Monogamy-of-entanglement games

Theory Seminar

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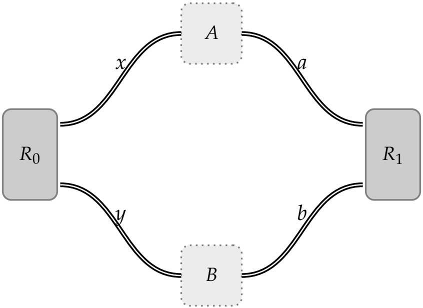
Outline

Nonlocal games

Nonlocal games

A *nonlocal game* is a cooperative game played between *Alice* and *Bob* against a *referee*.



1. Question and answer sets: $(\Sigma_\mathsf{A}, \Sigma_\mathsf{B})$ and $(\Gamma_\mathsf{A}, \Gamma_\mathsf{B})$,
2. Distributions on question pairs: $\pi : \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to [0, 1]$,
3. A predicate $V : \Gamma_\mathsf{A} \times \Gamma_\mathsf{B} \times \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to \{0, 1\}$, where

$$V(a, b|x, y) = \begin{cases} 1 & \text{if Alice and Bob win,} \\ 0 & \text{if Alice and Bob lose.} \end{cases}$$

Strategies and values for nonlocal games

Alice and Bob could use different types of *strategies*:

- *Classical strategies:* Alice and Bob answer deterministically, determined by functions of $f : \Sigma_\mathsf{A} \rightarrow \Gamma_\mathsf{A}$ and $g : \Sigma_\mathsf{B} \rightarrow \Gamma_\mathsf{B}$.

- *Quantum strategies:* Alice and Bob share a joint quantum system $\rho \in \mathrm{D}(\mathcal{A} \otimes \mathcal{B})$ and allow their answers to be outcomes of measurements on this shared system.

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The *value* of a nonlocal game is the maximal winning probability for the players to win over all strategies of a specified type.

For a nonlocal game, G, we denote the classical and quantum values as

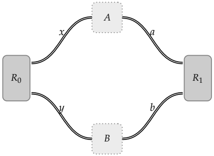
- Classical value: $\omega(G)$,
- Quantum value: $\omega^*(G)$.

Example: The CHSH game

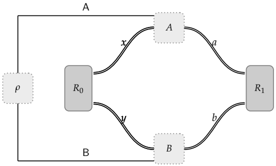
The CHSH game (G_{CHSH}). Question and answer sets over $\{0, 1\}$. Question pairs $\{00, 01, 10, 11\}$ selected with equal probability. Winning condition iff $a \oplus b = x \wedge y$.

$$\omega(G_{\text{CHSH}}) < \omega^*(G_{\text{CHSH}})$$

- $\omega(G_{\text{CHSH}}) = \frac{3}{4} = 0.75$:



- $\omega^*(G_{\text{CHSH}}) = \cos^2(\frac{\pi}{8}) \approx 0.8536$:



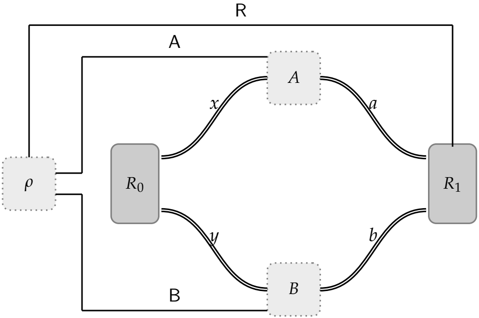
Demo Time: CHSH game in QETLAB

CHSH_GAME.M

Extended nonlocal games

Extended nonlocal games

An *extended nonlocal game* is a nonlocal game where the *referee also holds a quantum system* that he measures provided by Alice and Bob.



1. Question and answer sets $(\Sigma_{\mathsf{A}}, \Sigma_{\mathsf{B}})$ and $(\Gamma_{\mathsf{A}}, \Gamma_{\mathsf{B}})$.
2. Distribution on question pairs: $\pi : \Sigma_{\mathsf{A}} \times \Sigma_{\mathsf{B}} \to [0, 1]$.
3. A measurement operator $V : \Gamma_{\mathsf{A}} \times \Gamma_{\mathsf{B}} \times \Sigma_{\mathsf{A}} \times \Sigma_{\mathsf{B}} \to \mathrm{Pos}(\mathcal{R})$.

Extended nonlocal games: Winning and losing probabilities

At the end of the protocol, the referee has:

1. The state at the end of the protocol:

$$\rho_{a,b}^{x,y} \in \mathrm{D}(\mathcal{R}).$$

2. A measurement the referee makes on its part of the state ρ:

$$V(a, b|x, y) \in \mathrm{Pos}(\mathcal{R}).$$

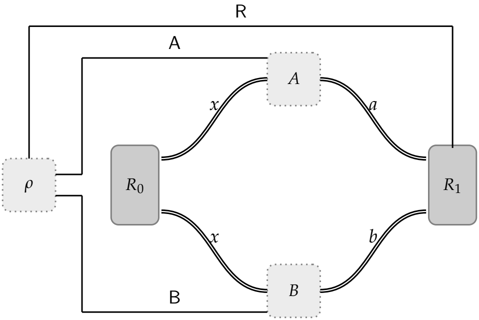
The respective winning and losing probabilities are given by

$$\left\langle V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle \quad \text{and} \quad \left\langle \mathbb{1} - V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle.$$

Monogamy-of-entanglement games

Monogamy-of-entanglement games

Monogamy-of-entanglement games¶, are a special type of extended nonlocal game.



1. Same question and answer sets: $\Sigma = \Sigma_\mathsf{A} = \Sigma_\mathsf{B}$ and $\Gamma = \Gamma_\mathsf{A} = \Gamma_\mathsf{B}$.
2. Alice and Bob get the same question: $\pi(x, y) = 0$ for $x \neq y$.
3. Referee's measurement operator: $R : \Sigma \times \Gamma \to \mathrm{Pos}(\mathcal{R})$.
4. Winning condition: Iff Alice's output, Bob's output, and the referee's output are all the *equal*.

Standard quantum strategies for monogamy-of-entanglement games

A *standard quantum strategy* consists of a tripartite state $\rho \in \mathrm{D}(\mathcal{R} \otimes \mathcal{A} \otimes \mathcal{B})$ and sets of local measurements for Alice and Bob.

- The winning probability for a monogamy-of-entanglement game using a standard quantum strategy is:

$$\sum_{x \in \Sigma} \pi(x) \sum_{a \in \Gamma} \left\langle R(a|x) \otimes A_a^x \otimes B_a^x, \rho \right\rangle.$$

The standard quantum value of a monogamy-of-entanglement game, G, denoted as $\omega^*(G)$, is the maximal winning probability for Alice and Bob over all standard quantum strategies.

Unentangled strategies for monogamy-of-entanglement games

In an *unentangled strategy*, the state ρ prepared by Alice and Bob is fully separable, that is

$$\{\rho_j^{\mathsf{R}} : j \in \Delta\} \subseteq \mathrm{D}(\mathcal{R}), \quad \{\rho_j^{\mathsf{A}} : j \in \Delta\} \subseteq \mathrm{D}(\mathcal{A}), \quad \{\rho_j^{\mathsf{B}} : j \in \Delta\} \subseteq \mathrm{D}(\mathcal{B}),$$

such that

$$\rho = \sum_{j \in \Delta} p(j) \rho_j^{\mathsf{R}} \otimes \rho_j^{\mathsf{A}} \otimes \rho_j^{\mathsf{B}}.$$

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such that

$$\rho = \sum_{j \in \Delta} p(j) \rho_j^{\mathsf{R}} \otimes \rho_j^{\mathsf{A}} \otimes \rho_j^{\mathsf{B}}.$$

Winning probability for an unentangled strategy is given by:

$$\sum_{x \in \Sigma} \pi(x) \sum_{a \in \Gamma} \left\langle R(a|x) \otimes A_a^x \otimes B_a^x, \rho \right\rangle$$

where ρ is separable.

The *unentangled value*, denoted as $\omega(G)$, is the supremum of the winning probability over all unentangled strategies.

Unentangled value of monogamy-of-entanglement games

For an unentangled strategy, we have that Alice, Bob, and the referee share

$$\rho = \sum_{j \in \Delta} p(j) \rho_j^{\mathsf{R}} \otimes \rho_j^{\mathsf{A}} \otimes \rho_j^{\mathsf{B}}.$$

- For $\omega(G)$, we want the *best* Alice and Bob can do.
- Since ρ is separable (no quantum correlations) pick *best* j:

$$\rho = \rho^{\mathsf{R}} \otimes \rho^{\mathsf{A}} \otimes \rho^{\mathsf{B}}.$$

Unentangled strategies for monogamy-of-entanglement games

Alice and Bob only win when their outputs agree, and we assume that the measurements of the referee are positive semidefinite (from the definition for monogamy-of-entanglement games).

- For any monogamy-of-entanglement game, G, the unentangled value is:

$$\omega(G) = \max_{f:\Sigma \to \Gamma} \left\| \sum_{x \in \Sigma} \pi(x) R(f(x)|x) \right\|,$$

where the maximum is over all functions $f : \Sigma \to \Gamma$.

The BB84 monogamy-of-entanglement game

The BB84 game (G_{BB84} for short)¶ is defined by:

1. Question and answer sets:

$$\Sigma = \Gamma = \{0, 1\},$$

2. Uniform probability for questions:

$$\pi(0) = \pi(1) = \frac{1}{2}$$

3. Measurements defined by the BB84 bases:

$$\begin{array}{lll} \text{For } x = 0: & R(0|0) = |0\rangle\langle 0|, & R(1|0) = |1\rangle\langle 1| \\ \text{For } x = 1: & R(0|1) = |+\rangle\langle +|, & R(1|1) = |-\rangle\langle -| \end{array}$$

The *unentangled* and *standard quantum* values for G_{BB84} coincide:

$$\omega(G_{\mathrm{BB84}}) = \omega^*(G_{\mathrm{BB84}}) = \cos^2(\pi/8) \approx 0.8536$$

¶ G_{BB84} was introduced in [Tomamichel, Fehr, Kaniewski, Wehner, (2013)].

Demo Time: BB84 game

BB84_GAME.M

A natural question for monogamy-of-entanglement games

- *Question:* For any monogamy-of-entanglement game, G, is it true that the *unentangled* and *standard quantum* values always coincide? In other words is it true that:

  $$\omega(G) = \omega^*(G)$$

  for all monogamy-of-entanglement games G?

Demo Time: Random monogamy-of-entanglement games

RANDOM_MOE_GAMES.M

A natural question for monogamy-of-entanglement games

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  for all monogamy-of-entanglement games G?

- *Answer:*
  - For certain cases: Yes.
  - In general: No.

$$\omega(G) = \omega^*(G)$$

In general No

Monogamy-of-entanglement games where $\omega(G) \neq \omega^*(G)$

There exists a monogamy-of-entanglement game, G, with $|\Sigma| = 4$ and $|\Gamma| = 3$ such that

$$\omega(G) < \omega^*(G).$$

1. Question and answer sets:

$$\Sigma = \{0, 1, 2, 3\}, \quad \Gamma = \{0, 1, 2\}.$$

2. Uniform probability for questions:

$$\pi(0) = \pi(1) = \pi(2) = \pi(3) = \frac{1}{4}.$$

3. Measurements defined by a mutually unbiased basis¶:

$$\{R(0|x), R(1|x), R(2|x)\}.$$

¶ $|u_x(a)^* u_{x'}(a)|^2 = 1/|\Gamma|$ for $R(a|x) = u_x(a)u_x(a)^*, R(a|x') = u_{x'}(a)u_{x'}(a)^*$

Demo Time: Mutually unbiased basis game

MUB_4_3_GAME.M

Monogamy-of-entanglement games where $\omega(G) \neq \omega^*(G)$

- An exhaustive search over all unentangled strategies reveals an optimal unentangled value:

$$\omega(G) = \frac{3 + \sqrt{5}}{8} \approx 0.6545.$$

- Alternatively, a computer search over standard quantum strategies and a heuristic approximation for the upper bound of $\omega^*(G)$ reveals that:

$$2/3 \geq \omega^*(G) \geq 0.6609.$$

This ability to compute upper bounds for extended nonlocal games is obtained from an adaptation of a technique known as the *NPA hierarchy.*

$$\omega(G) = \omega^*(G)$$

For certain classes, Yes.

Monogamy games that obey $\omega(G) = \omega^*(G)$

Theorem (Johnston, Mittal, R, Watrous)

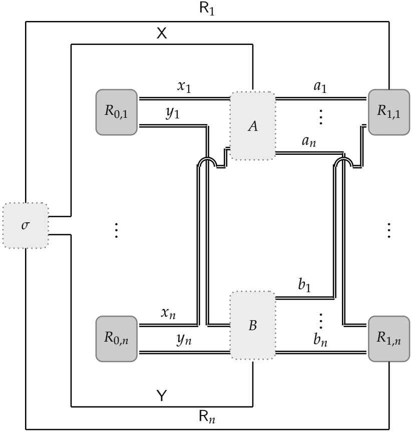
For any monogamy-of-entanglement game, G, for which $|\Sigma| = 2$:

$$\omega(G) = \omega^*(G).$$

Parallel repetition of monogamy-of-entanglement games

Parallel repetition of monogamy-of-entanglement games

- *Parallel repetition:* Run a monogamy-of-entanglement game, G, for n times in parallel, denoted as G^n.
- *Strong parallel repetition:* $\omega(G^n) = \omega(G)^n$



Question: Do all monogamy-of-entanglement games obey strong parallel repetition?

Parallel repetition of monogamy-of-entanglement games

- Recall:

$$\omega(G_{\text{BB84}}) = \omega^*(G_{\text{BB84}}) = \cos^2(\pi/8) \approx 0.8536.$$

- G_{BB84} obeys strong parallel repetition[¶]:

$$\omega^*(G^n_{\text{BB84}}) = \omega^*(G_{\text{BB84}})^n = \left(\cos^2(\pi/8)\right)^n.$$

[¶] [Tomamichel, Fehr, Kaniewski, Wehner, (2013)]

Demo Time: Strong parallel repetition of BB84

bb84_parallel_rep.m

Upper bounds on strong parallel repetition for monogamy games

Theorem (Tomamichel, Fehr, Kaniewski, Wehner)

Let $G = (\pi, R)$ be a monogamy game where π is uniform over Σ. It holds that

$$\omega^*(G^n) \leq \left(\frac{1}{|\Sigma|} + \frac{|\Sigma| - 1}{|\Sigma|}\sqrt{c(G)}\right)^n,$$

where $c(G)$ is the "maximal overlap of measurements" of the referee

$$c(G) = \max_{\substack{x,y\in\Sigma \\ x\neq y}} \max_{a,b\in\Gamma} \left\| \sqrt{R(a|x)}\sqrt{R(b|y)} \right\|^2.$$

Strong parallel repetition for certain monogamy games

Theorem (Johnston, Mittal, R, Watrous)

Let $G = (\pi, R)$ be a monogamy game where $|\Sigma| = 2$, π is uniform over Σ, and $R(a|x)$ are projective operators. It holds that

$$\omega^*(G^n) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n.$$

A key proposition and lemma

Lemma

Let Π_0 and Π_1 be nonzero projection operators on $\mathbb{C}^n$. It holds that

$$\| \Pi_0 + \Pi_1 \| = 1 + \| \Pi_0 \Pi_1 \|.$$

Proposition

Let $G = (\pi, R)$ be a monogamy-of-entanglement game for which $\Sigma = \{0, 1\}$, π is uniform over Σ, and $R(a|x)$ is a projection operator for each $x \in \Sigma$ and $a \in \Gamma$. It holds that

$$\omega(G) = \frac{1}{2} + \frac{1}{2} \max_{a,b \in \Gamma} \Big\| R(a|0) R(b|1) \Big\|.$$

Proof of proposition

Recall that the unentangled value for any monogamy game G is written as

$$\omega(G) = \max_{f:\Sigma\to\Gamma} \left\| \sum_{x\in\Sigma} \pi(x) R(f(x)|x) \right\|.$$

Assuming the lemma stating $\| \Pi_0 + \Pi_1 \| = 1 + \| \Pi_0 \Pi_1 \|$, we have

$$\omega(G) = \max_{a,b\in\Gamma} \left\| \frac{R(a|0) + R(b|1)}{2} \right\| = \frac{1}{2} + \frac{1}{2} \max_{a,b\in\Gamma} \left\| R(a|0) R(b|1) \right\|.$$

Proof of theorem

From the proposition that

$$\omega(G) = \frac{1}{2} + \frac{1}{2}\sqrt{c(G)}.$$

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Since this is an unentangled strategy, we can assume that Alice and Bob just play every instance optimally (since there is no quantum correlation). It follows then that

$$\omega(G^n) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n.$$

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Recall that the theorem from [Tomamichel, Fehr, Kaniewski, Wehner, (2013)] gives us

$$\omega^*(G^n) \leq \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n,$$

which gives us that $\omega^*(G^n) \leq \omega(G^n)$.

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$$\omega^*(G^n) \leq \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n,$$

which gives us that $\omega^*(G^n) \leq \omega(G^n)$. Finally,

$$\omega^*(G^n) \geq \omega(G^n) \geq \left(\frac{1}{2} + \frac{1}{2}\max_{a,b\in\Gamma}\left\| R(a|0)R(b|1) \right\|\right)^n = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n.$$

Open questions

Unentangled vs. standard quantum strategies for monogamy-of-entanglement games

Inputs ($\lvert\Sigma\rvert$)	Outputs ($\lvert\Gamma\rvert$)	$\omega^*(G) = \omega(G)$	$\omega^*(G^n) = \omega^*(G)^n$	$\omega_{ns}(G^n) = \omega_{ns}(G)^n$
2	$\lvert\Gamma\rvert \geq 1$	yes	yes¶	no
3	$\lvert\Gamma\rvert \geq 1$	?	?	no
4	3	no	?	no

Question: What about $|\Sigma| = 3$?

- Proof technique fails for $|\Sigma| > 2$.
- Computational search:
  - Generate random monogamy-of-entanglement games where $|\Sigma| = 3$ and $|\Gamma| \geq 2$.
  - 10^8 random games generates, no counterexamples found.

¶So long as the measurements used by the referee are projective and the probability distribution, π, from which the questions are asked is uniform.