MATHS DA-2

Q1 $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3$ $\qquad$ $f_x = 0 \Rightarrow 3x^2 - 3 = 0$

$f_y = 3y^2 - 12$ $\qquad$ $x^2 - 1 = 0$

$f_{xx} = 6x$ $\qquad$ $x = 1, -1$

$f_{yy} = 6y$

$f_{xy} = 0$ $\qquad$ $f_y = 0 \Rightarrow 3y^2 - 12 = 0$

$y^2 = 4$

$\Rightarrow y = 2, -2$

Stationary points : $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$

Points	A f_{xx}	B f_{xy}	C f_{yy}	$\Delta = AC - B^2$
(1,2)	6	0	12	72 $\rightarrow$ minima $= \boxed{2}$
(1,-2)	6	0	-12	-72 $\rightarrow$ saddle point
(-1,2)	-6	0	12	-72 $\rightarrow$ saddle point.
(-1,-2)	-6	0	-12	72 $\rightarrow$ maxima $= \boxed{38}$

Q2 $V = 32\ ft^3$

$x \rightarrow$ length, $y \rightarrow$ breadth, $z \rightarrow$ height

$V = xyz$

T.S.A. $= xy + 2yz + 2xz = (g)$

For minimum TSA,

$\nabla(g) = \lambda \nabla V$

$g_x = y + 2z \qquad V_x = yz$

$g_y = x + 2z \qquad V_y = xz$

$g_z = 2y + 2x \qquad V_z = xy$

$\Rightarrow g_x = \lambda V_x$

$g_y = \lambda V_y$

$g_z = \lambda V_z$

$\Rightarrow y + 2z = \lambda(yz) - ①$

$x + 2z = \lambda(xz) - ②$

$2y + 2x = \lambda(xy) - ③$

$\lambda(xyz) = xy + 2xz - ④$

$\lambda(xyz) = xy + 2yz - ⑤$

$\lambda(xyz) = 2xz + 2yz - ⑥$

From ④ and ⑤

$\cancel{xy} + \cancel{2}xz = \cancel{xy} + \cancel{2}yz$

$\Rightarrow [x = y]$

From ⑤ and ⑥

$xy + 2y\cancel{z} = 2xz + 2\cancel{yz}$

$\Rightarrow y = 2z$

$\therefore [x = y = 2z]$

Now, $V = 32 = xyz$

$\therefore \frac{x^3}{2} = 32$

$\Rightarrow x^3 = 64$

$\Rightarrow \boxed{x = 4}$

$\therefore$ $x = 4$ feet

$y = 4$ ft.

$z = 2$ ft.

Q3 $F(x,y,z) = xy^2z^3$, $g(x,y,z) = x+y+z = 6$, $x>0, y>0, z>0$

$F_x = y^2z^3$ $\quad g_x = 1 \quad F_x = \lambda g_x$

$F_y = 2xyz^3$ $\quad g_y = 1 \quad F_y = \lambda g_y$

$F_z = 3xy^2z^2$ $\quad g_z = 1 \quad F_z = \lambda g_z$

$\Rightarrow y^2z^3 = \lambda \quad -①$

$2xyz^3 = \lambda \quad -②$

$3xy^2z^2 = \lambda \quad -③$

From ① and ③

$\cancel{y^2}z^{\cancel{3}} = 3x\cancel{y^2}\cancel{z^2}$

$\Rightarrow \boxed{z = 3x} \Rightarrow \boxed{z = \frac{3y}{2}}$

$\therefore \cancel{\left[x = \frac{y}{2} = \frac{z}{3}\right]}$

From ① and ②

$y^{\cancel{2}}\cancel{z^3} = 2x\cancel{y}\cancel{z^3}$

$\boxed{y = 2x}$ ④

$\therefore \left[2x = y = \frac{2z}{3}\right]$

$\cancel{x+y+z=6}$

$\cancel{\Rightarrow x + 2x + 3x = 6}$

$\cancel{\Rightarrow}$

$x+y+z = 6$

$\Rightarrow \frac{y}{2} + y + \frac{3y}{2} = 6$

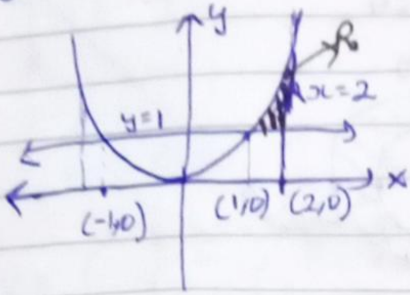
$\Rightarrow 6y = 12 \Rightarrow \boxed{y=2}$

$\therefore (x,y,z) = (1,2,3)$

$\therefore F(x,y,z) = 1 \times 2^2 \times 3^3$

$= \boxed{108}$

Q4 a) $y=x^2,\ x=2,\ y=1$



b) c) $\iint_R (x^2+y^2)\,dx\,dy$

$$= \int_1^4 \int_{\sqrt{y}}^{2} (x^2+y^2)\,dx\,dy$$

$$= \int_1^4 \left[\frac{x^3}{3}+y^2x\right]_{\sqrt{y}}^{2} dy$$

$$= \int_1^4 \left(\frac{8}{3}+2y^2\right)-\left(\frac{y^{3/2}}{3}+y^{5/2}\right) dy$$

$$= \left[\frac{8}{3}y+\frac{2y^3}{3}-\frac{2}{15}y^{5/2}-\frac{2}{7}y^{7/2}\right]_1^4$$

$$= \left[\frac{32}{3}+\frac{128}{3}-\frac{2}{15}\times 32-\frac{2}{7}\times 128\right]-\left[\frac{8}{3}+\frac{2}{3}-\frac{2}{15}-\frac{2}{7}\right]$$

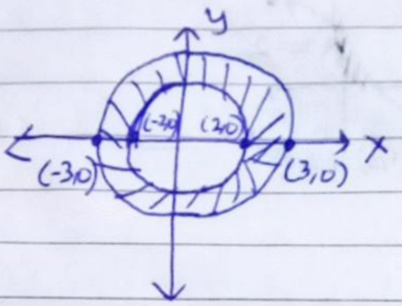
$$= \frac{160}{3}-\frac{64}{15}-\frac{256}{7}-\frac{10}{3}+\frac{2}{15}+\frac{2}{7}$$

$$= \frac{150}{3}-\frac{62}{15}-\frac{254}{7} = \frac{5250-434-3810}{105}$$

$$= \boxed{\frac{1006}{105}}$$

Q5 $\iint \sqrt{x^2+y^2}\, dx\, dy$ $\quad x^2+y^2=4,$

$x^2+y^2=9$

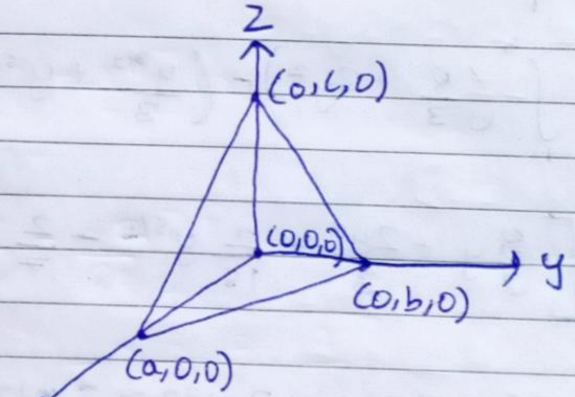


$$\int_0^{2\pi}\int_2^3 r \times r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\right]_3^2 d\theta = \frac{19}{3} \times 2\pi = \boxed{\frac{38\pi}{3}}$$

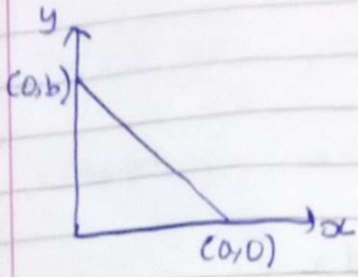
Q6 $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$V = \iiint_V dz\, dy\, dx$$



$\frac{z}{c} = 1 - \frac{x}{a} - \frac{y}{b} \Rightarrow z = c - \frac{cx}{a} - \frac{cy}{b}$

$\frac{y}{b} = 1 - \frac{x}{a} - \frac{z}{c} \Rightarrow y = b - \frac{bx}{a} - \frac{bz}{c}$



$$m = \frac{b}{-a}$$

$$\Rightarrow y - b = -\frac{b}{a}x$$

$$\Rightarrow y = b - \frac{bx}{a}$$

$$V = \int_0^a \int_0^{b-\frac{bx}{a}} \int_0^{c-\frac{cx}{a}-\frac{cy}{b}} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b-bx/a} [z]_0^{c-\frac{cx}{a}-\frac{cy}{b}}\,dy\,dx$$

$$= \int_0^a \int_0^{b-\frac{bx}{a}} \left(c - \frac{cx}{a} - \frac{cy}{b}\right) dy\,dx$$

$$= \int_0^a \left[cy - \frac{cxy}{a} - \frac{cy^2}{2b}\right]_0^{b-\frac{bx}{a}} dx$$

$$= \int_0^a cb - \frac{cbx}{a} - \frac{cx}{a}\left(b - \frac{bx}{a}\right) - \frac{c}{2b}\left(b - \frac{bx}{a}\right)^2 dx$$

$$= \int_0^a cb - \frac{cbx}{a} - \frac{cbx}{a} + \frac{cbx^2}{a^2} - \frac{c}{2b}\left(b^2 + \frac{b^2x^2}{a^2} - 2\frac{b^2x}{a}\right) dx$$

$$= \left[cbx - \frac{cbx^2}{a} + \frac{cbx^3}{3a^2} - \frac{cbx}{2} - \frac{cbx^3}{6a^2} + \frac{cbx^2}{2a}\right]_0^a$$

$$= \cancel{abc} - \cancel{abc} + \frac{abc}{3} - \cancel{\frac{abc}{2}} - \frac{abc}{6} + \cancel{\frac{abc}{2}}$$

$$= \boxed{\frac{abc}{6}}$$

Q7 $$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=\sqrt{x^2+y^2}}^{2} xyz\,dz\,dy\,dx$$

$$= \frac{1}{2}\int_{0}^{1}\int_{0}^{1}\left[xyz^2\right]_{\sqrt{x^2+y^2}}^{2} dy\,dx$$

$$= \frac{1}{2}\int_{0}^{1}\int_{0}^{1} 4xy - xy(x^2+y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_{0}^{1}\int_{0}^{1}(4xy - x^3y + xy^3)\,dy\,dx$$

$$= \frac{1}{2}\int_{0}^{1}\left[2xy^2 - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_{0}^{1} dx$$

$$= \frac{1}{2}\int_{0}^{1}\left(2x - \frac{x^3}{2} - \frac{x}{4}\right) dx$$

$$= \frac{1}{2}\left[x^2 - \frac{x^4}{8} - \frac{x^2}{8}\right]_{0}^{1}$$

$$= \frac{1}{2}\left[1 - \frac{1}{8} - \frac{1}{8}\right] = \frac{1}{2}\times\frac{3}{4} = \boxed{\frac{3}{8}}$$

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Q8 $Z = \sqrt{x^2+y^2}$, $z = x^2+y^2$

Volume $= \iiint\limits_V dx\,dy\,dz$

$= \iiint\limits_V r\,dz\,dr\,d\theta$

$Z = \sqrt{r^2} = r$, $z = r^2$

$\Rightarrow V = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 r(r-r^2)\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{r^3}{3} - \frac{r^4}{4}\right]_0^1 d\theta$

$= \frac{1}{12} \times 2\pi = \boxed{\frac{\pi}{6}}$

Q9 $\displaystyle\iiint_R \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}} = I$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$ $\qquad a > b > 0$

$z = \rho \cos\theta$ $\qquad x^2+y^2+z^2 = \rho^2$

$$I = \int_0^{\pi}\int_0^{2\pi}\int_b^{a} \frac{\cancel{\rho^2}\sin\phi}{\rho^{\cancel{3}}}\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \left[\ln \rho\right]_a^b \sin\phi\, d\theta\, d\phi$$

$$= \ln\left(\frac{a}{b}\right)\int_0^{\pi} 2\pi \sin\phi\, d\phi$$

$$= 2\pi \ln\left(\frac{a}{b}\right)\left[-\cos\phi\right]_0^{\pi}$$

$$= \boxed{4\pi \ln\left(\frac{a}{b}\right)}$$

Q10 $f(x,y) = x^2+xy+y^2-3x+3y$

$f_x = 2x+y-3=0$ — (1)

$f_y = x+2y+3=0$ — (2)

(1) × (2) → $4x+2y-6=0$ — (3)

(3) − (2)

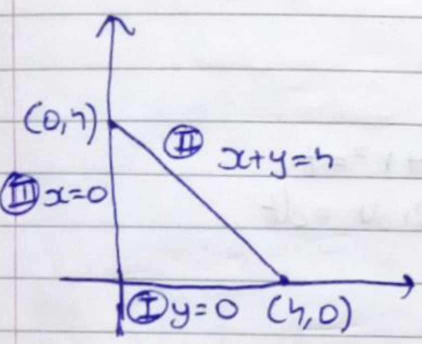
$\Rightarrow 4x+2\cancel{y}-x-\cancel{2y}-3-6=0$

$\Rightarrow 3x-9=0$

$\Rightarrow \boxed{x=3} \Rightarrow \boxed{y=-3}$

Stationary point → $(3,-3)$

$f(3,-3) = \cancel{9}-\cancel{9}+\cancel{9}-\cancel{9}-9 = \boxed{-9}$



(I) on $y=0$

$f'(x,0) = 2x-3=0$

$\Rightarrow \boxed{x=\frac{3}{2}}$

$f(4,0)=4$

$f(0,0)=0$

$f(\frac{3}{2},0) = \frac{9}{4} - 3\times\frac{3}{2} = \frac{9}{4}-\frac{9}{2} = \boxed{-\frac{9}{4}}$

(II) on $y=4-x$

$f(x,4-x) = x^2+x(4-x)+(4-x)^2-3x+3(4-x)$

$= \cancel{x^2}+4x-\cancel{x^2}+16+x^2-8x-3x+12-3x$

$= x^2-10x+28$

$\Rightarrow f'(x,4-x) = 2x-10=0$

$\Rightarrow \boxed{x=5}$

$f(5,-1) = 25-5+1-15-3$

$= \boxed{3}$

(III) on $x=0$

$f(0,y) = y^2+3y$

$f'(0,y) = 2y+3=0$

$\Rightarrow \boxed{y=-\frac{3}{2}}$

$f(0,\frac{3}{2}) = \frac{9}{4}-\frac{3}{2} = \boxed{-\frac{9}{4}}$

$f(0,4) = 28$

Absolute maximum $= 28$ at $(0,4)$

Absolute minimum $= -\frac{9}{4}$ at $(\frac{3}{2},0)$

Q11 $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\,dy\,dx}{(1+x^2+y^2)^2}$

$\displaystyle\int_{0}^{2\pi}\int_{0}^{1} \frac{2r\,dr\,d\theta}{(1+r^2)^2}$ $\qquad 1+r^2=t$

$2r\,dr = dt$

$\displaystyle\int_{0}^{2\pi}\int_{1}^{2} \frac{dt\,d\theta}{t^2}$

$\displaystyle = 2\pi\int_{0} \left[-\frac{1}{t}\right]_1^2 d\theta = \frac{1}{2}\times 2\pi = \boxed{\pi}$

Q12 $I = \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-(x^2+y^2)}^{(x^2+y^2)} 21xy^2 \, dz\, dy\, dx$

$= 21 \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{-r^2}^{r^2} r\cos\theta \, r^2 \sin^2\theta \, r \, dz\, dr\, d\theta$

$= 21 \int_{-\pi/2}^{\pi/2} \int_0^1 r^4 \times 2r^2 \cos\theta \sin^2\theta \, dr\, d\theta$

$= 42 \int_{-\pi/2}^{\pi/2} \left[\frac{r^7}{7}\right]_1^0 \cos\theta \sin^2\theta \, d\theta$

$= 6 \int_{-\pi/2}^{\pi/2} \cos\theta \sin^2\theta \, d\theta$

$\sin\theta = t$

$\cos\theta = dt$

$= 6 \int_{-1}^{1} t^2 dt = 2[t^3]_{-1}^{1} = \boxed{4}$

Q13 $I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} dz\, dy\, dx$

$= \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$

$= \int_0^{\pi/4} \int_0^{2\pi} \left[\frac{\rho^3}{3}\right]_0^{\sec\phi} \sin\phi \, d\theta\, d\phi$

$= \int_0^{\pi/4} \frac{\sec^3\phi}{3} \sin\phi \times 2\pi \, d\phi$

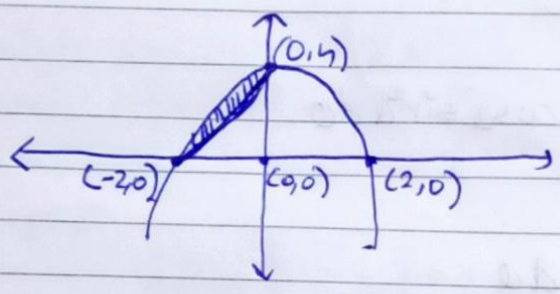
$$= \frac{2\pi}{3} \int_0^{\pi/4} \tan\phi \sec^2\phi \, d\phi$$

$\tan\phi = t$
$\sec^2\phi \, d\phi = dt$

$$= \frac{2\pi}{3} \int_0^1 t\,dt = \frac{2\pi}{3} \left[\frac{t^2}{2}\right]_0^1$$

$$= \frac{2\pi}{3} \times \frac{1}{2} = \boxed{\frac{\pi}{3}}$$

Q14 $y = 2x+4$, $y = 4-x^2$



$$A = \int_0^{-2} \int_{2x+4}^{4-x} dy\,dx = -\int_{-2}^{0} \int_{2x+4}^{4-x^2} dy\,dx$$

$$= \int_0^{-2} [y]_{2x+4}^{4-x^2} dx$$

$$= \int_0^{-2} 4 - x^2 - 2x - 4 \, dx$$

$$= \left[-\frac{x^3}{3} - x^2\right]_0^{-2} = \left|\frac{8}{3} - 4\right| = \boxed{\frac{4}{3} \text{ sq. units}}$$