Monogamy-of-entanglement games

Nonlocal Games Seminar

Vincent Russo

University of Waterloo

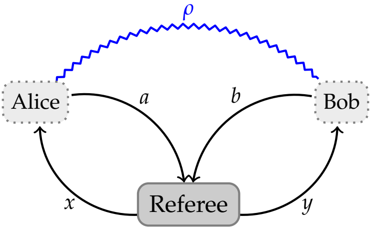
July 1, 2016

Outline

Nonlocal games

Nonlocal games

A *nonlocal game* is a cooperative game played between *Alice* and *Bob* against a *referee*.



1. Question and answer sets: $(\Sigma_\mathsf{A}, \Sigma_\mathsf{B})$ and $(\Gamma_\mathsf{A}, \Gamma_\mathsf{B})$
2. Distributions on question pairs: $\pi : \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to [0, 1]$
3. A predicate $V : \Gamma_\mathsf{A} \times \Gamma_\mathsf{B} \times \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to \{0, 1\}$, where

$$V(a, b|x, y) = \begin{cases} 1 & \text{if Alice and Bob win} \\ 0 & \text{if Alice and Bob lose} \end{cases},$$

Strategies for nonlocal games

Alice and Bob could use different types of *strategies*:

- *Classical strategies:* Alice and Bob answer deterministically, determined by functions of $x \in \Sigma_\mathsf{A}$ and $y \in \Sigma_\mathsf{B}$.

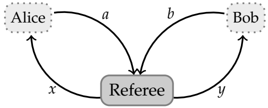
- *Quantum strategies:* Alice and Bob share a joint quantum system $\rho \in \mathrm{D}(\mathcal{A} \otimes \mathcal{B})$ and allow their answers to be outcomes of measurements on this shared system.

Example: The CHSH game

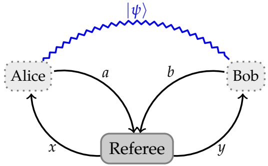
The CHSH game (G_{CHSH}). Winning condition iff $a \oplus b = x \wedge y$.

$$\omega(G_{\mathsf{CHSH}}) < \omega^*(G_{\mathsf{CHSH}})$$

- $\omega(G_{\mathsf{CHSH}}) = \frac{3}{4} = 0.75$:



- $\omega^*(G_{\mathsf{CHSH}}) = \cos^2(\frac{\pi}{8}) \approx 0.8536$:



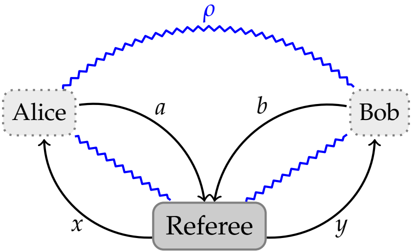
Demo Time: CHSH Game in QETLAB

CHSH_GAME.M

Extended nonlocal games

Extended nonlocal games

An *extended nonlocal game* is a nonlocal game where the *referee also holds a quantum system* the he measures provided by Alice and Bob.



1. Question and answer sets $(\Sigma_\mathsf{A}, \Sigma_\mathsf{B})$ and $(\Gamma_\mathsf{A}, \Gamma_\mathsf{B})$.
2. Distribution on question pairs: $\pi : \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to [0, 1]$.
3. A measurement operator $V : \Gamma_\mathsf{A} \times \Gamma_\mathsf{B} \times \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to \mathrm{Pos}(\mathcal{R})$.

Extended nonlocal games: Winning and losing probabilities

At the end of the protocol, the referee has:

1. The state at the end of the protocol:

$$\rho_{a,b}^{x,y} \in \mathrm{D}(\mathcal{R}).$$

2. A measurement the referee makes on its part of the state ρ:

$$V(a, b|x, y) \in \mathrm{Pos}(\mathcal{R}).$$

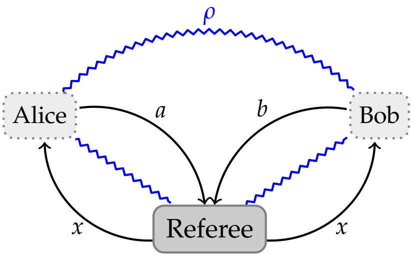
The respective winning and losing probabilities are given by

$$\left\langle V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle \quad \text{and} \quad \left\langle \mathbb{1} - V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle.$$

Monogamy-of-entanglement games

Monogamy-of-entanglement games

Monogamy-of-entanglement games[¶], are a special type of extended nonlocal game.



1. Same question and answer sets: $\Sigma = \Sigma_\mathsf{A} = \Sigma_\mathsf{B}$ and $\Gamma = \Gamma_\mathsf{A} = \Gamma_\mathsf{B}$.
2. Alice and Bob get the same question: $\pi(x, y) = 0$ for $x \neq y$.
3. Referee's measurement operator: $R : \Sigma \times \Gamma \rightarrow \mathrm{Pos}(\mathcal{R})$.
4. Winning condition: Iff Alice's output, Bob's output, and the referee's measurement output are the *same*.

¶[Tomamichel, Fehr, Kaniewski, Wehner, (2013)]

Standard quantum strategies for monogamy-of-entanglement games

- The expected *pay-off* for a monogamy-of-entanglement game, G using a standard quantum strategy is:

$$\sum_{x \in \Sigma} \pi(x) \sum_{a \in \Gamma} \left\langle R(a|x) \otimes A_a^x \otimes B_a^x, \rho \right\rangle.$$

- Since ρ just needs to be a valid density matrix, we use convexity to assume that ρ is pure (rank-one):

$$\omega^*(G) = \left\| \sum_{x \in \Sigma} \pi(x) \sum_{a \in \Gamma} R(a|x) \otimes A_a^x \otimes B_a^x \right\|$$

Unentangled strategies for monogamy-of-entanglement games

Alice and Bob only win when their outputs agree, and we assume that the measurements of the referee are positive semidefinite (from the definition for monogamy-of-entanglement games).

- For any monogamy-of-entanglement game, G, the unentangled value is:

$$\omega(G) = \max_{f:\Sigma\to\Gamma} \left\| \sum_{x\in\Sigma} \pi(x) R(f(x)|x) \right\|.$$

for choice of measurements $\{A_a^x\}$ for Alice and $\{B_b^y\}$ for Bob.

The BB84 monogamy-of-entanglement game

The BB84 game (G_{BB84} for short)¶ is defined by:

1. Question and answer sets:

$$\Sigma = \Gamma = \{0, 1\},$$

2. Uniform probability for questions:

$$\pi(0) = \pi(1) = \frac{1}{2}$$

3. Measurements defined by the BB84 bases:

$$\begin{array}{lll} \text{For } x = 0: & R(0|0) = |0\rangle\langle 0|, & R(1|0) = |1\rangle\langle 1| \\ \text{For } x = 1: & R(0|1) = |+\rangle\langle +|, & R(1|1) = |-\rangle\langle -| \end{array}$$

The *unentangled* and *standard quantum* values for G_{BB84} coincide:

$$\omega(G_{\mathrm{BB84}}) = \omega^*(G_{\mathrm{BB84}}) = \cos^2(\pi/8) \approx 0.8536$$

¶ G_{BB84} was introduced in [Tomamichel, Fehr, Kaniewski, Wehner, (2013)].

Demo Time: BB84 Game

BB84_GAME.M

Exhaustive search over unentangled strategies

Consider the following SDP:

Primal Problem: (γ)

$$\begin{aligned} \text{max:} \quad & \sum_{x \in \Sigma} \pi(x) \Big\langle R(f(x)|x), \rho \Big\rangle \\ \text{s.t.:} \quad & \mathrm{Tr}(\rho) = 1, \qquad (\rho \text{ is pure}). \\ & \rho \geq 0, \qquad (\rho \text{ is PSD}). \end{aligned}$$

We cycle over all possible choices of $f(x) \to a$ and run the above SDP. The best we can do is represented by $\max(\gamma)$ over all such choices.

- Calculating $\max(\gamma)$ is now not an SDP, but for small values of $|\Sigma|$ and $|\Gamma|$, we can brute force over every possible combination to obtain the maximum.

Demo Time: Calculating the unentangled value

unentangled_moe_2in_2out.m

A natural question for monogamy-of-entanglement games

- *Question:* For any monogamy-of-entanglement game, G, is it true that the *unentangled* and *standard quantum* values always coincide? In other words is it true that:

  $$\omega(G) = \omega^*(G)$$

  for all monogamy-of-entanglement games G?

Demo Time: Random Monogamy Games

RANDOM_MOE_GAMES.M

A natural question for monogamy-of-entanglement games

- *Question:* For any monogamy-of-entanglement game, G, is it true that the *unentangled* and *standard quantum* values always coincide? In other words is it true that:

  $$\omega(G) = \omega^*(G)$$

  for all monogamy-of-entanglement games G?

- *Answer:*
  - For certain cases: Yes.
  - In general: No.

$$\omega(G) = \omega^*(G)$$

In general No

Monogamy-of-entanglement games where $\omega(G) \neq \omega^*(G)$

There exists a monogamy-of-entanglement game, G, with $|\Sigma| = 4$ and $|\Gamma| = 3$ such that

$$\omega(G) < \omega^*(G).$$

1. Question and answer sets:

$$\Sigma = \{0, 1, 2, 3\}, \quad \Gamma = \{0, 1, 2\}.$$

2. Uniform probability for questions:

$$\pi(0) = \pi(1) = \pi(2) = \pi(3) = \frac{1}{4}.$$

3. Measurements defined by a mutually unbiased basis[¶]:

$$\{R(0|x), R(1|x), R(2|x)\}.$$

[¶] $|u_x(a)^* u_{x'}(a)|^2 = 1/|\Gamma|$ for $R(a|x) = u_x(a)u_x(a)^*, R(a|x') = u_{x'}(a)u_{x'}(a)^*$

Monogamy-of-entanglement games where $\omega(G) \neq \omega^*(G)$

- An exhaustive search over all unentangled strategies reveals an optimal unentangled value:

$$\omega(G) = \frac{3 + \sqrt{5}}{8} \approx 0.6545.$$

- Alternatively, a computer search over standard quantum strategies and a heuristic approximation for the upper bound of $\omega^*(G)$ reveals that

$$2/3 \geq \omega^*(G) \geq 0.6609$$

This ability to compute upper bounds for extended nonlocal games is obtained from an adaptation of a technique known as the *NPA hierarchy.*

Demo Time: MUB game

MUB_4_3_GAME.M

$$\omega(G) = \omega^*(G)$$

For certain classes, Yes.

Monogamy games that obey $\omega(G) = \omega^*(G)$

Theorem

For any monogamy-of-entanglement game, G, for which $|\Sigma| = 2$:

$$\omega(G) = \omega^*(G).$$

Proof: Monogamy games that obey $\omega(G) = \omega^*(G)$

Recall that for any monogamy-of-entanglement, G, the standard quantum value may be written as

$$\omega^*(G) = \left\| \lambda \sum_{a \in \Gamma} R(a|0) \otimes A_a^0 \otimes B_a^0 + (1-\lambda) \sum_{b \in \Gamma} R(b|1) \otimes A_b^1 \otimes B_b^1 \right\|$$

Proof: Monogamy games that obey $\omega(G) = \omega^*(G)$

Recall that for any monogamy-of-entanglement, G, the standard quantum value may be written as

$$\omega^*(G) = \left\| \lambda \sum_{a \in \Gamma} R(a|0) \otimes A_a^0 \otimes B_a^0 + (1-\lambda) \sum_{b \in \Gamma} R(b|1) \otimes A_b^1 \otimes B_b^1 \right\|$$

Since $\| P \| \leq \| Q \|$ if $P \leq Q$ for any $P, Q \geq 0$:

$$\omega^*(G) \leq \left\| \lambda \sum_{a \in \Gamma} R(a|0) \otimes A_a^0 \otimes \mathbb{1}_{\mathcal{B}} + (1-\lambda) \sum_{b \in \Gamma} R(b|1) \otimes \mathbb{1}_{\mathcal{A}} \otimes B_b^1 \right\|$$

Proof: Monogamy games that obey $\omega(G) = \omega^*(G)$

Recall that for any monogamy-of-entanglement, G, the standard quantum value may be written as

$$\omega^*(G) = \left\| \lambda \sum_{a \in \Gamma} R(a|0) \otimes A_a^0 \otimes B_a^0 + (1-\lambda) \sum_{b \in \Gamma} R(b|1) \otimes A_b^1 \otimes B_b^1 \right\|$$

Since $\|P\| \leq \|Q\|$ if $P \leq Q$ for any $P, Q \geq 0$:

$$\omega^*(G) \leq \left\| \lambda \sum_{a \in \Gamma} R(a|0) \otimes A_a^0 \otimes \mathbb{1}_{\mathcal{B}} + (1-\lambda) \sum_{b \in \Gamma} R(b|1) \otimes \mathbb{1}_{\mathcal{A}} \otimes B_b^1 \right\|$$

Since $\sum_{a \in \Gamma} A_a^x = \sum_{b \in \Gamma} B_b^y = \mathbb{1}$ the above quantity is equal to:

$$\left\| \lambda \sum_{(a,b) \in \Gamma} R(a|0) \otimes A_a^0 \otimes B_b^1 + (1-\lambda) \sum_{(a,b) \in \Gamma} R(b|1) \otimes A_a^0 \otimes B_b^1 \right\|.$$

Monogamy games that obey $\omega(G) = \omega^*(G)$

(Previous slide):

$$\left\| \lambda \sum_{(a,b)\in\Gamma} R(a|0) \otimes A_a^0 \otimes B_b^1 + (1-\lambda) \sum_{(a,b)\in\Gamma} R(b|1) \otimes A_a^0 \otimes B_b^1 \right\|.$$

Monogamy games that obey $\omega(G) = \omega^*(G)$

(Previous slide):

$$\left\| \lambda \sum_{(a,b)\in\Gamma} R(a|0) \otimes A_a^0 \otimes B_b^1 + (1-\lambda) \sum_{(a,b)\in\Gamma} R(b|1) \otimes A_a^0 \otimes B_b^1 \right\|.$$

Since $\{A_a^0 \otimes B_b^1 : a, b \in \Gamma\}$ are pairwise orthogonal projections ($\langle A_a^0 \otimes B_b^1, A_{a'} \otimes B_{b'} \rangle = 0$ for $a \neq a'$ and $b \neq b'$) we have

$$\left\| \sum_{(a,b)\in\Gamma} (\lambda R(a|0) + (1-\lambda) R(b|1)) \otimes A_a^0 \otimes B_b^1 \right\|.$$

Monogamy games that obey $\omega(G) = \omega^*(G)$

(Previous slide):

$$\left\| \lambda \sum_{(a,b)\in\Gamma} R(a|0) \otimes A_a^0 \otimes B_b^1 + (1-\lambda) \sum_{(a,b)\in\Gamma} R(b|1) \otimes A_a^0 \otimes B_b^1 \right\|.$$

Since $\{A_a^0 \otimes B_b^1 : a, b \in \Gamma\}$ are pairwise orthogonal projections ($\langle A_a^0 \otimes B_b^1, A_{a'} \otimes B_{b'} \rangle = 0$ for $a \neq a'$ and $b \neq b'$) we have

$$\left\| \sum_{(a,b)\in\Gamma} (\lambda R(a|0) + (1-\lambda) R(b|1)) \otimes A_a^0 \otimes B_b^1 \right\|.$$

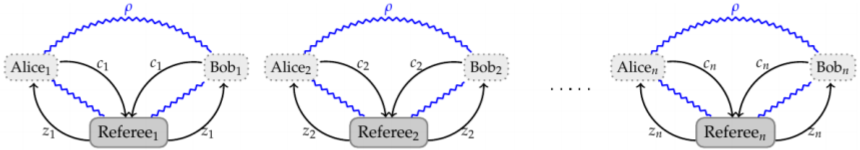
Every entangled strategy is equivalent to one where Alice and Bob use projective measurements so

$$\max_{a,b\in\Gamma} \left\| \lambda R(a|0) + (1-\lambda) R(b|1) \right\| = \omega(G).$$

Parallel Repetition of Monogamy-of-entanglement Games

Parallel repetition of monogamy-of-entanglement games

- *Parallel repetition:* Run a monogamy-of-entanglement game, G, for n times in parallel, denoted as G^n.
- *Strong parallel repetition:* $\omega(G^n) = \omega(G)^n$



Question: Do all monogamy-of-entanglement games obey strong parallel repetition?

Parallel repetition of monogamy-of-entanglement games

- Recall:

$$\omega(G_{\mathrm{BB84}}) = \omega^*(G_{\mathrm{BB84}}) = \cos^2(\pi/8) \approx 0.8536.$$

- G_{BB84} obeys strong parallel repetition[¶]:

$$\omega(G^n_{\mathrm{BB84}}) = \omega^*(G^n_{\mathrm{BB84}}) = \left(\cos^2(\pi/8)\right)^n.$$

[¶] [Tomamichel, Fehr, Kaniewski, Weher, (2013)]

Demo Time: Strong parallel repetition of BB84

bb84_parallel_rep.m

Strong parallel repetition for certain monogamy-of-entanglement games

Theorem (Johnston, Mittal, R, Watrous)

Let $G = (\pi, R)$ be a monogamy-of-entanglement game for which $\Sigma_{\mathsf{A}} = \{0, 1\}$, π is uniform over Σ_{A}, and $R(a|x)$ is a projection operator. It holds that

$$\omega^*(G^n) = \omega^*(G)^n = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n.$$

where $c(G)$ is the "maximal overlap of measurements" of the referee

$$c(G) = \max_{\substack{x,y \in \Sigma_{\mathsf{A}} \\ x \neq y}} \max_{a,b \in \Gamma_{\mathsf{A}}} \left\| \sqrt{R(a|x)}\sqrt{R(b|y)} \right\|^2$$

A key proposition and lemma

Proposition

Let $G = (\pi, R)$ be a monogamy-of-entanglement game for which $\Sigma = \{0, 1\}$, π is uniform over Σ, and $R(a|x)$ is a projection operator for each $x \in \Sigma$ and $a \in \Gamma$. It holds that

$$\omega(G) = \frac{1}{2} + \frac{1}{2} \max_{a,b \in \Gamma} \left\| R(a|0) R(b|1) \right\|.$$

Lemma

Let Π_0 and Π_1 be nonzero projection operators on $\mathbb{C}^n$. It holds that

$$\| \Pi_0 + \Pi_1 \| = 1 + \| \Pi_0 \Pi_1 \|.$$

Proof of proposition

Assuming the lemma stating $\|\Pi_0 + \Pi_1\| = 1 + \|\Pi_0\Pi_1\|$, we have that

$$\omega(G) = \max_{a,b\in\Gamma}\left\| \frac{R(a|0) + R(b|1)}{2} \right\| = \frac{1}{2} + \frac{1}{2}\max_{a,b\in\Gamma}\left\| R(a|0)R(b|1) \right\|.$$

The lower bound on $\omega(G)^*$ is obtained from [TFWK13][¶]. The upper bound follows from the fact that Alice and Bob can just play the optimal strategy for every n:

$$\omega^*(G^n) \geq \omega(G^n) \geq \left(\frac{1}{2} + \frac{1}{2}\max_{a,b\in\Gamma}\left\| R(a|0)R(b|1) \right\|\right)^n = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n.$$

[¶] [Tomamichel, Fehr, Kaniweski, Wehner (2013)]

Open questions

Unentangled vs. standard quantum strategies for monogamy-of-entanglement games

Inputs ($\lvert\Sigma\rvert$)	Outputs ($\lvert\Gamma\rvert$)	$\omega^*(G) = \omega(G)$	$\omega^*(G^n) = \omega^*(G)^n$	$\omega_{ns}(G^n) = \omega_{ns}(G)^n$
2	$\lvert\Gamma\rvert \geq 1$	yes	yes¶	no
3	$\lvert\Gamma\rvert \geq 1$	?	?	no
4	3	no	?	no

Question: What about $\lvert\Sigma\rvert = 3$?

- Proof technique fails for $\lvert\Sigma\rvert > 2$.
- Computational search:
  - Generate random monogamy-of-entanglement games where $\lvert\Sigma\rvert = 3$ and $\lvert\Gamma\rvert \geq 2$.
  - 10^8 random games generates, no counterexamples found.

¶So long as the measurements used by the referee are projective and the probability distribution, π, from which the questions are asked is uniform.

Supplementary material

Supplementary material:
Extended nonlocal games

Winning probability for standard quantum strategies

The winning probability is given by the following equation:

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \frac{\langle V(a,b|x,y), \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{B}}\,(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)\rangle}{\mathrm{Tr}(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)} \mathrm{Tr}(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)$$

The probabilities cancel giving

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \mathrm{Tr}\left(V(a,b|x,y)\, \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{B}}\left(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y\right) \rho\right)$$

The trace operator slips past the $\mathbb{1}_{\mathcal{R}}$ giving

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \mathrm{Tr}\left(V(a,b|x,y) \otimes (A_a^x \otimes B_b^y)\rho\right)$$

Writing the trace in terms of the inner product, we have that

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \left\langle V(a,b|x,y) \otimes A_a^x \otimes B_b^y, \rho \right\rangle.$$

Measurements may be assumed to be projective

WLOG, we may assume that Alice and Bob's measurements are projective since:

- Alice and Bob may extend the sizes of their Hilbert spaces,
- Naimark's theorem states that any strategy using non-projective measurements can be simulated by a strategy with projective measurements.

Supplementary material: Monogamy-of-entanglement games

Pure strategies are sufficient

Recall that the pay-off for a standard quantum strategy is given by

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} \left\langle V(a,b|x,y)\otimes A_a^x \otimes B_b^y, \rho \right\rangle.$$

WLOG, ρ can be assumed to be pure and the measurement operators projective since one could increase the dimension of the Hilbert space, i.e.:

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} u^* \left(V(a,b|x,y)\otimes A_a^x \otimes B_b^y \right) u,$$

where $\rho = uu^*$ with $u \in \mathcal{R}\otimes\mathcal{A}\otimes\mathcal{B}$. This follows from Naimark's theorem (next slide).

Pure strategies are sufficient: Naimark's theorem

Let $u \in \mathcal{R} \otimes \mathcal{A} \otimes \mathcal{B}$, $w_0 \in \mathbb{C}^{\Gamma_A}$. Define

$$v = u \otimes w_0 \in \mathcal{R} \otimes \mathcal{A} \otimes \mathcal{B} \otimes \mathbb{C}^{\Gamma_A} \quad \text{and} \quad \widetilde{A}_a^x = U^* \left(\mathbb{1}_{\mathcal{A}} \otimes E_{a,a}\right) U,$$

where $U \in \mathrm{U}(\mathcal{A} \otimes \mathbb{C}^{\Gamma_A})$ such that

$$U = w \otimes w_0 \to \sum_{a \in \Gamma_A} \sqrt{A_a^x} w \otimes e_a \qquad (\forall w \in \mathcal{A}).$$

It holds that $\widetilde{A}_a^x \in \mathrm{Proj}(\mathcal{A})$ and hence

$$\widetilde{A}_a^x v = U^* \left(\mathbb{1}_{\mathcal{A}} \otimes E_{a,a}\right) U \left(u \otimes w_0\right) = U^* \sqrt{A_a^x} u \otimes e_a,$$

and $\widetilde{A}_a^x = \left(\widetilde{A}_a^x\right)^* \left(\widetilde{A}_a^x\right)$. Note¶.

¶A similar argument holds for Bob's measurements.

The BB84 game: Unentangled value

For any monogamy-of-entanglement game, G, the unentangled value is:

$$\omega(G) = \max_{f:\Sigma\to\Gamma} \left\| \sum_{x\in\Sigma} \pi(x) R(f(x)|x) \right\|$$

Recall, G_{BB84} has

$$\pi(0) = \pi(1) = 1/2 \quad \text{and} \quad \Sigma = \Gamma = \{0, 1\}.$$

So we have that:

$$\begin{aligned} \omega(G_{\mathrm{BB84}}) &= \max_{f:\Sigma\to\Gamma} \left\| \frac{1}{2} R(f(x)|0) + \frac{1}{2} R(f(x)|1) \right\| \\ &= \frac{1}{2} \big\| |0\rangle\langle 0| + |+\rangle\langle +| \big\| + \frac{1}{2} \big\| |1\rangle\langle 1| + |-\rangle\langle -| \big\| \\ &= \frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos^2(\pi/8). \end{aligned}$$

The BB84 game: Standard quantum value

For any monogamy-of-entanglement game, G, the standard quantum value is:

$$\omega^*(G) = \left\| \sum_{x \in \Sigma} \pi(x) \sum_{a \in \Gamma} R(a|x) \otimes A_a^x \otimes B_a^x \right\|.$$

1. Alice and Bob send the following state to the referee:

$$v_\pm = \cos(\pi/8)|0\rangle \pm \sin(\pi/8)|1\rangle.$$

2. Alice and Bob always output $a = 0$ and have measurements:

$$A_0^0 = v_+ v_+^*, \quad A_0^1 = v_- v_-^*, \quad B_0^0 = B_0^1 = \mathbb{1}$$

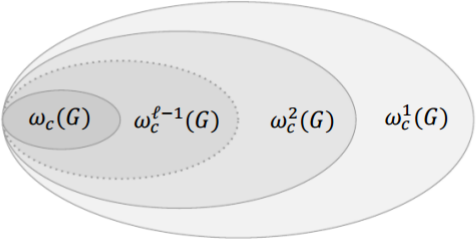
Plugging this into the standard quantum value formula:

$$\omega^*(G) = \left\| \frac{1}{2} \left(|0\rangle\langle 0| \otimes A_0^0 \otimes B_0^0 + |1\rangle\langle 1| \otimes A_0^0 \otimes B_0^0 \right) + \frac{1}{2} \left(|+\rangle\langle +| \otimes A_0^1 \otimes B_0^1 + |-\rangle\langle -| \otimes A_0^1 \otimes B_0^1 \right) \right\| = \cos^2(\pi/8)$$

Supplementary material:
Upper bounds for extended nonlocal games

Upper bounds for nonlocal games

- The NPA hierarchy[¶] is a method of placing *upper bounds* on the *quantum value* of nonlocal games.
- Hierarchy of semidefinite programs is *guaranteed* to converge to the commuting measurement value for some finite level, ℓ of the hierarchy.
- The commuting measurement value is an upper bound on the quantum value, $\omega^*(G) \leq \omega_c(G)$, for all nonlocal games, G.



¶ [Navascués, Pironio, and Acín, (2008)]

NPA hierarchy: Level 1

For a nonlocal game, the pay-off for a commuting measurement strategy is[¶]

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} V(a,b|x,y)\langle A_a^x B_b^y, \rho\rangle.$$

[¶] for $[A_a^x, B_b^y] = 0$, $\{A_a^x\}, \{B_b^y\} \subset \mathrm{Pos}(\mathcal{H})$, and $\rho \in \mathrm{D}(\mathcal{H})$.

NPA hierarchy: Level 1

For a nonlocal game, the pay-off for a commuting measurement strategy is¶

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} V(a,b|x,y)\langle A_a^x B_b^y, \rho\rangle.$$

Define a Gram matrix with entries:

$$C((x,a),(y,b)) = \langle A_a^x B_b^y, \rho\rangle.$$

¶ for $[A_a^x, B_b^y] = 0$, $\{A_a^x\}, \{B_b^y\} \subset \mathrm{Pos}(\mathcal{H})$, and $\rho \in \mathrm{D}(\mathcal{H})$.

NPA hierarchy: Level 1

For a nonlocal game, the pay-off for a commuting measurement strategy is¶

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} V(a,b|x,y)\langle A_a^x B_b^y, \rho\rangle.$$

Define a Gram matrix with entries:

$$C((x,a),(y,b)) = \langle A_a^x B_b^y, \rho\rangle.$$

The full block-matrix is

$$C = \left(\begin{array}{c|c} \langle A_a^x A_{a'}^{x'}, \rho\rangle & \langle A_a^x B_b^y, \rho\rangle \\ \hline \langle B_b^y A_a^x, \rho\rangle & \langle B_b^y B_{b'}^{y'}, \rho\rangle \end{array}\right)$$

- If the entries in C come from a commuting measurement strategy, C will satisfy *certain properties*.
- These properties are verifiable via an SDP.

¶ for $[A_a^x, B_b^y] = 0$, $\{A_a^x\}, \{B_b^y\} \subset \mathrm{Pos}(\mathcal{H})$, and $\rho \in \mathrm{D}(\mathcal{H})$.

Matrix constraints

As mentioned, the matrix C satisfies a number of constraints:

- It is positive semidefinite (by definition from the fact that it is a Gram matrix)
- Normalization: $C(1,1) = 1$.
- Commutation:

$$C((x,a),(y,b)) = C((y,b),(x,a))$$

- Measurements sum to identity:

$$\sum_{(x,a)} C((x,a),(y,b)) = C(1,(y,b))$$
$$\sum_{(y,b)} C((x,a),(y,b)) = C((x,a),1).$$

- Measurements are projective:

$$C(1,(y,b)) = C((y,b),(y,b))$$
$$C((x,a),1) = C((x,a),(x,a))$$

Pseudo commuting measurement assemblages

If these properties are satisfied, we say that C is a 1-*st order pseudo commuting measurement assemblage*.

- By imposing more structure on this matrix, we get closer to the set of commuting measurement operators.
- Indexing correspondence between strings and operators:

$$A_a^x \leftrightarrow (x, a) \quad \text{and} \quad A_{a_1}^{x_1} \cdots A_{a_n}^{x_n} \leftrightarrow (x_1, a_1) \cdots (x_n, a_n)$$

  Analogous for B_b^y operators.

Pseudo commuting measurement assemblages

If these properties are satisfied, we say that C is a 1-*st order pseudo commuting measurement assemblage*.

- By imposing more structure on this matrix, we get closer to the set of commuting measurement operators.
- Indexing correspondence between strings and operators:

$$A_a^x \leftrightarrow (x, a) \quad \text{and} \quad A_{a_1}^{x_1} \cdots A_{a_n}^{x_n} \leftrightarrow (x_1, a_1) \cdots (x_n, a_n)$$

  Analogous for B_b^y operators.

Define the alphabets

$$\Delta_{\mathsf{A}}^{\leq \ell} = \{\epsilon\} \cup (\Sigma_{\mathsf{A}} \times \Gamma_{\mathsf{A}})^{\leq \ell} \quad \text{and} \quad \Delta_{\mathsf{B}}^{\leq \ell} = \{\epsilon\} \cup (\Sigma_{\mathsf{B}} \times \Gamma_{\mathsf{B}})^{\leq \ell}.$$

The ℓ-th order pseudo commuting measurement assemblage is

$$C^{(\ell)} = \left(\begin{array}{c|c} \Delta_{\mathsf{A}}^{\leq \ell} \cup \Delta_{\mathsf{A}}^{\leq \ell} & \Delta_{\mathsf{A}}^{\leq \ell} \cup \Delta_{\mathsf{B}}^{\leq \ell} \\ \hline \Delta_{\mathsf{B}}^{\leq \ell} \cup \Delta_{\mathsf{A}}^{\leq \ell} & \Delta_{\mathsf{B}}^{\leq \ell} \cup \Delta_{\mathsf{B}}^{\leq \ell} \end{array} \right)$$

NPA hierarchy theorem

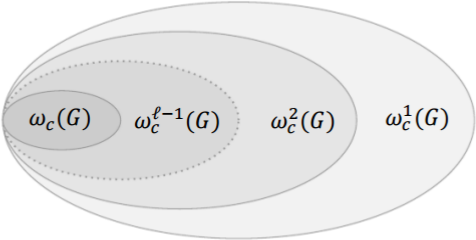
For some finite level ℓ, the pay-off for a commuting measurement strategy for a nonlocal game can be defined by

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B}\pi(x,y)\sum_{(a,b)\in\Gamma_A\times\Gamma_B}V(a,b|x,y)C^{(\ell)}.$$

Upper bounds for extended nonlocal games

Extended NPA hierarchy:

- Uses the same idea as the NPA hierarchy. (For $\dim(\mathcal{R}) = 1$, the NPA hierarchy is a special case.)
- Enables one to compute *upper bounds* on the *standard quantum value* for *extended nonlocal games*.
- Same idea as before, only now we need to take into account the actions of the referee.



Commuting measurement strategies

A *commuting measurement strategy* consists of a finite-dimensional complex Euclidean space $\mathcal{H}$ as well as the following:

- Shared state: $\rho \in \mathcal{R} \otimes \mathcal{H}$.
- Measurements: $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{H}), \quad \{B_b^y\} \subset \mathrm{Pos}(\mathcal{H})$, where $[A_a^x, B_b^y] = 0$ for all x, y, a, b.

Commuting measurement strategies

A *commuting measurement strategy* consists of a finite-dimensional complex Euclidean space $\mathcal{H}$ as well as the following:

- Shared state: $\rho \in \mathcal{R} \otimes \mathcal{H}$.
- Measurements: $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{H})$, $\quad \{B_b^y\} \subset \mathrm{Pos}(\mathcal{H})$,
  where $[A_a^x, B_b^y] = 0$ for all x, y, a, b.

The expected *pay-off* for a commuting measurement strategy is given by:

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} \left\langle V(a,b|x,y) \otimes A_a^x B_b^y, \rho \right\rangle$$

The *commuting measurement value*, denoted as $\omega_c(G)$, is the supremum of the pay-off over all commuting measurement strategies.

Extended NPA hierarchy

Same idea, but now we're taking into account the referee, and therefore have a larger matrix.
For each ℓ, now consider block matrices

$$M^{(\ell)} = \begin{pmatrix} M^{(\ell)}_{1,1} & \cdots & M^{(\ell)}_{1,m} \\ \vdots & \ddots & \vdots \\ M^{(\ell)}_{m,1} & \cdots & M^{(\ell)}_{m,m} \end{pmatrix}$$

where each block takes the form $M^{(\ell)}_{i,j} : \Sigma^{\leq \ell} \times \Sigma^{\leq \ell} \to \mathbb{C}$.

- Each submatrix has similar properties to what we saw for the NPA hierarchy.
- The overall matrix also has some structure, which is unique to this case.

Assemblages

- Another natural way to think about the commuting measurement value is in terms of *assemblages*:

$$K(a,b|x,y) = \mathrm{Tr}_{\mathcal{H}}\left((\mathbb{1}_{\mathcal{R}} \otimes A_a^x B_b^y)\rho\right)$$

- For a particular choice of x, y, a, b, an assemblage corresponds to the *unnormalized state* in the referee's hands at the end of the game.
- The function K completely determines the performance of Alice and Bob's strategy:

$$\sum_{(x,y)\in\Sigma_\mathrm{A}\times\Sigma_\mathrm{B}} \pi(x,y) \sum_{(a,b)\in\Gamma_\mathrm{A}\times\Gamma_\mathrm{B}} \Big\langle V(a,b|x,y), K(a,b|x,y) \Big\rangle.$$

Extended NPA hierarchy theorem

Let $\dim(\mathcal{R}) = m$. Then

$$\left\langle V(a,b|x,y), K(a,b|x,y) \right\rangle = \left\langle V(a,b|x,y), M^{(\ell)}((x,a),(y,b)) \right\rangle$$

for all $m, \ell \geq 1$, $(x,y) \in \Sigma_\mathsf{A} \times \Sigma_\mathsf{B}$ and $(a,b) \in \Gamma_\mathsf{A} \times \Gamma_\mathsf{B}$.

Supplementary material:
Lower bounds for extended nonlocal games

Lower bounds for extended nonlocal games

Key idea: Fixing measurements on one system yields the optimal measurements of the other system via an SDP¶

¶[Liang and Doherty (2007)]

Lower bounds for extended nonlocal games

Key idea: Fixing measurements on one system yields the optimal measurements of the other system via an SDP¶

Iterative "see-saw" algorithm between two SDPs:

- SDP-1: Fix Bob's measurements. Optimize over Alice's measurements.
- SDP-2: Fix Alice's measurements (from SDP-1). Optimize over Bob's measurements.
- Repeat.

Not guaranteed to give optimal value, as the algorithm can get stuck in a local minimum.

¶[Liang and Doherty (2007)]

Lower bounds for extended nonlocal games

Define $\{\rho_a^x : x \in \Sigma_\mathsf{A},\ a \in \Gamma_\mathsf{A}\} \subset \mathrm{Pos}(\mathcal{R} \otimes \mathcal{B})$ as the residual states acting on the referee and Bob's systems and let

$$f = \sum_{(x,y)\in\Sigma_\mathsf{A}\times\Sigma_\mathsf{B}} \pi(x,y) \sum_{(a,b)\in\Gamma_\mathsf{A}\times\Gamma_\mathsf{B}} \left\langle V(a,b|x,y) \otimes B_b^y, \rho_a^x \right\rangle$$

<u>Lower bound (SDP-1)</u>

$$\begin{aligned} \text{max:} \quad & f \\ \text{s.t.:} \quad & \sum_{a\in\Gamma_\mathsf{A}} \rho_a^x = \tau, \\ & \tau \in \mathrm{D}(\mathcal{R} \otimes \mathcal{B}). \end{aligned}$$

<u>Lower bound (SDP-2)</u>

$$\begin{aligned} \text{max:} \quad & f \\ \text{s.t.:} \quad & \sum_{b\in\Gamma_\mathsf{B}} B_b^y = \mathbb{1}_\mathcal{B}, \\ & B_b^y \in \mathrm{Pos}(\mathcal{B}). \end{aligned}$$

- Iterate between SDP-1 and SDP-2 until desired numerical precision is reached.

Supplementary material:
Parallel repetition for
monogamy-of-entanglement games

A key lemma (proof)

By definition

$$\|\Pi_0 + \Pi_1\| = \max\{v^*(\Pi_0 + \Pi_1)v : v \in \mathcal{S}\}.$$

A key lemma (proof)

By definition

$$\| \Pi_0 + \Pi_1 \| = \max\{v^*(\Pi_0 + \Pi_1)v : v \in \mathcal{S}\}.$$

Since $v \in \mathcal{S}$ are unit vectors,

$$v^*(\Pi_0 + \Pi_1)v = v^*\Pi_0 v + v^*\Pi_1 v = \| \Pi_0 \|^2 + \| \Pi_1 \|^2.$$

A key lemma (proof)

By definition

$$\|\Pi_0 + \Pi_1\| = \max\{v^*(\Pi_0 + \Pi_1)v : v \in \mathcal{S}\}.$$

Since $v \in \mathcal{S}$ are unit vectors,

$$v^*(\Pi_0 + \Pi_1)v = v^*\Pi_0 v + v^*\Pi_1 v = \|\Pi_0\|^2 + \|\Pi_1\|^2.$$

Recall the definition of the ∞-norm for some operator $A \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$:

$$\|A\| = \max\{\|Av\| : v \in \mathcal{S}(\mathcal{X})\}.$$

We can then rewrite our expression as

$$\max\{\|\Pi_0 v\|^2 + \|\Pi_1 v\|^2 : v \in \mathcal{S}\}.$$

A key lemma (proof)

Define unit vectors $u_0, u_1 \in \mathcal{S}(\mathbb{C}^n)$, we can write

$$\Pi_0 = u_0 u_0^* \quad \text{and} \quad \Pi_1 = u_1 u_1^*.$$

A key lemma (proof)

Define unit vectors $u_0, u_1 \in \mathcal{S}(\mathbb{C}^n)$, we can write

$$\Pi_0 = u_0 u_0^* \quad \text{and} \quad \Pi_1 = u_1 u_1^*.$$

Therefore

$$\Pi_0 v = u_0 u_0^* v = \langle u_0, v \rangle u_0$$

and that

$$\| \Pi_0 v \|^2 = |\langle u_0, v \rangle|^2 \| u_0 \| = |\langle u_0, v \rangle|^2$$

(and similarly for Π_1 and u_1).

A key lemma (proof)

Therefore, we have that

$$\max\{|\langle u_0, v\rangle| + |\langle u_1, v\rangle|^2 : v \in \mathcal{S}, u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\},$$

where $\mathcal{S}_0$ is unit sphere of $\mathrm{im}(\Pi_0)$ (similarly for $\mathcal{S}_1$ and Π_1).

A key lemma (proof)

Therefore, we have that

$$\max\{|\langle u_0, v\rangle| + |\langle u_1, v\rangle|^2 : v \in \mathcal{S}, u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\},$$

where $\mathcal{S}_0$ is unit sphere of $\mathrm{im}(\Pi_0)$ (similarly for $\mathcal{S}_1$ and Π_1). By Cauchy-Schwarz, we have that

$$|\langle u_0, v\rangle|^2 = \| u_0 \| \| v \|$$

(similarly for u_1 and v). Equality is achieved since u_0 and v are linearly dependent.

A key lemma (proof)

Therefore, we have that

$$\max\{|\langle u_0, v\rangle| + |\langle u_1, v\rangle|^2 : v \in \mathcal{S}, u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\},$$

where $\mathcal{S}_0$ is unit sphere of $\mathrm{im}(\Pi_0)$ (similarly for $\mathcal{S}_1$ and Π_1). By Cauchy-Schwarz, we have that

$$|\langle u_0, v\rangle|^2 = \| u_0 \| \| v \|$$

(similarly for u_1 and v). Equality is achieved since u_0 and v are linearly dependent. Since v is a unit eigenvector of $u_0 u_0^* + u_1 u_1^*$ that corresponds to $\lambda_{\max}$ we have that

$$v^*(u_0 u_0^* + u_1 u_1^*)v = \| u_0 u_0^* + u_1 u_1^* \|.$$

A key lemma (proof)

Therefore, we have that

$$\max\{|\langle u_0, v\rangle| + |\langle u_1, v\rangle|^2 : v \in \mathcal{S}, u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\},$$

where $\mathcal{S}_0$ is unit sphere of $\mathrm{im}(\Pi_0)$ (similarly for $\mathcal{S}_1$ and Π_1). By Cauchy-Schwarz, we have that

$$|\langle u_0, v\rangle|^2 = \| u_0 \| \| v \|$$

(similarly for u_1 and v). Equality is achieved since u_0 and v are linearly dependent. Since v is a unit eigenvector of $u_0 u_0^* + u_1 u_1^*$ that corresponds to $\lambda_{\max}$ we have that

$$v^*(u_0 u_0^* + u_1 u_1^*)v = \| u_0 u_0^* + u_1 u_1^* \|.$$

Again using definition of norm

$$\max\{v^*(u_0 u_0^* + u_1 u_1^*)v : v \in \mathcal{S}, u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\}.$$

A key lemma (proof)

Therefore

$$\max\{\| u_0 u_0^* + u_1 u_1^* \| : u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\}.$$

A key lemma (proof)

Therefore

$$\max\{\| u_0 u_0^* + u_1 u_1^* \| : u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\}.$$

Note that for every unit vector $u_0, u_1 \in \mathbb{C}^n$ it holds that

$$\| u_0 u_0^* + u_1 u_1^* \| = 1 + |\langle u_0, u_1 \rangle|,$$

since $u_0 u_0^* + u_1 u_1^*$ has at most two nonzero eigenvalues of $1 \pm |\langle u_0, u_1 \rangle|$. Therefore

$$\max\{1 + |\langle u_0, u_1 \rangle| : u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\}.$$

A key lemma (proof)

Therefore

$$\max\{\| u_0 u_0^* + u_1 u_1^* \| : u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\}.$$

Note that for every unit vector $u_0, u_1 \in \mathbb{C}^n$ it holds that

$$\| u_0 u_0^* + u_1 u_1^* \| = 1 + |\langle u_0, u_1 \rangle|,$$

since $u_0 u_0^* + u_1 u_1^*$ has at most two nonzero eigenvalues of $1 \pm |\langle u_0, u_1 \rangle|$. Therefore

$$\max\{1 + |\langle u_0, u_1 \rangle| : u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\}.$$

Finally, since

$$\| A \|_p = \max\{|\langle B, A \rangle| : B \in \mathrm{L}(\mathcal{X}, \mathcal{Y}),\ \| B \|_{p^*} \leq 1\},$$

it holds that

$$1 + \| \Pi_0 \Pi_1 \|.$$