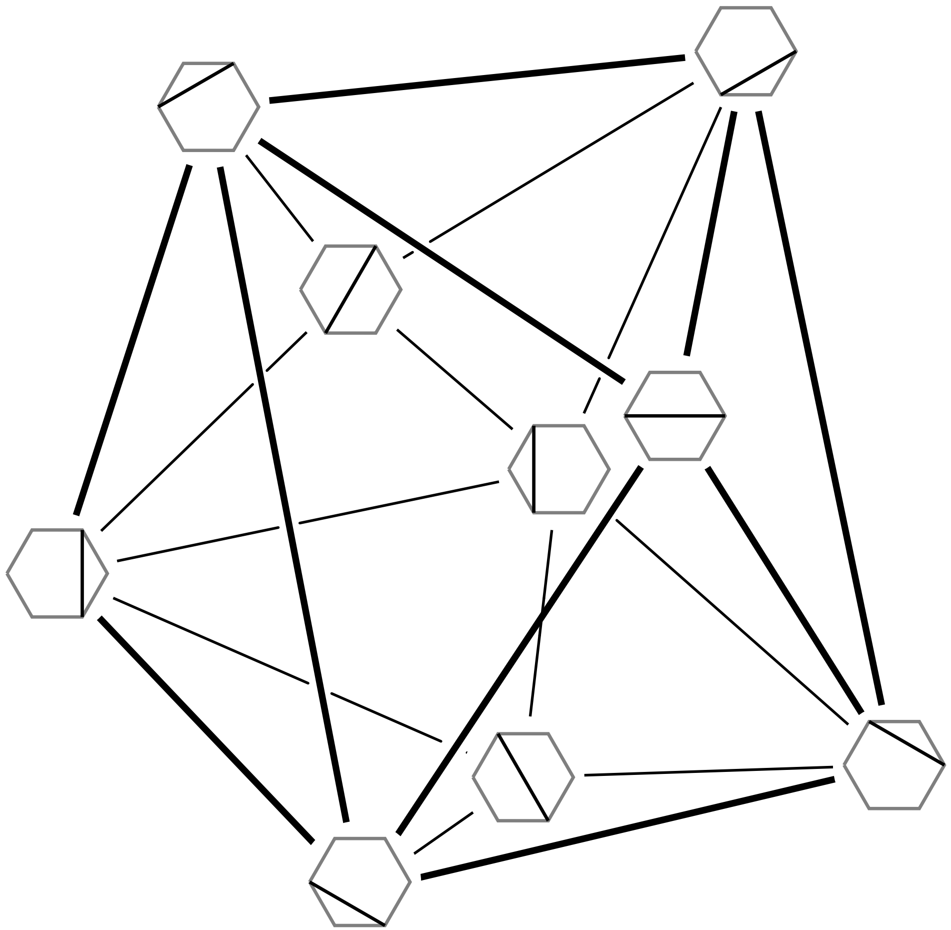
QUESTION. Is $\Delta_{n,k}$ the boundary complex of a simplicial $k(n-2k-1)$-polytope?

ASSOCIAHEDRON

$\boxed{k=1}$ Maximal elements of $\Delta_{n,1}$ = triangulations of the n-gon.

$\Delta_{n,1}$ = boundary complex of the dual of the $(n-3)$-dimensional associahedron.





L.J. Billera, P. Filliman & B. Sturmfels, Constructions and complexity of secondary polytopes, 1990.

ASSOCIAHEDRON

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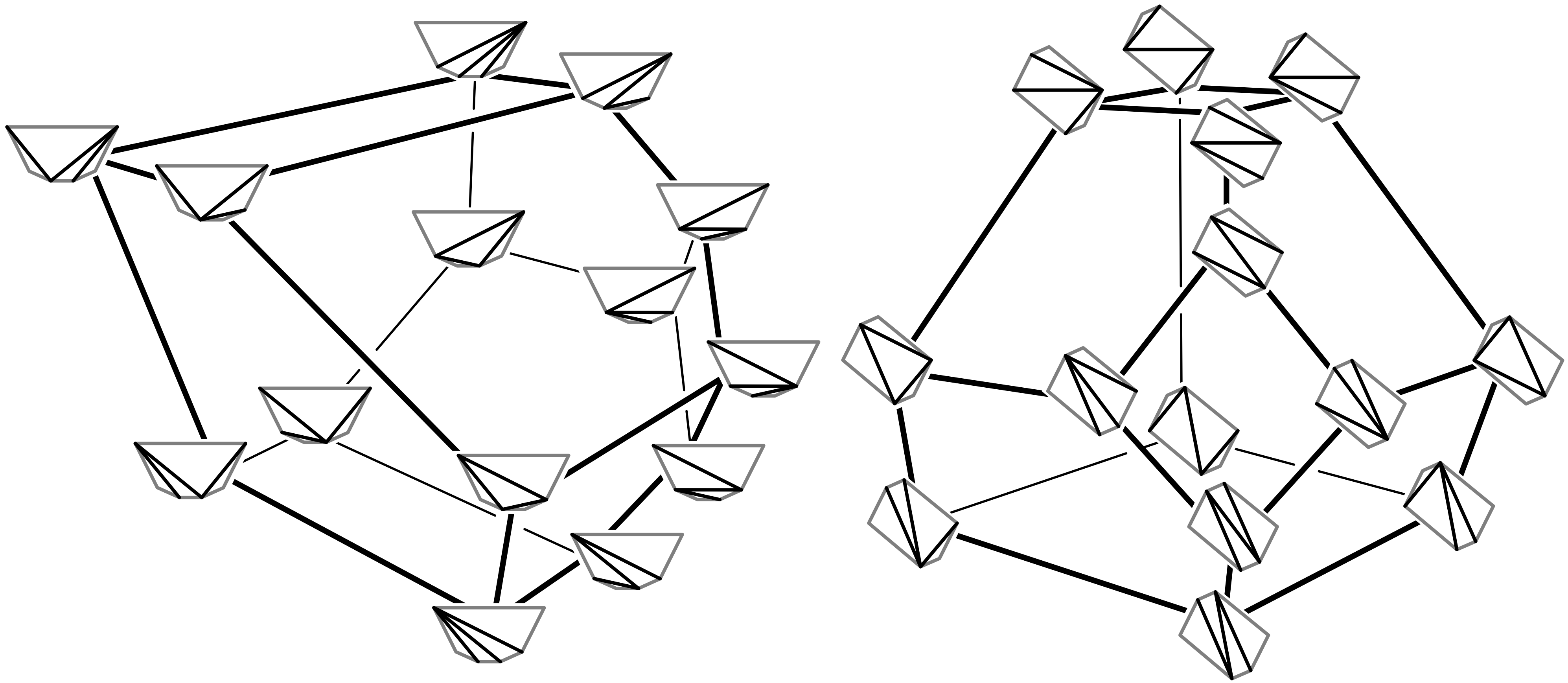

J.-L. Loday, Realization of the Stasheff polytope, 2004.

C. Hohlweg & C. Lange, Realizations of the associahedron and cyclohedron, 2007.

OTHER EXAMPLES

$\boxed{k = 1}$ Maximal elements of $\Delta_{n,1}$ = triangulations of the n-gon.
$\Delta_{n,1}$ = boundary complex of the dual of the $(n-3)$-dimensional associahedron.

$\boxed{n = 2k+1}$ $\Delta_{2k+1,k}$ = single k-triangulation.

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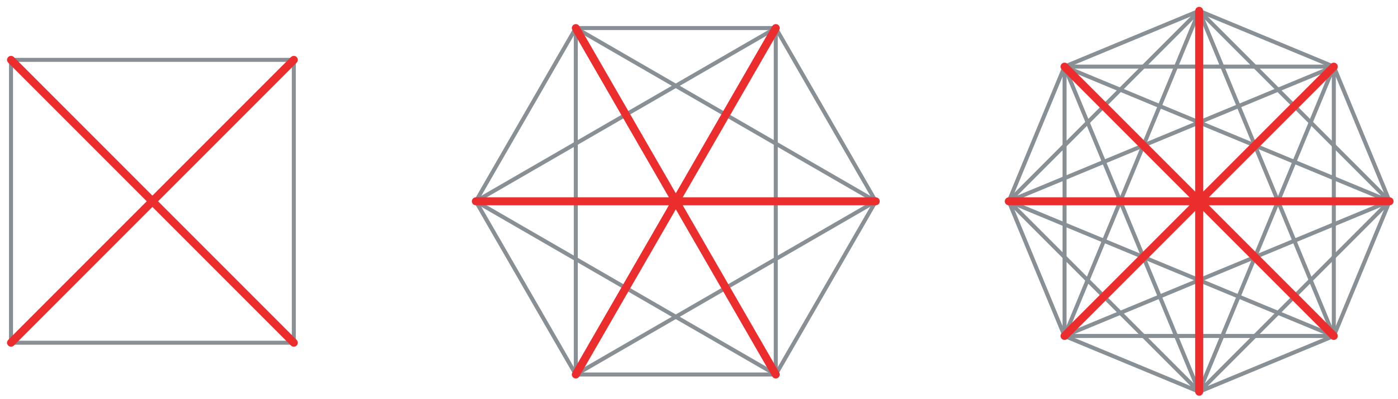

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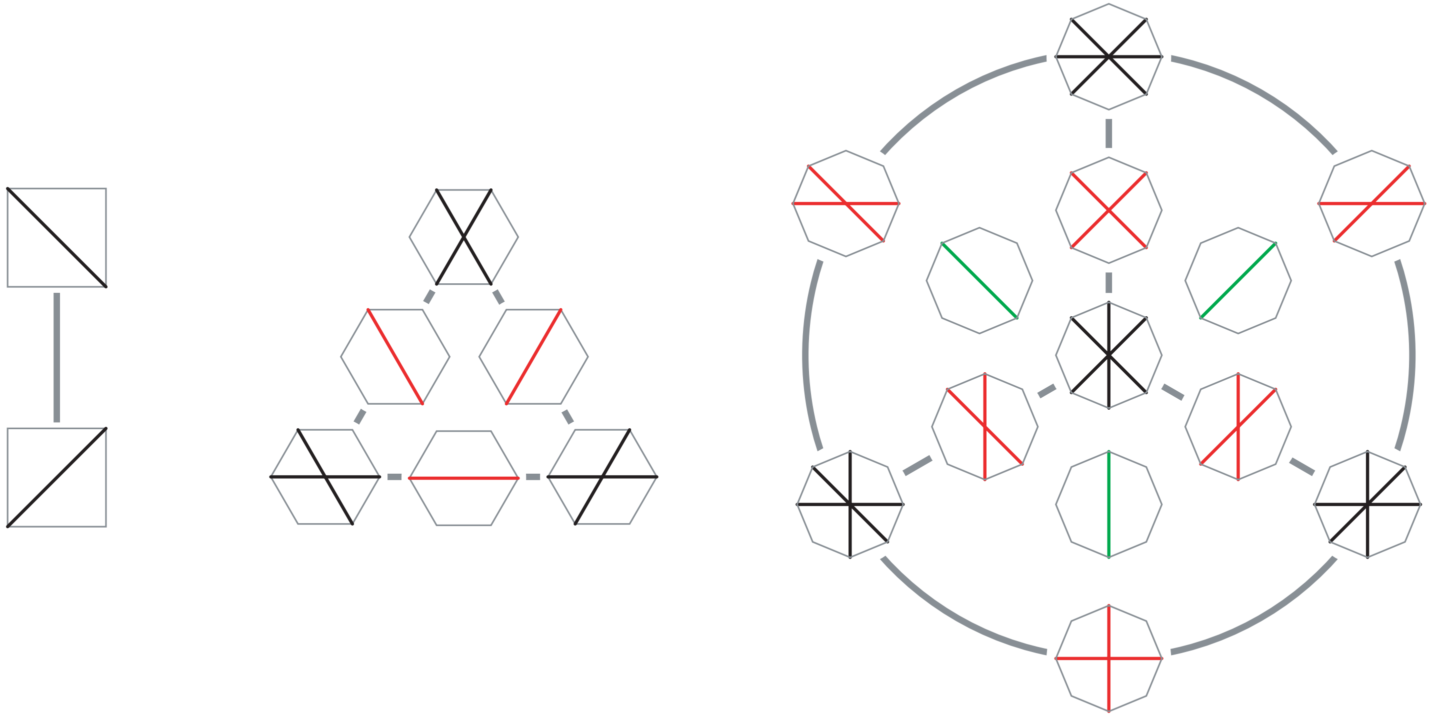

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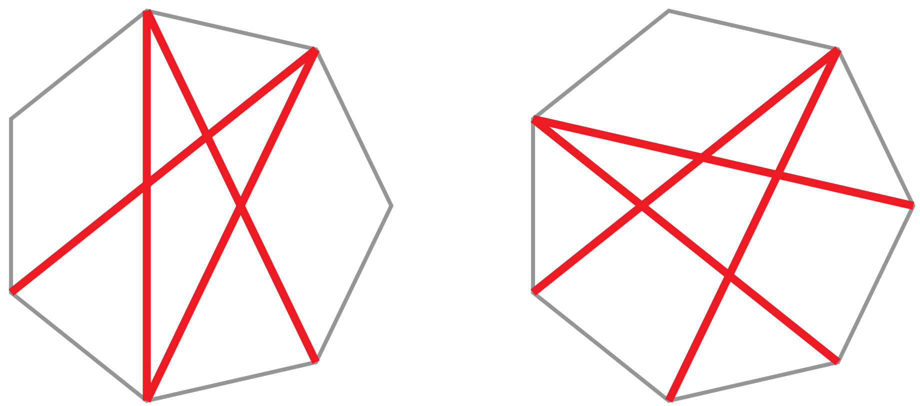
$\boxed{n=2k+3}$ $\Delta_{2k+3,k}$ = boundary complex of the cyclic polytope of dimension $2k$ with $2k+3$ vertices.



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$\boxed{n=8 \ \& \ k=2}$

f-vector of $\Delta_{8,2} = (12, 66, 192, 306, 252, 84)$

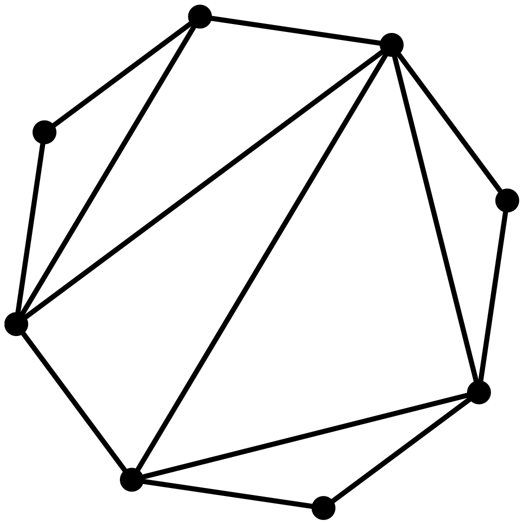
THEOREM. The space of symmetric realizations of $\Delta_{8,2}$ has dimension 4.

J. Bokowski & V. P., On symmetric realizations of the simplicial complex of 3-crossing-free sets of diagonals of the octagon, 2009.

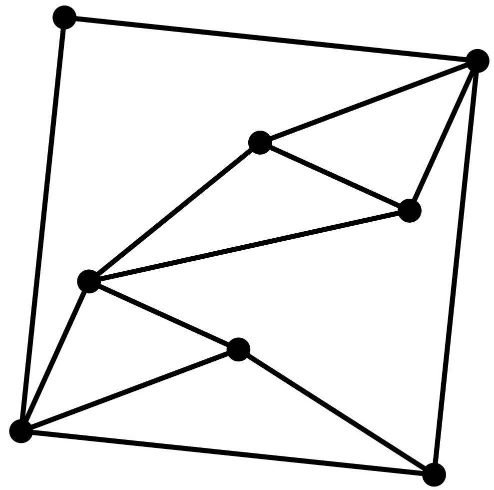
4. Multipseudotriangulations

THREE GEOMETRIC STRUCTURES

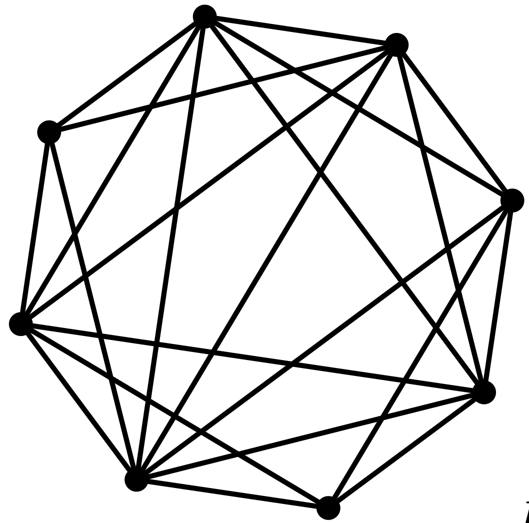
Triangulations



Pseudotriangulations



Multitriangulations



$k = 2$

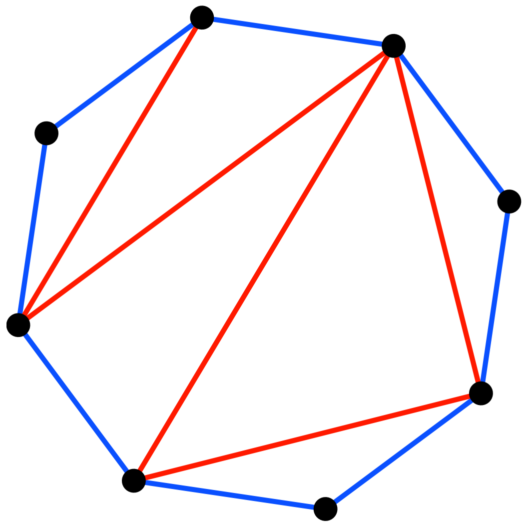
triangulation $=$ maximal crossing-free set of edges

pseudotriangulation $=$ maximal crossing-free pointed set of edges

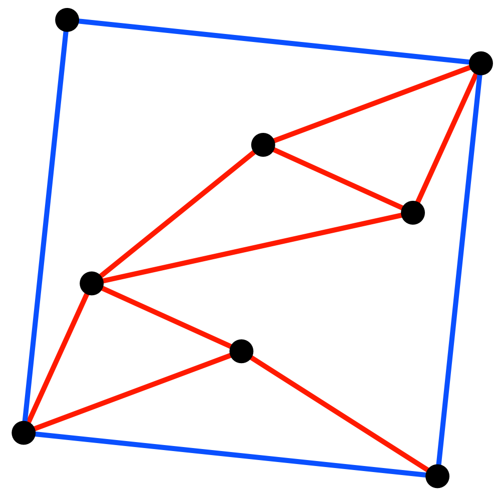
k-triangulation $=$ maximal $(k+1)$-crossing-free set of edges

THREE GEOMETRIC STRUCTURES

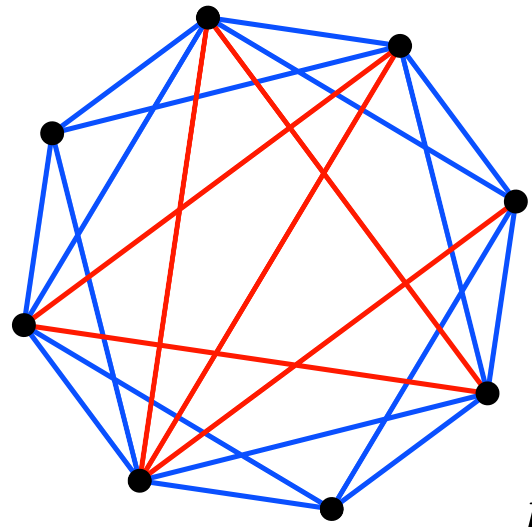
Triangulations



Pseudotriangulations



Multitriangulations



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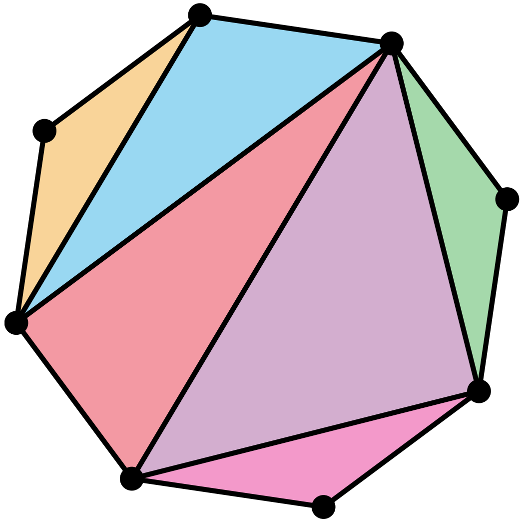
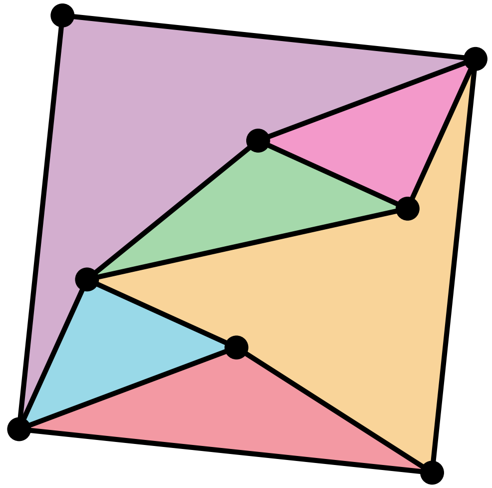
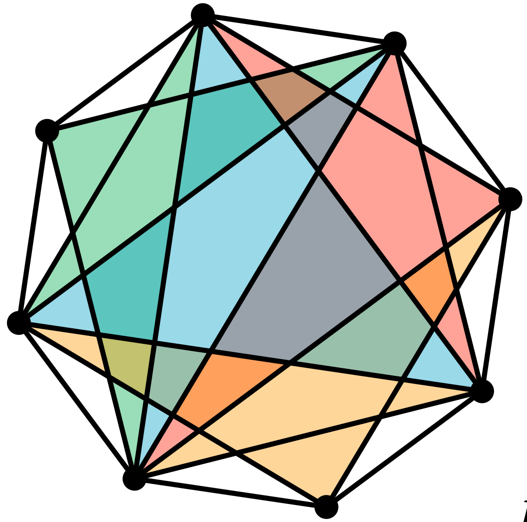
Triangulations



Pseudotriangulations



Multitriangulations



$k = 2$

triangulation = maximal crossing-free set of edges
= decomposition into triangles

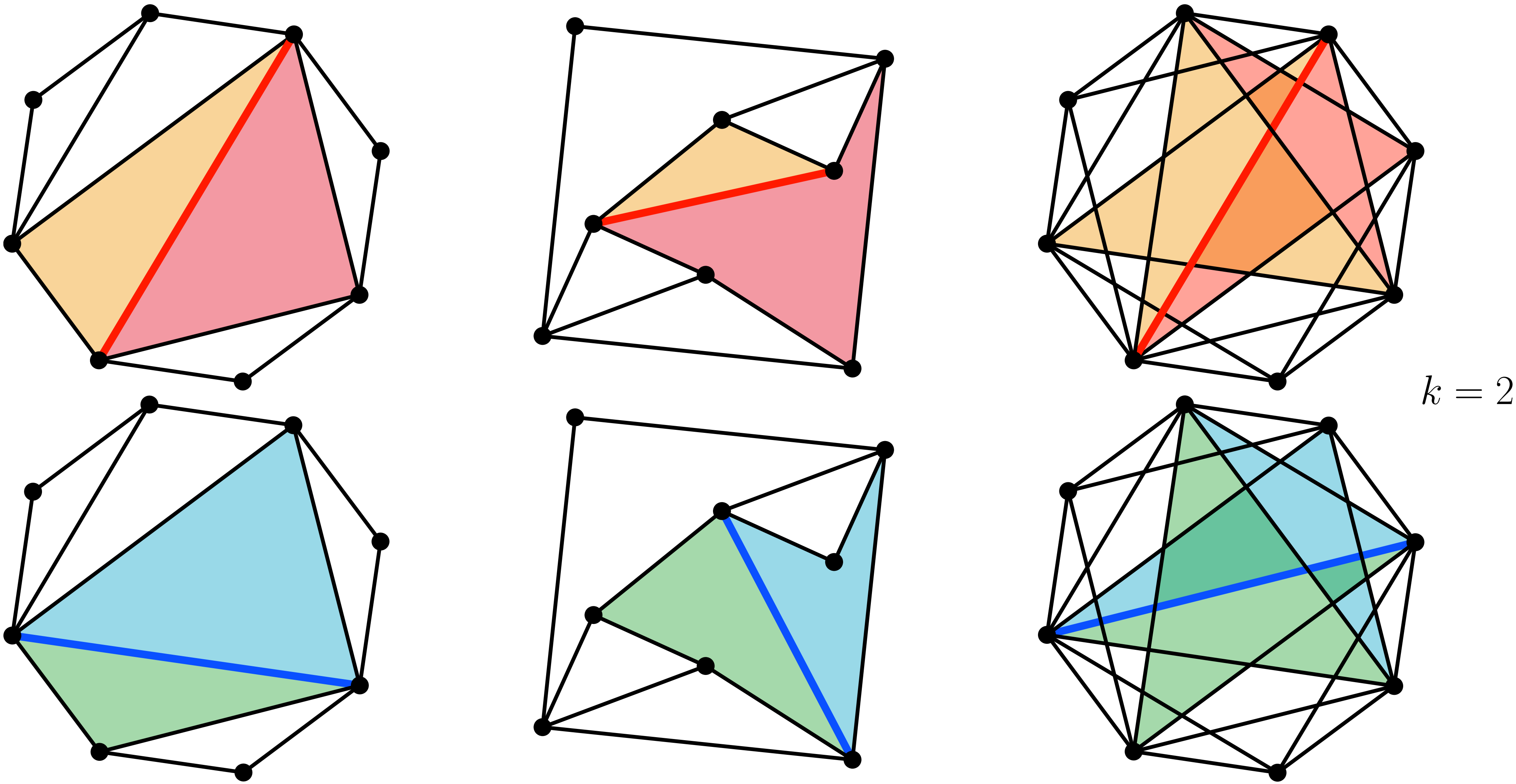
pseudotriangulation = maximal crossing-free pointed set of edges
= decomposition into pseudotriangles

k-triangulation = maximal $(k + 1)$-crossing-free set of edges
= decomposition into k-stars

VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.

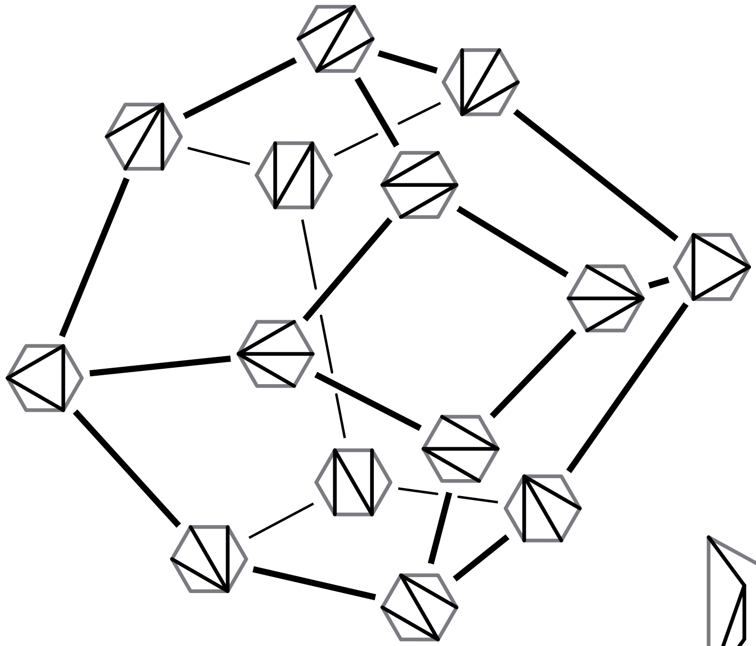
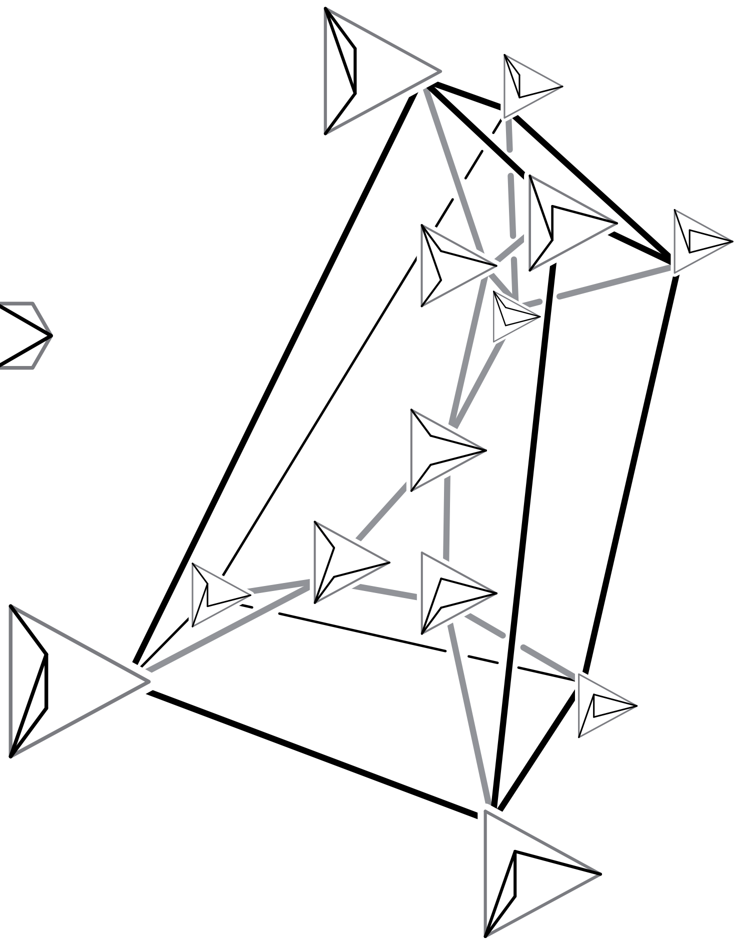
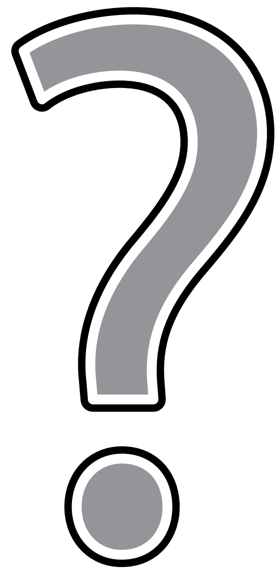
THREE GEOMETRIC STRUCTURES

Triangulations — Pseudotriangulations — Multitriangulations



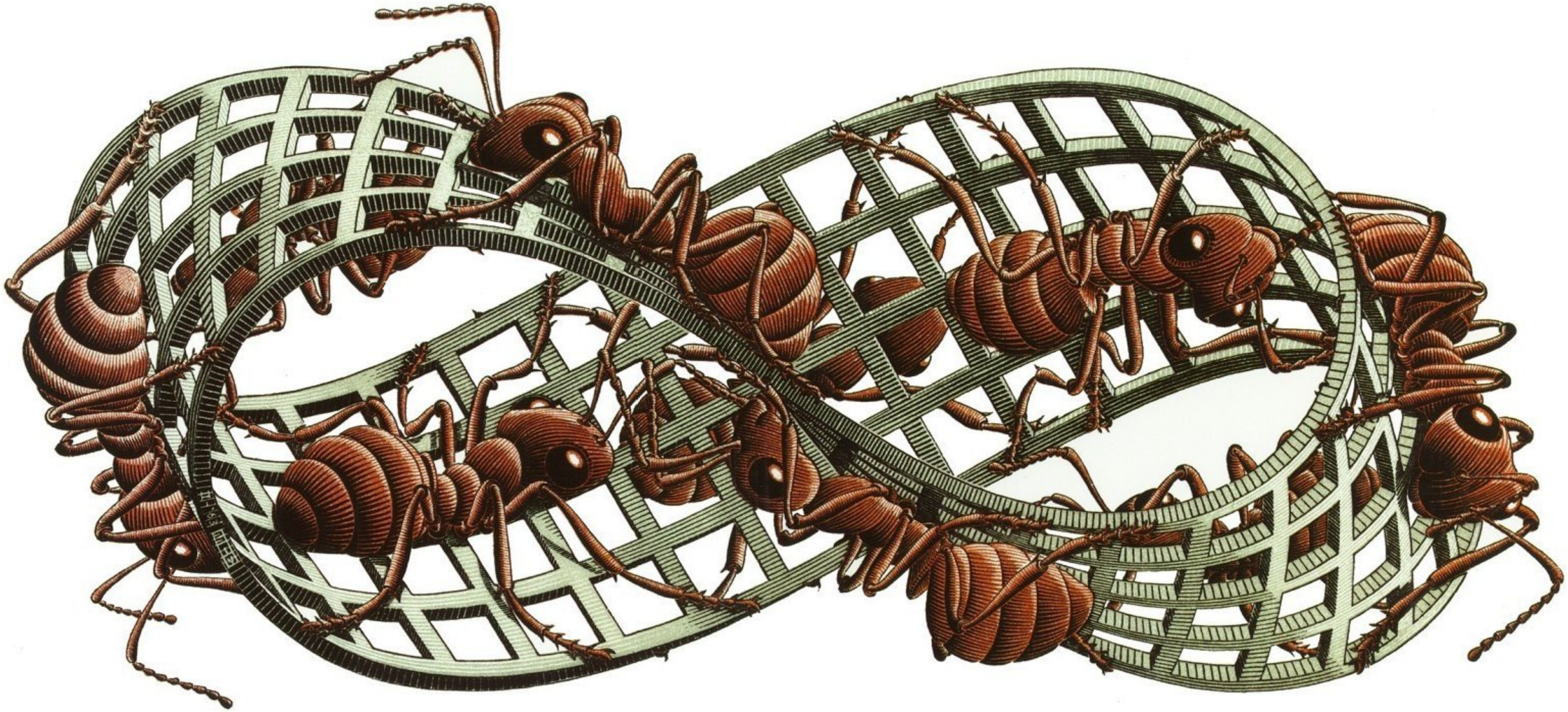
flip $=$ exchange an internal edge with the common bisector of the two adjacent cells

THREE GEOMETRIC STRUCTURES

Triangulations	Pseudotriangulations	Multitriangulations
		

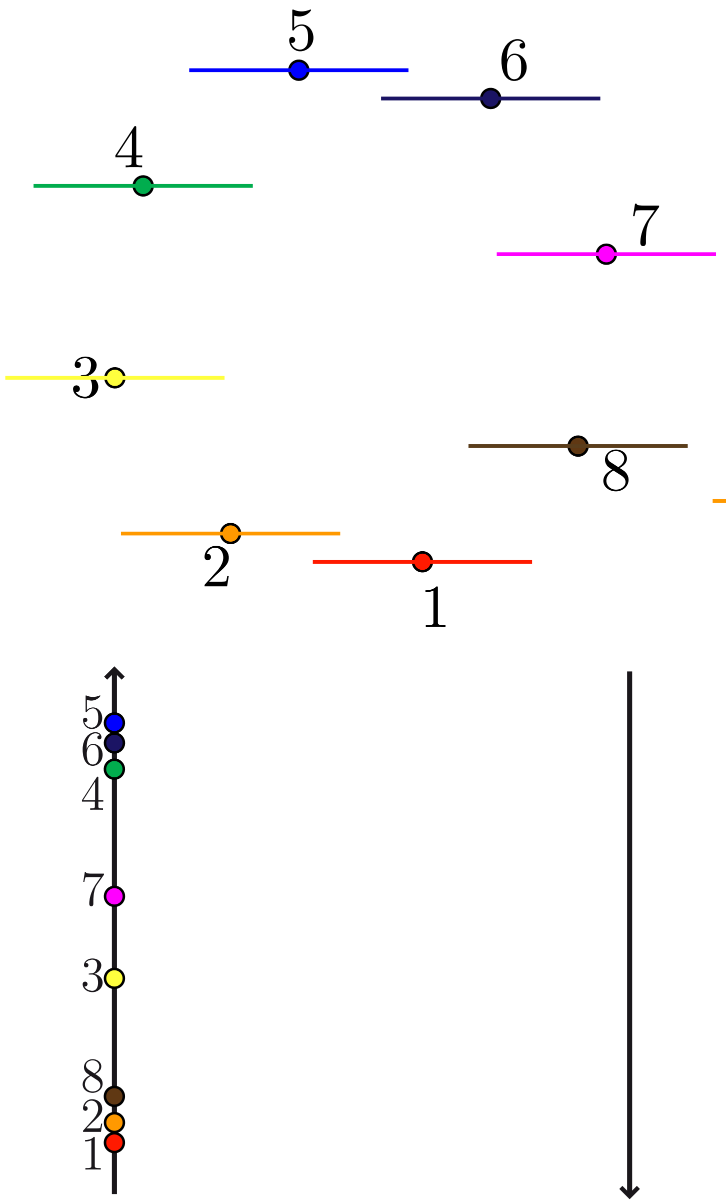
associahedron	$\longleftrightarrow$	crossing-free sets of internal edges
pseudotriangulations polytope	$\longleftrightarrow$	pointed crossing-free sets of internal edges
multiassociahedron	$\longleftrightarrow$	$(k+1)$-crossing-free sets of k-internal edges

LINE SPACE OF THE PLANE = MÖBIUS STRIP

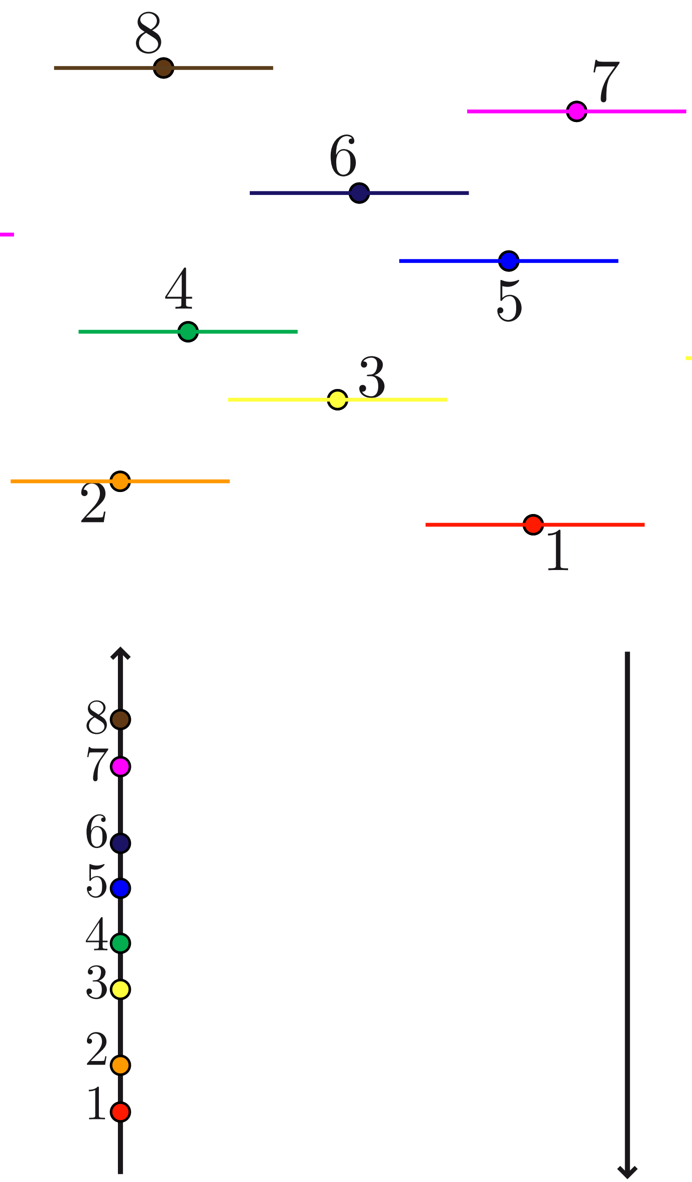


DUALITY

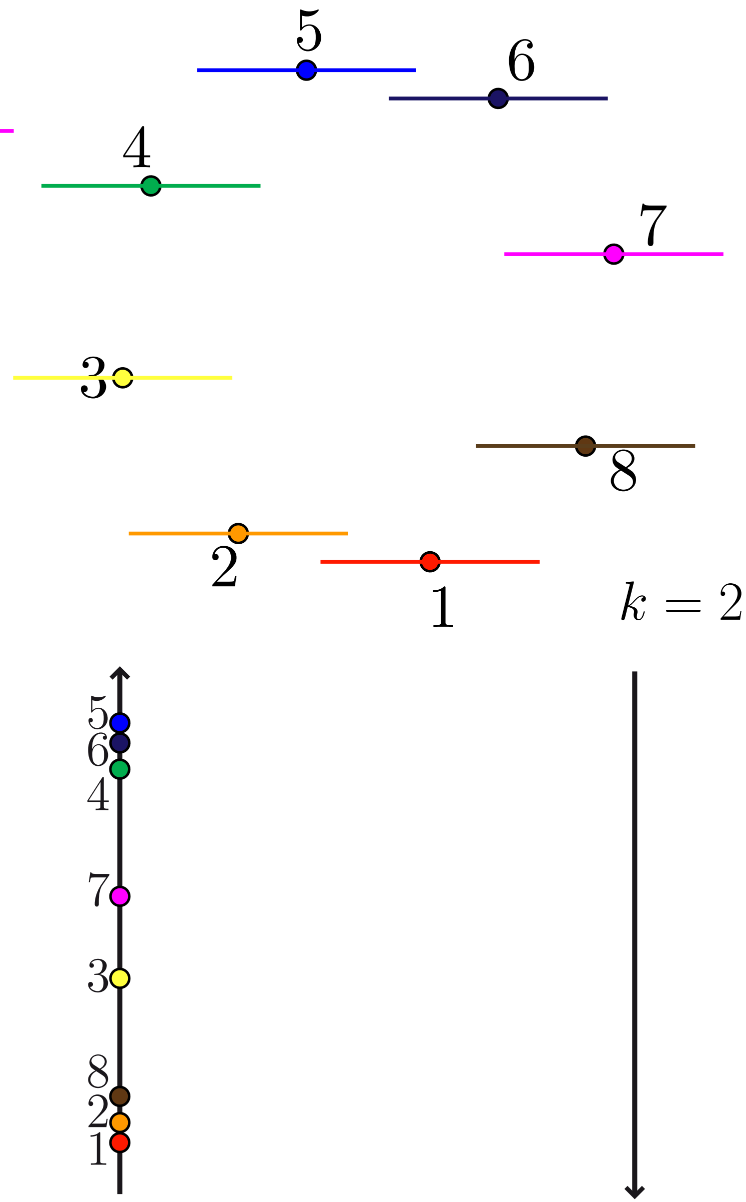
Triangulations



Pseudotriangulations



Multitriangulations



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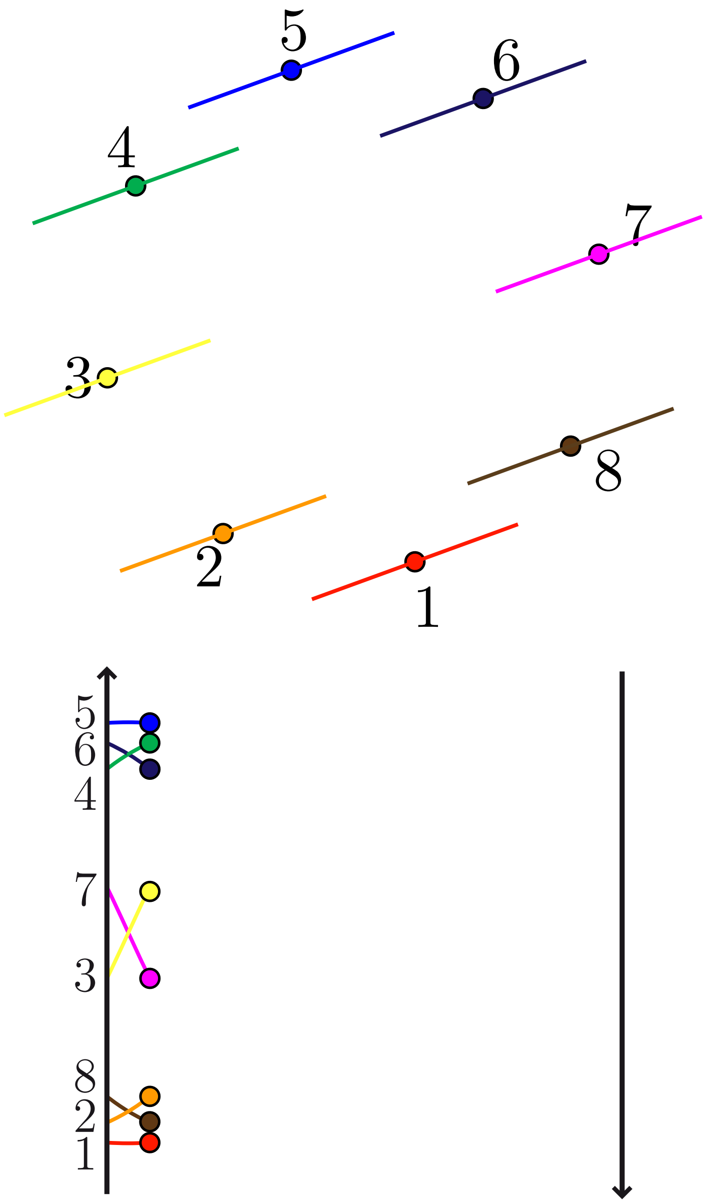

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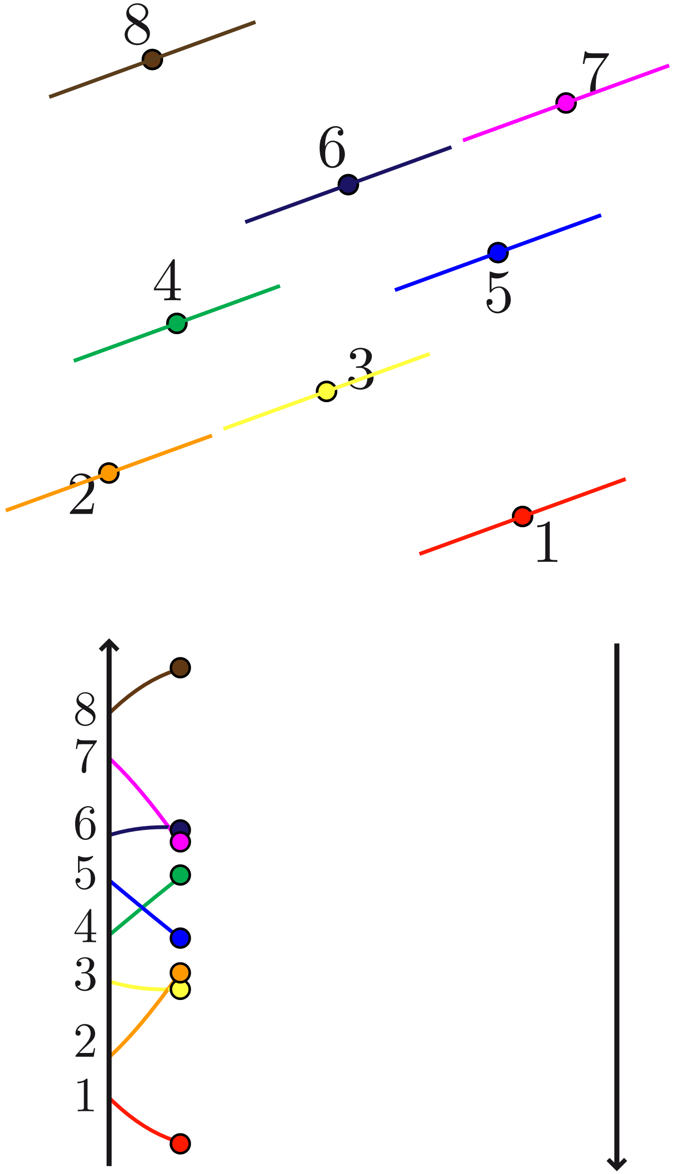
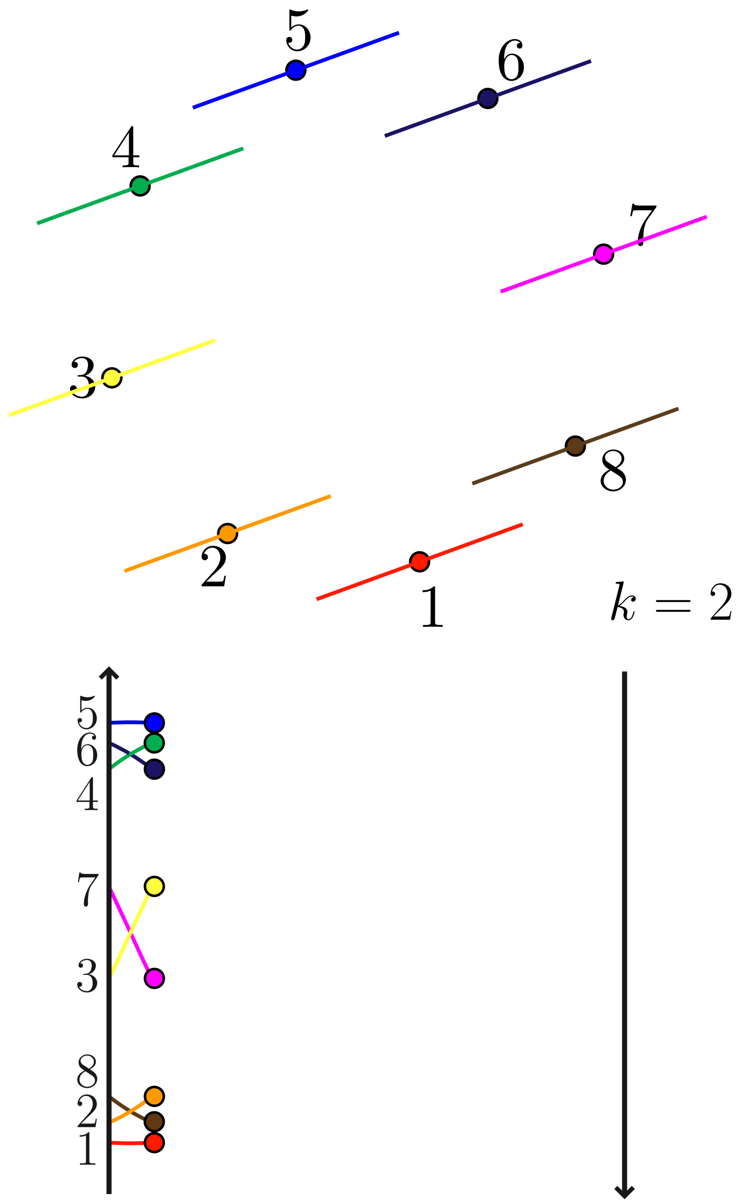


Multitriangulations



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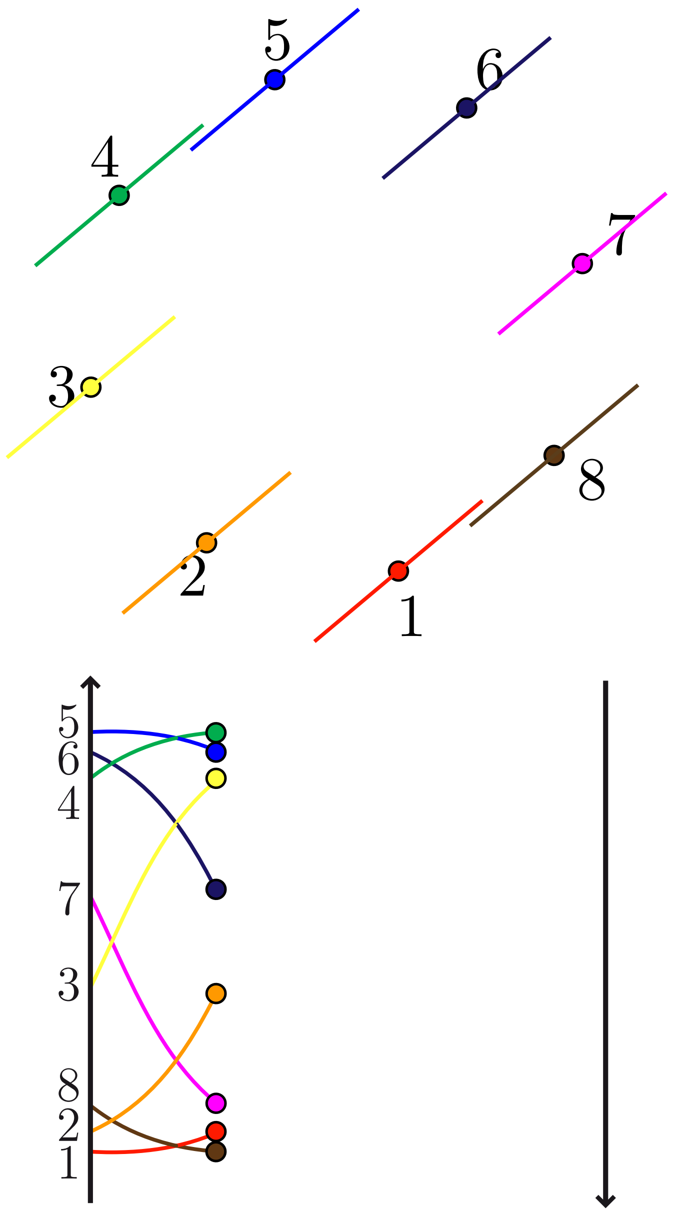
Triangulations



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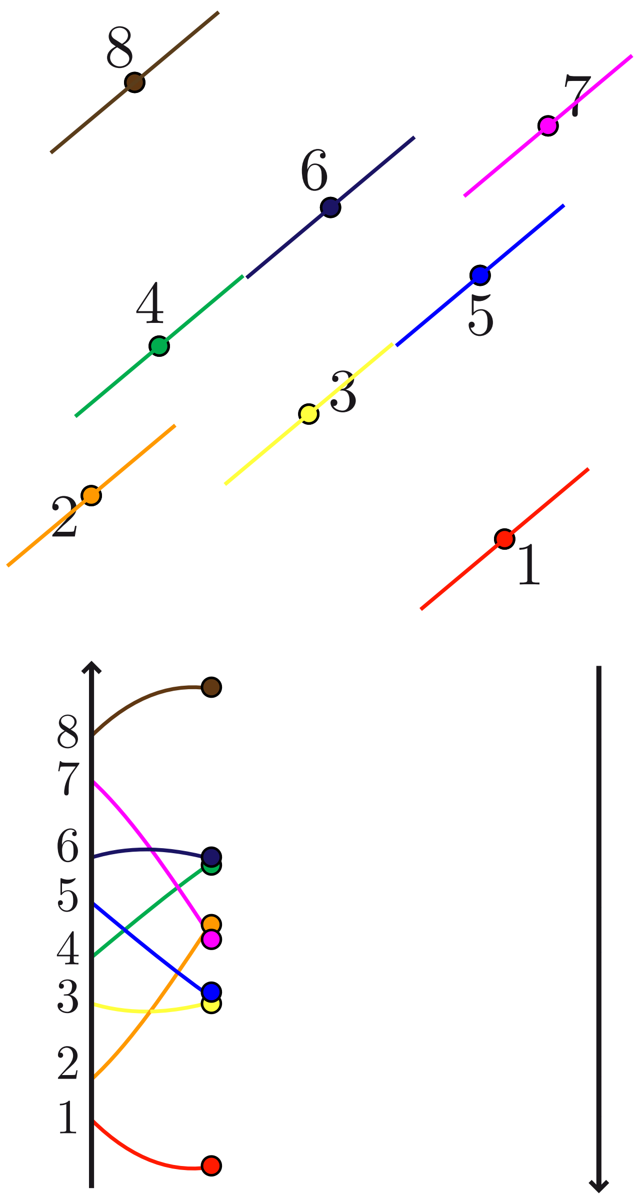

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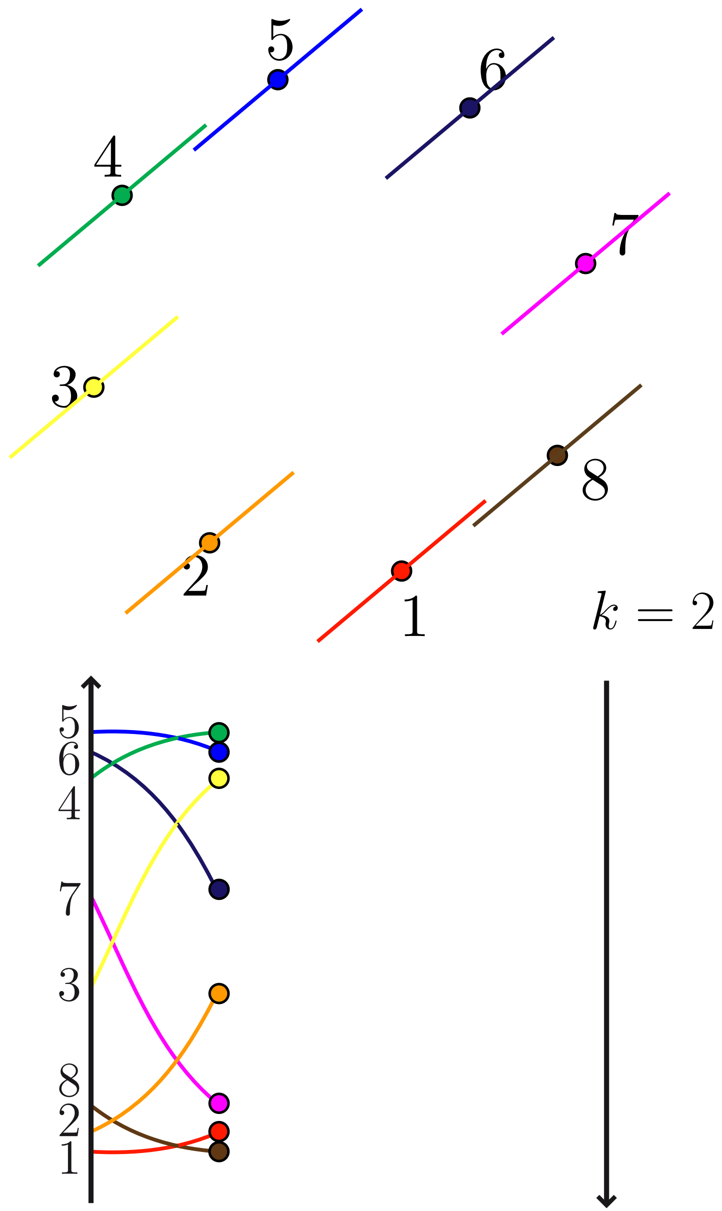

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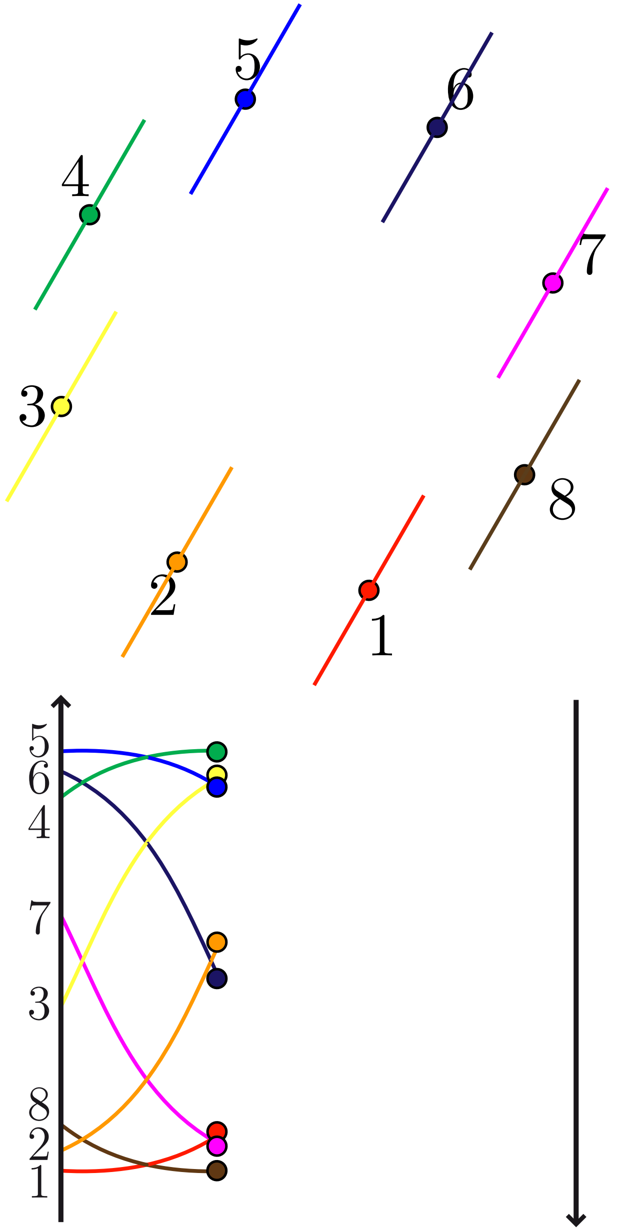

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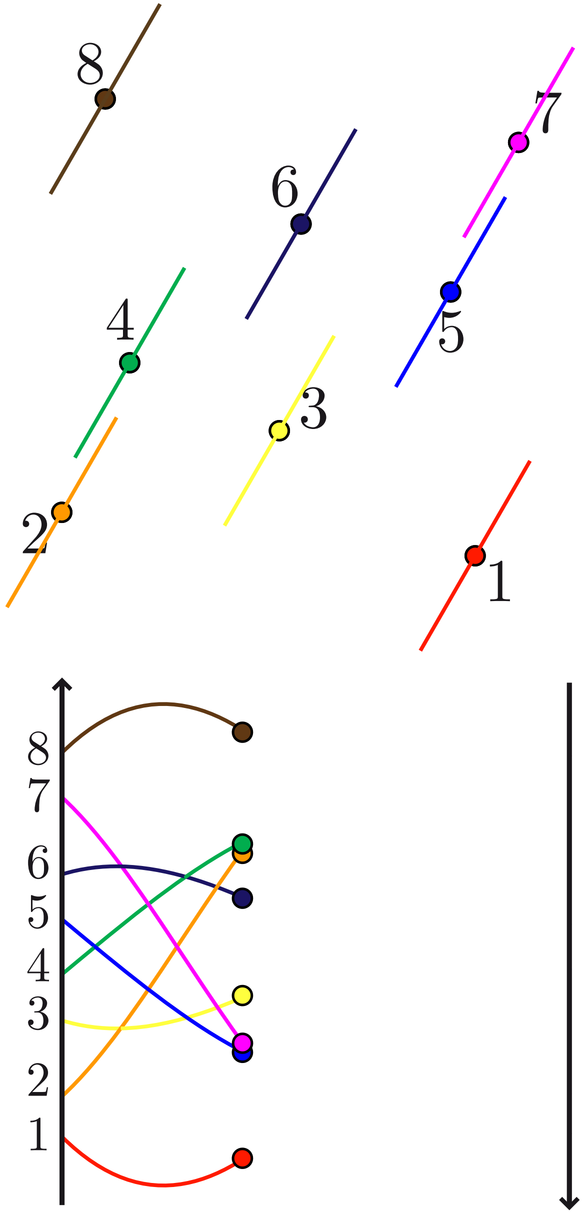

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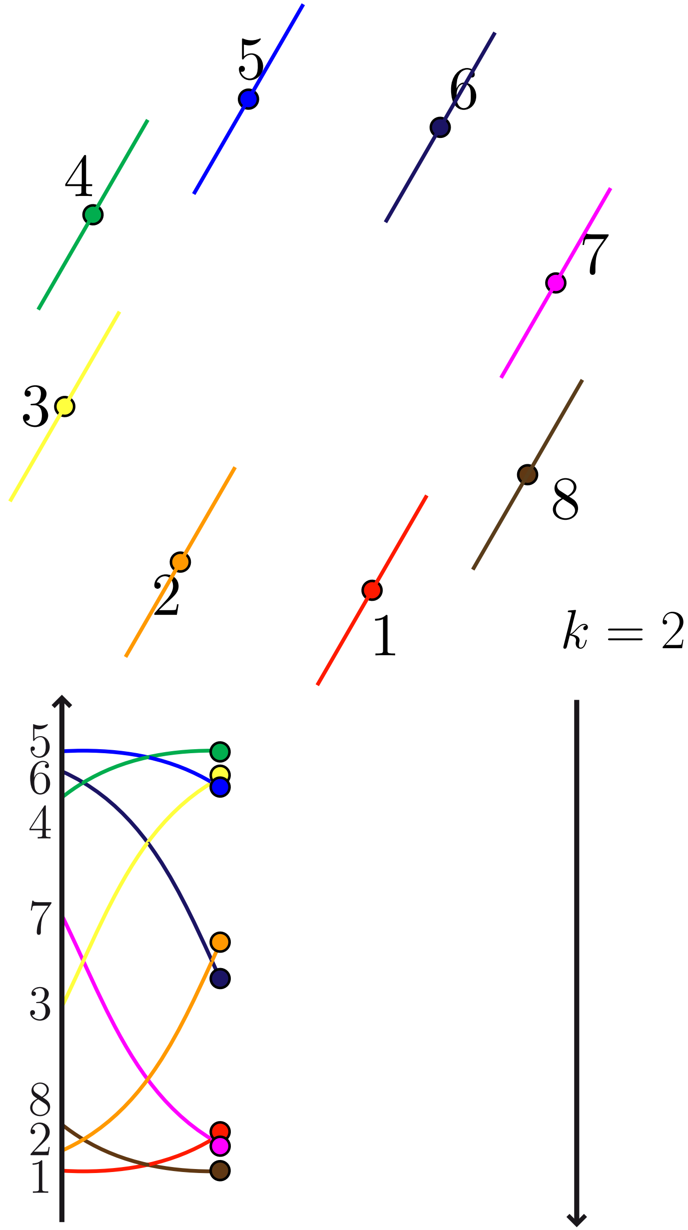

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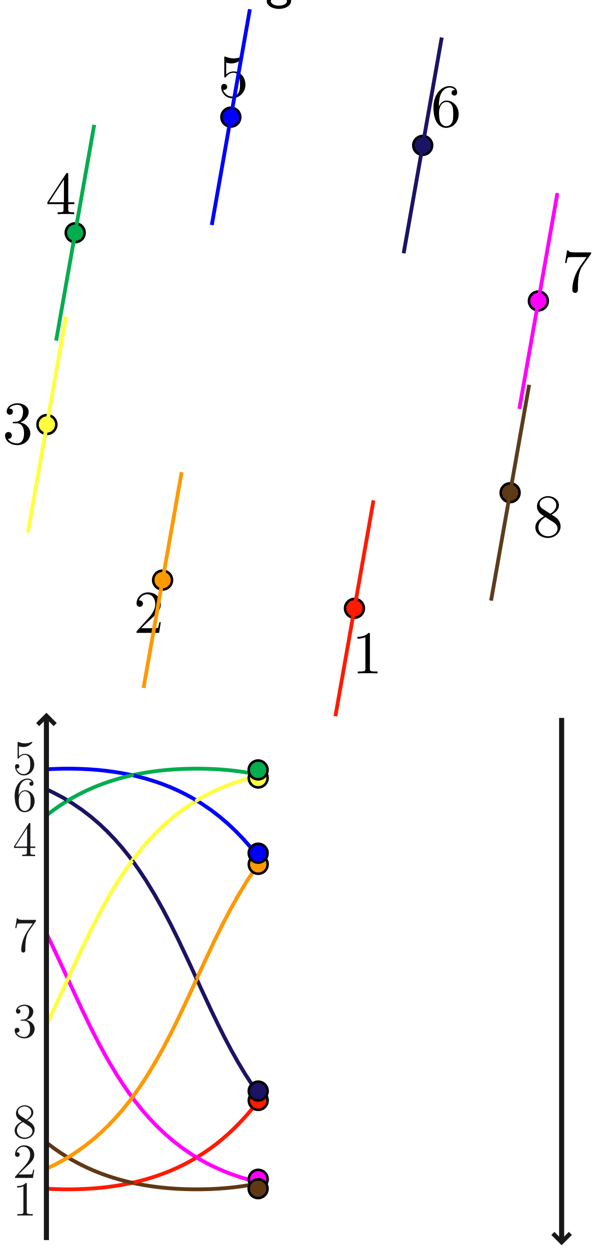

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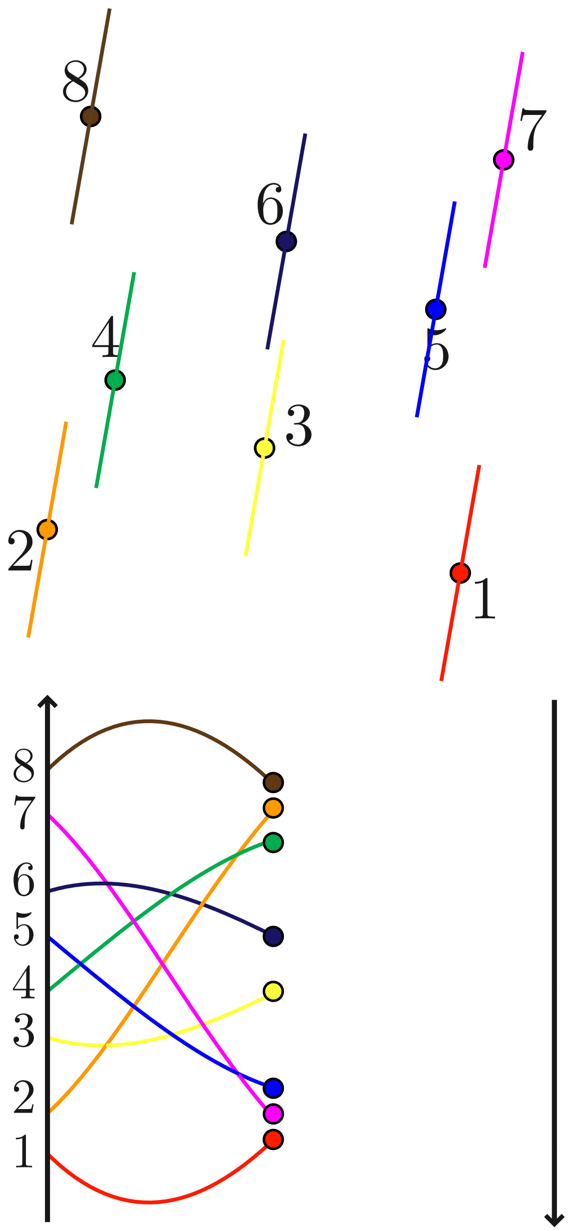

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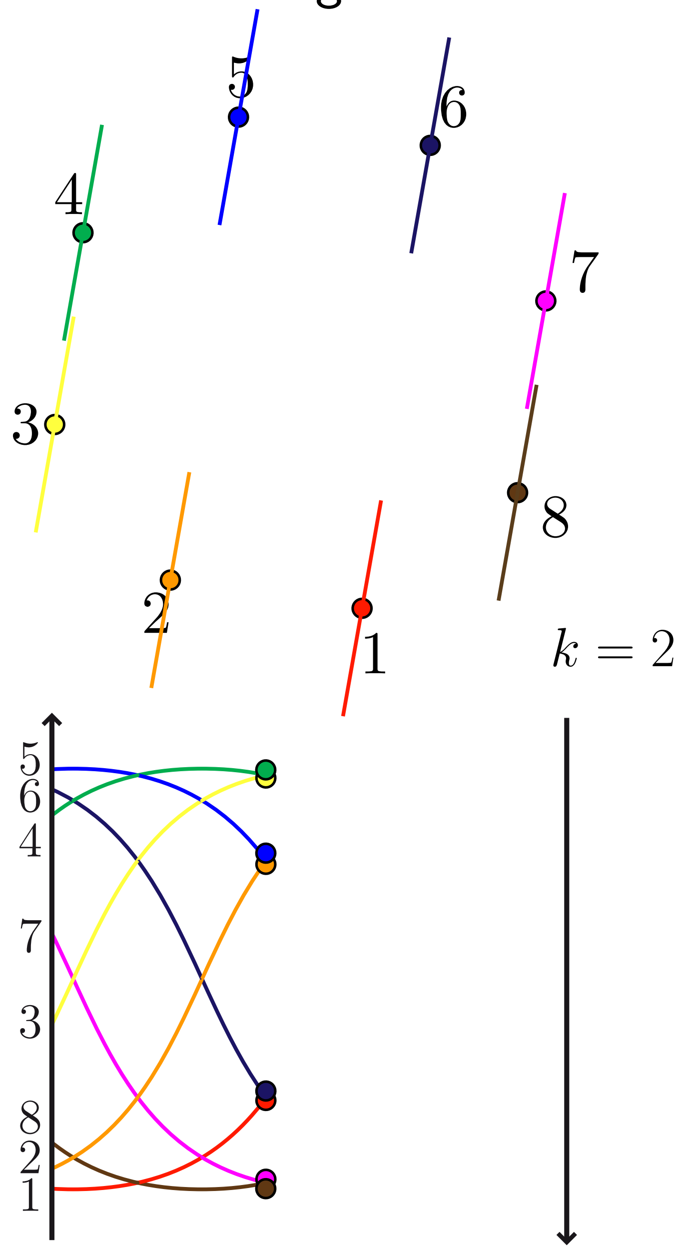

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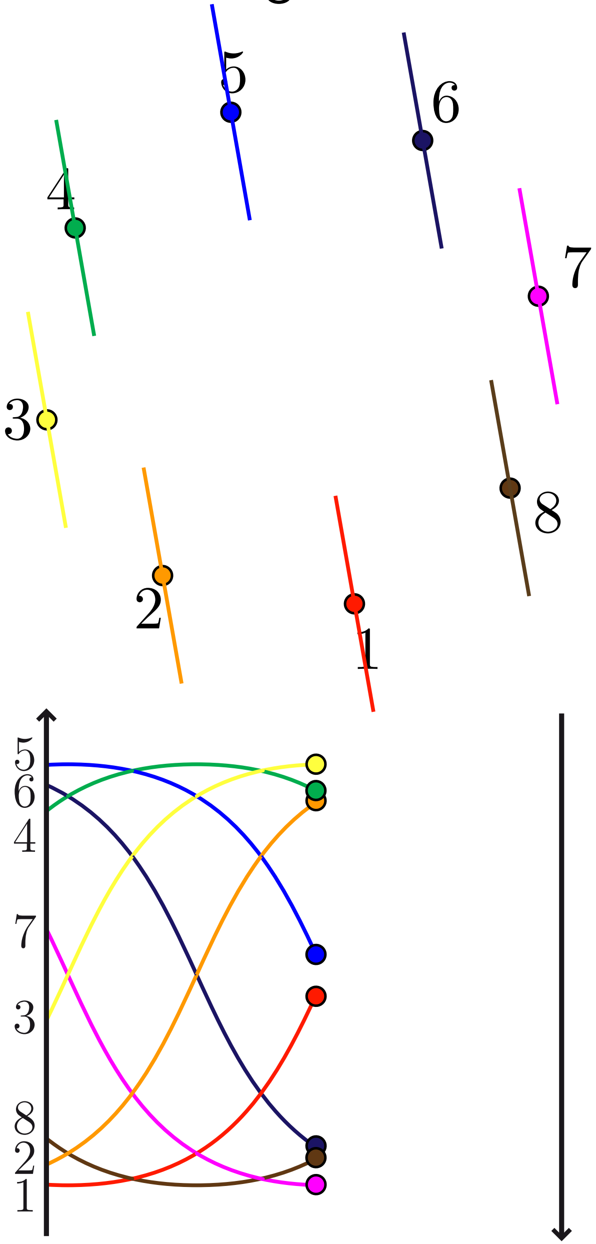

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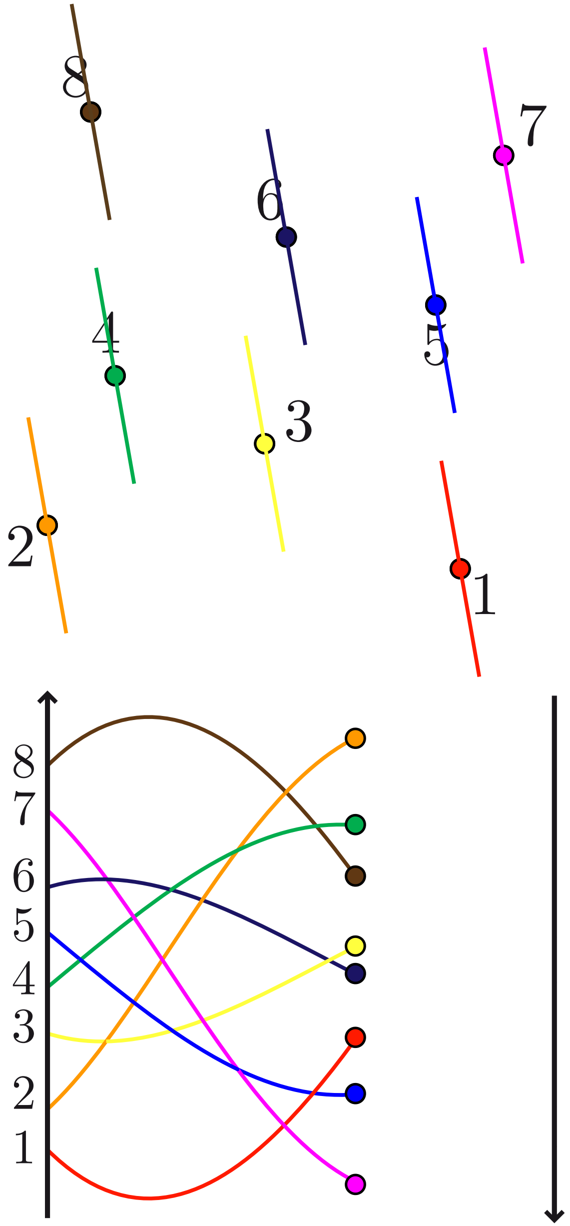

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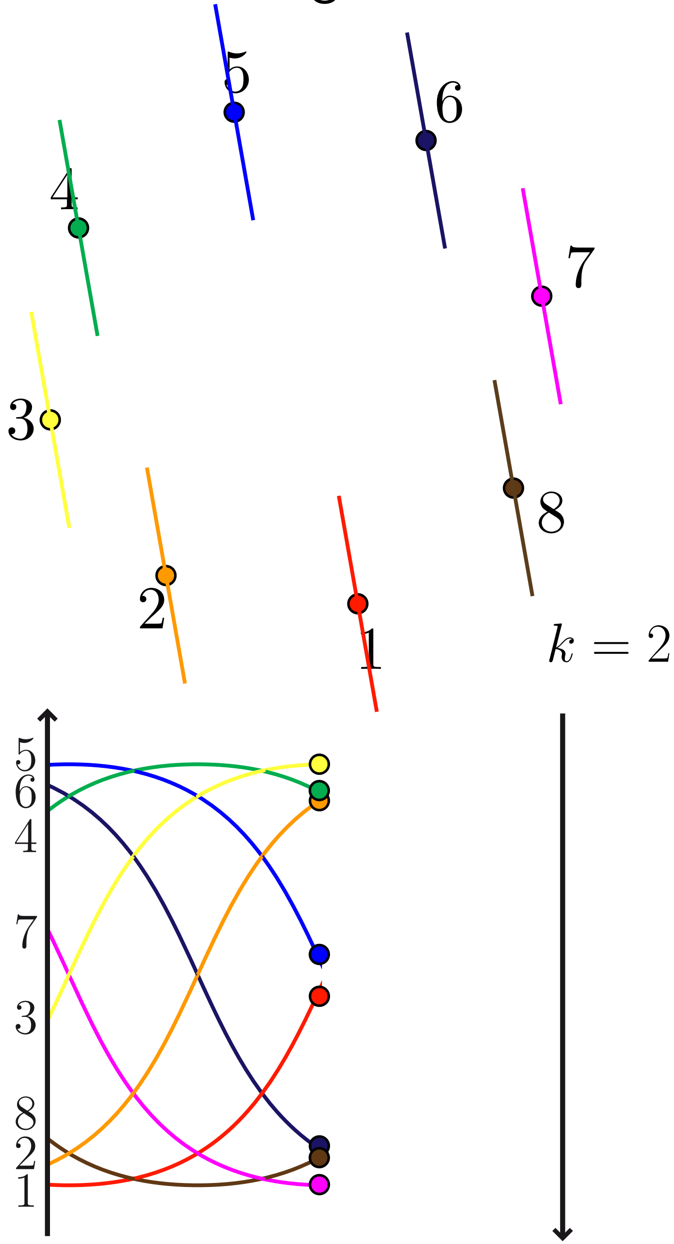

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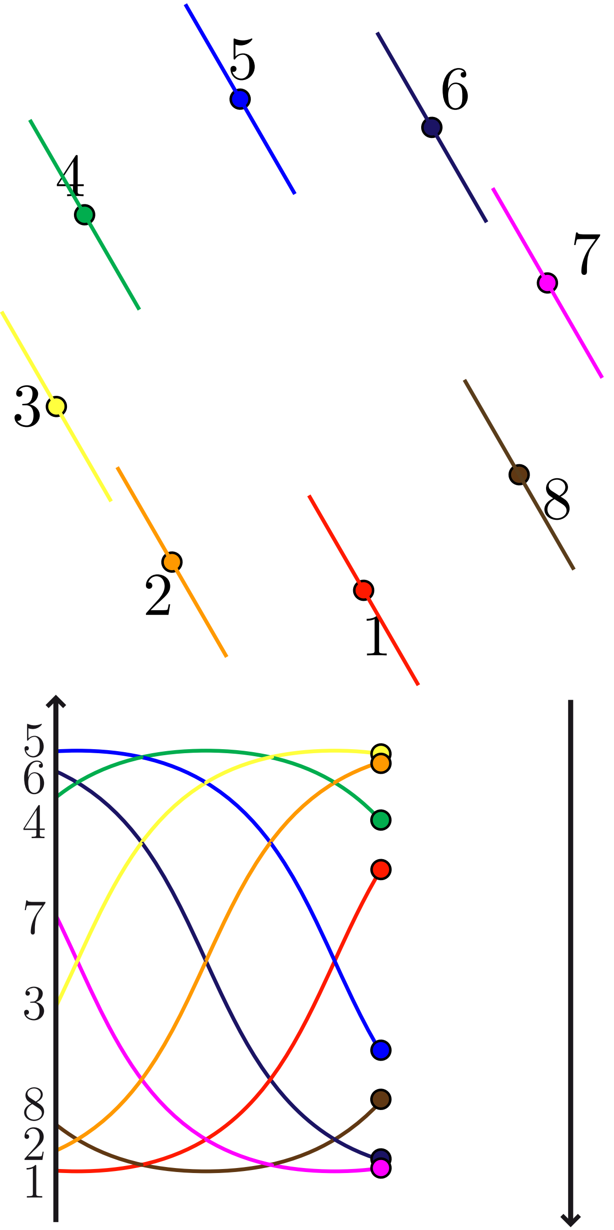
Triangulations



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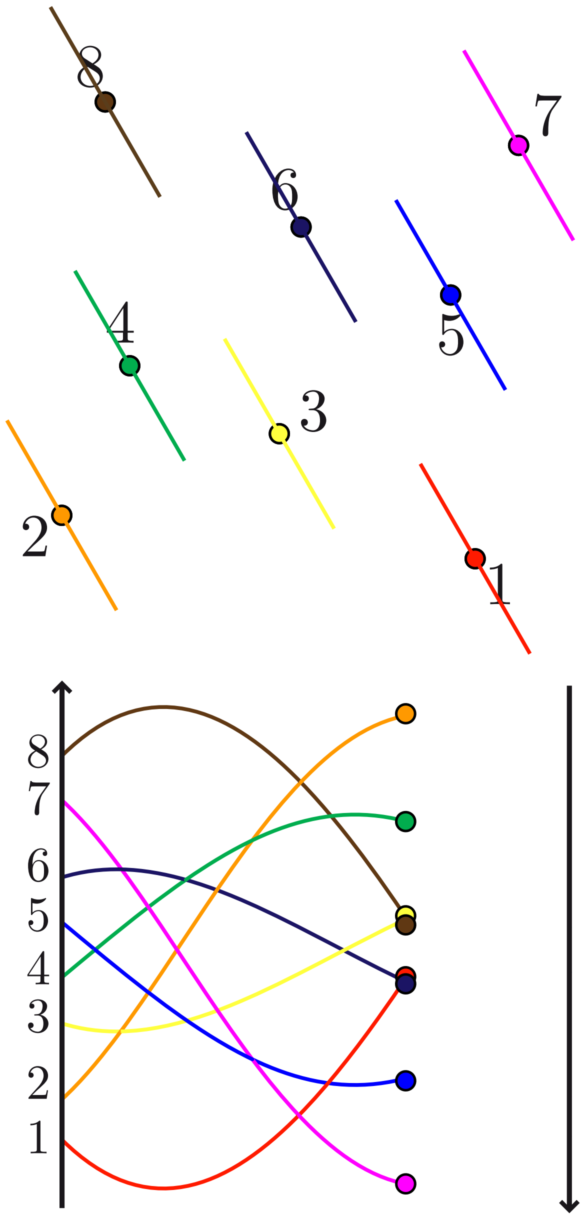
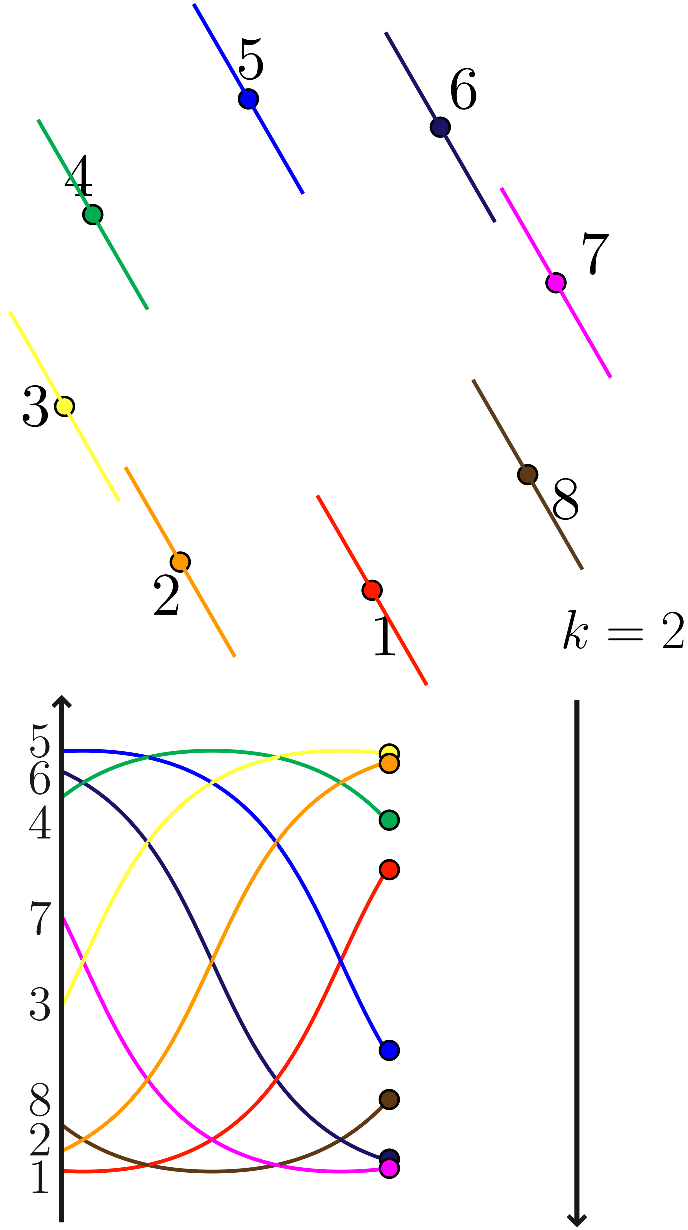


Multitriangulations

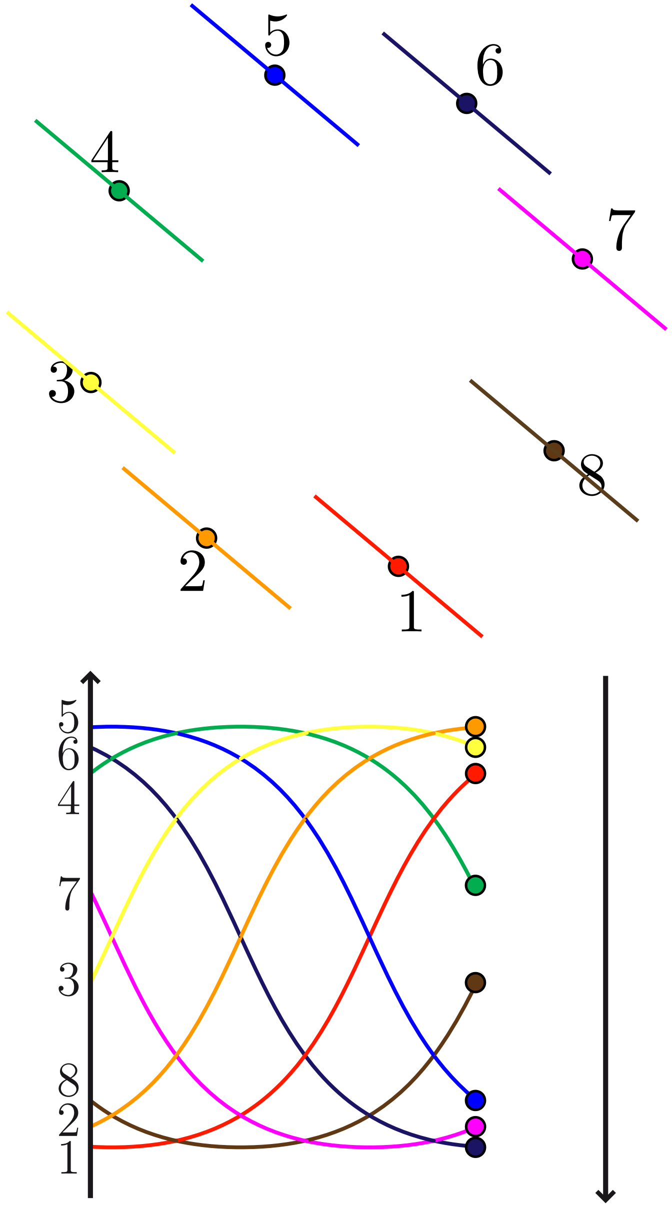


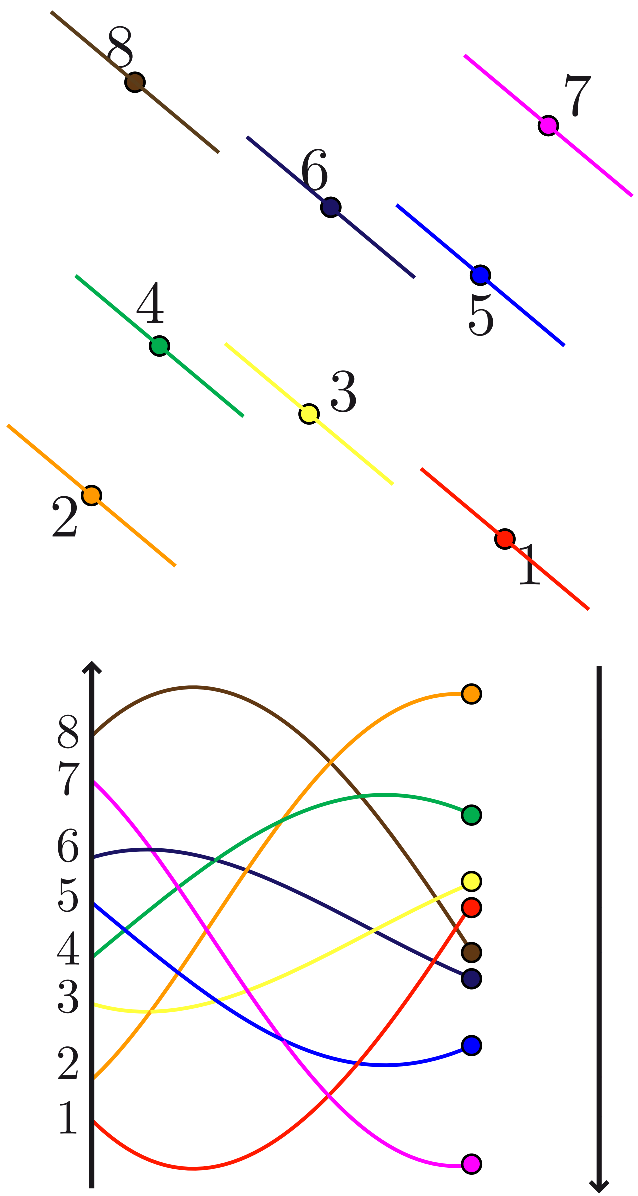
DUALITY

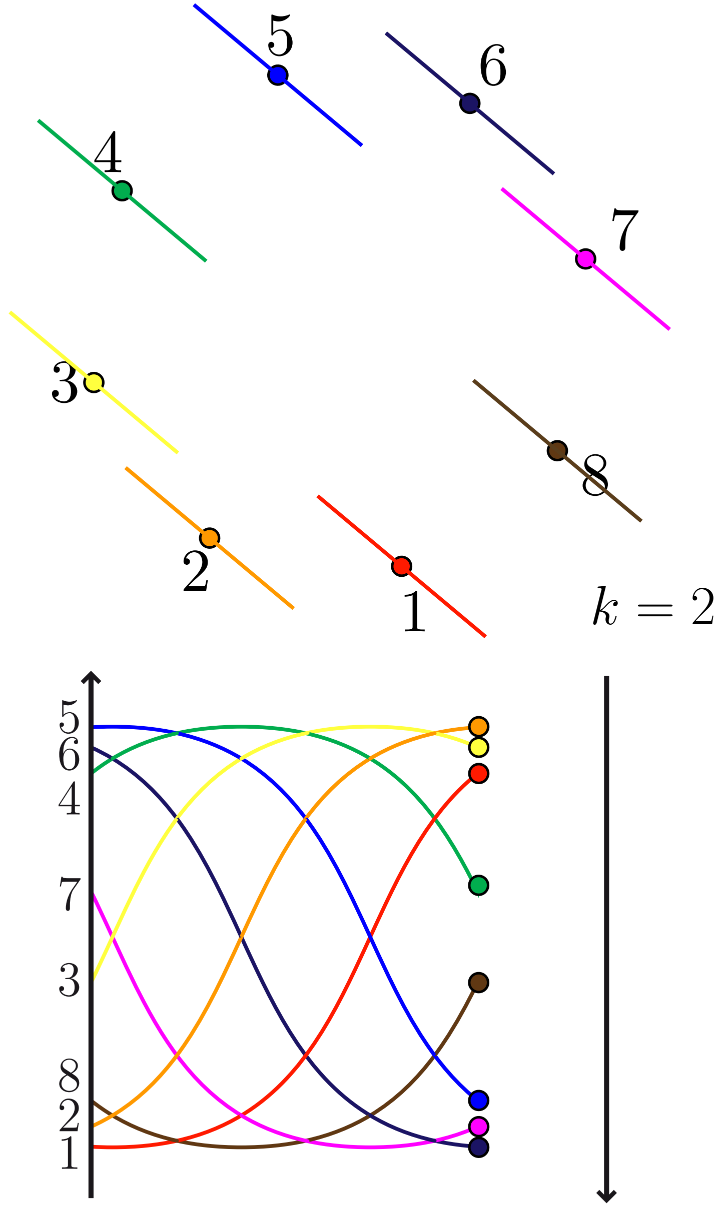
Triangulations

Pseudotriangulations

Multitriangulations

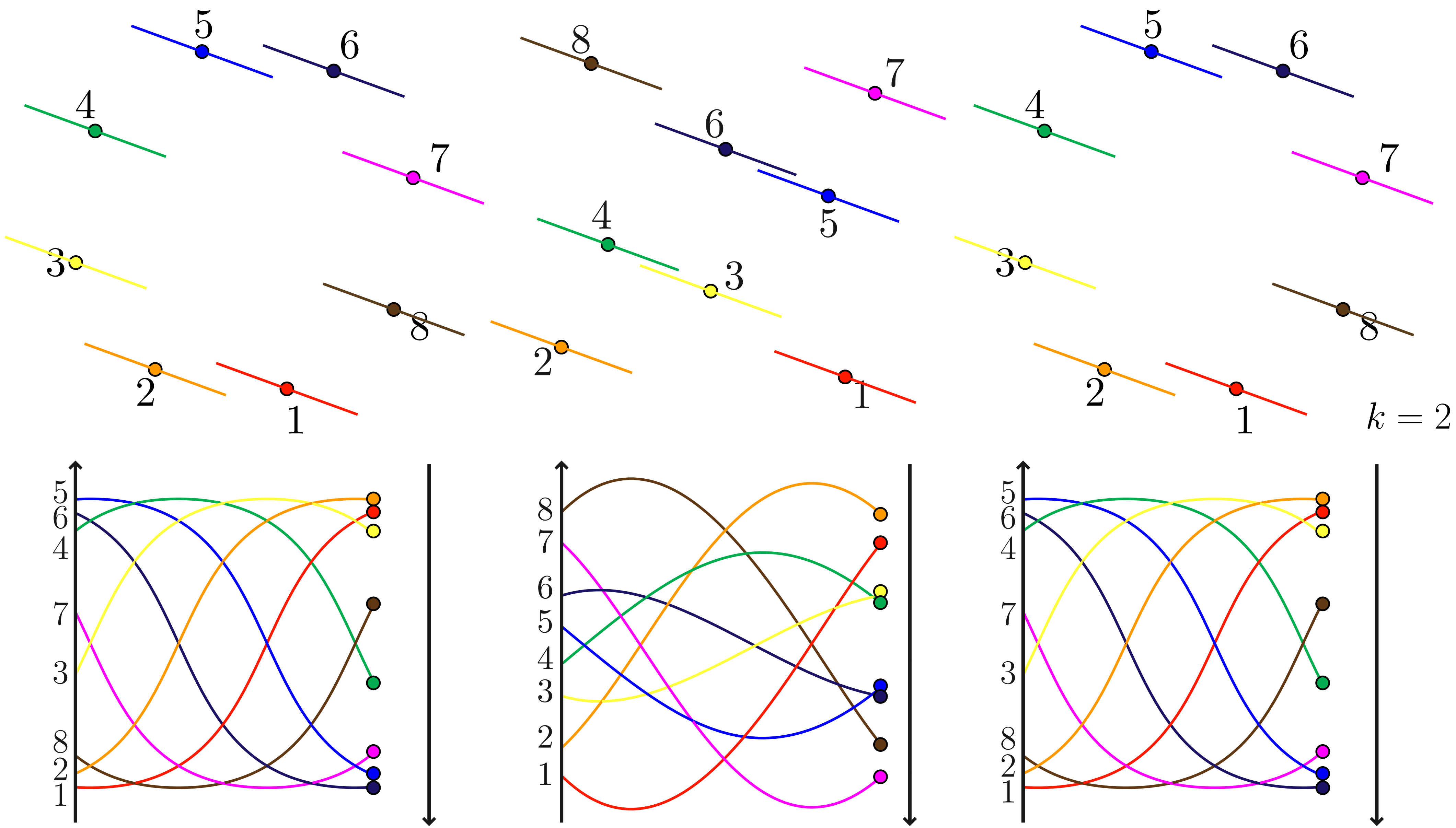






DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



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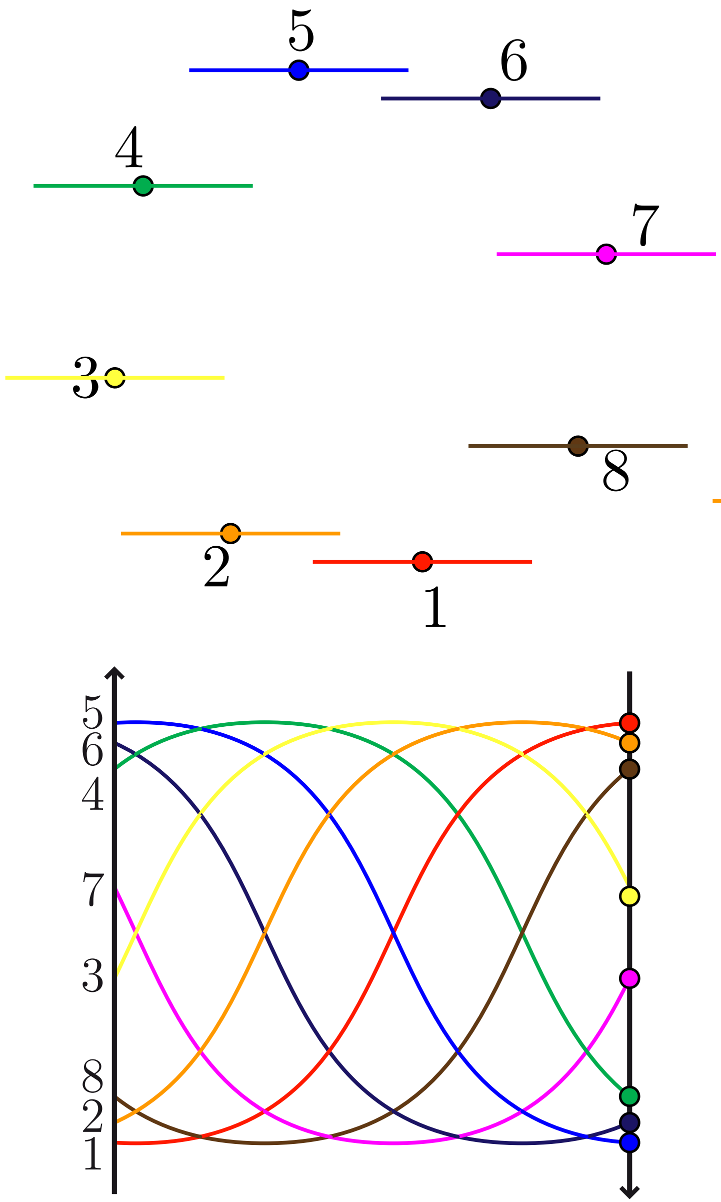
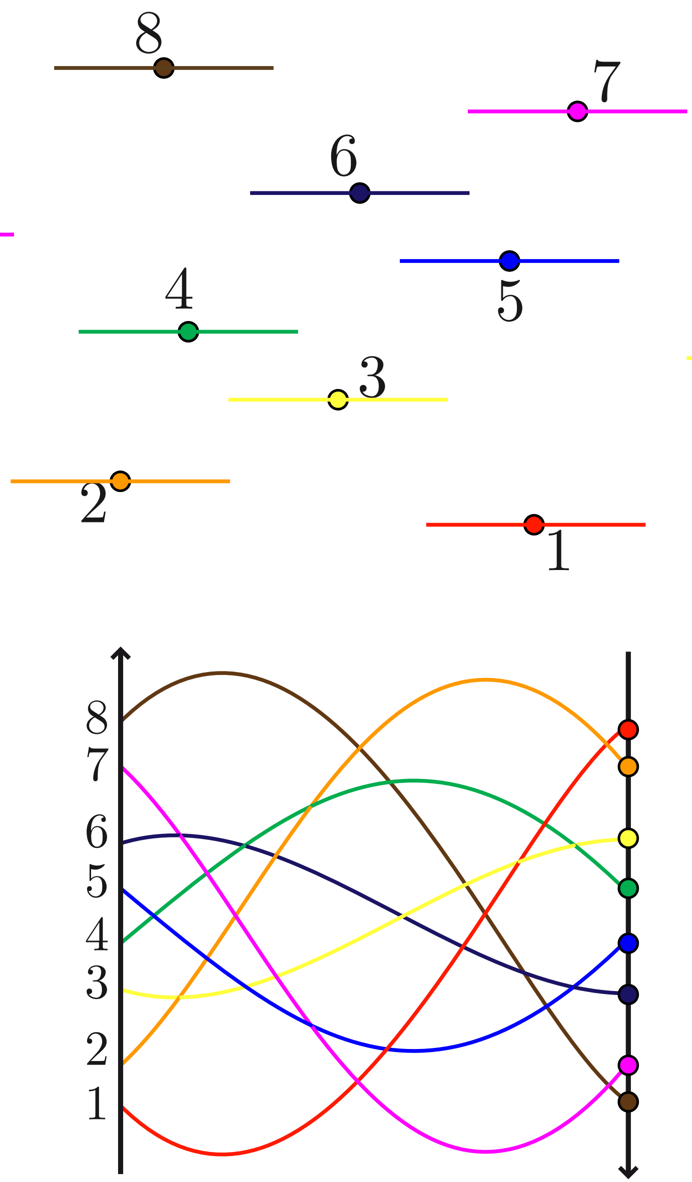
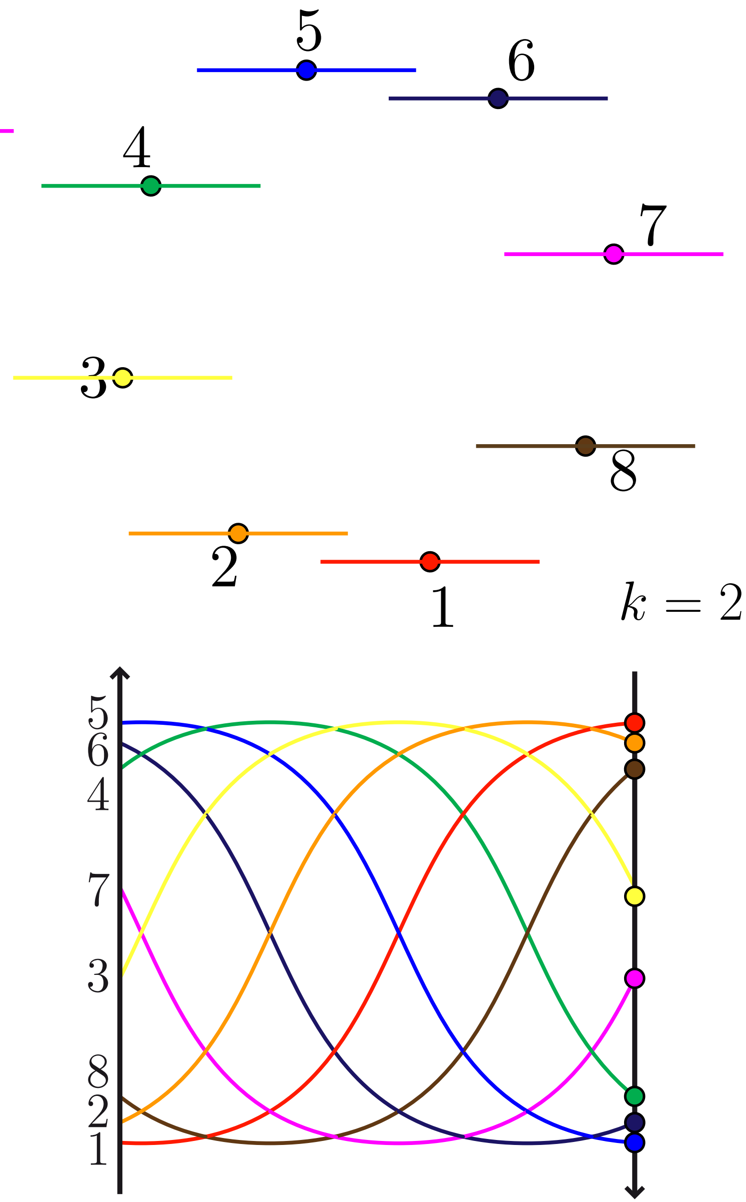
Triangulations



Pseudotriangulations

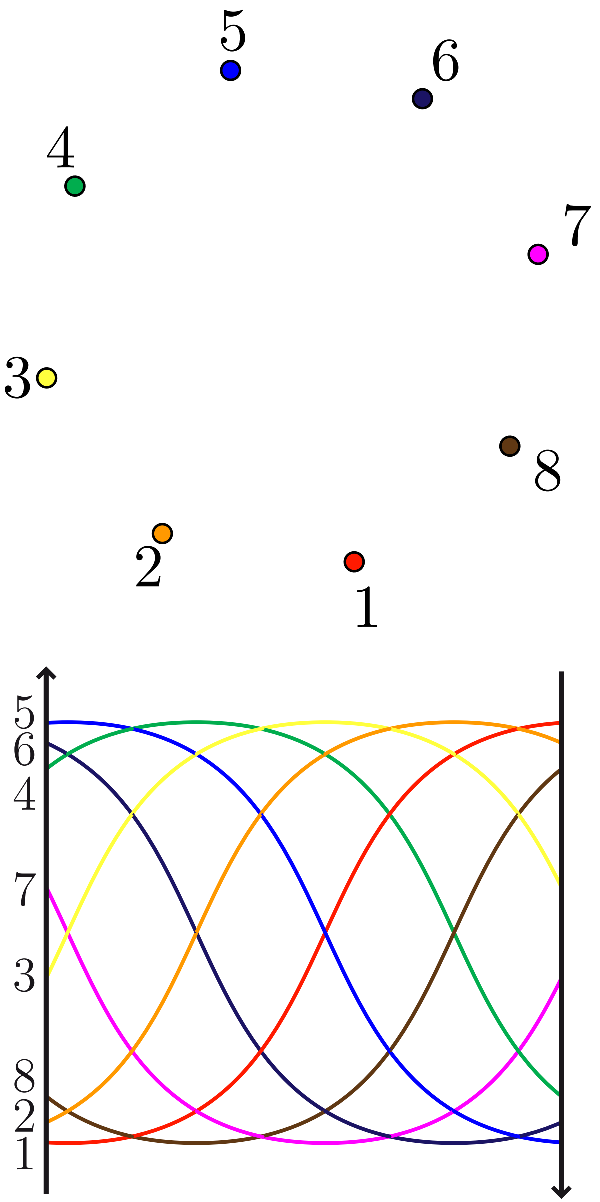


Multitriangulations

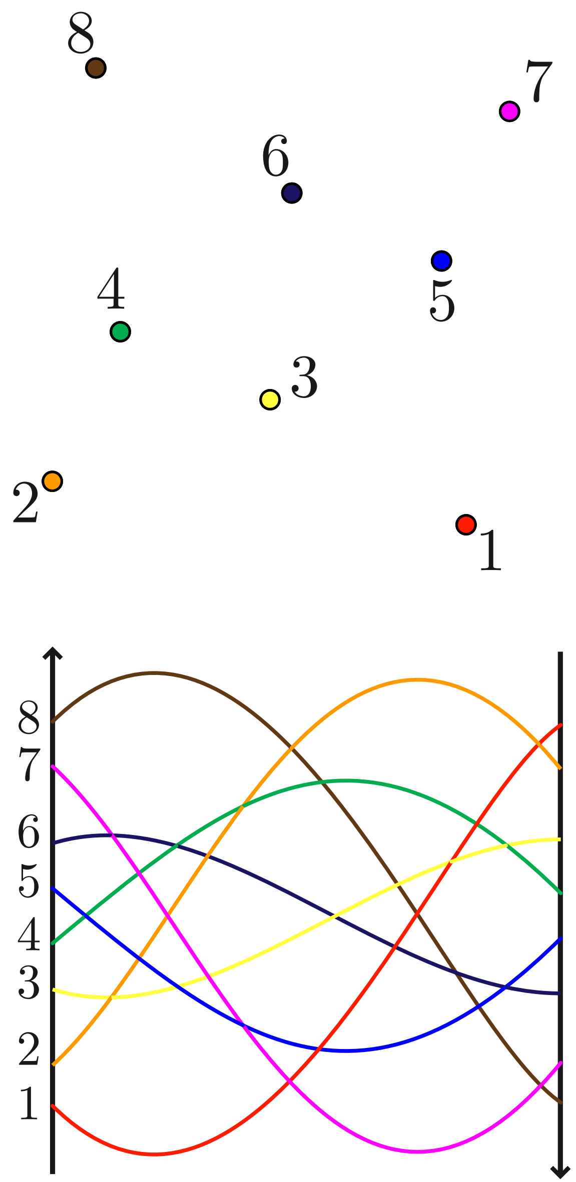


DUALITY

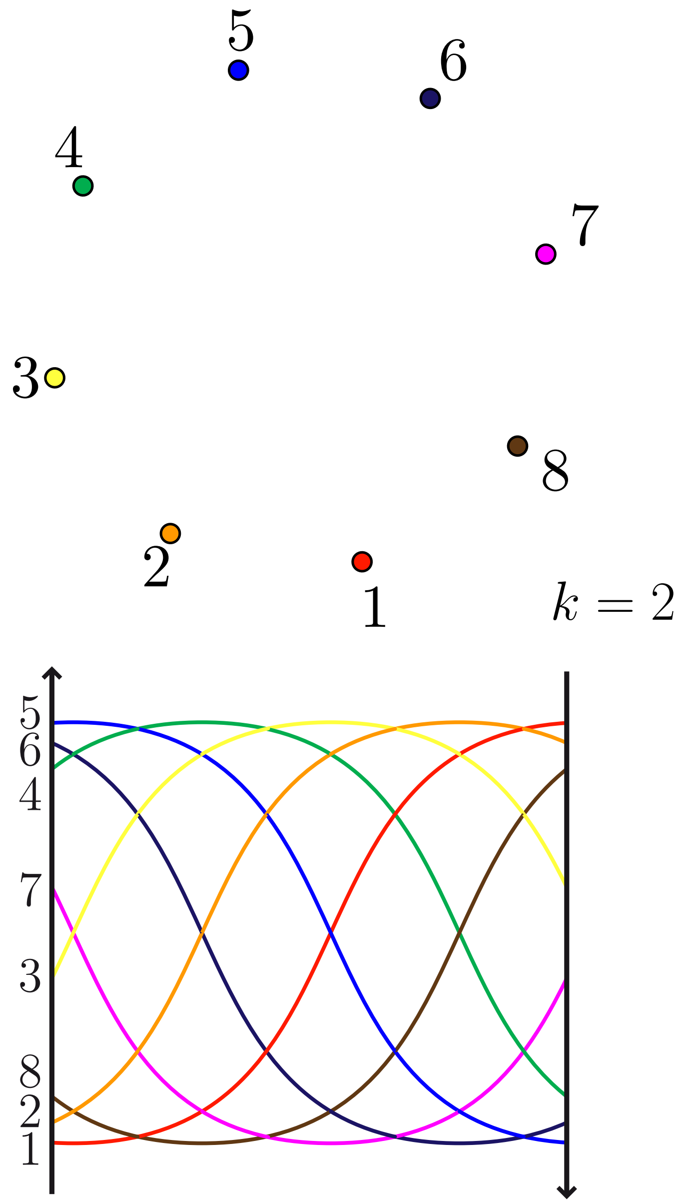
Triangulations



Pseudotriangulations



Multitriangulations



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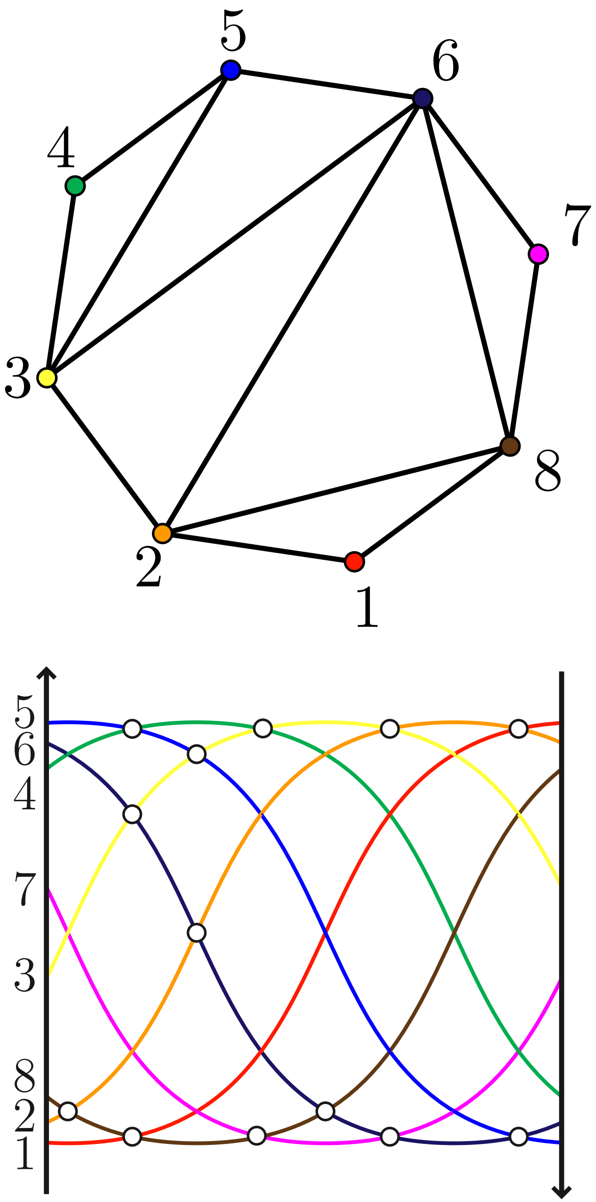
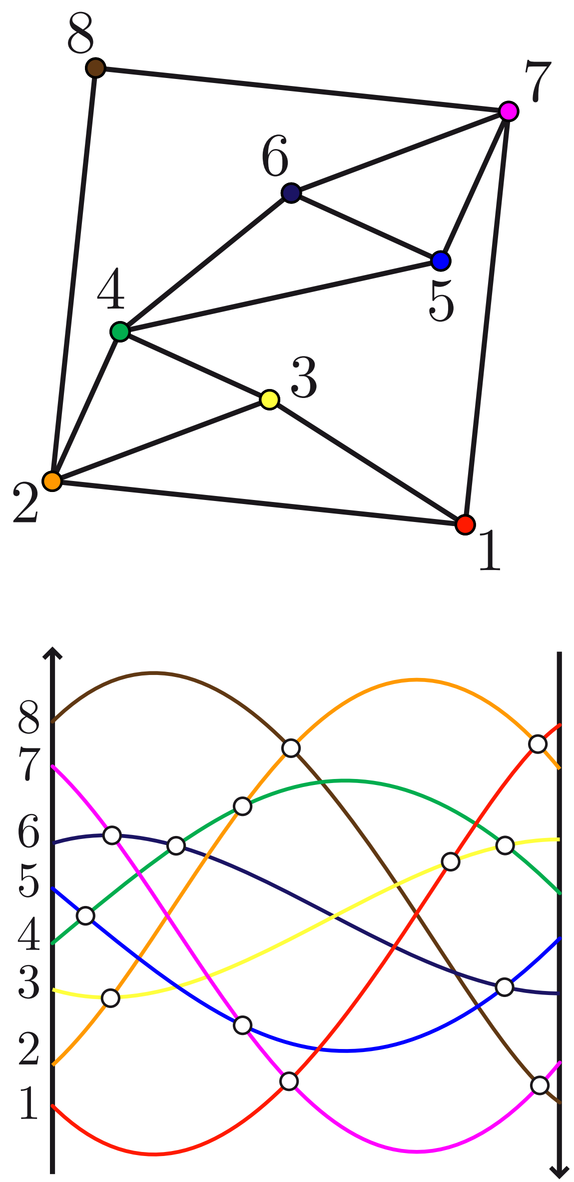
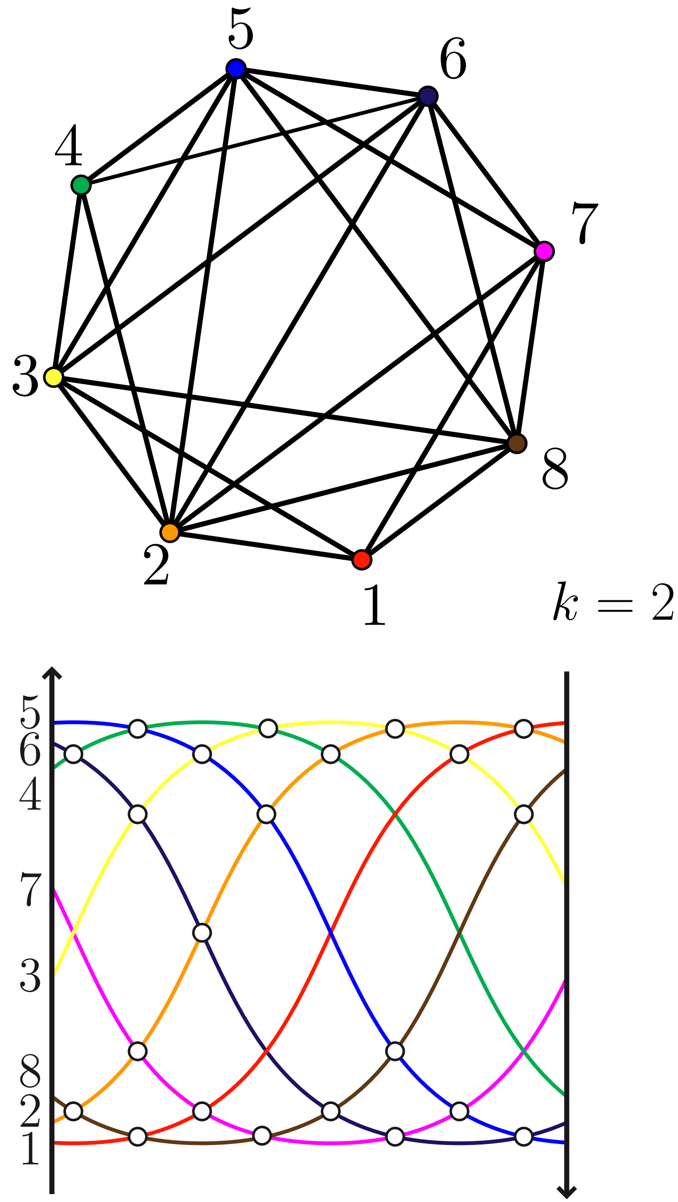
Triangulations



Pseudotriangulations



Multitriangulations



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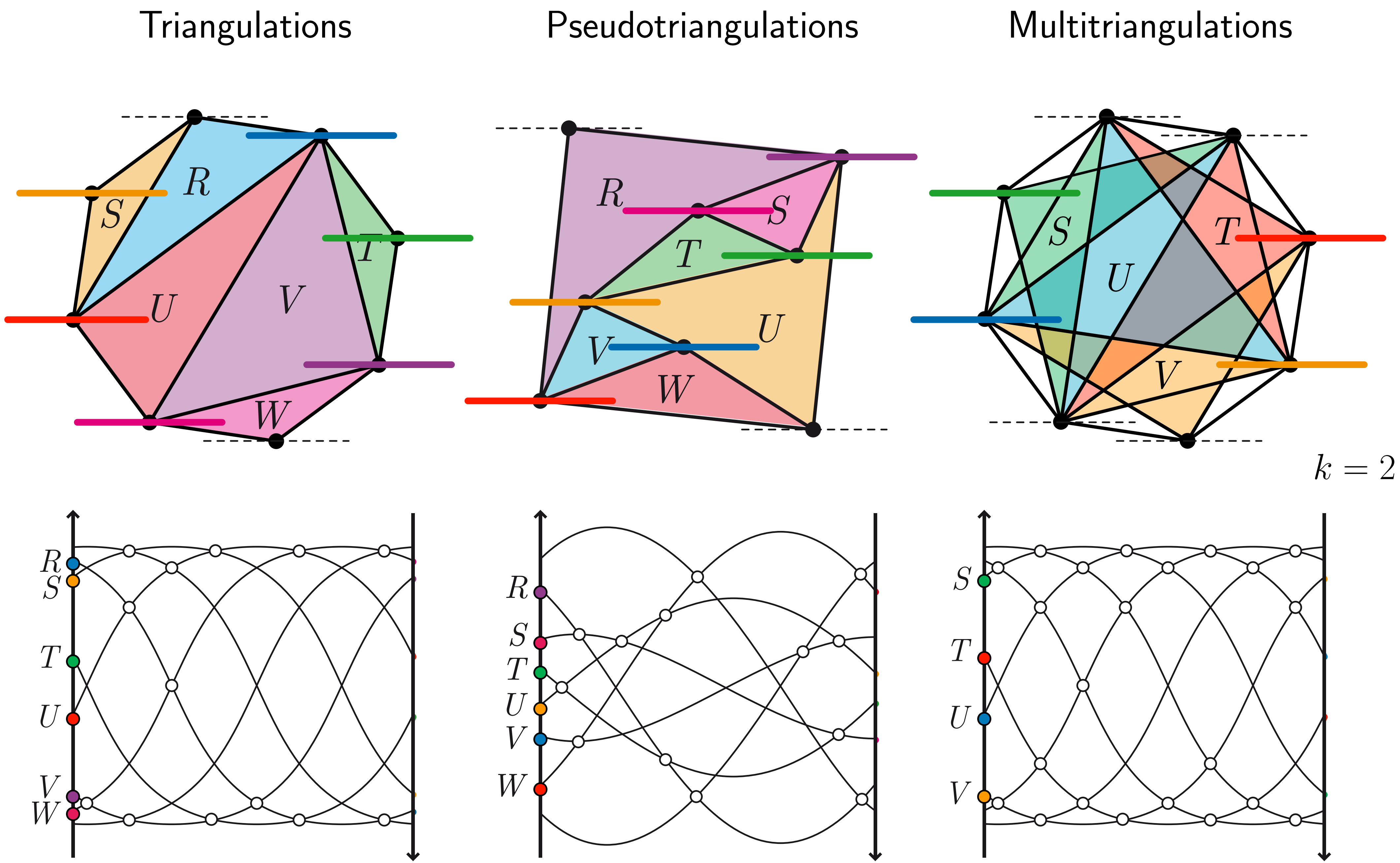
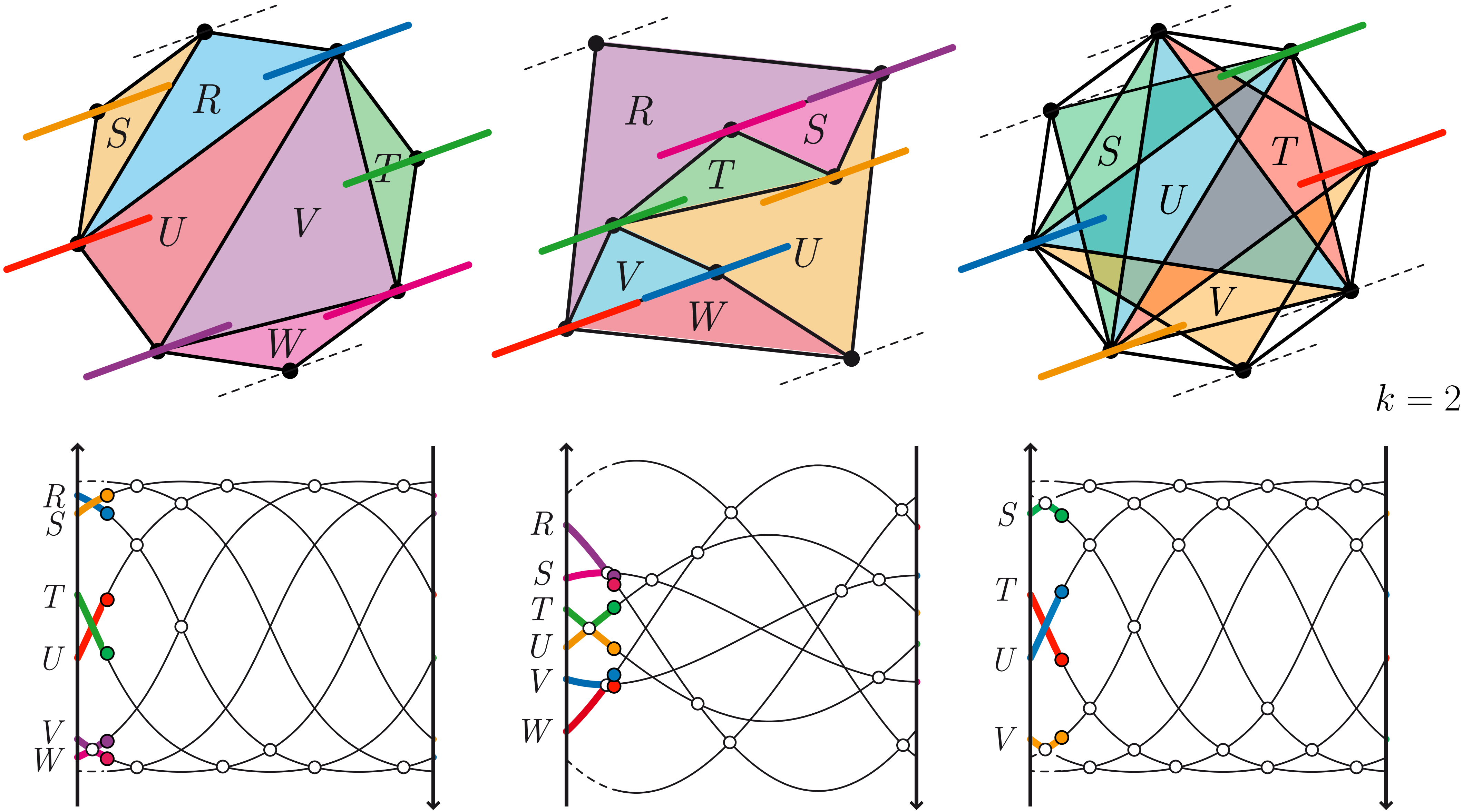

Triangulations
Pseudotriangulations
Multitriangulations
R
S
T
U
V
W
$k = 2$

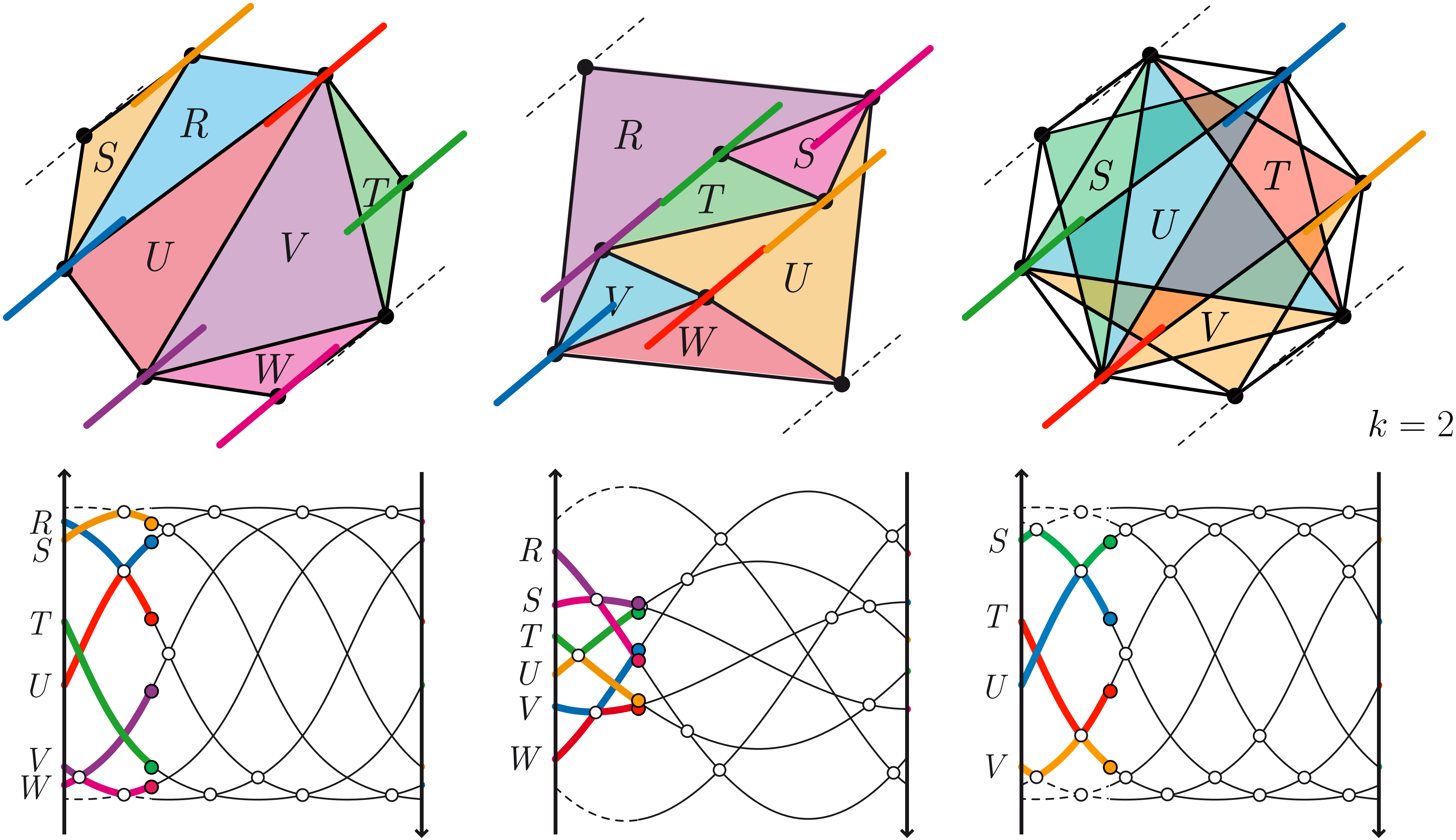
DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



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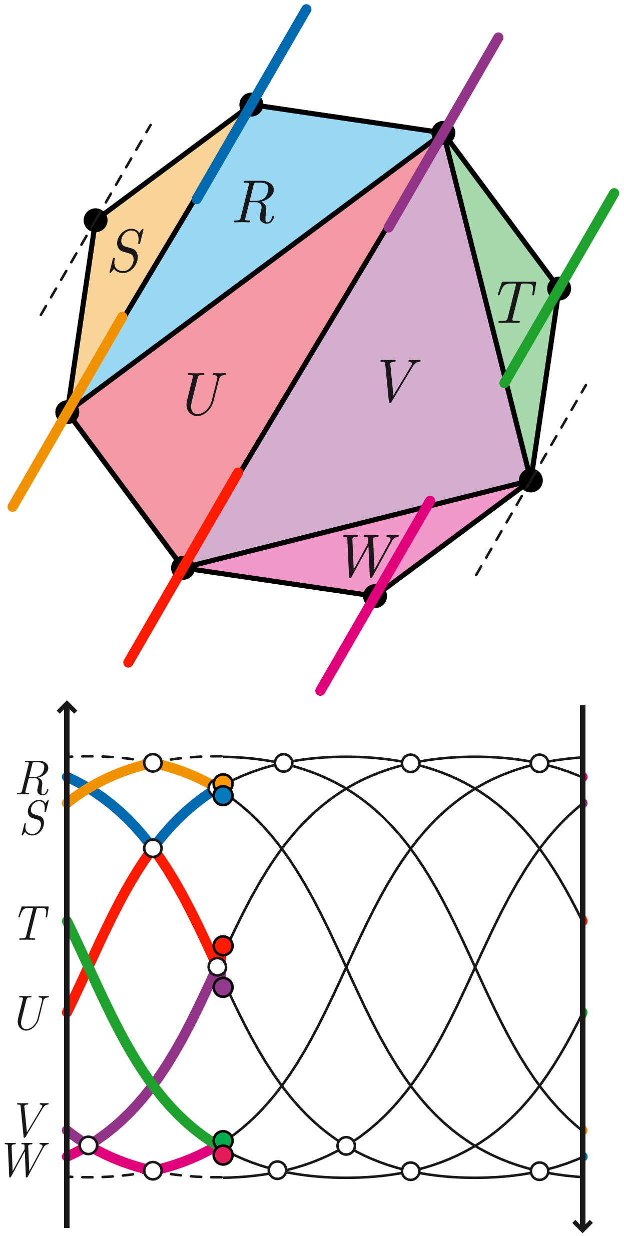
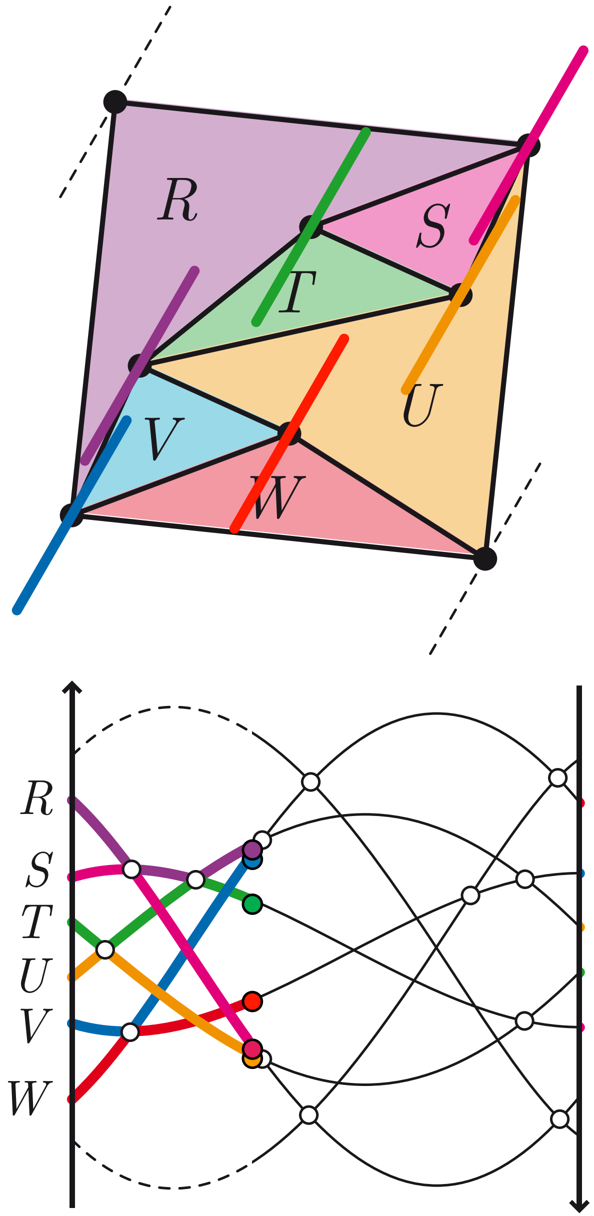
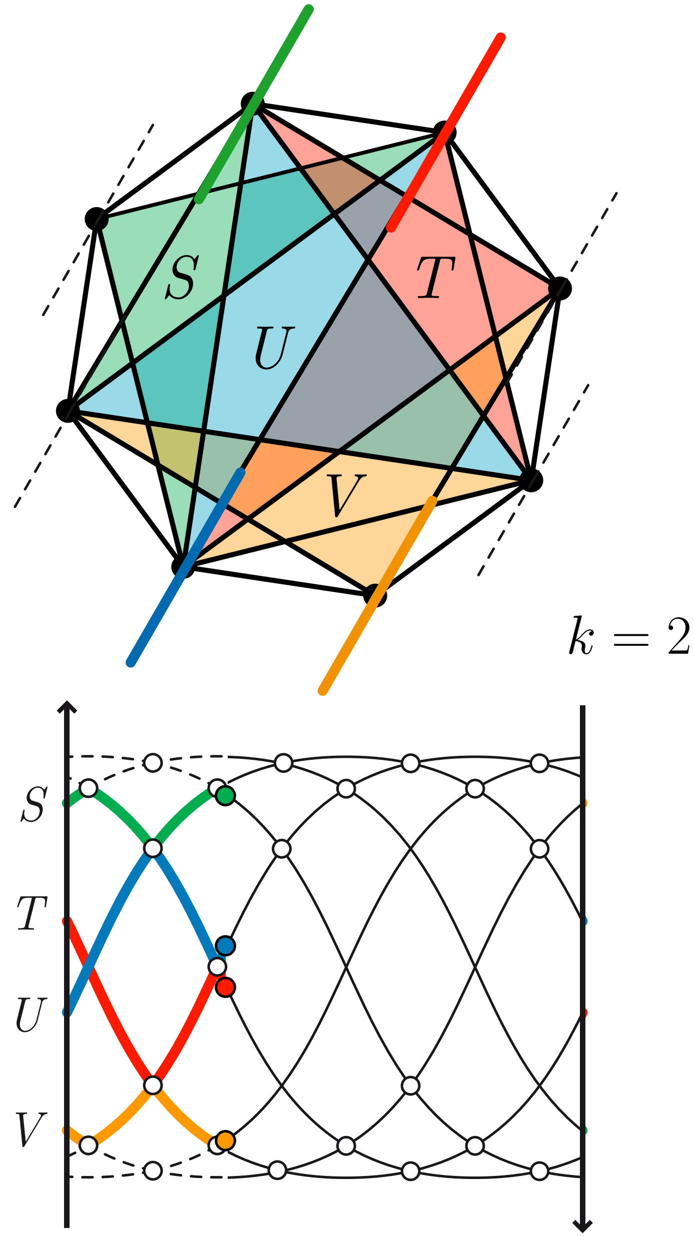
Triangulations



Pseudotriangulations



Multitriangulations



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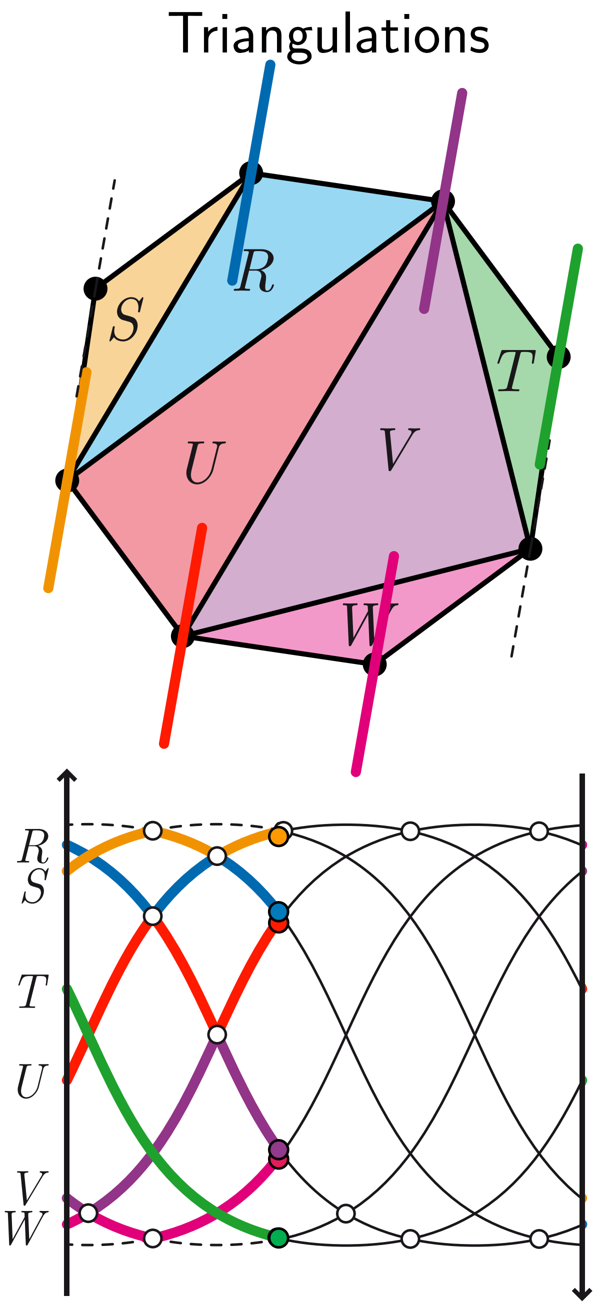
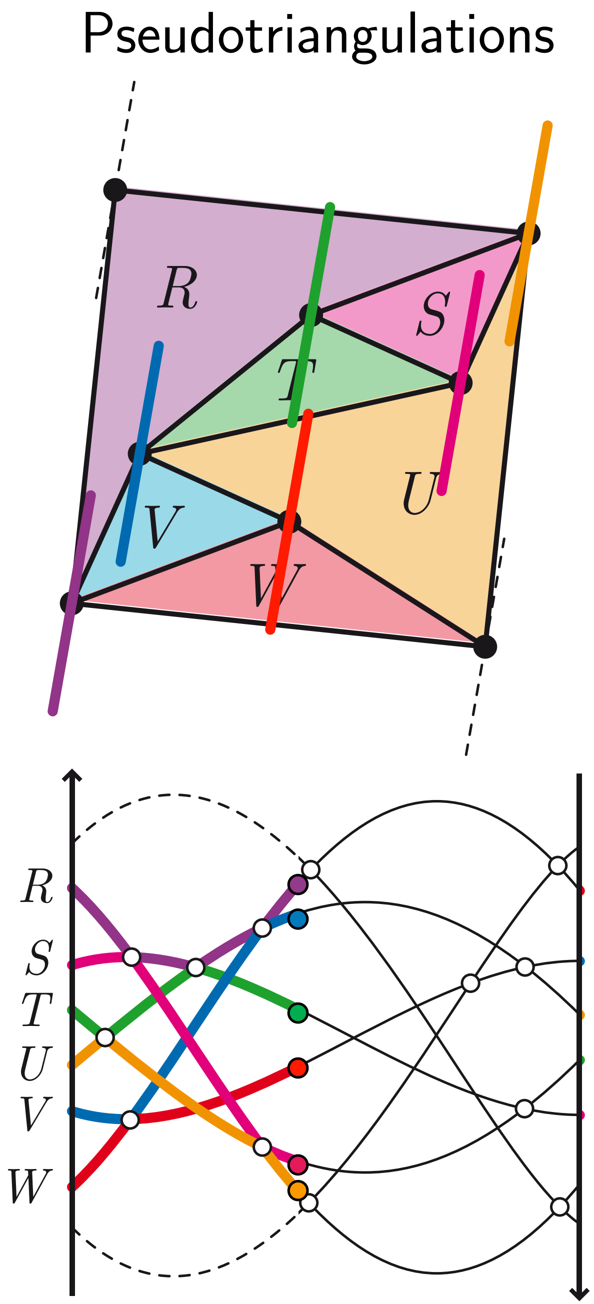
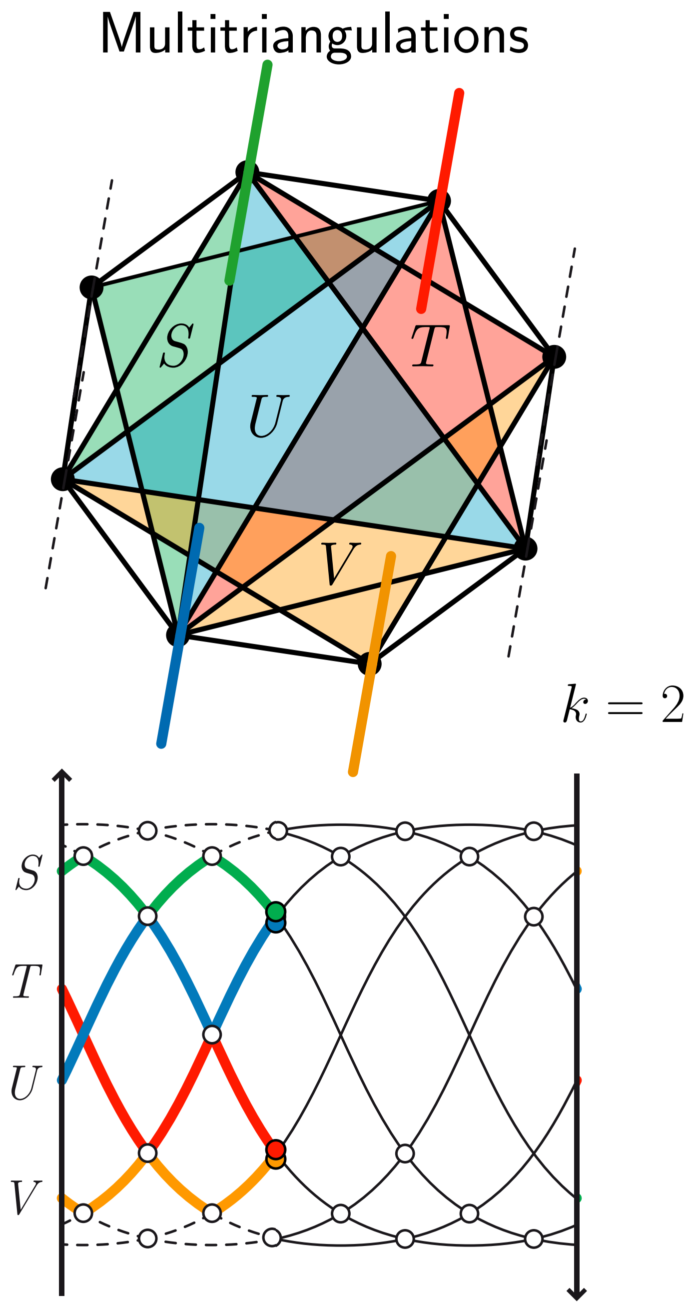


Triangulations
R
S
U
V
T
W
R
S
T
U
V
W



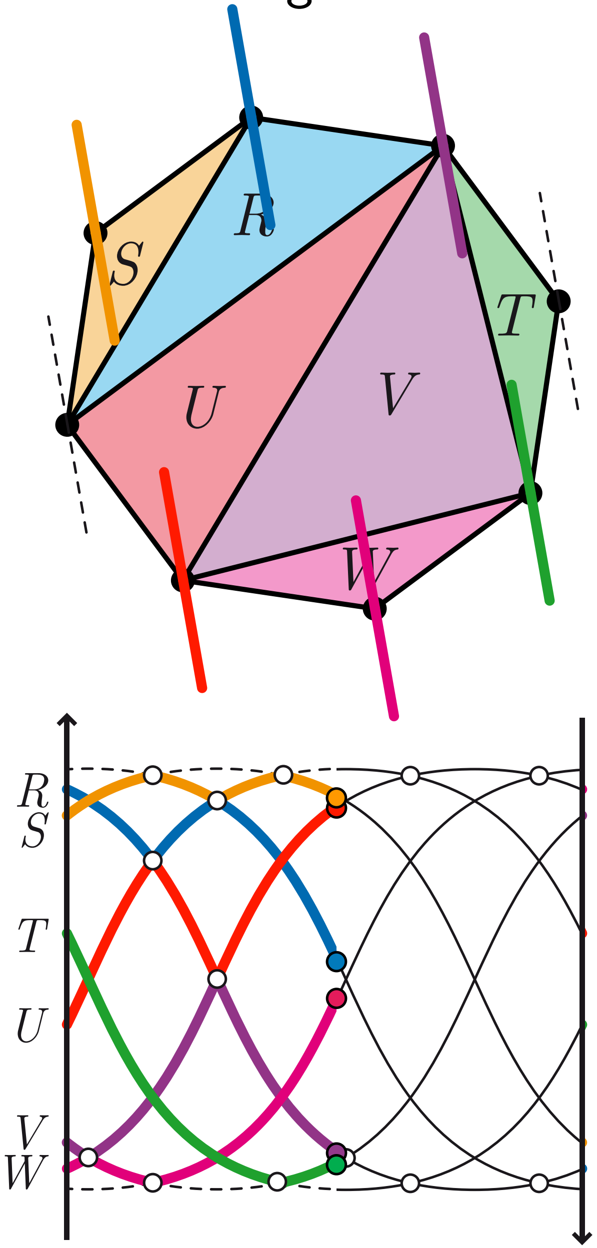
Pseudotriangulations
R
S
T
U
V
W
R
S
T
U
V
W



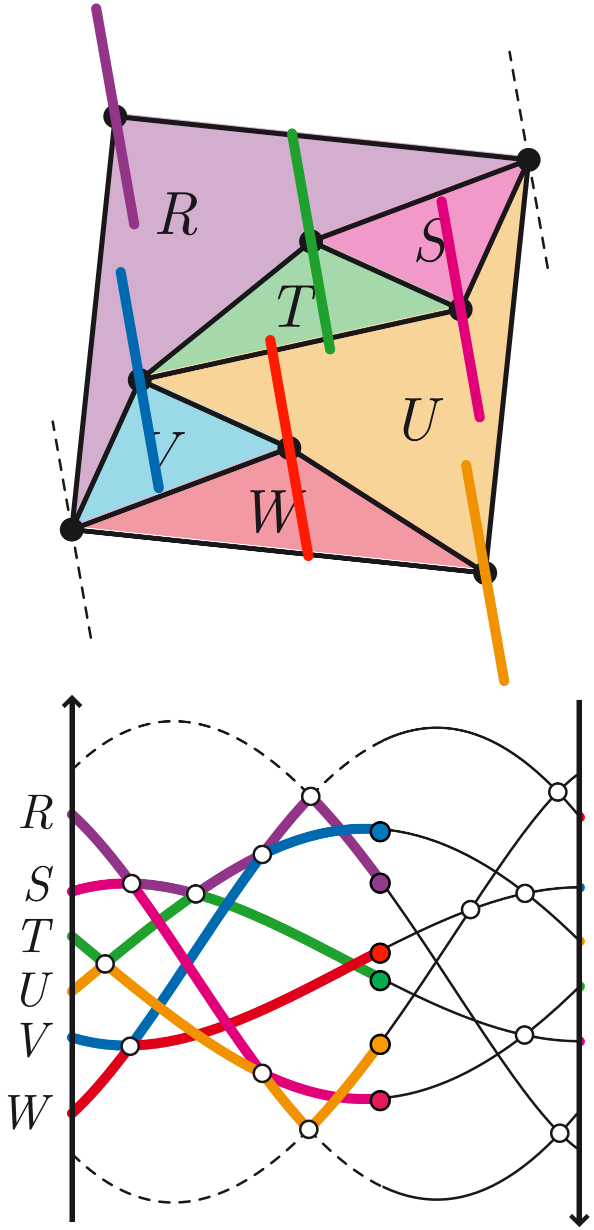
Multitriangulations
S
T
U
V
$k = 2$
S
T
U
V

DUALITY

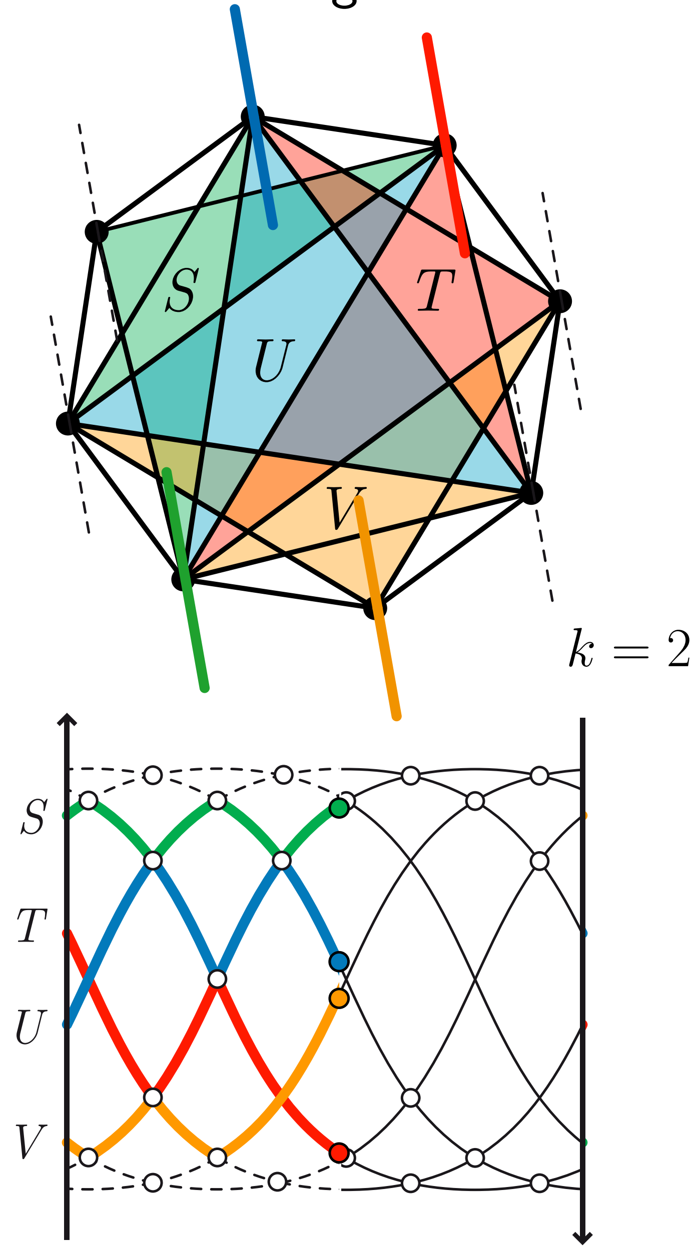
Triangulations



Pseudotriangulations



Multitriangulations



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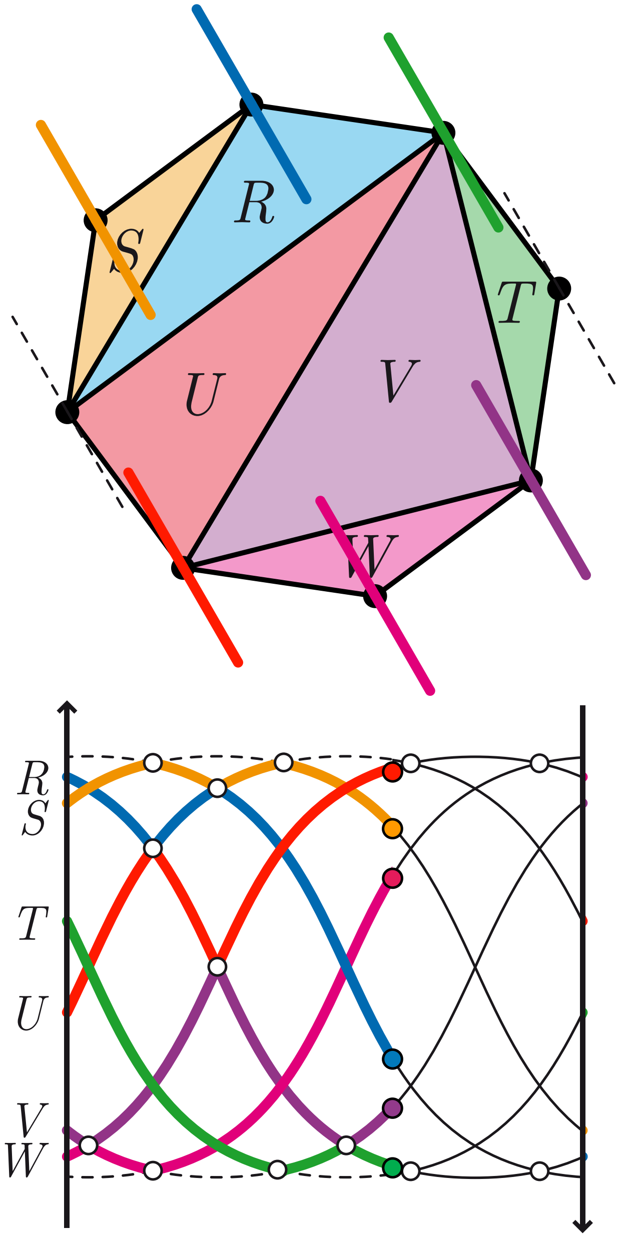
Triangulations



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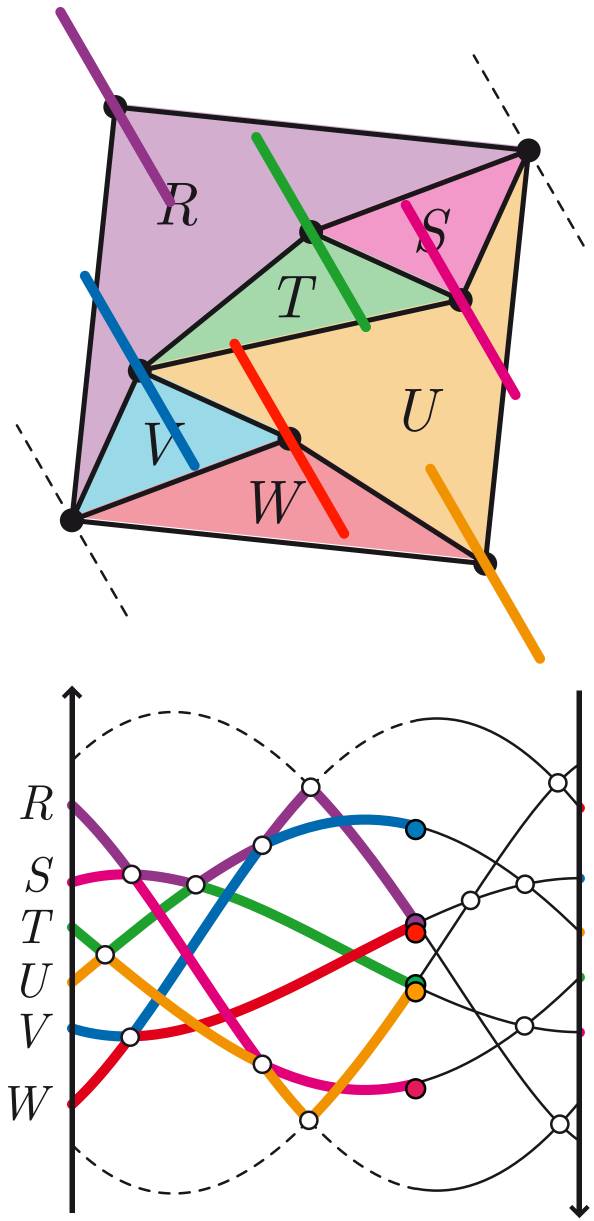

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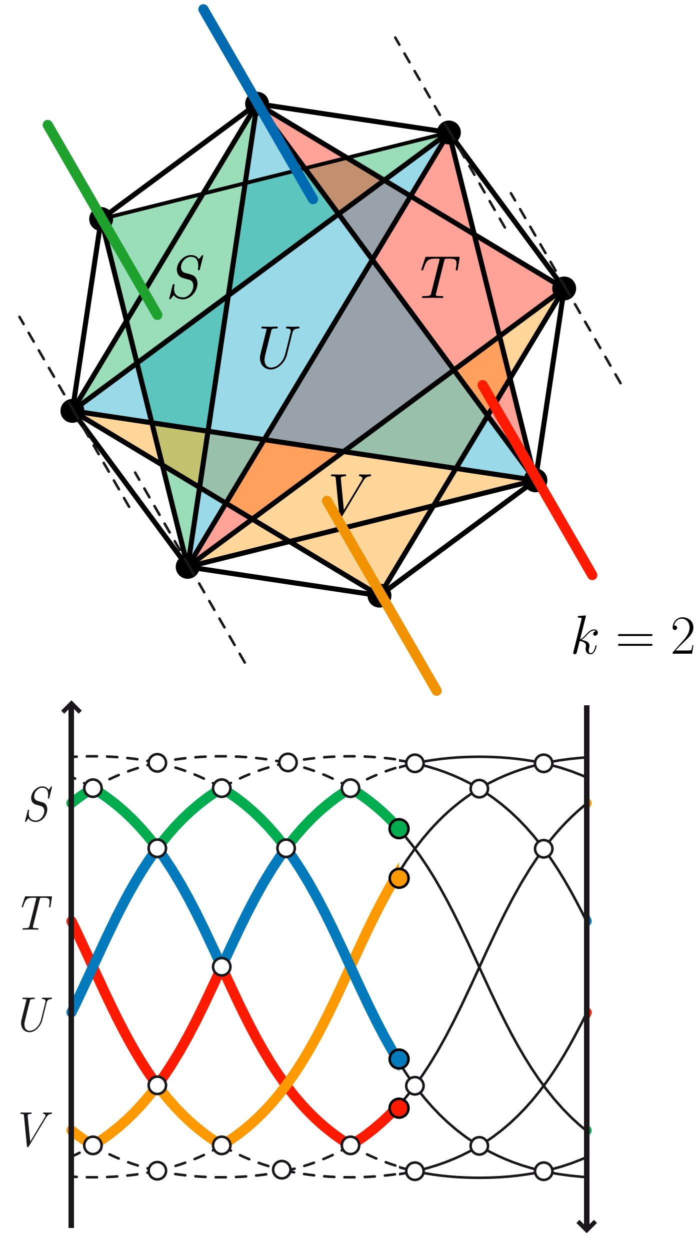
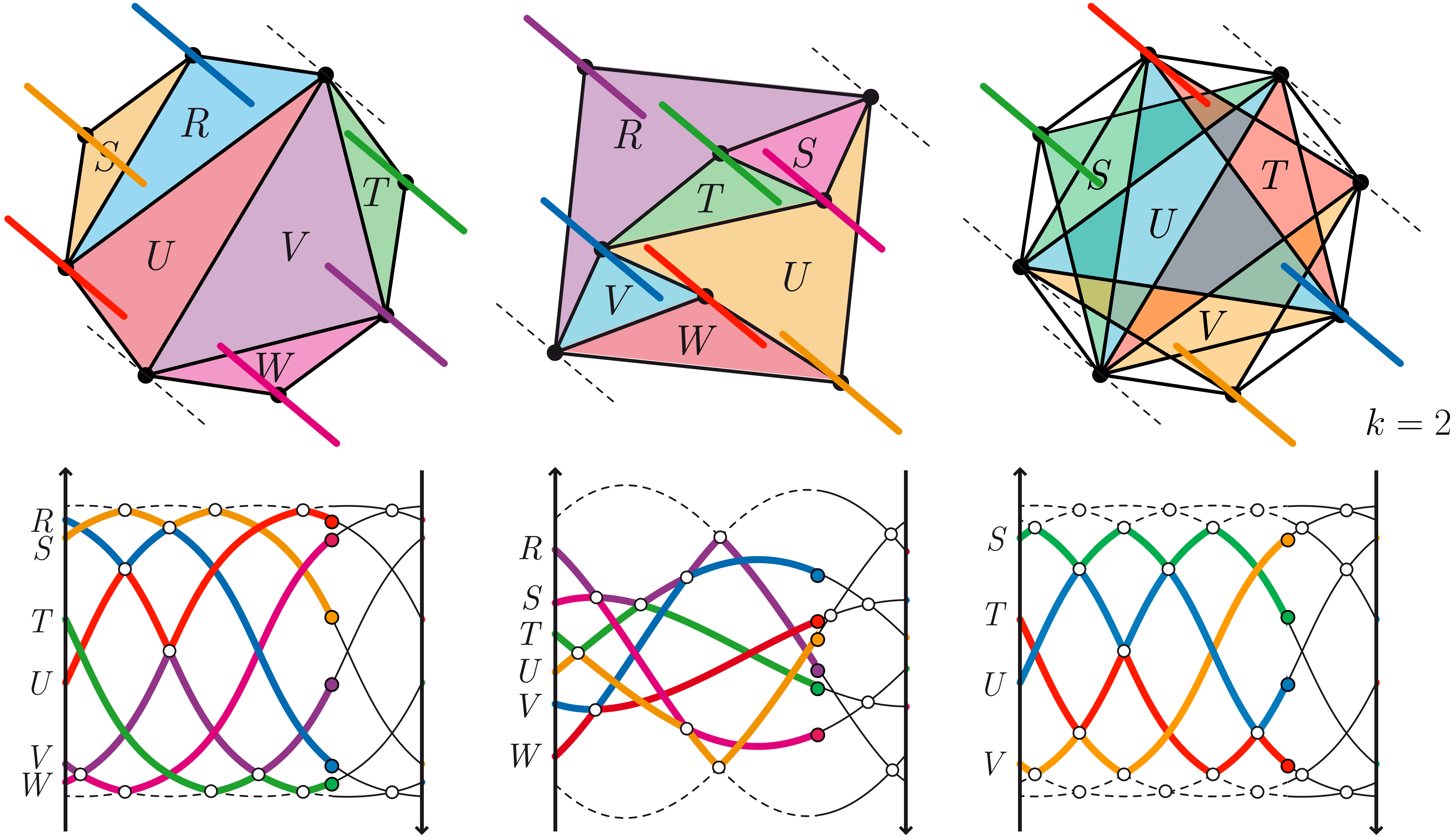


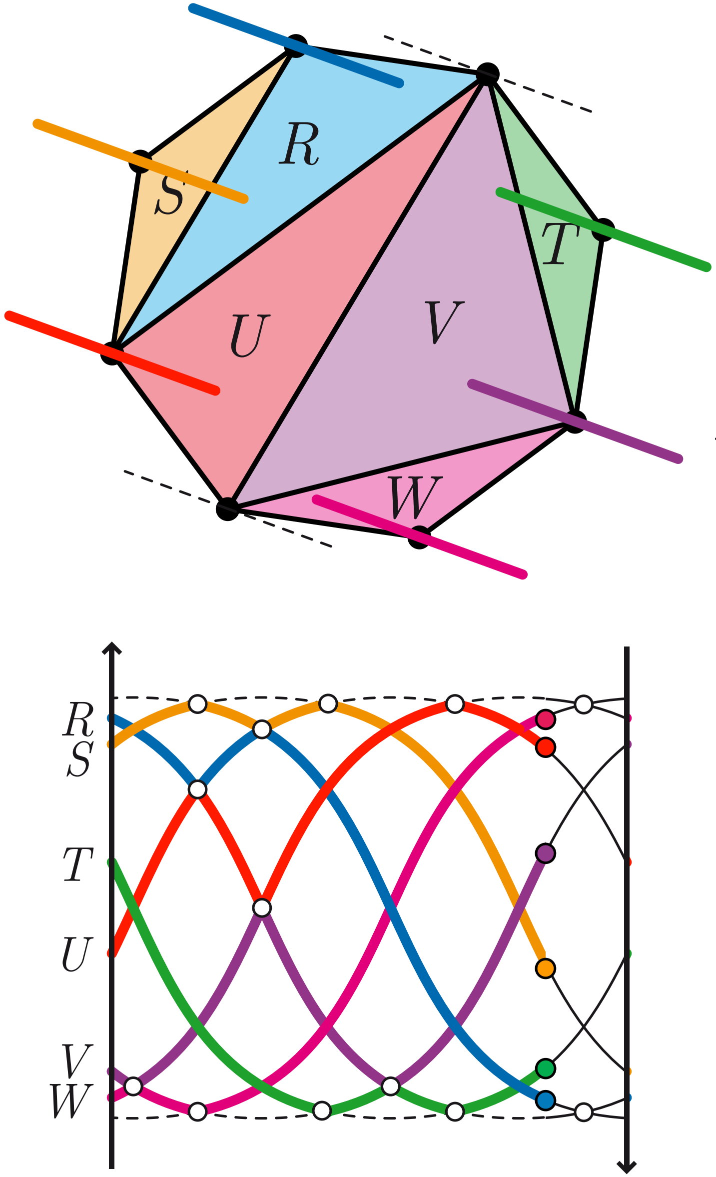
DUALITY

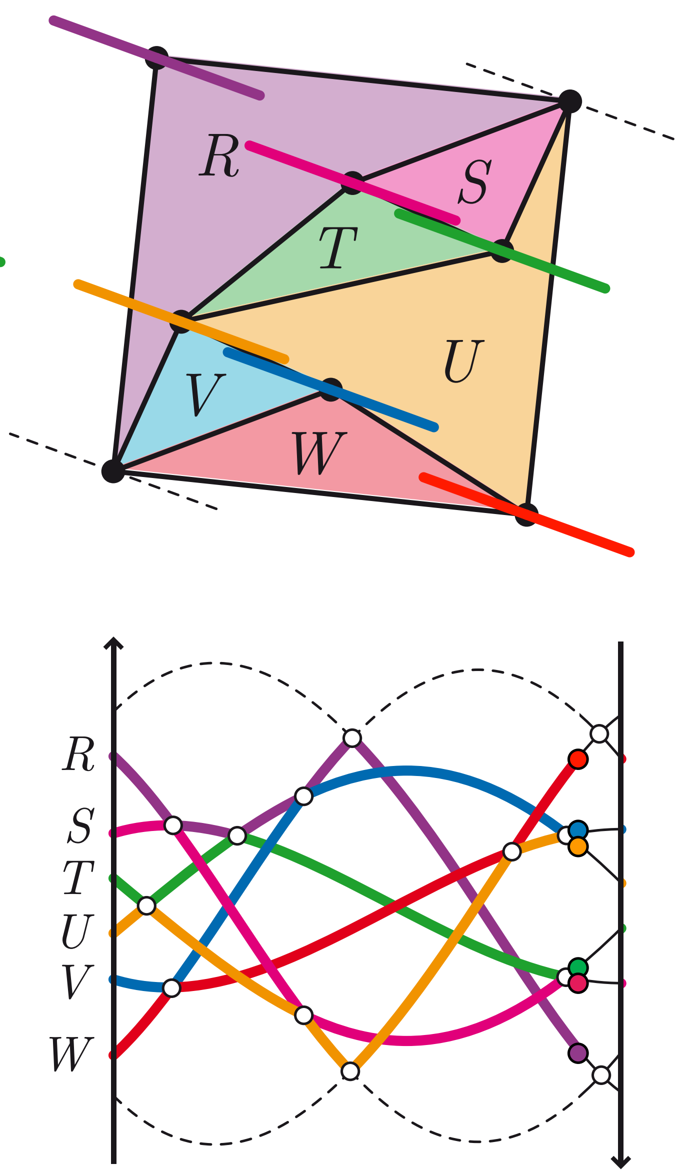
Triangulations | Pseudotriangulations | Multitriangulations

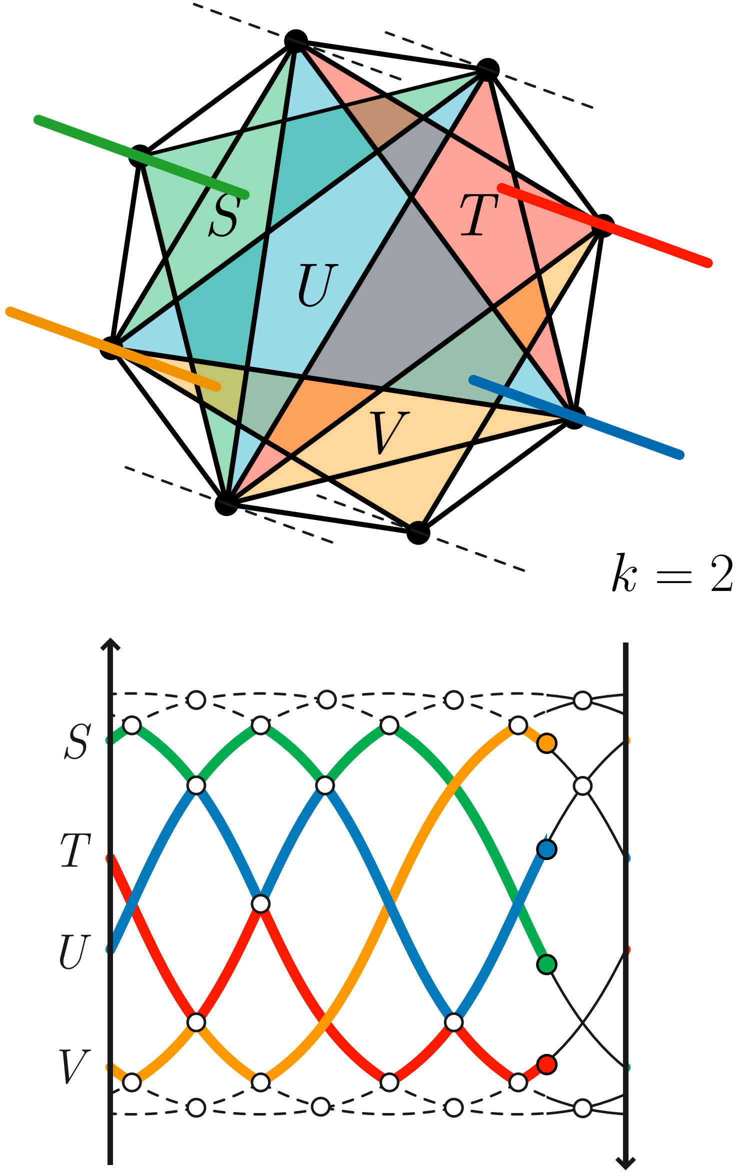


DUALITY

Triangulations	Pseudotriangulations	Multitriangulations







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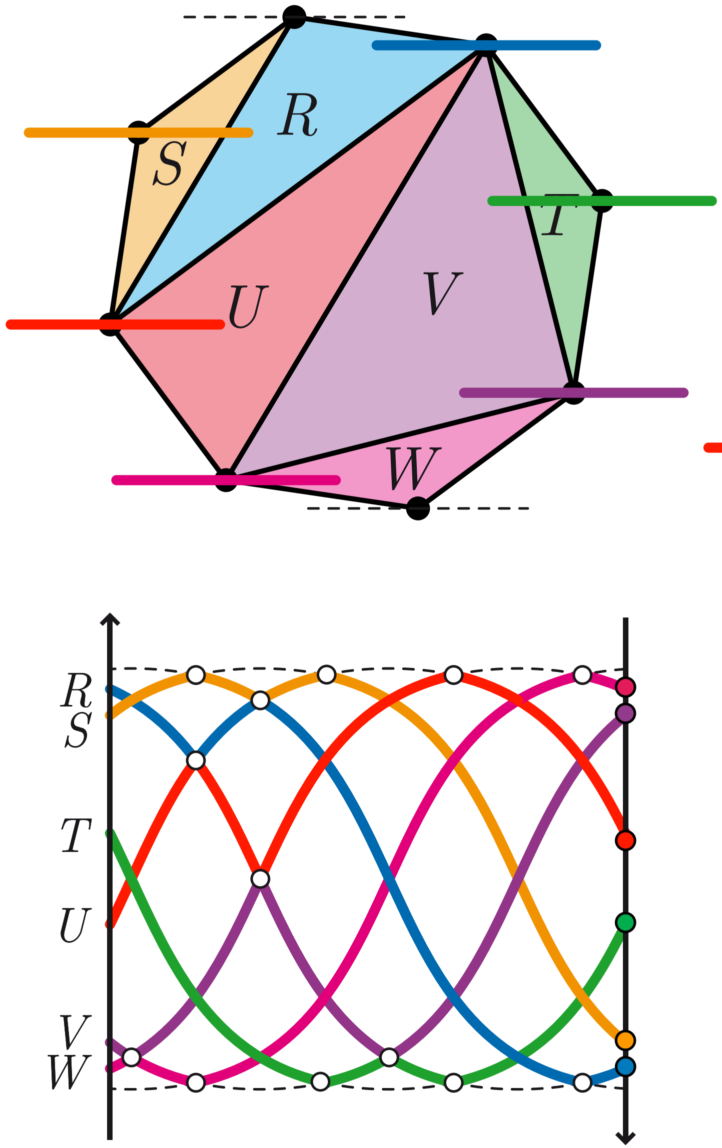
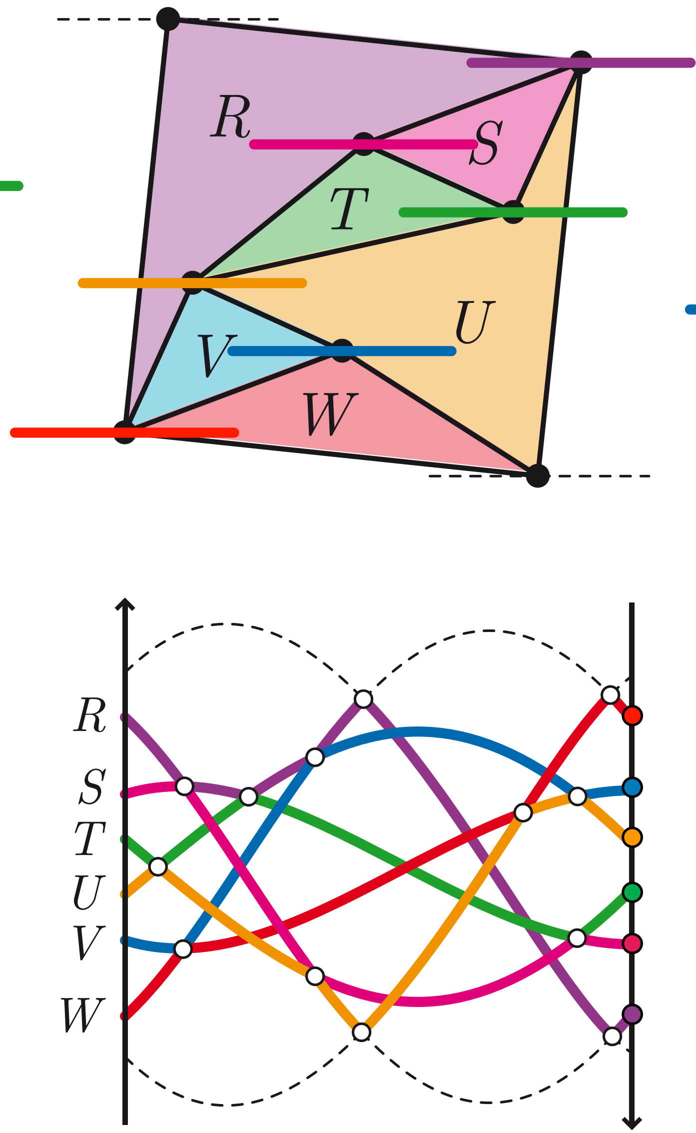
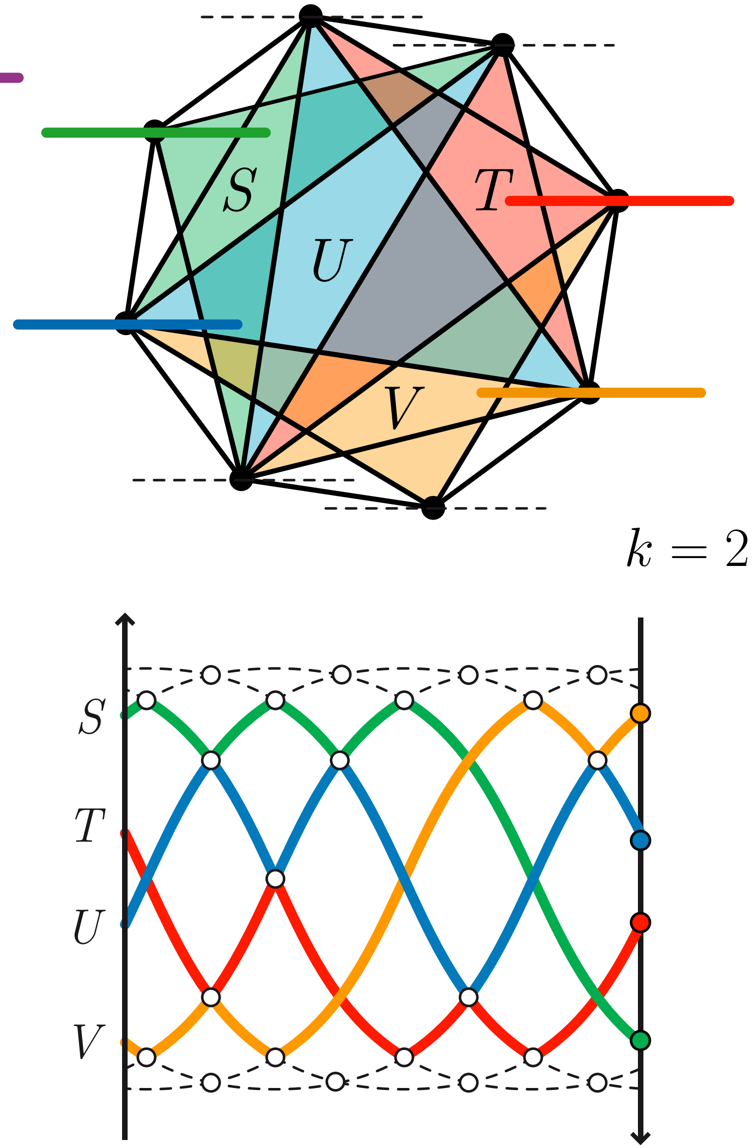
Triangulations



Pseudotriangulations



Multitriangulations



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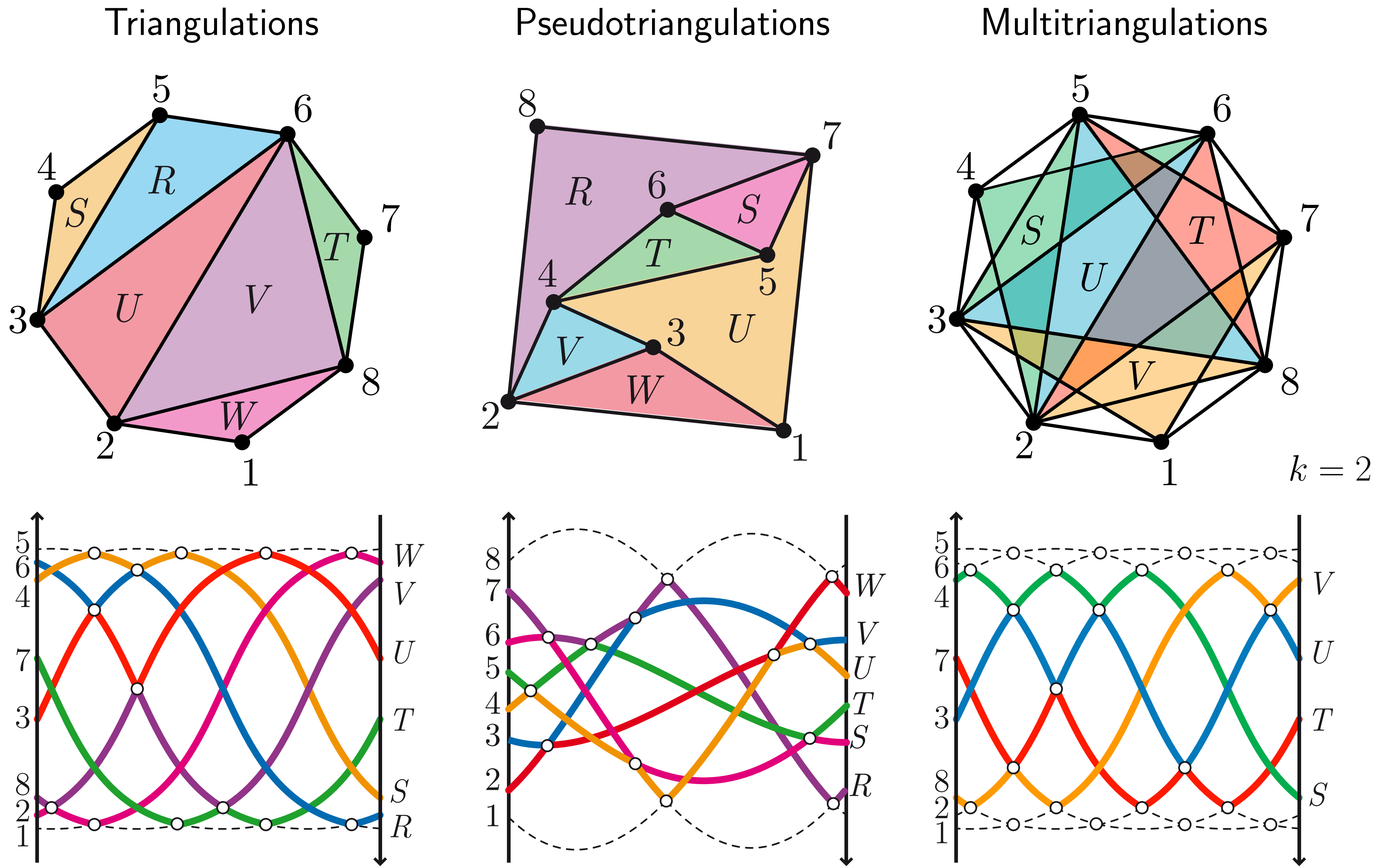
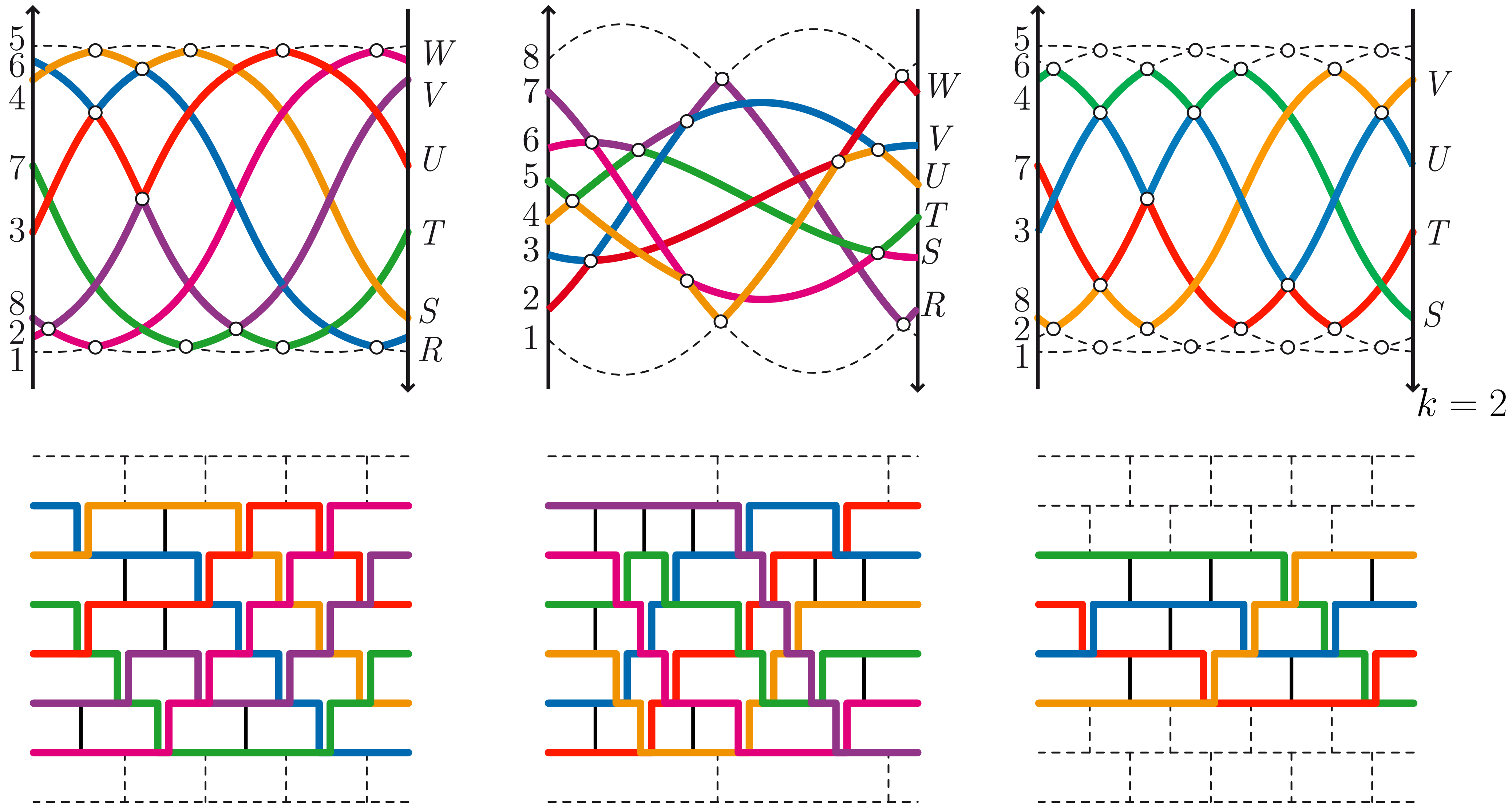


Triangulations
Pseudotriangulations
Multitriangulations
$k = 2$
R S T U V W
1 2 3 4 5 6 7 8

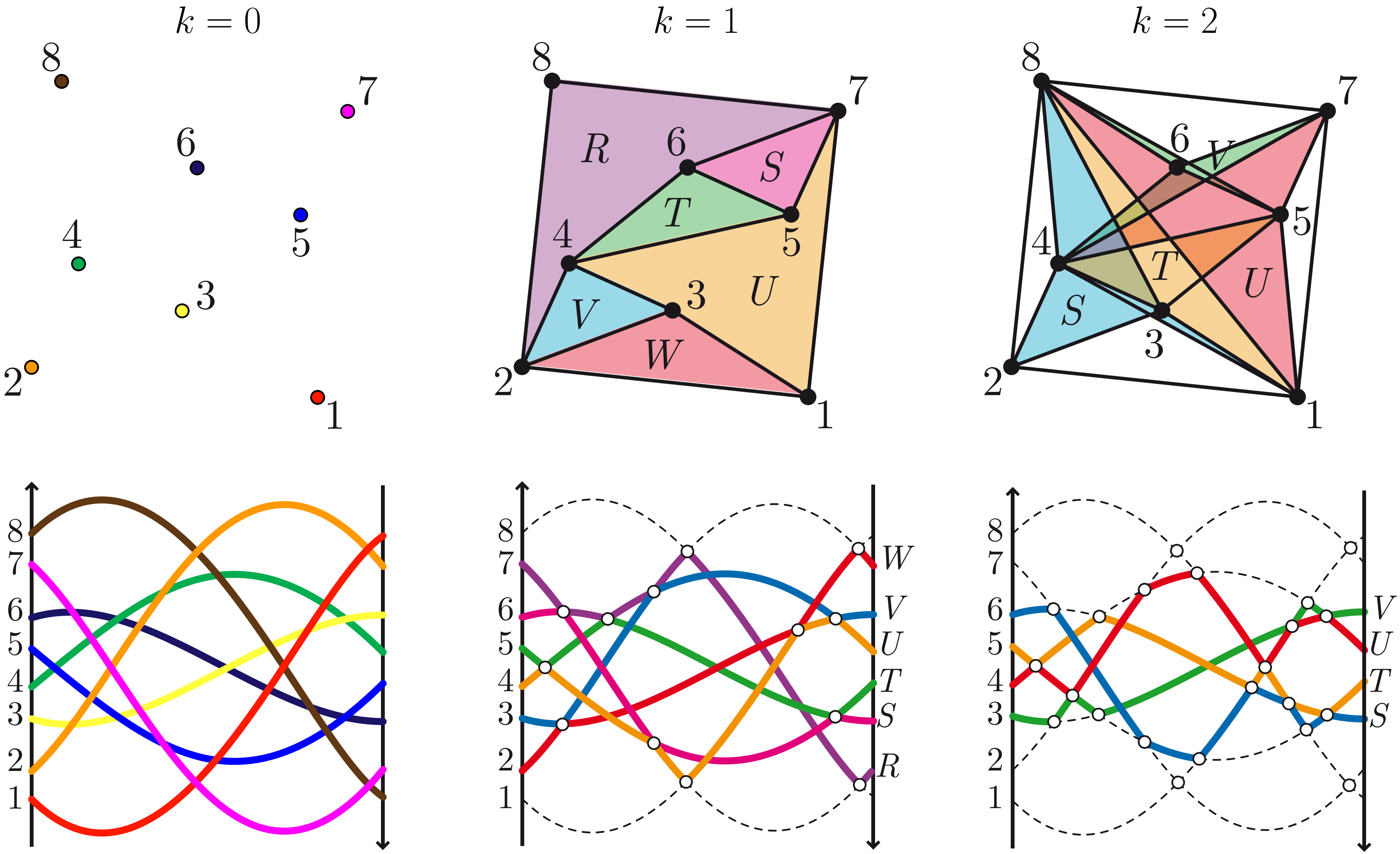
DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2012.

MULTIPSEUDOTRIANGULATIONS



VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2012.

MULTI-DYCK PATHS

THEOREM. The number of k-triangulations of the n-gon is

$$\det(C_{n-i-j})_{1\le i,j\le k} = \left| \begin{pmatrix} C_{n-2} & C_{n-3} & \dots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \dots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \dots & C_{n-2k+1} & C_{n-2k} \end{pmatrix} \right|,$$

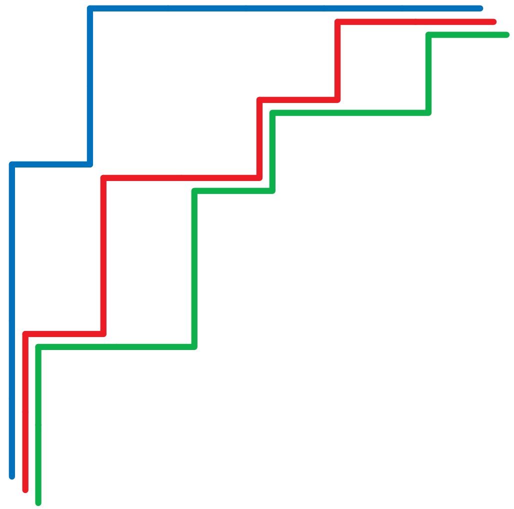
where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the mth Catalan number.

J. Jonsson, Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005.

There is even an explicit bijection

k-triangulations $\longleftrightarrow$ k-Dyck paths.

L. Serrano & C. Stump, Maximal fillings of moon polyominoes, simplicial complexes, and Schubert polynomials, 2011.

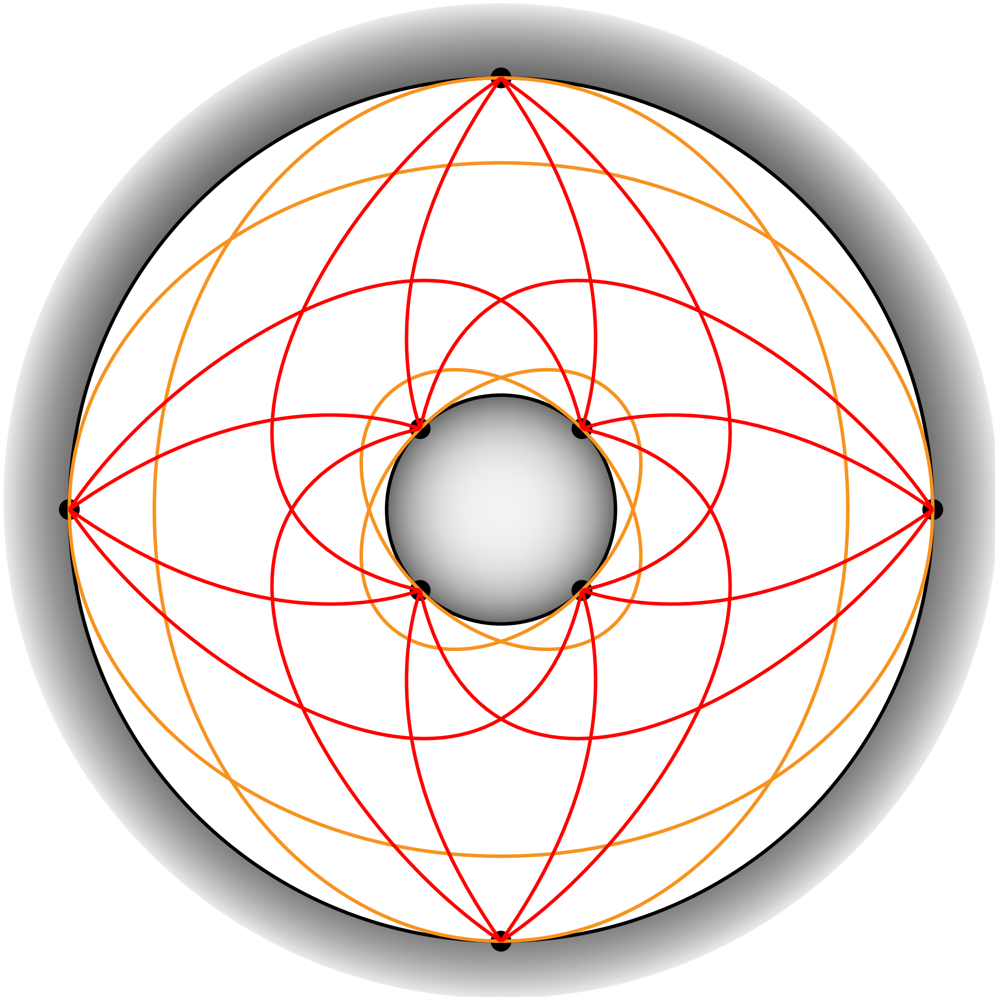


5. Multitriangulations of surfaces

MULTITRIANGULATIONS OF SURFACES

Fix a surface with boundaries, and pick a set of points on the boundaries.

k-triangulation $=$ maximal $(k+1)$-crossing-free set of arcs.

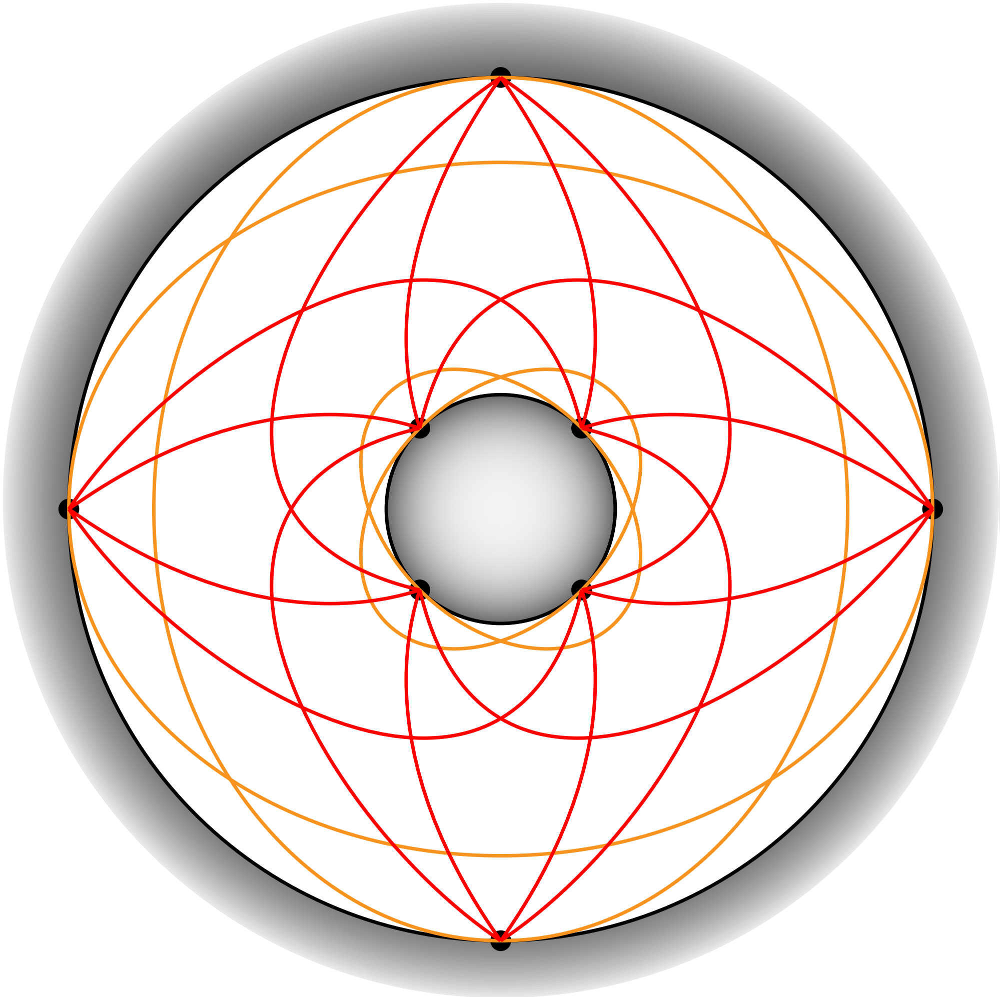


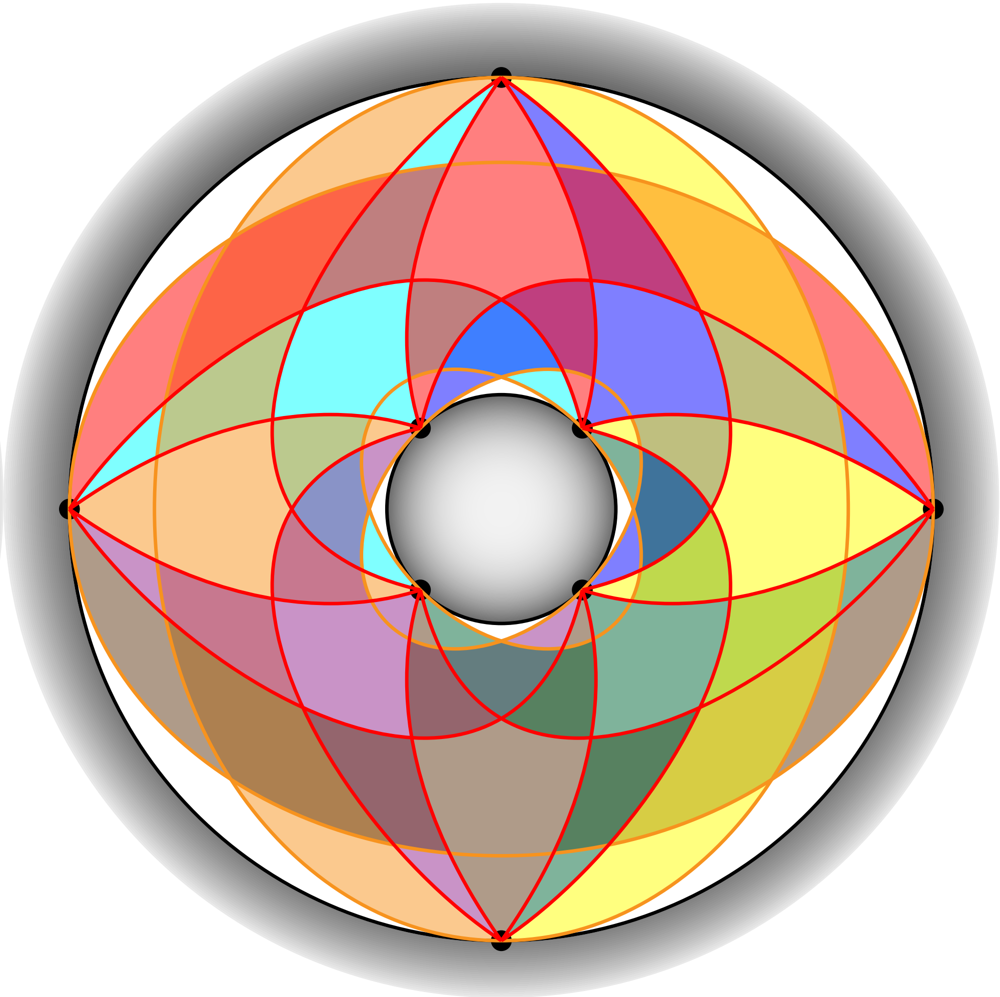
COMPLEX OF STARS

CONJECTURE. Decomposition into k-stars

$$\# \text{ stars} = n + 2k(2g + b - 2)$$

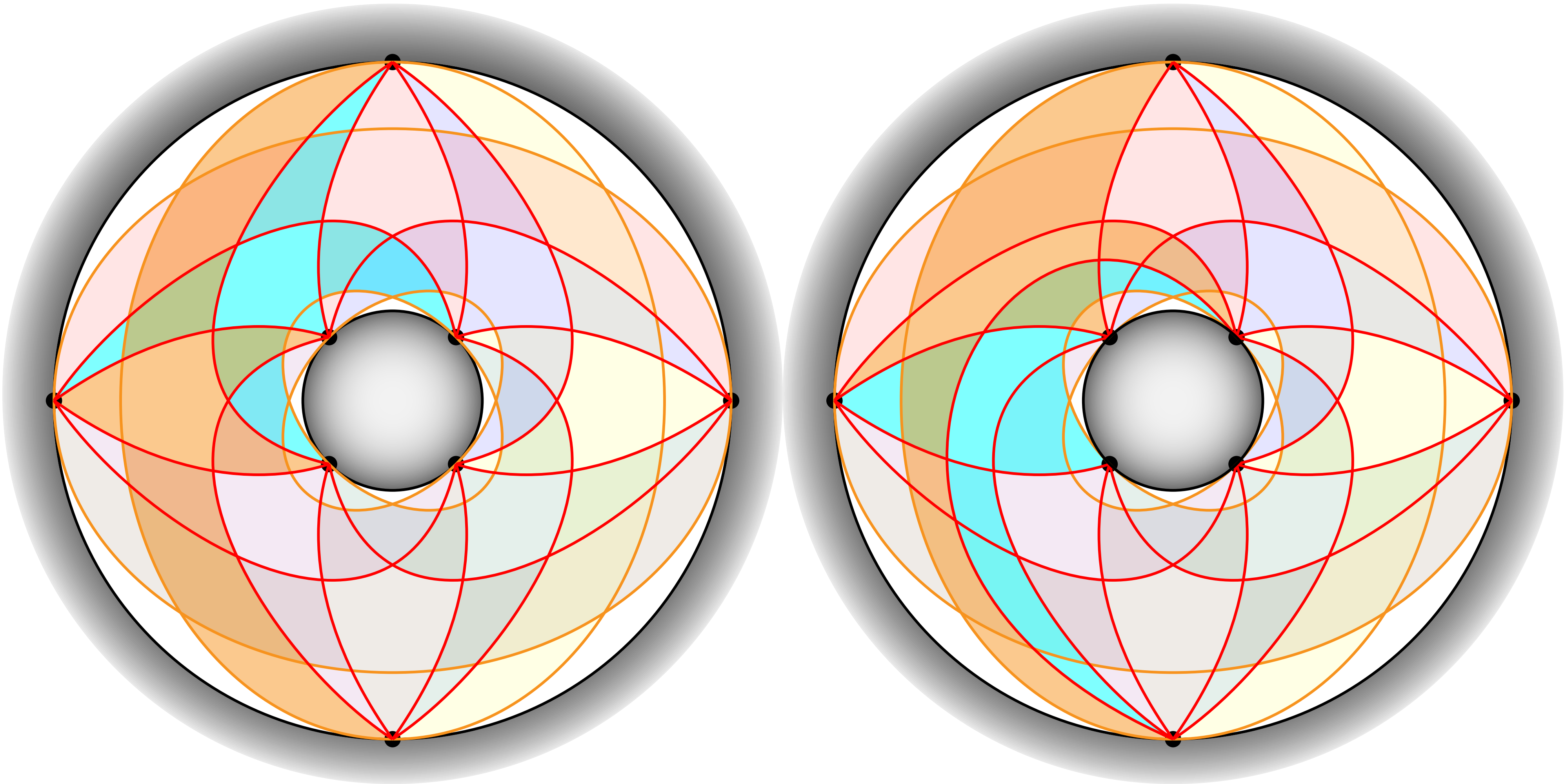
$$\# \text{ internal arcs} = kn + k(2k+1)(2g + b - 2)$$



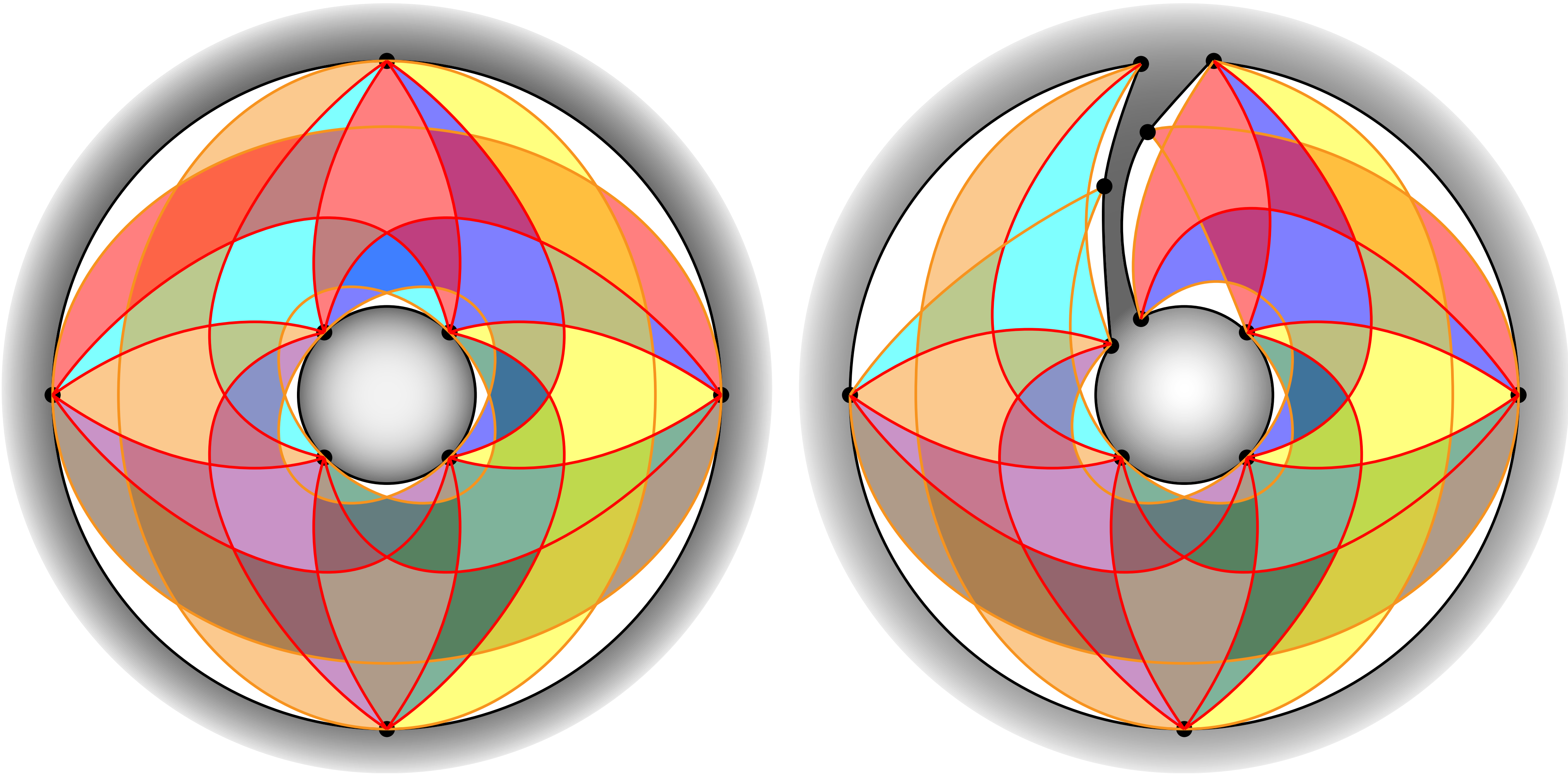


FLIP

CONJECTURE. The flip graph is $kn + k(2k+1)(b-2)$-regular and connected.



CUT OPERATION



cut a handle: $n \to n + 2k,\qquad g \to g - 1,\qquad b \to b + 1.$

cut a border: $n \to n + 2k,\qquad g \to g,\qquad b \to b - 1.$

$\#$ stars $= n + 2k(2g + b - 2)$

$\#$ internal arcs $= kn + k(2k + 1)(2g + b - 2)$

FURTHER QUESTIONS

(i) tamari order?

(ii) manifold ?

(iii) ????