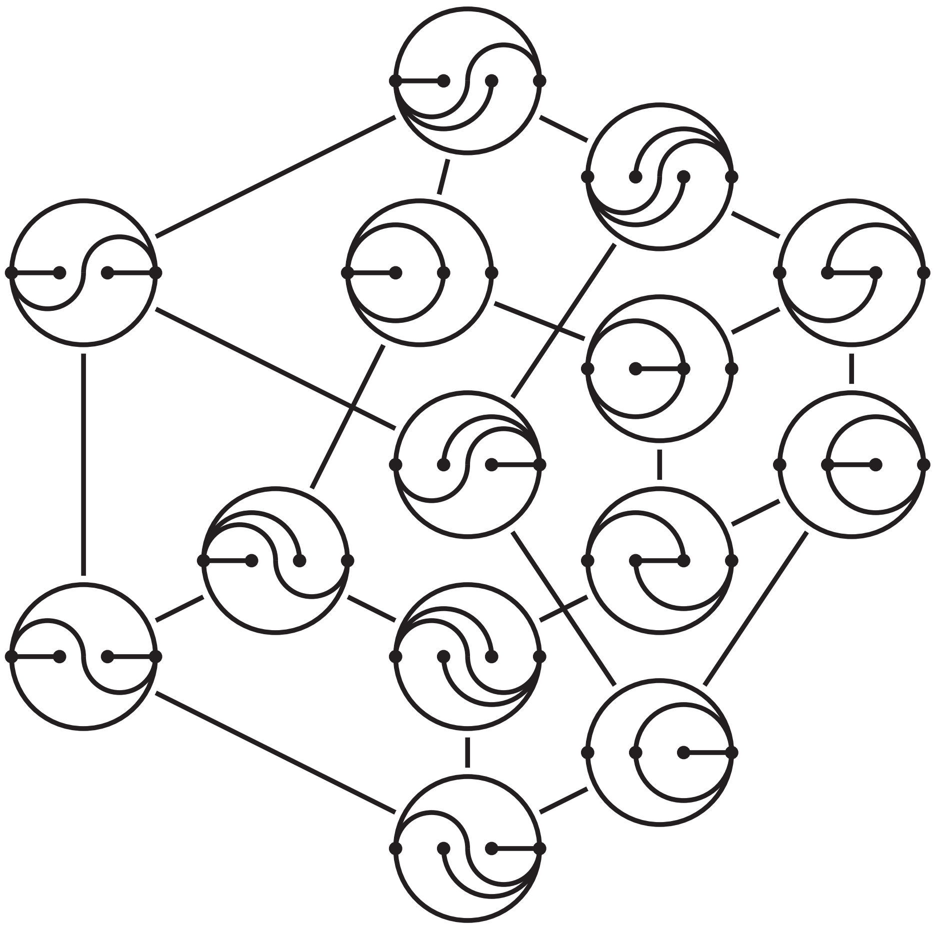
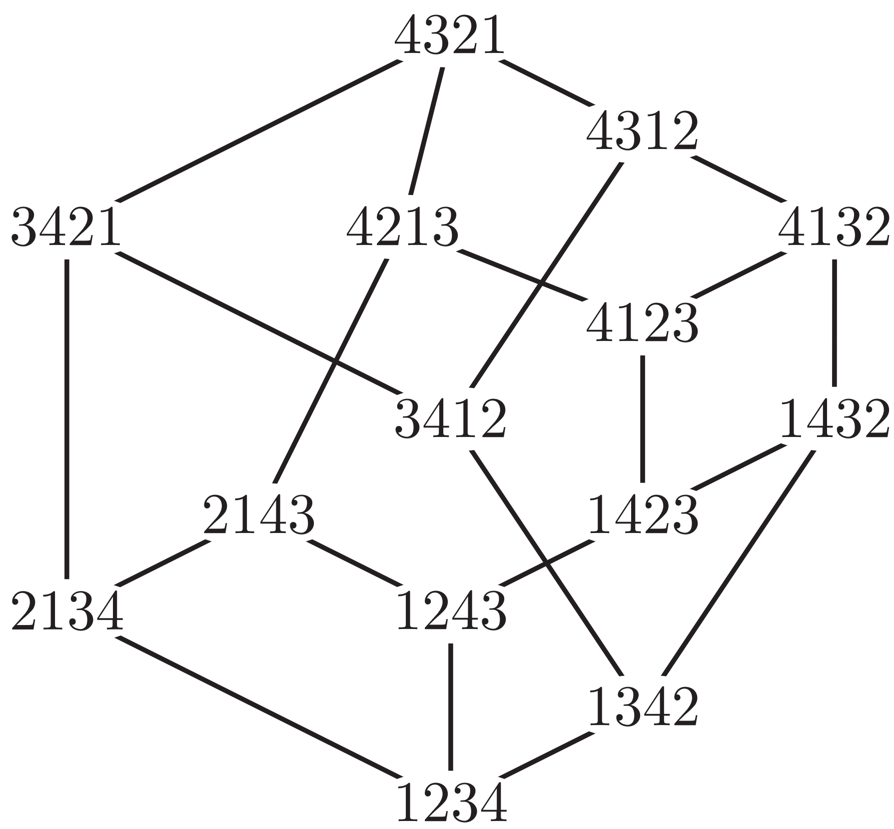
WIGGLYHEDRA

A. BAPAT (The Australian National University)
V. PILAUD (Universitat de Barcelona)

arxiv:2407.11632



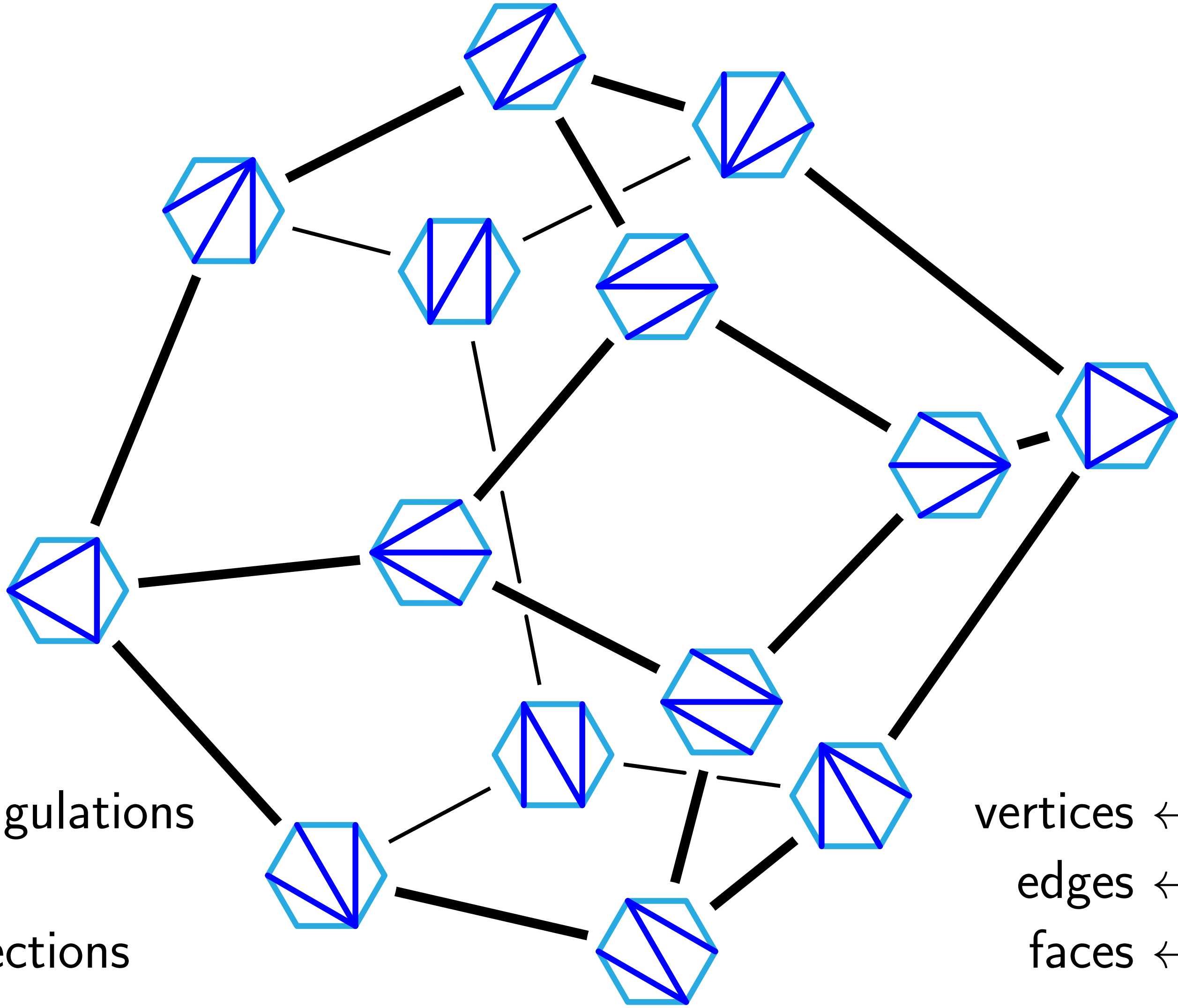


DMG Seminar Berlin — October 9, 2024

TRIANGULATIONS & ASSOCIAHEDRA

ASSOCIAHEDRON

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion

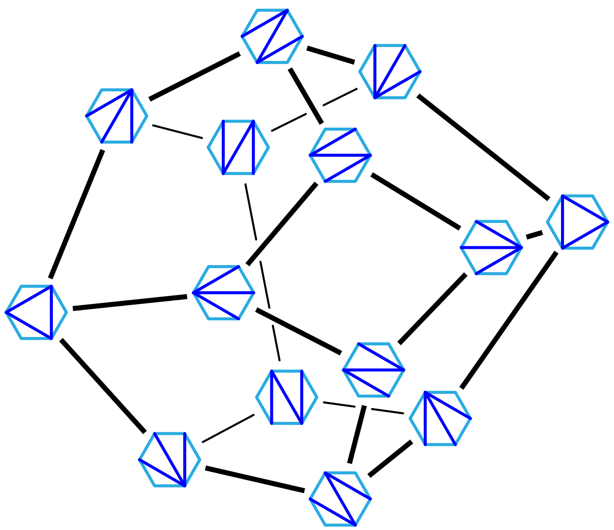


vertices $\leftrightarrow$ triangulations
edges $\leftrightarrow$ flips
faces $\leftrightarrow$ dissections

vertices $\leftrightarrow$ binary trees
edges $\leftrightarrow$ rotations
faces $\leftrightarrow$ Schröder trees

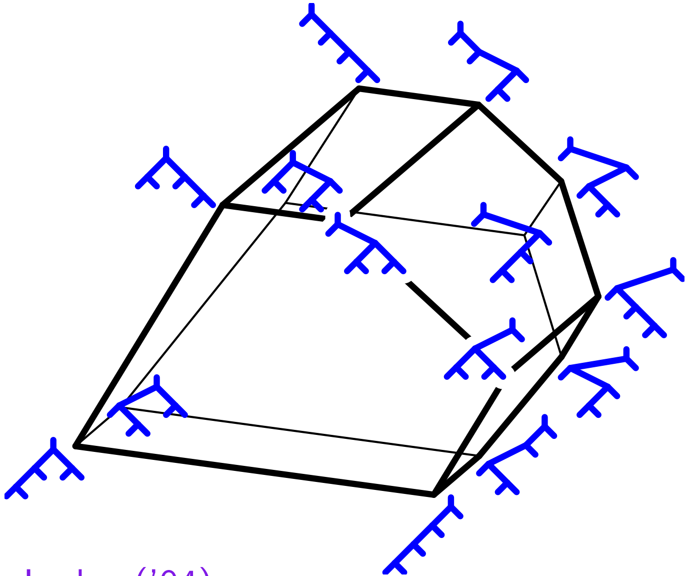
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



Gelfand–Kapranov–Zelevinsky ('94)
Billera–Filliman–Sturmfels ('90)

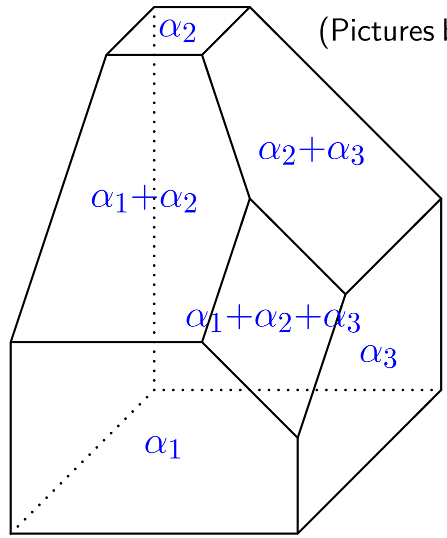
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg–Lange ('07)
Hohlweg–Lange–Thomas ('12)
Hohlweg–Pilaud–Stella ('18)
Pilaud–Santos–Ziegler ('24)

CHAP.–FOM.–ZEL.'S ASSOCIAHEDRON

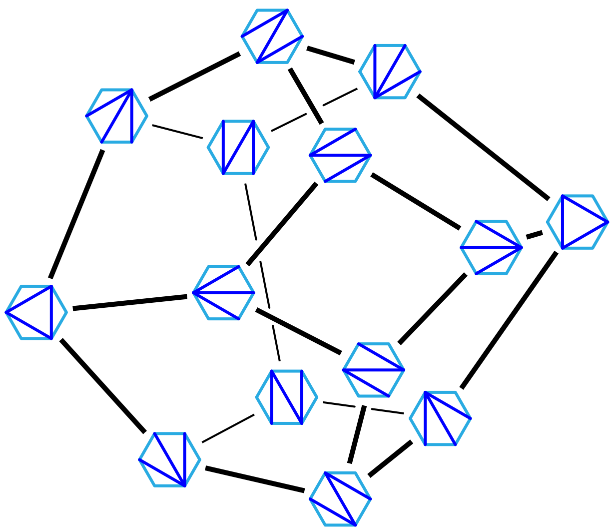
(Pictures by CFZ)



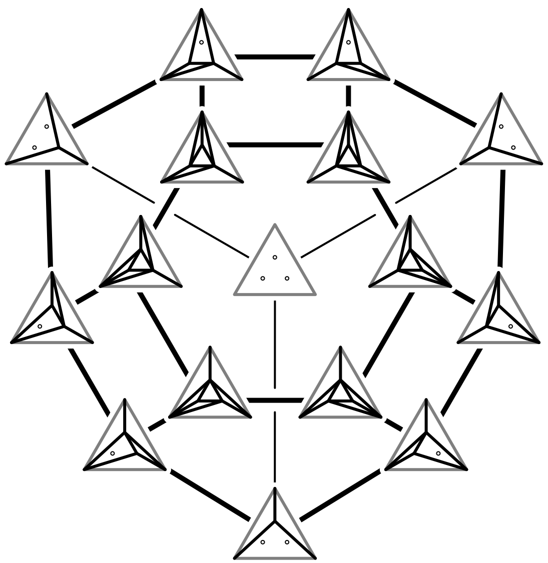
Chapoton–Fomin–Zelevinsky ('02)
Ceballos–Santos–Ziegler ('11)

THREE FAMILIES OF REALIZATIONS

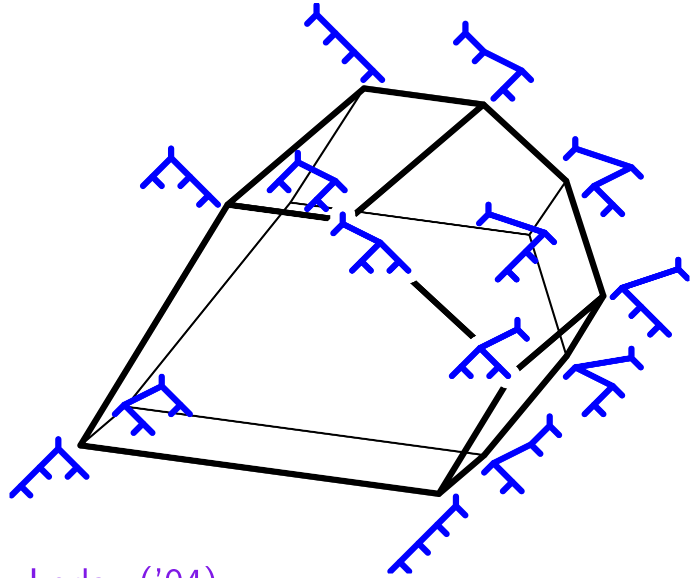
SECONDARY POLYTOPE



Gelfand–Kapranov–Zelevinsky ('94)
Billera–Filliman–Sturmfels ('90)

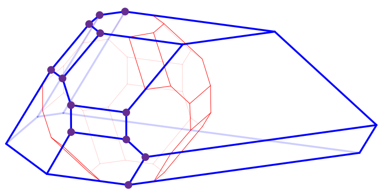


LODAY'S ASSOCIAHEDRON

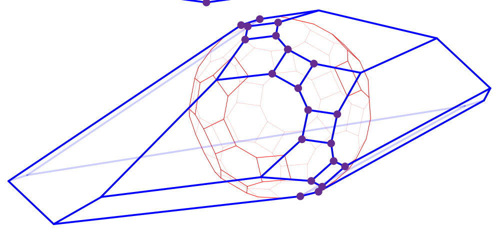


Loday ('04)
Hohlweg–Lange ('07)
Hohlweg–Lange–Thomas ('12)
Hohlweg–Pilaud–Stella ('18)
Pilaud–Santos–Ziegler ('24)

Hopf algebra

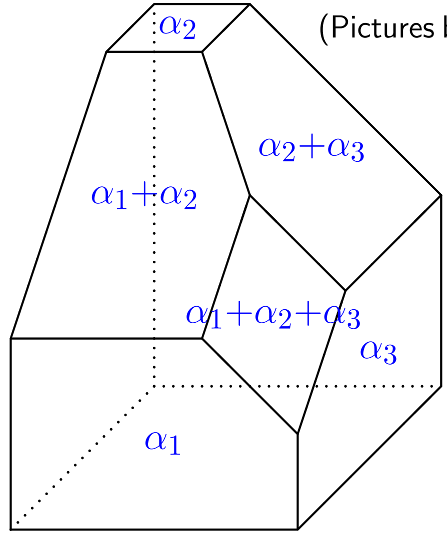


Cluster algebras



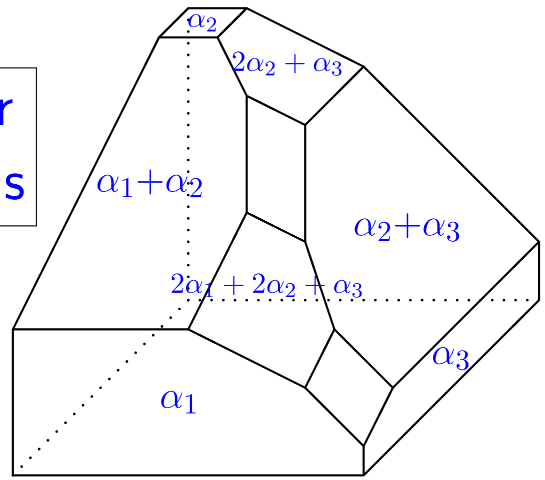
CHAP.–FOM.–ZEL.'S ASSOCIAHEDRON

(Pictures by CFZ)



Chapoton–Fomin–Zelevinsky ('02)
Ceballos–Santos–Ziegler ('11)

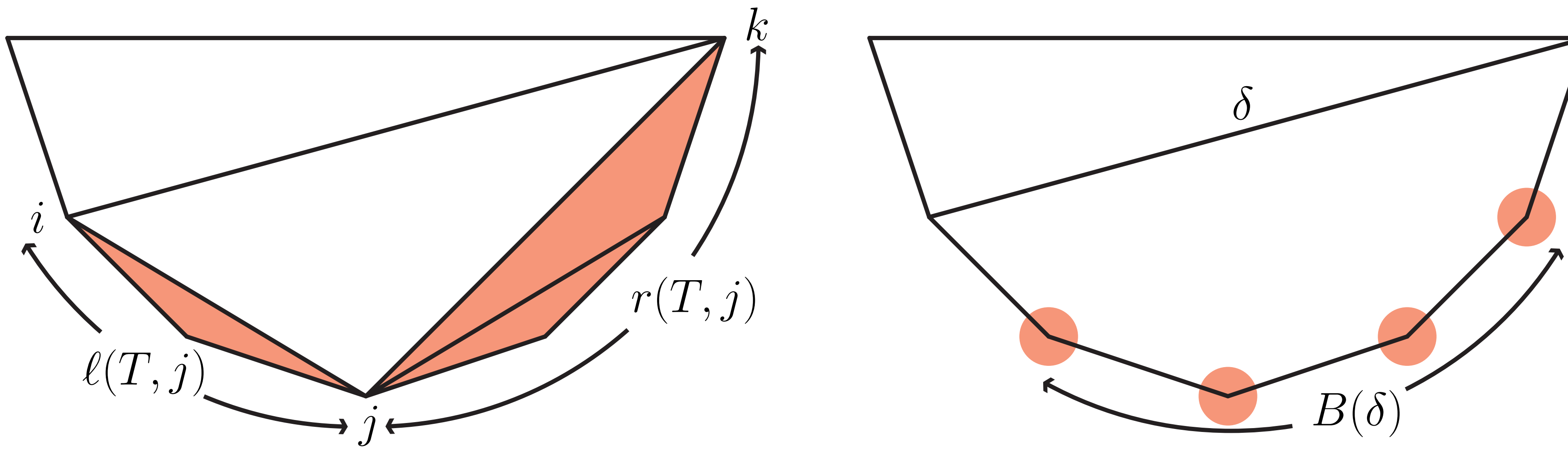
Cluster algebras



LODAY'S ASSOCIAHEDRON

Loday's associahedron $= \operatorname{conv}\{L(T) \mid T$ triangulation of the $(n+3)$-gon$\}$

$$= \mathbb{H} \cap \bigcap_{\substack{\delta \text{ diagonal} \\ \text{of the } (n+3)\text{-gon}}} \boldsymbol{H}^{\geq}(\delta)$$

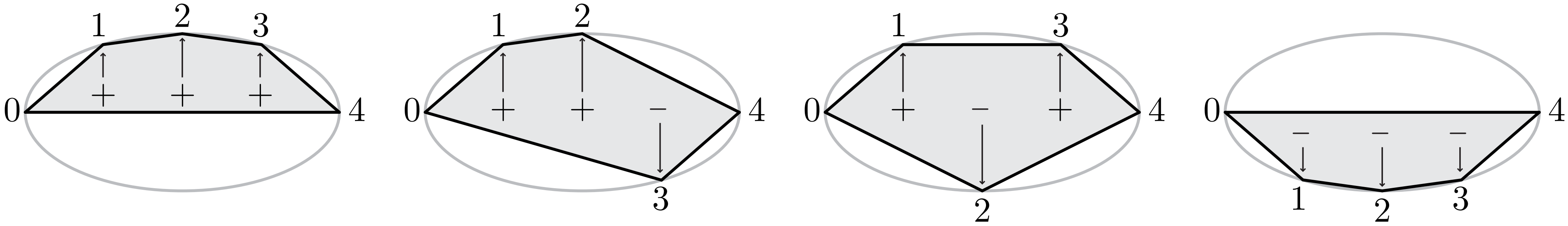


$$L(T) = \big(\ell(T,j) \cdot r(T,j)\big)_{j \in [n+1]} \qquad \boldsymbol{H}^{\geq}(\delta) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j \in B(\delta)} x_j \geq \binom{|B(\delta)|+1}{2} \right\}$$

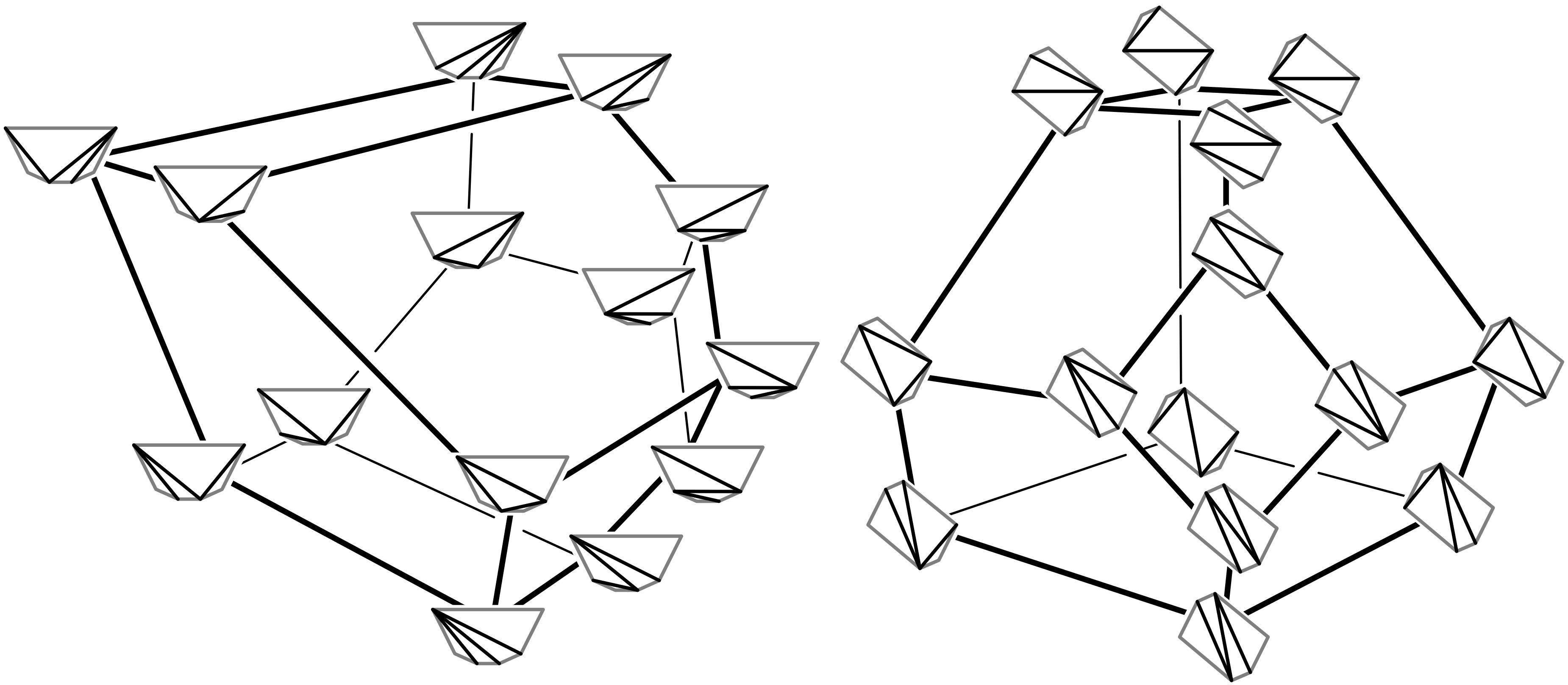
Loday ('04)

HOHLWEG & LANGE'S ASSOCIAHEDRA

Can also replace Loday's $(n+3)$-gon by others...



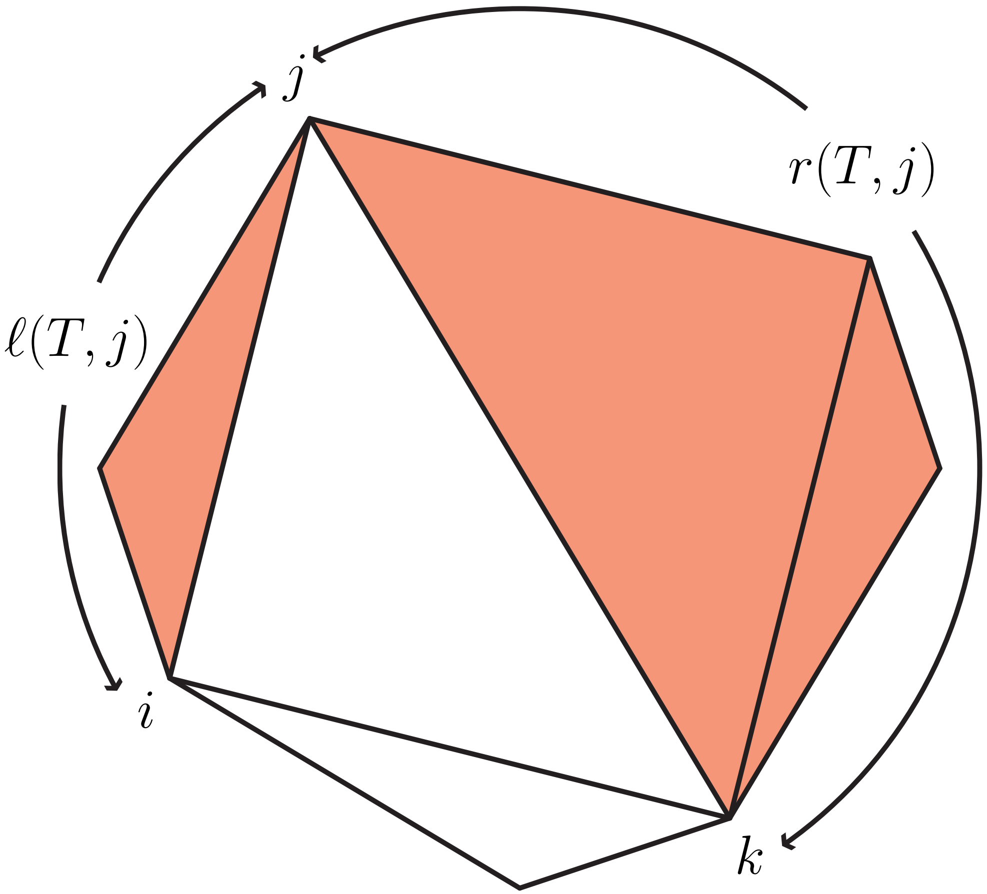
...to obtain different realizations of the associahedron

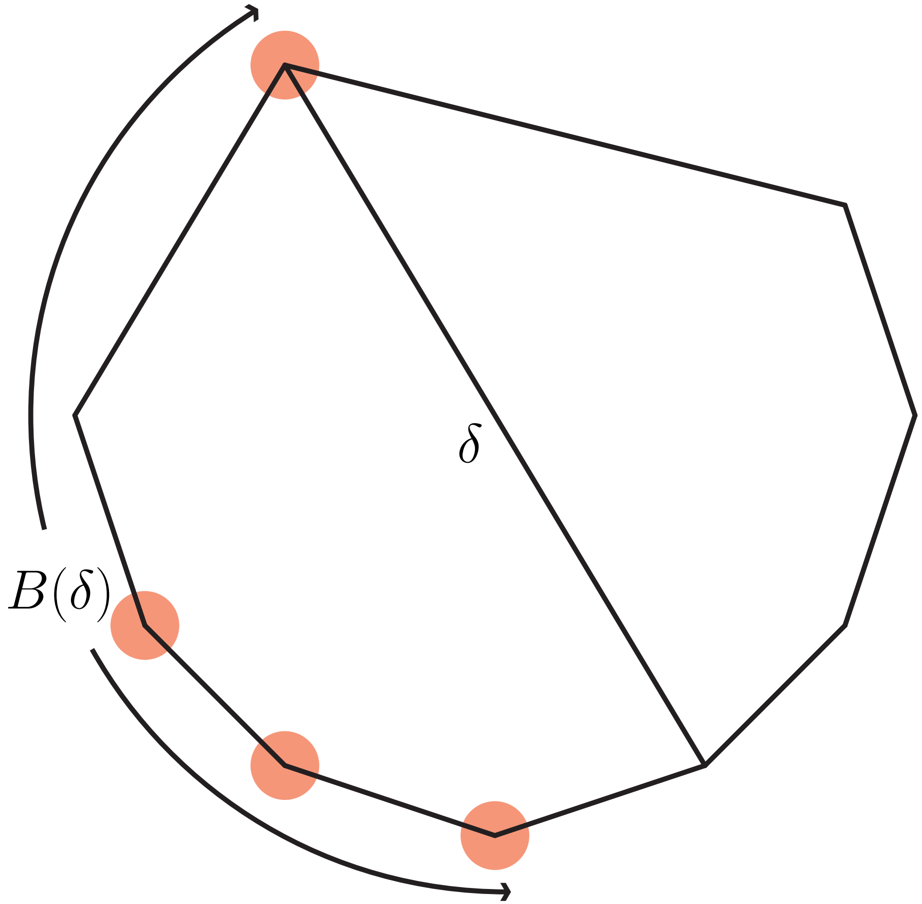


Hohlweg–Lange ('07)

HOHLWEG & LANGE'S ASSOCIAHEDRA

$$\mathbb{A}\mathsf{sso}(P) \;=\; \operatorname{conv}\{HL(T) \mid T \text{ triangulation of } P\} \;=\; \mathbb{H} \cap \bigcap_{\delta \text{ diagonal of } P} \boldsymbol{H}^{\geq}(\delta)$$





$$HL(T)_j = \begin{cases} \ell(T,j)\cdot r(T,j) & \text{if } j \text{ down} \\ n+2-\ell(T,j)\cdot r(T,j) & \text{if } j \text{ up} \end{cases}$$

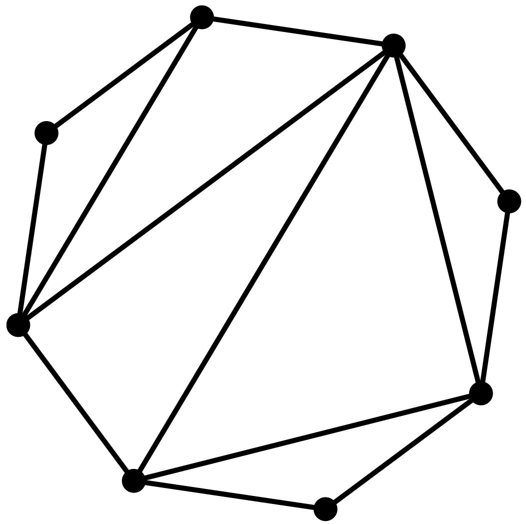
$$\boldsymbol{H}^{\geq}(\delta) = \left\{ \boldsymbol{x} \;\middle|\; \sum_{j\in B(\delta)} x_j \geq \binom{|B(\delta)|+1}{2} \right\}$$

Hohlweg–Lange ('07)

PSEUDOTRIANGULATIONS & PPT POLYTOPE

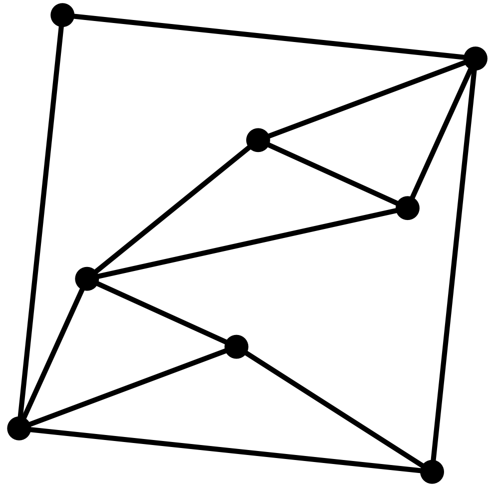
THREE GEOMETRIC STRUCTURES

triangulations



crossing-free

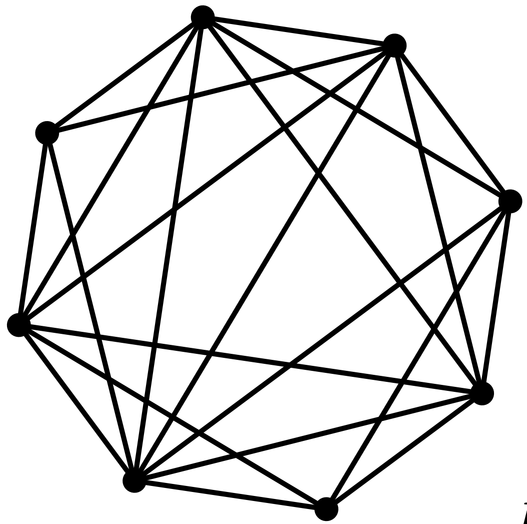
pseudotriangulations



crossing-free pointed

Pocchiola–Vegter
Rote–Santos–Streinu ('08)

multitriangulations



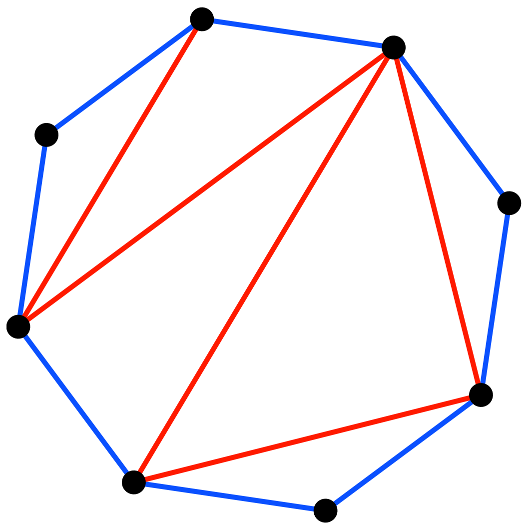
$k = 2$

$(k+1)$-crossing-free

Capoyleas–Pach ('92)
Jonsson ('05)

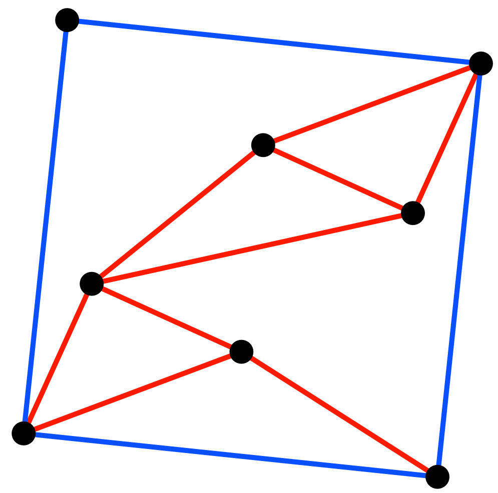
THREE GEOMETRIC STRUCTURES

triangulations



crossing-free

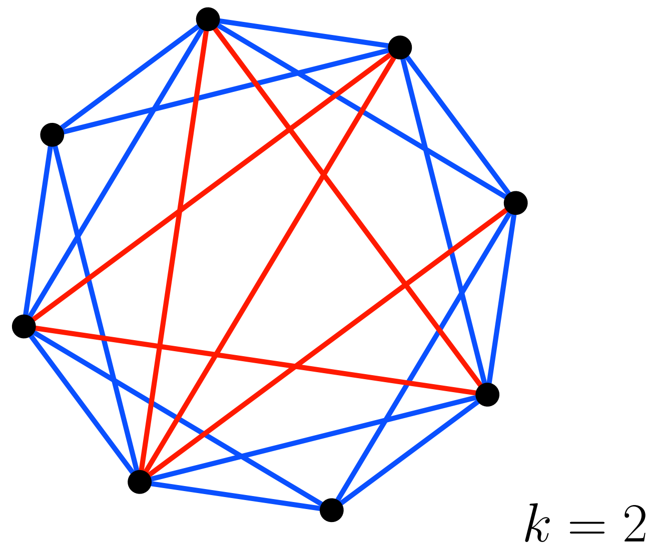
pseudotriangulations



crossing-free pointed

Pocchiola–Vegter
Rote–Santos–Streinu ('08)

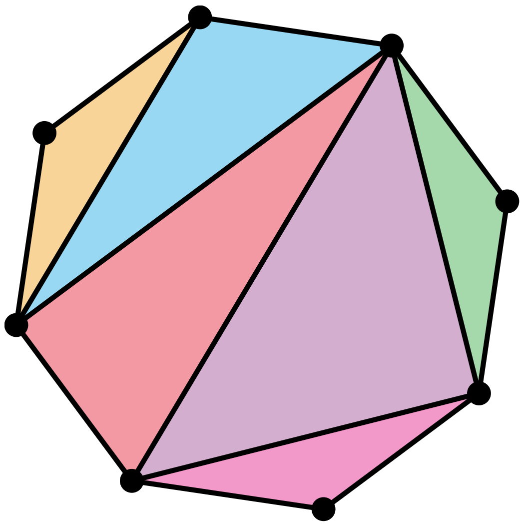
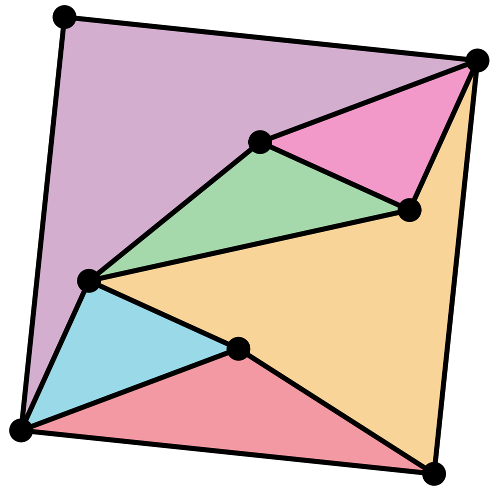
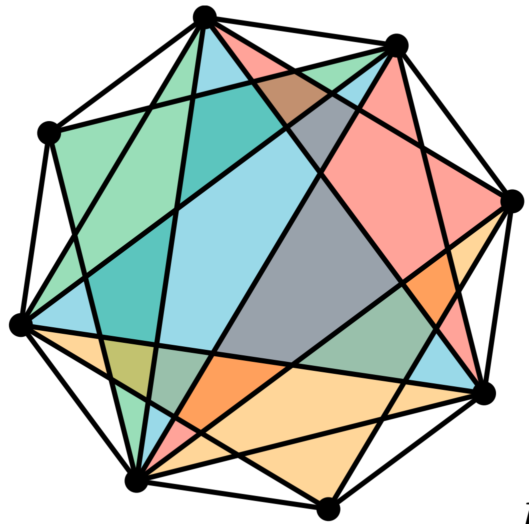
multitriangulations



$(k+1)$-crossing-free

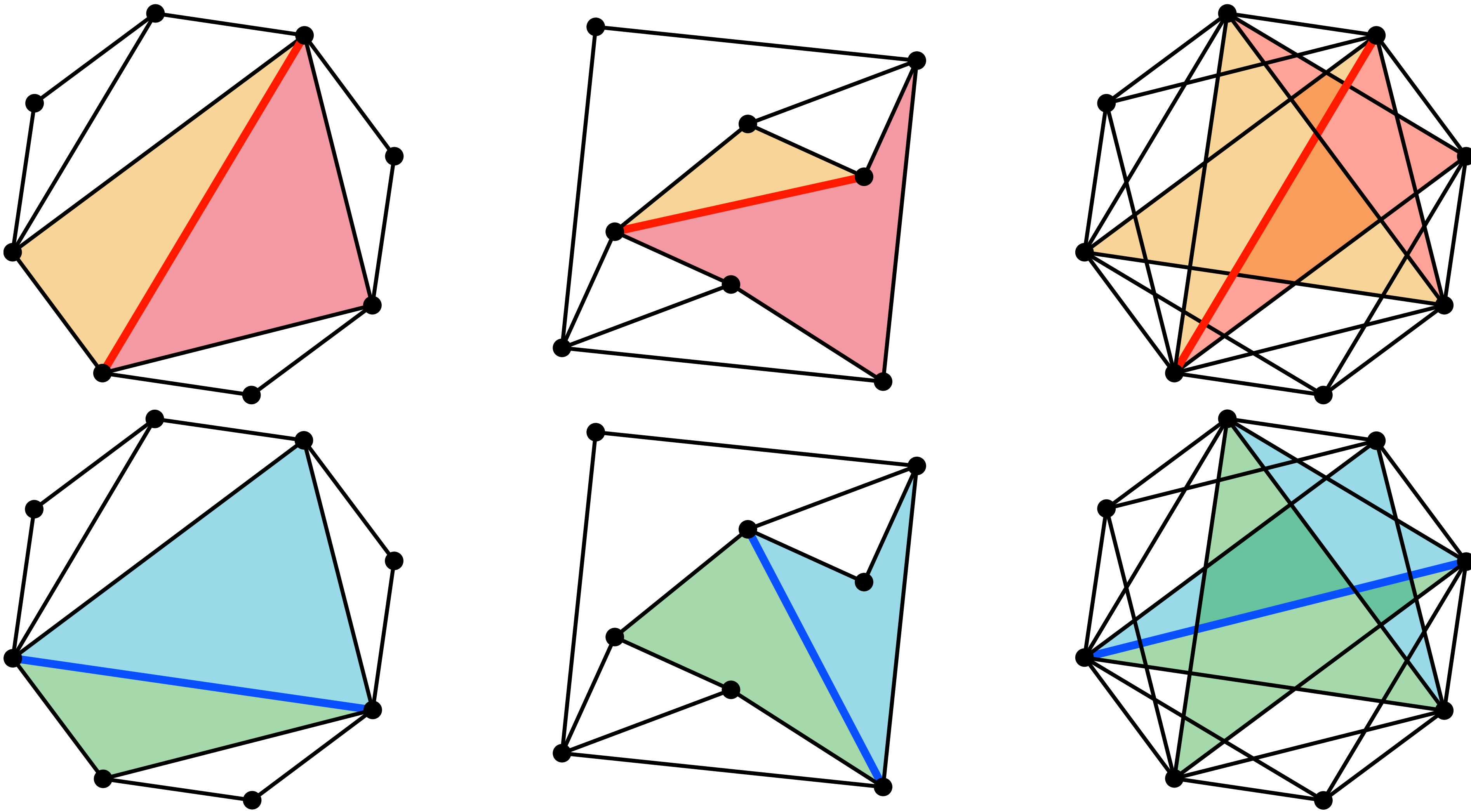
Capoyleas–Pach ('92)
Jonsson ('05)

THREE GEOMETRIC STRUCTURES

triangulations	pseudotriangulations	multitriangulations
		 $k = 2$
crossing-free	crossing-free pointed	$(k+1)$-crossing-free
	Pocchiola–Vegter Rote–Santos–Streinu ('08)	Capoyleas–Pach ('92) Jonsson ('05)
triangles	pseudotriangles	k-stars
		P.–Santos ('09)

THREE GEOMETRIC STRUCTURES

triangulations pseudotriangulations multitriangulations

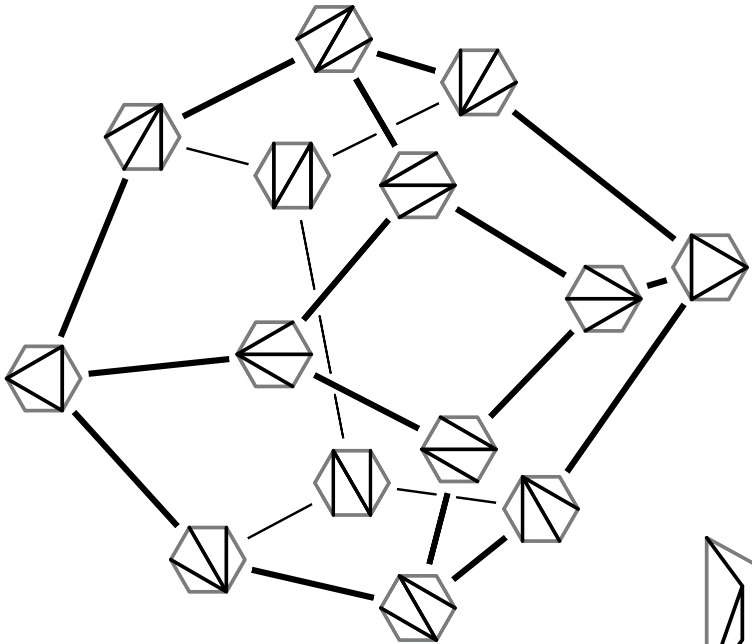


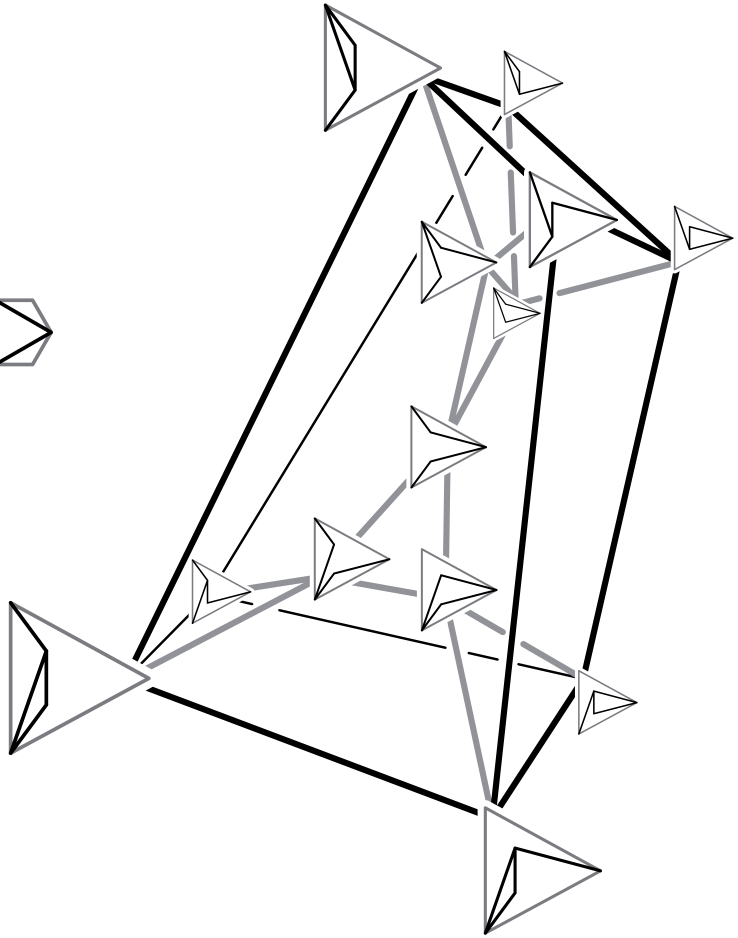
$k = 2$

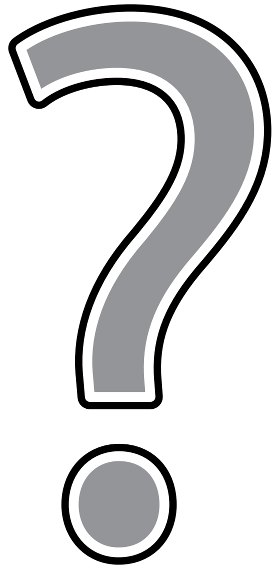
<u>flip</u> $=$ exchange an internal edge with the common bisector of the two adjacent cells

THREE GEOMETRIC STRUCTURES

triangulations | pseudotriangulations | multitriangulations

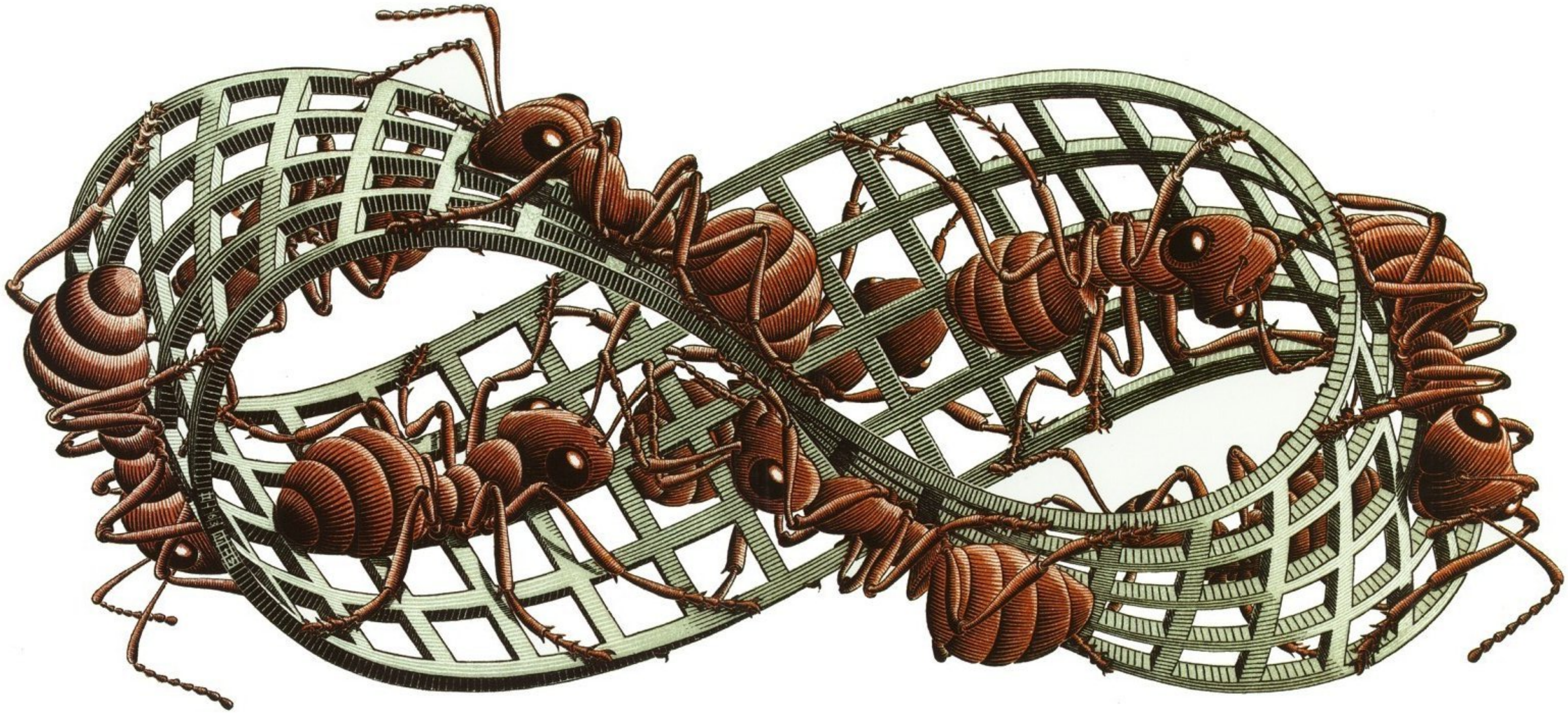






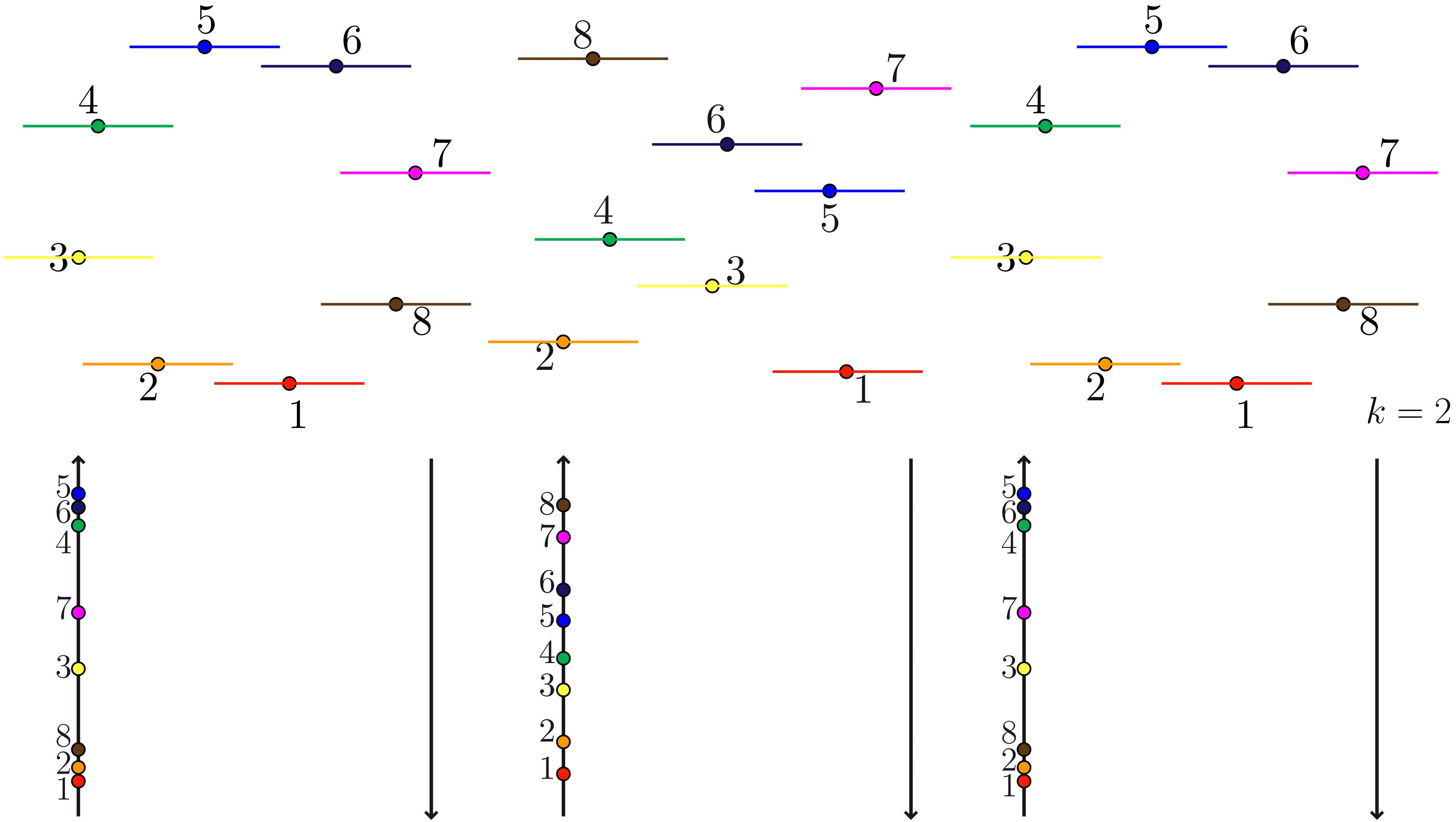
associahedron | pseudotriangulation polytope | multiassociahedron

LINE SPACE OF THE PLANE = MÖBIUS STRIP



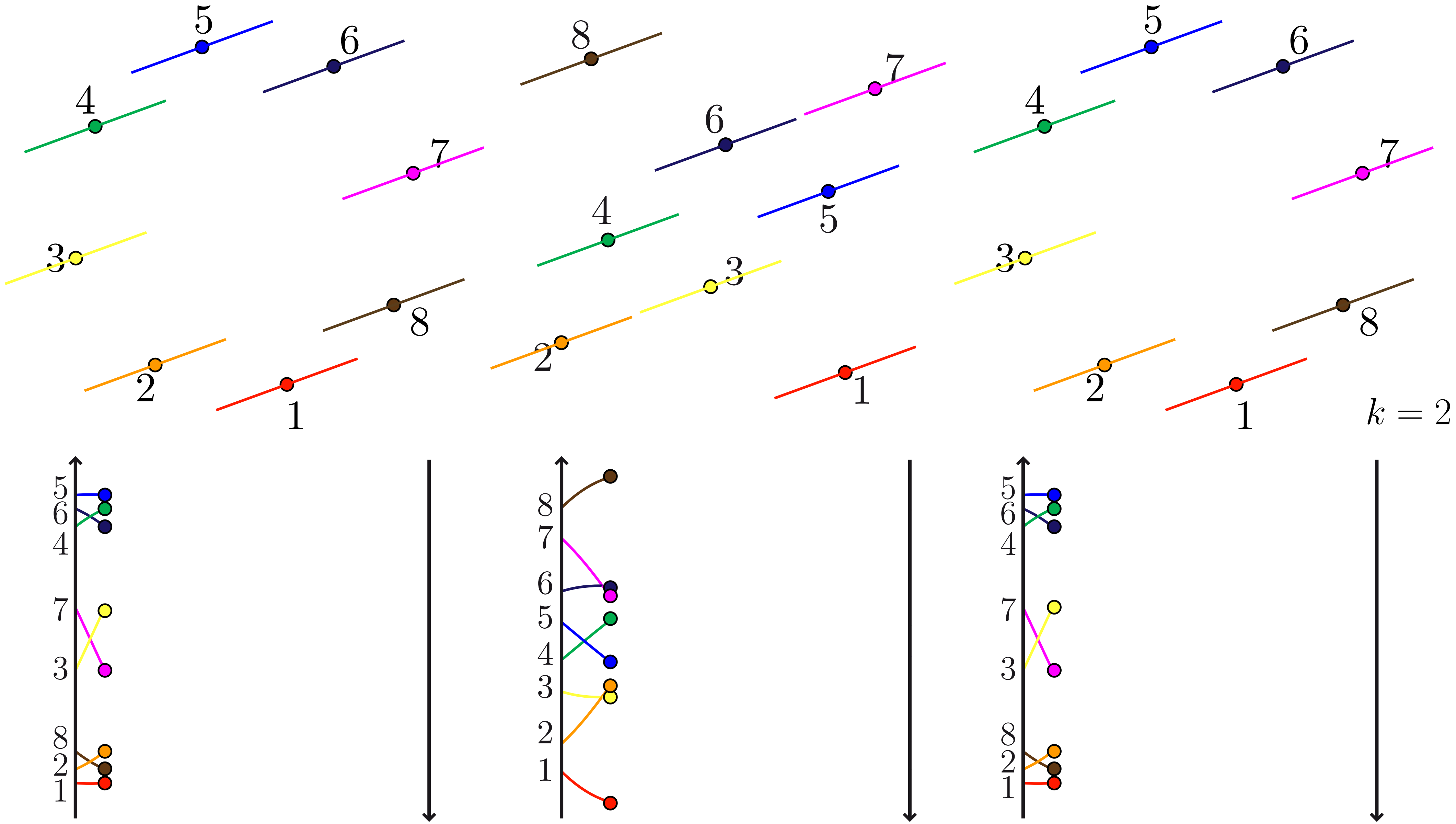
DUALITY

triangulations | pseudotriangulations | multitriangulations



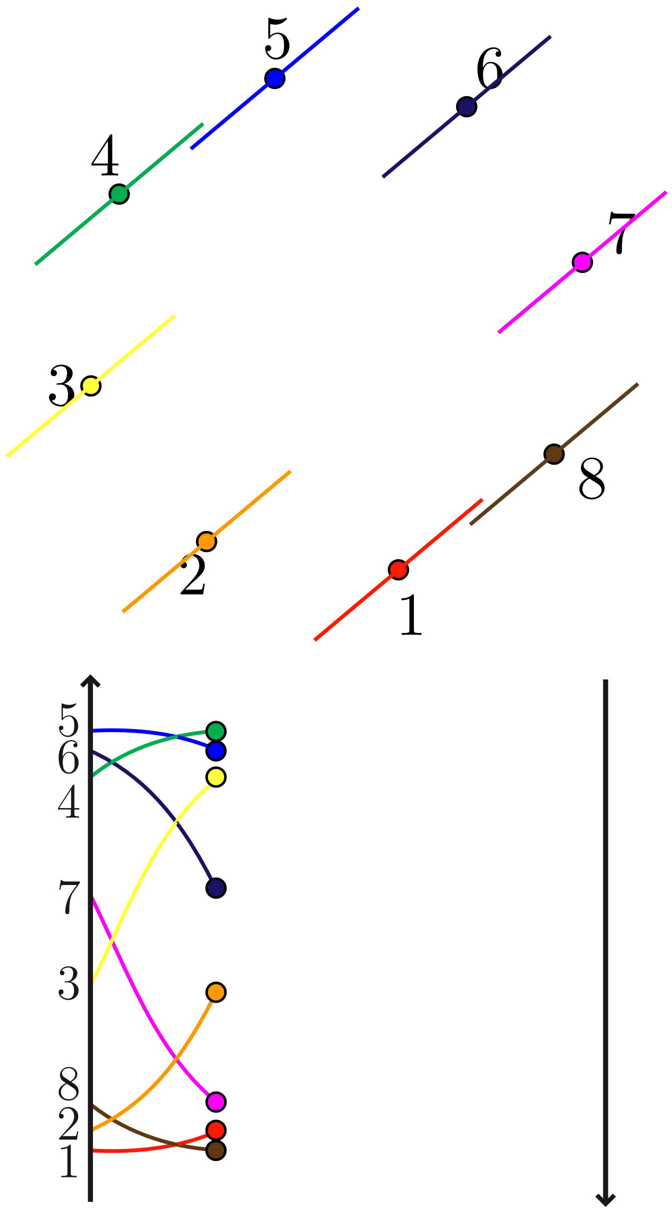
DUALITY

triangulations pseudotriangulations multitriangulations

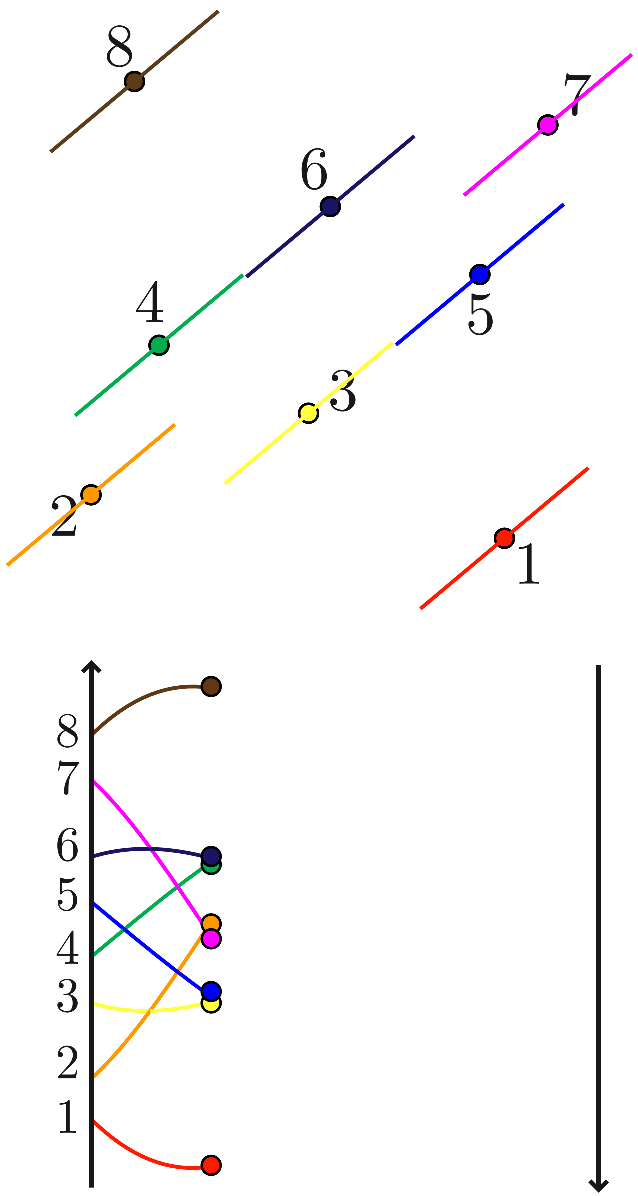


DUALITY

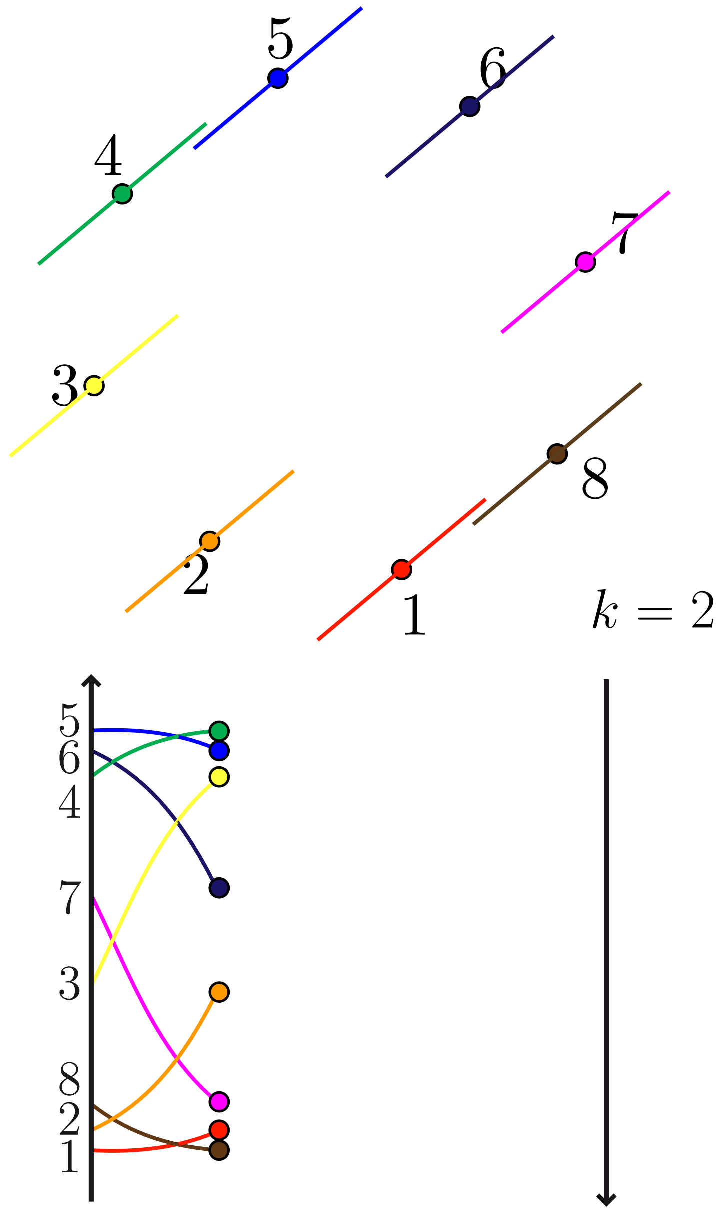
triangulations



pseudotriangulations

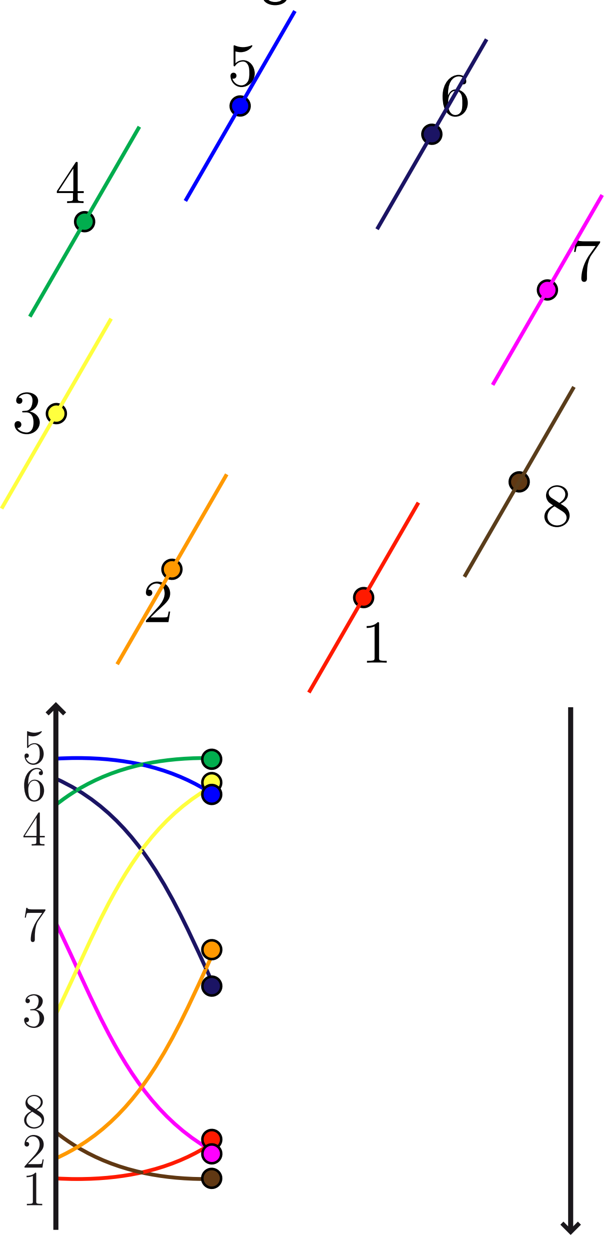


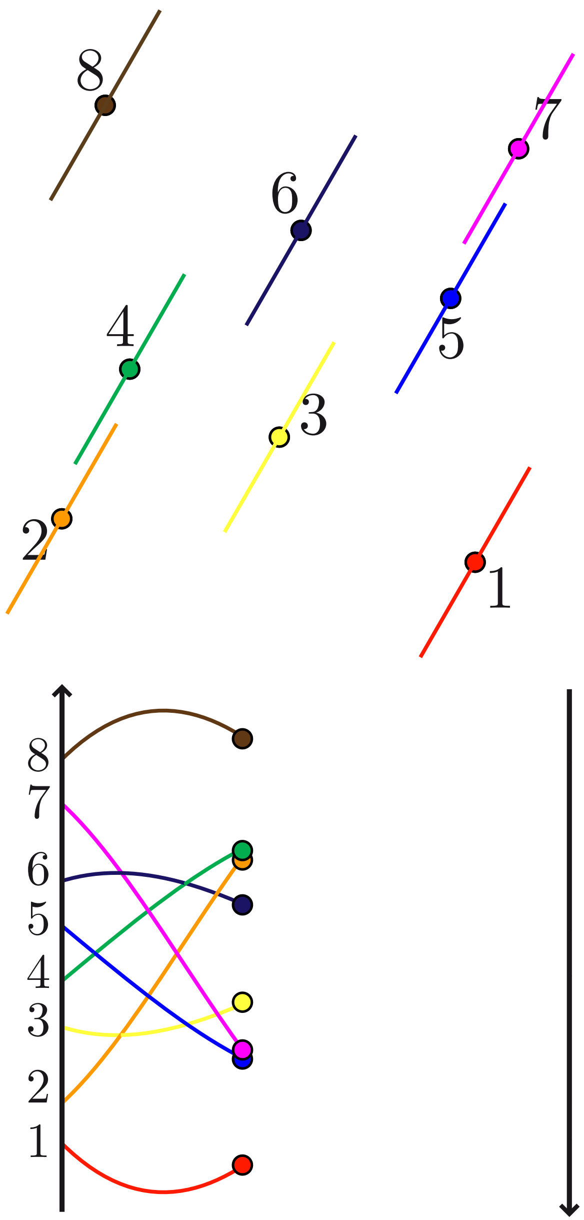
multitriangulations

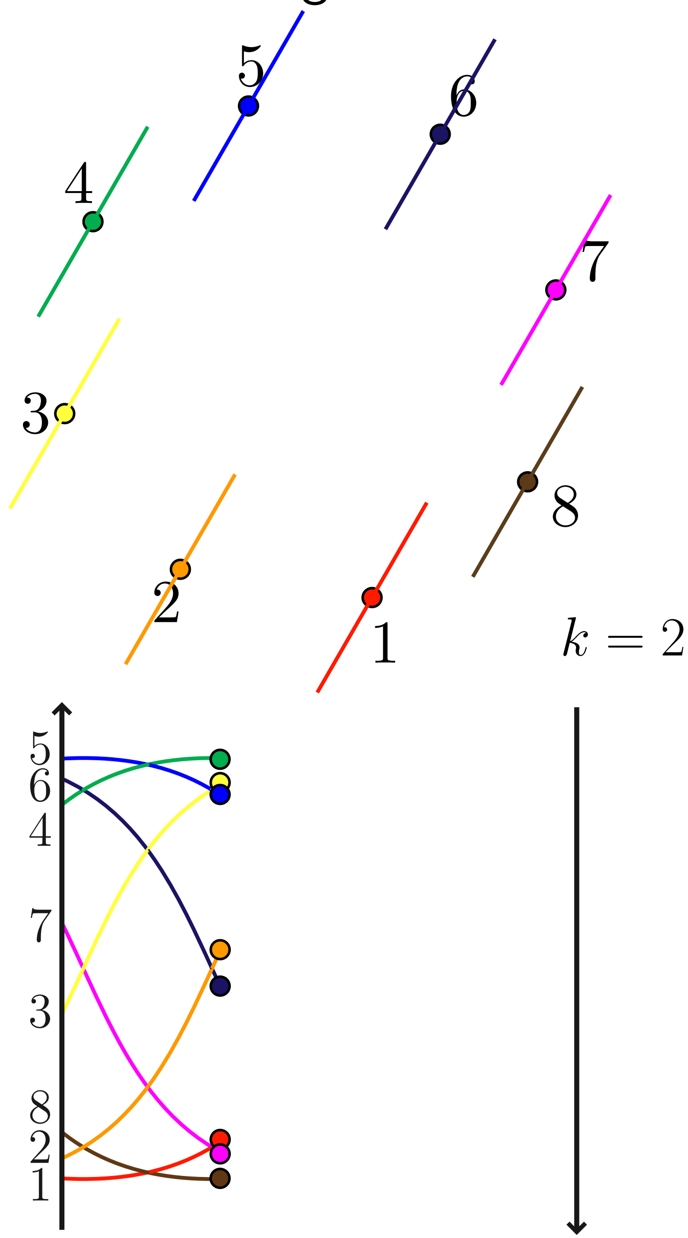


DUALITY

triangulations | pseudotriangulations | multitriangulations

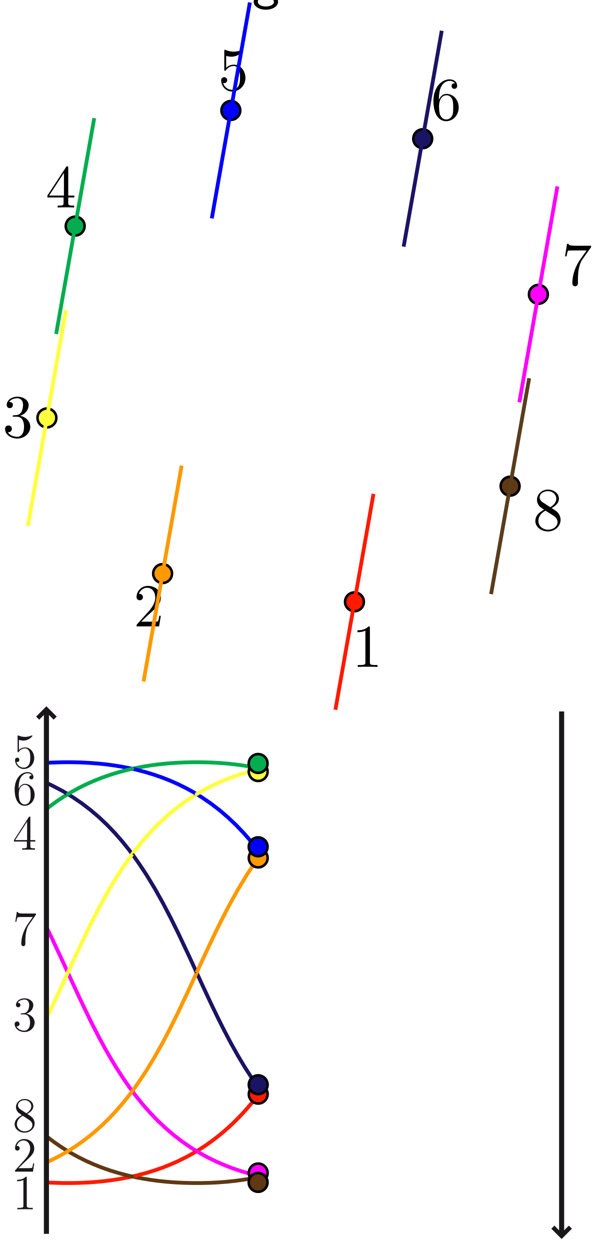




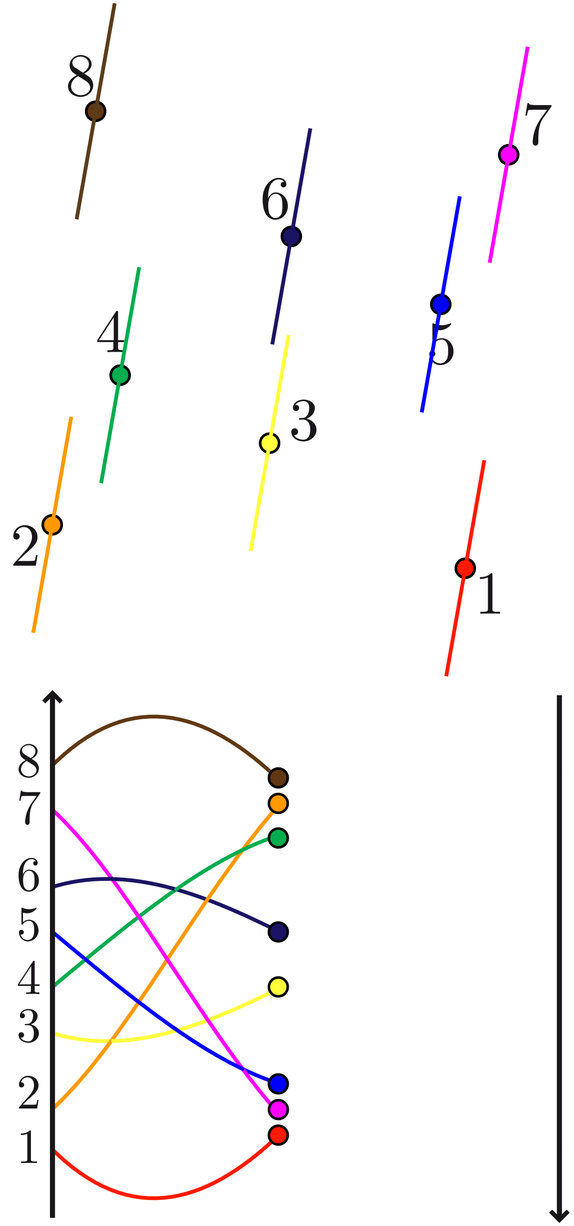


DUALITY

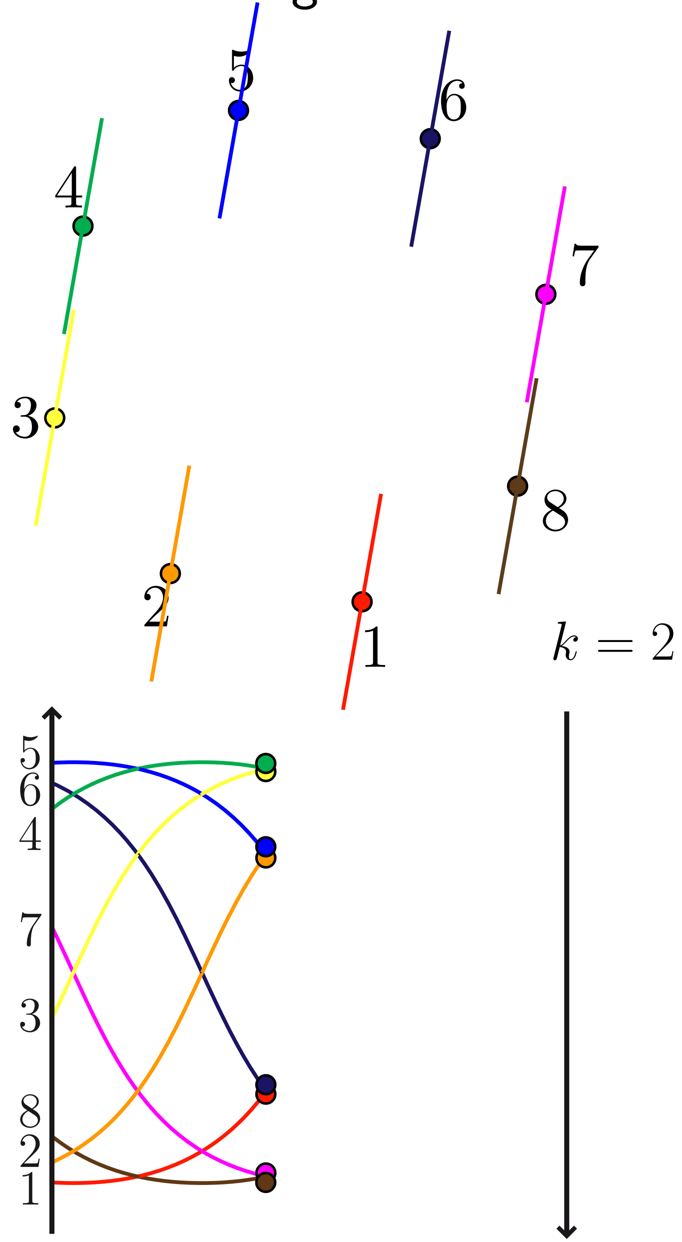
triangulations



pseudotriangulations

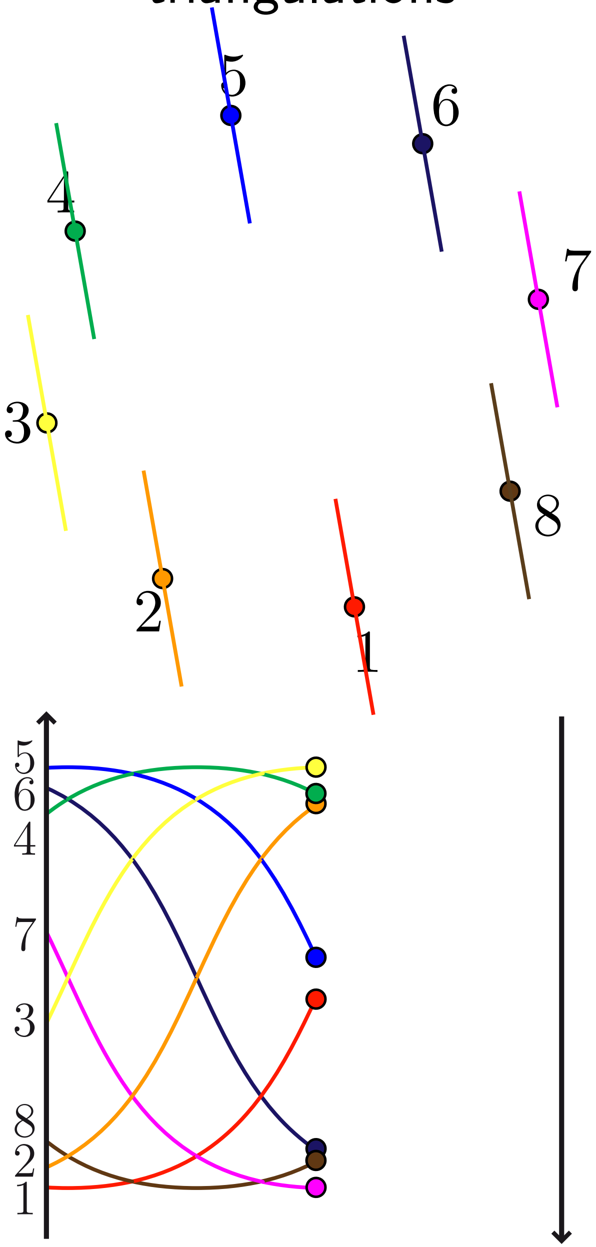


multitriangulations

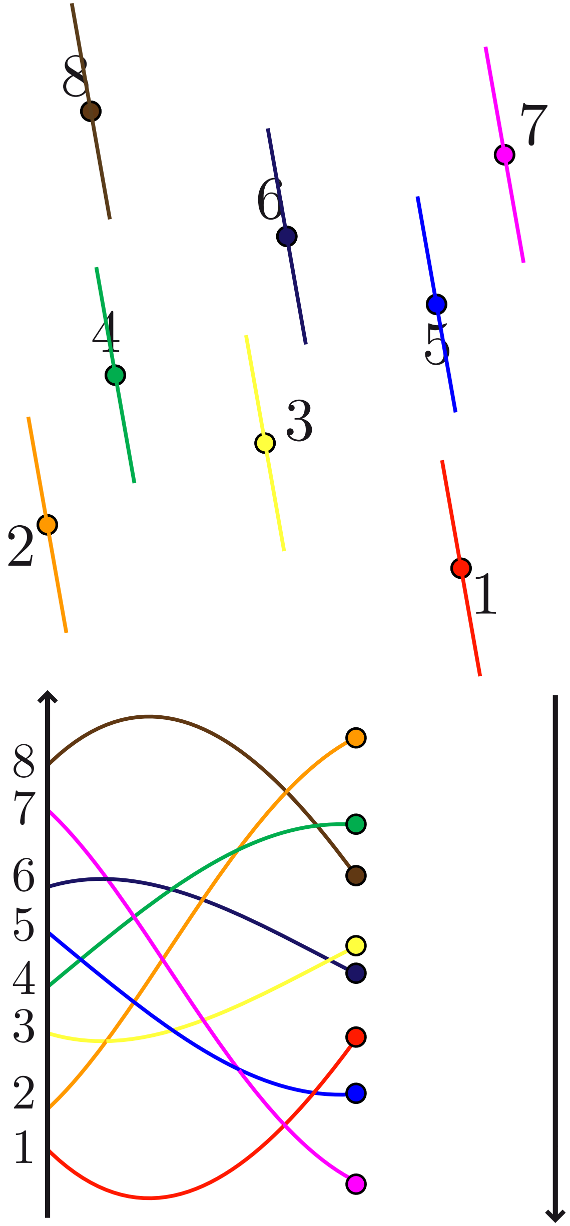


DUALITY

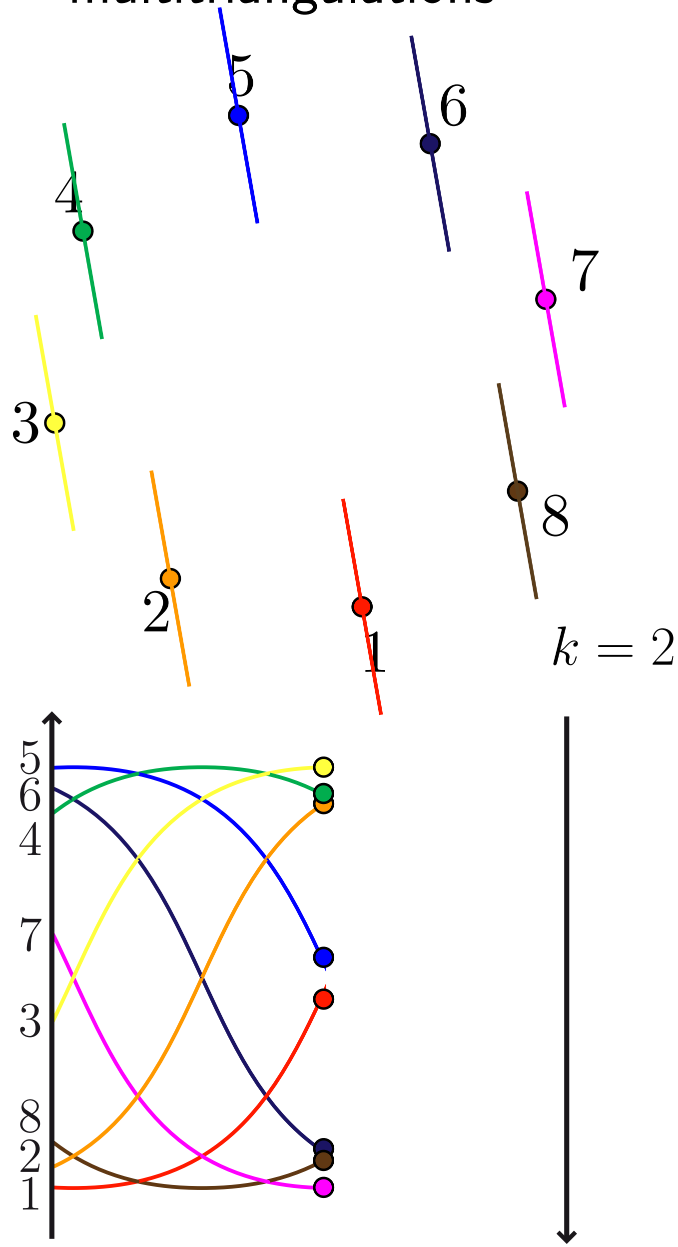
triangulations



pseudotriangulations

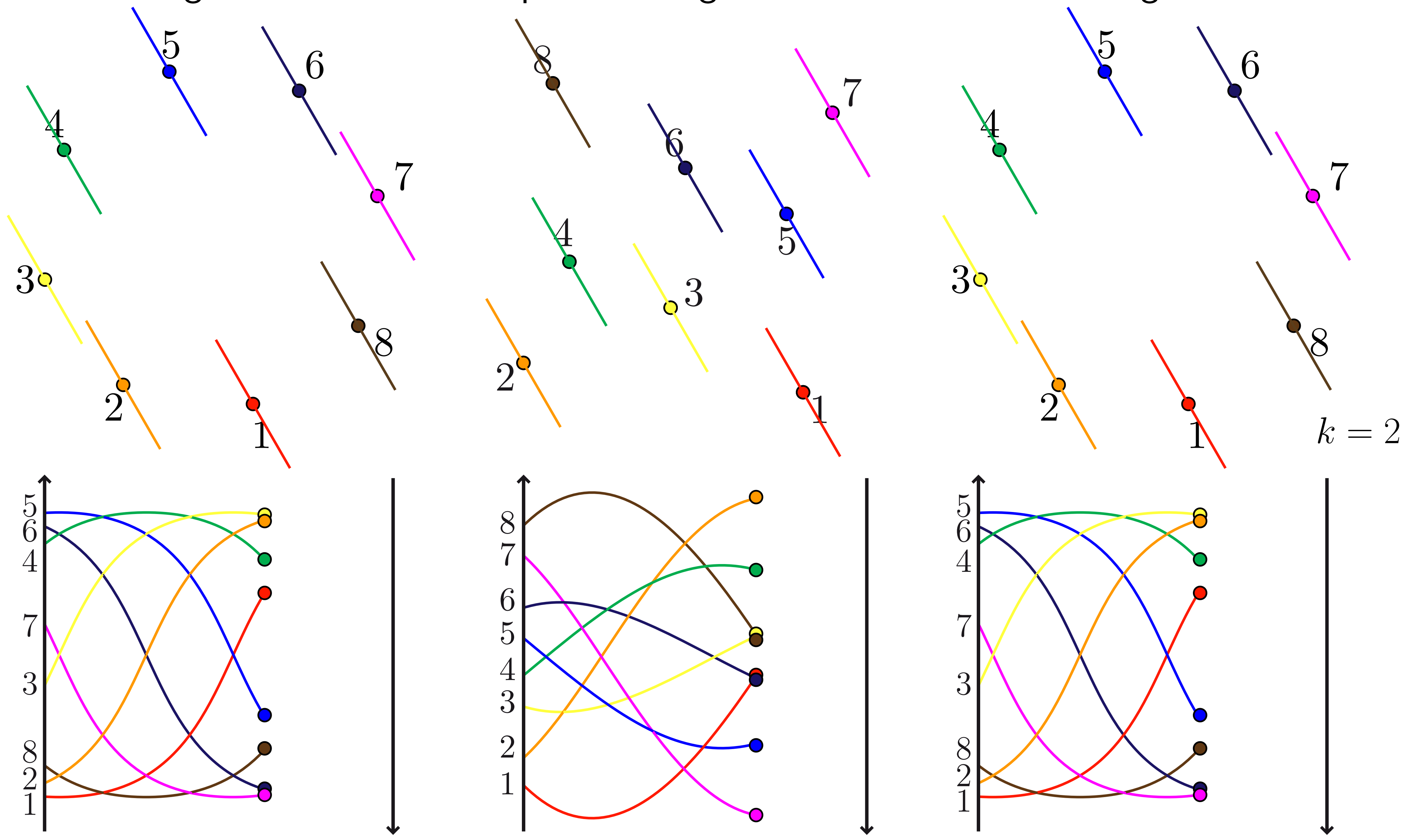


multitriangulations

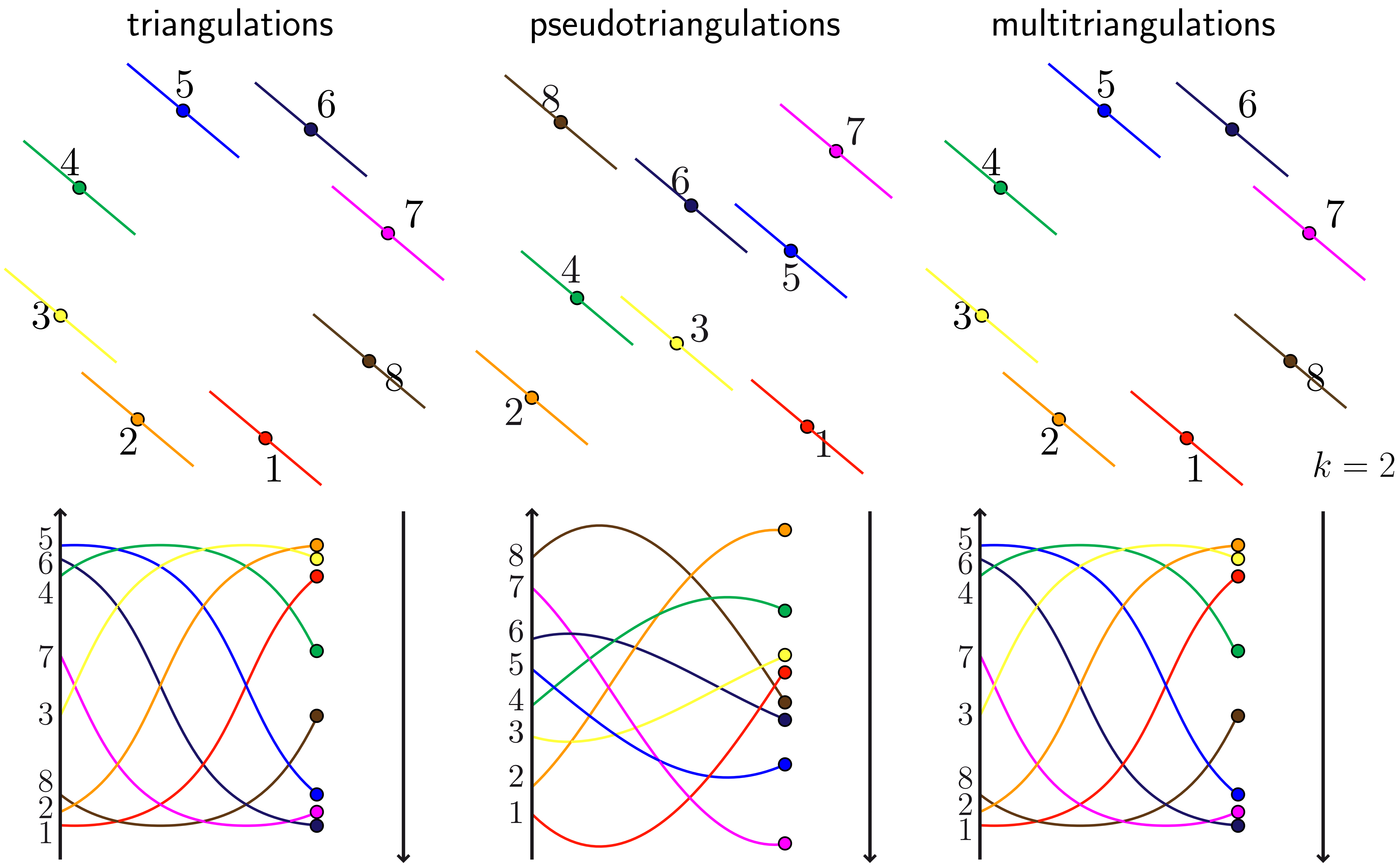


DUALITY

triangulations | pseudotriangulations | multitriangulations

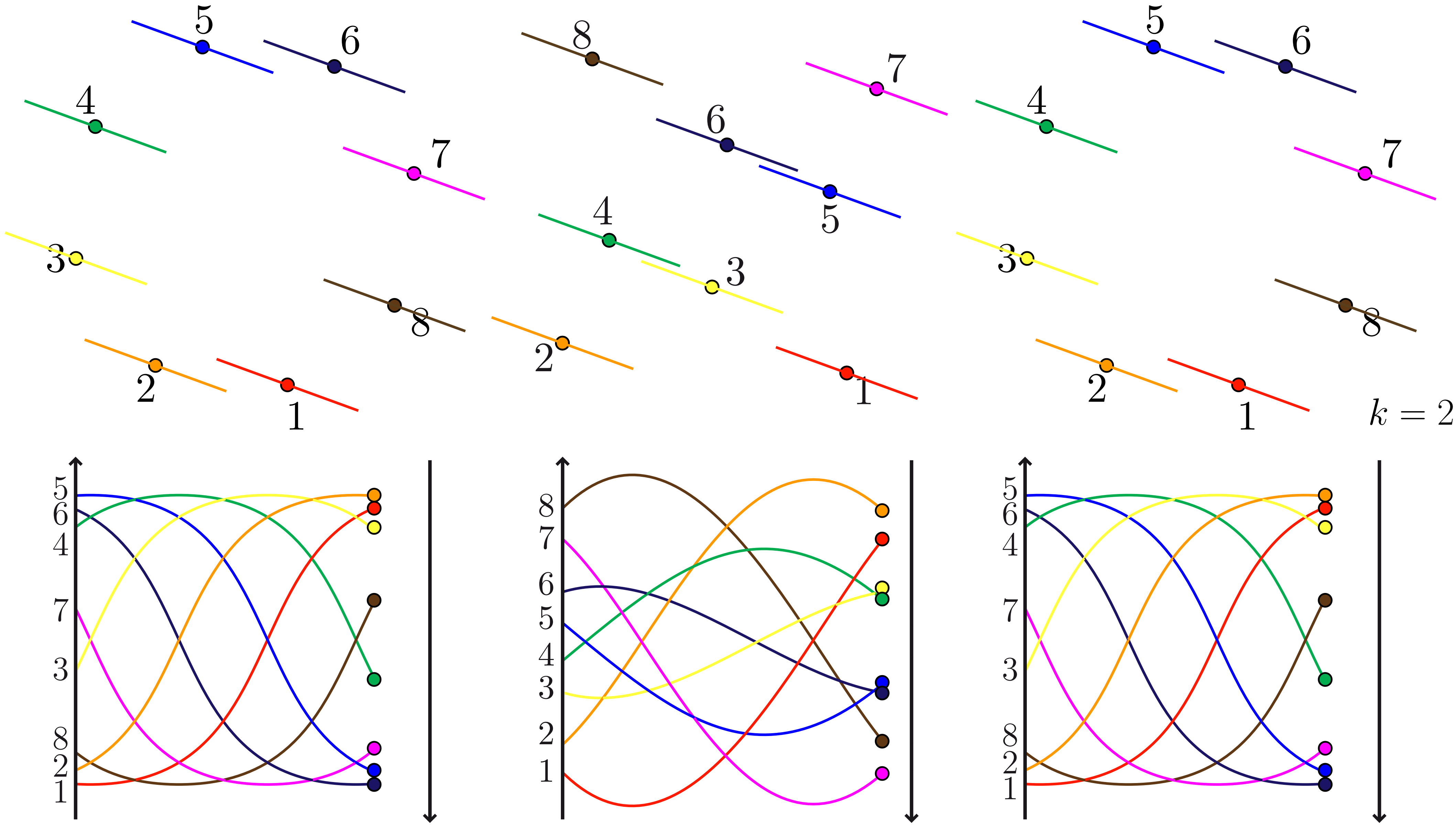


DUALITY

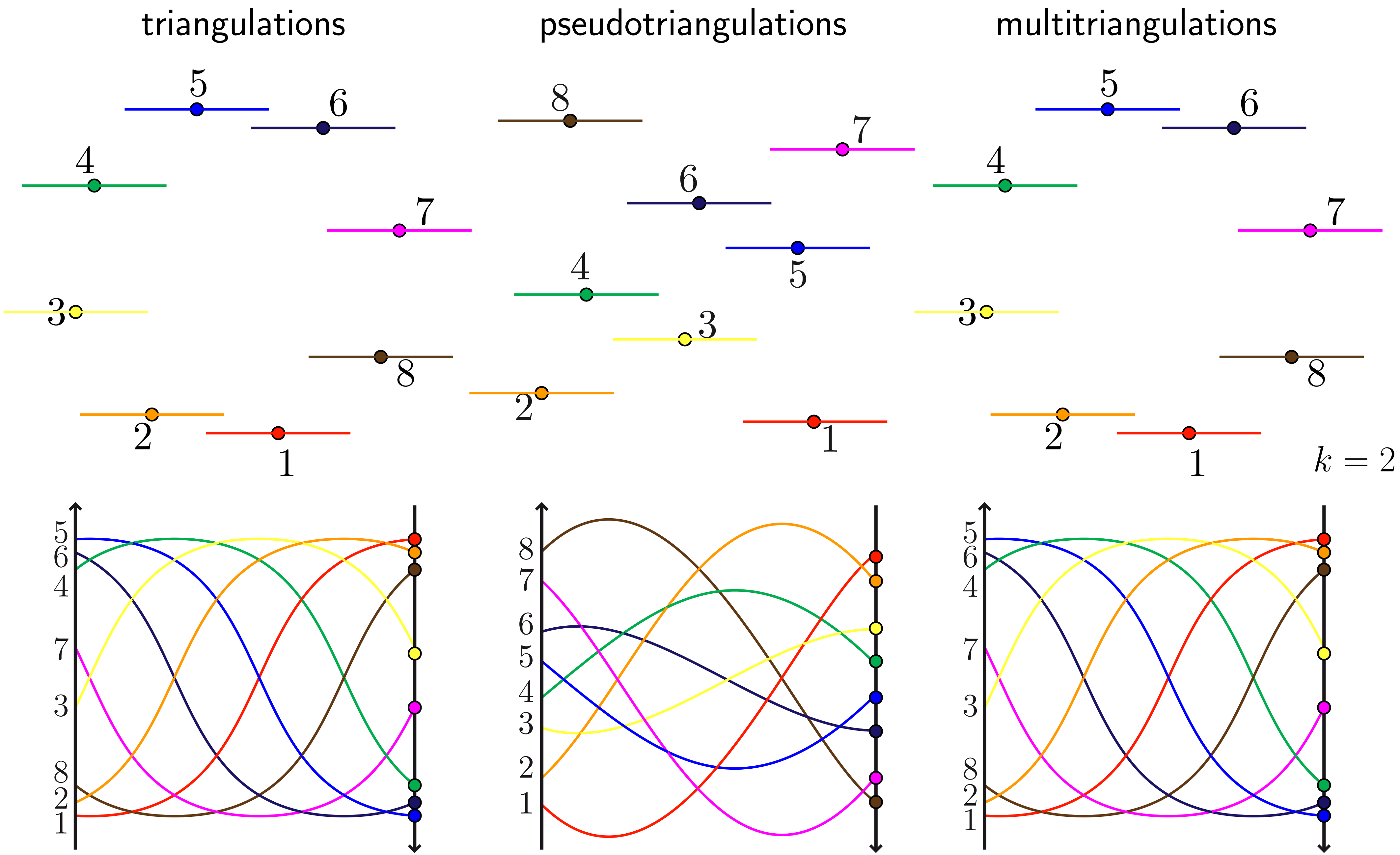

triangulations
pseudotriangulations
multitriangulations
1 2 3 4 5 6 7 8
$k = 2$

DUALITY

triangulations | pseudotriangulations | multitriangulations



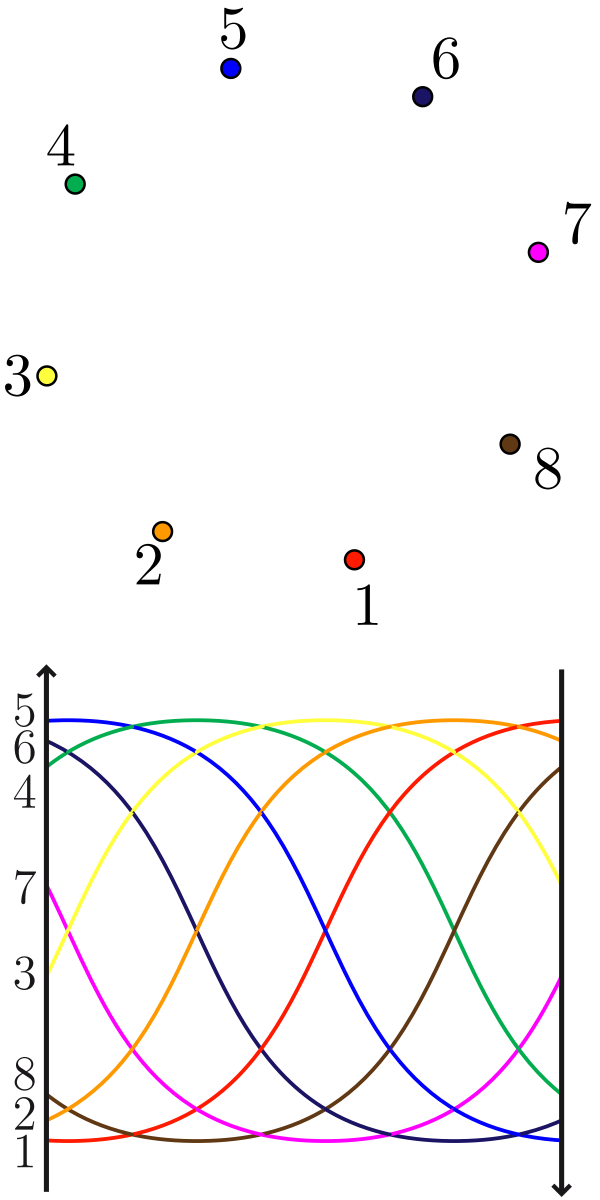
DUALITY



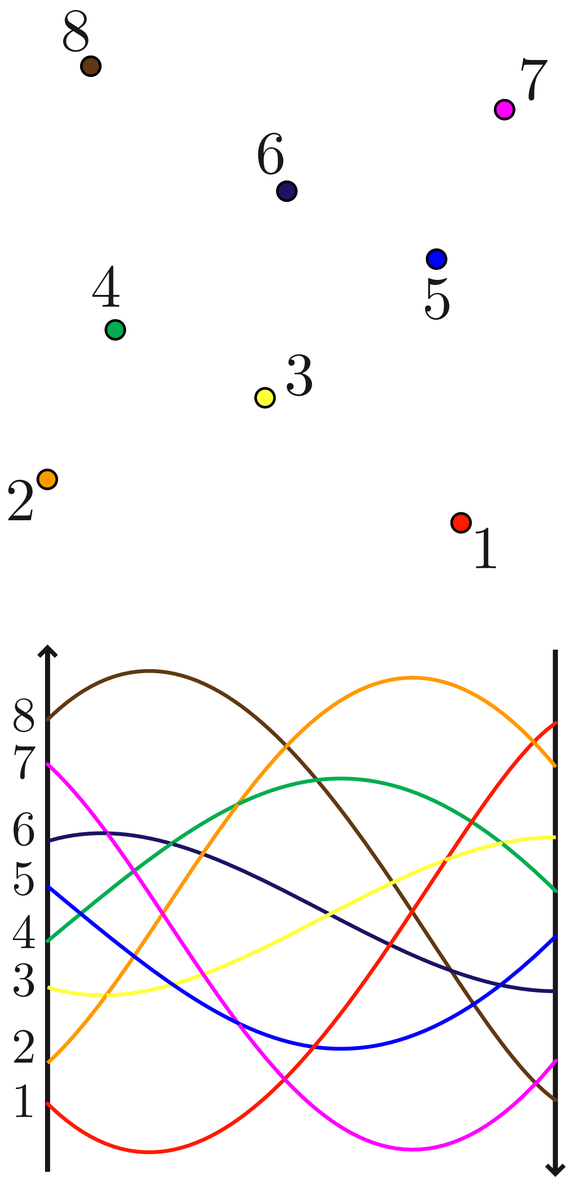
triangulations
pseudotriangulations
multitriangulations
5
6
4
7
3
8
2
1
$k = 2$

DUALITY

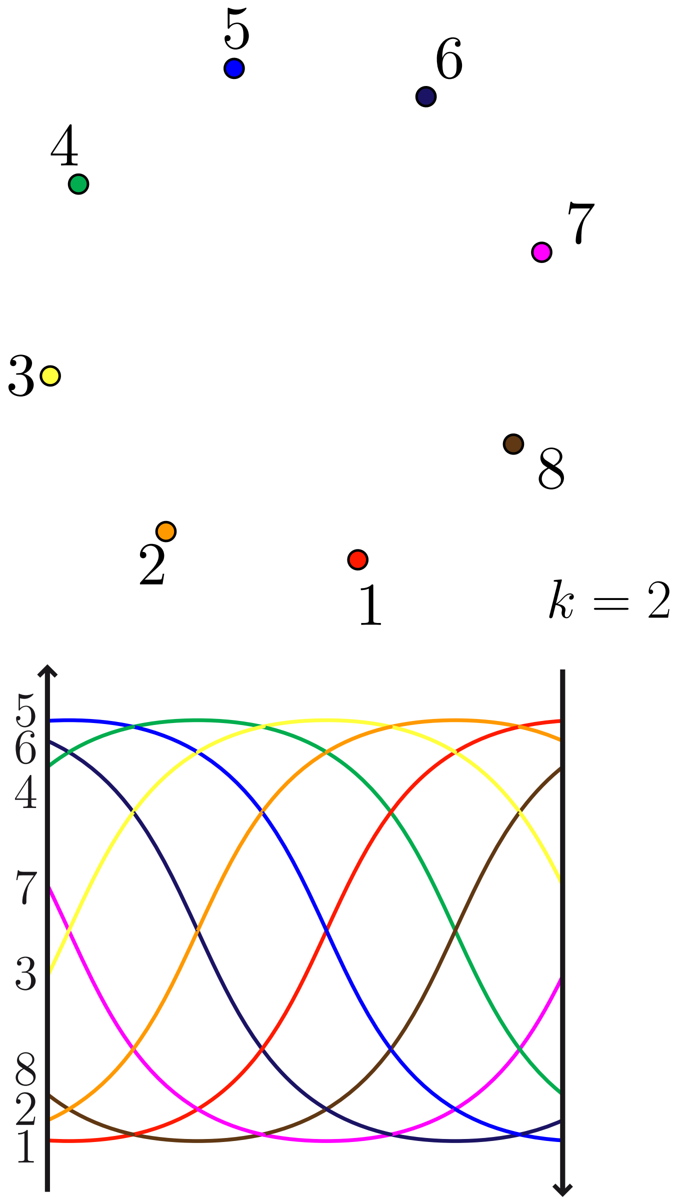
triangulations



pseudotriangulations

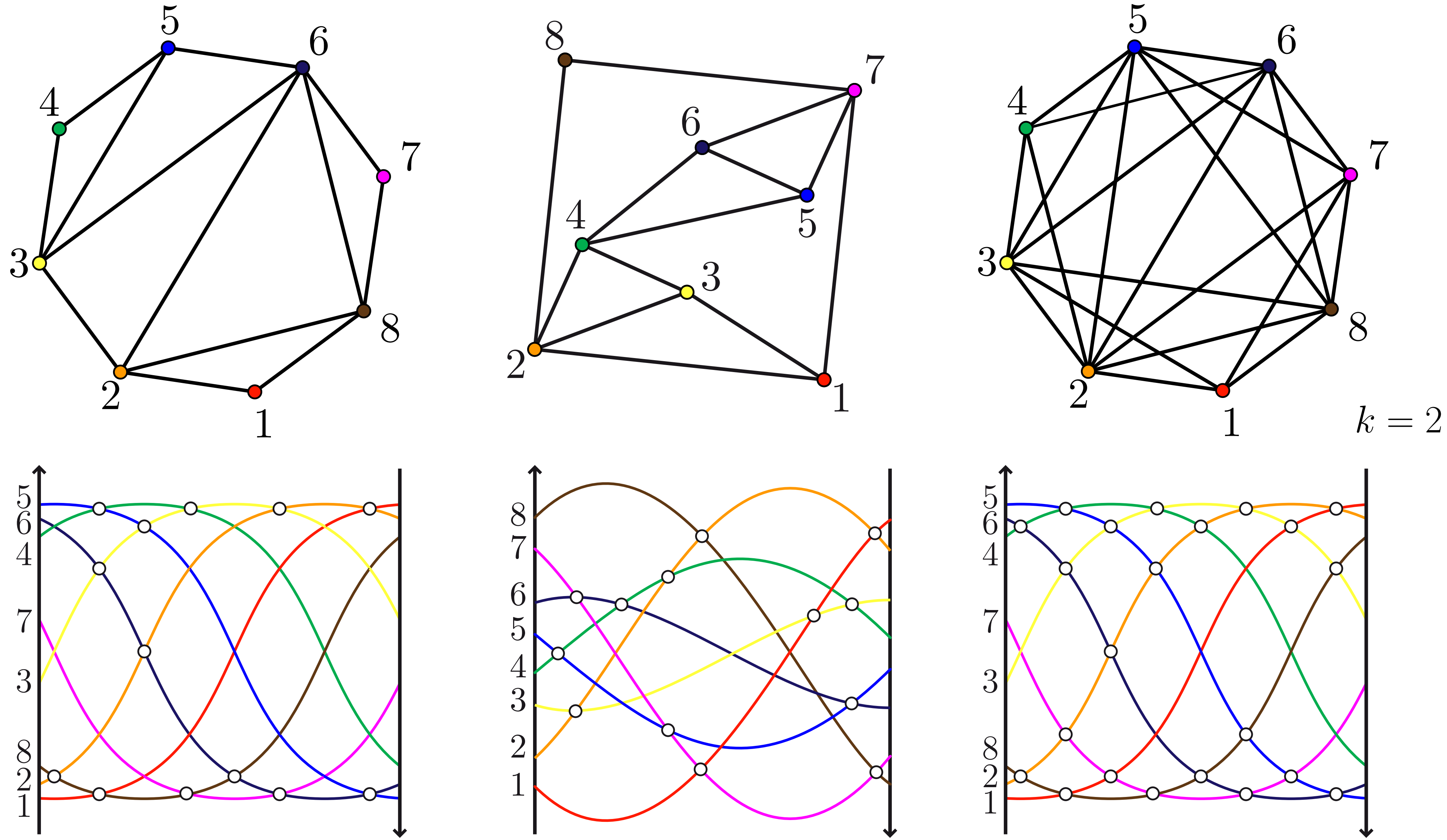


multitriangulations

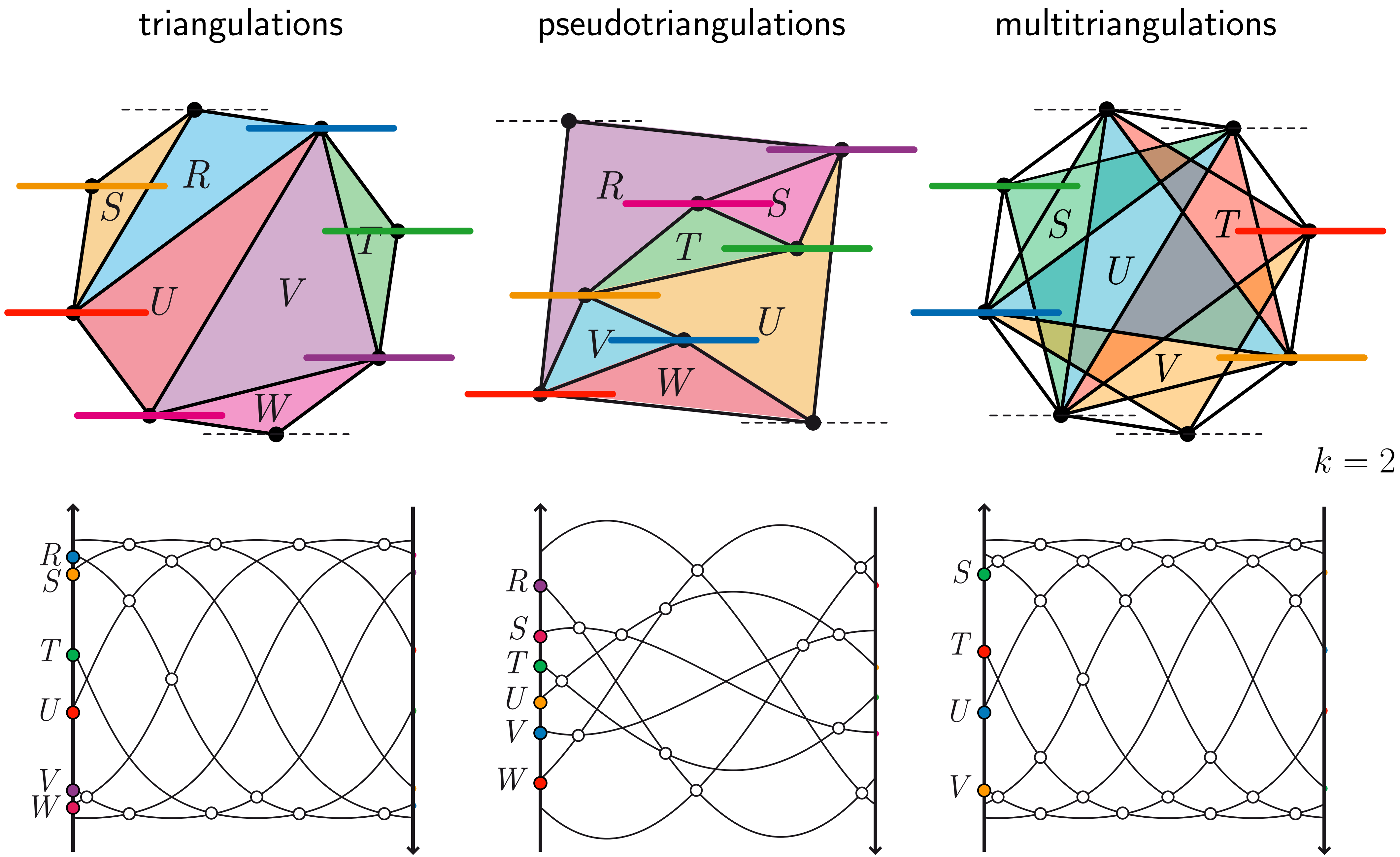


DUALITY

triangulations | pseudotriangulations | multitriangulations

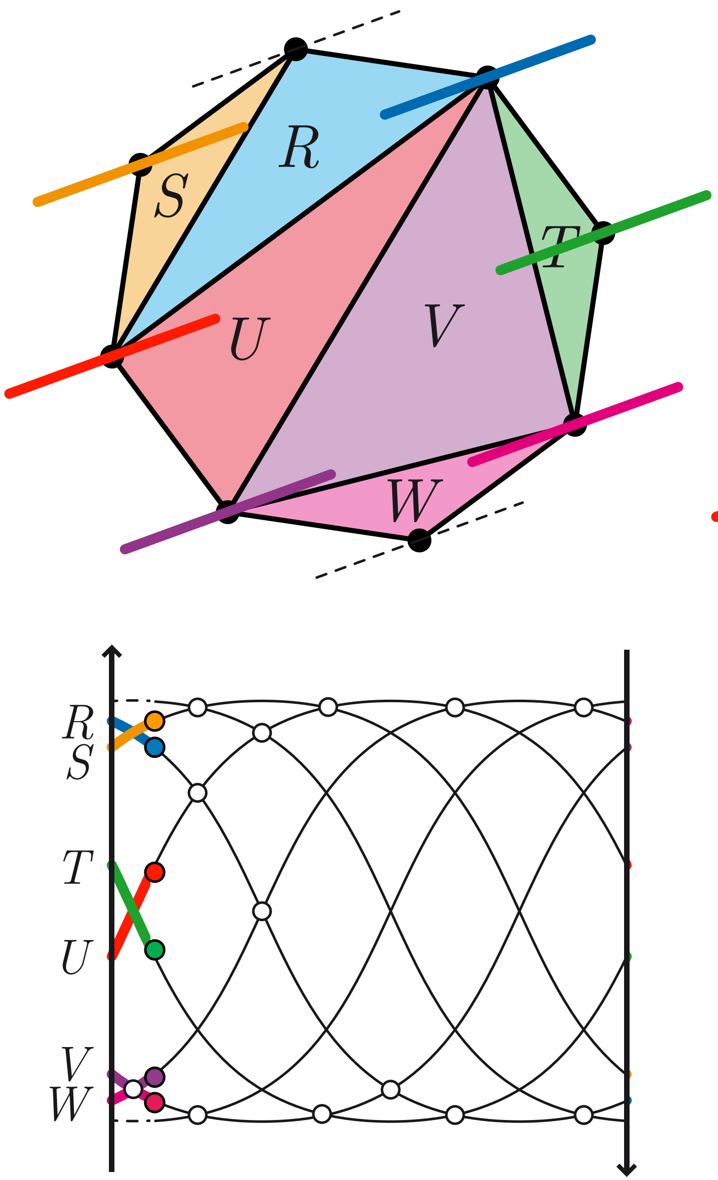


DUALITY

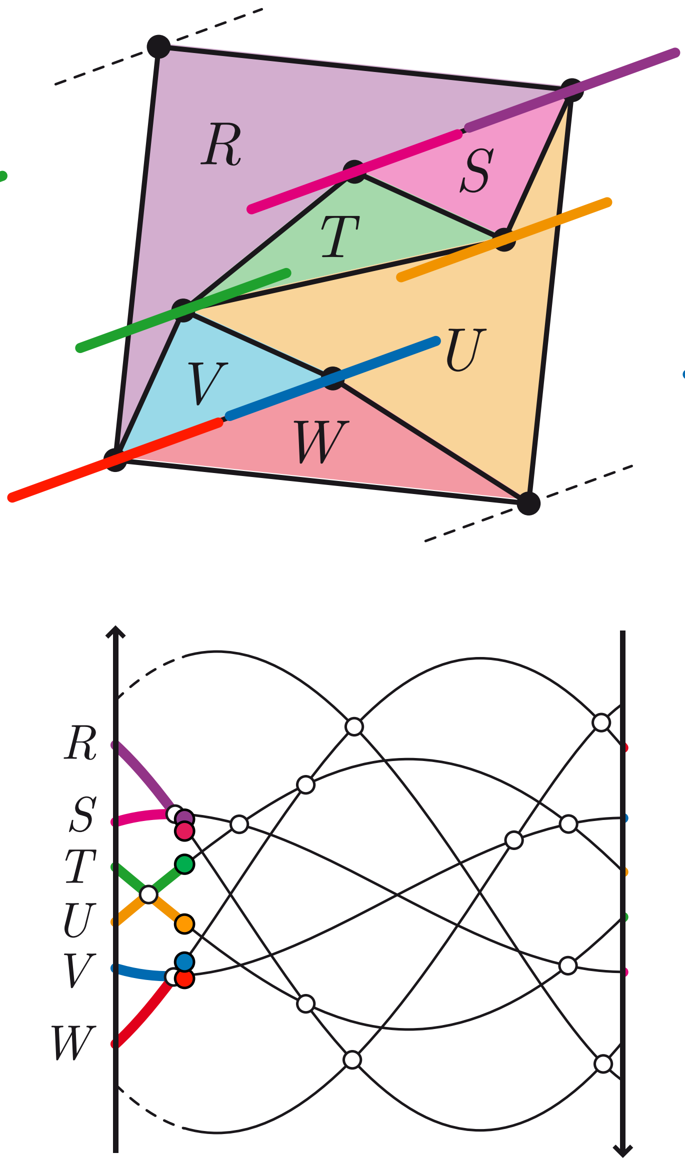

triangulations
pseudotriangulations
multitriangulations
R
S
T
U
V
W
$k = 2$

DUALITY

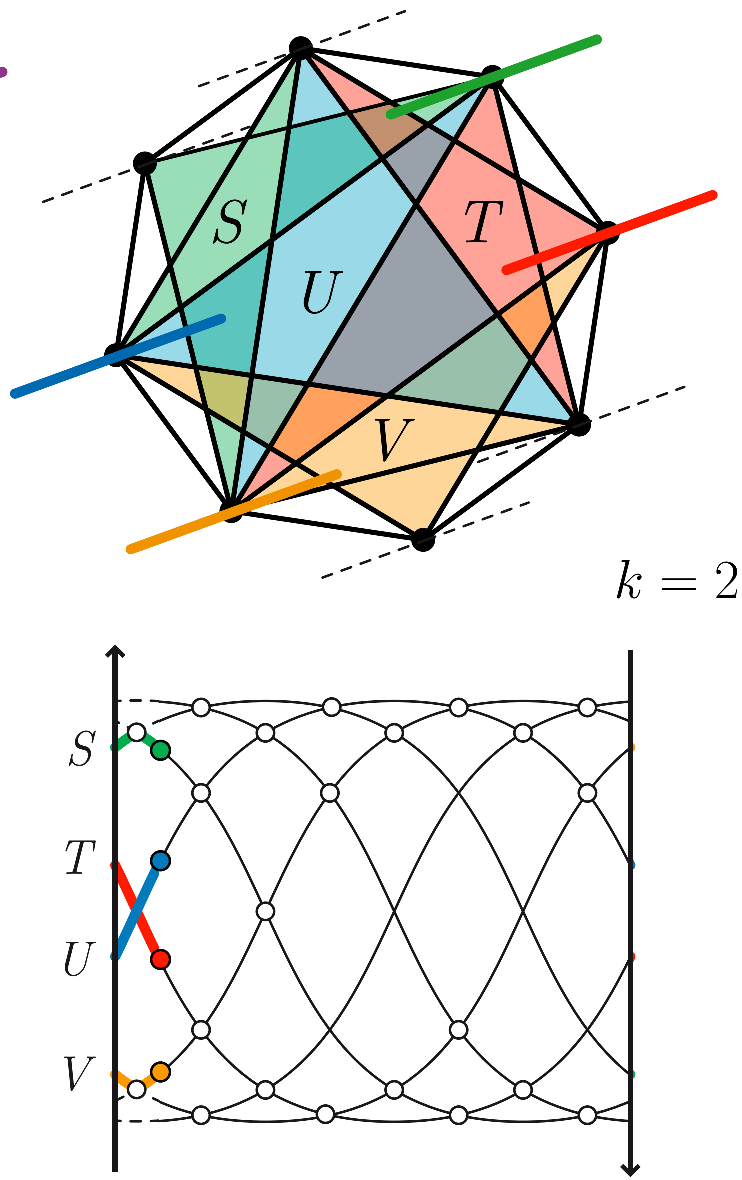
triangulations



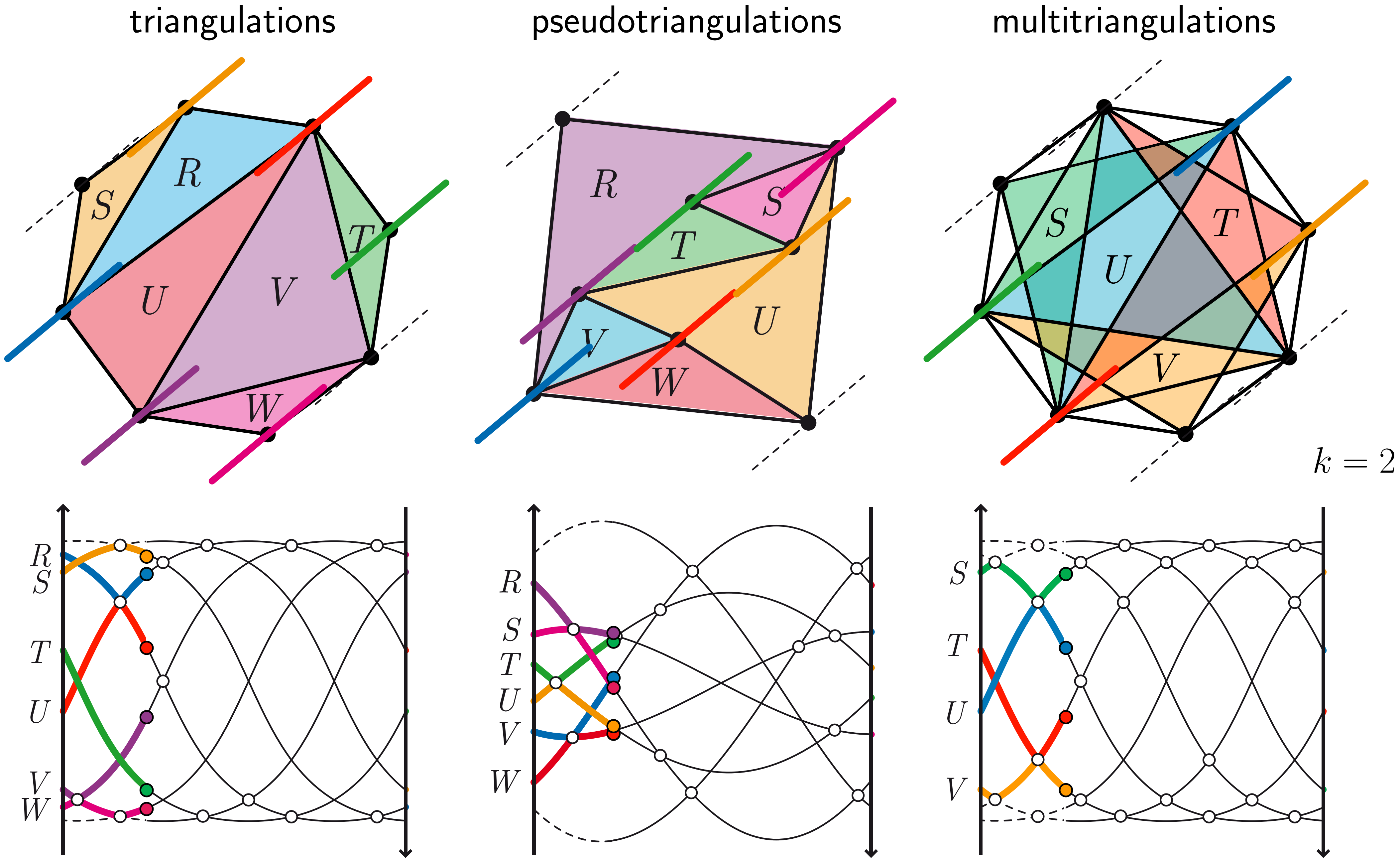
pseudotriangulations



multitriangulations



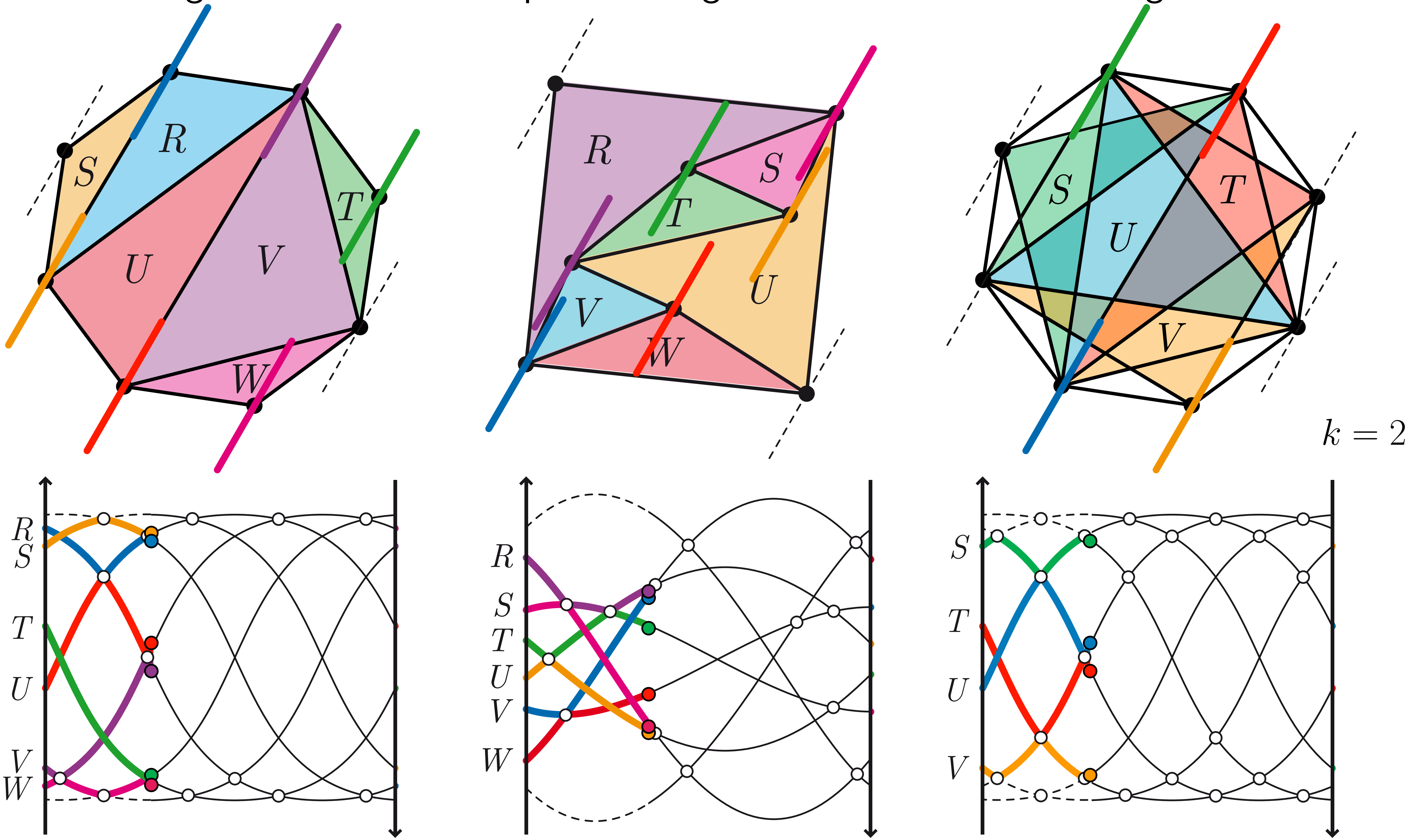
DUALITY



triangulations
pseudotriangulations
multitriangulations
R
S
T
U
V
W
k = 2

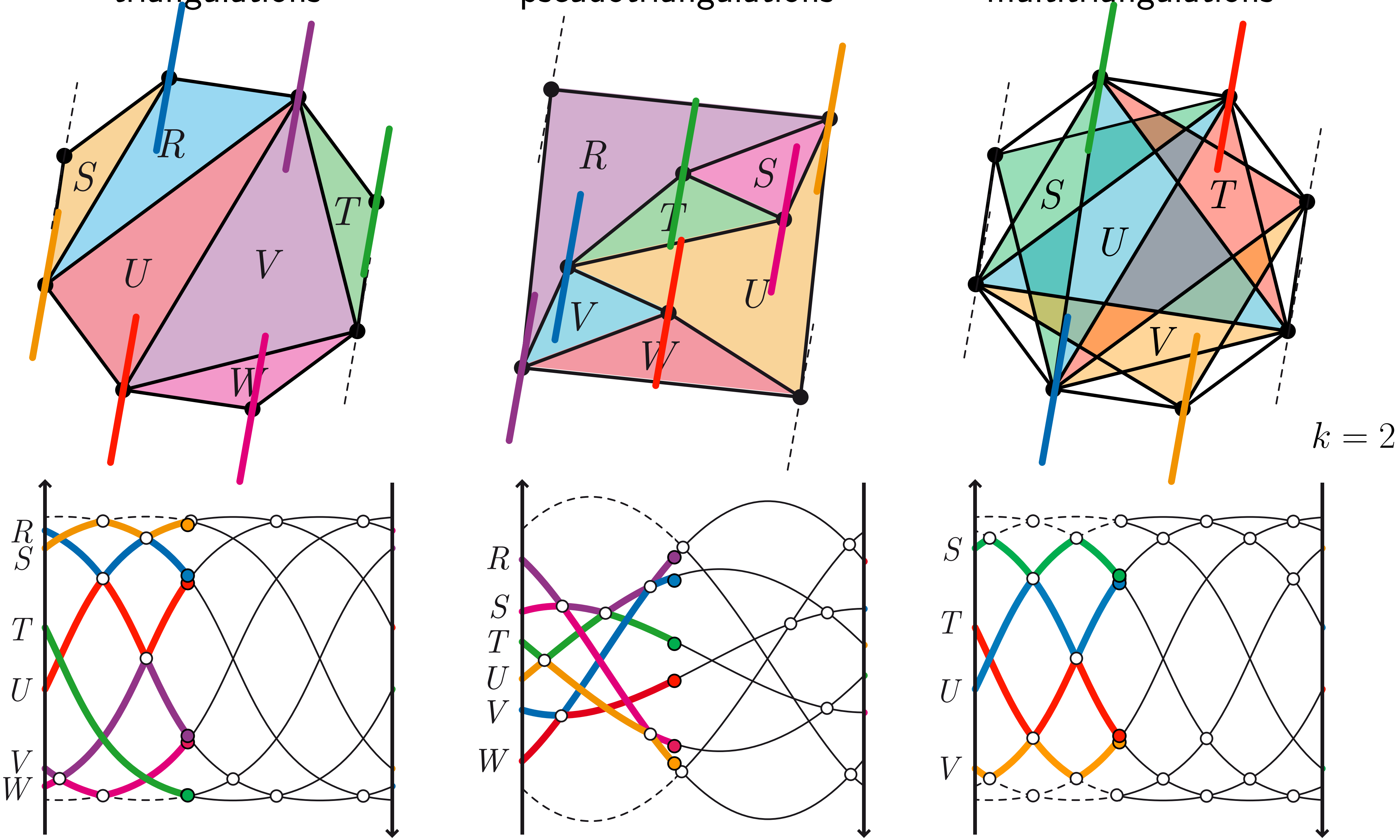
DUALITY

triangulations | pseudotriangulations | multitriangulations

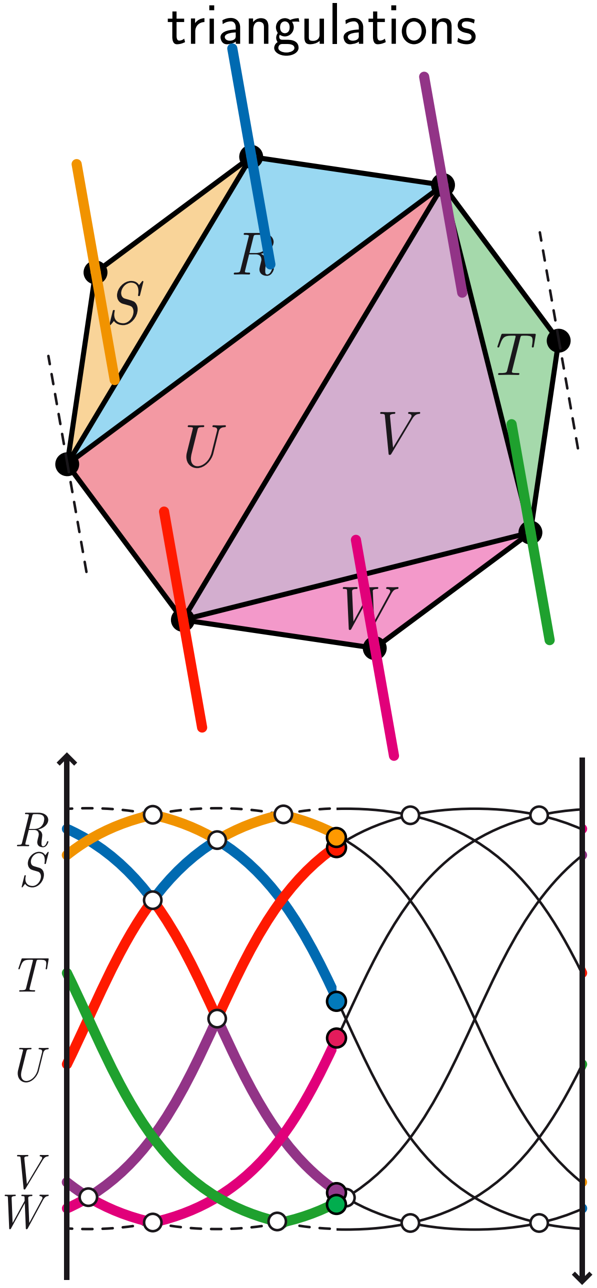


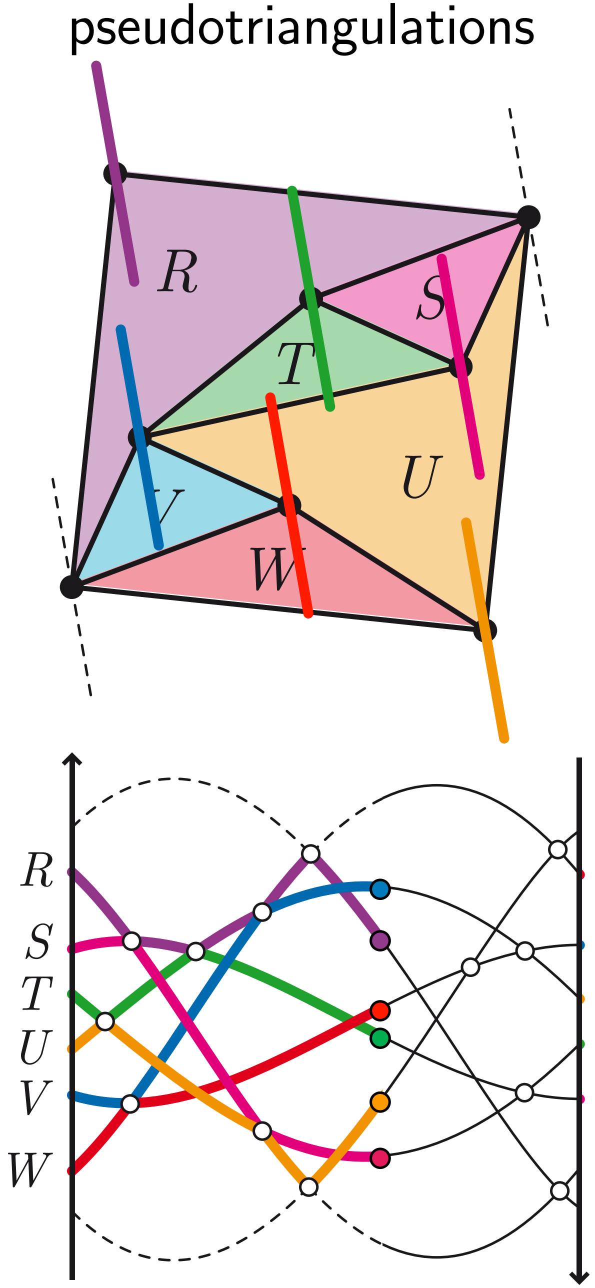
DUALITY

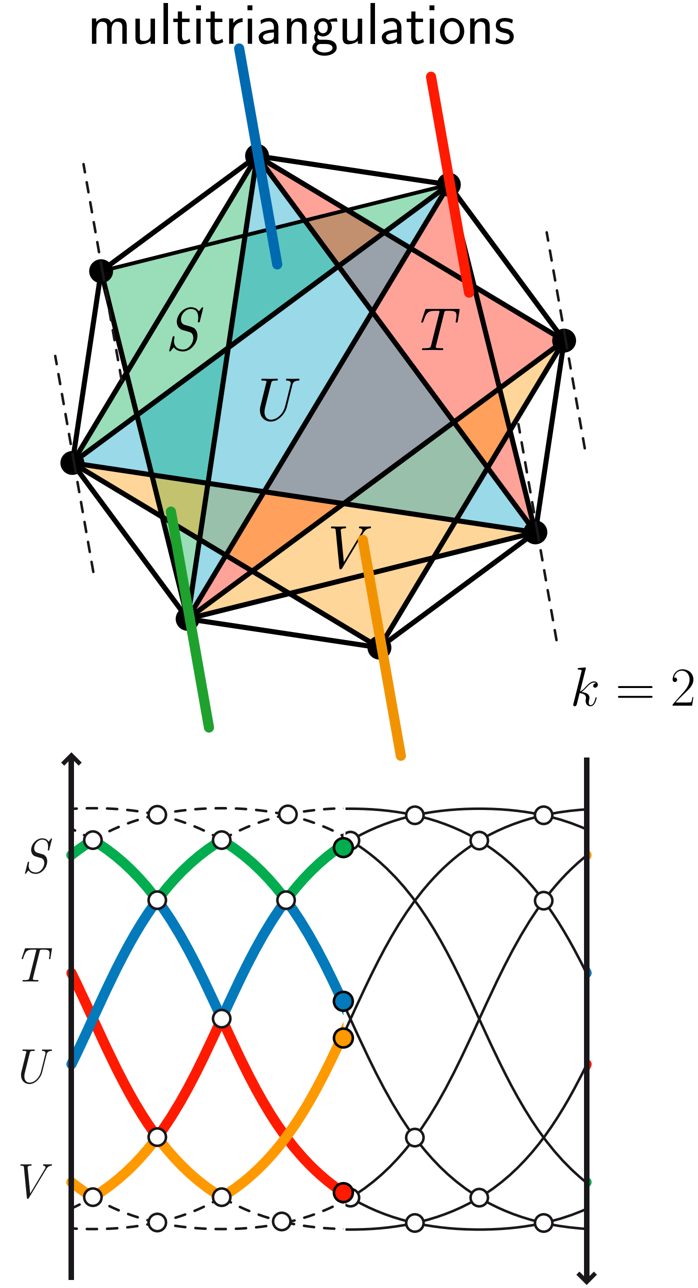
triangulations | pseudotriangulations | multitriangulations



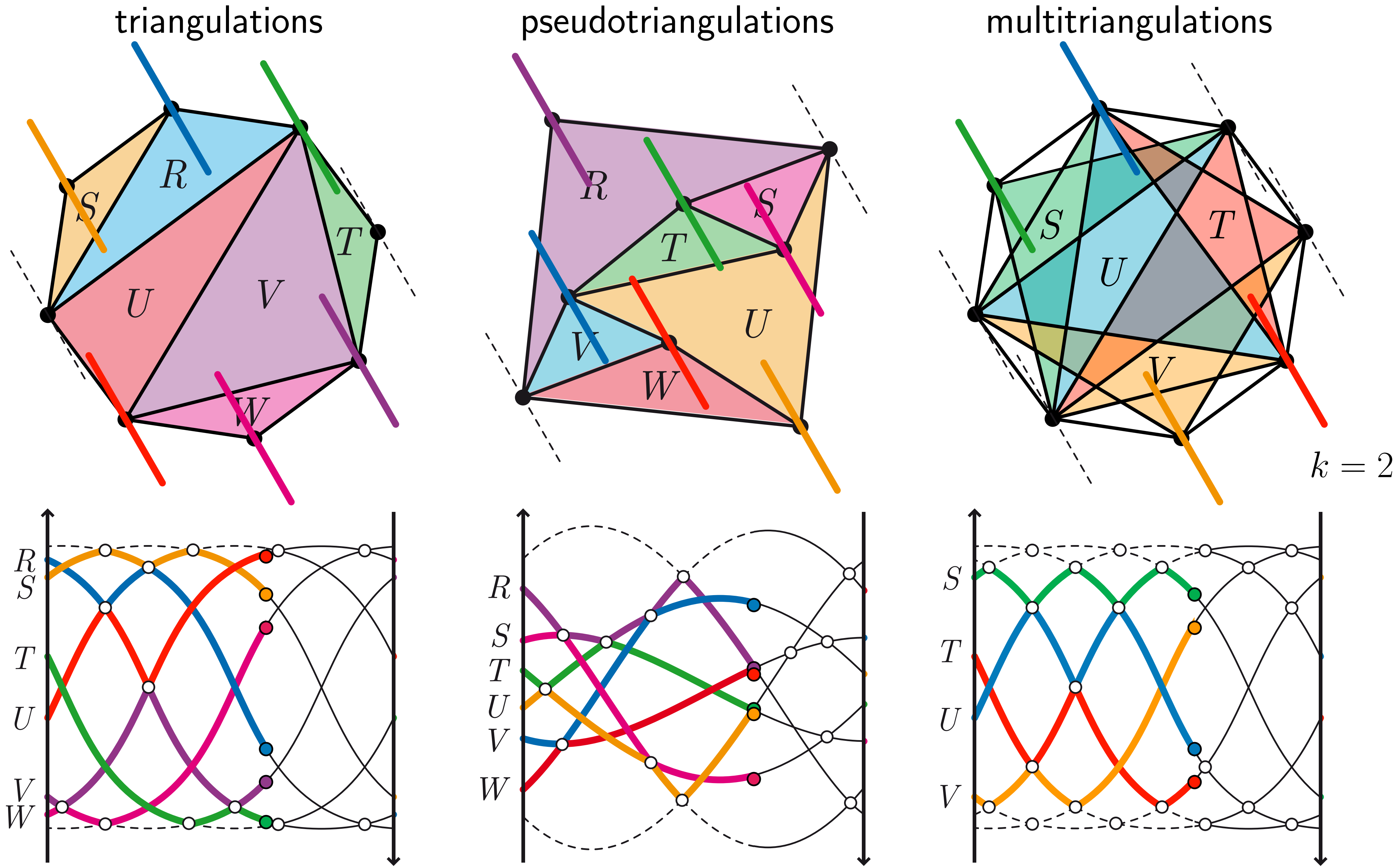
DUALITY


triangulations
R
S
T
U
V
W
R
S
T
U
V
W


pseudotriangulations
R
S
T
U
V
W
R
S
T
U
V
W


multitriangulations
S
T
U
V
k = 2
S
T
U
V

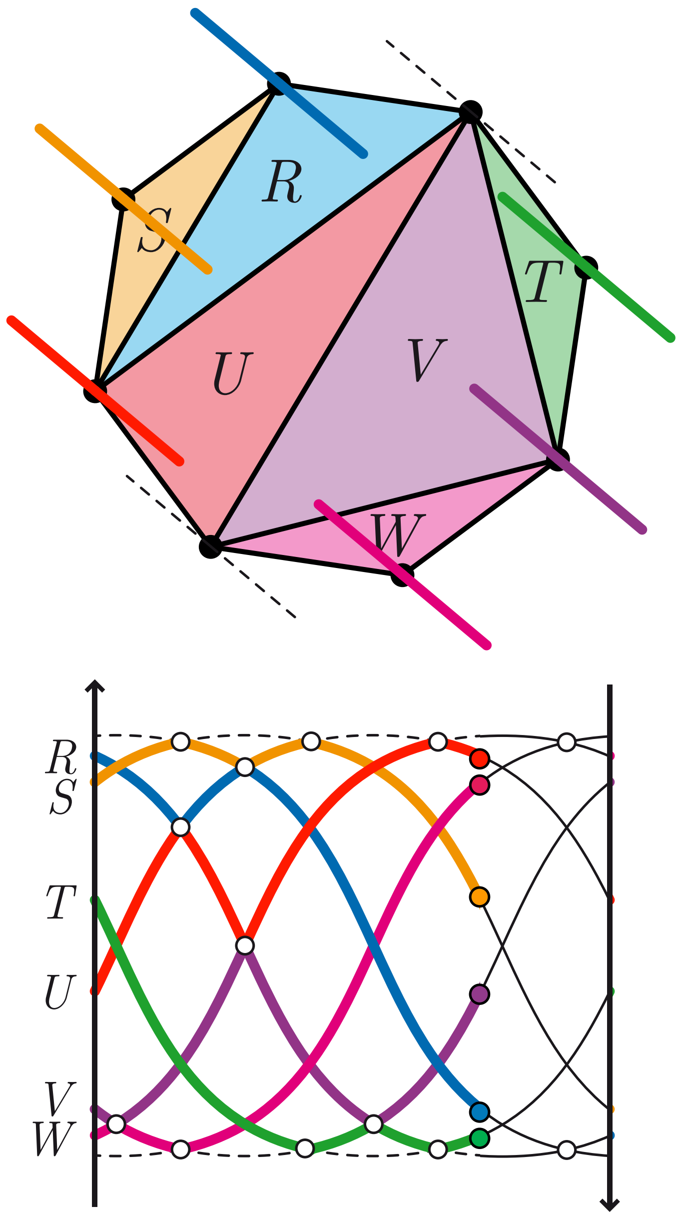
DUALITY



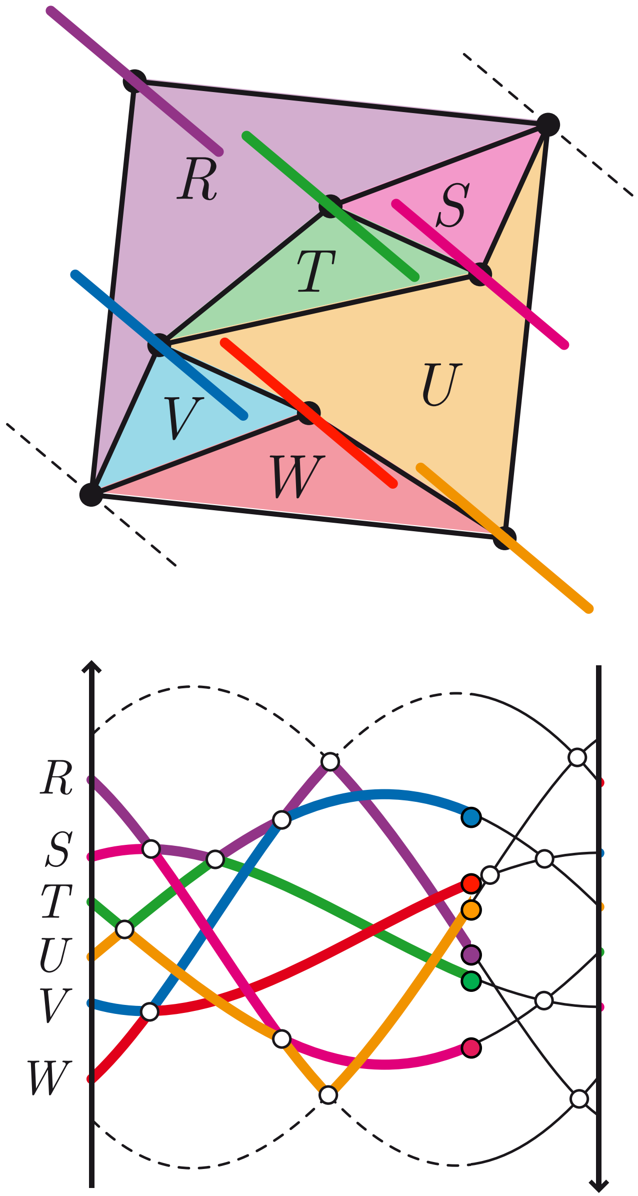
triangulations
pseudotriangulations
multitriangulations
R
S
T
U
V
W
$k = 2$

DUALITY

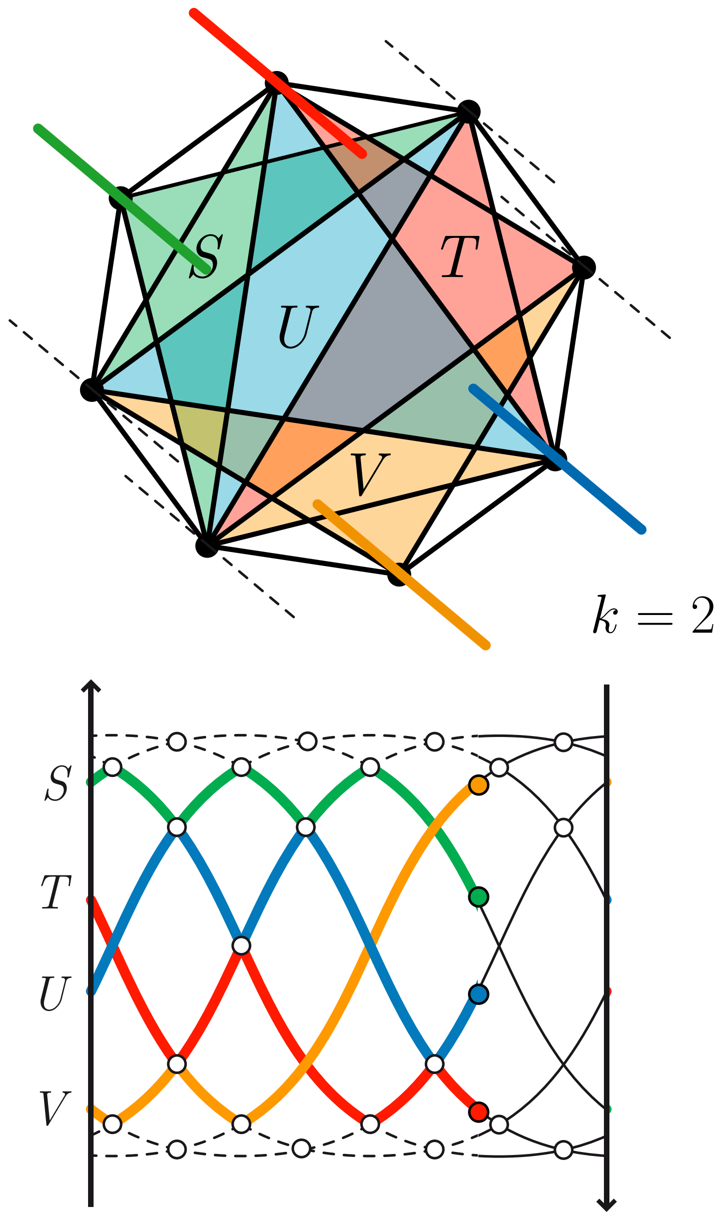
triangulations



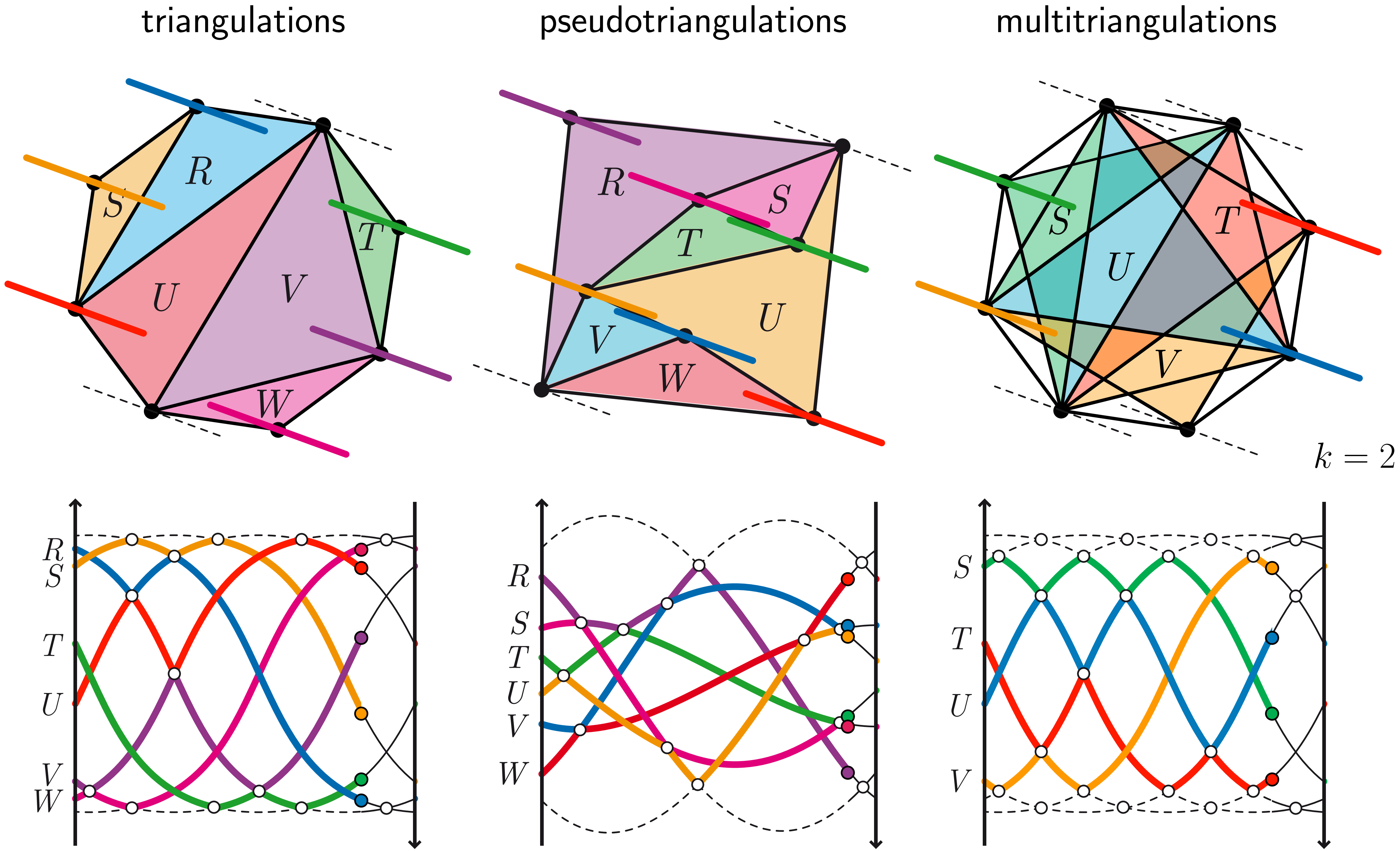
pseudotriangulations



multitriangulations

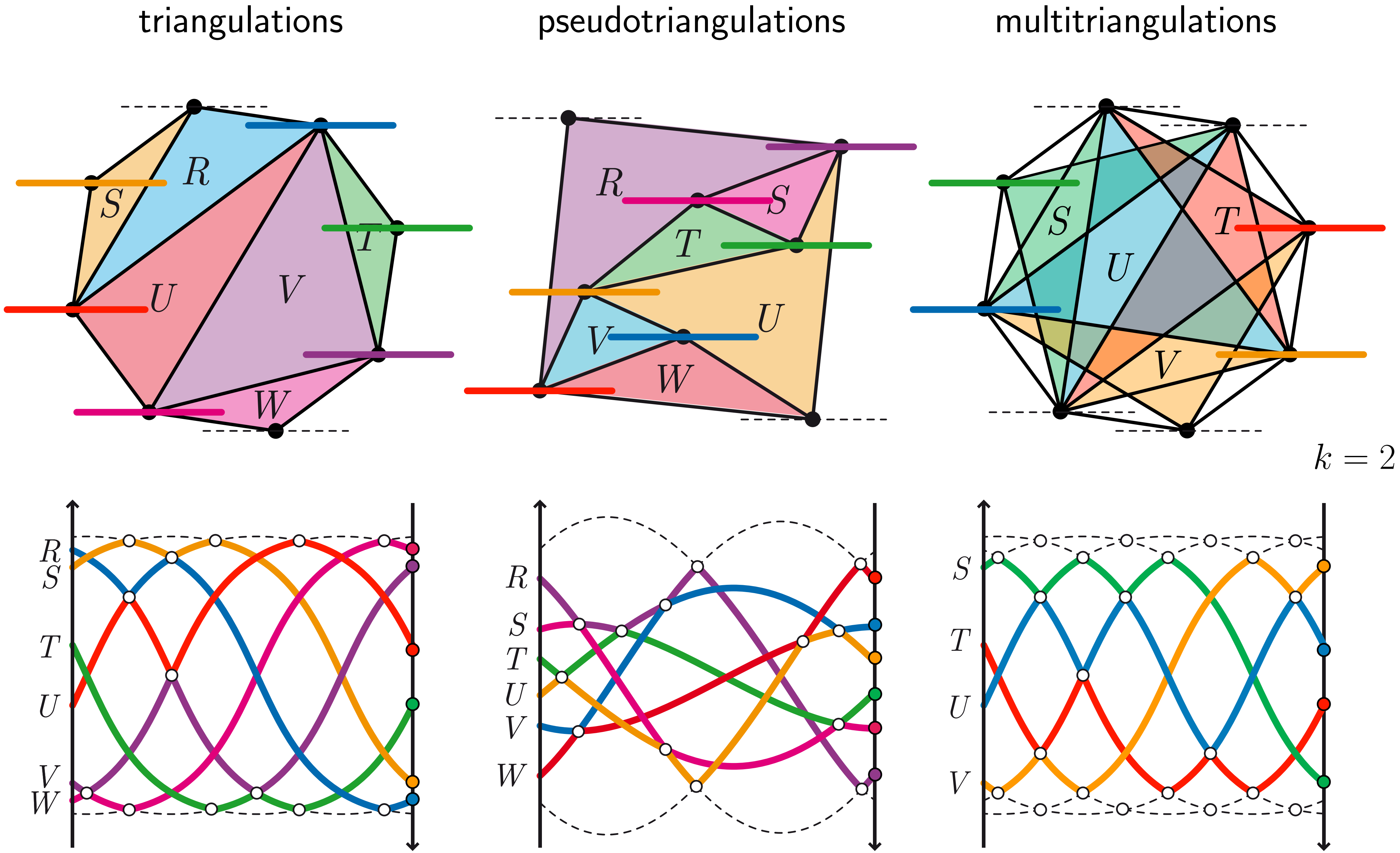


DUALITY



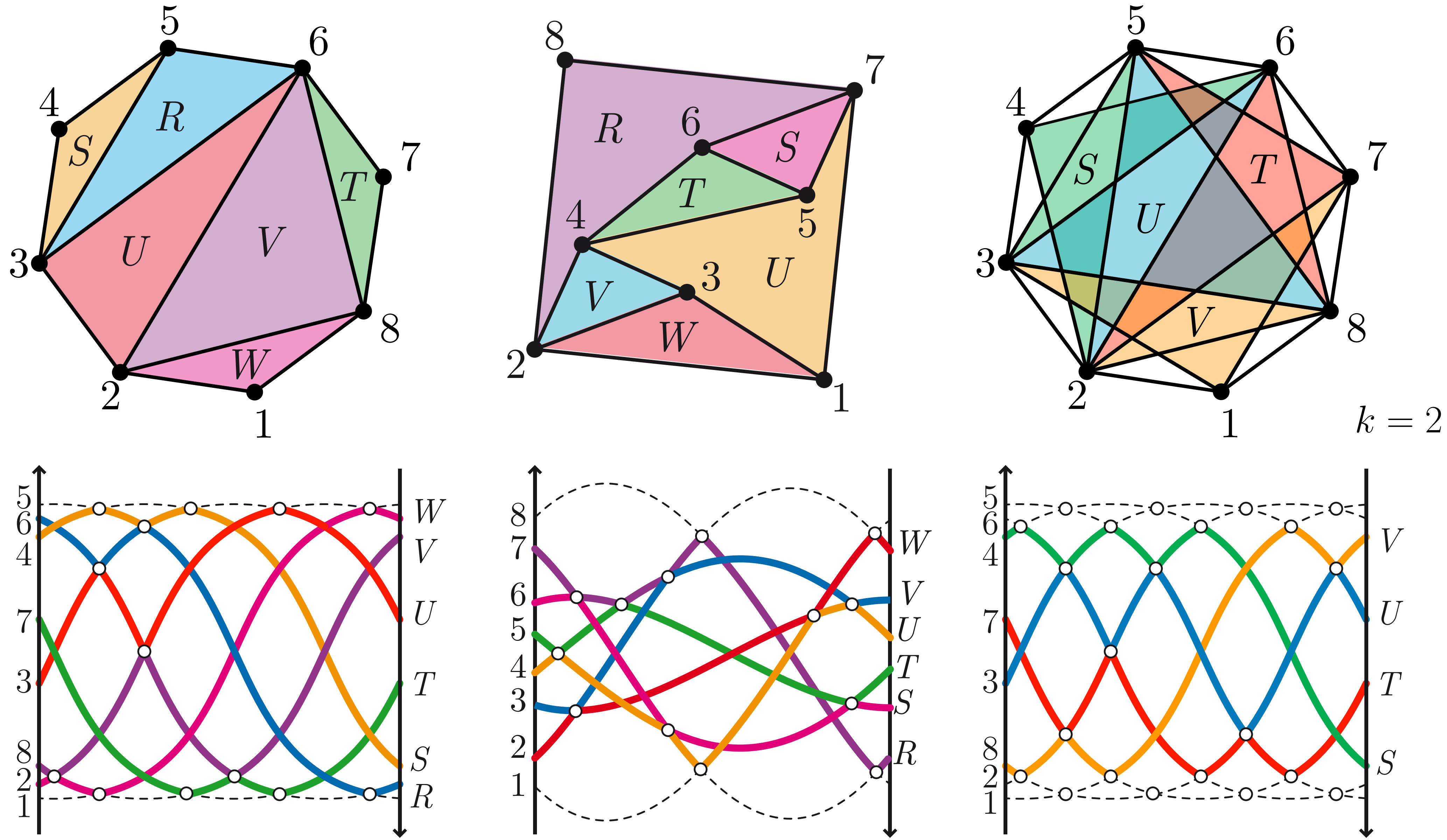
triangulations
pseudotriangulations
multitriangulations
R
S
T
U
V
W
$k = 2$

DUALITY


triangulations
pseudotriangulations
multitriangulations
R
S
T
U
V
W
$k = 2$

DUALITY

triangulations | pseudotriangulations | multitriangulations



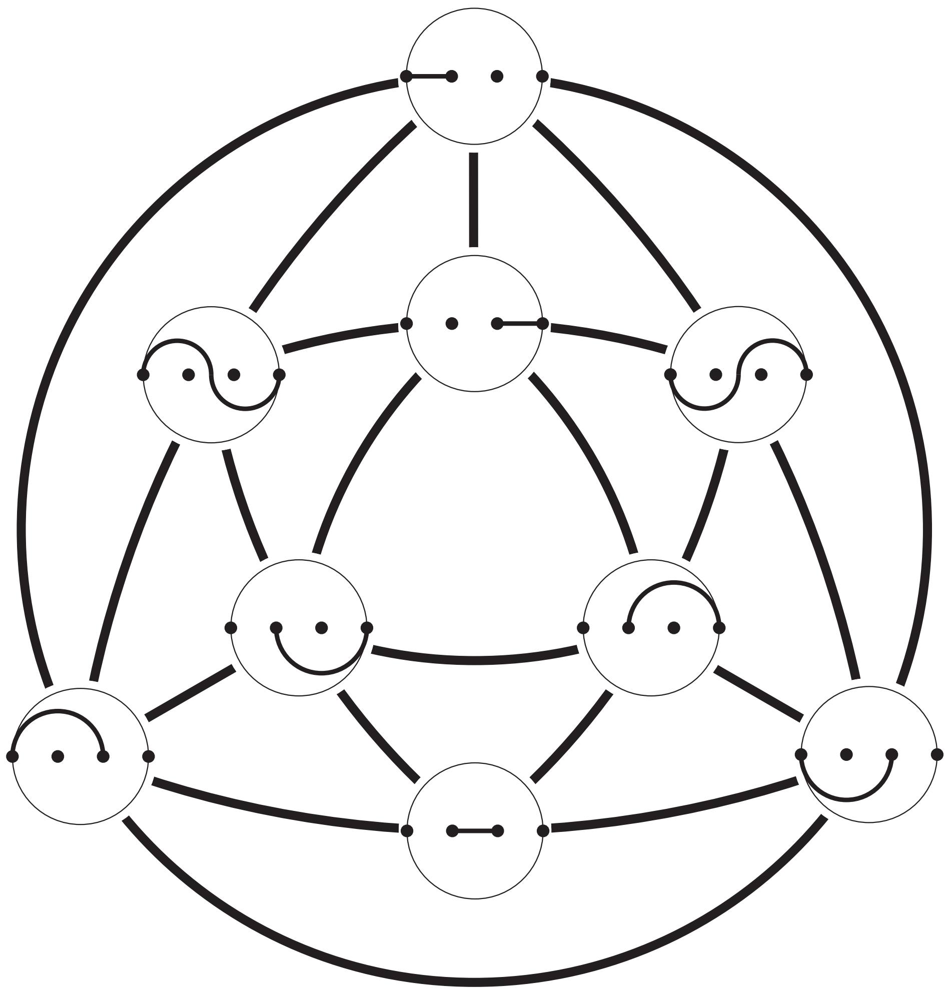
P.–Pocchiola ('12)

WIGGLY PSEUDOTRIANGULATIONS & WIGGLY COMPLEX

WIGGLY COMPLEX

wiggly dissection $=$ set of pairwise non-crossing and pointed wiggly arcs on $n+2$ points

wiggly complex WC_n $=$ simplicial complex of wiggly dissections



WIGGLY COMPLEX

wiggly dissection = set of pairwise non-crossing and pointed wiggly arcs on $n+2$ points

wiggly complex WC_n = simplicial complex of wiggly dissections

$$f(\mathrm{WC}_1) = (1, 2)$$
$$f(\mathrm{WC}_2) = (1, 9, 21, 14)$$
$$f(\mathrm{WC}_3) = (1, 24, 154, 396, 440, 176)$$
$$f(\mathrm{WC}_4) = (1, 55, 729, 4002, 10930, 15684, 11312, 3232)$$
$$f(\mathrm{WC}_5) = (1, 118, 2868, 28110, 140782, 400374, 673274, 662668, 352728, 78384)$$

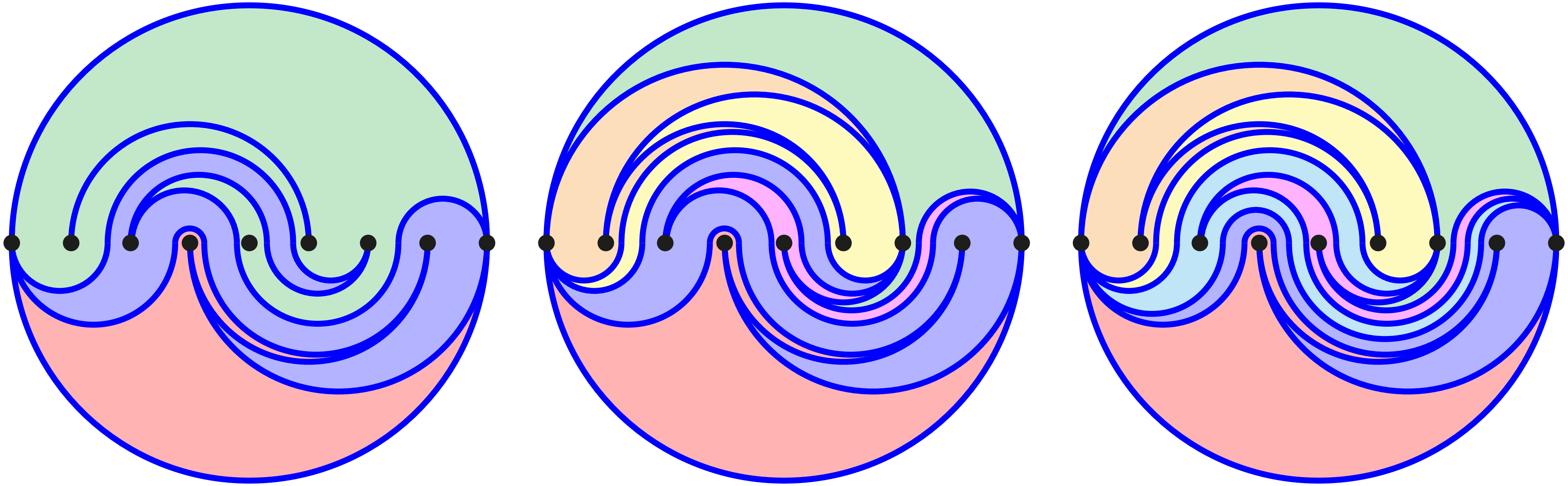
$$h(\mathrm{WC}_1) = (1, 1)$$
$$h(\mathrm{WC}_2) = (1, 6, 6, 1)$$
$$h(\mathrm{WC}_3) = (1, 19, 68, 68, 19, 1)$$
$$h(\mathrm{WC}_4) = (1, 48, 420, 1147, 1147, 420, 48, 1)$$
$$h(\mathrm{WC}_5) = (1, 109, 1960, 11254, 25868, 25868, 11254, 1960, 109, 1)$$

WIGGLY PSEUDOTRIANGULATIONS

c cell in a wiggly dissection with boundary ∂_c

degree $\delta_c = 1/2\,\#\text{arcs on } \partial_c + 2\,\#\text{connected components of } \partial_c - 1$

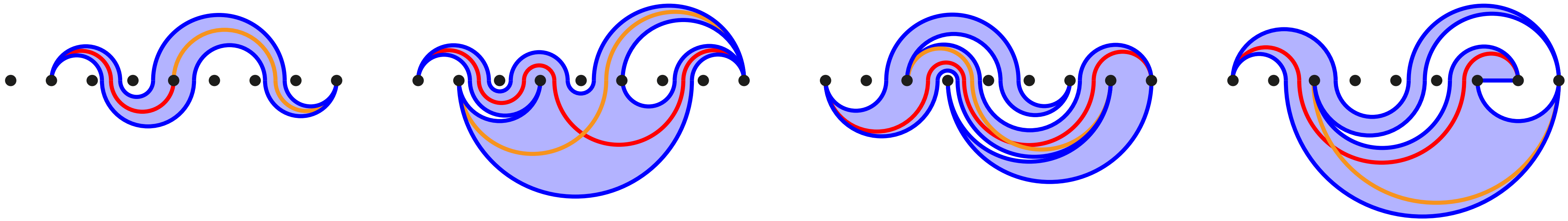
pseudotriangle = cell of degree 3 pseudoquadragle = cell of degree 4



PROP. The inclusion maximal wiggly pseudodissections are the pseudotriangulations, and contain $2n-1$ internal arcs and n cells. Bapat–P. (24$^+$)

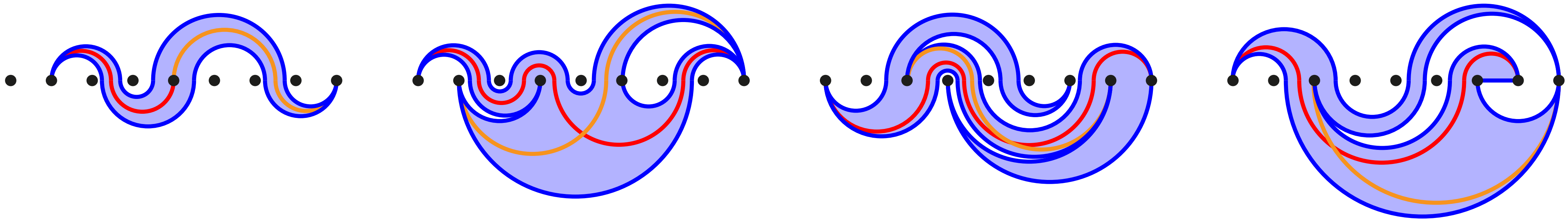
n	1	2	3	4	5	6	7	8	...
wp_n	2	14	176	3232	78384	2366248	85534176	3602770400	...

PROP. Any wiggly pseudoquadrangle has exactly two wiggly diagonals, and they either cross or are non pointed. Bapat–P. (24^+)

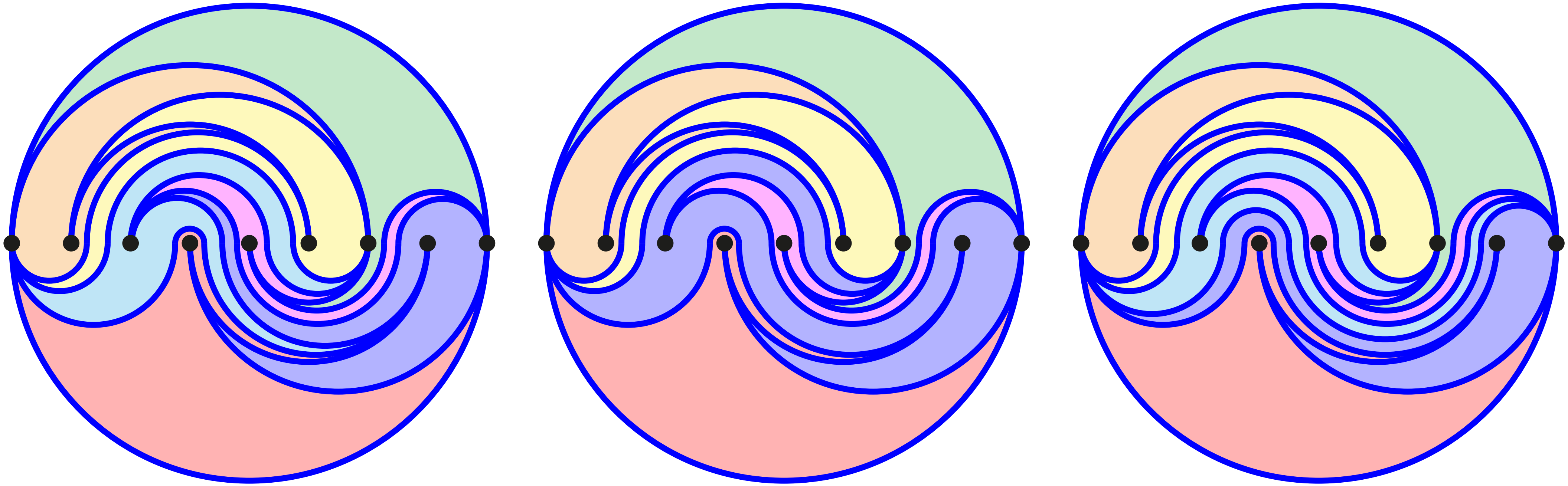


WIGGLY FLIP GRAPH

PROP. Any wiggly pseudoquadrangle has exactly two wiggly diagonals, and they either cross or are non pointed. Bapat–P. (24^+)

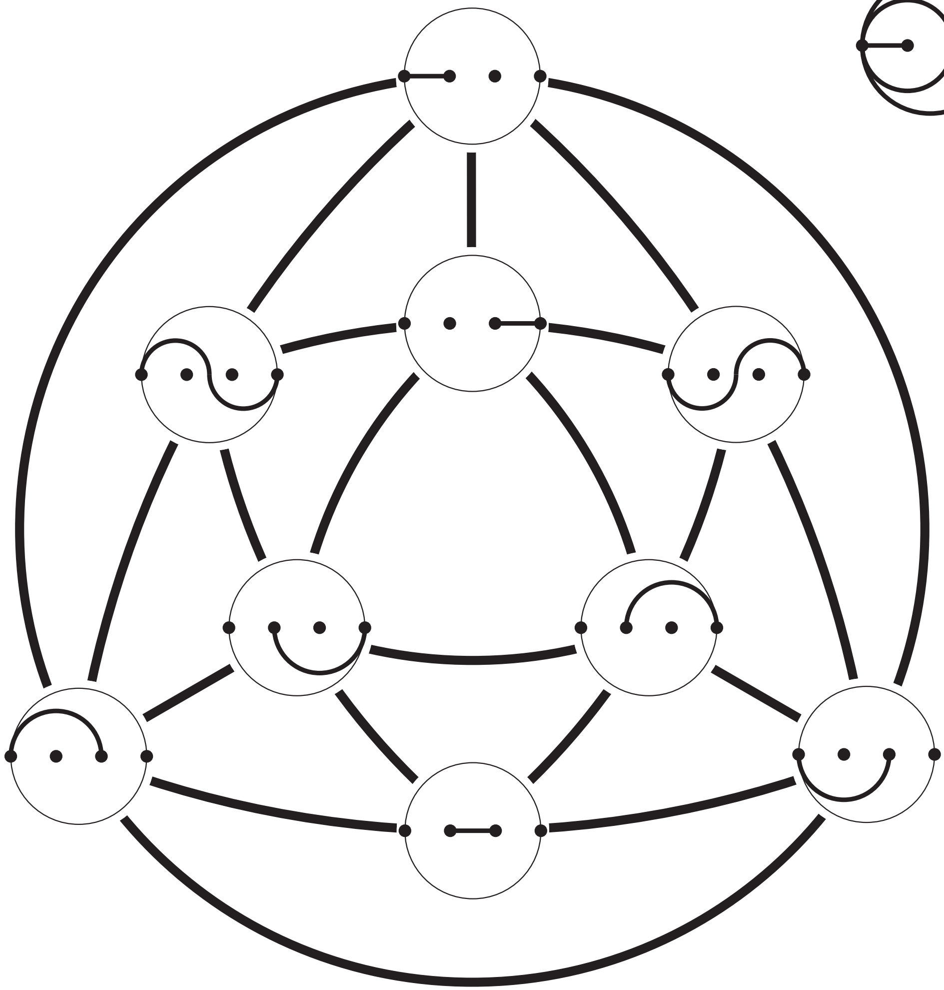


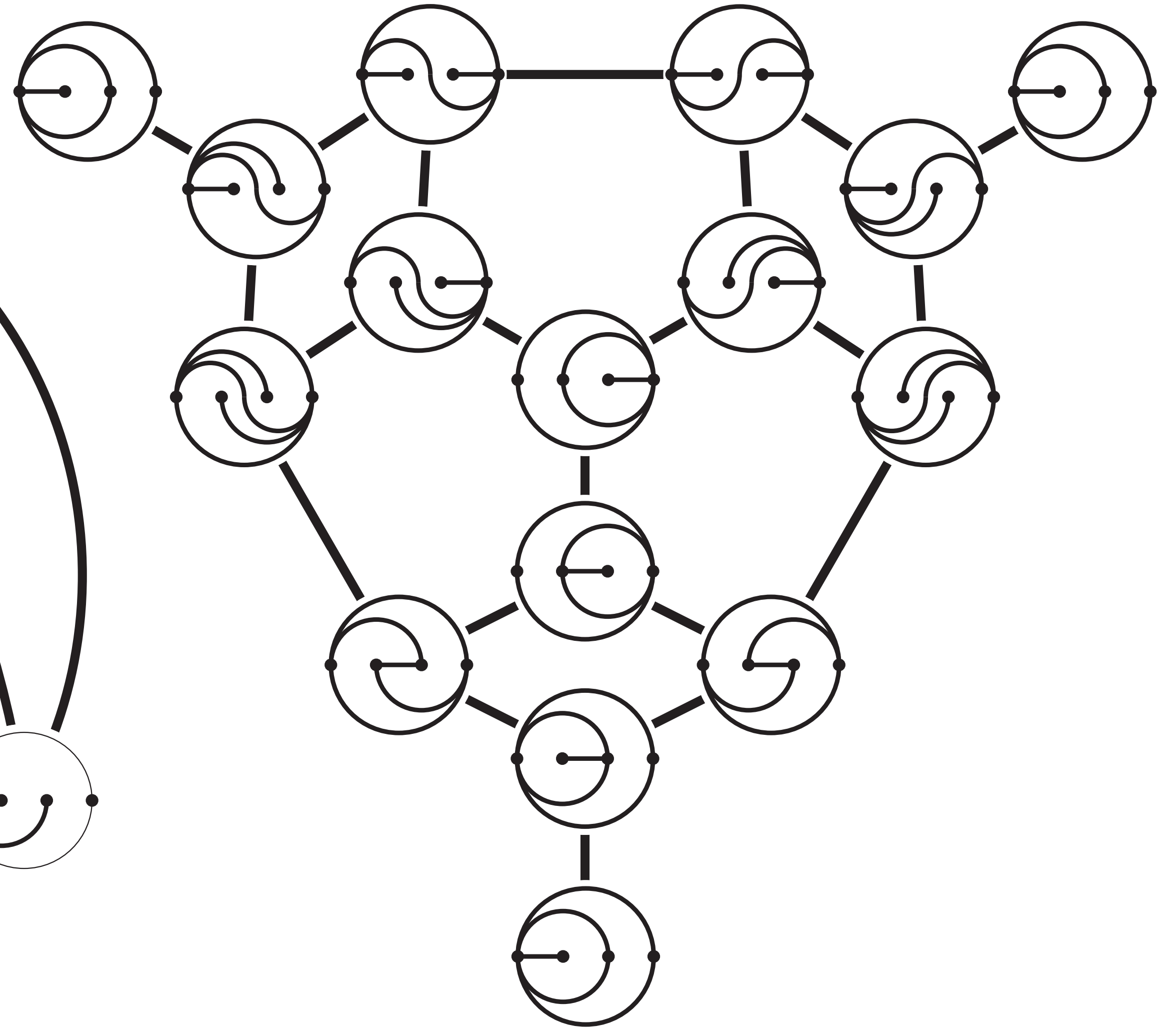
wiggly flip graph $\mathrm{WFG}_n =$ a vertex for each wiggly pseudotriangulation
an edge between T and T' if $T \smallsetminus \{\alpha\} = T' \smallsetminus \{\alpha'\}$



WIGGLY FLIP GRAPH

PROP. The wiggly flip graph WFG_n is $(2n-1)$-regular and connected. Bapat–P. (24^+)



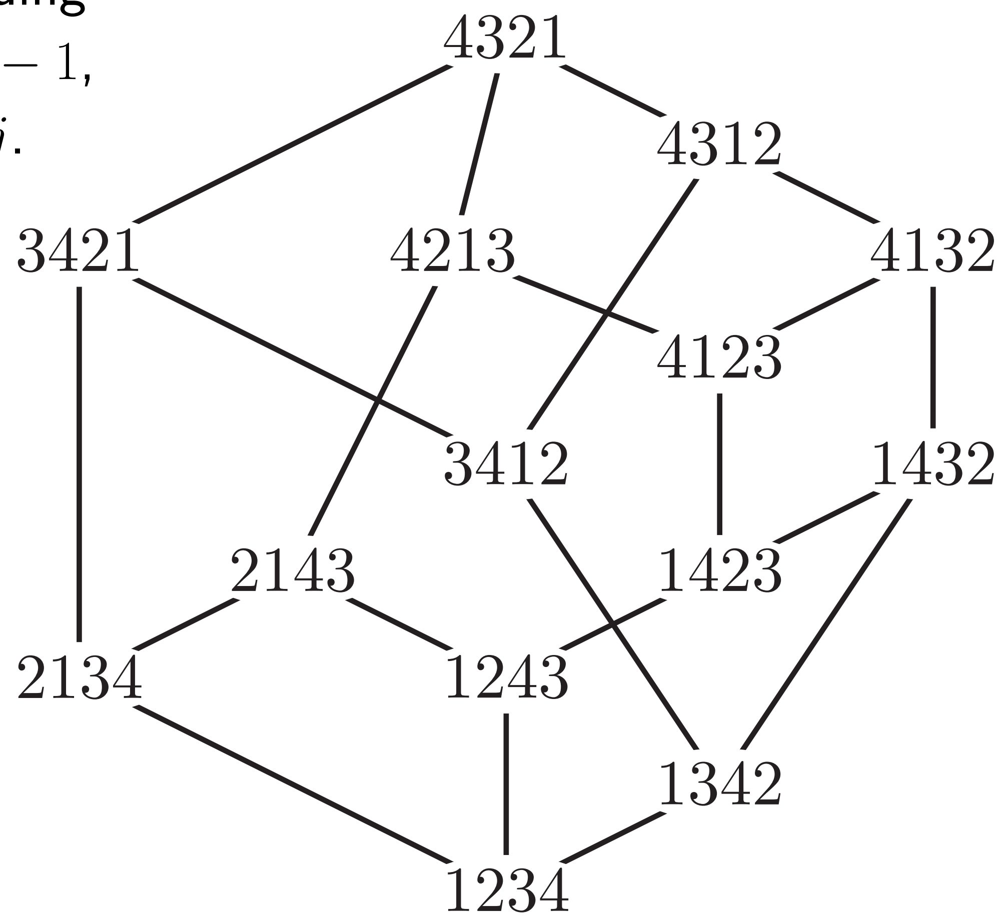


WIGGLY PERMUTATIONS & WIGGLY LATTICE

WIGGLY PERMUTATIONS

wiggly permutation $=$ permutation of $2n$ avoiding

- $(2j-1) \cdots i \cdots (2j)$ for $j \in [n]$ and $i < 2j-1$,
- $(2j) \cdots k \cdots (2j-1)$ for $j \in [n]$ and $k > 2j$.



PROP. The wiggly permutations induce a sublattice WL_n of the weak order on $\mathfrak{S}_{2n}$.

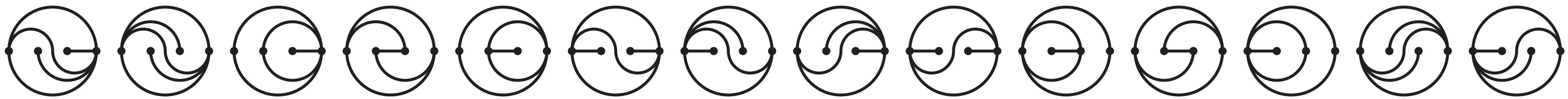
Bapat–P. (24^+)

PROP. The cover graph of the lattice WL_n is $(2n-1)$-regular and connected.

Bapat–P. (24^+)

n	1	2	3	4	5	6	7	8	...
wp_n	2	14	176	3232	78384	2366248	85534176	3602770400	...

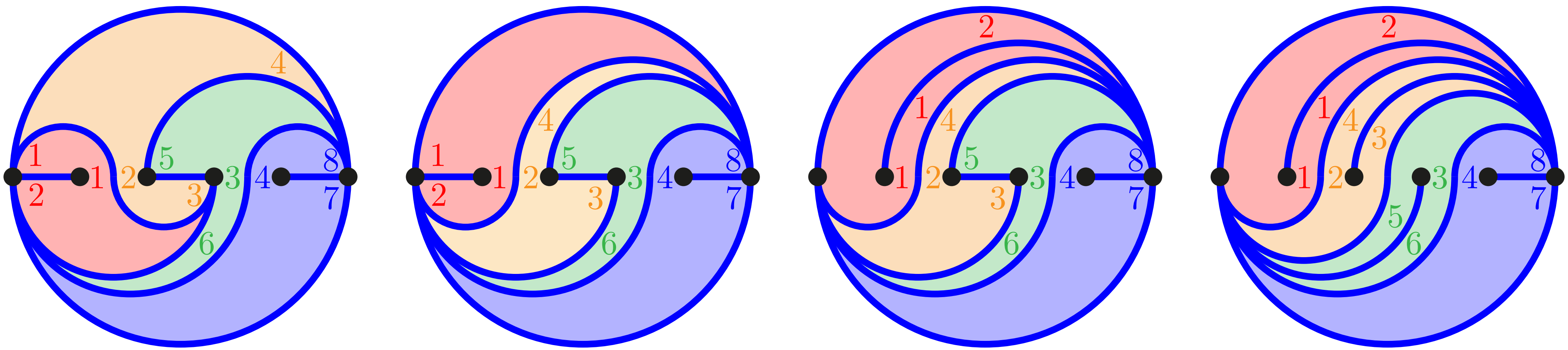
WIGGLY PSEUDOTRIANGULATIONS $\longleftrightarrow$ WIGGLY PERMUTATIONS



1234 1243 1342 1423 1432 2134 2143 3412 3421 4123 4132 4213 4312 4321

PROP. The wiggly pseudotriangulations and wiggly permutations are in bijection.

Bapat–P. (24^+)



78621354 78635421 78635412 78653412

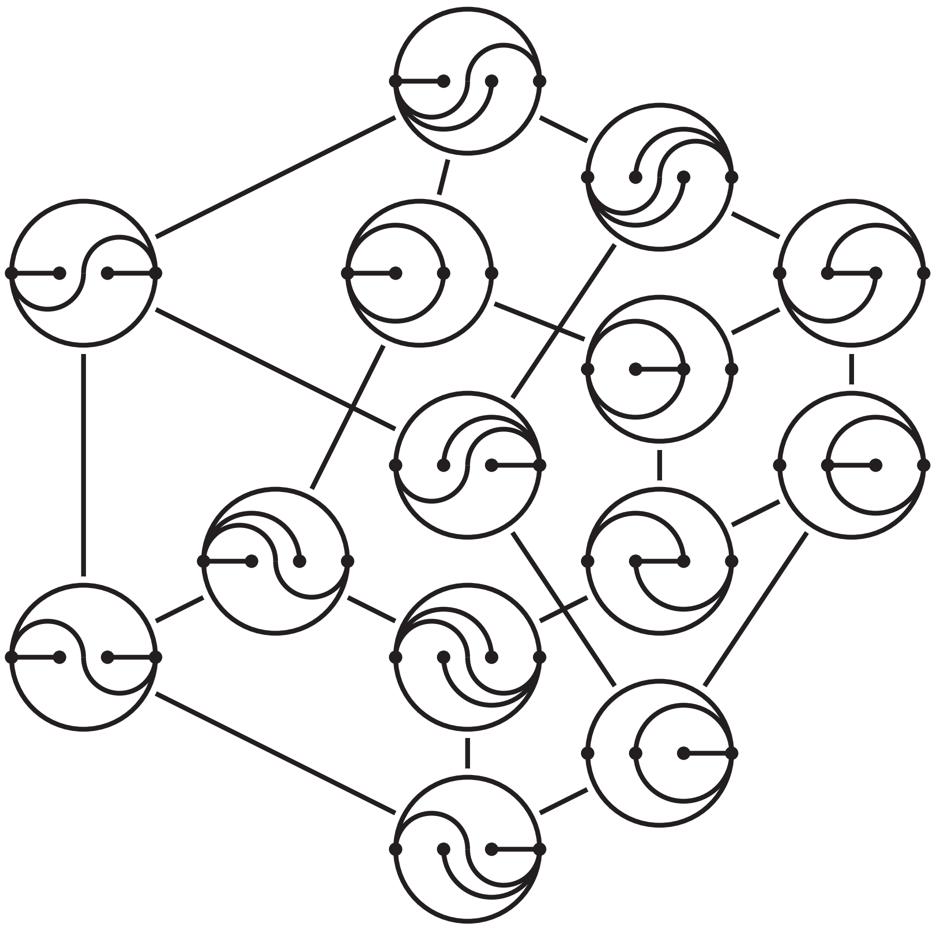
permutation of $2n$ avoiding $(2j-1)\cdots i\cdots(2j)$ for $j \in [n]$ and $i < 2j-1$

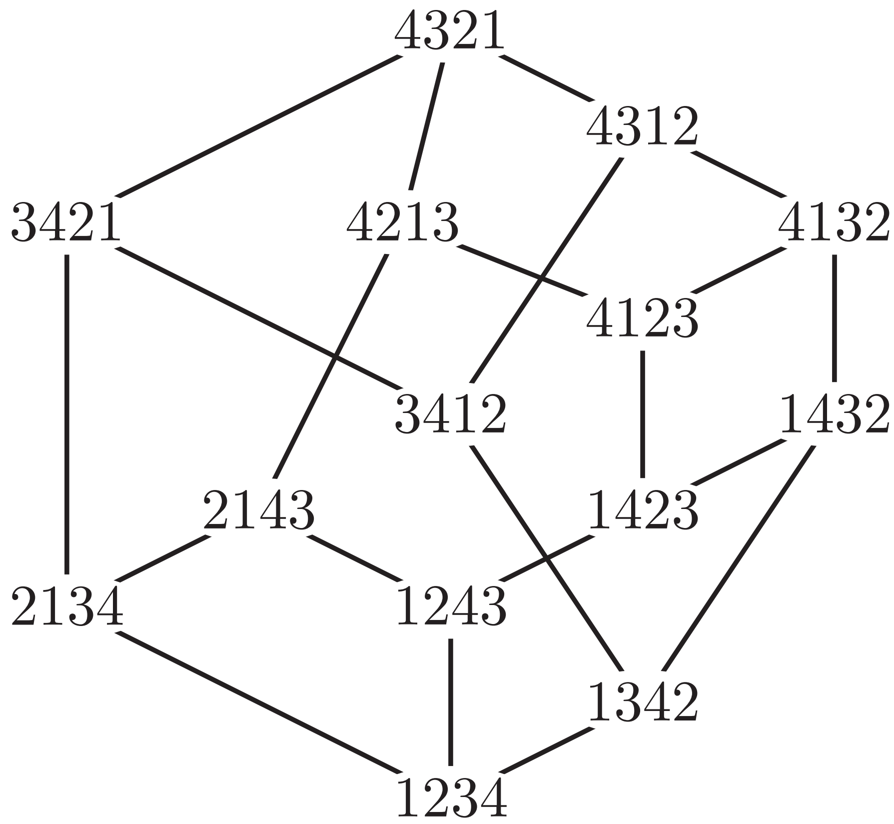
$(2j)\cdots k\cdots(2j-1)$ for $j \in [n]$ and $k > 2j$

WIGGLY PSEUDOTRIANGULATIONS $\longleftrightarrow$ WIGGLY PERMUTATIONS

PROP. This bijection induces a directed graph isomorphism between
- the wiggly increasing flip graph on wiggly pseudotriangulations,
- the Hasse diagram of the wiggly lattice on wiggly permutations.

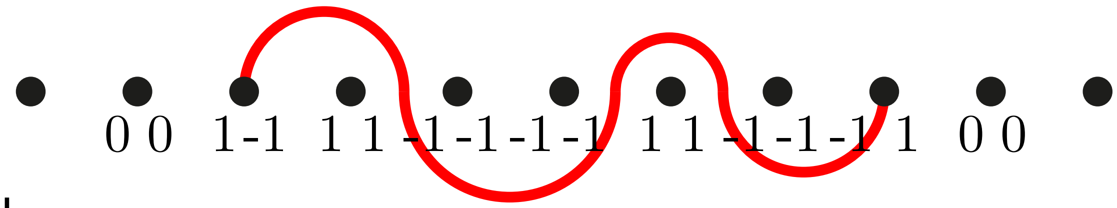
Bapat–P. (24^+)





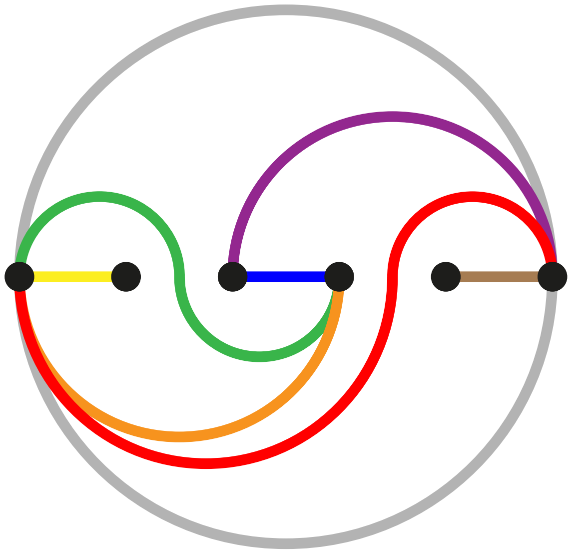
WIGGLY FAN

G- AND C-VECTORS

<u>$\boldsymbol{g}$-vector</u> of α = proj. on $\sum_{i=1}^{2n} x_i = 0$ of 

<u>$\boldsymbol{c}$-vector</u> of $\alpha \in T$ = you don't want to know...

> PROP. For any wiggly pseudotriangulation T, the $\boldsymbol{g}$-vectors $\{\boldsymbol{g}(\alpha) \mid \alpha \in T^\circ\}$ and the $\boldsymbol{c}$-vectors $\{\boldsymbol{c}(\alpha, T) \mid \alpha \in T^\circ\}$ form dual bases.
> Bapat–P. (24$^+$)



$$\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8 \end{array}\begin{pmatrix} 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 & 1 \\ 0 & -1 & -1 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & -1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

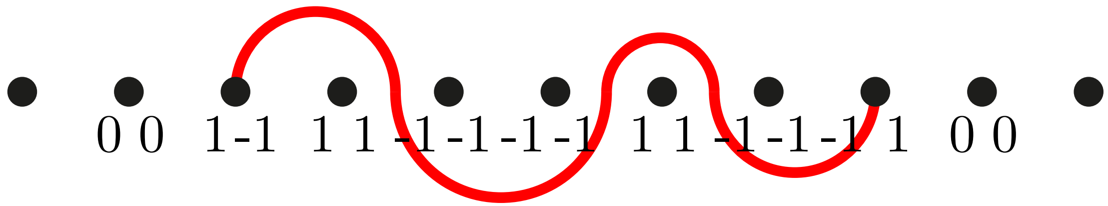
$\hat{\boldsymbol{g}}(T)$

$$\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8 \end{array}\begin{pmatrix} 0 & 0 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & -1 & 0 & 0 & 0 & -2 & 2 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$4\boldsymbol{c}(T)$

WIGGLY FAN

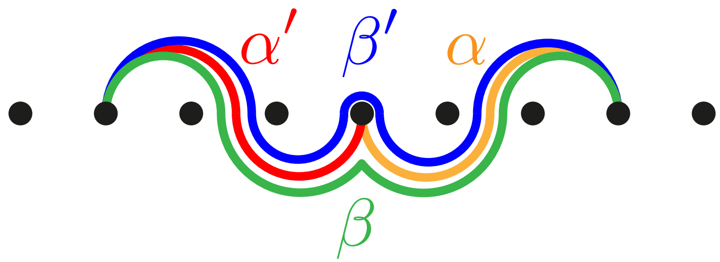
$\boldsymbol{g}$-vector of $\alpha =$ proj. on $\sum_{i=1}^{2n} x_i = 0$ of

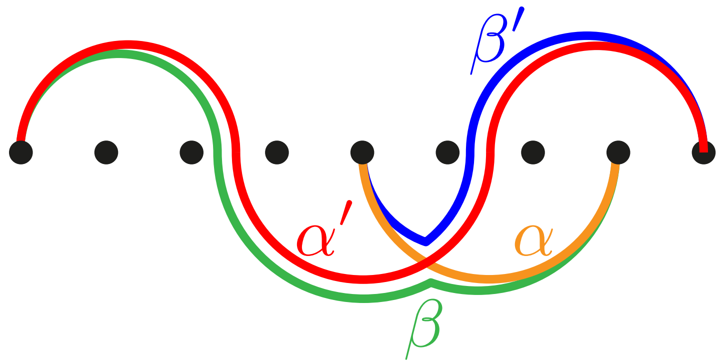


THM. The cones $\langle \boldsymbol{g}(\alpha) \mid \alpha \in D \rangle$ for all wiggly dissections D form a complete simplicial fan WF_n (in $\sum_{i=1}^{2n} x_i = 0$).

Bapat–P. (24+)

Main observation:





$$\boldsymbol{g}(\alpha) + \boldsymbol{g}(\alpha') = \big(\boldsymbol{g}(\beta) + \boldsymbol{g}(\beta')\big)/2 \qquad \boldsymbol{g}(\alpha) + \boldsymbol{g}(\alpha') = \boldsymbol{g}(\beta) + \boldsymbol{g}(\beta')$$

WIGGLYHEDRON

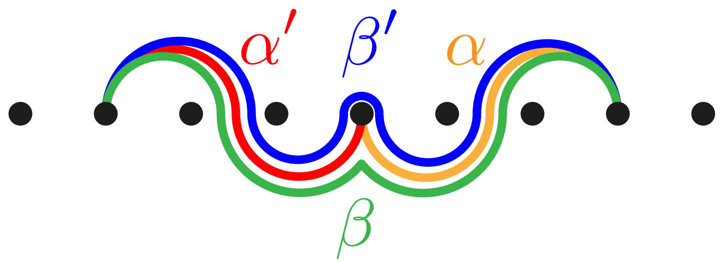
WIGGLYHEDRON

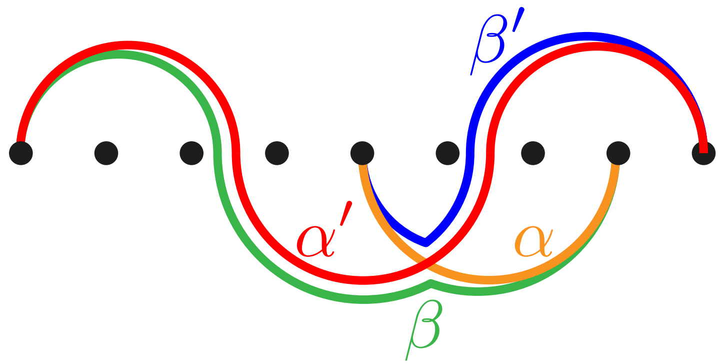
<u>incompatibility degree</u> $\delta(\alpha, \alpha') =$

- 0 if α and α' are pointed and non-crossing,
- 1 is α and α' are not pointed,
- the number of crossings of α and α' if they are crossing.

$\kappa(\alpha) =$ <u>incompatibility number</u> of $\alpha = \sum_{\alpha'} \delta(\alpha, \alpha')$.

Main observation:





$\kappa(\alpha) + \kappa(\alpha') > \big(\kappa(\beta) + \kappa(\beta')\big)/2 \qquad \kappa(\alpha) + \kappa(\alpha') > \kappa(\beta) + \kappa(\beta')$

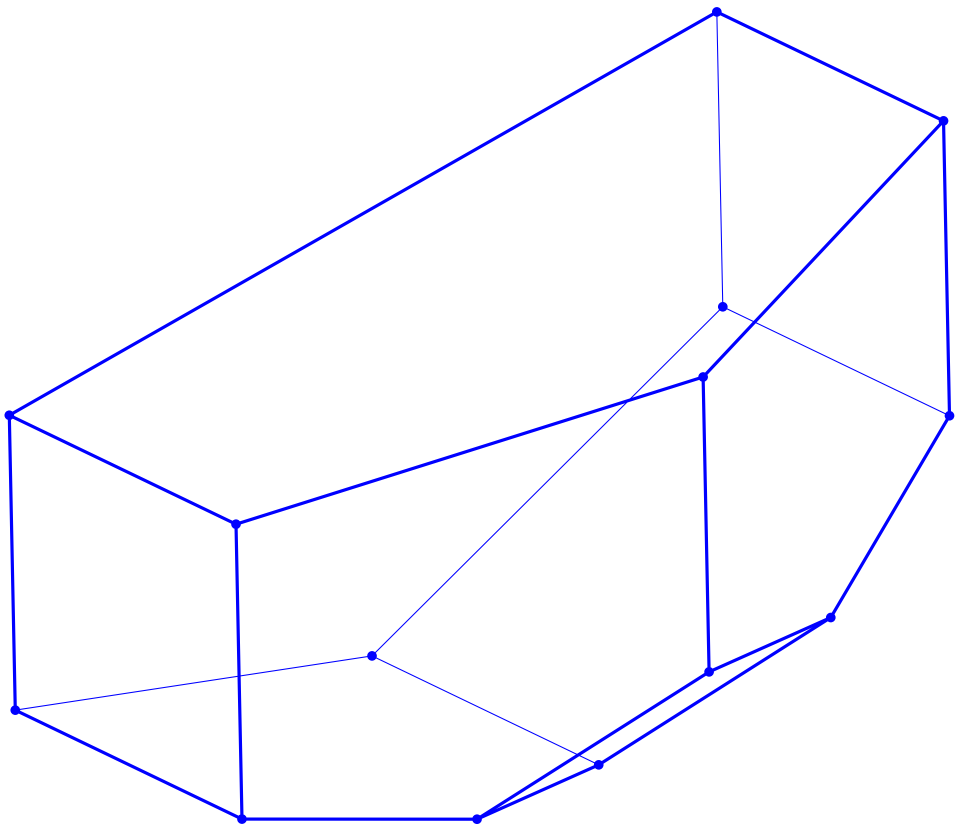
Hence, κ satisfies all wall-crossing inequalities of the wiggly fan...

WIGGLYHEDRON

THM. The wiggly fan WF_n is the normal fan of a simplicial $(2n-1)$-dimensional polytope, called the wigglyhedron $\mathbb{W}_n$, and defined equivalently as

- intersection of the halfspaces $\left\{\boldsymbol{x} \in \mathbb{R}^{2n} \;\middle|\; \langle\, \boldsymbol{g}(\alpha) \mid \boldsymbol{x}\,\rangle \le \kappa(\alpha)\right\}$ for all wiggly arcs α,
- convex hull of $\boldsymbol{p}(T) := \sum_{\alpha \in T} \kappa(\alpha)\, \boldsymbol{c}(\alpha, T)$ for all wiggly pseudotriangulations T.

Bapat–P. (24^{+})

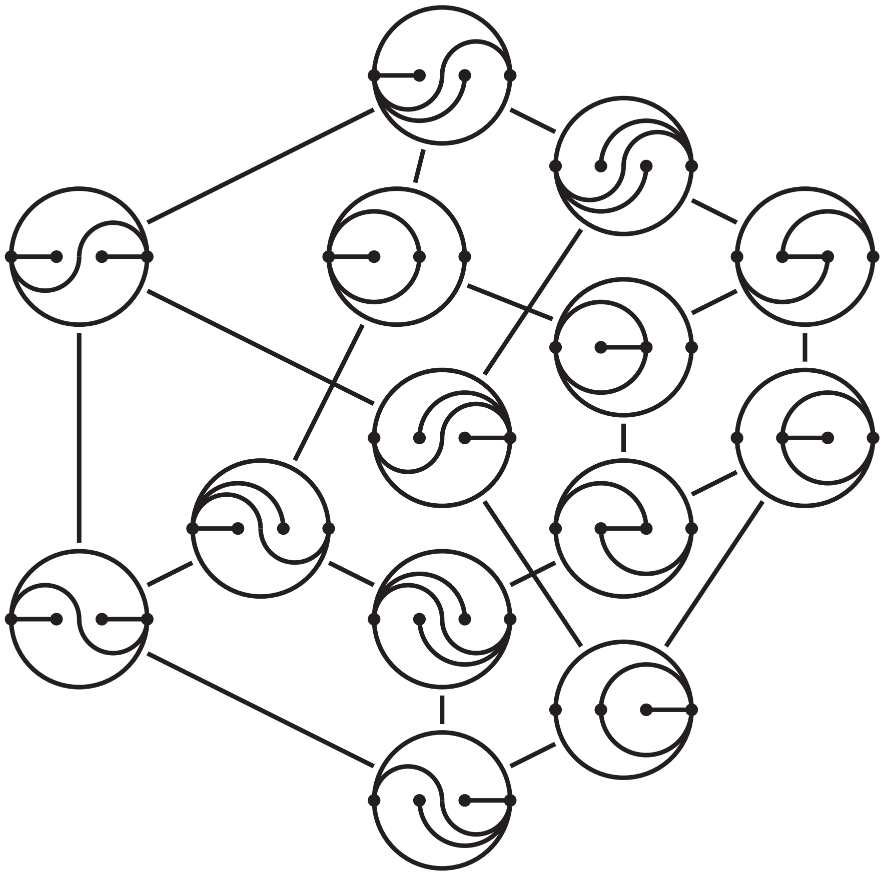


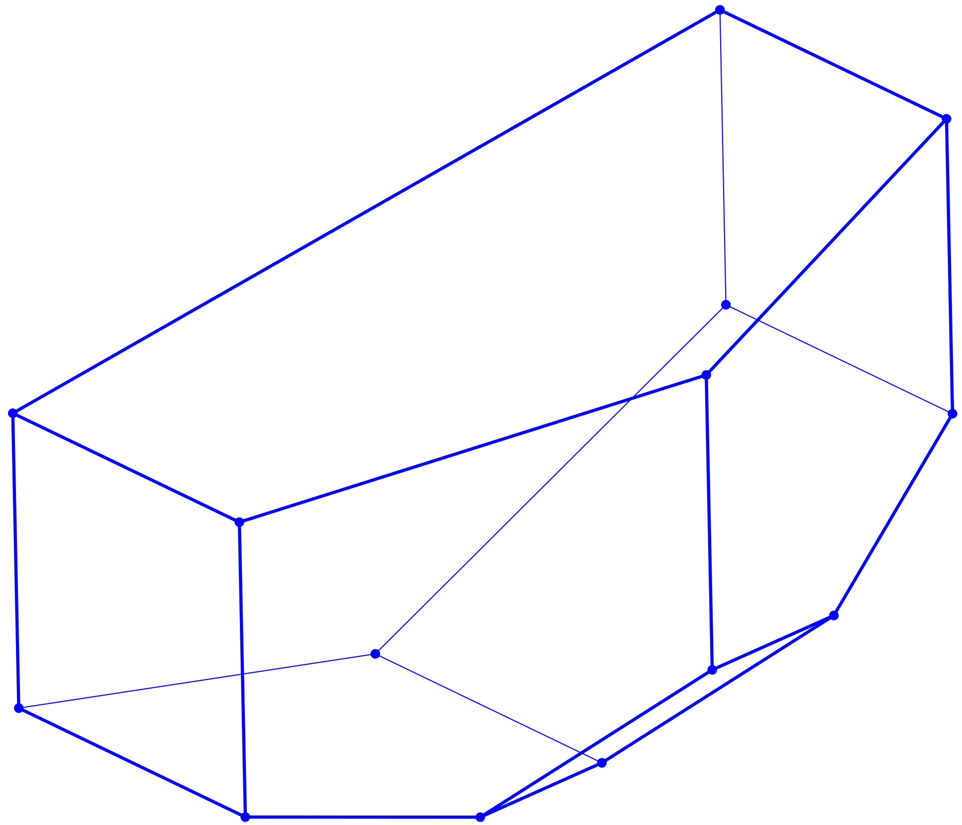
WIGGLYHEDRON

THM. The wigglyhedron $\mathbb{W}_n$ is a simple $(2n-1)$-dimensional polytope such that

- the wiggly complex WC_n is the boundary complex of the polar of $\mathbb{W}_n$,
- the Hasse diagram of the wiggly lattice is a linear orientation of the graph of $\mathbb{W}_n$.

Bapat–P. (24$^+$)





CAMBRIAN CONSIDERATIONS

CAMBRIAN CONSIDERATIONS

THM. For $\delta = \delta_1 \dots \delta_n \in \{+,-\}^n$, there are lattice isomorphisms between

- δ-triangulations = triangulation of the δ-gon, whose vertex at abscissa i has ordinate

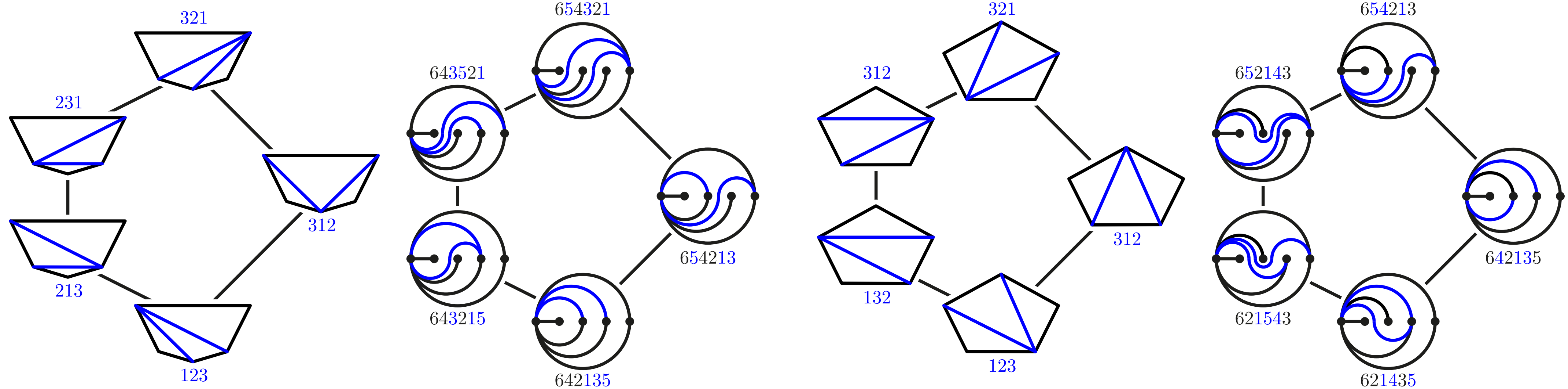
  positive if $\delta_j = +$ and negative if $\delta_j = -$
- δ-permutations = permutation of $[n]$ avoiding for $i < j < k$

  $\cdots j \cdots ki \cdots$ if $\delta_j = +$ and $\cdots ik \cdots j \cdots$ if $\delta_j = -$
- δ-wiggly pseudotriangulations = wiggly pseudotriangulation containing the arcs

  $(0, j, [1, j[, \varnothing)$ for $\delta_j = +$ and $(0, j, \varnothing, [1, j[)$ for $\delta_j = -$
- δ-wiggly permutations = wiggly permutation σ of $[2n]$ such that

  $\delta_j = + \implies \sigma^{-1}(i) \le \sigma^{-1}(2j-1)$ and $\delta_j = - \implies \sigma^{-1}(2j) \le \sigma^{-1}(i)$

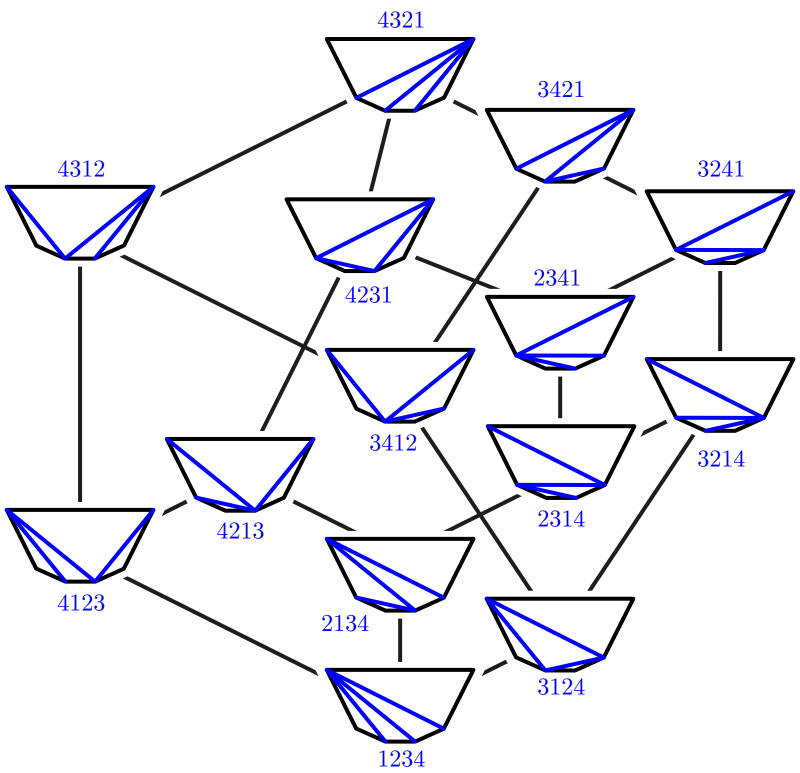
Bapat–P. (24^+)

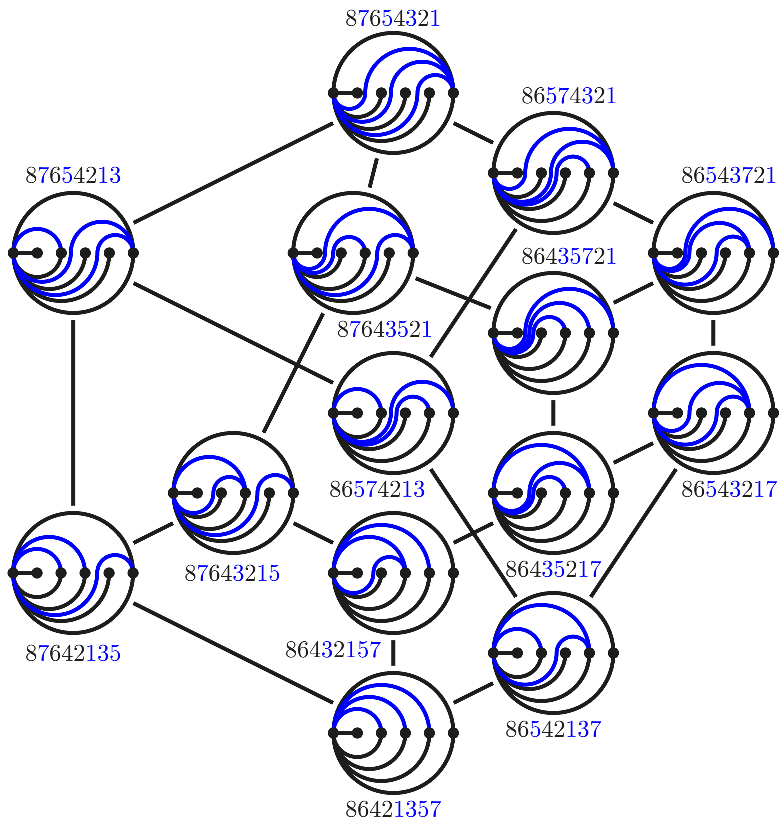


CAMBRIAN CONSIDERATIONS

THM. For $\delta = \delta_1 \dots \delta_n \in \{+, -\}^n$, there are lattice isomorphisms between
- δ-triangulations
- δ-permutations
- δ-wiggly pseudotriangulations
- δ-wiggly permutations

Bapat–P. (24^+)



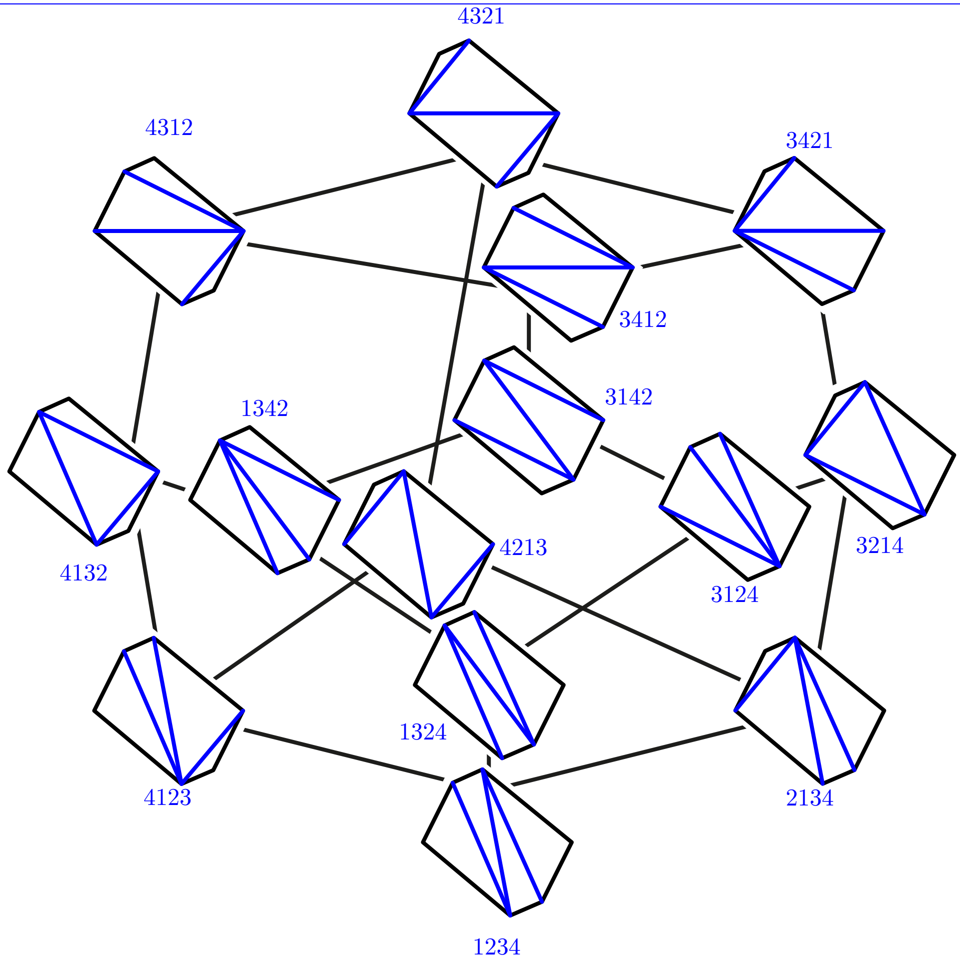


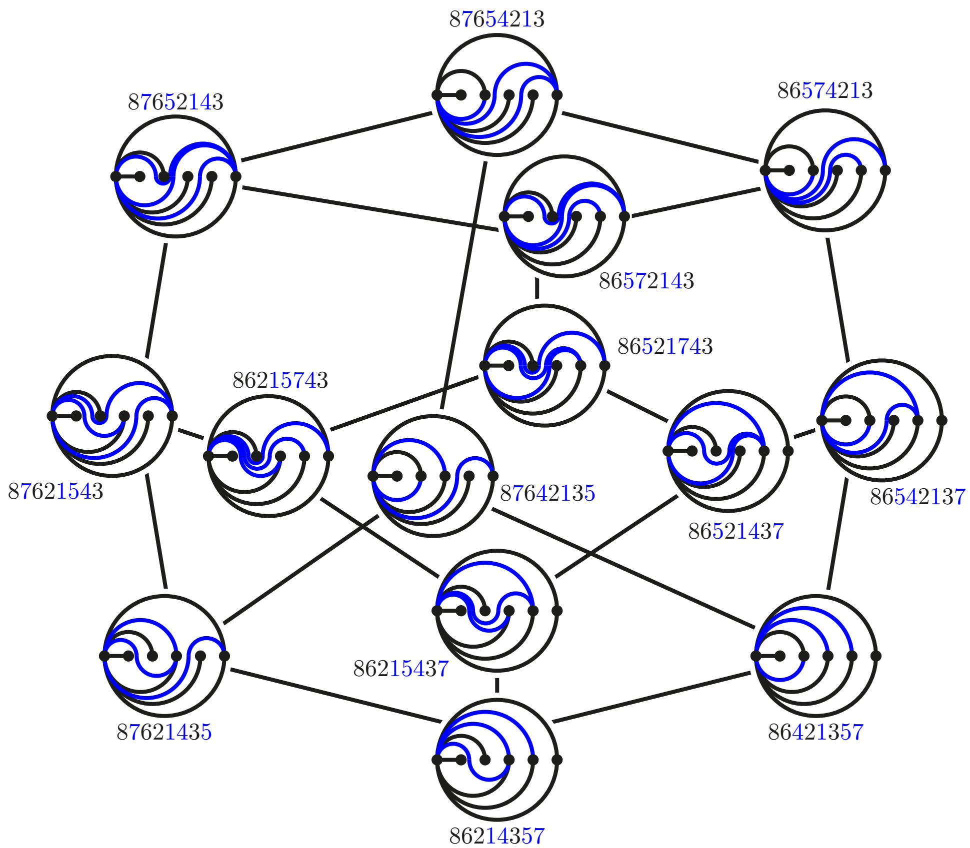
CAMBRIAN CONSIDERATIONS

THM. For $\delta = \delta_1 \dots \delta_n \in \{+,-\}^n$, there are lattice isomorphisms between

- δ-triangulations
- δ-permutations
- δ-wiggly pseudotriangulations
- δ-wiggly permutations

Bapat–P. (24^+)





CAMBRIAN CONSIDERATIONS

THM. For $\delta = \delta_1 \dots \delta_n \in \{+, -\}^n$, there are lattice isomorphisms between

- δ-triangulations
- δ-permutations
- δ-wiggly pseudotriangulations
- δ-wiggly permutations

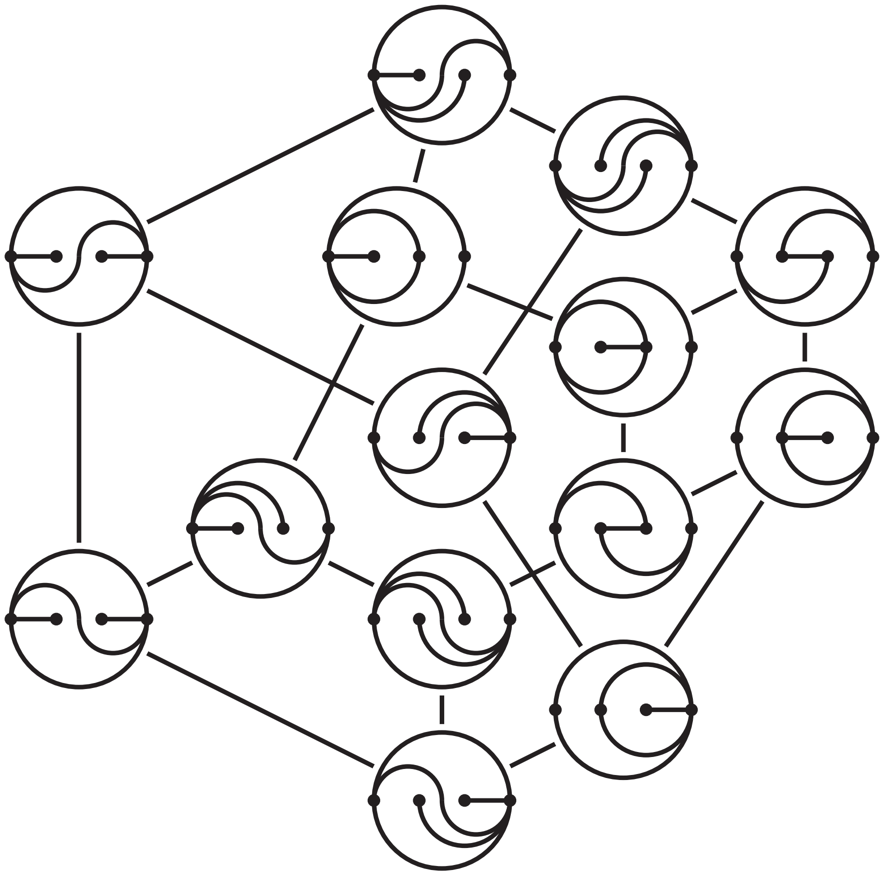
Bapat–P. (24^+)

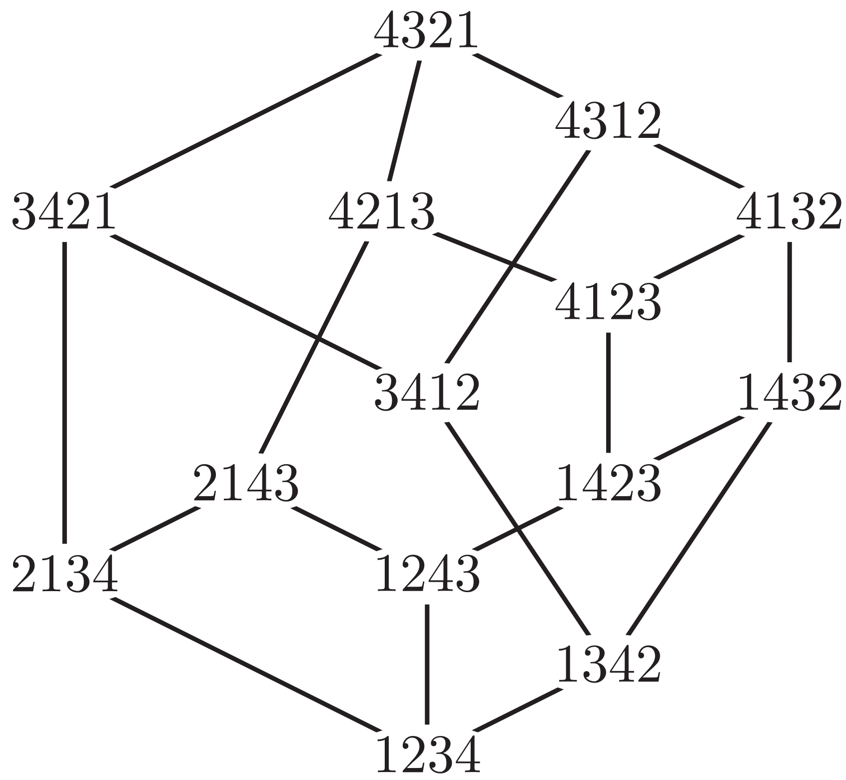
PROP. The δ-associahedron $\mathbb{A}\mathsf{sso}_\delta$ is normally equivalent to the face of the wigglyhedron $\mathbb{W}_n$ corresponding to the wiggly pseudodissection formed by the δ-wiggly arcs.

Bapat–P. (24^+)

SOME OPEN PROBLEMS

OPEN PROBLEM 1: GRAPH PROPERTIES OF WIGGLYHEDRON



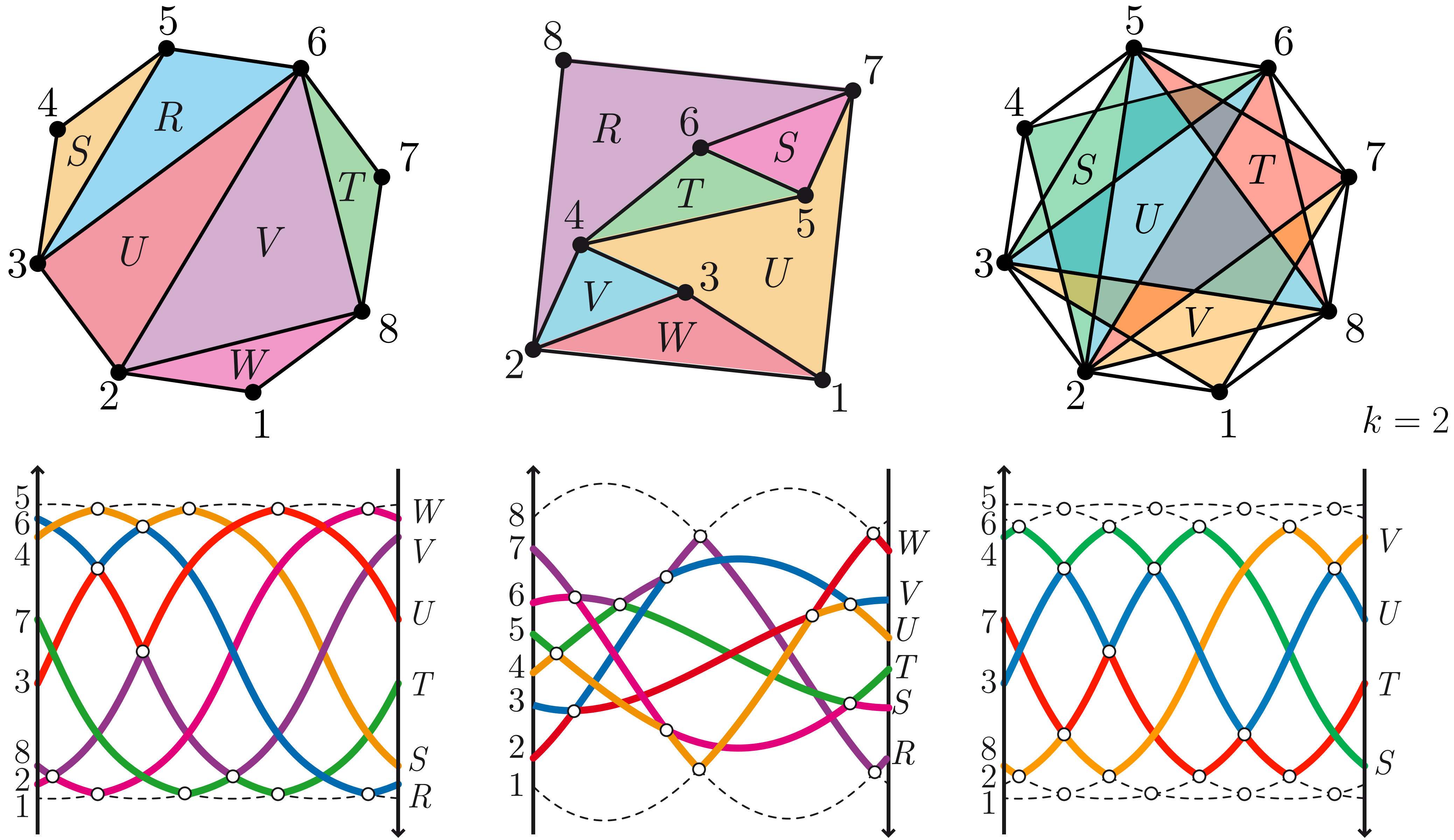


Q1a. Is the wiggly flip graph Hamiltonian?

[NB: wiggly permutations do not form a zigzag language, right?]

Q1b. What is the diameter of the wiggly flip graph?

OPEN PROBLEM 2: WIGGLY PSEUDOTRIANGULATIONS AND DUALITY



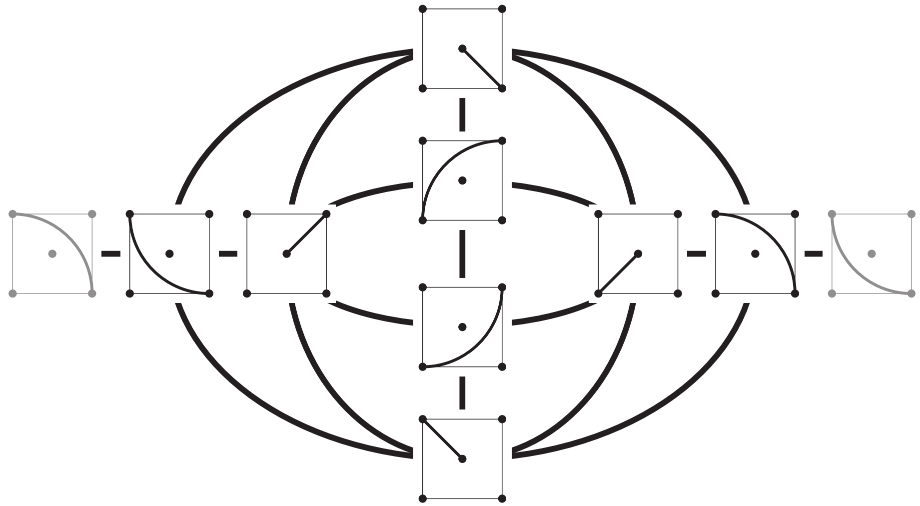
Q2. Is there a dual interpretation of wiggly pseudotriangulations?

OPEN PROBLEM 3: WIGGLY PSEUDOTRIANGULATIONS OF POINT SETS

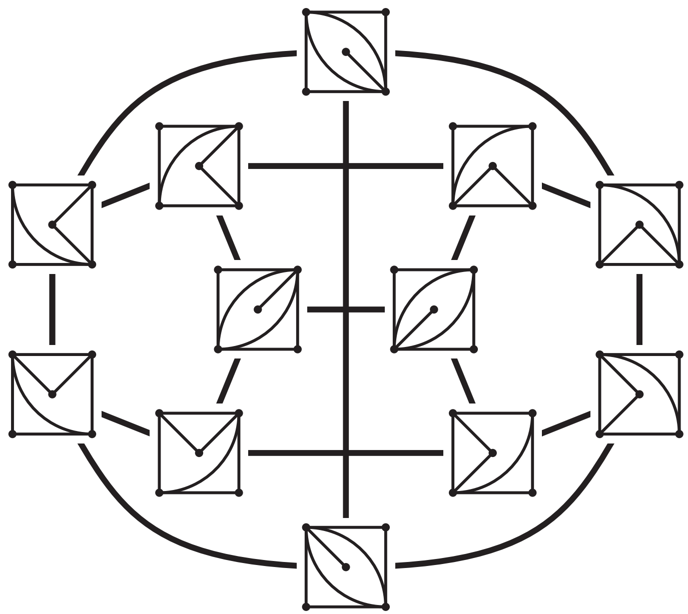
$\boldsymbol{P}$ arbitrary point set in the plane

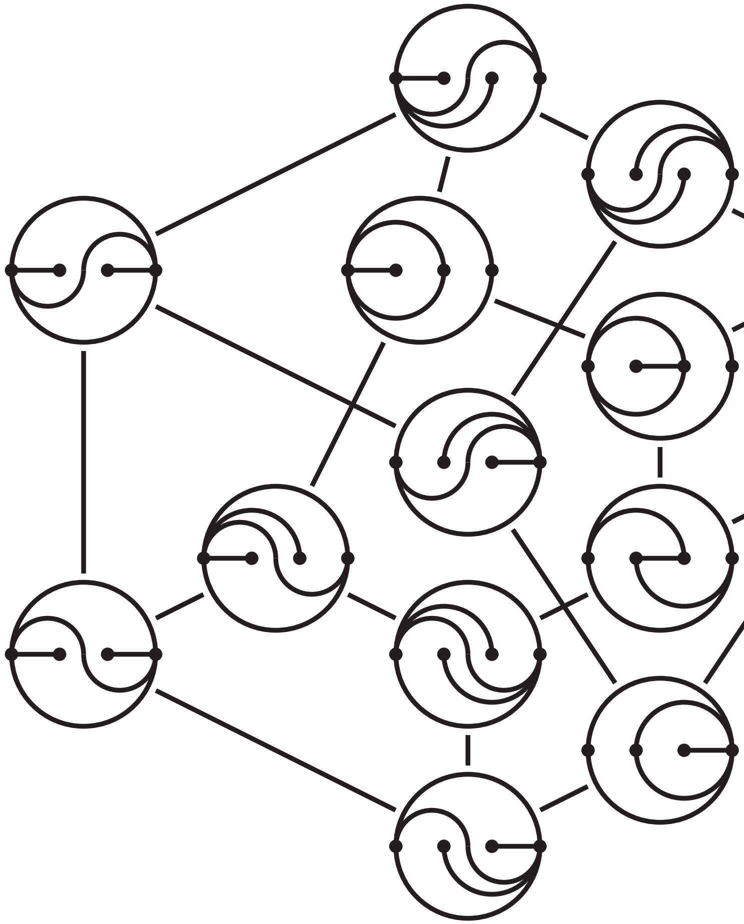
wiggly complex $\mathrm{WC}_{\boldsymbol{P}}$ = simplicial complex of non-crossing and pointed wiggly edges

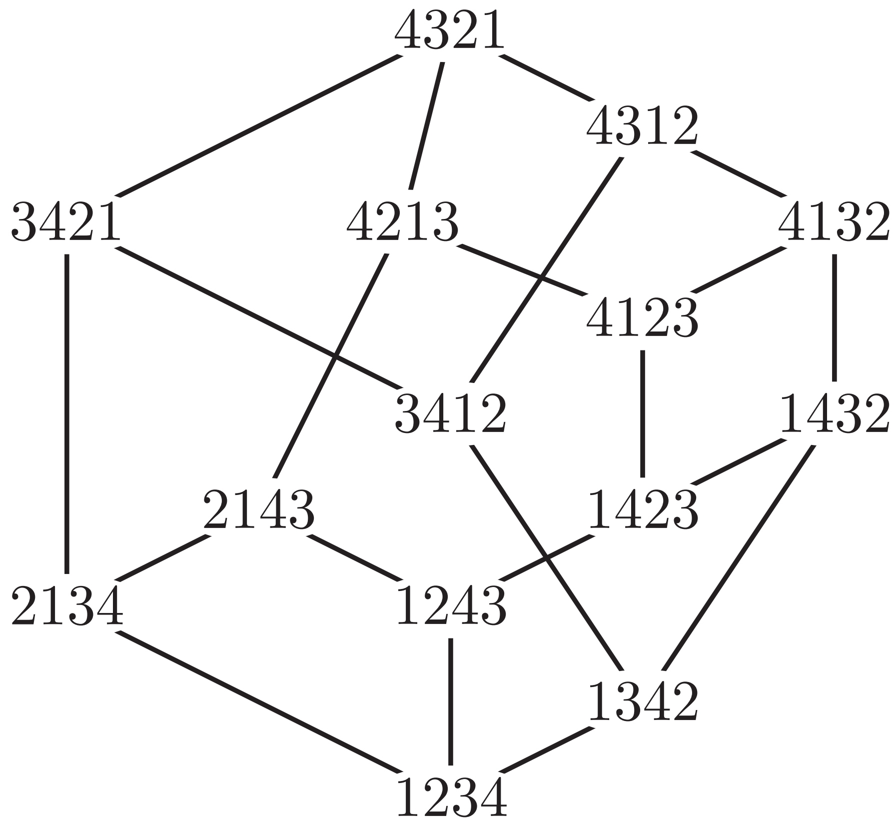
Q3a. Is $\mathrm{WC}_{\boldsymbol{P}}$ the boundary complex of a simplicial polytope?
[NB: aligned points $\Rightarrow$ wigglyhedron
general position $\Rightarrow$ Rote–Santos–Streinu]



Q3b. Is the graph of $\mathrm{WC}_{\boldsymbol{P}}$ Hamiltonian?






4321
4312
3421
4213
4132
4123
3412
1432
2143
1423
2134
1243
1342
1234

THANK YOU