Additional Questions : CAT 1

Q1 $f(x) = x^3 - 6x^2 + 9x + 75$

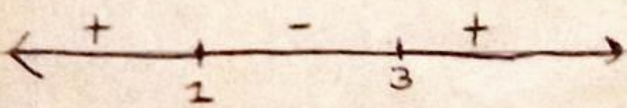
$f'(x) = 3x^2 - 12x + 9$

$f'(x) = 0 \Rightarrow 3x^2 - 12x + 9 = 0$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x-3)(x-1) = 0$

$\Rightarrow \boxed{x = 1, 3}$



$(-\infty, 1) \rightarrow$ Increasing

$(1, 3) \rightarrow$ Decreasing

$(3, \infty) \rightarrow$ Increasing

$f''(x) = 6x - 12$

$f''(1) = -6 < 0$

$f''(3) = 6 > 0$

$\therefore$ $f(x)$ has local minima at $x = 3$
and local maxima at $x = 1$

Q2 $y = x^4 - x^2, \ y = x^2$

$\Rightarrow x^4 - x^2 = x^2$

$\Rightarrow 2x^2 - x^4 = 0$

$\Rightarrow x^2(2 - x^2) = 0$

$\Rightarrow x^2(x^2 - 2) = 0$

$\Rightarrow \boxed{x = +\sqrt{2}, -\sqrt{2}}$

$$\text{Area} = \left| \int_{-\sqrt{2}}^{\sqrt{2}} (x^4 - 2x^2)\, dx \right|$$

$$= \left| \left[\frac{x^5}{5} - \frac{2x^3}{3} \right]_{-\sqrt{2}}^{\sqrt{2}} \right|$$

$$= \left| \frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} + \frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} \right|$$

$$= \left| \frac{8\sqrt{2}}{5} - \frac{8\sqrt{2}}{3} \right|$$

$$= \left| -\frac{16\sqrt{2}}{15} \right|$$

$\therefore \boxed{\text{Area enclosed} = \frac{16\sqrt{2}}{15}}$ sq. units

Q3 $y^2 = x, \ y = 4x, \ y = 0, \ y = \frac{1}{2}$

$$\text{Volume} = \pi \int_0^{1/2} \left(y^4 - \frac{y^2}{16} \right) dy$$

$$= \pi \left[\frac{y^5}{5} - \frac{y^3}{48} \right]_0^{1/2}$$

$$= \pi \left[\frac{1}{160} - \frac{1}{384} \right]$$

$$= \pi \left(\frac{224}{61440} \right)$$

$$\boxed{\text{Volume} = 0.01145 \text{ cubic units}}$$

Q4 $5x^2y + 4xu - 6yv = 0$ — (F)

$4y - 7x^2u - 5y^2v = 0$ — (G)

$$\frac{\partial(F,G)}{\partial(x,v)} = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 10xy + 4u & -6y \\ -14xu & -5y^2 \end{vmatrix}$$

$$= \boxed{-50xy^3 - 84uxy - 20uy^2}$$

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 4x & -6y \\ -7x^2 & -5y^2 \end{vmatrix}$$

$$= \boxed{-20xy^2 - 42x^2y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial(F,G)}{\partial(x,v)} \Big/ \frac{\partial(F,G)}{\partial(u,v)}$$

$$= -\frac{50xy^3 - 84uxy - 20uy^2}{-20xy^2 - 42x^2y} = \frac{-\cancel{xy(2)}}{-\cancel{xy(2)}}\cancel{\left(\frac{25y^2 + 7u}{10y + 21x}\right)}$$

$$\Rightarrow \cancel{\boxed{\frac{\partial u}{\partial x} \neq \frac{25y^2 + 7u}{10y + 21x}}}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{25xy^3 - 42uxy - 10uy^2}{10xy^2 - 21x^2y}}$$

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Q5 $z = e^{-xy}$, $x = s^2 - 5st$, $y = 3st + t^4$

$\frac{\partial z}{\partial x} = -ye^{-xy}$, $\frac{dz}{dy} = -xe^{-xy}$

$\frac{\partial x}{\partial s} = 2s - 5t$, $\frac{\partial y}{\partial s} = 3t$

$\frac{\partial x}{\partial t} = -5s$, $\frac{\partial y}{\partial t} = 3s + 4t^3$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$

$$= (-ye^{-xy})(2s-5t) + (-xe^{-xy})(3t)$$

$$= e^{-xy}\left[-2ys - 5yt - 3xt\right]$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial s} = -e^{-xy}(3xt + 5yt + 2ys)}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$

$$= (-ye^{-xy})(-5s) + (-xe^{-xy})(3s + 4t^3)$$

$$= -e^{-xy}\left[5ys + 3xs + 4xt^3\right]$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial t} = -e^{-xy}(3xs + 4xt^3 + 5ys)}$$

Q6 $f(x,y) = 4x^2y^{1/2} + y$, $P(1,3)$

$f_x = 8xy^{1/2}$, $f_x(1,3) = 8\sqrt{3}$

$f_y = 2x^2y^{-1/2} + 1$, $f_y(1,3) = \frac{2}{\sqrt{3}} + 1$

$f_{xx} = 8y^{1/2}$, $f_{xx}(1,3) = 8\sqrt{3}$

$f_{yy} = -x^2y^{-3/2}$; $f_{yy}(1,3) = \frac{1}{3\sqrt{3}}$

$f_{xy} = \frac{4x}{\sqrt{y}}$, $f_{xy}(1,3) = \frac{4}{\sqrt{3}} = f_{yx}$

$$T(x,y) = f(\alpha,\beta) + (x-\alpha)f_x(\alpha,\beta) + (y-\beta)f_y(\alpha,\beta)$$
$$+ \frac{1}{2!}\left[(x-\alpha)^2 f_{xx}(\alpha,\beta) + 2(x-\alpha)(y-\beta)f_{xy}(\alpha,\beta) + (y-\beta)^2 f_{yy}(\alpha,\beta)\right]$$

$$= 4\sqrt{3} + 3 + (x-1)(8\sqrt{3}) + (y-3)\left[\frac{2}{\sqrt{3}} + 1\right]$$
$$+ \frac{1}{2}\left[(x-1)^2(8\sqrt{3}) + 2(x-1)(y-3)\left(\frac{4}{\sqrt{3}}\right) + (y-3)^2\left(\frac{1}{3\sqrt{3}}\right)\right]$$

$$= 4\sqrt{3} + \cancel{3} + 8\sqrt{3}x - 8\sqrt{3} + \frac{2y}{\sqrt{3}} + y - 2\sqrt{3} - \cancel{3}$$
$$+ \frac{1}{2}\left[8\sqrt{3}x^2 - 16\sqrt{3}x + 8\sqrt{3} + \frac{8}{\sqrt{3}}(xy - 3x - y + 3) + \frac{y^2}{3\sqrt{3}} - \frac{2}{\sqrt{3}}y + \frac{3}{\sqrt{3}}\right]$$
$$= \cancel{8\sqrt{3}x} + \frac{2y}{\sqrt{3}} + y - 6\sqrt{3} + 4\sqrt{3}x^2 - \cancel{8\sqrt{3}x} + 4\sqrt{3} + \frac{4}{\sqrt{3}}xy$$
$$- 4\sqrt{3}x - \frac{4}{\sqrt{3}}y + 4\sqrt{3} + \frac{y^2}{6\sqrt{3}} - \frac{y}{\sqrt{3}} + \frac{1}{2\sqrt{3}}$$
$$= 4\sqrt{3}x^2 + \frac{y^2}{6\sqrt{3}} - 4\sqrt{3}x - \frac{3y}{\sqrt{3}} + y + \frac{4}{\sqrt{3}}xy + 2\sqrt{3} + \frac{1}{2\sqrt{3}}$$

$$\boxed{T(x,y) = \frac{1}{\sqrt{3}}\left(12x^2 + \frac{y^2}{6} + 4xy - 12x - 3y + \sqrt{3}y + \frac{13}{2}\right)}$$

Additional Questions: CAT 2

Q1 $f(x,y) = x^3 + y^3 - 12x + y^2 - 6y$

$f_x = 3x^2 - 12$

$f_y = 3y^2 + 2y - 6$

$f_{xx} = 6x$

$f_{yy} = 6y + 2$

$f_{xy} = 0$

$f_x = 0$

$\Rightarrow 3x^2 - 12 = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow \boxed{x = 2, -2}$

$f_y = 0$

$\Rightarrow 3y^2 + 2y - 6 = 0$

$y = \frac{-2 \pm \sqrt{4+72}}{6}$

$\Rightarrow y = \frac{-1 \pm \sqrt{19}}{3}$

$\Rightarrow y = -\frac{1+\sqrt{19}}{3}, -\frac{1-\sqrt{19}}{3}$

$\therefore$ Stationary points are: $\left(2, -\frac{1+\sqrt{19}}{3}\right), \left(2, -\frac{1-\sqrt{19}}{3}\right),$

$\left(-2, -\frac{1+\sqrt{19}}{3}\right), \left(-2, -\frac{1-\sqrt{19}}{3}\right)$

	Points	A	B	C	Δ
		f_{xx}	f_{xy}	f_{yy}	$AC-B^2$
Minimum	$\left(2, -\frac{1+\sqrt{19}}{3}\right)$	12	0	$2(-1+\sqrt{19})+2$	$24\sqrt{19}>0$
Saddle point	$\left(2, -\frac{1-\sqrt{19}}{3}\right)$	12	0	$2(-1-\sqrt{19})+2$	$-24\sqrt{19}<0$
Saddle point	$\left(-2, -\frac{1+\sqrt{19}}{3}\right)$	-12	0	$-2(-1+\sqrt{19})+2$	$24\sqrt{19}-48<0$
Maximum	$\left(-2, -\frac{1-\sqrt{19}}{3}\right)$	-12	0	$-2(-1-\sqrt{19})+2$	$-24\sqrt{19}-48<0$

Q2 $x+2y+3z=10$, $P(1,4,6)$

Distance $= \sqrt{(x-1)^2+(y-4)^2+(z-6)^2} = d$

$$d^2 = (x-1)^2+(y-4)^2+(z-6)^2$$

Minimising d^2 will minimize d:

$2(x-1) = \lambda \Rightarrow x = \frac{\lambda+2}{2}$

$2(y-4) = 2\lambda$

$\Rightarrow y = \lambda+4$

$2(z-6) = 3\lambda$

$\Rightarrow z = \frac{3\lambda+12}{2}$

$$x + 2y + 3z = 10$$

$$\Rightarrow \frac{\lambda+2}{2} + 2(\lambda+4) + 3\left(\frac{3\lambda+12}{2}\right) = 10$$

$$\Rightarrow \lambda + 2 + 4\lambda + 16 + 9\lambda + 36 = 10$$

$$\Rightarrow 14\lambda = -44$$

$$\boxed{\lambda = -\frac{22}{7}}$$

$$x = \frac{-8}{14} = \frac{-4}{7}$$

$$y = \frac{6}{7}$$

$$z = \frac{18}{14} = \frac{9}{7}$$

$$d^2 = \left(\frac{-4}{7} - 1\right)^2 + \left(\frac{6}{7} - 4\right)^2 + \left(\frac{9}{7} - 6\right)^2$$

$$= \left(\frac{-11}{7}\right)^2 + \left(-\frac{22}{7}\right)^2 + \left(-\frac{33}{7}\right)^2$$

$$d^2 = \frac{1694}{49}$$

$$d = \frac{1}{7}\sqrt{1694}$$

$$\Rightarrow \boxed{d = 5.88 \text{ units}}$$

Q3 $x^2+y^2=49 \Rightarrow r=7$

$x=r\cos\theta$, $y=r\sin\theta$

$$\iint xy(x^2+y^2)\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^7 r^4 \cos\theta\sin\theta \, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^5}{5}\right]_0^7 \cos\theta\sin\theta\, d\theta$$

$$= \frac{7^5}{5}\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta$$

$\sin\theta = t$

$\Rightarrow \cos\theta\, d\theta = dt$

$$= \frac{7^5}{5}\int_0^1 t\,dt$$

$$= \frac{7^5}{5}\left[\frac{t^2}{2}\right]_0^1$$

$$= \frac{7^5}{5}\times\frac{1}{2}$$

$$= \boxed{1680.7}$$

Q4 $\int_0^4 \int_0^{4-x} \int_0^{4-x-y} (xyz)\, dz\, dy\, dx$

$x+y+z \leq 4$

$\Rightarrow \frac{x}{4} + \frac{y}{4} + \frac{z}{4} \leq 1$

Let $u = \frac{x}{4}$, $v = \frac{y}{4}$, $w = \frac{z}{4}$

$\Rightarrow 4du = dx$ $\Rightarrow 4dv = dy$ $\Rightarrow 4dw = dz$

$u+v+w \leq 1$

$\Rightarrow v+w = 1-u = h$

$\Rightarrow 4^6 \int_0^1 \int_0^{1-u} \int_0^{1-u-v} uvw\, dw\, dv\, du$

$\Rightarrow 4^6 \int_0^1 u \left[\int_0^{u-1} \int_0^{u-1-v} vw\, dw\, dv \right] du$

$= 4^6 \int_0^1 u \left[(1-u)^4 \cdot \frac{\sqrt{2}\sqrt{2}}{\sqrt{2+2+1}} \right] du$

$= 4^6 \times \frac{1}{4!} \int_0^1 u(1-u)^4\, du$

$= \frac{4^5}{6} \beta(2,5)$

$= \frac{4^5}{6} \frac{\sqrt{2}\sqrt{5}}{\sqrt{7}}$

$= \frac{4^5}{6} \times 1 \times \frac{4!}{6!}$

$= \frac{256}{45} = \boxed{5.689}$

Q5 $\frac{x^3}{27} + \frac{y^3}{64} + \frac{z^3}{125} = 1$

$u \quad v \quad w$

$x = 3u^{1/3}, \quad y = 4v^{1/3}, \quad z = 5w^{1/3}$

$dx = u^{-2/3}du, \quad dy = \frac{4}{3}v^{-2/3}dv, \quad z = \frac{5}{3}w^{-2/3}dw$

$\therefore \iiint_E dx\,dy\,dz = \frac{20}{9}\iiint u^{-2/3}v^{-2/3}w^{-2/3}\,dw\,dv\,du$

According to Dirichlet's integral,

$= \frac{20}{9}\frac{\Gamma\frac{1}{3}\,\Gamma\frac{1}{3}\,\Gamma\frac{1}{3}}{\Gamma 2}$

$= \boxed{\frac{20}{9}\Gamma\frac{1}{3}\,\Gamma\frac{1}{3}\,\Gamma\frac{1}{3}}$