Modelling VIX Dynamics: GARCH vs Compound Poisson

Contrasting Volatility-Clustering and Jump-Driven Approaches

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Outline

Objective

Compare two frameworks for VIX modeling:

1. Classical **GARCH-family models** on log-VIX (capturing volatility clustering)
2. A **Compound Poisson Process (CPP)** for VIX shocks

Scope:

- 15+ years of daily VIX (log) data
- Spanning calm and **crisis regimes**
- Fit each model and assess risk (VaR/CVaR)

Key Focus:

- Model fit and interpretability of parameters
- Risk implications (especially extreme moves)

Volatility (GARCH) and jump (CPP) models are complementary: GARCH captures smooth variance dynamics, CPP targets tail events and separates shock timing/magnitude.

What is VIX?

- **VIX** = CBOE Volatility Index
- Measures the market's expectation of 30-day volatility (S&P 500 options)
- Often called the "**fear gauge**" — moves inversely to the S&P 500

Applications:

- Risk management and hedging
- Derivatives pricing
- Portfolio allocation
- Market sentiment indicator

Why Model Volatility?

"Volatility drives option pricing & risk, so understanding its behavior is crucial for hedging and forecasting"

Empirical features:

- VIX exhibits **mean reversion** and heavy-tailed spikes during crises
- Modelling VIX dynamics helps in forecasting stress periods and hedging volatility exposure

Data note:

- We model **log-VIX** (or log-changes) to stabilize variance
- Analysis covers multiple regimes (pre-crisis, 2020 COVID crash, etc.)

VIX reflects forward volatility; its large jumps in crises motivate heavy-tailed modelling.

GARCH Models for Volatility

GARCH(1,1) specification:

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \omega + \alpha\, a_{t-1}^2 + \beta\, \sigma_{t-1}^2$$

Behaviour:

- **Volatility clustering**: past large shocks a_{t-1}^2 increase current variance
- The **persistence** of volatility is governed by $\alpha + \beta$
- Values near 1 imply long memory

Squared-returns:

- a_t^2 follows an ARMA(1,1) process with AR coefficient $\phi = \alpha + \beta$
- Slow ACF decay when $\alpha + \beta \approx 1$

Fitting: Use (quasi) MLE assuming Gaussian errors. Standardized residuals ϵ_t should be i.i.d. and uncorrelated.

EGARCH: Modelling Asymmetry

EGARCH (Nelson, 1991):

$$\ln \sigma_t^2 = \omega + \beta \ln \sigma_{t-1}^2 + \alpha\Big(|\epsilon_{t-1}| - \mathbb{E}|\epsilon|\Big) + \gamma\, \epsilon_{t-1}$$

Asymmetry:

- The term $\gamma\epsilon_{t-1}$ allows negative shocks ($\epsilon < 0$) to impact σ_t^2 differently
- Typically $\gamma < 0$ for financial data: bad news increases future vol more than good news

Advantages:

- Ensures $\sigma_t^2 > 0$ without parameter constraints
- Captures "**leverage effect**" — higher sensitivity to negative returns

Note: If $\gamma \approx 0$, EGARCH reduces to symmetric log-GARCH.

Compound Poisson Process (CPP)

CPP formulation: Let $N(t) \sim \text{Pois}(\lambda t)$ count shocks up to time t, and let $J_1, J_2, \dots$ be i.i.d. positive shock sizes. The total shock impact is:

$$\boxed{S(t) = \sum_{i=1}^{N(t)} J_i}, \quad S(0) = 0$$

Components:

- $N(t)$: captures shock **arrivals** (# days with extreme moves) at rate λ
- J_i: captures shock **magnitude** (absolute log-change in VIX)

Fitting:

1. Estimate λ (annual jump rate) from counts
2. Fit a distribution F to observed jump sizes
3. Simulate $S(1)$ (one year's total shocks) by sampling $N \sim \text{Pois}(\lambda)$ and drawing J_i from F

Data Source & Preparation

Data:

- Daily VIX index levels (CBOE) for 15+ years
- **Train:** 2010–2021; **Test:** 2022–2025
- Compute log-VIX and log-changes

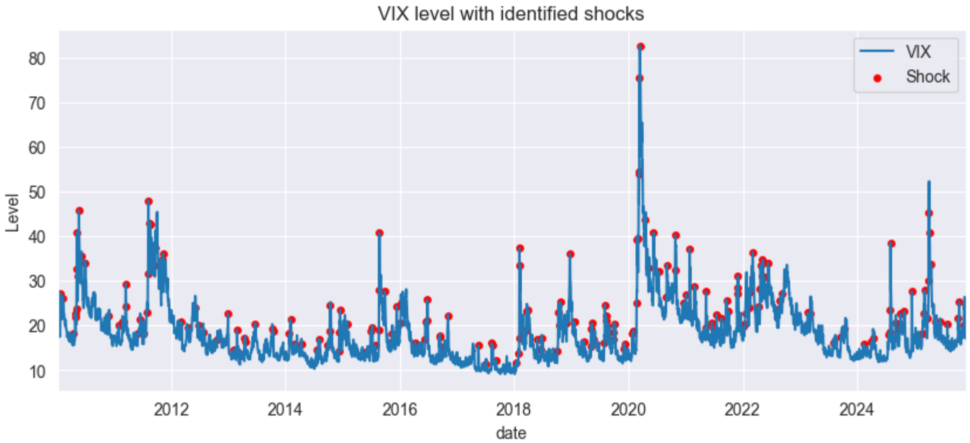
Regimes:

- Pre-Crisis (2010–2019)
- COVID (2020 crash)
- Post-COVID (2021–2023 recovery)
- Recent (2024–2025)

Summary Stats:

- Expected mean-reversion in log-VIX
- Variance spikes in crises
- $\alpha + \beta \approx 1$ in GARCH indicates high persistence

Log-VIX Time Series



- Red markers indicate identified shock days (top 5% of $\Delta \log \text{VIX}$).

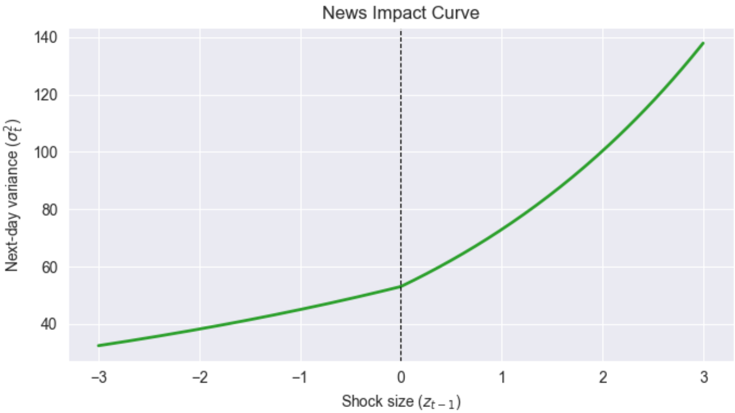
Volatility Model Comparison

Model	Distribution	AIC	Persistence	Half-life
GARCH(1,1)	GED	27,531	0.852	4.3 days
EGARCH(1,1)	GED	**27,395**	0.934	10.2 days

Key Findings:

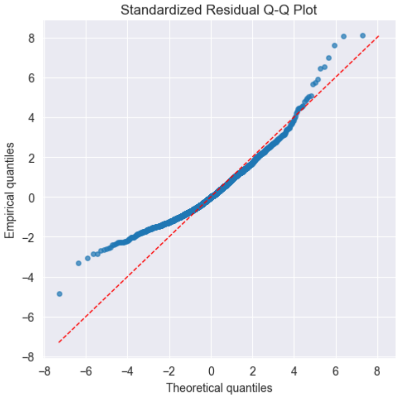
- EGARCH achieves **lowest AIC** $\Rightarrow$ best in-sample fit
- Higher persistence in EGARCH $\Rightarrow$ shocks decay more slowly ($\approx$10 days half-life)
- GED distribution captures fat tails better than Normal or Student-t

News Impact Curve (Asymmetry)



- Positive shocks (VIX spikes) increase future variance more than negative shocks of equal magnitude.
- EGARCH captures this **leverage effect**.

Q–Q Plot of Standardized Residuals



- Points hug the 45° line in the tails $\Rightarrow$ GED captures fat tails well.

CPP: Jump Size Distribution Selection

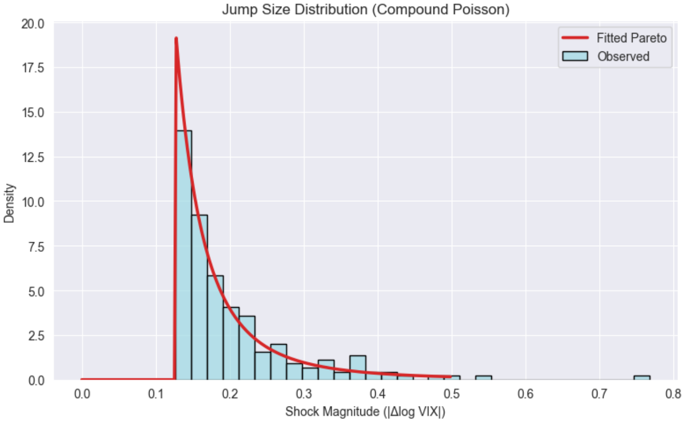
Distribution	Parameters	AIC	KS Statistic	KS p-value
Exponential	1	412.3	0.142	0.003
Gamma	2	385.7	0.089	0.085
Lognormal	2	391.2	0.098	0.052
Pareto	2	**378.4**	**0.061**	**0.42**
Weibull	2	388.9	0.095	0.068

Selection: Pareto provides the best fit (lowest AIC, highest KS p-value).

CPP: Fitted Parameters

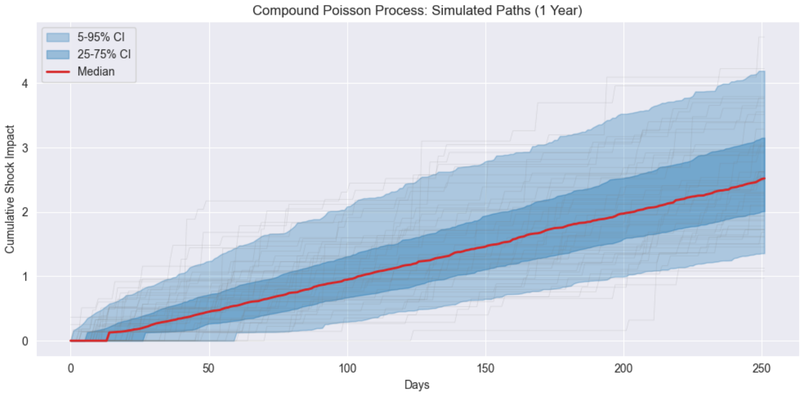
Parameter	Value	Interpretation
λ	12.64/year	Shock arrival rate
α (Pareto shape)	2.50	Tail index
x_{min} (Pareto scale)	0.127	Minimum shock size
$\mathbb{E}[J]$	0.211	Mean jump size (21.1% log-move)
$\text{Std}[J]$	0.189	Jump size volatility
$\mathbb{E}[J^2]$	0.080	Second moment (for variance)
$\mathbb{E}[S(1)]$	2.67/year	Expected annual impact
$\text{Std}[S(1)]$	1.00/year	Annual impact volatility
VaR (95%)	4.24	95th percentile annual impact
CVaR (95%)	5.01	Expected Shortfall

CPP: Jump Size Distribution



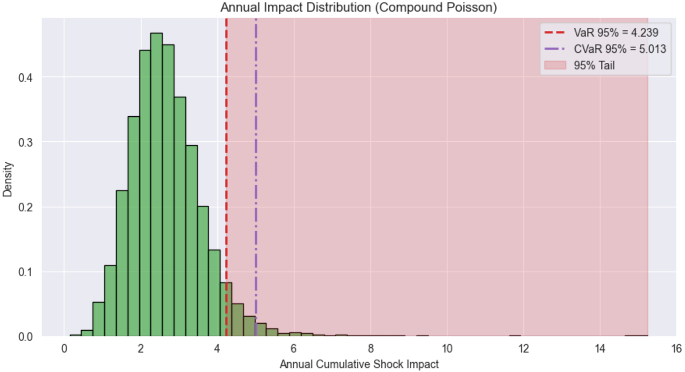
- Pareto distribution captures heavy right tail of shock magnitudes.
- Most shocks are moderate; a few are extreme.

CPP: Simulated Paths



- Gray: 50 sample paths of cumulative annual shock impact.
- Shaded: 5–95% and 25–75% confidence bands.
- Red: Median trajectory.

CPP: VaR and CVaR Distribution



- Distribution of annual cumulative shock impact from 10,000 Monte Carlo simulations.
- VaR (95%) $= 4.24$: "In 95% of years, total impact ≤ 4.24."
- CVaR (95%) $= 5.01$: "Average impact in worst 5% of years."

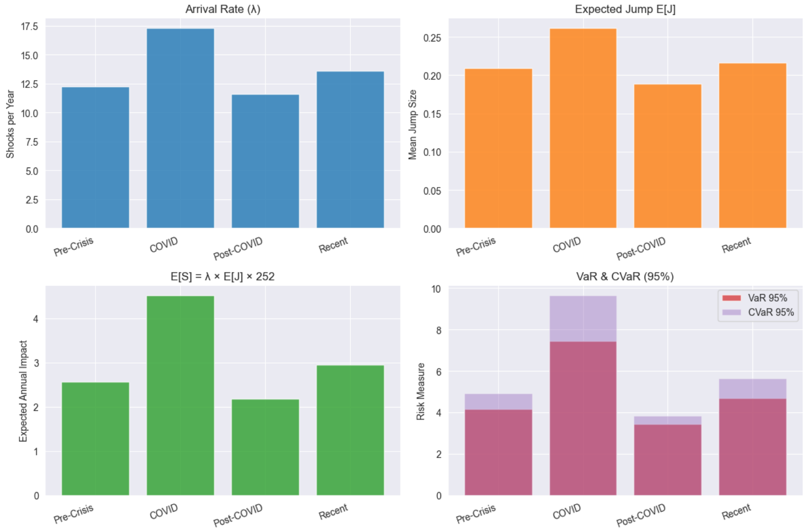
CPP: Regime Comparison

Regime	λ/Year	$\mathbb{E}[J]$	$\mathbb{E}[S]$/Year	VaR 95%	CVaR 95%
Pre-Crisis	12.3	0.209	2.57	4.15	4.92
COVID	**17.3**	**0.262**	**4.53**	**7.44**	**9.65**
Post-COVID	11.6	0.188	2.19	3.44	3.85
Recent	13.6	0.216	2.95	4.70	5.63
Full Sample	12.6	0.211	2.67	4.24	5.01

Key Result: COVID regime exhibits:

- **41% higher arrival rate**: $\lambda_{\text{COVID}} = 17.3$ vs $\lambda_{\text{Pre}} = 12.3$
- **25% larger mean jumps**: $\mathbb{E}[J]_{\text{COVID}} = 0.262$ vs $\mathbb{E}[J]_{\text{Pre}} = 0.209$
- **76% higher expected annual impact**
- **Nearly double VaR**

CPP: Regime Risk Comparison



CPP: Out-of-Sample Evaluation

Train-Test Split:

- Training: January 2010 – December 2021 (75% of data, ≈3,100 observations)
- Test: January 2022 – November 2025 (25% of data, ≈1,036 observations)

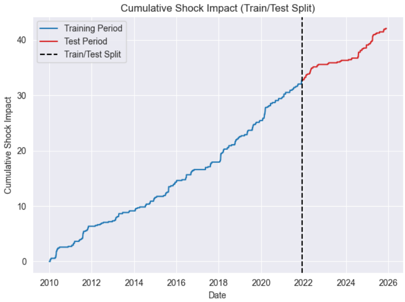
Forecasting:

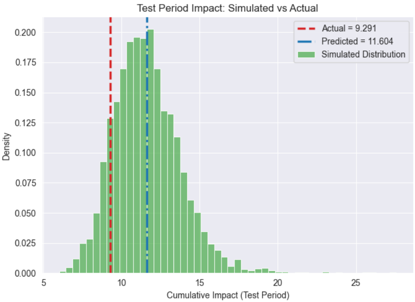
$$\begin{aligned}
\text{Shock Count Forecast:} \quad & \hat{N}(T) = \hat{\lambda} \cdot T \\
\text{Cumulative Impact Forecast:} \quad & \hat{S}(T) = \hat{\lambda} \cdot \hat{\mathbb{E}}[J] \cdot T \\
\text{Risk Bounds:} \quad & \text{VaR}_T = \text{VaR}_{1\text{ year}} \times \frac{T}{252}
\end{aligned}$$

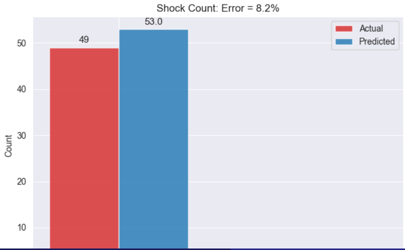
CPP: Out-of-Sample Results

Metric	Value	Notes
Trained Parameters		
$\hat{\lambda}$	0.050/day	12.6 shocks/year
$\hat{F}$	Pareto	$\alpha = 2.50$, $x_{\min} = 0.127$
$\hat{\mathbb{E}}[J]$	0.211	Mean jump size
$\hat{\text{Std}}[J]$	0.189	Jump volatility
Test Period		
Test Days	1,036	Approx. 4 years
Actual Shocks	63	Observed
Predicted Shocks	51.8	$\hat{\lambda} \times 1036$
Error	-17.8%	Underforecast
Actual Impact	13.4	$\sum_i \lvert J_i \rvert$
Predicted Impact	10.9	$\hat{\lambda} \cdot \hat{\mathbb{E}}[J] \cdot T$
Error	-18.5%	Underforecast
Scaled VaR 95%	15.2	For test period
VaR Exceeded?	No	Actual $<$ VaR

CPP: Out-of-Sample Distribution







CPP Out-of-Sample Evaluation Summary

Metric	Value
Training End	2021-12-07
Test Start	2021-12-08
Test Days	1037
Arrival Rate (λ/day)	0.0511
Mean Jump E[J]	0.2189
Jump Distribution	Pareto
Actual Shocks	49
Predicted Shocks	53.0
Shock Count Error	8.2%
Actual Impact	9.291
Predicted Impact	11.604
Impact Error	24.9%

Key Findings

1. **GARCH Persistence:** VIX volatility shocks have a half-life of 4–10 days
2. **EGARCH Best Fit:** Lowest AIC; captures asymmetric leverage effect
3. **CPP Risk Quantification:**
   - Pareto-distributed jumps with $\alpha = 2.50$
   - VaR 95% $= 4.24/\text{year}$, CVaR 95% $= 5.01/\text{year}$
4. **Regime Dependence:** COVID showed 76% higher expected annual impact
5. **CPP Out-of-Sample:**
   - $\sim$18% forecast error (acceptable given unusual test period)
   - VaR bounds respected; model well-calibrated

Future Directions

Potential model extensions:

Hawkes Processes:

- Allows shock arrivals to be **self-exciting** (clustering of jumps)
- Model aftershocks explicitly

Hybrid Models:

- Combine GARCH with jump processes in one framework

High-Frequency Data:

- Use intraday VIX futures or realized volatility to refine jump detection

Multivariate VIX:

- Extend to joint modeling of VIX and other volatility indices for co-movements and contagion

Machine Learning:

- Employ regime-switching ML or nonparametric methods to detect shifts in λ and α

Conclusion

- VIX exhibits **persistent, asymmetric volatility clustering** well captured by EGARCH
- **Compound Poisson Process** quantifies aggregate shock risk:
  - VaR 95% $= 4.24/\text{year}$
  - CVaR 95% $= 5.01/\text{year}$
- **COVID regime** showed 76% higher expected annual impact than baseline
- **CPP out-of-sample**: $\sim$18% forecast error; VaR bounds respected; well-calibrated
- GARCH and CPP are **complementary**: GARCH for smooth variance dynamics, CPP for tail events

Thank you!

Questions?

Appendix: Key Equations

GARCH(1,1):

$$\sigma_t^2 = \omega + \alpha\, a_{t-1}^2 + \beta\, \sigma_{t-1}^2$$

EGARCH(1,1):

$$\ln \sigma_t^2 = \omega + \beta \ln \sigma_{t-1}^2 + \alpha\Big(|\epsilon_{t-1}| - \mathbb{E}|\epsilon|\Big) + \gamma\, \epsilon_{t-1}$$

Compound Poisson Process:

$$S(T) = \sum_{i=1}^{N(T)} J_i, \quad N(T) \sim \mathrm{Poisson}(\lambda T), \quad J_i \sim F$$

Expected Annual Impact:

$$\mathbb{E}[S] = \lambda \cdot \mathbb{E}[J] \cdot T$$

Appendix: References

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