Monogamy-of-entanglement games

Nonlocal Games Seminar

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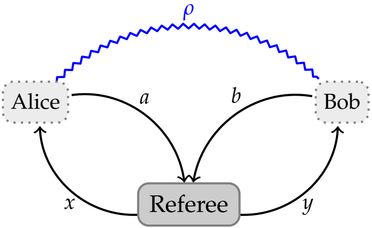
Outline

Nonlocal games

Nonlocal games

A *nonlocal game* is a cooperative game played between *Alice* and *Bob* against a *referee*.



1. Question and answer sets: $(\Sigma_\mathsf{A}, \Sigma_\mathsf{B})$ and $(\Gamma_\mathsf{A}, \Gamma_\mathsf{B})$
2. Distributions on question pairs: $\pi : \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to [0,1]$
3. A predicate $V : \Gamma_\mathsf{A} \times \Gamma_\mathsf{B} \times \Sigma_\mathsf{A} \times \Sigma_\mathsf{B} \to \{0,1\}$, where

$$V(a,b|x,y) = \begin{cases} 1 & \text{if Alice and Bob win} \\ 0 & \text{if Alice and Bob lose} \end{cases},$$

Strategies for nonlocal games

Alice and Bob could use different types of *strategies*:

- *Classical strategies:* Alice and Bob answer deterministically, determined by functions of $x \in \Sigma_A$ and $y \in \Sigma_B$.

- *Quantum strategies:* Alice and Bob share a joint quantum system $\rho \in \mathrm{D}(\mathcal{A} \otimes \mathcal{B})$ and allow their answers to be outcomes of measurements on this shared system.

- *Commuting measurement strategies:* Alice and Bob share a quantum system over a single Hilbert space $\rho \in \mathrm{D}(\mathcal{H})$ and allow their answers to be outcomes of measurements on this system.

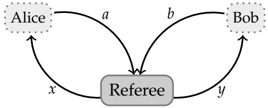
- *Non-signaling strategies:* No instantaneous communication between parties.

Example: The CHSH game

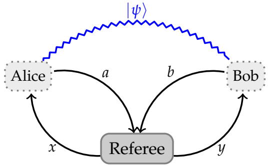
The CHSH game (G_{CHSH}). Winning condition iff $a \oplus b = x \wedge y$.

$$\omega(G_{\text{CHSH}}) < \omega^*(G_{\text{CHSH}})$$

- $\omega(G_{\text{CHSH}}) = \frac{3}{4} = 0.75$:



- $\omega^*(G_{\text{CHSH}}) = \cos^2(\frac{\pi}{8}) \approx 0.8536$:



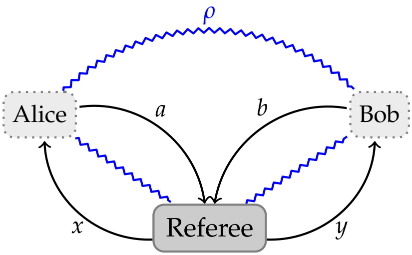
Demo Time: CHSH Game in QETLAB

CHSH_GAME.M

Extended nonlocal games

Extended nonlocal games

An *extended nonlocal game* is a nonlocal game where the *referee also holds a quantum system* the he measures provided by Alice and Bob.



1. Question and answer sets (Σ_A, Σ_B) and (Γ_A, Γ_B).
2. Distribution on question pairs: $\pi : \Sigma_A \times \Sigma_B \to [0, 1]$.
3. A measurement operator $V : \Gamma_A \times \Gamma_B \times \Sigma_A \times \Sigma_B \to \mathrm{Pos}(\mathcal{R})$.

Extended nonlocal games: Winning and losing probabilities

At the end of the protocol, the referee has:

1. The state at the end of the protocol:

$$\rho_{a,b}^{x,y} \in \mathrm{D}(\mathcal{R}).$$

2. A measurement the referee makes on its part of the state ρ:

$$V(a, b|x, y) \in \mathrm{Pos}(\mathcal{R}).$$

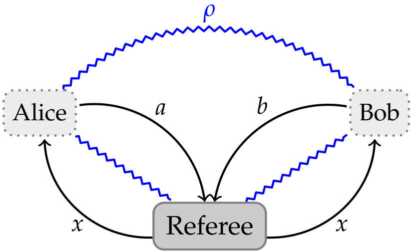
The respective winning and losing probabilities are given by

$$\left\langle V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle \quad \text{and} \quad \left\langle \mathbb{1} - V(a, b|x, y), \rho_{a,b}^{x,y} \right\rangle.$$

Monogamy-of-entanglement games

Monogamy-of-entanglement games

Monogamy-of-entanglement games¶, are a special type of extended nonlocal game.



1. Same question and answer sets: $\Sigma = \Sigma_A = \Sigma_B$ and $\Gamma = \Gamma_A = \Gamma_B$.
2. Alice and Bob get the same question: $\pi(x, y) = 0$ for $x \neq y$.
3. Referee's measurement operator: $R : \Sigma \times \Gamma \to \mathrm{Pos}(\mathcal{R})$.
4. Winning condition: Iff Alice's output, Bob's output, and the referee's measurement output are the *same*.

¶[Tomamichel, Fehr, Kaniewski, Wehner, (2013)]

Standard quantum strategies for monogamy-of-entanglement games

- The expected *pay-off* for a monogamy-of-entanglement game, G using a standard quantum strategy is:

$$\sum_{x \in \Sigma} \pi(x) \sum_{a \in \Gamma} \left\langle R(a|x) \otimes A_a^x \otimes B_a^x, \rho \right\rangle.$$

- Since ρ just needs to be a valid density matrix, we use convexity to assume that ρ is pure (rank-one):

$$\omega^*(G) = \left\| \sum_{x \in \Sigma} \pi(x) \sum_{a \in \Gamma} R(a|x) \otimes A_a^x \otimes B_a^x \right\|$$

Unentangled strategies for monogamy-of-entanglement games

Alice and Bob only win when their outputs agree, and we assume that the measurements of the referee are positive semidefinite (from the definition for monogamy-of-entanglement games).

- For any monogamy-of-entanglement game, G, the unentangled value is:

$$\omega(G) = \max_{f:\Sigma \to \Gamma} \left\| \sum_{x \in \Sigma} \pi(x) R(f(x)|x) \right\|.$$

for choice of measurements $\{A_a^x\}$ for Alice and $\{B_b^y\}$ for Bob.

The BB84 monogamy-of-entanglement game

The BB84 game (G_{BB84} for short)[¶] is defined by:

1. Question and answer sets:

$$\Sigma = \Gamma = \{0, 1\},$$

2. Uniform probability for questions:

$$\pi(0) = \pi(1) = \frac{1}{2}$$

3. Measurements defined by the BB84 bases:

$$\begin{array}{lll} \text{For } x = 0: & R(0|0) = |0\rangle\langle 0|, & R(1|0) = |1\rangle\langle 1| \\ \text{For } x = 1: & R(0|1) = |+\rangle\langle +|, & R(1|1) = |-\rangle\langle -| \end{array}$$

The *unentangled* and *standard quantum* values for G_{BB84} coincide:

$$\omega(G_{\text{BB84}}) = \omega^*(G_{\text{BB84}}) = \cos^2(\pi/8) \approx 0.8536$$

[¶] G_{BB84} was introduced in [Tomamichel, Fehr, Kaniewski, Wehner, (2013)].

Demo Time: BB84 Game

bb84_game.m

Exhaustive search over unentangled strategies

Consider the following SDP:

Primal Problem: (γ)

$$
\begin{aligned}
\text{max:} \quad & \sum_{x \in \Sigma} \pi(x) \left\langle R(f(x)|x), \rho \right\rangle \\
\text{s.t.:} \quad & \mathrm{Tr}(\rho) = 1, \qquad (\rho \text{ is pure}). \\
& \rho \geq 0, \qquad (\rho \text{ is PSD}).
\end{aligned}
$$

We cycle over all possible choices of $f(x) \rightarrow a$ and run the above SDP. The best we can do is represented by $\max(\gamma)$ over all such choices.

- Calculating $\max(\gamma)$ is now not an SDP, but for small values of $|\Sigma|$ and $|\Gamma|$, we can brute force over every possible combination to obtain the maximum.

Demo Time: Calculating the unentangled value

UNENTANGLED_MOE_2IN_2OUT.M

A natural question for monogamy-of-entanglement games

- *Question:* For any monogamy-of-entanglement game, G, is it true that the *unentangled* and *standard quantum* values always coincide? In other words is it true that:

  $$\omega(G) = \omega^*(G)$$

  for all monogamy-of-entanglement games G?

Demo Time: Random Monogamy Games

RANDOM_MOE_GAMES.M

A natural question for monogamy-of-entanglement games

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  for all monogamy-of-entanglement games G?

- *Answer:*
  - For certain cases: Yes.
  - In general: No.

$$\omega(G) = \omega^*(G)$$

In general No

Monogamy-of-entanglement games where $\omega(G) \neq \omega^*(G)$

There exists a monogamy-of-entanglement game, G, with $|\Sigma| = 4$ and $|\Gamma| = 3$ such that

$$\omega(G) < \omega^*(G).$$

1. Question and answer sets:

$$\Sigma = \{0, 1, 2, 3\}, \quad \Gamma = \{0, 1, 2\}.$$

2. Uniform probability for questions:

$$\pi(0) = \pi(1) = \pi(2) = \pi(3) = \frac{1}{4}.$$

3. Measurements defined by a mutually unbiased basis¶:

$$\{R(0|x), R(1|x), R(2|x)\}.$$

¶ $|u_x(a)^* u_{x'}(a)|^2 = 1/|\Gamma|$ for $R(a|x) = u_x(a)u_x(a)^*, R(a|x') = u_{x'}(a)u_{x'}(a)^*$

Monogamy-of-entanglement games where $\omega(G) \neq \omega^*(G)$

- An exhaustive search over all unentangled strategies reveals an optimal unentangled value:

$$\omega(G) = \frac{3 + \sqrt{5}}{8} \approx 0.6545.$$

- Alternatively, a computer search over standard quantum strategies and a heuristic approximation for the upper bound of $\omega^*(G)$ reveals that

$$2/3 \geq \omega^*(G) \geq 0.6609$$

This ability to compute upper bounds for extended nonlocal games is obtained from an adaptation of a technique known as the *NPA hierarchy.*

Demo Time: MUB game

mub_4_3_game.m

$$\omega(G) = \omega^*(G)$$

For certain classes, Yes.

Monogamy games that obey $\omega(G) = \omega^*(G)$

Theorem

For any monogamy-of-entanglement game, G, for which $|\Sigma| = 2$:

$$\omega(G) = \omega^*(G).$$

Proof: Monogamy games that obey $\omega(G) = \omega^*(G)$

Recall that for any monogamy-of-entanglement, G, the standard quantum value may be written as

$$\omega^*(G) = \left\| \lambda \sum_{a \in \Gamma} R(a|0) \otimes A_a^0 \otimes B_a^0 + (1 - \lambda) \sum_{b \in \Gamma} R(b|1) \otimes A_b^1 \otimes B_b^1 \right\|$$

Proof: Monogamy games that obey $\omega(G) = \omega^*(G)$

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Since $\| P \| \leq \| Q \|$ if $P \leq Q$ for any $P, Q \geq 0$:

$$\omega^*(G) \leq \left\| \lambda \sum_{a \in \Gamma} R(a|0) \otimes A_a^0 \otimes \mathbb{1}_{\mathcal{B}} + (1-\lambda) \sum_{b \in \Gamma} R(b|1) \otimes \mathbb{1}_{\mathcal{A}} \otimes B_b^1 \right\|$$

Proof: Monogamy games that obey $\omega(G) = \omega^*(G)$

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Since $\sum_{a \in \Gamma} A_a^x = \sum_{b \in \Gamma} B_b^y = \mathbb{1}$ the above quantity is equal to:

$$\left\| \lambda \sum_{(a,b) \in \Gamma} R(a|0) \otimes A_a^0 \otimes B_b^1 + (1-\lambda) \sum_{(a,b) \in \Gamma} R(b|1) \otimes A_a^0 \otimes B_b^1 \right\|.$$

Monogamy games that obey $\omega(G) = \omega^*(G)$

(Previous slide):

$$\left\| \lambda \sum_{(a,b)\in\Gamma} R(a|0) \otimes A_a^0 \otimes B_b^1 + (1-\lambda) \sum_{(a,b)\in\Gamma} R(b|1) \otimes A_a^0 \otimes B_b^1 \right\|.$$

Monogamy games that obey $\omega(G) = \omega^*(G)$

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Since $\{A_a^0 \otimes B_b^1 : a, b \in \Gamma\}$ are pairwise orthogonal projections ($\langle A_a^0 \otimes B_b^1, A_{a'} \otimes B_{b'} \rangle = 0$ for $a \neq a'$ and $b \neq b'$) we have

$$\left\| \sum_{(a,b)\in\Gamma} (\lambda R(a|0) + (1-\lambda) R(b|1)) \otimes A_a^0 \otimes B_b^1 \right\|.$$

Monogamy games that obey $\omega(G) = \omega^*(G)$

(Previous slide):

$$\left\| \lambda \sum_{(a,b)\in\Gamma} R(a|0) \otimes A_a^0 \otimes B_b^1 + (1-\lambda) \sum_{(a,b)\in\Gamma} R(b|1) \otimes A_a^0 \otimes B_b^1 \right\|.$$

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$$\left\| \sum_{(a,b)\in\Gamma} (\lambda R(a|0) + (1-\lambda) R(b|1)) \otimes A_a^0 \otimes B_b^1 \right\|.$$

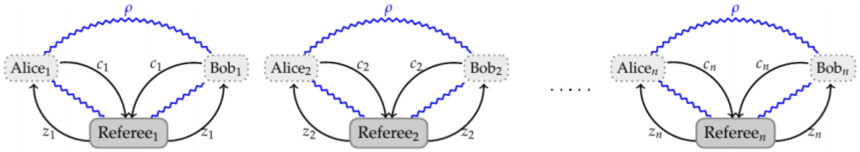
Every entangled strategy is equivalent to one where Alice and Bob use projective measurements so

$$\max_{a,b\in\Gamma} \left\| \lambda R(a|0) + (1-\lambda) R(b|1) \right\| = \omega(G).$$

Parallel Repetition of Monogamy-of-entanglement Games

Parallel repetition of monogamy-of-entanglement games

- *Parallel repetition:* Run a monogamy-of-entanglement game, G, for n times in parallel, denoted as G^n.
- *Strong parallel repetition:* $\omega(G^n) = \omega(G)^n$



Question: Do all monogamy-of-entanglement games obey strong parallel repetition?

Parallel repetition of monogamy-of-entanglement games

- Recall:

$$\omega(G_{\text{BB84}}) = \omega^*(G_{\text{BB84}}) = \cos^2(\pi/8) \approx 0.8536.$$

- G_{BB84} obeys strong parallel repetition¶:

$$\omega(G^n_{\text{BB84}}) = \omega^*(G^n_{\text{BB84}}) = \left(\cos^2(\pi/8)\right)^n.$$

¶[Tomamichel, Fehr, Kaniewski, Weher, (2013)] Demo Time: Strong parallel repetition of BB84

BB84_PARALLEL_REP.M

Strong parallel repetition for certain monogamy-of-entanglement games

Theorem (Johnston, Mittal, R, Watrous)

Let $G = (\pi, R)$ be a monogamy-of-entanglement game for which $\Sigma_{\mathsf{A}} = \{0, 1\}$, π is uniform over Σ_{A}, and $R(a|x)$ is a projection operator. It holds that

$$\omega^*(G^n) = \omega^*(G)^n = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n.$$

where $c(G)$ is the "maximal overlap of measurements" of the referee

$$c(G) = \max_{\substack{x,y \in \Sigma_{\mathsf{A}} \\ x \neq y}} \max_{a,b \in \Gamma_{\mathsf{A}}} \left\| \sqrt{R(a|x)}\sqrt{R(b|y)} \right\|^2$$

A key proposition and lemma

Proposition

Let $G = (\pi, R)$ be a monogamy-of-entanglement game for which $\Sigma = \{0, 1\}$, π is uniform over Σ, and $R(a|x)$ is a projection operator for each $x \in \Sigma$ and $a \in \Gamma$. It holds that

$$\omega(G) = \frac{1}{2} + \frac{1}{2} \max_{a,b \in \Gamma} \Big\| R(a|0) R(b|1) \Big\|.$$

Lemma

Let Π_0 and Π_1 be nonzero projection operators on $\mathbb{C}^n$. It holds that

$$\| \Pi_0 + \Pi_1 \| = 1 + \| \Pi_0 \Pi_1 \|.$$

Proof of proposition

Assuming the lemma stating $\|\Pi_0 + \Pi_1\| = 1 + \|\Pi_0\Pi_1\|$, we have that

$$\omega(G) = \max_{a,b\in\Gamma}\left\|\frac{R(a|0)+R(b|1)}{2}\right\| = \frac{1}{2} + \frac{1}{2}\max_{a,b\in\Gamma}\left\|R(a|0)R(b|1)\right\|.$$

The lower bound on $\omega(G)^*$ is obtained from [TFWK13][¶]. The upper bound follows from the fact that Alice and Bob can just play the optimal strategy for every n:

$$\omega^*(G^n) \geq \omega(G^n) \geq \left(\frac{1}{2} + \frac{1}{2}\max_{a,b\in\Gamma}\left\|R(a|0)R(b|1)\right\|\right)^n = \left(\frac{1}{2} + \frac{1}{2}\sqrt{c(G)}\right)^n.$$

[¶] [Tomamichel, Fehr, Kaniweski, Wehner (2013)]

Open questions

Unentangled vs. standard quantum strategies for monogamy-of-entanglement games

Inputs ($\|\Sigma\|$)	Outputs ($\|\Gamma\|$)	$\omega^*(G) = \omega(G)$	$\omega^*(G^n) = \omega^*(G)^n$	$\omega_{ns}(G^n) = \omega_{ns}(G)^n$
2	$\|\Gamma\| \geq 1$	yes	yes[¶]	no
3	$\|\Gamma\| \geq 1$	?	?	no
4	3	no	?	no

Question: What about $|\Sigma| = 3$?

- Proof technique fails for $|\Sigma| > 2$.
- Computational search:
  - Generate random monogamy-of-entanglement games where $|\Sigma| = 3$ and $|\Gamma| \geq 2$.
  - 10^8 random games generates, no counterexamples found.

[¶] So long as the measurements used by the referee are projective and the probability distribution, π, from which the questions are asked is uniform.

Supplementary material

Supplementary material:
Extended nonlocal games

Winning probability for standard quantum strategies

The winning probability is given by the following equation:

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \frac{\langle V(a,b|x,y), \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{B}} (\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)\rangle}{\mathrm{Tr}(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)} \mathrm{Tr}(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y)\rho)$$

The probabilities cancel giving

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \mathrm{Tr}\left(V(a,b|x,y)\, \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{B}} \left(\mathbb{1}_{\mathcal{R}} \otimes A_a^x \otimes B_b^y\right) \rho\right)$$

The trace operator slips past the $\mathbb{1}_{\mathcal{R}}$ giving

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \mathrm{Tr}\left(V(a,b|x,y) \otimes (A_a^x \otimes B_b^y)\rho\right)$$

Writing the trace in terms of the inner product, we have that

$$\sum_{x,y} \pi(x,y) \sum_{a,b} \left\langle V(a,b|x,y) \otimes A_a^x \otimes B_b^y, \rho \right\rangle.$$

Measurements may be assumed to be projective

WLOG, we may assume that Alice and Bob's measurements are projective since:

- Alice and Bob may extend the sizes of their Hilbert spaces,
- Naimark's theorem states that any strategy using non-projective measurements can be simulated by a strategy with projective measurements.

Supplementary material:
Monogamy-of-entanglement games

Pure strategies are sufficient

Recall that the pay-off for a standard quantum strategy is given by

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} \left\langle V(a,b|x,y)\otimes A_a^x \otimes B_b^y, \rho \right\rangle.$$

WLOG, ρ can be assumed to be pure and the measurement operators projective since one could increase the dimension of the Hilbert space, i.e.:

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} u^* \left(V(a,b|x,y)\otimes A_a^x \otimes B_b^y\right) u,$$

where $\rho = uu^*$ with $u \in \mathcal{R}\otimes\mathcal{A}\otimes\mathcal{B}$. This follows from Naimark's theorem (next slide).

Pure strategies are sufficient: Naimark's theorem

Let $u \in \mathcal{R} \otimes \mathcal{A} \otimes \mathcal{B}$, $w_0 \in \mathbb{C}^{\Gamma_A}$. Define

$$v = u \otimes w_0 \in \mathcal{R} \otimes \mathcal{A} \otimes \mathcal{B} \otimes \mathbb{C}^{\Gamma_A} \quad \text{and} \quad \widetilde{A}_a^x = U^* \left(\mathbb{1}_{\mathcal{A}} \otimes E_{a,a} \right) U,$$

where $U \in \mathrm{U}(\mathcal{A} \otimes \mathbb{C}^{\Gamma_A})$ such that

$$U = w \otimes w_0 \to \sum_{a \in \Gamma_A} \sqrt{A_a^x} w \otimes e_a \qquad (\forall w \in \mathcal{A}).$$

It holds that $\widetilde{A}_a^x \in \mathrm{Proj}(\mathcal{A})$ and hence

$$\widetilde{A}_a^x v = U^* \left(\mathbb{1}_{\mathcal{A}} \otimes E_{a,a} \right) U \left(u \otimes w_0 \right) = U^* \sqrt{A_a^x} u \otimes e_a,$$

and $\widetilde{A}_a^x = \left(\widetilde{A}_a^x \right)^* \left(\widetilde{A}_a^x \right)$. Note¶.

¶A similar argument holds for Bob's measurements.

The BB84 game: Unentangled value

For any monogamy-of-entanglement game, G, the unentangled value is:

$$\omega(G) = \max_{f:\Sigma\to\Gamma} \left\| \sum_{x\in\Sigma} \pi(x) R(f(x)|x) \right\|$$

Recall, G_{BB84} has

$$\pi(0) = \pi(1) = 1/2 \quad \text{and} \quad \Sigma = \Gamma = \{0, 1\}.$$

So we have that:

$$\begin{aligned}
\omega(G_{\mathrm{BB84}}) &= \max_{f:\Sigma\to\Gamma} \left\| \frac{1}{2} R(f(x)|0) + \frac{1}{2} R(f(x)|1) \right\| \\
&= \frac{1}{2} \left\| |0\rangle\langle 0| + |+\rangle\langle +| \right\| + \frac{1}{2} \left\| |1\rangle\langle 1| + |-\rangle\langle -| \right\| \\
&= \frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos^2(\pi/8).
\end{aligned}$$

The BB84 game: Standard quantum value

For any monogamy-of-entanglement game, G, the standard quantum value is:

$$\omega^*(G) = \left\| \sum_{x \in \Sigma} \pi(x) \sum_{a \in \Gamma} R(a|x) \otimes A_a^x \otimes B_a^x \right\|.$$

1. Alice and Bob send the following state to the referee:

$$v_\pm = \cos(\pi/8)|0\rangle \pm \sin(\pi/8)|1\rangle.$$

2. Alice and Bob always output $a = 0$ and have measurements:

$$A_0^0 = v_+ v_+^*, \quad A_0^1 = v_- v_-^*, \quad B_0^0 = B_0^1 = \mathbb{1}$$

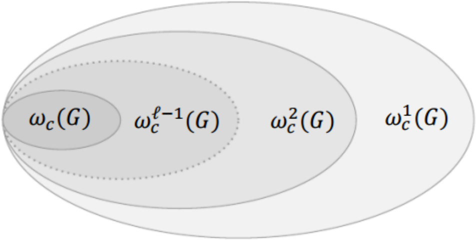
Plugging this into the standard quantum value formula:

$$\omega^*(G) = \left\| \frac{1}{2} \left(|0\rangle\langle 0| \otimes A_0^0 \otimes B_0^0 + |1\rangle\langle 1| \otimes A_0^0 \otimes B_0^0 \right) + \right.$$
$$\left. \frac{1}{2} \left(|+\rangle\langle +| \otimes A_0^1 \otimes B_0^1 + |-\rangle\langle -| \otimes A_0^1 \otimes B_0^1 \right) \right\| = \cos^2(\pi/8)$$

Supplementary material:
Upper bounds for extended nonlocal games

Upper bounds for nonlocal games

- The NPA hierarchy[¶] is a method of placing *upper bounds* on the *quantum value* of nonlocal games.
- Hierarchy of semidefinite programs is *guaranteed* to converge to the commuting measurement value for some finite level, ℓ of the hierarchy.
- The commuting measurement value is an upper bound on the quantum value, $\omega^*(G) \leq \omega_c(G)$, for all nonlocal games, G.



¶[Navascués, Pironio, and Acín, (2008)]

NPA hierarchy: Level 1

For a nonlocal game, the pay-off for a commuting measurement strategy is¶

$$\sum_{(x,y)\in\Sigma_\mathsf{A}\times\Sigma_\mathsf{B}} \pi(x,y) \sum_{(a,b)\in\Gamma_\mathsf{A}\times\Gamma_\mathsf{B}} V(a,b|x,y)\langle A_a^x B_b^y, \rho\rangle.$$

¶ for $[A_a^x, B_b^y] = 0$, $\{A_a^x\}, \{B_b^y\} \subset \mathrm{Pos}(\mathcal{H})$, and $\rho \in \mathrm{D}(\mathcal{H})$.

NPA hierarchy: Level 1

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Define a Gram matrix with entries:

$$C((x,a),(y,b)) = \langle A_a^x B_b^y, \rho\rangle.$$

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The full block-matrix is

$$C = \left(\begin{array}{c|c} \langle A_a^x A_{a'}^{x'}, \rho\rangle & \langle A_a^x B_b^y, \rho\rangle \\ \hline \langle B_b^y A_a^x, \rho\rangle & \langle B_b^y B_{b'}^{y'}, \rho\rangle \end{array}\right)$$

- If the entries in C come from a commuting measurement strategy, C will satisfy *certain properties*.
- These properties are verifiable via an SDP.

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Matrix constraints

As mentioned, the matrix C satisfies a number of constraints:

- It is positive semidefinite (by definition from the fact that it is a Gram matrix)
- Normalization: $C(1,1) = 1$.
- Commutation:

$$C((x,a),(y,b)) = C((y,b),(x,a))$$

- Measurements sum to identity:

$$\sum_{(x,a)} C((x,a),(y,b)) = C(1,(y,b))$$
$$\sum_{(y,b)} C((x,a),(y,b)) = C((x,a),1).$$

- Measurements are projective:

$$C(1,(y,b)) = C((y,b),(y,b))$$
$$C((x,a),1) = C((x,a),(x,a))$$

Pseudo commuting measurement assemblages

If these properties are satisfied, we say that C is a 1-*st order pseudo commuting measurement assemblage*.

- By imposing more structure on this matrix, we get closer to the set of commuting measurement operators.
- Indexing correspondence between strings and operators:

$$A_a^x \leftrightarrow (x, a) \quad \text{and} \quad A_{a_1}^{x_1} \cdots A_{a_n}^{x_n} \leftrightarrow (x_1, a_1) \cdots (x_n, a_n)$$

Analogous for B_b^y operators.

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  Analogous for B_b^y operators.

Define the alphabets

$$\Delta_{\mathrm{A}}^{\leq \ell} = \{\epsilon\} \cup (\Sigma_{\mathrm{A}} \times \Gamma_{\mathrm{A}})^{\leq \ell} \quad \text{and} \quad \Delta_{\mathrm{B}}^{\leq \ell} = \{\epsilon\} \cup (\Sigma_{\mathrm{B}} \times \Gamma_{\mathrm{B}})^{\leq \ell}.$$

The ℓ-th order pseudo commuting measurement assemblage is

$$C^{(\ell)} = \left(\begin{array}{c|c} \Delta_{\mathrm{A}}^{\leq \ell} \cup \Delta_{\mathrm{A}}^{\leq \ell} & \Delta_{\mathrm{A}}^{\leq \ell} \cup \Delta_{\mathrm{B}}^{\leq \ell} \\ \hline \Delta_{\mathrm{B}}^{\leq \ell} \cup \Delta_{\mathrm{A}}^{\leq \ell} & \Delta_{\mathrm{B}}^{\leq \ell} \cup \Delta_{\mathrm{B}}^{\leq \ell} \end{array} \right)$$

NPA hierarchy theorem

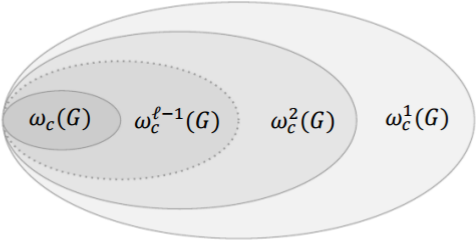
For some finite level ℓ, the pay-off for a commuting measurement strategy for a nonlocal game can be defined by

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B}\pi(x,y)\sum_{(a,b)\in\Gamma_A\times\Gamma_B}V(a,b|x,y)C^{(\ell)}.$$

Upper bounds for extended nonlocal games

Extended NPA hierarchy:

- Uses the same idea as the NPA hierarchy. (For $\dim(\mathcal{R}) = 1$, the NPA hierarchy is a special case.)
- Enables one to compute *upper bounds* on the *standard quantum value* for *extended nonlocal games*.
- Same idea as before, only now we need to take into account the actions of the referee.



Commuting measurement strategies

A *commuting measurement strategy* consists of a finite-dimensional complex Euclidean space $\mathcal{H}$ as well as the following:

- Shared state: $\rho \in \mathcal{R} \otimes \mathcal{H}$.
- Measurements: $\{A_a^x\} \subset \mathrm{Pos}(\mathcal{H}), \quad \{B_b^y\} \subset \mathrm{Pos}(\mathcal{H})$, where $[A_a^x, B_b^y] = 0$ for all x, y, a, b.

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The expected *pay-off* for a commuting measurement strategy is given by:

$$\sum_{(x,y) \in \Sigma_A \times \Sigma_B} \pi(x, y) \sum_{(a,b) \in \Gamma_A \times \Gamma_B} \left\langle V(a, b|x, y) \otimes A_a^x B_b^y, \rho \right\rangle$$

The *commuting measurement value*, denoted as $\omega_c(G)$, is the supremum of the pay-off over all commuting measurement strategies.

Extended NPA hierarchy

Same idea, but now we're taking into account the referee, and therefore have a larger matrix.
For each ℓ, now consider block matrices

$$M^{(\ell)} = \begin{pmatrix} M^{(\ell)}_{1,1} & \cdots & M^{(\ell)}_{1,m} \\ \vdots & \ddots & \vdots \\ M^{(\ell)}_{m,1} & \cdots & M^{(\ell)}_{m,m} \end{pmatrix}$$

where each block takes the form $M^{(\ell)}_{i,j} : \Sigma^{\leq \ell} \times \Sigma^{\leq \ell} \to \mathbb{C}$.

- Each submatrix has similar properties to what we saw for the NPA hierarchy.
- The overall matrix also has some structure, which is unique to this case.

Assemblages

- Another natural way to think about the commuting measurement value is in terms of *assemblages*:

$$K(a, b|x, y) = \mathrm{Tr}_{\mathcal{H}}\left((\mathbb{1}_{\mathcal{R}} \otimes A_a^x B_b^y)\rho\right)$$

- For a particular choice of x, y, a, b, an assemblage corresponds to the *unnormalized state* in the referee's hands at the end of the game.
- The function K completely determines the performance of Alice and Bob's strategy:

$$\sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x, y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} \Big\langle V(a, b|x, y), K(a, b|x, y)\Big\rangle.$$

Extended NPA hierarchy theorem

Let $\dim(\mathcal{R}) = m$. Then

$$\Big\langle V(a,b|x,y), K(a,b|x,y) \Big\rangle = \Big\langle V(a,b|x,y), M^{(\ell)}((x,a),(y,b)) \Big\rangle$$

for all $m, \ell \geq 1$, $(x,y) \in \Sigma_{\mathsf{A}} \times \Sigma_{\mathsf{B}}$ and $(a,b) \in \Gamma_{\mathsf{A}} \times \Gamma_{\mathsf{B}}$.

Supplementary material:
Lower bounds for extended nonlocal games

Lower bounds for extended nonlocal games

Key idea: Fixing measurements on one system yields the optimal measurements of the other system via an SDP¶

¶[Liang and Doherty (2007)]

Lower bounds for extended nonlocal games

Key idea: Fixing measurements on one system yields the optimal measurements of the other system via an SDP¶

Iterative "see-saw" algorithm between two SDPs:

- SDP-1: Fix Bob's measurements. Optimize over Alice's measurements.
- SDP-2: Fix Alice's measurements (from SDP-1). Optimize over Bob's measurements.
- Repeat.

Not guaranteed to give optimal value, as the algorithm can get stuck in a local minimum.

¶[Liang and Doherty (2007)]

Lower bounds for extended nonlocal games

Define $\{\rho_a^x : x \in \Sigma_A,\ a \in \Gamma_A\} \subset \mathrm{Pos}(\mathcal{R} \otimes \mathcal{B})$ as the residual states acting on the referee and Bob's systems and let

$$f = \sum_{(x,y)\in\Sigma_A\times\Sigma_B} \pi(x,y) \sum_{(a,b)\in\Gamma_A\times\Gamma_B} \left\langle V(a,b|x,y) \otimes B_b^y, \rho_a^x \right\rangle$$

<u>Lower bound (SDP-1)</u>

$$\begin{aligned} \text{max:} \quad & f \\ \text{s.t.:} \quad & \sum_{a\in\Gamma_A} \rho_a^x = \tau, \\ & \tau \in \mathrm{D}(\mathcal{R} \otimes \mathcal{B}). \end{aligned}$$

<u>Lower bound (SDP-2)</u>

$$\begin{aligned} \text{max:} \quad & f \\ \text{s.t.:} \quad & \sum_{b\in\Gamma_B} B_b^y = \mathbb{1}_{\mathcal{B}}, \\ & B_b^y \in \mathrm{Pos}(\mathcal{B}). \end{aligned}$$

- Iterate between SDP-1 and SDP-2 until desired numerical precision is reached.

Supplementary material:
Parallel repetition for
monogamy-of-entanglement games

A key lemma (proof)

By definition

$$\|\Pi_0 + \Pi_1\| = \max\{v^*(\Pi_0 + \Pi_1)v : v \in \mathcal{S}\}.$$

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Since $v \in \mathcal{S}$ are unit vectors,

$$v^*(\Pi_0 + \Pi_1)v = v^*\Pi_0 v + v^*\Pi_1 v = \|\Pi_0\|^2 + \|\Pi_1\|^2.$$

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Recall the definition of the ∞-norm for some operator $A \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$:

$$\| A \| = \max\{\| Av \| : v \in \mathcal{S}(\mathcal{X})\}.$$

We can then rewrite our expression as

$$\max\{\| \Pi_0 v \|^2 + \| \Pi_1 v \|^2 : v \in \mathcal{S}\}.$$

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Define unit vectors $u_0, u_1 \in \mathcal{S}(\mathbb{C}^n)$, we can write

$$\Pi_0 = u_0 u_0^* \quad \text{and} \quad \Pi_1 = u_1 u_1^*.$$

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Therefore

$$\Pi_0 v = u_0 u_0^* v = \langle u_0, v \rangle u_0$$

and that

$$\|\Pi_0 v\|^2 = |\langle u_0, v \rangle|^2 \|u_0\| = |\langle u_0, v \rangle|^2$$

(and similarly for Π_1 and u_1).

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Therefore, we have that

$$\max\{|\langle u_0, v\rangle| + |\langle u_1, v\rangle|^2 : v \in \mathcal{S}, u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\},$$

where $\mathcal{S}_0$ is unit sphere of $\mathrm{im}(\Pi_0)$ (similarly for $\mathcal{S}_1$ and Π_1).

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Again using definition of norm

$$\max\{v^*(u_0 u_0^* + u_1 u_1^*)v : v \in \mathcal{S}, u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\}.$$

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Note that for every unit vector $u_0, u_1 \in \mathbb{C}^n$ it holds that

$$\| u_0 u_0^* + u_1 u_1^* \| = 1 + |\langle u_0, u_1 \rangle|,$$

since $u_0 u_0^* + u_1 u_1^*$ has at most two nonzero eigenvalues of $1 \pm |\langle u_0, u_1 \rangle|$. Therefore

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$$\max\{1 + |\langle u_0, u_1 \rangle| : u_0 \in \mathcal{S}_0, u_1 \in \mathcal{S}_1\}.$$

Finally, since

$$\|A\|_p = \max\{|\langle B, A \rangle| : B \in \mathrm{L}(\mathcal{X}, \mathcal{Y}),\ \| B \|_{p^*} \leq 1\},$$

it holds that

$$1 + \| \Pi_0 \Pi_1 \|.$$