BMAT102P DA1

Q1 $Z = (x^2+a)(y^2+b)$

$Z = x^2y^2 + bx^2 + ay^2 + ab$ — ①

Differentiate ① partially w.r.t x

$\Rightarrow \frac{\partial z}{\partial x} = p = 2xy^2 + 2bx$

$\Rightarrow p = 2x(y^2+b)$

$\Rightarrow \boxed{y^2+b = \frac{p}{2x}}$ — ②

Differentiate ① partially w.r.t y

$\frac{\partial z}{\partial x} = q = 2x^2y + 2ay$

$q = 2y(x^2+a)$

$\Rightarrow \boxed{x^2+a = \frac{q}{2y}}$ — ③

Substitute ② and ③ in ①

$\Rightarrow Z = \frac{q}{2y} \cdot \frac{p}{2x}$

$\Rightarrow \boxed{pq = 4xyz}$

Q2 $z = (a \sin by) e^{bx}$ — ①

Differentiate ① partially w.r.t x

$\Rightarrow \frac{\partial z}{\partial x} = p = ab \sin by \, e^{bx}$

$\Rightarrow \frac{p}{b} = (a \sin by) e^{bx}$

$\Rightarrow \boxed{b = \frac{p}{z}}$ — ②

Differentiate ① partially w.r.t y

$\Rightarrow \frac{\partial z}{\partial y} = q = ab \cos by \, e^{bx}$

$\Rightarrow q = \frac{ab}{\tan by} \cdot \sin by \, e^{bx}$

$\Rightarrow \frac{q \tan by}{b} = (a \sin by) e^{bx}$

$\Rightarrow \frac{q \tan by}{z} = b$

$\Rightarrow \boxed{b = \frac{q \tan by}{z}}$ — ③

Compare ② and ③

$\Rightarrow p = q \tan by$

$\Rightarrow by = \tan^{-1}\left(\frac{p}{q}\right)$

$\Rightarrow \boxed{py = z \tan^{-1}\left(\frac{p}{q}\right)}$ $\left(\because b = \frac{p}{z}\right)$

Q3 $(x-a)^2 + (y-b)^2 + (z-c)^2 = k^2$

$(x-a)^2 + (y-b)^2 + (z-c)^2 = k^2 \quad -(1)$

Differentiate (1) partially w.r.t x

$\Rightarrow 2(x-a) + 2(z-c)\dfrac{\partial z}{\partial x} = 0$

$\Rightarrow x-a = -(z-c)p \quad -(2)$

Differentiate (1) partially w.r.t y

$\Rightarrow 2(y-b) + 2(z-c)\dfrac{\partial z}{\partial x} = 0$

$\Rightarrow y-b = -(z-c)q \quad -(3)$

Differentiate (2) partially w.r.t x

$1 = -\left[(z-c)\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial z}{\partial x}\right]$

$\Rightarrow 1 = -(z-c)r - p^2$

$\Rightarrow \boxed{z-c = -\left(\dfrac{p^2+1}{r}\right)} \quad -(5)$

Differentiate (3) partially w.r.t y

$\Rightarrow 1 = -\left[(z-c)\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial z}{\partial y}\right]$

$\Rightarrow 1 = -(z-c)t - q^2$

$\Rightarrow \boxed{z-c = -\left(\dfrac{q^2+1}{t}\right)} \quad -(5)$

Differentiate ② partially w.r.t y

$$\Rightarrow 0 = -\left[(z-c)\cdot\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial x}\right]$$

$$\Rightarrow 0 = -\left[(z-c)s + pq\right]$$

$$\Rightarrow \boxed{z-c = -\frac{pq}{s}} \quad - ⑥$$

Comparing ④ and ⑤

$$+\left(\frac{p^2+1}{r}\right) = +\left(\frac{q^2+1}{t}\right)$$

$$\Rightarrow \boxed{tp^2+t = rq^2+r}$$

Comparing ⑤ and ⑥

$$+\left(\frac{p^2+1}{r}\right) = +\frac{pq}{s}$$

$$\Rightarrow \boxed{\frac{pqr}{s} = p^2+1}$$

Comparing ⑤ and ⑥

$$+\left(\frac{q^2+1}{t}\right) = +\frac{pq}{s}$$

$$\Rightarrow \boxed{\frac{pqt}{s} = q^2+1}$$

$\therefore$ 3 PDEs formed

Q4 $xyz = \phi(x+y+z)$

$$xyz = \phi(x+y+z) \quad \text{--- (1)}$$

Differentiating (1) partially w.r.t x and y respectively

$$yz + xy\frac{\partial z}{\partial x} = \phi'(x+y+z)\left[1+\frac{\partial z}{\partial x}\right]$$

$$\Rightarrow \phi'(x+y+z) = \frac{yz+xyp}{1+p} \quad \text{--- (2)}$$

$$xz + xy\frac{\partial z}{\partial y} = \phi'(x+y+z)\left[1+\frac{\partial z}{\partial y}\right]$$

$$\Rightarrow \phi'(x+y+z) = \frac{xz+xyq}{1+q} \quad \text{--- (3)}$$

Comparing (2) and (3)

$$\frac{yz+xyp}{1+p} = \frac{xz+xyq}{1+q}$$

$$\Rightarrow yz + xyp + qyz + \cancel{xypq} = xz + xyq + pxz + \cancel{xypq}$$

$$\Rightarrow \boxed{px[y-z] + qy[z-x] = z[x-y]}$$

Q5 $z = x^2 + 2g\left(\frac{1}{y} + \log x\right)$

$z = x^2 + 2g\left(\frac{1}{y} + \log x\right)$ — (1)

Differentiate (1) partially w.r.t. x and y respectively

$$\frac{\partial z}{\partial x} = p = 2x + 2g'\left(\frac{1}{y} + \log x\right)\frac{1}{x}$$

$$\Rightarrow g'\left(\frac{1}{y} + \log x\right) = \frac{x(p-2x)}{2} \quad \text{— (1)}$$

$$\frac{\partial z}{\partial y} = q = 2g'\left(\frac{1}{y} + \log x\right)\left(-\frac{1}{y^2}\right)$$

$$\Rightarrow g'\left(\frac{1}{y} + \log x\right) = -\frac{qy^2}{2} \quad \text{— (2)}$$

Comparing (1) and (2)

$$\frac{x(p-2x)}{\cancel{2}} = -\frac{qy^2}{\cancel{2}}$$

$$\Rightarrow px - 2x^2 = -qy^2$$

$$\Rightarrow \boxed{px + qy^2 = 2x^2}$$

Q7 $\phi\left(\frac{x-y}{y-z},\ xy+yz+zx\right)=0$

$\Rightarrow \frac{x-y}{y-z} = f(xy+yz+zx) \quad - (1)$

Differentiate (1) partially w.r.t x

$\Rightarrow \frac{1}{y-z} = f'(xy+yz+zx)\left[y + yp + xp + z\right]$

$\Rightarrow f'(xy+yz+zx) = \frac{1}{(y-z)\left[z+y+p(x+y)\right]} \quad - (2)$

$\cancel{\Rightarrow f'(xy+yz+zx) = \frac{1}{(y-z)(1+p)(z+y)} \quad - (2)}$

Differentiate (1) partially w.r.t y

$\Rightarrow \frac{(z-y)-(x-y)}{(y-z)^2} = f'(xy+yz+zx)\left[x+qy+z+qx\right]$

$\Rightarrow f'(xy+yz+zx) = \frac{z-x}{(y-z)^2\left[x+z+q(x+y)\right]} \quad - (3)$

Compare (2) and (3)

$\Rightarrow \frac{1}{(y-z)\left[z+y+p(x+y)\right]} = \frac{z-x}{(y-z)^2\left[x+z+q(x+y)\right]}$

$\Rightarrow (y-z)\left[x+z+qx+yq\right] = (z-x)\left[z+y+px+py\right]$

$\Rightarrow yx + yz + qxy + y^2q - xz - z^2 - qxz - yqz$
$= z^2 + zy + zpx + zpy - xz - xy - px^2 - pxy$

$\Rightarrow y(x+z) + qy(y-z) + qx(y-z) - z^2 = px(z-x) + py(z-x) + zx$

$\Rightarrow \boxed{y(x+z) + q(x+y)(y-z) - z^2 = p(x+y)(z-x) + z(y+z) + xy}$

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Q8 $z = ax + by$

$\frac{\partial z}{\partial x} = p = a$

$\frac{\partial z}{\partial y} = q = b$

$\Rightarrow \boxed{z = px + qy}$

Q9 $z = y f\left(\frac{y}{x}\right) \quad \Rightarrow f\left(\frac{y}{x}\right) = \frac{z}{y}$

$\frac{\partial z}{\partial x} = p = y f'\left(\frac{y}{x}\right) \cdot \left(\frac{-y}{x^2}\right) \quad \Rightarrow f'\left(\frac{y}{x}\right) = -\frac{px^2}{y^2}$

$\frac{\partial z}{\partial y} = q = y f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + f\left(\frac{y}{x}\right) \quad \Rightarrow f'\left(\frac{y}{x}\right) = \left(q - \frac{z}{y}\right)\frac{x}{y}$

$\Rightarrow -\frac{px^2}{y^2} = \frac{(qy - z)x}{y^2}$

$\Rightarrow \boxed{z = px + qy}$

Q10 $pq = k$, k is a constant
$pq = k$ — (1)

Let $\underbrace{z = ax + by + c}_{(2)}$ be a trial solution

Differentiate (2) partially w.r.t. x and y respectively
$\Rightarrow p = \frac{\partial z}{\partial x} = a \Rightarrow \boxed{p = a}$

$\Rightarrow q = \frac{\partial z}{\partial y} = b \Rightarrow \boxed{q = b}$

Substitute value of p and q in eqⁿ (1)

$\Rightarrow ab = k$

$\Rightarrow \boxed{b = \frac{k}{a}}$

Substitute value of b in eqⁿ (2)

$\Rightarrow \boxed{z = ax + \frac{ky}{a} + c}$ → Complete solution — (3)

Differentiate (3) partially w.r.t. c

$\Rightarrow 0 = 0 + 0 + 1$

$\Rightarrow 0 = 1$ (which is absurd)

∴ <u>No singular solution</u>

<u>General Solution</u>:

Let $c = \phi(a)$

$\Rightarrow z = ax + \frac{ky}{a} + \phi(a)$ — (4)

Differentiate (4) partially w.r.t. a

$\Rightarrow 0 = x - \frac{ky}{a^2} + \phi'(a)$ — (5)

Solving (4) and (5) and eliminating a will give the general solⁿ

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Q11 $z = px + qy - 2\sqrt{pq}$

$z = px + qy - 2\sqrt{pq}$

$\Rightarrow p = a, \quad q = b$

$\Rightarrow \boxed{z = ax + by - 2\sqrt{ab}} \rightarrow$ Common Integral

$\hookrightarrow$ ①

Differentiate ① partially w.r.t. a

$0 = x - \frac{\cancel{2}}{\cancel{2}} \times \frac{1}{\sqrt{ab}} \cdot b^{\sqrt{b}}$

$\Rightarrow \boxed{x = \sqrt{\frac{b}{a}}}$

Differentiate ① partially w.r.t b

$0 = y - \frac{\cancel{2}}{\cancel{2}} \times \frac{1}{\sqrt{ab}} \cdot x^{\sqrt{a}}$

$\boxed{y = \sqrt{\frac{a}{b}}}$

$x - z = -\cancel{\sqrt{ab}} - \sqrt{ab} + 2\sqrt{ab} + \sqrt{\frac{b}{a}}$

$\Rightarrow \boxed{x - z = \sqrt{\frac{b}{a}}}$ — ②

$x - y = \sqrt{\frac{a}{b}} - \cancel{\sqrt{ab}} - \cancel{\sqrt{ab}} + 2\cancel{\sqrt{ab}}$

$\Rightarrow \boxed{x - y = \sqrt{\frac{a}{b}}}$ — ③

Multiplying eqⁿ ③ and ②

$\boxed{(x-z)(x-y) = 1} \rightarrow$ Singular solⁿ / Integral

Q12 $(p+q)(z-px-qy)=1$

$\Rightarrow z = px + qy + \frac{1}{p+q}$

$\Rightarrow p = a, \quad q = b$

$\therefore \boxed{z = ax + by + \frac{1}{a+b}} \rightarrow$ Common Integral $\rightarrow (1)$

Differentiate (1) partially w.r.t a

$0 = x - \frac{1}{(a+b)^2}$

$\Rightarrow x = \frac{1}{(a+b)^2} \Rightarrow a+b = \frac{1}{\sqrt{x}}$

Differentiate (1) partially w.r.t b

$0 = y - \frac{1}{(a+b)^2}$

$\Rightarrow y = \frac{1}{(a+b)^2} \Rightarrow a+b = \frac{1}{\sqrt{y}}$

$z = \frac{a}{(a+b)^2} + \frac{b}{(a+b)^2} + \frac{(a+b)}{(a+b)^2}$

$\Rightarrow z = \frac{2\cancel{(a+b)}}{(a+b)^{\cancel{2}}} \Rightarrow z = \frac{2}{a+b}$

Now, $x+y = \frac{2}{(a+b)^2}$

$\Rightarrow a+b = \frac{\sqrt{2}}{\sqrt{x+y}}$

$$\Rightarrow z = \frac{2}{\sqrt{\frac{2}{x+y}}}$$

$$\Rightarrow \boxed{z = \sqrt{2(x+y)}} \rightarrow \text{Singular Sol}^n\text{/ Integral}$$

Q13 $pqz = p^2(xq + p^2) + q^2(yp + q^2)$

$$\Rightarrow z = px + qy + \frac{p^3}{q} + \frac{q^3}{p}$$

$$\Rightarrow p = a, \; q = b$$

$$\therefore \boxed{z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}} \rightarrow \text{Common Integral} \quad (1)$$

Differentiate (1) partially w.r.t a

$$\Rightarrow 0 = x + \frac{3a^2}{b} - \frac{b^3}{a^2}$$

$$\Rightarrow x = \frac{b^3}{a^2} - \frac{3a^2}{b} \Rightarrow \boxed{x = \frac{b^4 - 3a^4}{a^2 b}}$$

Differentiate (1) partially w.r.t b

$$\Rightarrow 0 = y - \frac{a^3}{b^2} + \frac{3b^2}{a}$$

$$\Rightarrow y = \frac{a^3}{b^2} - \frac{3b^2}{a} \Rightarrow \boxed{y = \frac{a^4 - 3b^4}{ab^2}}$$

Q14 $(1-x^2)yp^2 + x^2q = 0$

$\Rightarrow (1-x^2)yp^2 = -x^2q$

$\Rightarrow (x^2-1)yp^2 = x^2q$

$\Rightarrow (x^2-1)\frac{p^2}{x^2} = \frac{q}{y} = a^2$ (say)

$\Rightarrow (x^2-1)\frac{p^2}{x^2} = a^2$ $\qquad\qquad$ $\Rightarrow \frac{q}{y} = a^2$

$\Rightarrow p^2 = \frac{a^2x^2}{x^2-1}$ $\qquad\qquad$ $\Rightarrow \boxed{q = a^2y}$

$\Rightarrow \boxed{p = \frac{ax}{\sqrt{x^2-1}}}$

Now, $dz = p\,dx + q\,dy$

$\Rightarrow dz = \frac{ax}{\sqrt{x^2-1}}dx + a^2y\,dy$

Integrating $\Rightarrow z = a\int \frac{x}{\sqrt{x^2-1}}dx + a^2\int y\,dy$

Now let $I_1 = \int \frac{x}{\sqrt{x^2-1}}dx$ $\qquad\qquad$ Let $I_2 = \int y\,dy$

Let $x^2-1 = t$ $\qquad\qquad$ $\Rightarrow I_2 = \frac{y^2}{2} + c_2$

$\Rightarrow 2x\,dx = dt$

$\Rightarrow x\,dx = dt/2$

$\therefore I_1 = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{\cancel{2}}{\cancel{2}}\sqrt{t} + c_1$

$\Rightarrow I_1 = \sqrt{x^2-1} + c_1$

$$\Rightarrow Z = \cancel{8}a\sqrt{x^2-1} + \frac{a^2y^2}{2} + C_1 + C_2$$

$$\Rightarrow \boxed{Z = a\sqrt{x^2-1} + \frac{a^2y^2}{2} + C} \quad -(1) \qquad (\text{where } C = C_1 + C_2)$$

↳ Common Integral

Differentiating (1) w.r.t c

$$\Rightarrow 0 = 0 + 0 + 1$$

$$\Rightarrow 0 = 1 \text{ (which is absurd)}$$

$\therefore$ No singular solution exist.

Q15 $(x^2-yz)p + (y^2-zx)q = z^2-xy$

Subsidiary eqn

$$\underset{(1)}{\frac{dx}{x^2-yz}} = \underset{(2)}{\frac{dy}{y^2-zx}} = \underset{(3)}{\frac{dz}{z^2-xy}} = \frac{d(x+y+z)}{x^2+y^2+z^2-xz-yz-xy} = \frac{d(y-z)}{(x+y+z)(y-z)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xz-yz-xy)}$$

Take (1, 1, 1) as multiplier to create a new ratio

$$\Rightarrow \frac{dx+dy+dz}{x^2-yz+y^2-zx+z^2-xy} = \frac{d(x+y+z)}{x^2+y^2+z^2-xz-yz-xy} \quad -(4)$$

Take (x, y, z) as multiplier to create a new ratio

$$\Rightarrow \frac{x\,dx+y\,dy+z\,dz}{x^3+y^3+z^3-3xyz} = \frac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xz-yz-xy)} \quad -(5)$$

Compare ratios (4) and (5)

$$\frac{d(x+y+z)}{\cancel{(x^2+y^2+z^2-xz-yz-xy)}} = \frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xz-yz-xy)}$$

$$\Rightarrow \int (x+y+z)\, d(x+y+z) = \int x\,dx + y\,dy + z\,dz$$

$$\Rightarrow \frac{(x+y+z)^2}{2} - \frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = c_1$$

$$\Rightarrow (x+y+z)^2 - (x^2+y^2+z^2) = 2c_1$$

$$\Rightarrow \boxed{(x+y+z)^2 - (x^2+y^2+z^2) = a} \quad (\text{where } a = 2c_1)$$

Take $(0, 1, -1)$ as multiplier to create a new ratio

$$\Rightarrow \frac{dy - dz}{y^2 - zx - z^2 + xy} = \frac{d(y-z)}{x(y-z) + (y-z)(y+z)}$$

$$= \frac{d(y-z)}{(x+y+z)(y-z)} \quad -(6)$$

Take $(1, -1, 0)$ as multiplier to create a new rati

$$\Rightarrow \frac{dx - dy}{x^2 - y^2 - yz + zx} = \frac{d(x-y)}{(x+y)(x-y) + z(x-y)} = \frac{d(x-y)}{(x+y+z)(x-y)} \ -$$

Compare ratio (6) and (7)

$$\Rightarrow \frac{d(y-z)}{\cancel{(x+y+z)}(y-z)} = \frac{d(x-y)}{\cancel{(x+y+z)}(x-y)}$$

$$\Rightarrow \int \frac{d(y-z)}{y-z} = \int \frac{d(x-y)}{x-y}$$

$$\Rightarrow \log(y-z) - \log(x-y) = \log b$$

$$\Rightarrow \log\left(\frac{y-z}{x-y}\right) = \log b$$

$$\therefore \boxed{\frac{y-z}{x-y} = b}$$

$$\therefore \boxed{\phi\left((x+y+z)^2 - (x^2+y^2+z^2), \frac{y-z}{x-y}\right) = 0}$$

↳ General Solution

Q16 $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$

Subsidiary Equation

$$\underset{①}{\frac{dx}{y^2(x-y)}} = \underset{②}{\frac{dy}{-x^2(x-y)}} = \underset{③}{\frac{dz}{z(x^2+y^2)}}$$

Comparing / Grouping ratios ① and ②

$$\frac{dx}{y^2\cancel{(x-y)}} = \frac{dy}{-x^2\cancel{(x-y)}}$$

$$\Rightarrow \cancel{\int \frac{dx}{-x^2} = \int \frac{dy}{y^2}} \Rightarrow \int -x^2 dx = \int y^2 dy$$

$$\Rightarrow \frac{x^3}{3} + \frac{y^3}{3} = c_1 \Rightarrow \boxed{x^3 + y^3 = a} \quad \text{where } a =$$

Take $(1,-1,0)$ as multiplier to create a new ra

$$\frac{dx - dy}{y^2(x-y) + x^2(x-y)} = \frac{d(x-y)}{(x^2+y^2)(x-y)} \quad -④$$

Comparing ratios ③ and ⑤

$$\frac{d(x-y)}{(x^2+y^2)(x-y)} = \frac{dz}{z(x^2+y^2)}$$

$$\Rightarrow \int \frac{d(x-y)}{(x-y)} = \int \frac{dz}{z}$$

$$\Rightarrow \log(x-y) - \log z = \log b$$

$$\Rightarrow \log\left(\frac{x-y}{z}\right) = \log b$$

$$\Rightarrow \boxed{\left(\frac{x-y}{z}\right) = b}$$

$$\Rightarrow \boxed{\phi\left(x^3+y^3, \frac{x-y}{z}\right) = 0} \rightarrow \text{General solution}$$

Q17 $(y^2+z^2-x^2)p - 2xyq = -2xz$

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

Subsidiary equation   ①   ②   ③

Grouping ratios ② and ③

$$\frac{dy}{\cancel{-2x}y} = \frac{dz}{\cancel{-2x}z} \Rightarrow \int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\Rightarrow \log y - \log z = \log a$$

$$\Rightarrow \log\left(\frac{y}{z}\right) = \log a$$

$$\Rightarrow \boxed{\frac{y}{z} = a}$$

Take (x, y, z) as multipliers to create a new ra

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{-x^3 - xy^2 - xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{-x(x^2 + y^2 + z^2)} \quad - (5)$$

Comparing ratios (2) and (5)

$$\Rightarrow \frac{dy}{+2\cancel{x}y} = \frac{x\,dx + y\,dy + z\,dz}{+\cancel{x}(x^2 + y^2 + z^2)}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2 + y^2 + z^2)}$$

$$\Rightarrow \log y - \log(x^2 + y^2 + z^2) = \log b$$

$$\Rightarrow \log\left(\frac{y}{x^2 + y^2 + z^2}\right) = \log(b)$$

$$\Rightarrow \boxed{\frac{y}{x^2 + y^2 + z^2} = b}$$

$$\Rightarrow \boxed{\phi\left(\frac{y}{z}, \frac{y}{x^2 + y^2 + z^2}\right) = 0} \rightarrow \text{General solution}$$

Q18 $xzp + yzq = xy$

subsidiary equation

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} = \frac{d(xy)}{2xyz}$$

(1), (2), (3)

Comparing ratios (1) and (2)

$$\frac{dx}{x\cancel{z}} = \frac{dy}{y\cancel{z}}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\Rightarrow \log x - \log y = \log a \Rightarrow \log\left(\frac{x}{y}\right) = \log a$$

$$\Rightarrow \boxed{\frac{x}{y} = a}$$

Taking $\left(\frac{1}{x}, \frac{1}{y}, 0\right)$ as multiplier to create a new ratio

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy}{2z} = \frac{y\,dx + x\,dy}{2xyz} = \frac{d(xy)}{2xyz} \quad \text{—(5)}$$

Comparing ratios (3) and (5)

$$\Rightarrow \frac{dz}{xy} = \frac{d(xy)}{2xyz}$$

$$\Rightarrow \int 2z\,dz = \int d(xy)$$

$$\Rightarrow \boxed{z^2 - xy = b}$$

$$\Rightarrow \boxed{\phi\left(\frac{x}{y}, z^2 - xy\right) = 0} \rightarrow \text{General solution}$$

Q19 $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x-y)$

Subsidiary equation

$$\frac{dx}{x^2 - y^2 - yz} \; (1) = \frac{dy}{x^2 - y^2 - zx} \; (2) = \frac{dz}{z(x-y)} \; (3) = \frac{d(x-y)}{z(x-y)} = \frac{2d(x-y)}{(x-y)(x+y)}$$

Take multipliers $(1, -1, -1)$ to make the denominator 0

$$\Rightarrow \frac{dx - dy - dz}{x^2 - y^2 - yz - x^2 + y^2 + zx - zx + zy}$$

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$$= \frac{dx - dy - dz}{0} \Rightarrow dx - dy - dz = 0$$

$$\Rightarrow \int dx - \int dy - \int dz = 0$$

$$\Rightarrow \boxed{x - y - z = a}$$

Take multiplier as $(1,-1,0)$ to create a new ratio

$$\frac{dx - dy}{x^2 - y^2 - x^2 + y^2 - yz + zx} = \frac{d(x-y)}{z(x-y)} \quad - (4)$$ (Not needed)

Take multiplier as $(x,-y,0)$ to create a new ratio

$$\Rightarrow \frac{x\,dx - y\,dy}{x(x^2-y^2) - \cancel{xyz} - y(x^2-y^2) + \cancel{xyz}} = \frac{x\,dx - y\,dy}{(x-y)(x^2-y^2)} \quad - (5)$$

Comparing ratios (3) and (5)

$$\frac{dz}{z\cancel{(x-y)}} = \frac{2x\,dx - 2y\,dy}{2\cancel{(x-y)}(x^2-y^2)}$$

$$\Rightarrow \int \frac{2\,dz}{z} = \int \frac{d(x^2-y^2)}{(x^2-y^2)}$$

$$\Rightarrow \log z^2 - \log(x^2-y^2) = \log b$$

$$\Rightarrow \log\left(\frac{z^2}{x^2-y^2}\right) = \log b$$

$$\Rightarrow \boxed{\frac{z^2}{x^2-y^2} = b}$$

$$\Rightarrow \boxed{\phi\left(x-y-z, \frac{z^2}{x^2-y^2}\right) = 0} \rightarrow \text{General solution}$$

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Q20 $u_x = 4u_y$; $u(0,y) = 8e^{-3y} + 4e^{-5y}$
$\hookrightarrow$ ①

Let $U(x,y) = X(x)\,Y(y)$
Where X is a function of x only
and Y is a function of y only

$\Rightarrow$ ~~$U = X_1 Y_1$~~ $U = X_0 Y_0$ — ②

Differentiate ② partially w.r.t x and y respectively

$\Rightarrow u_x = X_0' Y_0$
and $u_y = X_0 Y_0'$

Substituting in ①
$u_x = 4u_y$
$\Rightarrow X'Y = 4XY'$
$\Rightarrow \frac{X'}{4X} = \frac{Y'}{Y} = k$ (say)

$\Rightarrow \frac{X'}{4X} = k$

$\Rightarrow \int \frac{X'}{X} = \int 4k$

$\Rightarrow \log X = 4kx + \log c_1$
$\Rightarrow \log X = \log c_1 e^{4kx}$

$\Rightarrow \boxed{X = c_1 e^{4kx}}$

$\Rightarrow \int \frac{Y'}{Y} = \int k$

$\Rightarrow \log Y = ky + \log c_2$

$\Rightarrow \log Y = \log c_2 e^{ky}$

$\Rightarrow \boxed{Y = c_2 e^{ky}}$

Now, $U = X_1Y_1 + X_2Y_2$ is also a solution

because, $\frac{d(X_1+X_2)}{dx} - 4\frac{d(Y_1+Y_2)}{dy}$

$$= \frac{dX_1}{dx} - 4\frac{dY_1}{dy} + \frac{dX_2}{dx} - 4\frac{dY_2}{dy}$$

$$= \boxed{0} = \frac{dX}{dx} - 4\frac{dY}{dy} = u_x - 4u_y$$

$$\therefore U(x,y) = C_1C_2e^{k_1(4x+y)} + C_3C_4e^{k_2(4x+y)}$$

$$\Rightarrow U(x,y) = ae^{k_1(4x+y)} + be^{k_2(4x+y)} \quad (2)$$

where $C_1C_2 = a$ and $C_3C_4 = b$

Given $U(0,y) = 8e^{-3y} + 4e^{-5y}$ - (3)

$$\Rightarrow U(0,y) = ae^{k_1y} + be^{k_2y} \quad -(4)$$

Comparing (3) and (4)

$a = 8$, $b = 4$, $k_1 = -3$, $k_2 = -5$

$$\therefore \boxed{U(x,y) = 8e^{-3(4x+y)} + 4e^{-5(4x+y)}} \rightarrow \text{General solution}$$

Q21 $u_x = 2u_t + u$ where $u(x,0) = 6e^{-3x}$

$\hookrightarrow$ (1)

Let $u = X(x)\,T(t)$

Where X is a function of x only

and T is a function of t only

$$\Rightarrow u = XT \quad - (2)$$

Differentiate (2) partially w.r.t x ant t respectively

$$\Rightarrow u_x = X'T \quad , \quad u_t = XT'$$

Substituting in (1)

$$X'T = 2XT' + XT$$

$$\Rightarrow \frac{X'}{X} = \frac{2T'}{T} + 1 = k \text{ (say)}$$

$$\Rightarrow \int \frac{X'}{X} = \int k$$

$$\Rightarrow \log X = kx + \log C_1$$

$$\Rightarrow \log X = \log e^{kx} + \log C_1$$

$$\Rightarrow \log X = \log C_1 e^{kx}$$

$$\Rightarrow \boxed{X = C_1 e^{kx}}$$

$$\Rightarrow \int \frac{T'}{T} + \int \frac{1}{2} = \int \frac{K}{2}$$

$$\Rightarrow \log T = \left(\frac{K-1}{2}\right)t + \log C_2$$

$$\Rightarrow \log T = \log e^{\left(\frac{K-1}{2}\right)t} + \log C_2$$

$$\Rightarrow \log T = \log C_2 e^{\left(\frac{K-1}{2}\right)t}$$

$$\Rightarrow \boxed{T = C_2 e^{\left(\frac{K-1}{2}\right)t}}$$

Substituting in (2)

$$U = XT = C_1 C_2 e^{kx} e^{\left(\frac{K-1}{2}\right)t}$$

$$\Rightarrow U = a e^{kx} e^{\left(\frac{K-1}{2}\right)t} \quad \text{Where } C_1 C_2 = a$$

Now, $u(x,0) = ae^{kx} = 6e^{-3x}$ (given)

$$\Rightarrow a = 6, \ k = -3$$

$$\Rightarrow \frac{k-1}{2} = \frac{-3-1}{2} = -\frac{4}{2} = -2$$

$\therefore \boxed{u(x,t) = 6e^{-3x} e^{-2t}} \rightarrow$ General Solution