BMAT101L DA-3

Q1 $(D^2 + 3D + 2)y = e^{-6x}$

$\Rightarrow y'' + 3y' + 2y = e^{-6x}$

AE: $r^2 + 3r + 2 = 0$

$\Rightarrow r^2 + 2r + r + 2 = 0$

$\Rightarrow (r+2)(r+1) = 0$

$\Rightarrow r = -1, -2$

$\Rightarrow \boxed{y_{CF} = C_1 e^{-x} + C_2 e^{-2x}}$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-3x} + e^{-3x}$$

$\Rightarrow \boxed{W = -e^{-3x}}$

$$y_{PI} = -y_1 \int \frac{y_2 (RHS)}{W} dx + y_2 \int \frac{y_1 (RHS)}{W} dx$$

$y_1 = e^{-x}$, $y_2 = e^{-2x}$, $W = -e^{-3x}$, $RHS = e^{-6x}$

$$= -e^{-x} \int \frac{(e^{-2x})(e^{-6x})}{(-e^{-3x})} dx + e^{-2x} \int \frac{(e^{-x})(e^{-6x})}{(-e^{-3x})} dx$$

$$= +e^{-x} \int e^{-5x} dx - e^{-2x} \int e^{-4x} dx$$

$$\Rightarrow y_{PI} = \frac{e^{-6x}}{-5} - \frac{e^{-6x}}{-4}$$

$\Rightarrow \quad y_{PI} = e^{-6x}\left[\frac{1}{4} - \frac{1}{5}\right]$

$\Rightarrow \quad \boxed{y_{PI} = \frac{e^{-6x}}{20}}$

$y = y_{CF} + y_{PI}$

$\Rightarrow \quad \boxed{y = C_1 e^{-x} + C_2 e^{-2x} + \frac{e^{-6x}}{20}}$

Q2 $\frac{d^2y}{dt^2} + \frac{3dy}{dt} + 2y = e^{-6t}, \quad y(0), y'(0) = 1$

$\Rightarrow \quad L(y'') + 3L(y') + 2L(y) = L(e^{-6t})$

$\Rightarrow \quad s^2 y(s) - sy(0) - y'(0) + 3[sy(s) - y(0)] + 2y(s) = \frac{1}{s+6}$

$\Rightarrow \quad s^2 y(s) - s - 1 + 3[sy(s) - 1] + 2y(s) = \frac{1}{s+6}$

$\Rightarrow \quad s^2 y(s) - s - 1 + 3sy(s) - 3 + 2y(s) = \frac{1}{s+6}$

$\Rightarrow \quad y(s)[s^2 + 3s + 2] = \frac{1}{s+6} + s + 4$

$\Rightarrow \quad y(s)[(s+2)(s+1)] = \frac{1 + s^2 + 6s + 4s + 24}{(s+6)}$

$\Rightarrow \quad y(s) = \frac{s^2 + 10s + 25}{(s+2)(s+1)(s+6)} = \frac{A}{(s+2)} + \frac{B}{(s+1)} + \frac{C}{(s+6)}$

$\Rightarrow \quad A(s+1)(s+6) + B(s+2)(s+6) + C(s+1)(s+2) = s^2 + 10s + 25$

Put $s=-2$:

$\Rightarrow -4A = 9$

$\Rightarrow \boxed{A = -\frac{9}{4}}$

Put $s=-1$:

$5B = 16$

$\Rightarrow \boxed{B = \frac{16}{5}}$

Put $s=-6$:

$20C = 36-60+25$

$\Rightarrow 20C = 1$

$\Rightarrow \boxed{C = \frac{1}{20}}$

$\therefore \; y(s) = -\frac{9}{4}\frac{1}{(s+2)} + \frac{16}{5}\frac{1}{(s+1)} + \left(\frac{1}{20}\right)\left(\frac{1}{(s+6)}\right)$

Inverse Laplace transforms

$\Rightarrow \boxed{y_n = \frac{16}{5}e^{-t} - \frac{9}{4}e^{-2t} + \frac{1}{20}e^{-6t}}$

Q3 $y(k+2) + 3y(k+1) + 2y(k) = k \quad y(0)=0,\ y(1)=0$

$$y(z) = Z(y(k))$$

Applying Z-transform on both sides

$$Z[y(k+2) + 3y(k+1) + 2y(k)] = Z(k)$$

$$\Rightarrow Z(y(k+2)) + 3Z(y(k+1)) + Z(2y(k)) = Z(k)$$

Now, $Z(y(k+2)) = z^2\left(y(z) - y_0 - \frac{y_1}{z}\right)$

$Z(y(k+1)) = z(y(z) - y_0)$

$Z(y(k)) = y(z)$

$Z(k) = \frac{z}{(z-1)^2}$

$$\therefore z^2(y(z) - y_0 - y_1) + 3z(y(z) - y_0) + 2y(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow z^2 \cdot y(z) - 0 - 0 + 3z \cdot y(z) - 0 + 2 \cdot y(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow y(z)[z^2 + 3z + 2] = \frac{z}{(z-1)^2}$$

$$\Rightarrow y(z)[(z+1)(z+2)] = \frac{z}{(z-1)^2}$$

$$\Rightarrow y(z) = \frac{z}{(z+1)(z-1)^2(z+2)}$$

$$\Rightarrow \frac{y(z)}{z} = \frac{1}{(z+1)(z-1)^2(z+2)} = \frac{A}{(z+1)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2} + \frac{D}{(z+2)}$$

$$\Rightarrow A(z-1)^2(z+2) + B(z-1)(z+1)(z+2) + C(z+1)(z+2) + D(z-1)^2(z+1) = 1$$

<u>Put $z=-1$:</u>

$$\Rightarrow A(4)(1) = 1$$
$$\Rightarrow \boxed{A = \frac{1}{4}}$$

<u>Put $z=1$:</u>

$$\Rightarrow C(2)(3) = 1$$
$$\Rightarrow \boxed{C = \frac{1}{6}}$$

<u>Put $z=-2$:</u>

$$\Rightarrow D(9)(-1) = 1$$
$$\Rightarrow \boxed{D = \frac{-1}{9}}$$

<u>Put $z=0$:</u>

$$\Rightarrow A(+1)(2) + B(-1)(1)(2) + C(1)(2) + D(1)(1) = 1$$

$$\Rightarrow +\frac{2}{4} - 2B + \frac{2}{6} - \frac{1}{9} = 1$$

$$\Rightarrow 2B = +\frac{1}{2} + \frac{1}{3} - \frac{1}{9} - 1$$

$$\Rightarrow 2B = \frac{+9+6-2-18}{18}$$

$$\Rightarrow \boxed{B = \frac{-5}{36}}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{1}{4(z+1)} - \frac{5}{36(z-1)} + \frac{1}{6(z-1)^2} - \frac{1}{9(z+2)}$$

$$\Rightarrow Y(z) = \frac{1}{4}\left(\frac{z}{z+1}\right) - \frac{5}{36}\left(\frac{z}{z-1}\right) + \frac{1}{6}\left(\frac{z}{(z-1)^2}\right) - \frac{\cancel{1}z}{9(z+2)}$$

Inverse Z transform

$$\Rightarrow \boxed{y_k = \frac{1}{4}(-1)^k - \frac{5}{36} + \frac{k}{6} - \frac{1}{9}(-2)^k}$$

Q4 Inverse Z transform of $\frac{z}{(z+6)(z-2)}$

Let $U(z) = Z(u_n)$

$$U(z) = \frac{z}{(z+6)(z-2)}$$

$$\frac{U(z)}{z} = \frac{1}{(z+6)(z-2)} = \frac{A}{z+6} + \frac{B}{z-2}$$

$$\Rightarrow A(z-2) + B(z+6) = 1$$

Put $z=-6$:

$-8A = 1$

$\Rightarrow \boxed{A = -\frac{1}{8}}$

Put $z=2$:

$8B = 1$

$\Rightarrow \boxed{B = \frac{1}{8}}$

$$\Rightarrow \frac{U(z)}{z} = \frac{-1}{8(z+6)} + \frac{1}{8(z-2)}$$

$$\Rightarrow U(z) = \frac{-1}{8}\left(\frac{z}{z+6}\right) + \frac{1}{8}\left(\frac{z}{z-2}\right)$$

Taking Inverse Z transforma

$$u_n = \frac{-1}{8}\left[Z^{-1}\left(\frac{z}{z+6}\right)\right] + \frac{1}{8}\left[Z^{-1}\left(\frac{z}{z-2}\right)\right]$$

$$\Rightarrow \boxed{u_n = -\frac{1}{8}(-6)^n + \frac{1}{8}(2)^n}$$