Bounding the Value of Extended Nonlocal Games

Theory Seminar

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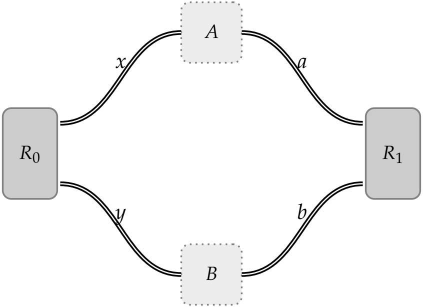
Outline

Nonlocal games

Nonlocal games

A *nonlocal game* is a cooperative game played between *Alice* and *Bob* against a *referee*.



1. Question and answer sets: (Σ_A, Σ_B) and (Γ_A, Γ_B),
2. Distributions on question pairs: $\pi : \Sigma_A \times \Sigma_B \to [0, 1]$,
3. A predicate $V : \Gamma_A \times \Gamma_B \times \Sigma_A \times \Sigma_B \to \{0, 1\}$, where

$$V(a, b|x, y) = \begin{cases} 1 & \text{if Alice and Bob win,} \\ 0 & \text{if Alice and Bob lose.} \end{cases}$$

Strategies for nonlocal games

Alice and Bob could use different types of *strategies*:

- *Classical strategies:* Alice and Bob answer deterministically, determined by functions of $f : \Sigma_\mathsf{A} \to \Gamma_\mathsf{A}$ and $g : \Sigma_\mathsf{B} \to \Gamma_\mathsf{B}$.
- *Quantum strategies:* Alice and Bob share a joint quantum system $\rho \in \mathrm{D}(\mathcal{A} \otimes \mathcal{B})$ and allow their answers to be outcomes of measurements on this shared system.
- *Commuting measurement strategies:* Alice and Bob share a quantum system over a single Hilbert space $\rho \in \mathrm{D}(\mathcal{H})$ and allow their answers to be outcomes of measurements on this system.
- *Non-signaling strategies:* No instantaneous communication between parties.

Values for nonlocal games

The *value* of a nonlocal game is the maximal winning probability for the players to win over all strategies of a specified type.

For a nonlocal game, G, we denote the classical and quantum values as

- Classical value: $\omega(G)$,
- Commuting measurement value: $\omega_c(G)$,
- Quantum value: $\omega^*(G)$,
- Non-signaling value: $\omega_{\text{ns}}(G)$.

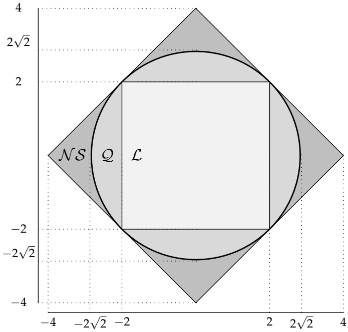
The values obey the following relationship for any nonlocal game:

$$\omega(G) \leq \omega^*(G) \leq \omega_c(G) \leq \omega_{\text{ns}}(G).$$

Optimizing over quantum strategies is hard

Want: Method to calculate the quantum value of a nonlocal game.

This figure shows representations of the space of joint distributions for fixed and finite number of possible questions and answers.

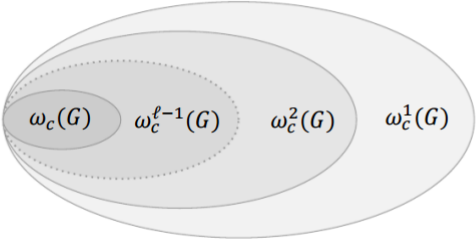


Unfortunately, the set $\mathcal{Q}$ is a non-polyhedral set with an infinite number of extreme points.

Upper bounding nonlocal games

Upper bounds for nonlocal games

- The NPA hierarchy[¶] is a method of placing *upper bounds* on the *quantum value* of nonlocal games.
- Hierarchy of semidefinite programs is *guaranteed* to converge to the commuting measurement value for some finite level, ℓ of the hierarchy.
- The commuting measurement value is an upper bound on the quantum value, $\omega^*(G) \leq \omega_c(G)$, for all nonlocal games, G.



¶[Navascués, Pironio, and Acín, (2008)]

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NPA theorem (Main idea)

- Finding a quantum state and measurements for a quantum strategy is a computationally difficult task.
- Instead then, let's think about a set of *weaker* conditions.
- In the NPA hierarchy, each condition amounts to verifying the existence of a PSD matrix with structure that depends on algebraic properties satisfied by a quantum strategy.
- If any of these conditions are violated, we know that

NPA theorem (Main idea)

NPA states that there exists some matrix $C^{(\ell)}$ that allows $\omega_c(G)$ to be calculated by maximizing

$$\sum_{x,y,a,b} \pi(x,y)V(a,b|x,y)C^{(\ell)}((x,a),(y,b))$$

for some finite level, ℓ, such that $C^{(\ell)}$ satisfies certain linear constraints.

- These linear constraints can be checked via SDP!
- The next few slides will describe how $C^{(\ell)}$ is defined.

PSD operator

[*** More on intuition behind the $C^{(\ell)}$ operator. ***]

Strings

In order to index into $C^{(\ell)}((x,a),(y,b))$, we will consider strings.

Define alphabets

$$\Sigma_{\mathsf{A}} = X \times A, \quad \Sigma_{\mathsf{B}} = Y \times B, \quad \Sigma^* = \{\epsilon\} \cup \Sigma_{\mathsf{A}} \cup \Sigma_{\mathsf{B}}.$$

For example, we can refer to operators (or products of operators) as tuples of concatenated strings. For example:

$$\begin{aligned} &A_a^x \to (x,a), \quad \text{and} \\ &A_{a_1}^{x_1} \cdots A_{a_k}^{x_k} \to (x_1,a_1)\cdots(x_k,a_k). \end{aligned}$$

Similarly for Bob.

Equivalence relations for strings

The measurements in a commuting measurement strategy are *projective* and they *commute*. This property can be conveyed in terms of a string relation:

For all strings $s, t \in \Sigma^*$,

1. Projective: $s\sigma t \sim s\sigma\sigma t$ for all $\sigma \in \Sigma$
2. Commute: $s\sigma\tau t \sim s\tau\sigma t$ for all $\sigma \in \Sigma_{\mathsf{A}}$ and $\tau \in \Sigma_{\mathsf{B}}$.

Admissible functions

The function

$$\phi : \Sigma^* \to \mathbb{C}$$

is *admissible* iff it satisfies the following conditions:

1. Measurements sum to identity:

$$\sum_a \phi(s(x,a)t) = \sum_b \phi(s(y,b)t) = \phi(st),$$

   for all $x, y \in X \times Y$.

2. Something:

$$\phi(s(x,a)(x,a')t) = \phi(s(y,b)(y,b')t) = 0$$

3. For all $s, t \in \Sigma^*$ where $s \sim t$

$$\phi(s) = \phi(t).$$

ℓ-th order admissible matrices

We call the matrix $C^{(\ell)}$ an *ℓ-th order admissible matrix* if

1. There exists an admissible function

$$\phi : \Sigma^{\leq 2\ell} \to \mathbb{C},$$

   such that

$$C^{(\ell)}(s,t) = \phi(s^R t) \quad \forall s,t \in \Sigma^{\leq \ell},$$

2. Normalization: $C^{(\ell)}(\epsilon, \epsilon) = 1$,
3. $C^{(\ell)}$ is positive semidefinite.

ℓ-th order pseudo commuting measurement assemblages

Define an ℓ*-th order pseudo commuting measurement assemblage*

$$K : A \times B \times X \times Y \to \mathrm{L}(\mathbb{C}),$$

for which there exists an ℓ-th order admissible matrix $C^{(\ell)}$ such that

$$K(a, b|x, y) = C^{(\ell)}((x, a), (y, b)) \quad \forall x, y, a, b.$$

Example

Consider a nonlocal game where $|X| = |Y| = |A| = |B| = 2$. Let's compute $C^{(1)}$:

$$C^{(1)} = \left(\begin{array}{c||cccc|ccc}
 & \mathbb{1} & A_0^0 & \cdots & A_1^1 & B_0^0 & \cdots & B_1^1 \\
\hline\hline
\mathbb{1} & & & & & & & \\
A_0^0 & & & & & & & \\
\vdots & & & & & & & \\
A_1^1 & & & & & & & \\
\hline
B_0^0 & & & & & & & \\
\vdots & & & & & & & \\
B_1^1 & & & & & & &
\end{array}\right)$$

- Fill in matrix with products of row and column to generate element Z.
- For each Z computed in this way, the entry refers to an inner product between Z and the shared state ρ, i.e. $\langle Z, \rho \rangle$.

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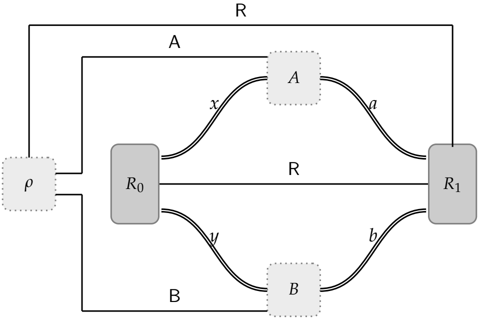
A_0^0 & A_0^0 & A_0^0 & \cdots & A_0^0 A_1^1 & A_0^0 B_0^0 & \cdots & A_0^0 B_1^1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_1^1 & A_1^1 & A_1^1 A_0^0 & \cdots & A_1^1 & A_1^0 B_0^0 & \cdots & A_1^1 B_1^1 \\
\hline
B_0^0 & B_0^0 & B_0^0 A_0^0 & \cdots & B_0^0 A_1^1 & B_0^0 & \cdots & B_0^0 B_1^1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
B_1^1 & B_1^1 & B_1^1 A_0^0 & \cdots & B_1^1 A_1^1 & B_1^1 B_0^0 & \cdots & B_1^1
\end{array}\right)$$

Extended nonlocal games

Extended nonlocal games

An *extended nonlocal game* is a nonlocal game where the *referee also holds a quantum system* that he measures provided by Alice and Bob.



1. Question and answer sets $(\Sigma_\mathsf{A}, \Sigma_\mathsf{B})$ and $(\Gamma_\mathsf{A}, \Gamma_\mathsf{B})$.
2. Distribution on question pairs: $\pi : \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to [0, 1]$.
3. A measurement operator $V : \Gamma_\mathsf{A} \times \Gamma_\mathsf{B} \times \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to \mathrm{Pos}(\mathcal{R})$.

Extended nonlocal games: Winning and losing probabilities

At the end of the protocol, the referee has:

1. The state at the end of the protocol:

$$\rho_{a,b}^{x,y} \in \mathrm{D}(\mathcal{R}).$$

2. A measurement the referee makes on its part of the state ρ:

$$V(a, b|x, y) \in \mathrm{Pos}(\mathcal{R}).$$

The respective winning and losing probabilities are given by

$$\left\langle V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle \quad \text{and} \quad \left\langle \mathbb{1} - V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle.$$

Standard quantum strategies

A *standard quantum strategy* consists of finite-dimensional complex Euclidean spaces $\mathcal{R}, \mathcal{A}$, and $\mathcal{B}$ as well as the following:

- Shared state: $\rho \in \mathcal{R} \otimes \mathcal{A} \otimes \mathcal{B}$.
- Measurements: $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{A}), \quad \{B_b^y\} \subset \mathrm{Pos}(\mathcal{B})$.

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Winning probability for a standard quantum strategy is given by:

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \left\langle V(a,b|x,y) \otimes A_a^x \otimes B_b^y, \rho \right\rangle$$

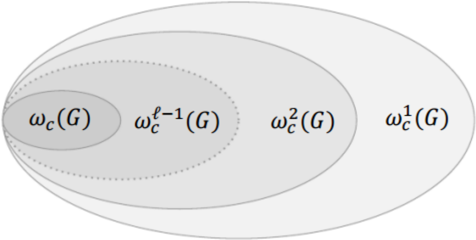
The *standard quantum value*, denoted as $\omega^*(G)$, is the supremum of the winning probability over all standard quantum strategies.

Upper bounding extended nonlocal games

Upper bounds for extended nonlocal games

Extended NPA hierarchy:

- Uses the same idea as the NPA hierarchy. (For $\dim(\mathcal{R}) = 1$, the NPA hierarchy is a special case.)
- Enables one to compute *upper bounds* on the *standard quantum value* for *extended nonlocal games*.
- Same idea as before, only now we need to take into account the actions of the referee.



Commuting measurement strategies (for ENLG)

A *commuting measurement strategy* consists of a finite-dimensional complex Euclidean space $\mathcal{H}$ as well as the following:

- Shared state: $\rho \in \mathcal{R} \otimes \mathcal{H}$.
- Measurements: $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{H}), \quad \{B_b^y\} \subset \mathrm{Pos}(\mathcal{H})$, where $[A_a^x, B_b^y] = 0$ for all x, y, a, b.

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The expected *pay-off* for a commuting measurement strategy is given by:

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} \left\langle V(a,b|x,y) \otimes A_a^x B_b^y, \rho \right\rangle$$

The *commuting measurement value*, denoted as $\omega_c(G)$, is the supremum of the pay-off over all commuting measurement strategies.

Extended NPA theorem

There exists some matrix $M^{(\ell)}$ that allows $\omega_c(G)$ (where G is an ENLG) to be calculated by maximizing

$$\sum_{x,y,a,b} \pi(x,y) \left\langle V(a,b|x,y), M^{(\ell)}((x,a),(y,b)) \right\rangle$$

Extended NPA hierarchy

Same idea, but now we're taking into account the referee, and therefore have a larger matrix.
For each ℓ, now consider block matrices

$$M^{(\ell)} = \begin{pmatrix} M_{1,1}^{(\ell)} & \cdots & M_{1,m}^{(\ell)} \\ \vdots & \ddots & \vdots \\ M_{m,1}^{(\ell)} & \cdots & M_{m,m}^{(\ell)} \end{pmatrix}$$

where each block takes the form $M_{i,j}^{(\ell)} : \Sigma^{\leq \ell} \times \Sigma^{\leq \ell} \to \mathbb{C}$.

- Each submatrix has similar properties to what we saw for the NPA hierarchy.
- The overall matrix also has some structure, which is unique to this case.

Supplementary material: Extended nonlocal games

Winning probability for standard quantum strategies

The winning probability is given by the following equation:

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \frac{\langle V(a,b|x,y), \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{B}}\left(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)\right\rangle}{\mathrm{Tr}(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)} \mathrm{Tr}(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)$$

The probabilities cancel giving

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \mathrm{Tr}\left(V(a,b|x,y)\, \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{B}}\left(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y\right) \rho\right)$$

The trace operator slips past the $\mathbb{1}_{\mathcal{R}}$ giving

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \mathrm{Tr}\left(V(a,b|x,y) \otimes (A_a^x \otimes B_b^y)\rho\right)$$

Writing the trace in terms of the inner product, we have that

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \left\langle V(a,b|x,y) \otimes A_a^x \otimes B_b^y, \rho \right\rangle.$$

Supplementary material:
Lower bounds for extended nonlocal games

Lower bounds for extended nonlocal games

Key idea: Fixing measurements on one system yields the optimal measurements of the other system via an SDP¶

¶[Liang and Doherty (2007)]

Lower bounds for extended nonlocal games

Key idea: Fixing measurements on one system yields the optimal measurements of the other system via an SDP¶

Iterative "see-saw" algorithm between two SDPs:

- SDP-1: Fix Bob's measurements. Optimize over Alice's measurements.
- SDP-2: Fix Alice's measurements (from SDP-1). Optimize over Bob's measurements.
- Repeat.

Not guaranteed to give optimal value, as the algorithm can get stuck in a local minimum.

¶[Liang and Doherty (2007)]

Lower bounds for extended nonlocal games

Define $\{\rho_a^x : x \in \Sigma_A, \ a \in \Gamma_A\} \subset \mathrm{Pos}(\mathcal{R} \otimes \mathcal{B})$ as the residual states acting on the referee and Bob's systems and let

$$f = \sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} \left\langle V(a,b|x,y) \otimes B_b^y, \rho_a^x \right\rangle$$

<u>Lower bound (SDP-1)</u>

$$\begin{aligned} \text{max:} \quad & f \\ \text{s.t.:} \quad & \sum_{a\in\Gamma_A} \rho_a^x = \tau, \\ & \tau \in \mathrm{D}(\mathcal{R} \otimes \mathcal{B}). \end{aligned}$$

<u>Lower bound (SDP-2)</u>

$$\begin{aligned} \text{max:} \quad & f \\ \text{s.t.:} \quad & \sum_{b\in\Gamma_B} B_b^y = \mathbb{1}_{\mathcal{B}}, \\ & B_b^y \in \mathrm{Pos}(\mathcal{B}). \end{aligned}$$

- Iterate between SDP-1 and SDP-2 until desired numerical precision is reached.