Optimal discrimination of noisy Bell states by local operations and classical communication requires maximal entanglement

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June 17, 2021

Abstract

Entangled states are useful resources for the task of quantum state discrimination by local operations and classical communication (LOCC). For example, a Bell state is necessary (and sufficient) to perfectly discriminate a set of either three or four Bell states by LOCC. In this paper it is proved that a Bell state is also required for optimal discrimination of a family of sets of noisy Bell states by LOCC. Notably, such sets, in general, do not contain any maximally entangled state, and in specific instances, do not even contain any entangled state.

1 Introduction

The paradigm of *local operations and classical communication* (LOCC) is of particular importance in quantum information theory [1]. LOCC protocols involve two or more parties sharing a composite quantum system who perform arbitrary quantum operations on the local subsystems and communicate only via classical channels. Note that quantum communication is not allowed between the parties. The framework of LOCC provides a natural way to study the resource theory of quantum entanglement [2], the nonlocal properties of quantum systems [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], and applications thereof [14, 15, 16, 17, 18, 19, 20, 21, 22].

Local state discrimination

One problem that has been extensively studied within the framework of LOCC is discrimination of quantum states [8, 9, 10, 12, 13, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 36, 38, 39,

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41, 43, 45]. The problem may be briefly described as follows. Let $\mathcal{E} = \{(p_i, \rho_i) : i = 1, \ldots, N\}$ be an ensemble of k-partite quantum states $\rho_1, \ldots, \rho_N$ with associated probabilities $p_1, \ldots, p_N$, where $k, N \geq 2$. Now suppose that k separated parties share a quantum system prepared in a state chosen from $\mathcal{E}$. The parties do not know the identity of the state but they do know from which ensemble the state has been chosen, so they have complete knowledge of $\mathcal{E}$. The goal is to gain as much knowledge about the state of the system by means of LOCC. For example, if the given states are mutually orthogonal, then they wish to find *which state* the system is in without error. This is evidently a state discrimination problem in which the allowed measurements are only those that are implementable by LOCC. So the question of interest here is the following: For a given set of states, does there exist a LOCC measurement that discriminates the states just as well as the best possible measurement that may be performed on the whole system?

In general, how well a given set of states can be discriminated can be quantified by the success probability for minimum-error state discrimination[1]. The success probability is the optimized value of the average probability of success, where the optimization is either over all measurements or some specific class of measurements. Thus for a given ensemble $\mathcal{E}$, let $p(\mathcal{E})$ and $p_{\mathrm{L}}(\mathcal{E})$ [42] denote the success probability (global optimum) and the local success probability (local optimum), respectively, where the corresponding optimizations are taken over all measurements and LOCC measurements.

Let us now come back to the question of whether the global optimum for a given set of states is always achievable by LOCC. The answer turns out to be "no" in general, i.e., there exist sets that cannot be optimally discriminated by LOCC, even if the states are all pure and mutually orthogonal. Once the initial results [8, 23] established this fact, most of the subsequent works were devoted towards identifying and characterizing the sets for which the global optimum is achievable by LOCC [24, 25, 32] and for which it is not (e.g., [9, 26, 29, 30, 31, 32]). For example, two pure states can be optimally discriminated by LOCC [24, 25] but an entangled orthogonal basis, such as the Bell basis, cannot be [10, 26, 29, 32]. So given a set of states that cannot be optimally discriminated by LOCC one must, therefore, consider using quantum entanglement as a resource for optimal discrimination.

Entanglement as a resource for local state discrimination

The limitations of LOCC protocols in discriminating quantum states can be overcome with shared entanglement used as a resource [46, 47, 48, 49, 50, 51]. Consider a simple example: The Bell

[1]There are other state discrimination measures as well, e.g., *fidelity* (see, [42] for discussions in this context).

basis $\mathcal{B}$, which is defined by the four Bell states,

$$
\begin{aligned}
|\Psi_1\rangle &= \tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \qquad & |\Psi_2\rangle &= \tfrac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right), \\
|\Psi_3\rangle &= \tfrac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right), \qquad & |\Psi_4\rangle &= \tfrac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right),
\end{aligned} \tag{1}
$$

cannot be perfectly discriminated by a LOCC measurement, even though the states are mutually orthogonal [26]. In particular, assuming the states are equally probable, one has (see e.g., [42])

$$
p_{\mathrm{L}}\left(\mathcal{B}\right) = \frac{1}{2}. \tag{2}
$$

Now suppose the parties are given a two-qubit ancillary state

$$
|\tau_\varepsilon\rangle = \sqrt{\frac{1+\varepsilon}{2}}\,|00\rangle + \sqrt{\frac{1-\varepsilon}{2}}\,|11\rangle \tag{3}
$$

for some $\varepsilon \in [0,1]$, where $|\tau_\varepsilon\rangle$ is entangled for $0 \leq \varepsilon < 1$. Once again assuming the Bell states are equiprobable, the local success probability for discriminating the set of states

$$
\mathcal{B} \otimes \tau_\varepsilon = \left\{|\Psi_i\rangle \otimes |\tau_\varepsilon\rangle : i = 1, \ldots, 4\right\}
$$

is given by [46]

$$
p_{\mathrm{L}}\left(\mathcal{B} \otimes \tau_\varepsilon\right) = \frac{1}{2}\left(1 + \sqrt{1-\varepsilon^2}\right) \tag{4}
$$

for all $\varepsilon \in [0,1]$. This value is achievable by a teleportation protocol[2]. Observe that the presence of $|\tau_\varepsilon\rangle$ enhances the local success probability; in particular, the more entangled $|\tau_\varepsilon\rangle$, the higher the local success probability. But perfect discrimination is possible if and only if $\varepsilon = 0$, i.e., when $|\tau_\varepsilon\rangle$ is maximally entangled.

Optimal resource states and entanglement cost

Suppose the states of a given bipartite or multipartite ensemble $\mathcal{E} = \{(p_i, \rho_i) : i = 1, \ldots, N\}$ cannot be optimally discriminated by LOCC. Let $\tau = |\tau\rangle\langle\tau|$ be a bipartite or multipartite ancilla

[2]The teleportation protocol is the following: Alice teleports her part of the unknown two-qubit state to Bob using $|\tau_\varepsilon\rangle$. After teleportation, Bob performs a two-qubit measurement to discriminate the states. If $|\tau_\varepsilon\rangle$ is maximally entangled, this protocol is guaranteed to achieve the global optimum for any set of states.

state such that

$$p_{\mathrm{L}}\left(\mathcal{E}\otimes\tau\right) > p_{\mathrm{L}}\left(\mathcal{E}\right), \tag{5}$$

where $\mathcal{E}\otimes\tau = \{(p_i, \rho_i\otimes\tau) : i = 1,\ldots,N\}$. We then say that $|\tau\rangle$ is a resource for discriminating the states of $\mathcal{E}$.

But ultimately though the goal is to find a $|\tau\rangle$ that enables optimal discrimination of $\mathcal{E}$ by LOCC and is also minimal in entanglement. Satisfying the first condition is easy because one can always use maximally entangled pair(s) and employ the teleportation protocol. For example, any set from $\mathbb{C}^d\otimes\mathbb{C}^d$ (see [50, 51] for discussions on multipartite systems) can be optimally discriminated using LOCC and a $\mathbb{C}^d\otimes\mathbb{C}^d$ maximally entangled state as a resource. But finding a $|\tau\rangle$ that not only discriminates the states optimally but is also minimal in entanglement is hard. That is because the teleportation protocol using a maximally entangled state or states may not be the most efficient strategy all the time, as one might do just as well with a clever protocol that consumes less entanglement.

For a given ensemble $\mathcal{E}$, we say that $|\tau\rangle\in\mathcal{H}_\tau$ is an *optimal* resource if it enables optimal discrimination of the states of $\mathcal{E}$ by LOCC, i.e., $p_{\mathrm{L}}\left(\mathcal{E}\otimes\tau\right) = p\left(\mathcal{E}\right)$, and is minimal in both entanglement and dimension [51]: for any other $|\tau'\rangle\in\mathcal{H}_{\tau'}$ satisfying $p_{\mathrm{L}}\left(\mathcal{E}\otimes\tau'\right) = p\left(\mathcal{E}\right)$, $E\left(\tau\right)\leq E\left(\tau'\right)$, where E is the entanglement entropy [2] and $\dim\left(\mathcal{H}_\tau\right)\leq\dim\left(\mathcal{H}_{\tau'}\right)$. The entanglement of an optimal resource is said to be the *entanglement cost* of discriminating the states under consideration. For example, a maximally entangled state is optimal for discriminating a maximally entangled basis on $\mathbb{C}^d\otimes\mathbb{C}^d$ by LOCC[3], so the entanglement cost here is $\log_2 d$ ebits. However, a maximally entangled state is not always an optimal resource: a two-qubit ensemble consisting of eight pure entangled states can be optimally discriminated by LOCC using a nonmaximally entangled state [47].

Motivation

In entanglement-assisted local state discrimination, we are mainly interested in finding the entanglement cost of discriminating the states (or simply, entanglement cost) of some given ensemble. So which factors determine the entanglement cost for a given ensemble? For a bipartite orthonormal basis, the average entanglement of the basis vectors provides a lower bound [47, 48]. This lower bound can be improved upon for almost all two-qubit entangled bases and was shown to be strictly greater than the average entanglement of the basis vectors [48]. However, finding the

[3]This follows from the argument that perfect discrimination of such a basis would lead to distillation of a $\mathbb{C}^d\otimes\mathbb{C}^d$ maximally entangled state across a bipartition where there was no entanglement to begin with [26, 29].

exact value for them remains an open problem.

The exact entanglement cost, however, is known for a few ensembles. The entanglement cost is 1 ebit for the Bell basis [26], a set of three Bell states [46], and a set containing a Bell state plus its orthogonal complement [44], but is less than an ebit for a set of nonorthogonal pure states and is given by their average entanglement [47]. In higher dimensions, to the best of our knowledge, exact results are known a maximally entangled basis and a set containing a maximally entangled state and its orthogonal complement [44].

Generally speaking, finding the entanglement cost, which is equivalent to the problem of finding an optimal resource state, for an arbitrary ensemble $\mathcal{E}$ seems quite hard even if the states are from $\mathbb{C}^2 \otimes \mathbb{C}^2$, the smallest composite state space. Alternatively, one might ask: For which bipartite ensembles a maximally entangled state is necessary for optimal discrimination by LOCC? Now the ensembles from $\mathbb{C}^2 \otimes \mathbb{C}^2$ that are known to require a Bell state have a common feature that is each contains at least one of the four Bell states. This raises a very basic question: Suppose that we are given a set of two-qubit states that does not contain a maximally entangled state and that cannot be optimally discriminated by LOCC. Do we still require a Bell state for optimal discrimination by LOCC? The present paper answers this question in affirmative and also shows that a Bell state is required for optimal discrimination of some sets that do not even contain an entangled state.

Problem statement and overview of results

Specifically, we study the problem of discriminating a set of "noisy" Bell states by LOCC assuming a uniform probability distribution, i.e., each state has an equal chance of being distributed to Alice and Bob. A noisy Bell state, in general, results from actions of quantum channels on one or both qubits of the Bell state in question. In fact, the task of LOCC discrimination of four Bell states in a realistic scenario boils down to LOCC discrimination of four noisy Bell states. That is because the unknown Bell state must be distributed to Alice and Bob through quantum channels that are noisy in practice.

In this paper we shall assume that a noisy Bell state results from mixing a Bell state with a two-qubit state with probabilities λ and $(1-\lambda)$, where $0 \leq \lambda \leq 1$, or as a consequence of the action of a quantum channel that leaves a Bell state unchanged with probability λ and converts it into a two-qubit state with probability $(1-\lambda)$.

Let $\mathrm{D}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$ be the set of all two-qubit density matrices. Let $\Psi_i = |\Psi_i\rangle \langle\Psi_i|$ denote the density operator corresponding to the Bell state $|\Psi_i\rangle$ given by (1) and ς be the density operator corresponding to a two-qubit state that could be either pure or mixed.

Consider a uniform collection of noisy Bell states

$$\mathcal{B}_{\lambda,\varsigma} = \{\varrho_i : i = 1, \ldots, 4\} \subset \mathrm{D}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right), \tag{6}$$

where

$$\varrho_i = \lambda \Psi_i + (1-\lambda)\,\varsigma \tag{7}$$

for $\lambda \in [0,1]$. The set $\mathcal{B}_{\lambda,\varsigma}$ is therefore completely determined by both λ and ς. Of course, the situation where $\lambda = 0$ is not interesting.

Observe that for a fixed $j \in \{1, \ldots, 4\}$ one has

$$\max_{i \in \{1,\ldots,4\}} \langle \Psi_j | \, \varrho_i \, | \Psi_j \rangle = \langle \Psi_j | \, \varrho_j \, | \Psi_j \rangle . \tag{8}$$

Eq. (8) means the following: Suppose a Bell measurement is performed on a two-qubit system that has been prepared with equal probability in one of $\{\varrho_i\}$. Then, given an outcome j, where $j \in \{1, \ldots, 4\}$, the system was most likely prepared in the state ϱ_j. Note that, in general,

$$\max_{j \in \{1,\ldots,4\}} \langle \Psi_j | \, \varrho_i \, | \Psi_j \rangle \neq \langle \Psi_i | \, \varrho_i \, | \Psi_i \rangle . \tag{9}$$

The distinction between (8) and (9) is important.

How well the states ϱ_i can be discriminated is quantified by the global optimum $p\,(\mathcal{B}_{\lambda,\varsigma})$. As the states are nonorthogonal except for $\lambda = 1$, it holds that $p\,(\mathcal{B}_{\lambda,\varsigma}) \leq 1$ where equality holds if and only if $\lambda = 1$, i.e., for the Bell basis. Our first result is a lower bound on $p\,(\mathcal{B}_{\lambda,\varsigma})$:

- For $\lambda \in [0,1]$ and any two-qubit state ς, it holds that

$$p\,(\mathcal{B}_{\lambda,\varsigma}) \geq \frac{1}{4}\,(1 + 3\lambda) . \tag{10}$$

  Later we will find that the lower bound is, in fact, the exact formula.

Next, we compute the local optimum.

- The local success probability of discriminating the states of $\mathcal{B}_{\lambda,\varsigma}$ is:

$$p_{\mathrm{L}}\,(\mathcal{B}_{\lambda,\varsigma}) = \frac{1}{4}\,(1 + \lambda) \tag{11}$$

  for $\lambda \in [0,1]$ and any two-qubit state ς.

Observe that

$$p_{\mathrm{L}}\left(\mathcal{B}_{\lambda,\varsigma}\right) < \frac{1}{4}\left(1+3\lambda\right) \leq p\left(\mathcal{B}_{\lambda,\varsigma}\right) \quad \forall\lambda \in (0,1]\,,$$

which proves that:

- The states of $\mathcal{B}_{\lambda,\varsigma}$ cannot be optimally discriminated by LOCC for any $\lambda \in (0,1]$ and any two-qubit ς.

So the next thing is to find the entanglement cost of discriminating the states of $\mathcal{B}_{\lambda,\varsigma}$ using LOCC. We assume that $|\tau_\varepsilon\rangle$ (given by (3)) is used as a resource.

First, we obtain the success probability of discriminating the states of $\mathcal{B}_{\lambda,\varsigma}$ using LOCC and $|\tau_\varepsilon\rangle$.

- The local success probability of distinguishing the states of

  $$\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon = \left\{\varrho_i \otimes \tau_\varepsilon : i = 1,\ldots,4\right\}$$

  is given by:

  $$p_{\mathrm{L}}\left(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon\right) = \frac{1}{4}\left(1+\lambda+2\lambda\sqrt{1-\varepsilon^2}\right) \tag{12}$$

  for $\varepsilon \in [0,1]$, $\lambda \in [0,1]$, and any two-qubit state ς. Eq. (12) can also be written as

  $$p_{\mathrm{L}}\left(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon\right) = p_{\mathrm{L}}\left(\mathcal{B}_{\lambda,\varsigma}\right) + \frac{1}{2}\lambda\sqrt{1-\varepsilon^2}.$$

  Observe the contribution of the resource in the above equation, which is given by the the second term on the right hand side for all $\varepsilon \in [0,1)$.

- Eq. (12) leads to the formula for the global optimum $p\left(\mathcal{B}_{\lambda,\varsigma}\right)$:

  $$p\left(\mathcal{B}_{\lambda,\varsigma}\right) = \frac{1}{4}\left(1+3\lambda\right) \tag{13}$$

  for $\lambda \in [0,1]$ and any two-qubit state ς. So the lower bound from (10) turns out to be exact.

Now observe that for $\lambda \in (0,1]$

$$p_{\mathrm{L}}\left(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon\right) \leq p\left(\mathcal{B}_{\lambda,\varsigma}\right), \tag{14}$$

where equality holds if and only if $\varepsilon = 0$. This gives us the entanglement cost.

- The entanglement cost of discriminating the states of $\mathcal{B}_{\lambda,\varsigma}$ by LOCC is 1 ebit for any $\lambda \in (0,1]$ and two-qubit state ς.

Thus a maximally entangled state is required for optimal discrimination of the states of $\mathcal{B}_{\lambda,\varsigma}$ by LOCC, although for any given value of $\lambda \in (0,1)$ the ensemble, in general, does not contain any maximally entangled state. To the best of our knowledge, this is the first example for which a maximally entangled state is required to optimally discriminate a set of states that does not contain a maximally entangled state.

Remark 1. The success probabilities $p_{\text{L}}\left(\mathcal{B}_{\lambda,\varsigma}\right)$, $p_{\text{L}}\left(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_{\varepsilon}\right)$, and $p\left(\mathcal{B}_{\lambda,\varsigma}\right)$ are all independent of ς. This is not something that was expected *a priori* but seems to be the consequence of the fact that the Bell states are all mixed with the same ς. We should not expect something similar if different two-qubit states are mixed with different Bell states.

Remark 2. The entanglement cost is seen to be independent of the gap between the local and the global optima that are given by (11) and (13), respectively. As long as the gap remains finite, no matter how small, the entanglement cost remains 1 ebit irrespective of the the entanglement or other properties of the states.

We illustrate the results with an example in which ς is taken to be $\frac{1}{4}\mathbf{1}$. The general result tells us that the entanglement cost is 1 ebit for $\lambda \in (0,1]$. In this case, the states are entangled for $\lambda \in \left(\frac{1}{3}, 1\right]$ but separable for $\lambda \in \left(0, \frac{1}{3}\right]$. So if we consider a set $\mathcal{B}_{\lambda,\varsigma}$ for some $\lambda \in \left(0, \frac{1}{3}\right]$ and $\varsigma = \frac{1}{4}\mathbf{1}$, then such a set contains only separable states. Nevertheless, optimal discrimination by LOCC requires a two-qubit maximally entangled state as resource.

2 Preliminaries

There is no tractable characterization of the set of LOCC measurements. In fact, even deciding whether a measurement on a composite system describes an LOCC measurement is computationally hard. For these reasons, LOCC state discrimination problems are often investigated by considering the more tractable classes: separable measurements (SEP) [37, 42, 46] and positive partial transpose (PPT) measurements [41, 43, 44, 46]. A separable measurement is where the measurement operators are all separable, and a PPT measurement is where the measurement operators are all positive under partial transposition. These measurements often yield useful results and insights. One accordingly defines $p_{\text{SEP}}\left(\mathcal{E}\right)$ as the separable success probability and $p_{\text{PPT}}\left(\mathcal{E}\right)$ as the PPT success probability. Since

$$\{\text{LOCC}\} \subset \{\text{SEP}\} \subset \{\text{PPT}\} \subset \{\text{all}\},$$

it holds that

$$p_{\mathrm{L}}(\mathcal{E}) \leq p_{\mathrm{SEP}}(\mathcal{E}) \leq p_{\mathrm{PPT}}(\mathcal{E}) \leq p(\mathcal{E}).$$

Let $\mathcal{X}$ and $\mathcal{Y}$ represent finite-dimensional Hilbert spaces associated with quantum systems that belong to Alice and Bob respectively. Let $\mathrm{Pos}(\mathcal{X})$, $\mathrm{Pos}(\mathcal{Y})$, and $\mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$ denote the sets of positive semidefinite operators acting on $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{X} \otimes \mathcal{Y}$, respectively. An operator $P \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is PPT if $\mathrm{T}_{\mathcal{X}}(P) \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$, where $\mathrm{T}_{\mathcal{X}}$ represents partial transposition taken in the standard basis $\{|0\rangle, \ldots, |d-1\rangle\}$ of $\mathcal{X}$. A PPT measurement is defined by a collection of measurement operators $\{P_1, \ldots, P_N\}$ where each operator is PPT.

Let us denote the set of all PPT operators acting on $\mathcal{X} \otimes \mathcal{Y}$ by $\mathrm{PPT}(\mathcal{X} : \mathcal{Y})$. The set $\mathrm{PPT}(\mathcal{X} : \mathcal{Y})$ is a closed, convex cone. For a given ensemble $\mathcal{E} = \{(p_1, \rho_1), \ldots, (p_N, \rho_N)\} \subset \mathcal{X} \otimes \mathcal{Y}$ the problem of finding $p_{\mathrm{PPT}}(\mathcal{E})$ can be expressed as a semidefinite program [41]:

$\texttt{Primal problem}:$

$$\begin{aligned}
\texttt{maximize}: \quad & \sum_{i=1}^{N} p_i \mathrm{Tr}(\rho_i P_i) \\
\texttt{subject to}: \quad & \sum_{i=1}^{N} P_i = \mathbf{1}_{\mathcal{X} \otimes \mathcal{Y}} \\
& P_k \in \mathrm{PPT}(\mathcal{X} : \mathcal{Y}) \quad (\text{for each } k = 1, \ldots, N)
\end{aligned}$$

$\texttt{Dual problem}:$

$$\begin{aligned}
\texttt{minimize}: \quad & \mathrm{Tr}(H) \\
\texttt{subject to}: \quad & H - p_k \rho_k \in \mathrm{PPT}(\mathcal{X} : \mathcal{Y}) \quad (\text{for each } k = 1, \ldots, N) \\
& H \in \mathrm{Herm}(\mathcal{X} \otimes \mathcal{Y}),
\end{aligned}$$

where $\mathrm{Herm}(\mathcal{X} \otimes \mathcal{Y})$ is the set of Hermitian operators acting on $\mathcal{X} \otimes \mathcal{Y}$. By weak duality every feasible solution of the dual problem provides an upper bound on $p_{\mathrm{PPT}}(\mathcal{E})$.

3 LOCC discrimination of $\mathcal{B}_{\lambda,\varsigma}$

In this section we prove that the states of a set $\mathcal{B}_{\lambda,\varsigma}$ for $\lambda \in (0, 1]$ and any choice of $\varsigma \in \mathrm{D}(\mathcal{X}_1 \otimes \mathcal{Y}_1)$ cannot be optimally discriminated by LOCC.

Let $\mathcal{X}_1 = \mathbb{C}^2$ and $\mathcal{Y}_1 = \mathbb{C}^2$ denote the Hilbert spaces of Alice and Bob respectively. First, we give a lower bound on $p(\mathcal{B}_{\lambda,\varsigma})$.

Lemma 3. *$p(\mathcal{B}_{\lambda,\varsigma}) \geq \frac{1}{4}(1+3\lambda)$ for $\lambda \in [0,1]$ and any $\varsigma \in D(\mathcal{X}_1 \otimes \mathcal{Y}_1)$.*

Proof. For any quantum measurement $\{M_a\}$ on $\mathcal{X}_1 \otimes \mathcal{Y}_1$, it holds that

$$p(\mathcal{B}_{\lambda,\varsigma}) \geq \frac{1}{4} \sum_a \max_i \operatorname{Tr}(\varrho_i M_a).$$

Choosing $\{M_a\}$ as the Bell measurement $\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, we get

$$p(\mathcal{B}_{\lambda,\varsigma}) \quad \geq \quad \frac{1}{4} \sum_{a=1}^{4} \max_i \langle \Psi_a | \varrho_i | \Psi_a \rangle. \tag{15}$$

Noting that $\max_{i \in \{1,\ldots,4\}} \langle \Psi_a | \varrho_i | \Psi_a \rangle = \langle \Psi_a | \varrho_a | \Psi_a \rangle$, we can write (15) as

$$\begin{aligned} p(\mathcal{B}_{\lambda,\varsigma}) &\geq \frac{1}{4} \sum_{a=1}^{4} \langle \Psi_a | \varrho_a | \Psi_a \rangle \\ &= \lambda + \left(\frac{1-\lambda}{4}\right) \sum_{a=1}^{4} \langle \Psi_a | \varsigma | \Psi_a \rangle \\ &= \frac{1}{4}(1+3\lambda). \end{aligned} \tag{16}$$

To arrive at the last line we have used $\sum_{a=1}^{4} \langle \Psi_a | \varsigma | \Psi_a \rangle = 1$. Clearly, (16) holds for all $\lambda \in [0,1]$ and any ς. ☐

Lemma 4. *$p_{\mathrm{L}}(\mathcal{B}_{\lambda,\varsigma}) = \frac{1}{4}(1+\lambda)$ for $\lambda \in [0,1]$ and any $\varsigma \in D(\mathcal{X}_1 \otimes \mathcal{Y}_1)$.*

Proof. The proof contains two parts. First, we show that $p_{\mathrm{PPT}}(\mathcal{B}_{\lambda,\varsigma}) \leq \frac{1}{4}(1+\lambda)$ and then we will give a local protocol that achieves this bound.

Let $\lambda \in [0,1]$. Consider the operator

$$H_\lambda = \frac{1}{8}\left[\lambda \mathbf{1}_{\mathcal{X}_1 \otimes \mathcal{Y}_1} + 2(1-\lambda)\varsigma\right] \in \operatorname{Herm}(\mathcal{X}_1 \otimes \mathcal{Y}_1), \tag{17}$$

where $\mathbf{1}_{\mathcal{X}_1 \otimes \mathcal{Y}_1}$ is the identity operator acting on $\mathcal{X}_1 \otimes \mathcal{Y}_1$. Then

$$\operatorname{Tr}(H_\lambda) = \frac{1}{4}(1+\lambda). \tag{18}$$

We will now show that H_λ is a feasible solution of the dual of the PPT state discrimination problem. In particular, we will show that

$$\mathrm{T}_{\mathcal{X}_1}\left(H_\lambda - \frac{1}{4}\varrho_i\right) \in \mathrm{Pos}\left(\mathcal{X}_1 \otimes \mathcal{Y}_1\right)\ \forall i = 1, \ldots, 4, \tag{19}$$

which is a sufficient condition for dual feasibility.

Observe that

$$\begin{aligned} H_\lambda - \frac{1}{4}\varrho_i &= \frac{1}{8}\left(\lambda \mathbf{1}_{\mathcal{X}_1\otimes\mathcal{Y}_1} - 2\lambda\Psi_i\right) \\ &= \frac{\lambda}{4}\left(\frac{1}{2}\mathbf{1}_{\mathcal{X}_1\otimes\mathcal{Y}_1} - \Psi_i\right) \\ &= \frac{\lambda}{4}\mathrm{T}_{\mathcal{X}_1}\left(\Psi_{5-i}\right) \qquad \text{(for every } i = 1, \ldots, 4) \end{aligned}$$

Hence

$$\mathrm{T}_{\mathcal{X}_1}\left(H_\lambda - \frac{1}{4}\varrho_i\right) = \frac{\lambda}{4}\Psi_{i-5} \in \mathrm{Pos}\left(\mathcal{X}_1 \otimes \mathcal{Y}_1\right)$$

for every $i = 1, \ldots, 4$. So by weak duality we have

$$p_{\mathrm{PPT}}\left(\mathcal{B}_{\lambda,\varsigma}\right) \leq \mathrm{Tr}\left(H_\lambda\right) = \frac{1}{4}\left(1+\lambda\right). \tag{20}$$

Consequently,

$$p_{\mathrm{L}}\left(\mathcal{B}_{\lambda,\varsigma}\right) \leq p_{\mathrm{PPT}}\left(\mathcal{B}_{\lambda,\varsigma}\right) \leq \frac{1}{4}\left(1+\lambda\right). \tag{21}$$

We will now show that the above upper bound is also a lower bound on $p_{\mathrm{L}}\left(\mathcal{B}_{\lambda,\varsigma}\right)$. Choosing the local measurement in the computational basis $\{|a\rangle : a \in \{00, 01, 10, 11\}\}$ we get

$$\begin{aligned} p_{\mathrm{L}}\left(\mathcal{B}_{\lambda,\varsigma}\right) &\geq \frac{1}{4}\sum_a \max_i \langle a\,|\varrho_i|\,a\rangle \\ &= \frac{\lambda}{2} + \left(\frac{1-\lambda}{4}\right)\sum_a \langle a\,|\varsigma|\,a\rangle \\ &= \frac{1}{4}\left(1+\lambda\right). \end{aligned} \tag{22}$$

From (21) and (22) it follows that

$$p_{\mathrm{L}}\left(\mathcal{B}_{\lambda,\varsigma}\right) = \frac{1}{4}\left(1+\lambda\right) \tag{23}$$

for $\lambda \in [0,1]$ and any two-qubit state ς. $\square$

Lemmas 3 and 4 together imply:

$$p_{\mathrm{L}}\left(\mathcal{B}_{\lambda,\varsigma}\right) < \frac{1}{4}\left(1+3\lambda\right) \leq p\left(\mathcal{B}_{\lambda,\varsigma}\right) \quad \text{for } \lambda \in (0,1]\,,$$

which proves the following theorem.

Theorem 5. *The states of a set $\mathcal{B}_{\lambda,\varsigma}$, as defined by (6), cannot be optimally discriminated by LOCC for any $\lambda \in (0,1]$ and any two-qubit state $\varsigma \in D\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$.*

In the next section, we take up the question of finding the entanglement cost: the amount of entanglement one must consume to optimally discriminate the states of a set $\mathcal{B}_{\lambda,\varsigma}$, where $\lambda \in (0,1]$, by LOCC.

4 The entanglement cost of discriminating $\mathcal{B}_{\lambda,\varsigma}$

Let us now assume that Alice and Bob share an additional resource state $|\tau_\varepsilon\rangle \in \mathcal{X}_2 \otimes \mathcal{Y}_2$ defined by (3), where $\mathcal{X}_2 = \mathbb{C}^2$ and $\mathcal{Y}_2 = \mathbb{C}^2$ are the Hilbert spaces associated with the ancilla systems. That means we now consider the task of LOCC discrimination of the states corresponding to the set

$$\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon = \{\varrho_i \otimes \tau_\varepsilon : i = 1,\ldots,4\} \subset \left(\mathcal{X}_1 \otimes \mathcal{Y}_1\right) \otimes \left(\mathcal{X}_2 \otimes \mathcal{Y}_2\right), \tag{24}$$

where the states are all equally probable, and $\tau_\varepsilon = |\tau_\varepsilon\rangle\langle\tau_\varepsilon| \subset \mathrm{D}\left(\mathcal{X}_2 \otimes \mathcal{Y}_2\right)$. The main result of this section is the following:

Theorem 6. *The local success probability of discriminating the states of $\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon$ is given by*

$$p_L\left(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon\right) = \frac{1}{4}\left(1+\lambda+2\lambda\sqrt{1-\varepsilon^2}\right) \tag{25}$$

for $\varepsilon \in [0,1]$, $\lambda \in [0,1]$, and any $\varsigma \in D\left(\mathcal{X}_1 \otimes \mathcal{Y}_1\right)$.

Proof. Let $\varepsilon \in [0,1]$, $\lambda \in [0,1]$, and $\varsigma \in \mathrm{D}\left(\mathcal{X}_1 \otimes \mathcal{Y}_1\right)$. First, we will prove that

$$p_{\mathrm{PPT}}\left(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon\right) \leq \frac{1}{4}\left(1+\lambda+2\lambda\sqrt{1-\varepsilon^2}\right), \tag{26}$$

and then we will give a local protocol that achieves this upper bound.

Define the operator:

$$H_{\lambda,\varepsilon} = \lambda H_\varepsilon + \left(\frac{1-\lambda}{4}\right)\varsigma \otimes \tau_\varepsilon \in \mathrm{Herm}\left(\mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2\right), \tag{27}$$

where

$$H_\epsilon = \frac{1}{8}\left[\mathbf{1}_{\mathcal{X}_1\otimes\mathcal{Y}_1} \otimes \tau_\varepsilon + \sqrt{1-\varepsilon^2}\mathbf{1}_{\mathcal{X}_1\otimes\mathcal{Y}_1} \otimes \mathrm{T}_{\mathcal{X}_2}\left(\Psi_4\right)\right] \in \mathrm{Herm}\left(\mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2\right). \tag{28}$$

It holds that

$$\mathrm{Tr}\left(H_{\lambda,\varepsilon}\right) = \frac{1}{4}\left(1 + \lambda + 2\lambda\sqrt{1-\varepsilon^2}\right). \tag{29}$$

We now show that $H_{\lambda,\varepsilon}$ is a feasible solution of the dual problem of discriminating the states $\varrho_i \otimes \tau_\varepsilon$, $i = 1, \ldots, 4$, by PPT measurements. In particular, we will prove that

$$\left(\mathrm{T}_{\mathcal{X}_1} \otimes \mathrm{T}_{\mathcal{X}_2}\right)\left(H_{\lambda,\varepsilon} - \frac{1}{4}\varrho_i \otimes \tau_\varepsilon\right) \in \mathrm{Pos}\left(\mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2\right)\ \forall i = 1, \ldots, 4, \tag{30}$$

which is a sufficient condition for dual feasibility. The proof is, in fact, almost immediate. Observe that

$$H_{\lambda,\varepsilon} - \frac{1}{4}\varrho_i \otimes \tau_\varepsilon = \lambda\left(H_\varepsilon - \frac{1}{4}\Psi_i \otimes \tau_\varepsilon\right) \ \ (\text{for every } i = 1, \ldots, 4).$$

Therefore,

$$\left(\mathrm{T}_{\mathcal{X}_1} \otimes \mathrm{T}_{\mathcal{X}_2}\right)\left(H_{\lambda,\varepsilon} - \frac{1}{4}\varrho_i \otimes \tau_\varepsilon\right) = \lambda\left(\mathrm{T}_{\mathcal{X}_1} \otimes \mathrm{T}_{\mathcal{X}_2}\right)\left(H_\varepsilon - \frac{1}{4}\Psi_i \otimes \tau_\varepsilon\right) \ \ (\text{for every } i = 1, \ldots, 4)$$

which is positive semidefinite [46]. So we have

$$\left(\mathrm{T}_{\mathcal{X}_1} \otimes \mathrm{T}_{\mathcal{X}_2}\right)\left(H_{\lambda,\varepsilon} - \frac{1}{4}\varrho_i \otimes \tau_\varepsilon\right) \in \mathrm{Pos}\left(\mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2\right)$$

for every $i = 1, \ldots, 4$. By weak duality

$$p_{\mathrm{PPT}}\left(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon\right) \leq \mathrm{Tr}\left(H_{\lambda,\varepsilon}\right) = \frac{1}{4}\left(1 + \lambda + 2\lambda\sqrt{1-\varepsilon^2}\right). \tag{31}$$

Since $p_L(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon) \leq p_{\text{PPT}}(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon)$, it holds that

$$p_{\text{L}}(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon) \leq \frac{1}{4}\left(1+\lambda+2\lambda\sqrt{1-\varepsilon^2}\right). \tag{32}$$

We now give a lower bound on $p_{\text{L}}(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon)$. This lower bound is based on the teleportation protocol: Alice teleports her qubit to Bob using τ_ε as the teleportation channel. This results in Bob holding one of the four two-qubit states from

$$\begin{aligned}
\varrho_1' &= \lambda\tau_\varepsilon + (1-\lambda)\varsigma', \\
\varrho_2' &= \lambda(\mathbf{1}\otimes\sigma_z)\tau_\varepsilon(\mathbf{1}\otimes\sigma_z) + (1-\lambda)\varsigma', \\
\varrho_3' &= \lambda(\mathbf{1}\otimes\sigma_x)\tau_\varepsilon(\mathbf{1}\otimes\sigma_x) + (1-\lambda)\varsigma', \\
\varrho_4' &= \lambda(\mathbf{1}\otimes\sigma_y)\tau_\varepsilon(\mathbf{1}\otimes\sigma_y) + (1-\lambda)\varsigma',
\end{aligned}$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices and ς' is the post-teleportation ς. Subsequently, Bob performs the Bell measurement $\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$ on the two qubits, which are in one of the above four states. This strategy leads to

$$\begin{aligned}
p_{\text{L}}(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon) &\geq \frac{1}{4}\sum_{a=1}^{4}\max_i \left\langle \Psi_a \left| \varrho_i' \right| \Psi_a \right\rangle \\
&= \frac{\lambda}{2}\left(1+\sqrt{1-\varepsilon^2}\right) + \left(\frac{1-\lambda}{4}\right)\sum_{a=1}^{4}\left\langle \Psi_a \left| \varsigma' \right| \Psi_a \right\rangle \\
&= \frac{1}{4}\left(1+\lambda+2\lambda\sqrt{1-\varepsilon^2}\right).
\end{aligned} \tag{33}$$

From (32) and (33) we obtain the desired result. This completes the proof. □

We see that

$$p_{\text{L}}(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_\varepsilon) \leq \frac{1}{4}(1+3\lambda) \leq p(\mathcal{B}_{\lambda,\varsigma}) \quad \lambda \in (0,1],$$

where the first inequality is an equality for $\varepsilon = 0$. In other words, the best possible local success probability is obtained when $|\tau_\varepsilon\rangle$ is maximally entangled and that must also be, in this case, the global optimum, i.e.,

$$p(\mathcal{B}_{\lambda,\varsigma}) = p_{\text{L}}(\mathcal{B}_{\lambda,\varsigma} \otimes \tau_{\varepsilon=0}) = \frac{1}{4}(1+3\lambda). \tag{34}$$

So the lower bound in Lemma 3 is, in fact, the global optimum. Now, by definition, the entanglement cost is the amount of entanglement of an optimal resource state that enables optimal discrimination by LOCC and is also minimal in both entanglement and dimension. Since the resource state $|\tau_\varepsilon\rangle$ is from $\mathbb{C}^2 \otimes \mathbb{C}^2$, from (34) it follows that:

Theorem 7. *The entanglement cost of optimal discrimination of the states of* $\mathcal{B}_{\lambda,\varsigma}$ *by LOCC is* 1 *ebit for any* $\lambda \in (0,1]$ *and any two-qubit state* ς.

Remark 8. So the entanglement cost in this case is independent of the entanglement of the constituent states and also the choice of the two-qubit state ς. In fact, the entanglement cost is 1 ebit as long as the gap between the local and the global optima is nonzero. Note that a set $\mathcal{B}_{\lambda,\varsigma}$ for any $\lambda \in (0,1)$, in general, does not contain a maximally entangled state. So such sets, to the best of our knowledge, are the first examples of sets that do not contain a maximally entangled state but still requires a maximally entangled state for optimal discrimination by LOCC.

Example: Bell states mixed with white noise

Let us now consider a concrete example in which $\varsigma = \frac{1}{4}\mathbf{1}_{\mathcal{X}_1 \otimes \mathcal{Y}_1}$. Then we have the following set of noisy Bell states:

$$\mathcal{B}_{\lambda,\frac{1}{4}\mathbf{1}} = \{\Omega_i : i = 1, \ldots, 4\},$$

where

$$\Omega_i = \lambda \Psi_i + \frac{1-\lambda}{4}\mathbf{1}_{\mathcal{X}_1 \otimes \mathcal{Y}_1}$$

for $\lambda \in (0,1]$. The results obtained earlier apply straightaway. But now the range of λ has a clear interpretation in terms of the entanglement of the states: each state Ω_i , where $i = 1, \ldots, 4$ is entangled if and only if $\lambda \in \left(\frac{1}{3}, 1\right]$. So a set $\mathcal{B}_{\lambda,\frac{1}{4}\mathbf{1}}$ contains entangled states for $\lambda \in \left(\frac{1}{3}, 1\right]$ and separable states for $\lambda \in \left(0, \frac{1}{3}\right]$. But for any such set we now know that the entanglement cost of discrimination by LOCC is 1 ebit. So the entanglement cost, in this case, is independent of the entanglement of the states. Furthermore, one requires a full ebit even when the states are separable.

5 Conclusions

A set of bipartite or multipartite quantum states cannot always be optimally discriminated by LOCC. So, given a set of states that cannot be optimally discriminated by LOCC, a basic question is, how much entanglement, as a resource, must one consume to perform the task of optimal discrimination by LOCC? For instance, a set of three or four Bell states can only be perfectly discriminated by LOCC if a Bell state is used as a resource.

In this paper we considered the problem of LOCC discrimination of a uniform collection $\mathcal{B}_{\lambda,\varsigma}$ of noisy Bell states that are obtained by mixing the Bell states with a two-qubit state ς with probabilities λ and $(1-\lambda)$. First, we showed that the states of $\mathcal{B}_{\lambda,\varsigma}$ cannot be optimally discriminated by LOCC for any $\lambda \in (0,1]$ and ς, so optimal discrimination will require an ancillary entangled state. Since, such sets, in general, do not contain a maximally entangled state, it was interesting to find out whether optimal discrimination is possible without using a maximally entangled state. We, however, proved that optimal discrimination by LOCC is possible if and only if a two-qubit maximally entangled state is used as a resource for any $\lambda \in (0,1]$ and ς. So the result holds regardless of the entanglement of the states that could even be separable in some cases. More specifically, the result holds as long as the gap between the local and the global optima is nonzero, no matter how small. To prove our results we have utilized the fact that determining the optimal value of discriminating via PPT measurements can be represented as a semidefinite program [41].

An interesting open question is whether a nonmaximally entangled, orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be perfectly discriminated with LOCC using a nonmaximally entangled state. One may, for instance, consider working with the bases in [47, 48] for which the lower bound was proved to be strictly larger than the entropy bound (the average entanglement of the states assuming they are equally probable).

Finally, we hope the results presented in this paper and particularly the techniques [41, 43, 46] used to prove the results, will be useful for future research works in this area.

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