Combinatorics of Rectangulations

Jean Cardinal
Université libre de Bruxelles (ULB)

Jagiellonian TCS Seminar – April 2024

Collaborators

Andrei Asinowski   Stefan Felsner   Éric Fusy   Vincent Pilaud



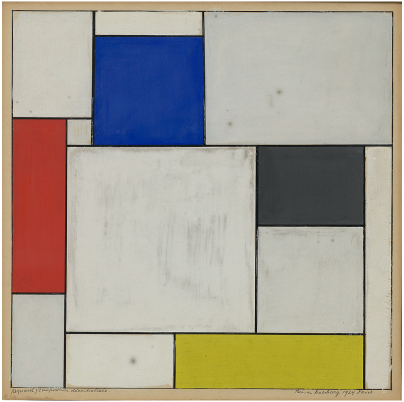

Order & Geometry 2022. September 13-18, 2022, Ciążeń, Poland

Stefan Felsner and Piotrek Micek

Combinatorics, Algorithms, and Geometry. March 4-8, 2024, Dresden, Germany

Namrata and Torsten Mütze

Rectangulations and Combinatorics



Bijective Permutation classes

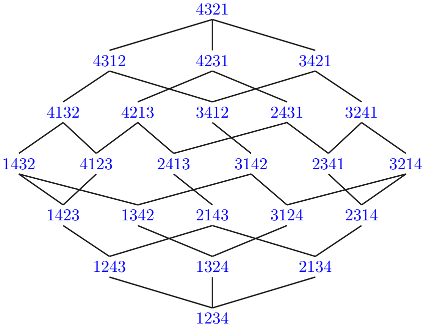
Order-theoretic Lattice congruences

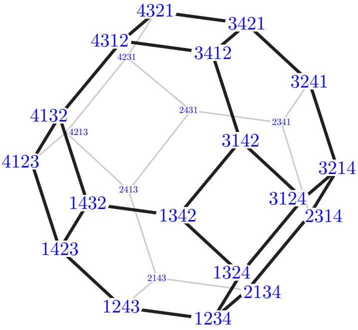
Polyhedral Generalized permutahedra

Permutations

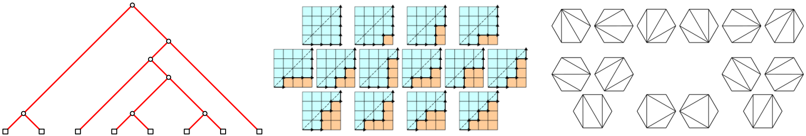
Order-theoretic Weak order

Polyhedral Permutahedron





Binary trees



(Wikimedia commons)

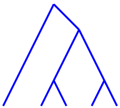
Bijective Catalan families

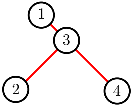
Order-theoretic Tamari lattice

Polyhedral Associahedra

Tamari lattice

Linear extensions of binary tree posets.

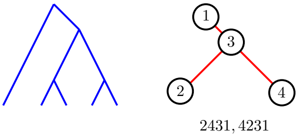


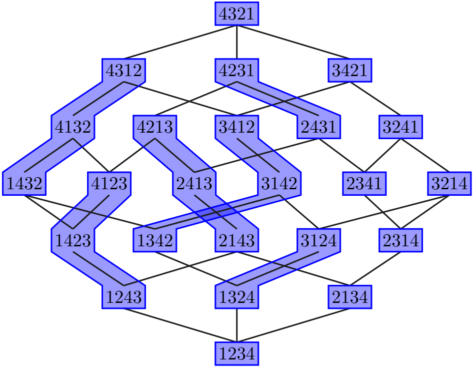


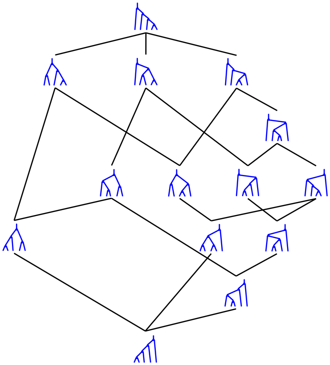
$2431, 4231$

Tamari lattice

Linear extensions of binary tree posets.

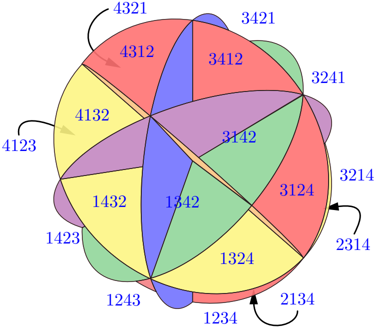


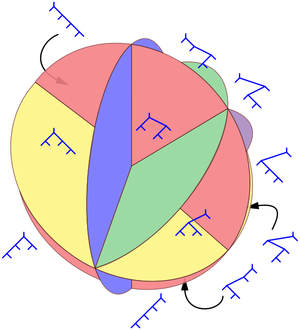




Tamari 1951

The sylvester congruence





Lattice congruences

An equivalence relation $\equiv$ is a lattice congruence if it respects joins and meets:

$$(x \equiv x' \text{and } y \equiv y') \implies (x \vee y \equiv x' \vee y' \text{and } x \wedge y \equiv x' \wedge y')$$

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Theorem

For any congruence of the weak order, gluing together the cones of the braid fan corresponding to the same congruence class yields the normal fan of a polytope.

Pilaud-Santos 2019

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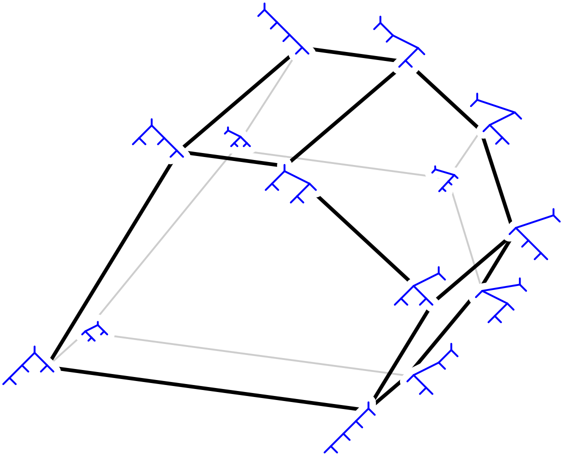
Pilaud-Santos 2019

The cover graph of the lattice quotient is the skeleton of the polytope, called a quotientope.

Associahedra



Stasheff 1963

Loday coordinates

Theorem

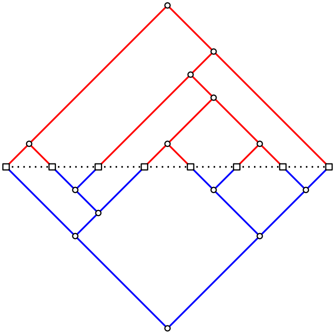
The associahedron is realized by the convex hull of the points

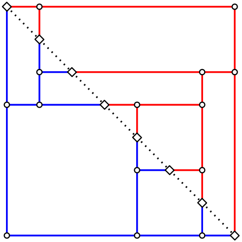
$$\sum_{i \in [n]} \ell_i^T \cdot r_i^T \cdot \mathbf{e}_i$$

for all binary trees T with n internal nodes, where ℓ_i^T and r_i^T denote the number of leaves in the left and right subtrees of i in T.

Loday 2004

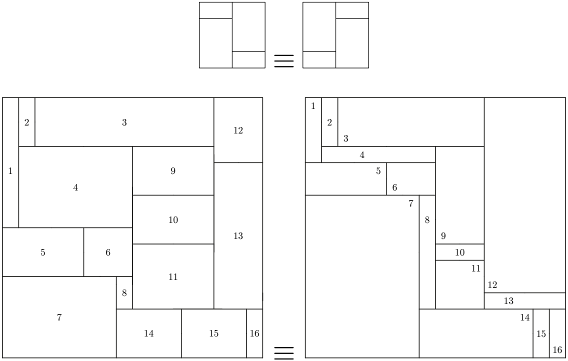
Diagonal rectangulations and twin binary trees





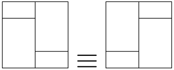
Gluing two “twin” binary trees yields a diagonal rectangulation.

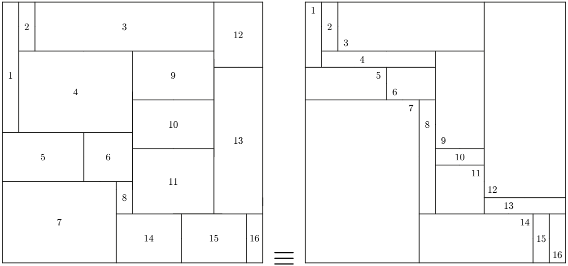
Weak equivalence relation:



Weak rectangulations

Weak equivalence relation:

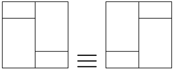


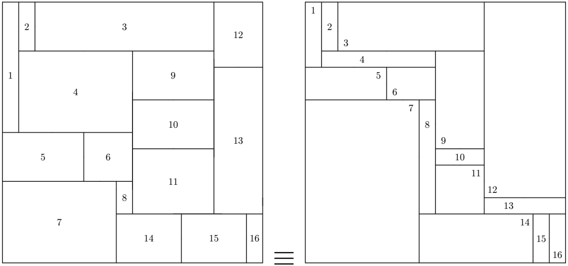


Bijective Baxter permutations, avoiding 2413 and 3142

Weak rectangulations

Weak equivalence relation:



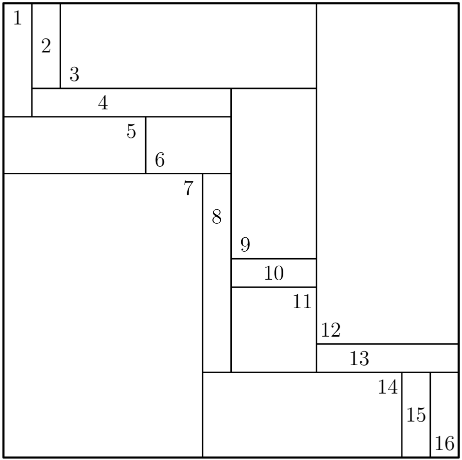


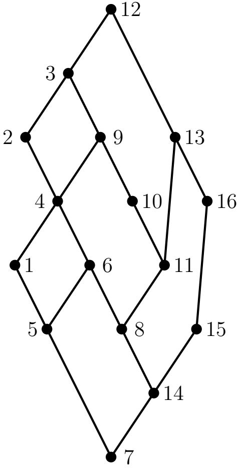
Bijective Baxter permutations, avoiding 2413 and 3142

Order-theoretic Weak rectangulation congruence

Polyhedral Weak rectangulotopes

Adjacency poset

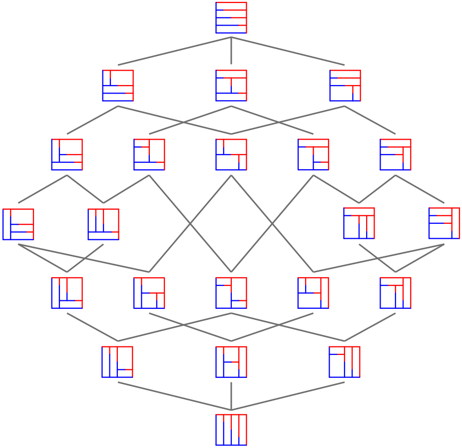




Meehan 2016

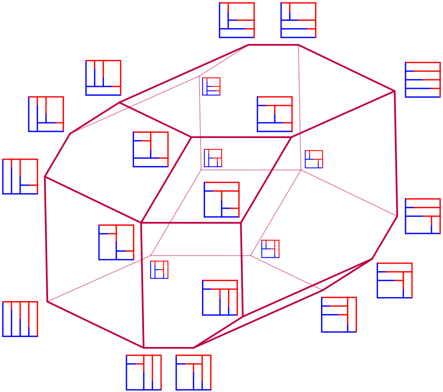
The weak rectangulation congruence

Linear extensions of adjacency posets.



Law-Reading 2010

Weak rectangulotopes



Theorem

Weak rectangulotopes are Minkowski sums of two opposite associahedra.

Law-Reading 2010

Loday coordinates for weak rectangulotopes

Theorem

The weak rectangulotope is realized by the convex hull of the points

$$\sum_{i \in [n]} \left(\overset{\frown}{w}{}_i^R - \overset{\smile}{w}{}_i^R \right) \cdot \boldsymbol{e}_i$$

for all weak rectangulations R of size n, with

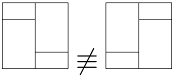
$$\overset{\frown}{w}{}_i^R := h_i^T \cdot v_i^T \qquad \textit{and} \qquad \overset{\smile}{w}{}_i^R := h_i^S \cdot v_i^S,$$

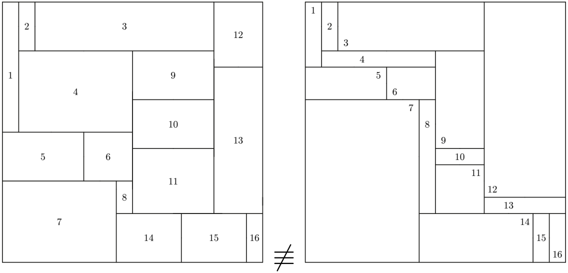
where S and T are the twin binary trees of the rectangulation, and h_i^T and v_i^T denote the number of leaves in the horizontal and vertical subtrees of i in T.

Law-Reading 2010, C.-Pilaud 2024

Strong rectangulations

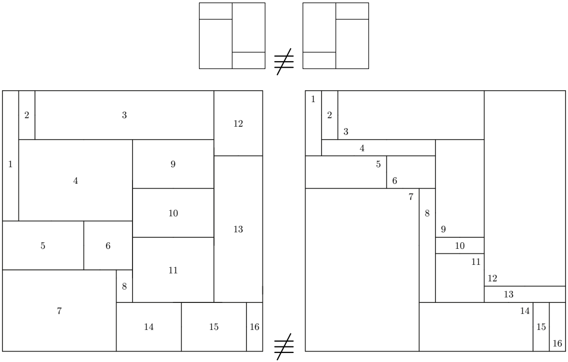
Strong equivalence relation:





Strong rectangulations

Strong equivalence relation:

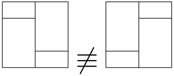


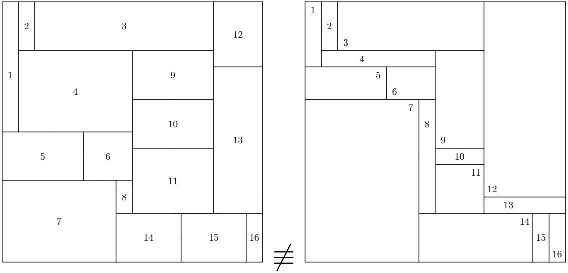
Bijective 2-clumped permutations – forbidding the patterns
35124, 35142, 24513, and 42513

Reading 2012

Strong rectangulations

Strong equivalence relation:





Bijective 2-clumped permutations – forbidding the patterns 35124, 35142, 24513, and 42513

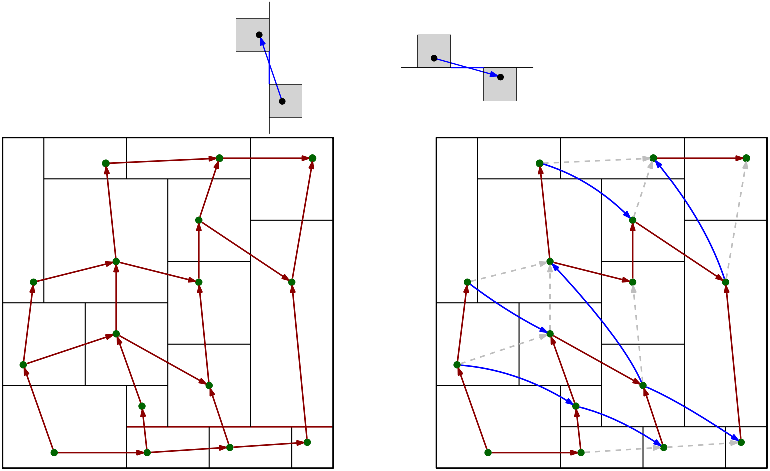
Reading 2012

Order-theoretic strong rectangulation lattice

Polyhedral strong rectangulotopes

Strong poset

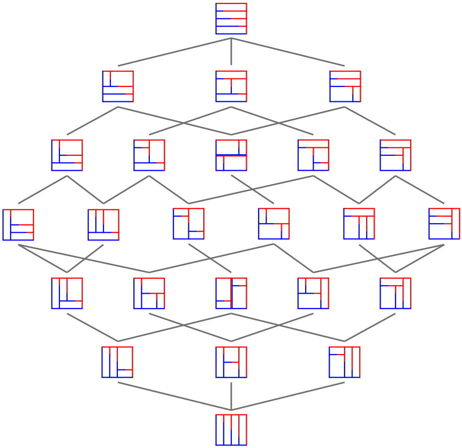
Adjacency, and two “special relations”:



Asinowski, C., Felsner, Fusy 2024

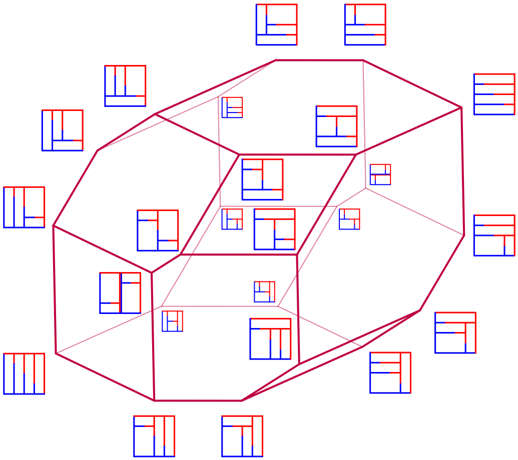
The strong rectangulation congruence

Linear extensions of strong posets.



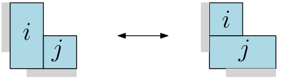
Reading 2012, Asinowski, C., Felsner, Fusy 2024

Strong rectangulotopes

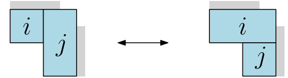


Exist thanks to Pilaud-Santos 2019

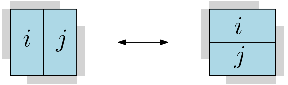
Flip graph



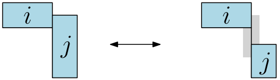
left-bottom pivoting



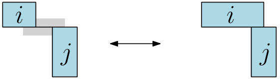
right-top pivoting



simple flip



vertical wall slide



horizontal wall slide

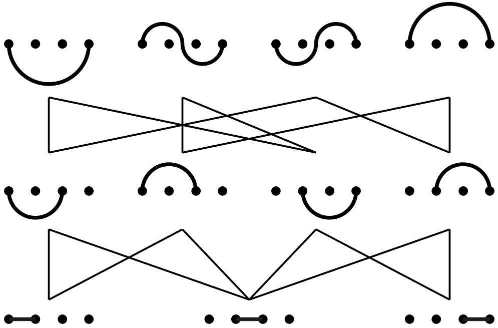
Meehan 2019

Congruences and arc diagrams

Theorem

Congruences $\equiv$ of the weak order are one-to-one with arc ideals $\mathcal{A}_{\equiv}$.

Reading 2015

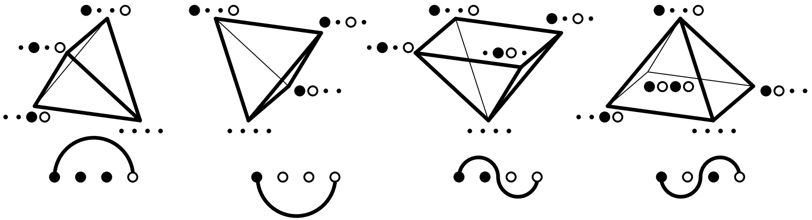


Shard polytopes

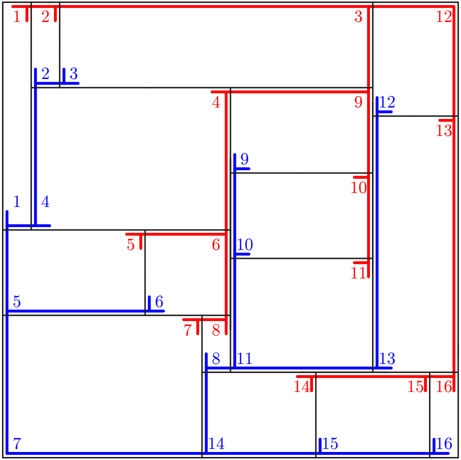
Theorem

The quotientope of a congruence $\equiv$ is the Minkowski sum of the shard polytope of each arc in $\mathcal{A}_{\equiv}$.

Padrol, Pilaud, Ritter 2023



Intertwined binary trees



C.-Pilaud 2024

Loday coordinates for strong rectangulotopes

Theorem

The $(n-1)$-dimensional strong rectangulotope is realized by the convex hull of the points

$$\sum_{i<j} (\widetilde{w}^R_{i,j} - \overset{\frown}{w}^R_{i,j}) \cdot (\boldsymbol{e}_i - \boldsymbol{e}_j),$$

for all strong rectangulations R of size n, with

$$\widetilde{w}^R_{i,j} := h^T_i \cdot cv^{T,S}_{i,j} \cdot h^S_j \cdot [\![\neg\, i \setminus j]\!] \qquad \text{and} \qquad \overset{\frown}{w}^R_{i,j} := v^S_i \cdot ch^{S,T}_{i,j} \cdot v^T_j \cdot [\![\neg\, j \setminus i]\!],$$

where $ch^{T,S}_{i,j}$ (resp. $cv^{T,S}_{i,j}$) denote the number of common leaves of the horizontal (resp. vertical) subtree of i in T and the horizontal (resp. vertical) subtree of j in S,

C.-Pilaud 2024

Conclusion

- Unified descriptions of bijections between equivalence classes of rectangulations and classes of permutations. See ACFF for more bijections.

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- Complete descriptions of two families of polytopes encoding equivalence classes of rectangulations, which are natural and inistructive examples of quotientopes. See CP (in preparation) for details.

Conclusion

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Thank you!