BMAT101L DA2

Q) $y_{n+2} - 4y_{n+1} - 5y_n = 24n - 8$; $y_0 = 3$, $y_1 = -5$

let $y(z) = Z(y_n)$

Applying Z transform on both the sides

$Z(y_{n+2} - 4y_{n+1} - 5y_n) = Z(24n-8)$

$\Rightarrow Z(y_{n+2}) - 4Z(y_{n+1}) - 5Z(y_n) = Z(24n) - Z(8)$ — ①

Now, $Z(y_n) = y(z)$

$Z(y_{n+1}) = z.(y(z) - y_0)$

$Z(y_{n+2}) = z^2.\left(y(z) - y_0 - \frac{y_1}{z}\right)$

$Z(n) = \frac{z}{(z-1)^2}$; $Z(1) = \frac{z}{z-1}$

- Substitute these values in eqn ①

$\Rightarrow z^2\left(y(z) - y_0 - \frac{y_1}{z}\right) - 4z(y(z) - y_0) - 5(y(z)) = \frac{24z}{(z-1)^2} - \frac{8z}{z-1}$

$\Rightarrow z^2.y(z) - 3z^2 + 5z - 4z.y(z) + 12z - 5y(z) = \frac{24z}{(z-1)^2} - \frac{8z}{z-1}$

$\Rightarrow y(z)[z^2 - 4z - 5] = \frac{24z}{(z-1)^2} - \frac{8z}{z-1} + 3z^2 - 17z$

$\Rightarrow y(z)[z^2 - 4z - 5] = \frac{24z - 8z^2 + 8z + (3z^2 - 17z)(z^2 - 2z + 1)}{(z-1)^2}$

$\Rightarrow y(z)[(z-5)(z+1)] = \frac{32z - 8z^2 + 3z^4 - 6z^3 + 3z^2 - 17z^3 + 34z^2 - 17z}{(z-1)^2}$

$\Rightarrow y(z)[(z-5)(z+1)] = \frac{3z^4 - 23z^3 + 29z^2 + 15z}{(z-1)^2}$

2) $y(z) = \dfrac{z[3z^3 - 23z^2 + 29z + 15]}{(z-1)^2 (z-5)(z+1)}$

$\Rightarrow \dfrac{y(z)}{z} = \dfrac{(3z^3 - 23z^2 + 29z + 15)}{(z-1)^2(z-5)(z+1)} = \dfrac{A}{(z-1)^2} + \dfrac{B}{(z-5)} + \dfrac{C}{(z+1)} + \dfrac{D}{(z-1)}$

$\Rightarrow A(z-5)(z+1) + B(z-1)^2(z+1) + C(z-1)^2(z-5)$
$+ D(z-1)(z-5)(z+1) = 3z^3 - 23z^2 + 29z + 15$

$\Rightarrow$ <u>Put $z=1$:</u>

$A(-8) = 3 - 23 + 29 + 15.$

$\Rightarrow A = \dfrac{24}{-8} \Rightarrow \boxed{A = -3}$

<u>Put $z = 5$:</u>

$B(16)(6) = 375 - 575 + 145 + 15$

$\Rightarrow C = -\dfrac{\cancel{40}\,\cancel{10}\,5}{4\times 4\times \cancel{6}\,3} \Rightarrow \boxed{B = -\dfrac{5}{12}}$

<u>Put $z=-1$:</u>

$C(-24) = -3 - 23 - 29 + 15$

$\Rightarrow C = +\dfrac{\cancel{40}\,5}{+3\times \cancel{8}} \Rightarrow \boxed{C = \dfrac{5}{3}}$

Put $z=0$:

$\Rightarrow -5A + B - 5C + 5D = 15$

$\Rightarrow 5D = 15 + 5A - B + 5C$

$= 1\cancel{8} - 18 + \frac{5}{12} + \frac{25}{3}$

$D = \frac{\overset{217}{\cancel{105}}}{\underset{4}{\cancel{12}} \times \cancel{5}} \Rightarrow \boxed{D = \frac{7}{4}}$

$$\Rightarrow \frac{y(z)}{z} = \frac{-3}{(z-1)^2} - \frac{5}{12}\frac{1}{(z-5)} + \frac{5}{3}\frac{1}{(z+1)} + \frac{7}{4}\frac{1}{(z-1)}$$

$$\Rightarrow y(z) = \frac{-3z}{(z-1)^2} - \frac{5z}{12(z-5)} + \frac{5z}{3(z+1)} + \frac{7z}{4(z-1)}$$

Applying Inverse Z-transform

$$y_n = -3n - \frac{5}{12}(5)^n + \frac{5}{3}(-1)^n + \frac{7}{4}$$

$$\Rightarrow \boxed{y_n = \frac{5}{3}(-1)^n - \frac{5}{12}(5)^n - 3n + \frac{7}{4}}$$

Q2 $y_{n+2} + 4y_{n+1} - 5y_n = 24;\ y_0 = 0,\ y_0 = 1$

$$Z(y_n) = y(z)$$

Applying z transform on both sides

$$Z(y_{n+2} + 4y_{n+1} - 5y_n) = Z(24)$$

$$\Rightarrow Z(y_{n+2}) + 4Z(y_{n+1}) - 5Z(y_n) = Z(24)$$

Now, $Z(y_n) = y(z)$

$$Z(y_{n+1}) = z(y(z) - y_0)$$

$$Z(y_{n+2}) = z^2\left(y(z) - y_0 - \frac{y_1}{z}\right)$$

$$z(1) = \frac{z}{z-1}$$

$$\Rightarrow z^2\left(y(z) - y_0 - \frac{y_1}{z}\right) + 4z\left(y(z) - y_0\right) - 5y(z) = \frac{24z}{z-1}$$

$$\Rightarrow z^2 y(z) - 0 - 0 + 4z.y(z) - 0 - 5\cdot y(z) = \frac{24z}{z-1}$$

$$\Rightarrow y(z)\left[z^2 + 4z - 5\right] = \frac{24z}{z-1}$$

$$\Rightarrow y(z) = \frac{24z}{(z-1)^2(z+5)}$$

$$\Rightarrow \frac{y(z)}{z} = \frac{24}{(z-1)^2(z+5)} = \frac{A}{(z-1)^2} + \frac{B}{(z+5)} + \frac{C}{(z-1)}$$

$$\Rightarrow A(z+5) + B(z-1)^2 + C(z-1)(z+5) = 24$$

Put $z=1$:

$\Rightarrow 6A=24 \Rightarrow \boxed{A=4}$

Put $z=-5$:

$36B=24$

$B=\frac{24}{36} \Rightarrow \boxed{B=\frac{2}{3}}$

Put $z=0$:

$5A+B-5C=24$

$\Rightarrow -5C=24-20-\frac{2}{3}$

$\Rightarrow -5C=\frac{10}{3} \Rightarrow \boxed{C=-\frac{2}{3}}$

$$\Rightarrow \frac{y(z)}{z}=\frac{4}{(z-1)^2}+\frac{2}{3}\left(\frac{1}{z+5}\right)-\frac{2}{3}\left(\frac{1}{z-1}\right)$$

$$\Rightarrow y(z)=\frac{4z}{(z-1)^2}+\frac{2z}{3(z+5)}-\frac{2z}{3(z-1)}$$

Applying Inverse z transform

$$\Rightarrow \boxed{y_n=4n+\frac{2}{3}(-5)^n-\frac{2}{3}}$$

Q3 $\int_{-\pi}^{\pi} (2\sin^2 5x - \cos 3x)^2 \, dx$

Parseval's identity: $\frac{1}{2l}\left[\int_{l}^{2+l} (f(x))^2 dx\right] = \frac{a_0^2}{4} + \frac{1}{2}\left[\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} 2\sin^2 5x - \cos 3x$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} (1 - \cos 10x - \cos 3x) dx$

$= \frac{1}{\pi}\left[[x]_{-\pi}^{\pi} - \left[\frac{\sin 10x}{10}\right]_{-\pi}^{\pi} - \left[\frac{\sin 3x}{3}\right]_{-\pi}^{\pi}\right.$

$= \frac{1}{\cancel{\pi}}[2\cancel{\pi}]$

$\boxed{a_0 = 2}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos(nx) dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} [(2\sin^2 5x \cos nx) - (\cos 3x \cos nx)] dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} ([1 - \cos 10x]\cos nx - \cos 3x \cos nx) dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx - \frac{1}{2}[\cos(10+n)x + \cos(10-n)x] + \cos(3+n)x + \cos(3-n)x]$

$$= \frac{1}{\pi}\left[\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{1}{2}\left[\frac{\sin(10+n)x}{(10+n)} + \frac{\sin(10-n)x}{(10-n)} + \frac{\sin(3+n)x}{(3+n)} + \frac{\sin(3-n)x}{(3-n)}\right]_{-\pi}^{\pi}\right]$$

$$\Rightarrow \boxed{a_n = 0}$$

By parseval's identity,

$$\text{RMS Value} = \sqrt{\frac{a_0^2}{4} + \frac{1}{2}\left[\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]} = \sqrt{\frac{1}{2\pi}\left[\int_{-\pi}^{\pi}(f(x))^2\,dx\right]}$$

$$\Rightarrow \frac{2^2}{4} + 0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(2\sin^2 5x - \cos 3x)^2\,dx$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}(2\sin^2 5x - \cos 3x)^2\,dx = 2\pi}$$

Q4

$$f(x) = \begin{cases} 1 & -1 \leq x < 0 \\ \frac{1}{2} & x = 0 \\ x & 0 < x \leq 1 \end{cases} \quad \text{on } [-1, 1]$$

$$a_0 = \frac{2}{2}\left[\int_{-1}^{0} 1\,dx + 0 + \int_{0}^{1} x\,dx\right]$$

$$\Rightarrow a_0 = [x]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1$$

$$\Rightarrow \boxed{a_0 = \frac{3}{2}}$$

$$a_n = \frac{2}{2}\left[\int_{-1}^{0} \cos(n\pi x)\,dx + 0 + \int_{0}^{1} x \cos(n\pi x)\,dx\right]$$

$$a_n = \left[\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{0} + \left[\frac{+x\sin(n\pi x)}{n\pi} + \frac{1}{n\pi}\frac{\cos n\pi x}{n\pi}\right]_{0}^{1}$$

$$\cancel{a_n = 0 - \frac{\cos(n\pi)}{n\pi}} \Rightarrow a_n = \frac{\cos(n\pi)}{(n\pi)^2} - \frac{1}{(n\pi)^2}$$

$$\Rightarrow \cancel{\boxed{a_n = \frac{(-1)^{n+1}}{n\pi}}} \qquad \Rightarrow \boxed{a_n = \frac{(-1)^n - 1}{(n\pi)^2}}$$

$$b_n = \frac{2}{2}\left[\int_{-1}^{0} \sin(n\pi x)\,dx + 0 + \int_{0}^{1} x\sin(n\pi x)\,dx\right]$$

$$= \left[-\frac{\cos(n\pi x)}{n\pi}\right]_{-1}^{0} + \left[\frac{-x\cos(n\pi x)}{n\pi} + 1.\frac{\sin(n\pi x)}{(n\pi)^2}\right]_{0}^{1}$$

$$\Rightarrow b_n = \frac{-1}{n\pi} + \cancel{\frac{\cos(n\pi)}{n\pi}} + \left[\cancel{-\frac{\cos(n\pi)}{n\pi}}\right]$$

$$\Rightarrow \boxed{b_n = \frac{-1}{n\pi}}$$

$$\Rightarrow \boxed{f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{(n\pi)^2}\right] + \sum_{n=1}^{\infty}\left[\frac{-1}{n\pi}\right]}$$

Q5 $f : [-\pi, \pi] \to R$

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{2^n} = \frac{\sin x}{2} + \frac{\sin 2x}{4} + \frac{\sin 3x}{8} + \frac{\sin 4x}{16} + \ldots$$

i) $\displaystyle\int_{-\pi}^{\pi} f(x) \sin 3x \, dx$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin\left(\frac{\cancel{2}(n)\cancel{\pi} x}{\cancel{2\pi}}\right) dx$$

$$\Rightarrow b_3 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin 3x \, dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \sin 3x \, dx = \pi \cdot b_3$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} f(x) \sin 3x \, dx = \frac{\pi}{8}}$$

ii) $\displaystyle\int_{-\pi}^{\pi} (f(x))^2 \, dx$

Since $(f(x))^2$ is an even function,

$$\therefore \int_{-\pi}^{\pi} (f(x))^2 \, dx = 2 \int_{0}^{\pi} (f(x))^2 \, dx$$

$$2 \int_{0}^{\pi} (f(x))^2 \, dx = \frac{2\pi}{2} \sum_{n=1}^{\infty} b_n^2$$

$$= \pi \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)^2$$

$$\Rightarrow \int_{-\pi}^{\pi} (f(x))^2\,dx = \pi \sum_{n=1}^{\infty} \left(\frac{1}{2^{2n}}\right)$$

$$= \pi \left[\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots \right]$$

$$= \pi \left[\frac{1/4}{1 - 1/4} \right]$$

$$= \pi \left[\frac{1}{3} \right]$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} (f(x))^2\,dx = \frac{\pi}{3}}$$

Q6 $(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$

Eqn ① $\rightarrow$ $y - 2 = \left(\frac{-4}{4-0}\right)(x - 0)$

$\Rightarrow y = -x + 2$ — ① $x \in [0, 4]$

Eqn ② $\rightarrow$ $y = x + 2$ — ② $x \in [-4, 0]$

Eqn ③ $\rightarrow$ $y + 2 = \left(\frac{0+2}{2}\right)(x - 4)$

$\Rightarrow y + 2 = x - 4 \Rightarrow y = x - 6$ — ③ $x \in [4, 6]$

Eqn ④ $\rightarrow$ $y = -x - 6$ — ④ $x \in [-6, -4]$

Eqn ⑤ $\rightarrow$ $y = 0$ — ⑤ $x \in [6, 10]$

Function:

$$f(x) = \begin{cases} -(x+6) & -6 \le x < -4 \\ x+2 & -4 \le x < 0 \\ -x+2 & 0 \le x < 4 \\ x-6 & 4 \le x < 6 \\ 0 & 6 \le x \le 10 \end{cases}$$

Time period: $\boxed{16}$

$$a_0 = \frac{2}{16} \int_{-6}^{10} f(x)\,dx$$

$$= \frac{1}{8} \int_{-6}^{6} f(x)\,dx + \int_{6}^{10} 0\,dx$$

$$= \frac{1}{8} \times 2 \int_{0}^{6} f(x)\,dx$$

$$= \frac{1}{4} \left[\int_{0}^{4} (-x+2)\,dx + \int_{4}^{6} (x-6)\,dx \right]$$

$$a_0 = \frac{1}{4} \left[\left[-\frac{x^2}{2} + 2x \right]_0^4 + \left[\frac{x^2}{2} + 6x \right]_4^6 \right]$$

$$a_0 = \frac{1}{4} \left[-\cancel{8} + \cancel{8} + 18 + 36 - 8 - 24 \right]$$

$$a_0 = \frac{22}{4} \Rightarrow \boxed{a_0 = \frac{11}{2}}$$

$$a_n = \frac{2}{16} \times 2 \int_0^6 f(x)\,dx$$

$$\Rightarrow a_n = \frac{1}{4}\left[\int_0^4 (-x+2)\cos\left(\frac{n\pi x}{8}\right)dx + \int_4^6 (x-6)\cos\left(\frac{n\pi x}{8}\right)dx\right]$$

$$\Rightarrow a_n = \frac{1}{4}\left[2\int_0^4 \cos\left(\frac{n\pi x}{8}\right)dx - 6\int_4^6 \cos\left(\frac{n\pi x}{8}\right)dx - \int_0^4 x\cos\left(\frac{n\pi x}{8}\right) + \int_4^6 x\cos\left(\frac{n\pi x}{8}\right)\right]$$

$$\Rightarrow a_n = \frac{1}{4}\left[\left[\frac{2\times 8}{n\pi}\sin\left(\frac{n\pi x}{8}\right)\right]_0^4 - \left[\frac{6\times 8}{n\pi}\sin\left(\frac{n\pi x}{8}\right)\right]_4^6 - \left[\frac{x\times 8}{n\pi}\sin\left(\frac{n\pi x}{8}\right) + \frac{8^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{8}\right)\right]_0^4 + \left[\frac{x\times 8}{n\pi}\sin\left(\frac{n\pi x}{8}\right) + \frac{8^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{8}\right)\right]_4^6\right]$$

$$\Rightarrow a_n = \frac{1}{4}\left[\cancel{\frac{16}{n\pi}\sin\left(\frac{n\pi}{2}\right)} - \cancel{\frac{48}{n\pi}\sin\left(\frac{3n\pi}{2}\right)} + \cancel{\frac{48}{n\pi}\sin\left(\frac{n\pi}{2}\right)} - \cancel{\frac{32}{n\pi}\sin\left(\frac{n\pi}{2}\right)} - \frac{64}{(n\pi)^2} + \cancel{\frac{48}{n\pi}\sin\left(\frac{3n\pi}{2}\right)} \cancel{+ \frac{64}{(n\pi)^2}\cos} - \cancel{\frac{32}{n\pi}\sin\left(\frac{n\pi}{2}\right)}\right]$$

$$\Rightarrow a_n = \frac{\overset{16}{\cancel{64}}}{\cancel{4}(n\pi)^2} \quad \Rightarrow \boxed{a_n = \frac{16}{n^2\pi^2}}$$

The function is even

$\therefore \boxed{b_n = 0}$ $\boxed{a_0 = \frac{11}{2}}$