DA-1 - MATHS

Q1 $y = x^4 - 4x^3$

$y' = 4x^3 - 12x^2$

$y'(x) = 0$

$\Rightarrow 4x^2(x-3) = 0$

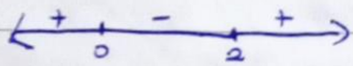
$\therefore x = 0, 3$

$y'' = 12x^2 - 24x$

$y''(x) = 0$

$\Rightarrow 12x(x-2) = 0$

$\therefore x = 0, 2$



Interval	Concavity
$(-\infty, 0)$	Upwards
$(0, 2)$	downwards
$(2, \infty)$	Upwards

→ The point of ~~intersections~~ inflections are at $x = 0$ and $x = 2$

$y''(0) = 0$

$y''(3) = 12(9) - 24(3)$

$= 108 - 72 = 36$

$\therefore$ y has local minima at $x = 3$.

Q2 (i) $f(x) = 2x^3 + 3x^2 - 36x$

$f'(x) = 6x^2 + 6x - 36$

$f'(x) = 0$

$\Rightarrow 6x^2 + 6x - 36 = 0$

$\Rightarrow x^2 + x - 6 = 0$

$\Rightarrow (x-2)(x+3) = 0$

$\therefore x = 2, -3$

$(-\infty, -3) \rightarrow$ Increasing

$(-3, 2) \rightarrow$ Decreasing

$(2, \infty) \rightarrow$ Increasing

→ Here, the critical points are at $x=2$ and $x=-3$

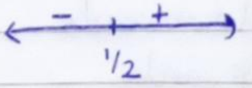
$f''(x) = 12x+6$

$f''(2) = 30$

$f''(-3) = -30$

$\therefore$ $f(x)$ has local minima at $x=2$ and
local maxima at $x=-3$

$f''(x) = 0$



$\Rightarrow 2x+1=0$

$\therefore x = -\frac{1}{2}$

Interval	Concavity
$(-\infty, -1/2)$	downwards
$(-1/2, \infty)$	upwards

(ii) $f(x) = 4x^3 + 3x^2 - 6x + 1$

$f'(x) = 12x^2 + 6x - 6 = 0$

$= 2x^2 + x - 1 = 0$

→ Critical points are $x = -1, 1/2$

$f'(x) = 0$

$\Rightarrow 2x^2 + x - 1 = 0$

$\Rightarrow (2x-1)(x+1) = 0$

$\therefore x = \frac{1}{2}, -1$

$(-\infty, -1)$ → Increasing

$(-1, 1/2)$ → decreasing

$(1/2, \infty)$ → Increasing

→ $f''(x) = 24x + 6$

$f''(-1) = -24 + 6 = -18$

$f''(1/2) = 12 + 6 = 18$

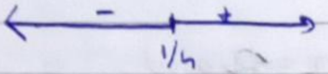
$\Rightarrow$ $f(x)$ has local maxima at $x=-1$ and
local minima at $x = 1/2$

$f''(x) = 0$

$\Rightarrow 4x + 1 = 0$

$\Rightarrow x = -\frac{1}{4}$

→ Here, the point of inflection is at $x = -1/4$



<u>interval</u>

$(-\infty, -1/4) \rightarrow$ downwards

$(-1/4, \infty) \rightarrow$ Upwards

(iii) $f(x) = x^4 - 2x^2 + 3$

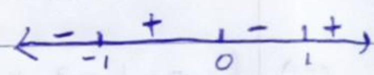
$f'(x) = 4x^3 - 4x$

$f'(x) = 0$

$\Rightarrow 4x(x^2 - 1) = 0$

$\therefore x = 0, 1, -1$

→ Here the critical points are 0, 1, −1



$(-\infty, -1) \rightarrow$ decreasing

$(0, 1) \rightarrow$ decreasing

$(-1, 0) \rightarrow$ increasing

$(1, \infty) \rightarrow$ increasing

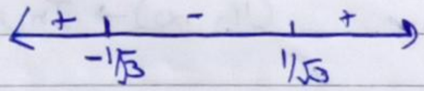
$f''(x) = 0$

$\Rightarrow 12x^2 - 4 = 0$

$\Rightarrow x^2 = 1/3$

$\therefore x = +\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

→ Here the point of inflection are at $x = \frac{1}{\sqrt{3}}$, $x = -\frac{1}{\sqrt{3}}$



<u>Interval</u>	<u>Concavity</u>
$(-\infty, -1/\sqrt{3})$	Upwards
$(-1/\sqrt{3}, 1/\sqrt{3})$	downwards
$(1/\sqrt{3}, \infty)$	Upwards

Q3 (i) $f(x) = 5 - 12x + 3x^2, \quad x \in [1,3]$

(1) $f(x)$ is continuous on $[1,3]$
as it is a polynomial

(2) $f(x)$ is differentiable on $(1,3)$
$f'(x) = -12 + 6x$

(3) $f(1) = f(3)$
$f(1) = 5 - 12 + 3 = -4$
$f(3) = 5 - 36 + 27 = -4$

$\therefore$ According to Rolle's Theorem,
$f'(c) = 0$ for some $x \in [0,4]$

$\Rightarrow f'(c) = -12 + 6c = 0$
$\Rightarrow \boxed{c = 2}$

(ii) $f(x) = x^3 - x^2 - 6x + 2, \quad x \in [0,3]$

(1) $f(x)$ is continuous on $[0,3]$
as it is a polynomial

(2) $f(x)$ is differentiable on $(0,3)$
$f'(x) = 3x^2 - 2x - 6$

(3) $f(0) = f(3)$
$f(0) = 2$
$f(3) = 27 - 9 - 18 + 2 = 2$

$\therefore$ According to Rolle's Theorem,
$f'(c) = 0$ for some $c \in [0,3]$

$$\Rightarrow f'(c) = 3c^2 - 2c - 6 = 0$$

$$c = \frac{2 \pm \sqrt{76}}{6}$$

$$\Rightarrow c = \frac{1 \pm \sqrt{19}}{3}$$

$$\therefore c = \frac{1+\sqrt{19}}{3}, \quad c = \frac{1-\sqrt{19}}{3}$$

$$\because c \in (0,3)$$

$$\Rightarrow \boxed{c = \frac{1+\sqrt{19}}{3}}$$

Q4 (i) $f(x) = 3x^2 + 2x + 5$, $x \in [-1, 1]$

(1) $f(x)$ is continuous on $[-1, 1]$
as it is a polynomial

(2) $f(x)$ is differentiable on $(-1, 1)$
$f'(x) = 6x + 2$

According to Mean value theorem,
$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad c \in (-1, 1)$$

$$\Rightarrow f'(c) = \frac{10 - 6}{1 - (-1)} = \frac{4}{2}$$

$$\Rightarrow f'(c) = 2$$

$$\Rightarrow \quad 6c + 2 = 2$$

$$\Rightarrow \boxed{c = 0}$$

(ii) $f(x) = x^3 + x - 1, \quad x \in [0, 2]$

(i) $f(x)$ is continuous on $[0, 2]$
~~(2) for~~ as it is a polynomial

(2) $f(x)$ is differentiable on $(0, 2)$
$f'(x) = 3x^2 + 1$

According to mean value theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad c \in (0, 2)$$

$$\Rightarrow f'(c) = \frac{9 + 1}{2} = \frac{10}{2}$$

$$\Rightarrow f'(c) = 5$$

$$\Rightarrow 3c^2 + 1 = 5$$

$$\Rightarrow c^2 = 4/3$$

$$\Rightarrow c = \pm \frac{2}{\sqrt{3}}$$

$\because c \in (0, 2)$

$\therefore \boxed{c = \frac{2}{\sqrt{3}}}$

Q5 (i) $f(x) = x^3 - 6x^2 + 9x + 1, \quad x \in [2,4]$

$f'(x)\ 3x^2 - 12x + 9$

$f'(x) = 0$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x-1)(x-3) = 0$

$\Rightarrow x = 1 \text{ OR } x = 3$

$x \in [2,4] \Rightarrow x = 3$

$f''(x) = 6x - 12$

$f''(3) = 18 - 12 = 6$

$\therefore f(x)$ has local minima at $x = 3$ in the given interval

$f(2) = 8 - 24 + 18 + 1 = 3$

$f(3) = 27 - 54 + 27 + 1 = 1$

$f(4) = 64 - 96 + 36 + 1 = 5$

$\therefore f(x)$ has absolute maximum at $x = 4$ and absolute minimum at $x = 3$

(ii) $f(x) = x\sqrt{1-x}, \quad x \in [-1,1]$

$f'(x) = \sqrt{1-x} + x \times \frac{1}{2} \times \frac{1}{\sqrt{1-x}} (-1)$

$\Rightarrow f'(x) = \sqrt{1-x} - \frac{x}{2\sqrt{1-x}}$

$\Rightarrow f'(x) = \frac{2 - 2x - x}{2\sqrt{1-x}}$

$\Rightarrow f'(x) = \frac{2 - 3x}{2\sqrt{1-x}}$

$f'(x) = 0$

$\Rightarrow \dfrac{2-3x}{2\sqrt{1-x}} = 0$

$\Rightarrow x = \dfrac{2}{3}$

$$f''(x) = \frac{2\sqrt{1-x}\cdot(-3) - (2-3x)\left(\frac{-\cancel{2}}{\cancel{2}\sqrt{1-x}}\right)}{4(1-x)}$$

$$\Rightarrow f'(x) = \frac{-6\sqrt{1-x} + \left(\frac{2-3x}{\sqrt{1-x}}\right)}{4(1-x)}$$

$$f''\left(\frac{2}{3}\right) = \frac{-6\times 1/\sqrt{3} + \frac{2-(3\times 2/3)}{1/\sqrt{3}}}{4\times 1/3}$$

$$\Rightarrow f''\left(\frac{2}{3}\right) = \frac{-6/\sqrt{3}}{4/3} = \frac{-\overset{3}{\cancel{6}}\times 3\sqrt{3}}{\underset{2}{\cancel{4}}\times\sqrt{3}} = -\frac{3\sqrt{3}}{2}$$

$\therefore$ $f(x)$ has local maxima at $x = \dfrac{2}{3}$

$f(2/3) = \dfrac{2}{3\sqrt{3}}$

$f(1) = 0$

$f(-1) = -\sqrt{2}$

$\therefore$ $f(x)$ has its absolute maximum at $x = 2/3$ and absolute minimum at $x = -1$

Q6 i) $f(x,y) = \frac{-x}{\sqrt{x^2+y^2}}$

(a) Calculating the limit along the path $y=x$ for $x>0$,

$$\lim_{x\to 0^+} \frac{-x}{\sqrt{x^2+y^2}}$$

$$= \lim_{x\to 0^+} \frac{-x}{\sqrt{2x^2}}$$

$$= \lim_{x\to 0^+} \frac{-x}{\sqrt{2}x}$$

$$= \lim_{x\to 0^+} -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

(b) Now, along the path $y=x$ for $x<0$,

$$\lim_{x\to 0^-} = \frac{-x}{\sqrt{x^2+y^2}}$$

$$= \lim_{x\to 0^-} = \frac{-x}{\sqrt{2x^2}}$$

$$= \lim_{x\to 0^-} = \frac{-x}{\sqrt{2}(-x)}$$

$$= \lim_{x\to 0^-} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

As the 2 limits are not equal

$\therefore$ limit does not exist for the given function.

(ii) $f(x,y) = \dfrac{x^4}{x^4+y^2}$

(a) Calculating the limit along the path $y = mx$ for $x > 0$

$$f(x, mx) = \frac{x^4}{x^2(x^2+m^2)} \Rightarrow \cancel{\frac{x^2}{(x^2+m^2)}}$$

$$\lim_{x\to 0^+} = \frac{x^2 - m^2}{x^2+m^2}$$

$$\lim_{x\to 0^+} = -\frac{m^2}{m^2} = -1$$

(b) Now, along the path $y = x^2$

$$f(x, x^2) = \frac{x^4 - x^4}{x^4+x^4} = 0$$

→ As limits are different limit does not exist.

(iii) $f(x,y) = \dfrac{x^4 - y^2}{x^4+y^2}$

(a) Limit along the path $y = x^2$

$$\lim_{x\to 0^+} \frac{x^4 - x^4}{x^4+x^4} = 0$$

(b) Limit along the path $x = y^2$

$$f(y^2, y) = \frac{y^8 - y^2}{y^8+y^2} = \frac{y^6 - 1}{y^6+1}$$

$$\lim_{y\to 0^+} \frac{y^6-1}{y^6+1} = -1$$

→ As two limits are not same, the limit does not exist

(iv) $f(x,y) = \frac{xy}{|xy|}$

(a) Limit when $x>0$ and $y>0$
$f(x,y)=1$

(b) Limit when $x>0$ and $y<0$
$f(x,y)=-1$

(c) Limit when $x<0$ and $y>0$
$f(x,y)=-1$

(d) Limit when $x<0$ and $y<0$

$f(x,y)=1$

As results are not ~~so~~ equal, limit does not exist

Q7

(i) a) $f(x,y)=\sin(x+y)$
→ Continuous at all points on the plane

b) $f(x,y)=\ln(x^2+y^2)$
→ Continuous at all points on the plane except $(0,0)$

(ii) a) $f(x,y)=\frac{x+y}{x-y}$

→ Continuous at all points on the plane except points where $x=y$

b) $f(x,y) = \frac{y}{x^2+1}$

→ Continuous at all points on the plane

$g(x,y) = \sin\frac{1}{xy}$

→ Continuous at all points except $(0,0)$, $(0,y)$, $(x,0)$

~~$g(x,y) = \ln(x+y)$~~

$g(x,y) = \frac{x+y}{2+\cos x}$

→ Continuous at all points on the plane

Q8 i) $f(x,y) = x + y + xy = f$

→ $f_x = \frac{\partial f}{\partial x} = 1 + y$

→ $f_y = \frac{\partial f}{\partial y} = 1 + x$

→ $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 0$

→ $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 0$

→ $f_{xy} = 1 = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

→ $f_{yx} = 1$

ii) $f(x,y) = \sin xy = f$

$f_x = y\cos xy$
$f_{xx} = -y^2 \sin xy$
$f_{xy} = -xy\sin xy + \cos xy$

$f_y = x\cos y$
$f_{yy} = -x^2 \sin xy$
$f_{yx} = \cos xy - xy\sin xy$

(iii) $g(x,y) = x^2y + \cos y + y\sin x = g$

$g_x = 2xy + y\cos x$

$g_{xx} = 2y - y\sin x$

$g_{xy} = 2x + \cos x$

$g_y = x^2 - \sin y + \sin x$

$g_{yy} = -\cos y$

$g_{yx} = 2x + \cos x$

(iv) $h(x,y) = xe^y + y + 1 = h$

$h_x = e^y$
$h_{xx} = 0$
$h_{xy} = e^y$

$h_y = xe^y + 1$
$h_{yy} = xe^y$
$h_{yx} = e^y$

(v) $r(x,y) = \ln(x+y) = r$

$r_x = \frac{1}{x+y}$

$r_{xx} = -\frac{1}{(x+y)^2}$

$r_{xy} = \frac{-1}{(x+y)^2}$

$r_y = \frac{1}{x+y}$

$r_{yy} = \frac{-1}{(x+y)^2}$

$r_{yx} = \frac{-1}{(x+y)^2}$

Q9 (i) $w = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$, $t = 0$

$\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y$

$\frac{\partial x}{\partial t} = \cos t - \sin t$ $\quad$ $\frac{\partial y}{\partial t} = -\sin t - \cos t$

$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial t}\right)$

$\frac{\partial w}{\partial t} = 2x(\cos t - \sin t) + 2y(-\sin t - \cos t)$

at $t=0$, $x=1$, $y=1$

Putting t, x and y in $\frac{dw}{dt}$

$\Rightarrow \frac{\partial w}{\partial t} = 2(1) + 2(-1) = 2-2$

$\Rightarrow \boxed{\frac{\partial w}{\partial t} = 0}$

(ii) $w = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = \frac{1}{t}$, $t = 3$

$\frac{\partial w}{\partial x} = \frac{1}{z}$, $\frac{\partial w}{\partial y} = \frac{1}{z}$, $\frac{\partial w}{\partial z} = \frac{-x-y}{z^2}$

$\frac{\partial x}{\partial t} = -\sin 2t$, $\frac{\partial y}{\partial t} = \sin 2t$, $\frac{\partial z}{\partial t} = \frac{-1}{t^2}$

$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial t}\right)$

$$\frac{dw}{dt} = -\frac{\sin \cancel{2}t}{\cancel{z}} + \frac{\sin \cancel{2}t}{\cancel{z}} + \frac{x+y}{z^2t^2} = \frac{x+y}{z^2t^2}$$

at $t=3$, $x = \cos^2(3)$, $y = \sin^2(3)$, $z = 1/3$

Putting t, x, y and z in $\frac{dw}{dt}$

$$\Rightarrow \frac{dw}{dt} = \frac{\cos^2(3) + \sin^2(3)}{1/9 \times 9}$$

$$\Rightarrow \boxed{\frac{dw}{dt} = 1}$$

(iii) $w = \ln(x^2+y^2+z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$, $t = 3$

$$\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+z^2}, \quad \frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2}, \quad \frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2}$$

$$\frac{\partial x}{\partial t} = -\sin t, \quad \frac{\partial y}{\partial t} = \cos t, \quad \frac{\partial z}{\partial t} = \frac{2}{\sqrt{t}}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial t}\right)$$

$$= \frac{-2x\sin t + 2y\cos t + 4z/\sqrt{t}}{x^2+y^2+z^2}$$

$$= \frac{-2x\sin t + 2y\cos t + 4z/\sqrt{t}}{1+z^2}$$

Put $t=3$ in $\frac{dw}{dt}$

$$\Rightarrow \frac{dw}{dt} = \frac{-2(\cos 3)(\sin 3) + 2\sin(3)\cos(3) + 4 \times \frac{4\sqrt{t}}{\sqrt{t}}}{1 + (16 \times 3)} \Rightarrow \boxed{\frac{dw}{dt} = \frac{16}{49}}$$

Q11 $z = 2x^2 + y^2 = f$, $P(1,1,3)$

Tangent plane eqⁿ: $[f_x(x_0,y_0)](x-x_0) + [f_y(x_0,y_0)](y-y_0) - (z-z_0) = 0$

$f_x = 4x$ $\qquad$ $f_y = 2y$ $\qquad$ $x_0 = 1$ $\quad$ $z_0 = 3$

$\Rightarrow f_x(1,1) = 4$ $\qquad$ $f_y(1,1) = 2$ $\qquad$ $y_0 = 1$

$\Rightarrow$ Tangent plane eqⁿ: $4(x-1) + 2(y-1) - (z-3) = 0$

$\Rightarrow 4x - 4 + 2y - 2 - z + 3 = 0$

$\Rightarrow \boxed{4x + 2y - z = 3}$

Q12 $z = e^x \sin y$, $x = st^2$, $y = s^2 t$

$\rightarrow \dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\left(\dfrac{\partial x}{\partial s}\right) + \dfrac{\partial z}{\partial y}\left(\dfrac{\partial y}{\partial s}\right)$

$= e^x \sin y\,(t^2) + e^x \cos y\,(2st)$

$$\boxed{\frac{\partial z}{\partial s} = e^x t\,(t\sin y + 2s\cos y)}$$

$\rightarrow \dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\left(\dfrac{\partial x}{\partial t}\right) + \dfrac{\partial z}{\partial y}\left(\dfrac{\partial y}{\partial t}\right)$

$= e^x \sin y\,(2st) + e^x \cos y\,(s^2)$

$$\boxed{\frac{\partial z}{\partial t} = s e^x (2t\sin y + s\cos y)}$$

Q13 i) $x^2+y^2+z^2 = 3xyz \Rightarrow x^2+y^2+z^2-3xyz=0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(2x-3yz)}{2z-3xy}$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{3yz-2x}{2z-3xy}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(2y-3xz)}{2z-3xy}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{3xz-2y}{2z-3xy}}$$

ii) $xyz = \cos(x+y+z) \Rightarrow xyz-\cos(x+y+z)=0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-(yz+\sin(x+y+z))}{xy+\sin(x+y+z)}$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-yz-\sin(x+y+z)}{xy+\sin(x+y+z)}}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \boxed{\frac{-xz-\sin(x+y+z)}{xy+\sin(x+y+z)}}$$

iii) $x-z = \arctan(yz) \Rightarrow x-z-\arctan(yz)=0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-1}{\frac{-y}{\sqrt{1+y^2z^2}}-1}$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{\sqrt{1+y^2+z^2}}{y+\sqrt{1+y^2+z^2}}}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{z/\sqrt{1+y^2+z^2}}{\frac{-y}{\sqrt{1+y^2+z^2}}-1}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{z}{-y-\sqrt{1+y^2+z^2}}}$$

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iv) $yz = \ln(x+z) \Rightarrow yz - \ln(x+z) = 0$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{\frac{1}{x+z}}{y - \frac{1}{x+z}}$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{1}{xy+zy-1}}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-z}{y - \frac{1}{x+z}}$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-z(x+z)}{xy+yz-1}}$$

Q14 $u = f(x,y)$, $x = e^s \cos t$, $y = e^s \sin t$

To prove: $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$

$$\frac{\partial x}{\partial s} = e^s \cos t, \quad \frac{\partial y}{\partial s} = e^s \sin t$$

$$\frac{\partial x}{\partial t} = -e^s \sin t, \quad \frac{\partial y}{\partial t} = e^s \cos t$$

Now, $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial s}\right)$

$$\Rightarrow \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} e^s \cos t + \frac{\partial u}{\partial y} e^s \sin t \quad -(1)$$

Now,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}(-e^s \sin t) + \frac{\partial u}{\partial y}(e^s \cos t) \quad -(2)$$

Now,

$$\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2(e^{2s}\cos^2 t)+\left(\frac{\partial u}{\partial y}\right)^2(e^{2s}\sin^2 t)$$

$$+\left(\frac{\partial u}{\partial x}\right)^2(e^{2s}\sin^2 t)+\left(\frac{\partial u}{\partial y}\right)^2(e^{2s}\cos^2 t)$$ (from ① and ③)

$$\Rightarrow \left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2\left[e^{2s}(1)\right]+\left(\frac{\partial u}{\partial y}\right)^2\left[e^{2s}(1)\right]$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\frac{1}{e^{2s}}\left[\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]$$

(Hence proved)

Q15 $z=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$

To prove: $$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}$$

$$\frac{\partial^2 z}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)=\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial r}\right)+\frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial r}\right)\right]$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)$$

$$=\cos\theta\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial r}\right)+\sin\theta\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial r}\right)$$

$$=\cos\theta\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right]+\sin\theta\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial r^2}=\cos^2\theta\frac{\partial^2 z}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 z}{\partial y^2} \quad -①$$

$$\Rightarrow \frac{\partial^2 z}{\partial \theta^2} = -r\frac{\partial z}{\partial r} + r^2\sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{-1}{r}\frac{\partial z}{\partial r} + \sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 z}{\partial y^2} \quad -(2)$$

Adding (1) and (2) (and preplace $\sin^2\theta + \cos^2\theta$ by 1)

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = -\frac{1}{r}\frac{\partial z}{\partial r} + \frac{\partial^2 z}{\partial x^2}(1) + \frac{\partial^2 z}{\partial x^2}(1) + \cancel{2\sin\theta\cos\theta\frac{\partial z^2}{\partial x\partial y}} - \cancel{2\sin\theta\cos\theta\frac{\partial z^2}{\partial x\partial y}}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

(Hence proved)

Q16 i) $x = 5u - v, \quad y = u + 3v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix}$$

$$= \boxed{16}$$

ii) $x = e^{-r}\sin\theta, \quad y = e^{r}\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} -e^{-r}\sin\theta & e^{-r}\cos\theta \\ e^{r}\cos\theta & -e^{r}\sin\theta \end{vmatrix}$$

$$= \sin^2\theta - \cos^2\theta = \boxed{-\cos 2\theta}$$

iii) $x = e^{s+t}, \quad y = e^{s-t}$

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}$$

$$= \begin{vmatrix} e^{s}e^{t} & e^{s}e^{t} \\ e^{s}e^{-t} & -e^{s}e^{-t} \end{vmatrix} = -e^{2s} - e^{2s}$$

$$= \boxed{-2e^{2s}}$$

iv) $x = \frac{u}{v}, \; y = \frac{v}{w}, \; z = \frac{w}{u}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1/w & -v/w^2 \\ -w/u^2 & 0 & 1/u \end{vmatrix}$$

$$= \frac{1}{v}\left(\frac{1}{wu}\right) + \frac{\cancel{u}}{v^{\cancel{2}}}\left(-\frac{\cancel{v}}{wu^{\cancel{2}}}\right)$$

$$= \frac{1}{uvw} - \frac{1}{uvw} = \boxed{0}$$

Q17 $xy + x^2u - vy^2 = 0 \quad - (F)$

$3x - 4uy - x^2v = 0 \quad - (G)$

(i) $\boxed{\frac{\partial u}{\partial x}}$:

$$\frac{\partial(F,G)}{\partial(x,v)} = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial x} & \frac{\partial F}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} y+2xu & -y^2 \\ 3-2xv & -x^2 \end{vmatrix} = -x^2y - 2x^3u + 3y^2 - 2xvy^2$$

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & -y^2 \\ -4y & -x^2 \end{vmatrix} = -x^4 - 4y^3$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial(F,G)}{\partial(x,v)} \Big/ \frac{\partial(F,G)}{\partial(u,v)}$$

$$\boxed{\therefore \frac{\partial u}{\partial x} = \frac{-x^2y - 2x^3u + 3y^2 - 2xvy^2}{-x^4 - 4y^3}}$$

ii) $\boxed{\frac{\partial v}{\partial x}}$:

$$\frac{\partial(F,G)}{\partial(u,x)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial x} \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & y+2ux \\ -4y & 3-2xv \end{vmatrix} = 3x^2 - 2x^3v + 4y^2\ 8xyu$$

$$\frac{\partial(F,G)}{\partial(\cancel{v,u})} = x^4 + 4y^3$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial(F,G)}{\partial(u,x)} \Big/ \frac{\partial(F,G)}{\partial(v,u)}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{3x^2 - 2x^3v + 4y^2 + 8xyu}{x^4 + 4y^3}}$$

Q18 $u = \dfrac{x}{y-z}$, $y = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$

For functional dependence $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{1}{y-z} & \frac{-x}{(y-z)^2} & \frac{x}{(y-z)^2} \\ \frac{y}{(z-x)^2} & \frac{1}{(z-x)} & \frac{-y}{(z-x)^2} \\ \frac{-z}{(x-y)^2} & \frac{z}{(x-y)^2} & \frac{1}{x-y} \end{vmatrix}$$

$$= \frac{1}{y-z}\left[\frac{1}{(z-x)(x-y)} + \frac{yz}{(x-y)^2(z-x)^2}\right] + \frac{x}{(y-z)^2}\left[\frac{y}{(z-x)^2(x-y)} - \frac{yz}{(x-y)^2(z-x)^2}\right]$$

$$+ \frac{x}{(y-z)^2}\left[\frac{yz}{(z-x)^2(x-y)^2} + \frac{z}{(x-y)^2(z-x)}\right]$$

$$= \frac{1}{(x-y)(y-z)(z-x)} + \frac{yz}{(x-y)^2(y-z)(z-x)^2} + \frac{xy}{(x-y)(y-z)^2(z-x)^2}$$

$$\cancel{- \frac{xyz}{(x-y)^2(y-z)^2(z-x)^2}} \cancel{+ \frac{xyz}{(x-y)^2(y-z)^2(z-x)^2}} + \frac{xz}{(x-y)^2(z-x)(y-z)^2}$$

$$= \frac{(x-y)(y-z)(z-x) + yz(y-z) + xy(x-y) + xz(z-x)}{(x-y)^2(y-z)^2(z-x)^2}$$

$$= \frac{(x-y)(yz - yx - z^2 + zx) + y^2 z - yz^2 + x^2 y - xy^2 + xz^2 - x^2 z}{(x-y)^2(y-z)^2(z-x)^2}$$

$$= \frac{xyz - x^2 y - xz^2 + x^2 z - y^2 z + y^2 x + yz^2 - xyz + y^2 z - yz^2 + x^2 y - xy^2 + xz^2 - x^2 z}{(x-y)^2(y-z)^2(z-x)^2}$$

$$= \boxed{0}$$

$\therefore$ The functions satisfy necessary condition for functional dependence

Q19 $u = e^{xy}$, $v = e^{yz}$, $w = e^{zx}$, $x = r$, $y = s^2$, $z = t^3$

$$\frac{\partial(u,v,w)}{\partial(r,s,t)} = \frac{\partial(u,v,w)}{\partial(x,y,z)} \times \frac{\partial(x,y,z)}{\partial(r,s,t)} \quad -①$$

$$\frac{\partial(u,v,w)}{\partial(r,s,t)} = \begin{vmatrix} e^{rs^2} & 2se^{rs^2} & 0 \\ 0 & 2se^{s^2t^3} & 3t^2e^{s^2t^3} \\ e^{t^3r} & 0 & 3t^2e^{t^3r} \end{vmatrix}$$

$$= e^{rs^2}\left(6st^2e^{s^2t^3 + t^3r}\right) - 2se^{rs^2}\left(-3t^2e^{s^2t^3+t^3r}\right)$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} e^{xy} & e^{xy} & 0 \\ 0 & e^{yz} & e^{yz} \\ e^{zx} & 0 & e^{zx} \end{vmatrix}$$

$$= e^{xy}\left(e^{yz+zx}\right) - e^{xy}\left(e^{zx+yz}\right)$$

$$\frac{\partial(x,y,z)}{\partial(r,s,t)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2s & 0 \\ 0 & 0 & 3t^2 \end{vmatrix} = 6st^2$$

substituting values in eqn ①

$$e^{rs^2}\left(6st^2 e^{s^2t^3+t^3r}\right) - 2se^{rs^2}\left(-3t^2 e^{s^2t^3+t^3r}\right)$$
$$= 6st^2 \left[e^{xy}\left(e^{yz+2x}\right) + e^{xy}\left(e^{zx+yz}\right)\right.$$

$\rightarrow$ <u>LHS</u> $= 6st^2 e^{rs^2}\left[e^{s^2t^3+t^3r} + e^{s^2t^3+t^3r}\right]$

<u>LHS</u> $= 12st^2 e^{rs^2+s^2t^3+t^3r}$ — ②

$\rightarrow$ <u>RHS</u> $= 12st^2\left[e^{xy}\left(e^{yz+zx}\right)\right]$

$$= 12st^2\left[e^{rs^2}\left(e^{s^2t^3+t^3r}\right)\right]$$

$$= 12st^2 e^{rs^2+s^2t^3+t^3r}$$ — ③

From ② and ③

LHS = RHS

∴ Chain for the Jacobians for the above functions is verified.

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Q20 i) $z = x^3 \ln(y^2)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= 3x^2 \ln(y^2) dx + x^3 \cdot \frac{(2y)}{y^2} dy$$

$$\therefore \boxed{dz = 3x^2 \ln(y^2) dx + \frac{2x^3}{y} dy}$$

ii) $v = y \cos xy$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\boxed{dv = (-y^2 \sin xy) dx + (\cos xy - xy \sin xy) dy}$$

iii) $m = p^5 q^3$

$$dm = \frac{\partial m}{\partial p} dp + \frac{\partial m}{\partial q} dq$$

$$\boxed{dm = 3p^5 q^3 dq + 5p^4 q^3 dp}$$

iv) $T = \frac{v}{1+uvw}$

$$dT = \frac{\partial T}{\partial u} du + \frac{\partial T}{\partial v} dv + \frac{\partial T}{\partial w} dw$$

$$= \frac{-v(vw)}{(1+uvw)^2} du + \left(\frac{1+uvw - v(uw)}{(1+uvw)^2}\right) dv + \left(\frac{-v(uv)}{(1+uvw)^2}\right) dw$$

$$\therefore \boxed{dT = \frac{dv - v^2 w\, du - v^2 u\, dw}{(1+uvw)^2}}$$