Automated Deduction

Laura Kovács

Outline

Inference Systems - Soundness (Recap)

- An inference is sound if the conclusion of this inference is a logical consequence of its premises.
- An inference system is sound if every inference rule in this system is sound.

Inference Systems - Soundness (Recap)

- An inference is sound if the conclusion of this inference is a logical consequence of its premises.
- An inference system is sound if every inference rule in this system is sound.

$\mathbb{BR}$ is sound.

Consequence of soundness: let S be a set of clauses. If $\square$ can be derived from S in $\mathbb{BR}$, then S is unsatisfiable.

Lecture 2 - Exercise recap : infinite number of $\mathbb{BR}$ derivations of $\square$

Can this be used for checking (un)satisfiability (Recap)

1. What happens when the empty clause cannot be derived from S?
2. How can one search for possible derivations of the empty clause?

Can this be used for checking (un)satisfiability (Recap)

1. Completeness.
   Let S be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from S in $\mathbb{BR}$.

Can this be used for checking (un)satisfiability (Recap)

1. Completeness.
   Let S be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from S in $\mathbb{BR}$.
2. We have to formalize search for derivations.

.

Can this be used for checking (un)satisfiability (Recap)

1. Completeness.
   Let S be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from S in $\mathbb{BR}$.
2. We have to formalize search for derivations.

We introduced well-behaved selection functions for selecting literals in clauses and applying inferences only over selected literals.

.

Can this be used for checking (un)satisfiability (Recap)

1. Completeness.
   Let S be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from S in $\mathbb{BR}$.
2. We have to formalize search for derivations.

We introduced well-behaved selection functions for selecting literals in clauses and applying inferences only over selected literals.

Binary resolution $\mathbb{BR}$ with selection is complete for every well-behaved selection function.

Outline

How to Establish Unsatisfiability?

Completeness is formulated in terms of derivability of the empty clause $\square$ from a set S_0 of clauses in an inference system $\mathbb{I}$. However, this formulations gives no hint on how to search for such a derivation.

How to Establish Unsatisfiability?

Completeness is formulated in terms of derivability of the empty clause $\square$ from a set S_0 of clauses in an inference system $\mathbb{I}$. However, this formulations gives no hint on how to search for such a derivation.

Idea:

- Take a set of clauses S (the search space), initially $S = S_0$. Repeatedly apply inferences in $\mathbb{I}$ to clauses in S and add their conclusions to S, unless these conclusions are already in S.
- If, at any stage, we obtain $\square$, we terminate and report unsatisfiability of S_0.

How to Establish Satisfiability?

When can we report satisfiability?

How to Establish Satisfiability?

When can we report satisfiability?

When we build a set S such that any inference applied to clauses in S is already a member of S. Any such set of clauses is called saturated (with respect to $\mathbb{I}$).

How to Establish Satisfiability?

When can we report satisfiability?

When we build a set S such that any inference applied to clauses in S is already a member of S. Any such set of clauses is called saturated (with respect to $\mathbb{I}$).

In first-order logic it is often the case that all saturated sets are infinite (due to undecidability), so in practice we can never build a saturated set.

The process of trying to build one is referred to as saturation.

Saturated Set of Clauses

Let $\mathbb{I}$ be an inference system on formulas and S be a set of formulas.

- S is called saturated with respect to $\mathbb{I}$, or simply $\mathbb{I}$-saturated, if for every inference of $\mathbb{I}$ with premises in S, the conclusion of this inference also belongs to S.

- The closure of S with respect to $\mathbb{I}$, or simply $\mathbb{I}$-closure, is the smallest set S' containing S and saturated with respect to $\mathbb{I}$.

Inference Process

Inference process: sequence of sets of formulas $S_0, S_1, \ldots$, denoted

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

$(S_i \Rightarrow S_{i+1})$ is a step of this process.

Inference Process

Inference process: sequence of sets of formulas $S_0, S_1, \ldots$, denoted

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

$(S_i \Rightarrow S_{i+1})$ is a step of this process.

We say that this step is an $\mathbb{I}$-step if

1. there exists an inference

$$\frac{F_1 \quad \ldots \quad F_n}{F}$$

   in $\mathbb{I}$ such that $\{F_1, \ldots, F_n\} \subseteq S_i$;
2. $S_{i+1} = S_i \cup \{F\}$.

Inference Process

Inference process: sequence of sets of formulas $S_0, S_1, \ldots$, denoted

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

$(S_i \Rightarrow S_{i+1})$ is a step of this process.

We say that this step is an $\mathbb{I}$-step if

1. there exists an inference

$$\frac{F_1 \quad \ldots \quad F_n}{F}$$

   in $\mathbb{I}$ such that $\{F_1, \ldots, F_n\} \subseteq S_i$;
2. $S_{i+1} = S_i \cup \{F\}$.

An $\mathbb{I}$-inference process is an inference process whose every step is an $\mathbb{I}$-step.

Property

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an $\mathbb{I}$-inference process and a formula F belongs to some S_i. Then S_i is derivable in $\mathbb{I}$ from S_0. In particular, every S_i is a subset of the $\mathbb{I}$-closure of S_0.

Limit of a Process

The limit of an inference process $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ is the set of formulas $\bigcup_i S_i$.

Limit of a Process

The limit of an inference process $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ is the set of formulas $\bigcup_i S_i$.

In other words, the limit is the set of all derived formulas. We denote the limit by S_∞.

Limit of a Process

The limit of an inference process $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ is the set of formulas $\bigcup_i S_i$.

In other words, the limit is the set of all derived formulas. We denote the limit by S_∞.

Suppose that we have an infinite inference process such that S_0 is unsatisfiable and we use the binary resolution inference system.

Limit of a Process

The limit of an inference process $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ is the set of formulas $\bigcup_i S_i$.

In other words, the limit is the set of all derived formulas. We denote the limit by S_∞.

Suppose that we have an infinite inference process such that S_0 is unsatisfiable and we use the binary resolution inference system.

Question: does completeness imply that the limit of the process contains the empty clause?

Fairness

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an inference process with the limit S_∞.
The process is called fair if for every $\mathbb{I}$-inference

$$\frac{F_1 \quad \ldots \quad F_n}{F},$$

if $\{F_1, \ldots, F_n\} \subseteq S_\infty$, then there exists i such that $F \in S_i$.

Limit of a Fair Inference Process

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an fair inference process using a sound inference system $\mathbb{I}$.

Exercise: Show that the limit of S_∞ is the $\mathbb{I}$-closure of S_0.

Completeness, reformulated

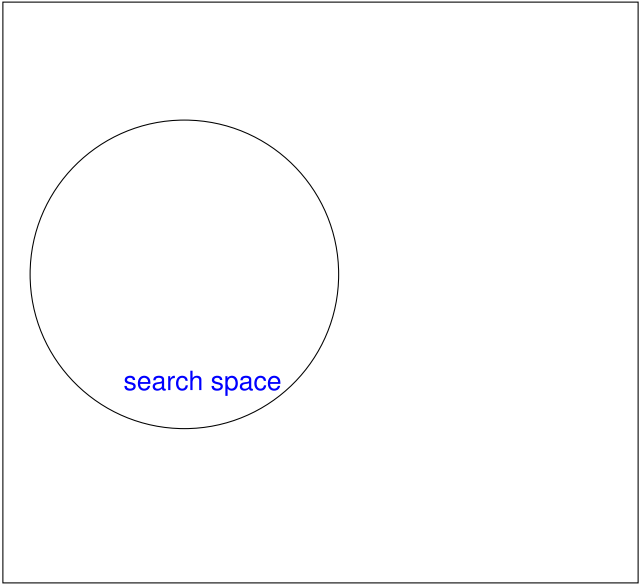
Theorem Let $\mathbb{I}$ be an inference system. The following conditions are equivalent.

1. $\mathbb{I}$ is complete.

2. For every unsatisfiable set of formulas S_0 and any fair $\mathbb{I}$-inference process with the initial set S_0, the limit S_∞ of this inference process contains $\square$.

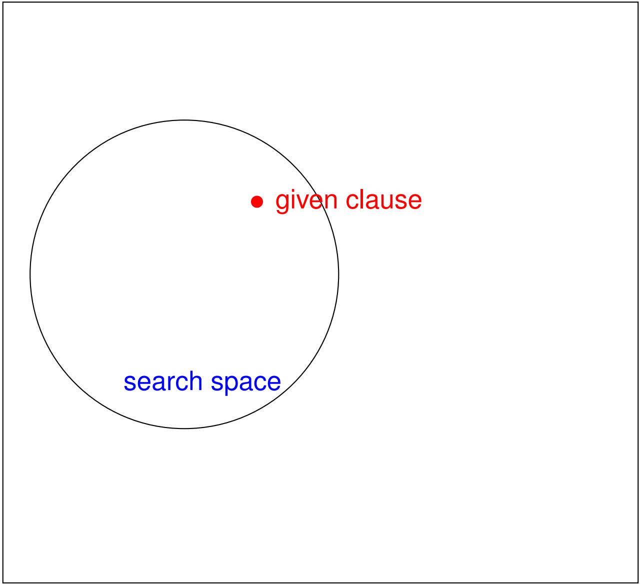
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



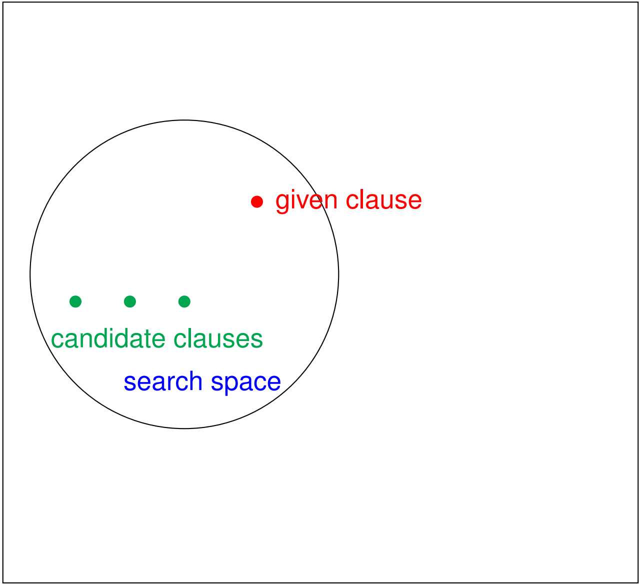
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



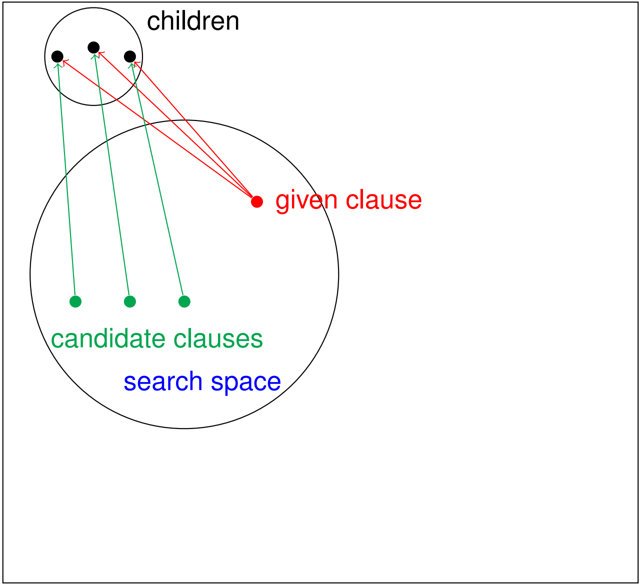
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



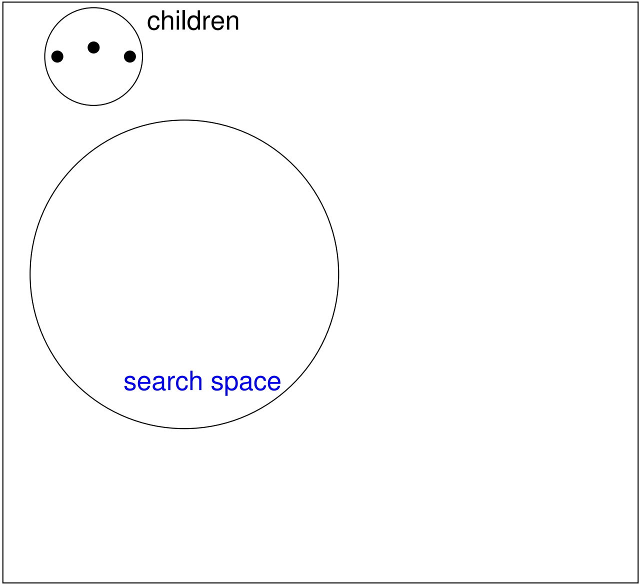
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



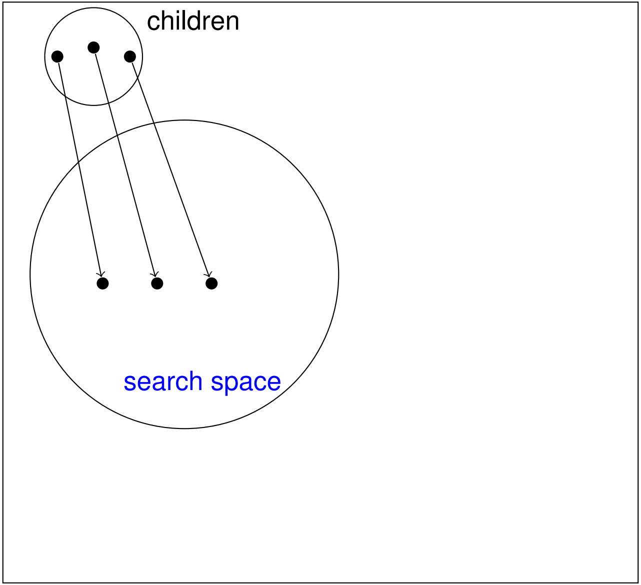
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



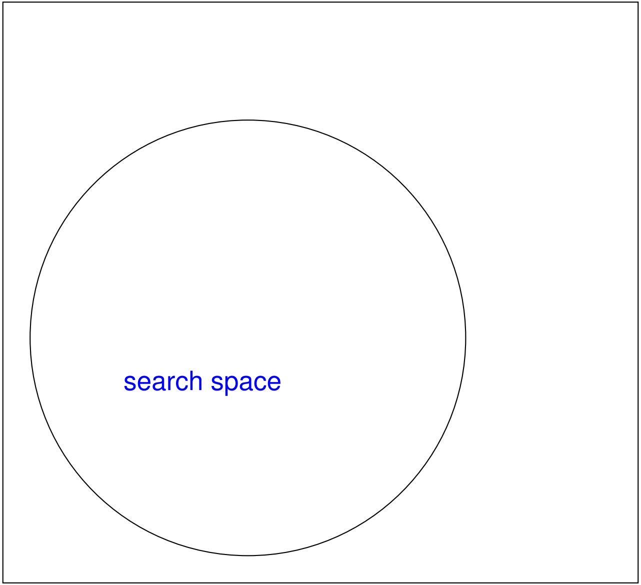
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



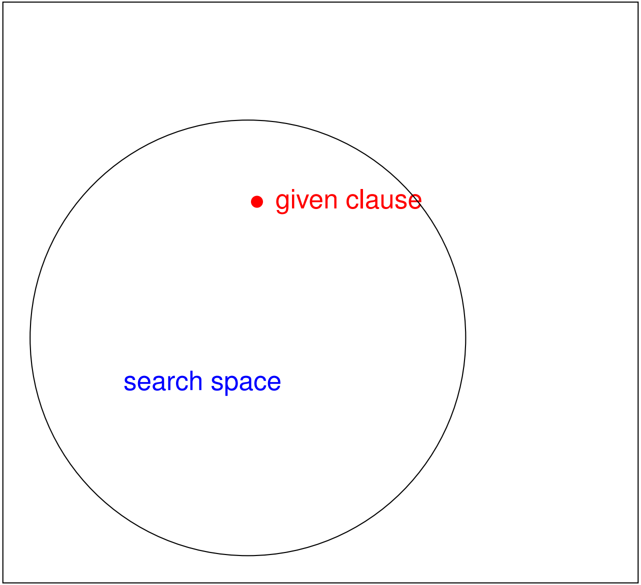
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



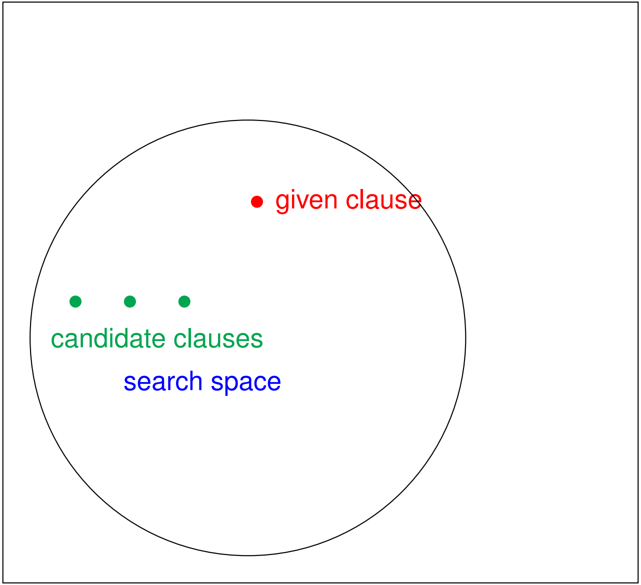
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



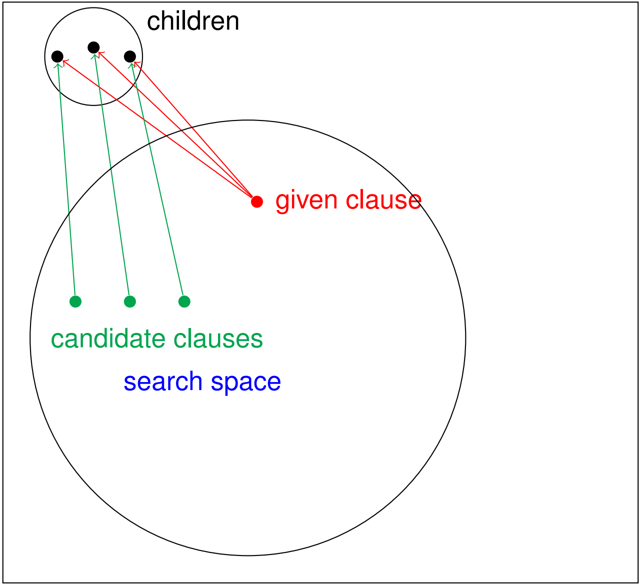
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



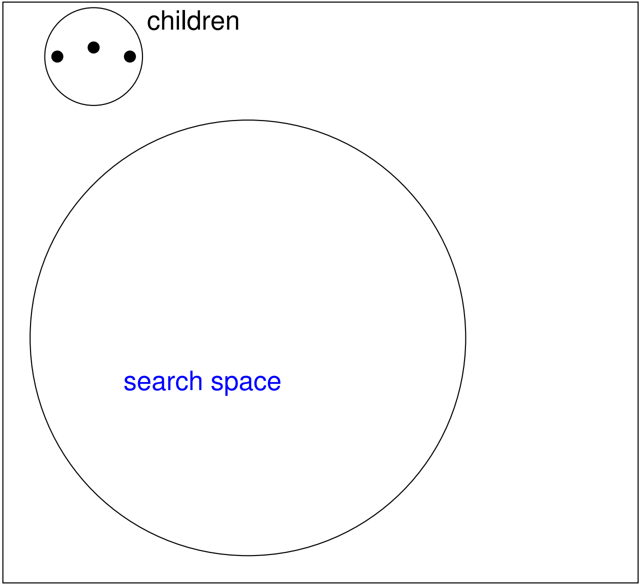
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



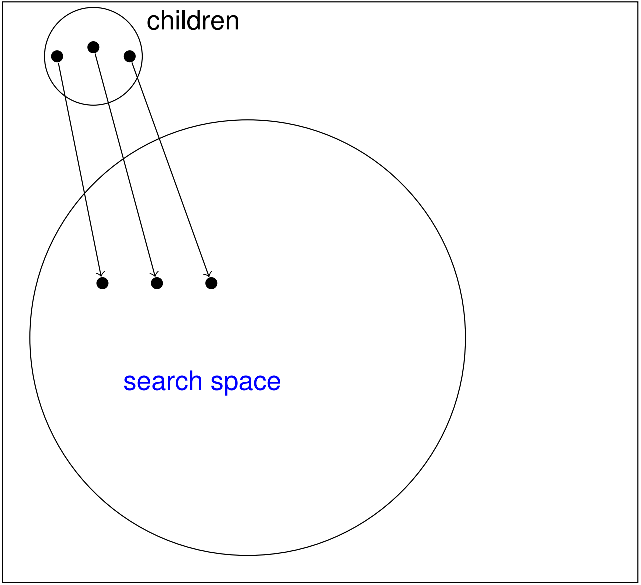
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



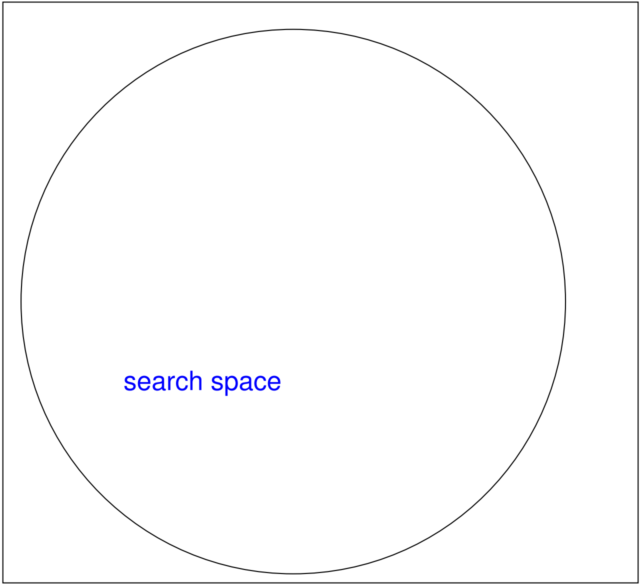
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



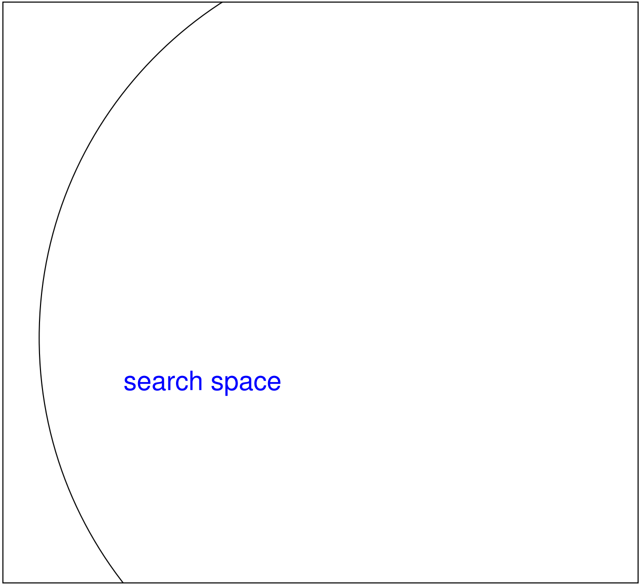
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



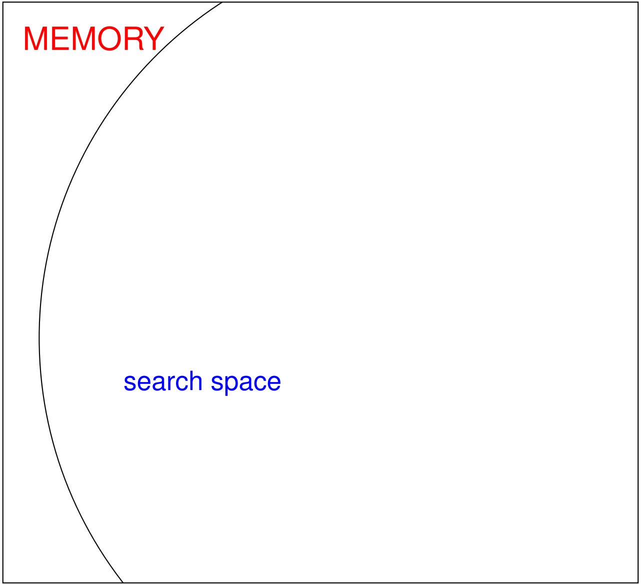
Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



Fair Saturation Algorithms: Inference Selection by Clause Selection

(Given-Clause Algorithm)



Saturation Algorithm

A saturation algorithm tries to saturate a set of clauses with respect to a given inference system.

In theory there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run forever, but without generating $\square$. In this case the input set of clauses is satisfiable.

Saturation Algorithm in Practice

In practice there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run until we run out of resources, but without generating $\square$. In this case it is unknown whether the input set is unsatisfiable.

Saturation Algorithm

Even when we implement inference selection by clause selection, there are too many inferences, especially when the search space grows.

Saturation Algorithm

Even when we implement inference selection by clause selection, there are too many inferences, especially when the search space grows.

Solution: only apply inferences to the selected clause and the previously selected clauses.

Saturation Algorithm

Even when we implement inference selection by clause selection, there are too many inferences, especially when the search space grows.

Solution: only apply inferences to the selected clause and the previously selected clauses.

Thus, the search space is divided in two parts:

- active clauses, that participate in inferences;
- passive clauses, that do not participate in inferences.

Observation: the set of passive clauses is usually considerably larger than the set of active clauses, often by 2-4 orders of magnitude (depending on the saturation algorithm and the problem).

Outline

Subsumption and Tautology Deletion

A clause is a propositional tautology if it is of the form $p \vee \neg p \vee C$, that is, it contains a pair of complementary literals.
There are also equational tautologies, for example
$a \neq b \vee b \neq c \vee f(c, c) = f(a, a)$.

Subsumption and Tautology Deletion

A clause is a propositional tautology if it is of the form $p \vee \neg p \vee C$, that is, it contains a pair of complementary literals.
There are also equational tautologies, for example
$a \neq b \vee b \neq c \vee f(c, c) = f(a, a)$.

A clause C subsumes any clause $C \vee D$, where D is non-empty.

Subsumption and Tautology Deletion

A clause is a propositional tautology if it is of the form $p \vee \neg p \vee C$, that is, it contains a pair of complementary literals.
There are also equational tautologies, for example
$a \neq b \vee b \neq c \vee f(c, c) = f(a, a)$.

A clause C subsumes any clause $C \vee D$, where D is non-empty.

It was known since 1965 that subsumed clauses and propositional tautologies can be removed from the search space.

Problem

How can we prove that completeness is preserved if we remove subsumed clauses and tautologies from the search space?

Problem

How can we prove that completeness is preserved if we remove subsumed clauses and tautologies from the search space?

Solution: general theory of redundancy.