# Automated Deduction

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# Outline

Inference Systems

Selection Functions

# Inference System

$\overline{T}$

- inference has the form

$$\frac{F_1 \quad \dots \quad F_n}{G},$$

where $n \geq 0$ and $F_1, \dots, F_n, G$ are formulas.

- The formula $G$ is called the conclusion of the inference;
- The formulas $F_1, \dots, F_n$ are called its premises.
- An inference rule $R$ is a set of inferences.
- Every inference $I \in R$ is called an instance of $R$.
- An Inference system $\mathbb{I}$ is a set of inference rules.
- Axiom: inference rule with no premises.

# Inference System: Example

Represent the natural number $n$ by the string $\underbrace{|\ldots|}_{n \text{ times}}\varepsilon$.

The following inference system contains 6 inference rules for deriving equalities between expressions containing natural numbers, addition $+$ and multiplication $\cdot$.

$$\frac{}{\varepsilon = \varepsilon}\,(\varepsilon) \qquad \frac{x = y}{|x = |y}\,(|)$$

$$\frac{}{\varepsilon + x = x}\,(+_1) \qquad \frac{x + y = z}{|x + y = |z}\,(+_2)$$

$$\frac{}{\varepsilon \cdot x = \varepsilon}\,(\cdot_1) \qquad \frac{x \cdot y = u \quad y + u = z}{|x \cdot y = z}\,(\cdot_2)$$

# Derivation, Proof

- Derivation in an inference system $\mathbb{I}$: a tree built from inferences in $\mathbb{I}$.
- If the root of this derivation is $E$, then we say it is a derivation of $E$.
- Proof of $E$: a finite derivation whose leaves are axioms.
- Derivation of $E$ from $E_1, \ldots, E_m$: a finite derivation of $E$ whose every leaf is either an axiom or one of the expressions $E_1, \ldots, E_m$.

# Examples

For example,

$$\frac{||\varepsilon + |\varepsilon = |||\varepsilon}{|||\varepsilon + |\varepsilon = ||||\varepsilon}\ (+_2)$$

is an inference that is an instance (special case) of the inference rule

$$\frac{x + y = z}{|x + y = |z}\ (+_2)$$

$x = ||\varepsilon$  $y = |\varepsilon$

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It has one premise $||\varepsilon + |\varepsilon = |||\varepsilon$ and the conclusion $|||\varepsilon + |\varepsilon = ||||\varepsilon$.

The axiom

$$\frac{}{\varepsilon + |||\varepsilon = |||\varepsilon}\ (+_1)$$

$x = |||\varepsilon$

is an instance of the rule

$$\frac{}{\varepsilon + x = x}\ (+_1)$$

# Proof, in this Inference System

Proof of $||\varepsilon \cdot ||\varepsilon = ||||\varepsilon$ (that is, $2 \cdot 2 = 4$).

$$
\dfrac{
  \dfrac{
    \dfrac{}{\varepsilon \cdot ||\varepsilon = \varepsilon}\,(\cdot_1)
    \qquad
    \dfrac{\dfrac{\dfrac{}{\varepsilon + \varepsilon = \varepsilon}\,(+_1)}{|\varepsilon + \varepsilon = |\varepsilon}\,(+_2)}{||\varepsilon + \varepsilon = ||\varepsilon}\,(+_2)
  }{|\varepsilon \cdot ||\varepsilon = ||\varepsilon}\,(\cdot_2)
  \qquad
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}{||\varepsilon \cdot ||\varepsilon = ||||\varepsilon}\,(\cdot_2).
$$

# Proof, Derivation in this Inference System

Proof of $||\varepsilon \cdot ||\varepsilon = ||||\varepsilon$ (that is, $2 \cdot 2 = 4$).

Derivation of $|\varepsilon \cdot ||\varepsilon = ||\varepsilon$ from $\varepsilon \cdot ||\varepsilon = \varepsilon$ and $|\varepsilon + \varepsilon = |\varepsilon$.

$$\dfrac{\dfrac{\dfrac{}{\varepsilon \cdot ||\varepsilon = \varepsilon}\,(\cdot_1) \qquad \dfrac{\dfrac{\dfrac{}{\varepsilon + \varepsilon = \varepsilon}\,(+_1)}{|\varepsilon + \varepsilon = |\varepsilon}\,(+_2)}{||\varepsilon + \varepsilon = ||\varepsilon}\,(+_2)}{|\varepsilon \cdot ||\varepsilon = ||\varepsilon}\,(\cdot_2) \qquad \dfrac{\dfrac{\dfrac{}{\varepsilon + ||\varepsilon = ||\varepsilon}\,(+_1)}{|\varepsilon + ||\varepsilon = |||\varepsilon}\,(+_2)}{||\varepsilon + ||\varepsilon = ||||\varepsilon}\,(+_2)}{||\varepsilon \cdot ||\varepsilon = ||||\varepsilon}\,(\cdot_2).$$

# Arbitrary First-Order Formulas

- A first-order signature (vocabulary): function symbols (including constants), predicate symbols. Equality is part of the language.
- A set of variables.
- Terms are buit using variables and function symbols. For example, $f(x) + g(x)$.
- Atoms, or atomic formulas are obtained by applying a predicate symbol to a sequence of terms. For example, $p(a, x)$ or $f(x) + g(x) \geq 2$.
- Formulas: built from atoms using logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and quantifiers $\forall$, $\exists$. For example, $(\forall x)x = 0 \vee (\exists y)y > x$.

# Clauses

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- Clause: a disjunction $L_1 \vee \ldots \vee L_n$ of literals, where $n \geq 0$.

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- Empty clause, denoted by $\square$: clause with 0 literals, that is, when $n = 0$.
- A formula in Clausal Normal Form (CNF): a conjunction of clauses.
  - quantifier-free
- From now on: A clause is ground if it contains no variables.
- If a clause contains variables, we assume that it implicitly universally quantified. That is, we treat $p(x) \vee q(x)$ as $\forall x(p(x) \vee q(x))$.

# Binary Resolution Inference System

The binary resolution inference system, denoted by $\mathbb{BR}$ is an inference system on propositional clauses (or ground clauses). It consists of two inference rules:

- Binary resolution, denoted by BR:

$$\frac{p \vee C_1 \quad \neg p \vee C_2}{C_1 \vee C_2} \text{ (BR)}.$$

- Factoring, denoted by Fact:

$$\frac{L \vee L \vee C}{L \vee C} \text{ (Fact)}.$$

# Soundness

- An inference is sound if the conclusion of this inference is a logical consequence of its premises.
- An inference system is sound if every inference rule in this system is sound.

$$\frac{C_1 \quad \ldots \quad C_n}{C} \quad \text{sound}$$

$$C_1 \wedge \ldots \wedge C_n \Rightarrow C$$

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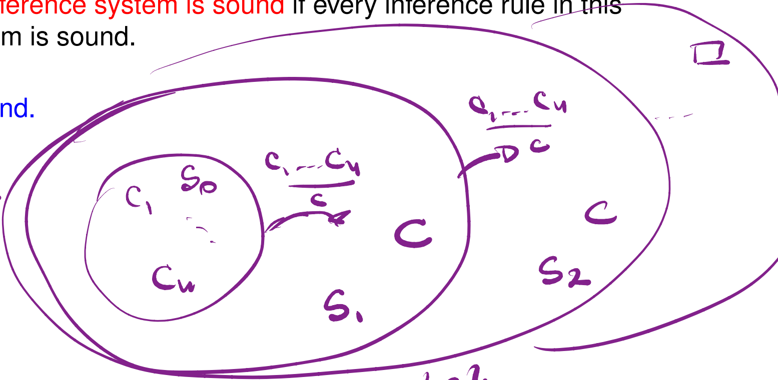
$\mathbb{BR}$ is sound.

$$S_0 \Rightarrow S_1 \Rightarrow S_2$$
$$\Downarrow$$
$$\vdots$$
$$\Downarrow$$
$$\square$$




$S_1 = S_0 \cup \{C\}$

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Consequence of soundness: let $S$ be a set of clauses. If $\square$ can be derived from $S$ in $\mathbb{BR}$, then $S$ is unsatisfiable.

# Example

Consider the following set $S$ of clauses

$$\{\neg p \vee \neg q,\ \neg p \vee q,\ p \vee \neg q,\ p \vee q\}.$$

Is $S$ unsatisfiable?

$$\dfrac{\dfrac{\dfrac{\neg p \vee \neg q \quad \neg p \vee q}{\neg p \vee \neg p}\,\text{BR}}{\neg p}\,\text{FACT} \qquad \dfrac{\dfrac{p \vee \neg q \quad p \vee q}{p \vee p}\,\text{BR}}{p}\,\text{FACT}}{\Box}\,\text{BR}$$

# Example

Consider the following set $S$ of clauses

$$\{\neg p \lor \neg q,\ \neg p \lor q,\ p \lor \neg q,\ p \lor q\}.$$

Is $S$ unsatisfiable?

The following derivation derives the empty clause from this set:

$$\dfrac{\dfrac{\dfrac{p \lor q \quad p \lor \neg q}{p \lor p}\,(\mathrm{BR})}{p}\,(\mathrm{Fact}) \qquad \dfrac{\dfrac{\neg p \lor q \quad \neg p \lor \neg q}{\neg p \lor \neg p}\,(\mathrm{BR})}{\neg p}\,(\mathrm{Fact})}{\square}\,(\mathrm{BR})$$

Hence, this set $S$ of clauses is unsatisfiable.

# Exercise

Consider the following set $S$ of clauses

$$\{\neg p \vee \neg q,\ \neg p \vee q,\ p \vee \neg q,\ p \vee q\}.$$

Show that there exists an infinite number of different $\mathbb{BR}$ derivations of the empyt clause $\Box$ from the clauses of $S$.

# Soundness - Summarized

- An inference is sound if the conclusion of this inference is a logical consequence of its premises.
- An inference system is sound if every inference rule in this system is sound.

$\mathbb{BR}$ is sound.

Consequence of soundness: let $S$ be a set of clauses. If $\square$ can be derived from $S$ in $\mathbb{BR}$, then $S$ is unsatisfiable.

# Can this be used for checking (un)satisfiability?

1. What happens when the empty clause cannot be derived from $S$?

2. How can one search for possible derivations of the empty clause?

# Can this be used for checking (un)satisfiability?

Refutational

1. Completeness.
   *Let $S$ be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from $S$ in $\mathbb{BR}$.*

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1. Completeness.
   *Let $S$ be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from $S$ in $\mathbb{BR}$.*

2. We have to formalize search for derivations.

However, before doing this we will introduce a slightly more refined inference system.

# Outline

# Selection Function

A literal selection function selects literals in a clause.

- If $C$ is non-empty, then at least one literal is selected in $C$.

$r \vee \underline{q}$ $\qquad\qquad$ $p \vee q \vee \underline{\neg r}$

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Note: selection function does not have to be a function. It can be any oracle that selects literals.

# Binary Resolution with Selection

We introduce a family of inference systems, parametrised by a literal selection function $\sigma$.

The binary resolution inference system, denoted by $\mathbb{BR}_\sigma$, consists of two inference rules:

- Binary resolution, denoted by BR

$$\frac{\underline{p} \vee C_1 \quad \underline{\neg p} \vee C_2}{C_1 \vee C_2} \text{ (BR).}$$

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$$\frac{\boxed{p \vee q \vee \underline{p}}}{p \vee q} \text{ Fact}$$

- Positive factoring, denoted by Fact:

$$\frac{\underline{p} \vee \underline{p} \vee C}{p \vee C} \text{ (Fact).}$$

# Completeness?

Binary resolution with selection may be incomplete, even when factoring is unrestricted (also applied to negative literals).

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Consider this set of clauses:

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\begin{array}{ll}
(1) & \neg q \vee \underline{r} \\
(2) & \neg p \vee \underline{q} \\
(3) & \neg r \vee \underline{\neg q} \\
(4) & \neg q \vee \underline{\neg p} \\
(5) & \neg p \vee \underline{\neg r} \\
(6) & \neg r \vee \underline{p} \\
(7) & r \vee q \vee \underline{p}
\end{array}
$$

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It is unsatisfiable:

| | | |
|---|---|---|
| (8) | $q \vee p$ | $(6, 7)$ |
| (9) | $q$ | $(2, 8)$ |
| (10) | $r$ | $(1, 9)$ |
| (11) | $\neg q$ | $(3, 10)$ |
| (12) | $\square$ | $(9, 11)$ |

Note the linear representation of derivations (used by Vampire and many other provers).

However, any inference with selection applied to this set of clauses give either a clause in this set, or a clause containing a clause in this set.

# Literal Orderings

$p > q > r$

Take any well-founded ordering $\succ$ on atoms, that is, an ordering such that there is no infinite decreasing chain of atoms:

$$A_0 \succ A_1 \succ A_2 \succ \cdots$$

In the sequel $\succ$ will always denote a well-founded ordering.

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Extend it to an ordering on literals by:

- If $p \succ q$, then $p \succ \neg q$ and $\neg p \succ q$;
- $\neg p \succ p$.

Ex1 $p > q$

$\neg p > p > \neg q > q$

Ex2 $p > q > r$

$\neg p > p > \neg q > q > \neg r > r$

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- $\neg p \succ p$.

Example: Given $p_6 \succ p_5 \succ p_4 \succ p_3 \succ p_2 \succ p_1$. What is the extended ordering on literals?

$\neg p_6 > p_6 > \neg p_5 > p_5 > \neg p_4 > p_4 > \neg p_3 > p_3 > \neg p_2 > p_2 >$
$\neg p_1 > p_1$

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**Exercise:** prove that the induced ordering on literals is well-founded too.

# Orderings and Well-Behaved Selections

well-founded

Fix an ordering $\succ$. A literal selection function is well-behaved if

- either a negative literal is selected,
  or all maximal literals (w.r.t. $\succ$) must be selected in $C$.

$p > q > r$

$\underline{p} \vee q \vee r$

$\underline{r}$

$\underline{p} \vee \neg q$

$p \vee \underline{\neg q}$

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$\underline{q} \vee r$

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- either a negative literal is selected,
  or all maximal literals (w.r.t. $\succ$) must be selected in $C$.

To be well-behaved, we sometimes must select more than one different literal in a clause. Example: $p \vee p$ or $p(x) \vee p(y)$.

That is: - either a negative literal is selected

or

- if no negative literal is selected, then (only) all maximal literals are selected.

# Completeness of Binary Resolution with Selection

Binary resolution with selection is complete for every well-behaved selection function.

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Consider our previous example:

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A well-behave selection function must satisfy:

1. $r \succ q$, because of (1)
2. $q \succ p$, because of (2)
3. $p \succ r$, because of (6)

There is no ordering that satisfies these conditions.

$Q \supset P$

$\underline{\neg P}$ $\qquad$ $P \vee \underline{Q}$

---

$$\frac{\neg P \qquad P \vee Q}{Q}$$

$S \to$ finite set of atoms

$\downarrow$

well-founded $>$ on atom

$\downarrow$

$>$ on literals

$\downarrow$

selection fct $\sigma$ well-behaved

$\downarrow$

$BR_\sigma$

# Example

$p > q > p > q > p > q > p > q > \cdots$

Let $p, q$ be boolean atoms and let $S$ be the following set of ground formulas:

$$\{\neg p \vee \neg q, \quad \neg p \vee q, \quad p \vee \neg q, \quad p \vee q\}$$

Take any ordering such that $p \succ q$ and any selection function $\sigma$ over $S$ such that

$$\{\neg p \vee \underline{\neg q}, \quad \underline{\neg p} \vee q, \quad p \vee \underline{\neg q}, \quad \underline{p} \vee q\}$$

(a) Is $\sigma$ a well-behaved selection function over $S$? Yes

(b) How many inferences of $\mathbb{BR}_\sigma$ are applicable to $S$?

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only these clauses can be used in one application of BR

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