NOTE AND CONJECTURES ON MULTIVARIATE DIAGONAL HARMONICS

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Abstract. This merges the summary of Paris 2019 with the new stuff from Graz 2023. The main heuristic is that for any concave partition there is a multivariate (diagonal) symmetric functions such that when specialize in two parameter (q, t) we get the symmetric function of BHMPS. There should be a module of harmonics behind all this, but the general construction is not yet clear. There is a section on triangular, concave, convex partition on higher rank at the end. (incomplete)

Possible Content:

(1) Triangular and concave partitions
(2) We investigate symmetric functions $\mathcal{E}_\gamma(\boldsymbol{q};\boldsymbol{z})$ associated to triangular and concave partitions γ (conjectured to be $s\otimes s$–positive). The function $\mathcal{E}_\gamma(q,t;\boldsymbol{z})$ should be equal to BHMPS. There should be a module of harmonics behind each partition γ, but the general construction is not yet clear.
(3) The $Q+1$ transformation: $\mathcal{F}_\gamma(\boldsymbol{q};\boldsymbol{z}) = \overline{\mathcal{E}_\gamma(\boldsymbol{q}+1;\boldsymbol{z})}$, $s\otimes e$–positivity conjecture and the stabilization for n.
(4) The γ-Tamari lattice: we conjectured that there exists a wonderful expression for $\mathcal{F}_\gamma(\boldsymbol{q};\boldsymbol{z})$ as a sum over the elements of the γ-Tamari lattice $\mathcal{T}_\gamma$ where

$$\mathcal{F}_\gamma(\boldsymbol{q};\boldsymbol{z}) = \sum_{\mu\in\mathcal{T}_\gamma} \sigma_\mu(\boldsymbol{q}) \otimes e_{type(\mu)},$$

where $\sigma_\mu(\boldsymbol{q})$ is schur positive. We conjecture many properties of the $\sigma_\mu(\boldsymbol{q})$. Some of the property are derived from the module of harmonics we expect behind these symmetric functions. [we talk about polarization in particular].
(5) One of our main advancement is that given γ, let $\mu_1, \mu_2, \ldots, \mu_d$ be the cover of the bottom $\mathbf{0} \in \mathcal{T}_\gamma$. The interval $[\mu_i, 1] \subset \mathcal{T}_\gamma$ gives us a copy of a γ_i-Tamari lattice for some partition γ_i. This decompose $\mathcal{T}_\gamma$ into pieces $\mathcal{T}_{\gamma_i}$ for $1 \le i \le d$. We conjecture that $\mathcal{F}_\gamma(\boldsymbol{q};\boldsymbol{z})$ can be constructed recursively from the $\mathcal{F}_{\gamma_i}(\boldsymbol{q};\boldsymbol{z})$.
(6) We explored the partitions in higher dimensions. We show that convex and concave partitions form a lattice, and conjecture that the intersection of these two lattice is also a lattice.

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1.1. The square case $n = m$. We are studying the Frobenius image of the character of multivariate diagonal harmonics. François shows that it is possible to find a formula

$$\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z}) = \sum_{\substack{\lambda\vdash n\\ \nu\vdash\ell<\binom{n}{2}}} c_{\nu,\lambda} s_\nu(\boldsymbol{q})\otimes s_\lambda(\boldsymbol{z}), \tag{1}$$

where the $c_{\mu,\lambda}$ are positive integers and the specialization $\boldsymbol{q}\mapsto q_1+q_2+\cdots+q_r$ gives the r-graded Frobenius image of the character of the multivariate diagonal harmonics with r set of variables of cardinality n with the diagonal action of S_n. This formula is computed explicitly by computer for $n\in\{1,2,3,4,5,6\}$.

Guided by our heuristic that such formula such is constructed from the Tamari lattice using Hopf chain, we came to the conclusion that it is possible to write $\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z})$ in different ways. Let $\mu\subseteq\delta_n$ be a partition included in the staircase represent Dyck path in the Tamari lattice. Possibly adding zero parts, we require that $\ell(\mu)=n$ to encode n.

Conjecture 1 (Multi-shuffle conjecture)**.** *Let t be a variable, our multishuffle conjecture is*

$$\mathcal{E}_n(\boldsymbol{q}+t;\boldsymbol{z}) = \sum_{\mu\subseteq\delta_n} \sigma_\mu(\boldsymbol{q})\otimes \mathbb{L}_\mu(t;\boldsymbol{z}). \tag{2}$$

Using Hopf chains, and the expression computed by François for $n\leq 6$ we arrive at the conclusion that $\sigma_\mu(\boldsymbol{q})$ is not Schur positive but it should be monomial positive. In fact, for any $\mu\subseteq\delta_n$, we conjecture that it should be possible to find a subset G_μ of good chain among the Hopf chain with top element μ, and a multistatistic composition $\mathrm{mcol}(\mathbf{c})\models n$ for any $\mathbf{c}\in G_\mu$ such that

$$\sigma_\mu(\boldsymbol{q}) = \sum_{\mathbf{c}\in G_\mu} M_{\mathrm{mcol}(\mathbf{c})}(\boldsymbol{q}), \tag{3}$$

where M_α are the monomial quasisymmetric functions.

Using some reasonable additional heuristic, we deduce the following properties of the $\sigma_\mu(\boldsymbol{q})$. For a symmetric function $f=\sum c_\nu s_\nu$, let $\ell(f)=\max\{\ell(\nu)\mid c_\nu\neq 0\}$ and $\deg(f)=\max\{|\nu|\mid c_\nu\neq 0\}$. The area of a $\mu\subseteq\delta_n$ is $\mathrm{area}(\mu)=\binom{n}{2}-|\mu|$.

Property 1. $\ell(\sigma_\mu)=n-|\{k\mid k\in\mu\}|$ *and* $\deg(\sigma_\mu)\leq\mathrm{area}(\mu)$.

Here, $|\{k\mid k\in\mu\}|$ is the number of distinct parts in μ including 0's. This property follows from the definition of the module of multivariate diagonal harmonics. We also derive from the module that

Property 2. $\sigma_{\delta_n}=1$ *and* $\sigma_{0^n}=\langle\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z})\mid s_{1^k}(\boldsymbol{z})\rangle=\mathcal{A}_n(\boldsymbol{q})$ *the* GL*-character of the space of alternating multidiagonal harmonics.*

We decided **not to use** the property $\sigma_\mu=\sigma_{\mu'}$, since we cannot really justify it.

The next property is inspired from the behaviour of (Hopf) chains in the Tamari lattice. For $\mu\subseteq\delta_n$, suppose that $\mu_i=n-i$, that is the Dyck path return to the diagonal, then we can write $\mu=\alpha\odot\beta=[(n-i)^{n-i}+\beta]\cdot\alpha$, that is we decompose the dyck path of μ as the concatenation of the path of α followed by the path of β.

Property 3. *If $\mu = \alpha \odot \beta$, then $\sigma_\mu = \sigma_\alpha \sigma_\beta$.*

We can take the scalar product of $\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z})$ with different symmetric functions, this gives us identities among the σ_μ. It is enough to take the scalar product with Schur function (as it is a basis and we get all possible information. For this we use the $\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z})$ computed with the program $n \le 6$ in the left-hand side and use Equation (2).

Property 4. *For any $\lambda \vdash n$,*

$$\langle\, \mathcal{E}_n(\boldsymbol{q}+t;\boldsymbol{z}) \mid s_\lambda(\boldsymbol{z})\,\rangle = \sum_{\mu\subseteq\delta_n} \langle\, \mathbb{L}_\mu(t;\boldsymbol{z}) \mid s_\lambda(\boldsymbol{z})\,\rangle\; \sigma_\mu(\boldsymbol{q}).$$

Notice that this gives us a property for each $\lambda \vdash n$. Our old property **9** can be derived using s_{1^n}. We have

$$\sigma_{0^n}(\boldsymbol{q}+t) = \langle\, \mathcal{E}_n(\boldsymbol{q}+t;\boldsymbol{z}) \mid s_{1^n}(\boldsymbol{z})\,\rangle = \sum_{\mu\subseteq\delta_n} t^{\mathrm{dinv}(\mu)}\sigma_\mu(\boldsymbol{q}). \tag{4}$$

This follows from the fact that $\langle\, \mathbb{L}_\mu(t;\boldsymbol{z}) \mid s_\lambda(\boldsymbol{z})\,\rangle = t^{\mathrm{dinv}(\mu)}$, since there is a unique parking function π of shape $(\mu+1^n)/\mu$ that will give us $\mathrm{comp}(\pi) = 1^n$ and this is the coefficient of $F_{1^n} = s_{1^n}$ which appear only there. The $\mathrm{dinv}(\pi)$ of that special parking function is what we define to be $\mathrm{dinv}(\mu)$.

1.2. **On Δ conjecture and Tamari localization.** Now we will consider a local version of the Δ-conjecture. Haglund stated that

$$(\Delta'_{e_k} e_n)(q,t;\boldsymbol{z}) = \sum_{\mu\subseteq\delta_n} \Big(\sum_{\substack{J\subseteq[n-1] \\ |J|=k,\ \mathrm{desc}(\mu)\subseteq J}} q^{\sum_{i\in J} a(\mu,i)} \Big) \mathbb{L}_\mu(t;\boldsymbol{z}) \tag{5}$$

where $a(\mu,i) = n - i - \mu_i$ is the contribution to the area in row i, and $\mathrm{desc}(\mu) := \{i \in [n-1] \mid \mu_i > \mu_{i+1}\}$. On the other hand, François conjectured that $e^{\perp_1}_{n-1-k}\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z})$ should be a generalization of the Δ conjecture [We have more on this later as discovered by Vincent]. More precisely

$$\Delta'_{e_k} e_n = (e^{\perp_1}_{n-1-k}\mathcal{E}_n)(q+t;\boldsymbol{z}) \tag{6}$$

If we expand the right-hand side using Equation (2), and use (6) for the left-hand side we get

$$\sum_{\mu\subseteq\delta_n} \Big(\sum_{\substack{J\subseteq[n-1] \\ |J|=k,\ \mathrm{desc}(\mu)\subseteq J}} q^{\sum_{i\in J} a(\mu,i)} \Big) \mathbb{L}_\mu(t;\boldsymbol{z}) = \sum_{\mu\subseteq\delta_n} (e^{\perp}_{n-1-k}\sigma_\mu)(q)\mathbb{L}_\mu(t;\boldsymbol{z}) \tag{7}$$

WARNING The $\mathbb{L}_\mu(t;\boldsymbol{z})$ are not linearly independent but we will assume that their coefficient are equal. Nantel justify this as the equations must be shadows of noncommutative version of LLT polynomials defined in the Catalan Hopf algebra related to Tamari lattice. We will do the same later with non-commutative symmetric functions related to the (dual) boolean lattice.

Taking the coefficient of $\mathbb{L}_\mu(t;\boldsymbol{z})$ on both side of Equation (7) give what we call the Tamari localization of the Δ-conjecture:

Property 5. *For $\mu \subseteq \delta_n$*

$$(e^{\perp}_{n-1-k}\sigma_\mu)(q) = \sum_{\substack{J\subseteq[n-1] \\ |J|=k,\, \mathrm{desc}(\mu)\subseteq J}} q^{\sum_{i\in J} a(\mu,i)}$$

1.3. **Projection of Tamari on the (dual) boolean lattice.** For the next properties, we derive a very surprising identity from Equation (2). Before we need to set a few more notations. Given $\mu \subseteq \delta_n$ we get its descent set $\mathrm{desc}(\mu) \subseteq [n-1]$. It is well known that for a fix $A \subseteq [n-1]$, the set $\{\mu \mid \mathrm{desc}(\mu) = A\} = [\mu^0_A, \mu_A]$ is an interval in the Tamari lattice:

- μ^0_A denotes the bottom of the interval, it is the bounce path defined by A,
- μ_A denotes the top of the interval, it is the steep path defined by A.

The map desc is in fact a lattice projection from the Tamari lattice to the boolean lattice. We sometime use composition to describe $A \subseteq [n-1]$, this has the advantage that n is understood from the composition whereas A alone does not know n. We will denote by $c(\mu)$ the composition subset $\mathrm{desc}(\mu) \subseteq [n-1]$.

Let $r_\alpha(\boldsymbol{z})$ be the ribbon-schur symmetric function defined by the composition $\alpha \models n$. For example, if $n = 6$ and $A = \{2, 4, 5\}$, then $\alpha = 2211$ is the composition associated to A. The ribbon-Schur $r_\alpha = s_{3332/221}$ is a skew Shur function, the shape is a connected ribbon shape with α_i cell in row i.

When we let $t = 0$, the polynomial LLT behave very nicely.

Proposition 1.1.

$$\mathbb{L}_\mu(0; \boldsymbol{z}) = \begin{cases} 0 & \text{if } \mu < \mu_{\mathrm{desc}(\mu)}, \\ r_{c(\mu)}(\boldsymbol{z}) & \text{otherwise.} \end{cases} \tag{8}$$

where Π_μ is the set of parking function of shape $(\mu + 1^n)/u$ (standard tableau).

Proof. Recall that the definition of LLT is

$$\mathbb{L}_\mu(t; \boldsymbol{z}) = \sum_{\pi\in\Pi_\mu} t^{\mathrm{dinv}(\pi)} F_{\mathrm{comp}(\pi)}(\boldsymbol{z})$$

where $\mathrm{dinv}(\pi)$ is XXXX(hahaha), $\mathrm{comp}(\pi)$ is the composition associated with the descent set of the inverse of the diagonal reading the the entry of π [WARNING: do not confuse this with $c(\mu)$] and F are the Gessel fundamental quasisymmetric functions. Now, if we compute $\mathrm{dinv}(\pi)$ we realize that $\mathrm{dinv}(\pi) > 0$ always if $\mu < \mu_{\mathrm{desc}(\mu)}$. That is a consequence to the fact that if we have two columns that are not consecutive than there must be at least one inversion in $\mathrm{dinv}(\pi)$.

Now if we have μ_A for some A, then the columns of $(\mu + 1^n)/u$ are consecutive. The parking functions of shape $(\mu + 1^n)/u$ that have no dinv are exactly the filling of $(\mu + 1^n)/u$ that fit the associated ribbon shape. That shows the second part of the proposition. □

We can now evaluate both sides of Equation (2) with $t = 0$ and obtain a very surprising identity:

Conjecture 2. *We have an equation of $\mathcal{E}$ on the (dual) boolean lattice, namely:*

$$\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z}) = \sum_{A\subseteq[n-1]} \sigma_{\mu_A}(\boldsymbol{q}) \otimes r_{c(\mu_A)}(\boldsymbol{z}). \tag{9}$$

This is a totally new expression and will have many deep consequence. Francois conjecture something a bit crazy about the coefficient above, but this would be fantastic. In particular it would clarify what we want as a generalization of the zeta map and its role with our statistics.

Conjecture 3. *There exists a map ζ_A, a generalization of the zeta map when (exactly the zeta map when to $A=\emptyset$), such that*

$$\Delta\sigma_{\mu_A} = \sum_{\nu\leq\mu_A} \sigma_\nu \otimes \sigma_{\zeta_A(\mu)} \tag{10}$$

The conjectures above should be the shadow on the (dual) boolean lattice of the expression on the Tamari lattice, via a Galois correspondence. As a special case, for all $n\leq 4$, wit our current values of σ_μ we have:

$$\Delta\sigma_0 = \sum_\nu \sigma_\nu \otimes \sigma_{\zeta(\nu)},$$

where ζ stands for the usual "Zeta map". For $n=5$ it also almost works with our current values, since

$$\Delta\sigma_0 - \sum_\nu \sigma_\nu \otimes \sigma_{\zeta(\nu)} = (s_4 - s_3)\otimes s_{22}.$$

Better still, if we had a similar expression for all μ:

$$\Delta\sigma_\mu = \sum_{\nu\preceq\mu} \sigma_\nu \otimes \sigma_{\psi(\nu)}, \tag{11}$$

for some function $\psi=\psi_\mu$, then we would have a natural candidate for "the" multivariate LLT polynomial $\mathbb{L}_\mu = \mathbb{L}_\mu(\boldsymbol{t};\boldsymbol{z})$. Indeed,

$$\begin{aligned}
\mathcal{E}_n(\boldsymbol{q}+\boldsymbol{t}+t;\boldsymbol{z}) &= \sum_\mu \sigma_\mu(\boldsymbol{q}+\boldsymbol{t})\otimes \mathbb{L}_\mu(t;\boldsymbol{z}),\\
&= \sum_\mu \sum_{\nu\preceq\mu} \sigma_\nu(\boldsymbol{q})\otimes \sigma_{\psi_\mu(\nu)}(\boldsymbol{t})\otimes \mathbb{L}_\mu(t;\boldsymbol{z}),\\
&= \sum_\nu \sigma_\nu(\boldsymbol{q})\otimes \sum_{\mu\succeq\nu} \sigma_{\psi_\mu(\nu)}(\boldsymbol{t})\otimes \mathbb{L}_\mu(t;\boldsymbol{z}),
\end{aligned}$$

so that we could then set

$$\mathbb{L}_\mu(\boldsymbol{t};z) = \sum_{\nu\succeq\mu} \sigma_{\psi_\mu(\nu)}(\boldsymbol{t}-t)\otimes \boldsymbol{L}_\nu(t;\boldsymbol{z}),$$

to get a multivariate extension of LLT-polynomial for all μ. By another method (previous to the coproduct approach), François and Vincent had already successfully calculated such extensions of LLT-polynomials for all μ with $n\leq 4$, and part of $n=5$.

(FRANÇOIS SAYS: I AM LOST HERE. The next sentence does not seem to relate to a link between Conjecture 1 and Conjecture 3. I think that it goes with Conjecture 2 and that Nantel forgot to move it to the right place when he added the above coproduct-zeta conjecture.) This conjecture is a consequence of Conjecture 1. We now put $t=1$ in Equation (2) and $\boldsymbol{q}+1$ in (10). We have to remark that LLT symmetric function at $t=1$ is a product of elementary symmetric function, but we want to expand it as a positive sum of ribbon Schur function (WARNING: this is not unique, we do it in a non-commutative way). Let $B = \mathrm{desc}(\nu)$. We have

$$\mathbb{L}_\nu(1;\boldsymbol{z}) = e_\alpha = \sum_{A\subset B} r_{c(A)}(\boldsymbol{z}).$$

The second equality is a direct consequence of jeux de taquin. Let $\alpha = c(A)$, let $t=1$ in Equation (2) and take the coefficient of r_α (WARNING: the r_β are not linearly independent but we imagine that this in written in non-commutative symmetric functions.). We compare this to the coefficient of r_α in Equation (10) with $\boldsymbol{q}+1$. We get

$$\sigma_{\mu_A}(\boldsymbol{q}+1) = \sum_{\nu : A\subseteq \mathrm{desc}(\nu)} \sigma_\nu(\boldsymbol{q}) = \sum_{\nu\le\mu_A} \sigma_\nu(\boldsymbol{q}).$$

A (dual) boolean lattice localization (coefficient of $r_{c(A)}$) at $\boldsymbol{q}+1$ gives our old Property $\mathbf{9}'$:

Property 6. *For any $A\subseteq[n-1]$, let μ_A be the steep shape corresponding to A:*

$$\sigma_{\mu_A}(\boldsymbol{q}+1) = \sum_{\nu\le\mu_A} \sigma_\nu(\boldsymbol{q})$$

1.4. **Revisiting Δ conjecture with the (dual) boolean lattice equation.** Another use of (10) is via the following identity that François proved:

$$e_k^\perp \sigma_{0^n}(\boldsymbol{q}) = \left\langle\, \mathcal{E}_n(\boldsymbol{q};\boldsymbol{z}) \;\middle|\; s_{(k|n-k-1)}(\boldsymbol{z}) \,\right\rangle,$$

using Frobenius notation $(k|n-k-1)$ for the hook shape partition $(k+1)1^{n-k-1}$. We remark that

$$\left\langle\, r_{c(A)} \;\middle|\; s_{(k|n-k-1)} \,\right\rangle = \delta_{|A|=k},$$

hence taking the scalar product $\left\langle\, \mathcal{E}_n(\boldsymbol{q};\boldsymbol{z}) \;\middle|\; s_{(k|n-k-1)}(\boldsymbol{z}) \,\right\rangle$ using (10) gives

Property 7. *For $k\le n-1$*

$$e_k^\perp \sigma_{0^n}(\boldsymbol{q}) = \sum_{A:\ |A|=k} \sigma_{\mu_A}$$

This is a reflection of what should be a complete general Δ-conjecture. From François expression above and the Equation (6), we say that $e_k^{\perp 1}\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z})$ should be expressible as a generalization of the Δ-conjecture. Before we do so, let us use now the Equation (10) in Equation (6):

$$\Delta'_{e_k} e_n = (e^{\perp_1}_{n-1-k}\mathcal{E}_n)(q,t;\boldsymbol{z}) = \sum_{A\subseteq[n-1]} (e^\perp_{n-1-k}\sigma_{\mu_A})(q,t)\, r_{c(\mu_A)} \tag{12}$$

Property 8. *For $k\le n-1$*

$$\sum_{A\subseteq[n-1]} \Big(e^\perp_{n-1-k}\sigma_{\mu_A}\Big)(q,t)\, r_{c(\mu_A)}(\boldsymbol{z}) = \sum_{\mu\subseteq\delta_n} \Bigg(\sum_{\substack{J\subseteq[n-1]\\ |J|=k,\, \mathrm{desc}(\mu)\subseteq J}} q^{\sum_{i\in J} a(\mu,i)} \Bigg) \mathbb{L}_\mu(t;\boldsymbol{z})$$

The Property 5 is in spirit a Tamari localization of $t = 0$ in the above expression. We can also set $t = 1$ and do a (dual) boolean lattice localization at $A \subseteq [n-1]$:

$$\left(e^{\perp}_{n-1-k}\sigma_{\mu_A}\right)(q,1) = \sum_{\nu \leq \mu_A} \Bigg(\sum_{\substack{J \subseteq [n-1] \\ |J|=k,\, \mathrm{desc}(\nu) \subseteq J}} q^{\sum_{i\in J} a(\mu,i)} \Bigg) \tag{13}$$

1.5. **Anklet and generalized Δ-conjecture.** The anklet $\mathrm{ank}(\mathbf{c})$ of a Hopf chain $\mathbf{c} = (c_1, \dots, c_r)$ is the maximal number of paths that can be inserted between the first two paths c_1 and c_2 of the chain $\mathbf{c}$ so that it remains a Hopf chain. Note that the anklet looks like the area of the second path c_2, but it is not since the rest of the chain matters.

The anklet statistic can be decomposed according to the level of the flips. Namely, given a Hopf chain $\mathbf{c} = (c_1, \dots, c_r)$, consider a Hopf chain $\mathbf{C}$ containing $\mathbf{c}$ together with $\mathrm{ank}(\mathbf{c})$ additional paths in between the first two paths c_1 and c_2 of $\mathbf{c}$. Let $\mathrm{ank}_p(\mathbf{c})$ be the number of flips at the horizontal level p that were made in the part of $\mathbf{C}$ between the first two paths of $\mathbf{c}$. This decomposes the anklet into $\mathrm{ank}(\mathbf{c}) = \mathrm{ank}_1(\mathbf{c}) + \cdots + \mathrm{ank}_n(\mathbf{c})$. (WARNING: it is not clear that this decomposition is independent of the choice of the superchain $\mathbf{C}$.) It yields the following generalized Δ-conjecture:

$$(e^{\perp}_{n-1-k}\mathcal{E}_n)(q,t,1^{r-2};\boldsymbol{z}) = \sum_{\substack{\mathbf{c}\text{ good chain} \\ \text{of length } r}} \sum_{\substack{J \subseteq [n-1] \\ |J|=k,\, \mathrm{desc}(\mathbf{c}) \subseteq J}} q^{\sum_{p\in J} \mathrm{ank}_p(\mathbf{c})} \otimes \mathbb{L}_{c_r}(t;\boldsymbol{z}),$$

where $\mathrm{desc}(\mathbf{c}) = \{i \mid \mathrm{ank}_i(\mathbf{c}) < \mathrm{ank}_{i-1}(\mathbf{c})\}$. Note that this can also be computed using strict chains as follows:

$$(e^{\perp}_{n-1-k}\mathcal{E}_n)(q,t,1^{r-2};\boldsymbol{z}) = \sum_{\ell} \sum_{\substack{\mathbf{c}\text{ strict good} \\ \text{chain of length } \ell}} \sum_{\substack{J \subseteq [n-1] \\ |J|=k,\, \mathrm{desc}(\mathbf{c}) \subseteq J}} \left(\mathbf{1}_{k=0}\binom{r-2}{\ell-1} + q^{\sum_{p\in J} \mathrm{ank}_p(\mathbf{c})}\binom{r-2}{\ell-2} \right) \otimes \mathbb{L}_{c_r}(t;\boldsymbol{z}).$$

This was checked by computer for $k < n \leq 4$. For $n = 5$, we do not have the right notion of good chains.

1.6. **Some new combinatorial identities.** When we evaluate Equation (10) at $\boldsymbol{q} \mapsto q$ or $\boldsymbol{q} \mapsto q + t$ we get some new combinatorial identities that one should be able to prove independently of our conjecture. For $\boldsymbol{q} \mapsto q$, the symmetric function $\mathcal{E}_n(q;\boldsymbol{z})$ is the graded action of S_n on the cohomology of the Flag manifold, or the graded action of S_n on the quotient of polynomials in one set of variables by the ideals of symmetric function. This is given by the Hall-Littlewood symmetric function indexed by 1^n. Now evaluate Equation (10) at $\boldsymbol{q} \mapsto q$, you get

$$\frac{h_n\big[\boldsymbol{z}/(1-q)\big]}{h_n\big[1/(1-q)\big]} = \sum_{A \subseteq [n-1]} q^{\mathrm{area}(\mu_A)} r_{c(\mu_A)}(\boldsymbol{z}). \tag{14}$$

Is this expression known? Nantel has done this expression before (unpublished?) and think Lascoux has done it also a long time ago. This follow from the shuffle theorem in q, t, putting $t = 0$. Or equivalently, put $k = n - 1$ and $t = 0$ in Formula (6).

Now at $\boldsymbol{q} \mapsto q + t$ this is **really** new. Evaluating Equation (10) at $\boldsymbol{q} \mapsto q$ gives us

$$\nabla e_n = \sum_{A \subseteq [n-1]} \sigma_{\mu_A}(q,t) r_{c(\mu_A)}. \tag{15}$$

But even more combinatorially, if we use Equation (3) evaluated at $q+t$ we get at most the two chains:

$$\sigma_{\mu_A}(q,t) = \sum_{\nu\leq\mu_A} q^{\mathrm{ank}(\delta_n,\nu,\mu_A)} t^{\beta(\delta_n,\nu,\mu_A)}. \tag{16}$$

The statistic $\mathrm{ank}(\delta_n,\nu,\mu_A)$ is the anklet statistic. It is almost the area of ν? At least when we set $t=1$ and $k=n-1$ in Property 8 we get

$$\sigma_{\mu_A}(q,1) = \sum_{\nu\leq\mu_A} q^{\mathrm{area}(\nu)},$$

and this agrees with Property 6. Hence we conjecture

Conjecture 4. *Given $A\subseteq[n-1]$ and $\nu\leq\mu_A$*

$$\mathrm{ank}(\delta_n,\nu,\mu_A) = \mathrm{area}(\nu).$$

Idea of proof. There is a unique maximal Hopf chain $(\delta_n, c_1, \dots, c_{\ell-1}, \nu)$ such that $\ell = \mathrm{area}(\nu)$. We know that chain well. We just need to show that $(\delta_n, c_1, \dots, c_{\ell-1}, \nu, \mu_A)$ is Hopf!?

SECOND ARGUMENT: If we use Conjecture 3 something nice happen.

$$\sigma_{\mu_A}(q+t) = \sum_{\nu\leq\mu_A} \sigma_\nu(q)\sigma_{\zeta_A(\nu)}(t) = \sum_{\nu\leq\mu_A} q^{\mathrm{area}(\nu)} t^{\mathrm{area}(\zeta_A(\nu))}.$$

This not only show Conjecture 4, but also say that $\beta(\delta_n,\nu,\mu_A) = \mathrm{area}(\zeta_A(\nu))$ □

Regardless of Conjecture 4 being true or false, we should work hard to understand the values of ank and β for this case and show

$$\nabla e_n = \sum_{A\subseteq[n-1]} \Big[\sum_{\nu\leq\mu_A} q^{\mathrm{ank}(\delta_n,\nu,\mu_A)} t^{\beta(\delta_n,\nu,\mu_A)} \Big] r_{c(\mu_A)}. \tag{17}$$

It would give us so much insight on the statistic we are seeking and ∇e_n is well studied. We should look at $\sigma_{\mu_A}(1,t)$... put

$$\sum_{A\subseteq[n-1]} \sigma_{\mu_A}(1,t) r_{c(\mu_A)}(\boldsymbol{z}) = \sum_{\mu} \mathbb{L}_\mu(t;\boldsymbol{z}) = (\nabla e_n)(1,t;\boldsymbol{z}).$$

The expression on the right is well known (but Nantel forgot).

1.7. Some open questions and dreaming bigger.

(1) Can we do localization for other lattices in between Tamari and the (dual) boolean lattice? [Nantel expec this to work if there is an associated Hopf algebra]. We can push the idea in the other direction, Is there an analogue of Formula (2) over permutations? We can then localize over the Bruhat order? That would give us even more refined properties
(2) Our exploration suggests that we should be able to construct the σ_μ via operators applied to $\mathbf{1}$. These operators could be "dual" to the Carlson-Mellit operator?
(3) We feel there should be a collar version of dinv

(4) We have to understand better what happens when the variable move from one side of the tensor to the other side in the expression $\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z})$. François' experiments suggest (explicit expressions for $n \leq 4$, and partial for $n = 5$) that there is in fact a nice expansion of the form

$$\mathcal{E}_n(\boldsymbol{q}+\boldsymbol{t};\boldsymbol{z}) = \sum_{\mu\subseteq\delta_n} \sigma_\mu(\boldsymbol{q})\mathbb{L}_\mu(\boldsymbol{t};\boldsymbol{z}), \tag{18}$$

with $\boldsymbol{t}$ a general set of variables. To get these it is interesting to exploit the specialization that corresponds to Conjecture 1 (all left terms of length 1 in $\mathbb{L}_\mu(\boldsymbol{t};\boldsymbol{z})$ correspond to classical LLT: $\mathbb{L}_\mu(t;\boldsymbol{z})$), as well as the fact that the formula has to be $(\boldsymbol{q},\boldsymbol{t})$-symmetric, so that one has

$$\sum_{\mu\subseteq\delta_n} \sigma_\mu(\boldsymbol{q})\mathbb{L}_\mu(\boldsymbol{t};\boldsymbol{z}) = \sum_{\mu\subseteq\delta_n} \sigma_\mu(\boldsymbol{t})\mathbb{L}_\mu(\boldsymbol{q};\boldsymbol{z}).$$

Calculation suggest that there are nice stabilities (to be described later) in the "coefficients".

2. Tables

Here are some values of σ_μ, for $n \leq 5$.

$n = 1$

$$\sigma_0 = 1;$$

$n = 2$

$$\sigma_{00} = s_1, \qquad \sigma_{10} = 1;$$

$n = 3$

$$\sigma_{000} = s_{11} + s_3, \qquad \sigma_{100} = s_2, \qquad \sigma_{200} = s_1, \qquad \sigma_{110} = s_1, \qquad \sigma_{210} = 1;$$

$n = 4$

$$\begin{aligned}
&\sigma_{0000} = s_{111} + s_{31} + s_{41} + s_6,\\
&\sigma_{1000} = s_{31} + s_5, \qquad \sigma_{2000} = s_{21} + s_4, \qquad \sigma_{3000} = s_{11} + s_3,\\
&\sigma_{1100} = s_{21} + s_4, \qquad \sigma_{2200} = s_{11} + s_2, \qquad \sigma_{1110} = s_{11} + s_3,\\
&\sigma_{3100} = s_2, \qquad \sigma_{2100} = s_3, \qquad \sigma_{2110} = s_2, \qquad \sigma_{3200} = s_1,\\
&\sigma_{3110} = s_1, \qquad \sigma_{2210} = s_1, \qquad \sigma_{3210} = 1.
\end{aligned}$$

$n = 5$

$$\sigma_{00000} = s_{1111} + s_{311} + s_{411} + s_{42} + s_{43} + s_{511} + s_{61} + s_{62} + s_{71} + s_{81} + s_{10.},$$
$$\sigma_{10000} = s_{33} + s_{411} + s_{52} + s_{61} + s_{71} + s_{9},$$
$$\sigma_{20000} = s_{311} + s_{42} + s_{51} + s_{61} + s_{8},$$
$$\sigma_{30000} = s_{211} + s_{32} + s_{41} + s_{51} + s_{7},$$
$$\sigma_{40000} = s_{111} + s_{31} + s_{41} + s_{6},$$
$$\sigma_{11000} = s_{311} + s_{42} + s_{51} + s_{61} + s_{8},$$
$$\sigma_{22000} = s_{211} + s_{22} + 2s_{41} + s_{6}, \qquad \sigma_{33000} = s_{111} + s_{21} + s_{31} + s_{4},$$
$$\sigma_{21000} = -s_{22} + s_{32} + s_{51} + s_{7}, \qquad \sigma_{31000} = s_{22} + s_{41} + s_{6}, \qquad \sigma_{41000} = s_{31} + s_{5},$$
$$\sigma_{11100} = s_{211} + s_{32} + s_{41} + s_{51} + s_{7}, \qquad \sigma_{11110} = s_{111} + s_{31} + s_{41} + s_{6},$$
$$\sigma_{21100} = s_{22} + s_{41} + s_{6}, \qquad \sigma_{31100} = s_{31} + s_{5}, \qquad \sigma_{41100} = s_{21} + s_{4},$$
$$\sigma_{32000} = s_{31} + s_{5}, \qquad \sigma_{42000} = s_{21} + s_{4}, \qquad \sigma_{43000} = s_{11} + s_{3},$$
$$\sigma_{21110} = s_{31} + s_{5}, \qquad \sigma_{22100} = s_{31} + s_{5},$$
$$\sigma_{32100} = s_{4}, \qquad \sigma_{31110} = s_{21} + s_{4},$$
$$\sigma_{22200} = s_{111} + s_{21} + s_{31} + s_{4}, \qquad \sigma_{22110} = s_{21} + s_{4},$$
$$\sigma_{42100} = s_{3}, \qquad \sigma_{41110} = s_{11} + s_{3}, \qquad \sigma_{33100} = s_{21} + s_{3}, \qquad \sigma_{32200} = s_{21} + s_{3},$$
$$\sigma_{32110} = s_{3}, \qquad \sigma_{22210} = s_{11} + s_{3}, \qquad \sigma_{43100} = s_{2}, \qquad \sigma_{42200} = s_{11} + s_{2},$$
$$\sigma_{42110} = s_{2}, \qquad \sigma_{33200} = s_{11} + s_{2}, \qquad \sigma_{33110} = s_{11} + s_{2}, \qquad \sigma_{32210} = s_{2},$$
$$\sigma_{43200} = s_{1}, \qquad \sigma_{43110} = s_{1}, \qquad \sigma_{42210} = s_{1}, \qquad \sigma_{33210} = s_{1}, \qquad \sigma_{43210} = 1.$$

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