Bounding the Value of Extended Nonlocal Games

Theory Seminar

Vincent Russo

University of Waterloo

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Outline

Nonlocal games

Upper bounding nonlocal games

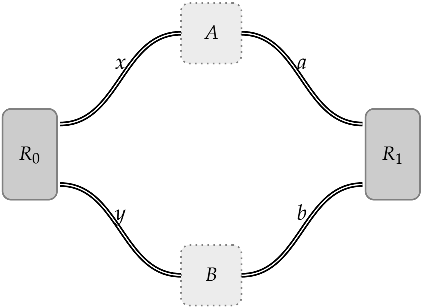
Extended nonlocal games

Upper bounding extended nonlocal games

Nonlocal games

Nonlocal games

A *nonlocal game* is a cooperative game played between *Alice* and *Bob* against a *referee*.



1. Question and answer sets: (Σ_A, Σ_B) and (Γ_A, Γ_B),
2. Distributions on question pairs: $\pi : \Sigma_A \times \Sigma_B \to [0, 1]$,
3. A predicate $V : \Gamma_A \times \Gamma_B \times \Sigma_A \times \Sigma_B \to \{0, 1\}$, where

$$V(a, b|x, y) = \begin{cases} 1 & \text{if Alice and Bob win,} \\ 0 & \text{if Alice and Bob lose.} \end{cases}$$

Strategies for nonlocal games

Alice and Bob could use different types of *strategies*:

- *Classical strategies:* Alice and Bob answer deterministically, determined by functions of $f : \Sigma_{\mathsf{A}} \to \Gamma_{\mathsf{A}}$ and $g : \Sigma_{\mathsf{B}} \to \Gamma_{\mathsf{B}}$.

- *Quantum strategies:* Alice and Bob share a joint quantum system $\rho \in \mathrm{D}(\mathcal{A} \otimes \mathcal{B})$ and allow their answers to be outcomes of measurements on this shared system.

- *Commuting measurement strategies:* Alice and Bob share a quantum system over a single Hilbert space $\rho \in \mathrm{D}(\mathcal{H})$ and allow their answers to be outcomes of measurements on this system.

Winning probabilities for Alice and Bob

For any type of strategy, the output probability distribution produced by Alice and Bob may be described by a *correlation operator*:

$$C \in \mathrm{L}(\mathbb{R}^{\Sigma_\mathsf{A} \times \Gamma_\mathsf{A}}, \mathbb{R}^{\Sigma_\mathsf{B} \times \Gamma_\mathsf{B}}),$$

where $C((x,a),(y,b))$ is the probability that Alice and Bob output (a,b) given questions (x,y).

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where $C((x, a), (y, b))$ is the probability that Alice and Bob output (a, b) given questions (x, y).

For a nonlocal game, the probability that Alice and Bob win for a given choice of x, y, a, b is

$$\sum_{x,y} \pi(x, y) \sum_{a,b} V(a, b|x, y) C((x, a), (y, b)).$$

Values for nonlocal games

The *value* of a nonlocal game is the maximal winning probability for the players to win over all strategies of a specified type.

For a nonlocal game, G, we denote the classical and quantum values as

- Classical value: $\omega(G)$,
- Commuting measurement value: $\omega_c(G)$,
- Quantum value: $\omega^*(G)$.

The values obey the following relationship for any nonlocal game:

$$\omega(G) \leq \omega^*(G) \leq \omega_c(G).$$

Quantum strategies for nonlocal games

A *quantum strategy* consists of finite-dimensional complex Euclidean spaces $\mathcal{A}$ and $\mathcal{B}$ as well as the following:

- Shared state: $\rho \in \mathcal{A} \otimes \mathcal{B}$.
- Measurements $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{A})$ and $\{B_b^y\} \subset \mathrm{Pos}(\mathcal{B})$.

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Winning probability for a quantum strategy is given by:

$$\sum_{x,y} \pi(x,y) \sum_{a,b} V(a,b|x,y) \langle A_a^x \otimes B_b^y, \rho \rangle.$$

The *quantum value*, denoted as as $\omega^*(G)$, is the supremum of the winning probability over all quantum strategies.

Commuting measurement strategies for nonlocal games

A *commuting measurement strategy* consists of a single (possibly infinite-dimensional) complex Euclidean space $\mathcal{H}$, as well as the following:

- Shared state: $\rho \in \mathcal{H}$.
- Measurements $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{H})$ and $\{B_b^y\} \subset \mathrm{Pos}(\mathcal{H})$,

such that $\left[A_a^x, B_b^y\right] = 0$ for all x, y, a, b.

Winning probability for a commuting measurement strategy is given by:

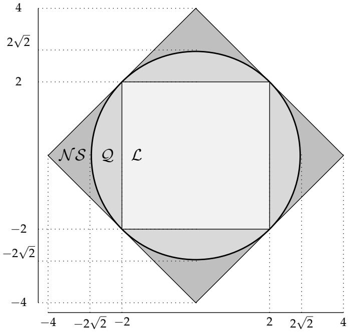
$$\sum_{x,y} \pi(x, y) \sum_{a,b} V(a, b|x, y) \langle A_a^x B_b^y, \rho \rangle.$$

The *commuting measurement value*, denoted as as $\omega_c(G)$, is the supremum of the winning probability over all commuting measurement strategies.

Optimizing over quantum strategies is hard

Want: Method to calculate the quantum value of a nonlocal game.

This figure shows representations of the space of correlations for fixed and finite number of possible questions and answers.

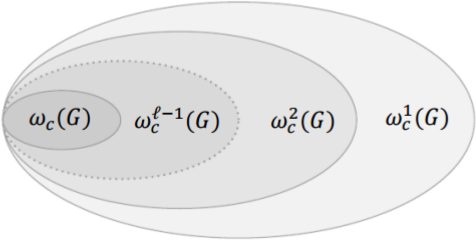


Unfortunately, the set $\mathcal{Q}$ is a non-polyhedral set with an infinite number of extreme points.

Upper bounding nonlocal games

Upper bounds for nonlocal games

- The NPA hierarchy[¶] is a method of placing *upper bounds* on the *quantum value* of nonlocal games.
- Hierarchy of semidefinite programs is *guaranteed* to converge to the commuting measurement value for some finite level, ℓ of the hierarchy.
- The commuting measurement value is an upper bound on the quantum value, $\omega^*(G) \leq \omega_c(G)$, for all nonlocal games, G.



¶ [Navascués, Pironio, and Acín, (2008)]

NPA theorem (Main idea)

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NPA theorem (Main idea)

- Finding a quantum state and measurements for a quantum strategy is a computationally difficult task.
- Instead then, let's think about a set of *weaker* conditions that correspond to a commuting measurement strategy.
- In the NPA hierarchy, each condition amounts to verifying the existence of a positive semidefinite matrix with structure that depends on algebraic properties satisfied by a commuting measurement strategy.
- If any of these conditions are violated, we may conclude that there does not exist an adequate state and sets of measurements.

NPA theorem (Main idea)

NPA states that there exists some matrix $C^{(\ell)}$ that allows $\omega_c(G)$ to be calculated by maximizing

$$\sum_{x,y,a,b} \pi(x,y)V(a,b|x,y)C^{(\ell)}((x,a),(y,b))$$

for some finite level, ℓ, such that $C^{(\ell)}$ satisfies certain linear constraints.

- These linear constraints can be checked via SDP!
- The next few slides will describe how $C^{(\ell)}$ is defined.

PSD operator

[*** More on intuition behind the $C^{(\ell)}$ operator. ***]

Strings

In order to index into $C^{(\ell)}((x,a),(y,b))$, we will consider strings.

Define alphabets

$$\Sigma_{\mathsf{A}} = X \times A, \quad \Sigma_{\mathsf{B}} = Y \times B, \quad \Sigma^* = \{\epsilon\} \cup \Sigma_{\mathsf{A}} \cup \Sigma_{\mathsf{B}}.$$

For example, we can refer to operators (or products of operators) as tuples of concatenated strings. For example:

$$\begin{aligned} &A_a^x \to (x,a), \quad \text{and} \\ &A_{a_1}^{x_1} \cdots A_{a_k}^{x_k} \to (x_1,a_1)\cdots(x_k,a_k). \end{aligned}$$

Similarly for Bob.

Equivalence relations for strings

The measurements in a commuting measurement strategy are *projective* and they *commute*. This property can be conveyed in terms of a string relation:

For all strings $s, t \in \Sigma^*$,

1. Projective: $s\sigma t \sim s\sigma\sigma t$ for all $\sigma \in \Sigma$
2. Commute: $s\sigma\tau t \sim s\tau\sigma t$ for all $\sigma \in \Sigma_\mathsf{A}$ and $\tau \in \Sigma_\mathsf{B}$.

Admissible functions

The function

$$\phi : \Sigma^* \to \mathbb{C}$$

is *admissible* iff it satisfies the following conditions:

1. Measurements sum to identity:

$$\sum_a \phi(s(x,a)t) = \sum_b \phi(s(y,b)t) = \phi(st),$$

   for all $x, y \in X \times Y$.

2. Something:

$$\phi(s(x,a)(x,a')t) = \phi(s(y,b)(y,b')t) = 0$$

3. For all $s, t \in \Sigma^*$ where $s \sim t$

$$\phi(s) = \phi(t).$$

ℓ-th order admissible matrices

We call the matrix $C^{(\ell)}$ an *ℓ-th order admissible matrix* if

1. There exists an admissible function

$$\phi : \Sigma^{\leq 2\ell} \to \mathbb{C},$$

   such that

$$C^{(\ell)}(s,t) = \phi(s^R t) \quad \forall s,t \in \Sigma^{\leq \ell},$$

2. Normalization: $C^{(\ell)}(\epsilon,\epsilon) = 1$,
3. $C^{(\ell)}$ is positive semidefinite.

ℓ-th order pseudo commuting measurement assemblages

Define an ℓ*-th order pseudo commuting measurement assemblage*

$$K : A \times B \times X \times Y \to \mathrm{L}(\mathbb{C}),$$

for which there exists an ℓ-th order admissible matrix $C^{(\ell)}$ such that

$$K(a, b|x, y) = C^{(\ell)}((x, a), (y, b)) \quad \forall x, y, a, b.$$

Example

Consider a nonlocal game where $|X| = |Y| = |A| = |B| = 2$. Let's compute $C^{(1)}$:

$$C^{(1)} = \left(\begin{array}{c||cccc|ccc}
 & \mathbb{1} & A_0^0 & \cdots & A_1^1 & B_0^0 & \cdots & B_1^1 \\
\hline\hline
\mathbb{1} & & & & & & & \\
A_0^0 & & & & & & & \\
\vdots & & & & & & & \\
A_1^1 & & & & & & & \\
\hline
B_0^0 & & & & & & & \\
\vdots & & & & & & & \\
B_1^1 & & & & & & &
\end{array}\right)$$

- Fill in matrix with products of row and column to generate element Z.
- For each Z computed in this way, the entry refers to an inner product between Z and the shared state ρ, i.e. $\langle Z, \rho \rangle$.

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\hline\hline
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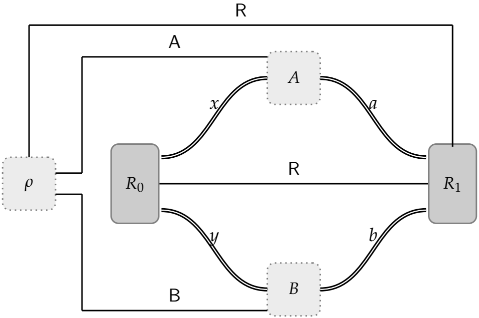
A_0^0 & A_0^0 & A_0^0 & \cdots & A_0^0 A_1^1 & A_0^0 B_0^0 & \cdots & A_0^0 B_1^1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_1^1 & A_1^1 & A_1^1 A_0^0 & \cdots & A_1^1 & A_1^0 B_0^0 & \cdots & A_1^1 B_1^1 \\
\hline
B_0^0 & B_0^0 & B_0^0 A_0^0 & \cdots & B_0^0 A_1^1 & B_0^0 & \cdots & B_0^0 B_1^1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
B_1^1 & B_1^1 & B_1^1 A_0^0 & \cdots & B_1^1 A_1^1 & B_1^1 B_0^0 & \cdots & B_1^1
\end{array}\right)
$$

Extended nonlocal games

Extended nonlocal games

An *extended nonlocal game* is a nonlocal game where the *referee also holds a quantum system* that he measures provided by Alice and Bob.



1. Question and answer sets $(\Sigma_\mathsf{A}, \Sigma_\mathsf{B})$ and $(\Gamma_\mathsf{A}, \Gamma_\mathsf{B})$.
2. Distribution on question pairs: $\pi : \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to [0, 1]$.
3. A measurement operator $V : \Gamma_\mathsf{A} \times \Gamma_\mathsf{B} \times \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to \mathrm{Pos}(\mathcal{R})$.

Extended nonlocal games: Winning and losing probabilities

At the end of the protocol, the referee has:

1. The state at the end of the protocol:

$$\rho_{a,b}^{x,y} \in \mathrm{D}(\mathcal{R}).$$

2. A measurement the referee makes on its part of the state ρ:

$$V(a, b|x, y) \in \mathrm{Pos}(\mathcal{R}).$$

The respective winning and losing probabilities are given by

$$\left\langle V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle \quad \text{and} \quad \left\langle \mathbb{1} - V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle.$$

Standard quantum strategies

A *standard quantum strategy* consists of finite-dimensional complex Euclidean spaces $\mathcal{R}, \mathcal{A}$, and $\mathcal{B}$ as well as the following:

- Shared state: $\rho \in \mathcal{R} \otimes \mathcal{A} \otimes \mathcal{B}$.
- Measurements: $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{A}), \quad \{B_b^y\} \subset \mathrm{Pos}(\mathcal{B})$.

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- Measurements: $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{A}), \quad \{B_b^y\} \subset \mathrm{Pos}(\mathcal{B})$.

Winning probability for a standard quantum strategy is given by:

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \left\langle V(a,b|x,y) \otimes A_a^x \otimes B_b^y, \rho \right\rangle$$

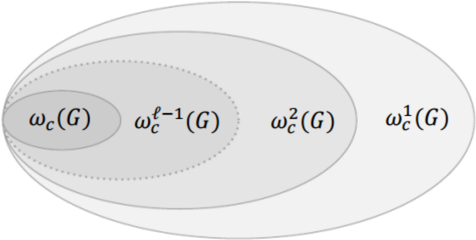
The *standard quantum value*, denoted as $\omega^*(G)$, is the supremum of the winning probability over all standard quantum strategies.

Upper bounding extended nonlocal games

Upper bounds for extended nonlocal games

Extended NPA hierarchy:

- Uses the same idea as the NPA hierarchy. (For $\dim(\mathcal{R}) = 1$, the NPA hierarchy is a special case.)
- Enables one to compute *upper bounds* on the *standard quantum value* for *extended nonlocal games*.
- Same idea as before, only now we need to take into account the actions of the referee.



Commuting measurement strategies (for ENLG)

A *commuting measurement strategy* consists of a finite-dimensional complex Euclidean space $\mathcal{H}$ as well as the following:

- Shared state: $\rho \in \mathcal{R} \otimes \mathcal{H}$.
- Measurements: $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{H}), \quad \{B_b^y\} \subset \mathrm{Pos}(\mathcal{H})$, where $[A_a^x, B_b^y] = 0$ for all x, y, a, b.

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The expected *pay-off* for a commuting measurement strategy is given by:

$$\sum_{(x,y)\in\Sigma_\mathsf{A}\times\Sigma_\mathsf{B}} \pi(x,y) \sum_{(a,b)\in\Gamma_\mathsf{A}\times\Gamma_\mathsf{B}} \Big\langle V(a,b|x,y) \otimes A_a^x B_b^y, \rho \Big\rangle$$

The *commuting measurement value*, denoted as $\omega_c(G)$, is the supremum of the pay-off over all commuting measurement strategies.

Extended NPA theorem

There exists some matrix $M^{(\ell)}$ that allows $\omega_c(G)$ (where G is an ENLG) to be calculated by maximizing

$$\sum_{x,y,a,b} \pi(x,y) \left\langle V(a,b|x,y), M^{(\ell)}((x,a),(y,b)) \right\rangle$$

Extended NPA hierarchy

Same idea, but now we're taking into account the referee, and therefore have a larger matrix.
For each ℓ, now consider block matrices

$$M^{(\ell)} = \begin{pmatrix} M^{(\ell)}_{1,1} & \cdots & M^{(\ell)}_{1,m} \\ \vdots & \ddots & \vdots \\ M^{(\ell)}_{m,1} & \cdots & M^{(\ell)}_{m,m} \end{pmatrix}$$

where each block takes the form $M^{(\ell)}_{i,j} : \Sigma^{\leq \ell} \times \Sigma^{\leq \ell} \to \mathbb{C}$.

- Each submatrix has similar properties to what we saw for the NPA hierarchy.
- The overall matrix also has some structure, which is unique to this case.

Supplementary material:
Extended nonlocal games

Winning probability for standard quantum strategies

The winning probability is given by the following equation:

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \frac{\langle V(a,b|x,y), \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{B}}\left(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho\right)\rangle}{\mathrm{Tr}(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)} \mathrm{Tr}(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)$$

The probabilities cancel giving

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \mathrm{Tr}\left(V(a,b|x,y)\,\mathrm{Tr}_{\mathcal{A}\otimes\mathcal{B}}\left(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y\right)\rho\right)$$

The trace operator slips past the $\mathbb{1}_{\mathcal{R}}$ giving

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \mathrm{Tr}\left(V(a,b|x,y) \otimes (A_a^x \otimes B_b^y)\rho\right)$$

Writing the trace in terms of the inner product, we have that

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \left\langle V(a,b|x,y) \otimes A_a^x \otimes B_b^y, \rho \right\rangle.$$

Supplementary material:
Lower bounds for extended nonlocal games

Lower bounds for extended nonlocal games

Key idea: Fixing measurements on one system yields the optimal measurements of the other system via an SDP[¶]

¶ [Liang and Doherty (2007)]

Lower bounds for extended nonlocal games

Key idea: Fixing measurements on one system yields the optimal measurements of the other system via an SDP¶

Iterative "see-saw" algorithm between two SDPs:

- SDP-1: Fix Bob's measurements. Optimize over Alice's measurements.
- SDP-2: Fix Alice's measurements (from SDP-1). Optimize over Bob's measurements.
- Repeat.

Not guaranteed to give optimal value, as the algorithm can get stuck in a local minimum.

¶[Liang and Doherty (2007)]

Lower bounds for extended nonlocal games

Define $\{\rho_a^x : x \in \Sigma_\mathsf{A},\ a \in \Gamma_\mathsf{A}\} \subset \mathrm{Pos}(\mathcal{R} \otimes \mathcal{B})$ as the residual states acting on the referee and Bob's systems and let

$$f = \sum_{(x,y)\in\Sigma_\mathsf{A}\times\Sigma_\mathsf{B}} \pi(x,y) \sum_{(a,b)\in\Gamma_\mathsf{A}\times\Gamma_\mathsf{B}} \left\langle V(a,b|x,y) \otimes B_b^y, \rho_a^x \right\rangle$$

<u>Lower bound (SDP-1)</u>

$$\begin{aligned} \text{max:} \quad & f \\ \text{s.t.:} \quad & \sum_{a\in\Gamma_\mathsf{A}} \rho_a^x = \tau, \\ & \tau \in \mathrm{D}(\mathcal{R} \otimes \mathcal{B}). \end{aligned}$$

<u>Lower bound (SDP-2)</u>

$$\begin{aligned} \text{max:} \quad & f \\ \text{s.t.:} \quad & \sum_{b\in\Gamma_\mathsf{B}} B_b^y = \mathbb{1}_\mathcal{B}, \\ & B_b^y \in \mathrm{Pos}(\mathcal{B}). \end{aligned}$$

- Iterate between SDP-1 and SDP-2 until desired numerical precision is reached.