Given that the momentum constraint (12.37) is linear, we can construct a *binary* black hole solution by superposition of single solutions

$$\boxed{\bar{A}^{ij} = \bar{A}^{ij}_{\mathbf{C}_1\mathbf{P}_1} + \bar{A}^{ij}_{\mathbf{C}_1\mathbf{S}_1} + \bar{A}^{ij}_{\mathbf{C}_2\mathbf{P}_2} + \bar{A}^{ij}_{\mathbf{C}_2\mathbf{S}_2}.} \tag{12.44}$$

This completes an analytic solution of the momentum constraint describing two black holes with arbitrary momenta and spins.

Exercise 12.6 Show that[9]

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 \tag{12.45}$$

is the total linear momentum of the solution (12.44) and

$$\mathbf{J} = \mathbf{C}_1 \times \mathbf{P}_1 + \mathbf{C}_2 \times \mathbf{P}_2 + \mathbf{S}_1 + \mathbf{S}_2 \tag{12.46}$$

its total angular momentum about the origin of the coordinate system.

For a binary system, $\mathbf{S}_1$ and $\mathbf{S}_2$ can be associated with the spin of the individual black holes only in the limit of *infinite* binary separation, but we nevertheless take the liberty to define the orbital angular momentum $\mathbf{L}$ as

$$\mathbf{L} \equiv \mathbf{J} - \mathbf{S}_1 - \mathbf{S}_2. \tag{12.47}$$

12.2.2 Solving the Hamiltonian constraint

With a solution to the momentum constraint at hand we can now proceed to solve the Hamiltonian constraint (12.3). Under the assumptions of conformal flatness and maximal slicing, this equation reduces to

$$\bar{D}^2\psi = -\frac{1}{8}\psi^{-7}\bar{A}_{ij}\bar{A}^{ij}. \tag{12.48}$$

We have reduced the construction of binary black hole initial data to solving a single nonlinear elliptic equation. On the right hand side, the term $\bar{A}_{ij}\bar{A}^{ij}$ can be computed analytically from (12.44). Unfortunately this term diverges at the black holes' centers $\mathbf{C}_i$. Dealing with this singularity requires some extra care.

Two different approaches have been adopted in the literature to solve this problem; they differ in the topology of the resulting solution. Recall our discussion in Chapter 3.1, which demonstrated that initial data sets representing multiple black holes are not unique. As a starting point, we can look for generalizations of the time-symmetric solution

$$\psi = 1 + \frac{\mathcal{M}_1}{r_{\mathbf{C}_1}} + \frac{\mathcal{M}_2}{r_{\mathbf{C}_2}}, \tag{12.49}$$

[9] To represent the momenta P^i and spins S^i of these solutions it is convenient to use bold-face notation $\mathbf{P}$ and $\mathbf{S}$ instead of index notation.

divergence. It is nevertheless useful to keep this decomposition in mind, since it makes the counting of freely specifiable variables more transparent. In particular we will choose $\bar{A}^{ij}_{TT} = 0$, which amounts to making two arbitrary choices for its two independent variables. We still have to find solutions $\bar{A}^{ij}_{L}$, but as we have seen in Chapter 3.2 we can construct these *analytically* given our assumptions of conformal flatness and maximal slicing. This, in fact, is the essence of this so-called *Bowen–York approach*: we can solve the momentum constraint analytically, leaving only the Hamiltonian constraint to be solved numerically.

In Chapter 3.2 we found two such analytical black hole solutions to (12.37); one that carries a linear momentum P^i,

$$\bar{A}^{ij}_{\mathrm{P}} = \frac{3}{2r^2}\left(P^i n^j + P^j n^i - (\eta^{ij} - n^i n^j) n^k P_k\right) \tag{12.38}$$

(see equation 3.80), and one that carries an angular momentum (spin) S^i,

$$\bar{A}^{ij}_{\mathrm{S}} = \frac{6}{r^3} n^{(i} \epsilon^{j)kl} S_k n_l \tag{12.39}$$

(see equation 3.66). Here we are assuming Cartesian coordinates, $r = \sqrt{x^2 + y^2 + z^2}$ is the coordinate distance to the center of the black hole (which is located at the origin), and $n^i = x^i/r$ is the normal vector pointing away from the black hole's center. In exercise 3.32 we evaluated the surface integral (3.195)

$$P^i = \frac{1}{8\pi} \oint_\infty K^{ij} dS_j \tag{12.40}$$

to show that the linear momentum associated with equation (12.38) is indeed P^i, and in exercise 3.29 we computed

$$J_i = \frac{\epsilon_{ijk}}{8\pi} \oint_\infty x^j K^{kl} dS_l \tag{12.41}$$

to verify that the angular momentum associated with equation (12.39) is S^i.

To allow for a black hole located at a point C^i we introduce a subscript (or occasionally superscript) **C**. We then have

$$\boxed{\bar{A}^{ij}_{\mathrm{CP}} = \frac{3}{2r_{\mathrm{C}}^2}\left(P^i n^j_{\mathrm{C}} + P^j n^i_{\mathrm{C}} - (\eta^{ij} - n^i_{\mathrm{C}} n^j_{\mathrm{C}}) n^k_{\mathrm{C}} P_k\right),} \tag{12.42}$$

and

$$\boxed{\bar{A}^{ij}_{\mathrm{CS}} = \frac{6}{r_{\mathrm{C}}^3} n^{(i}_{\mathrm{C}} \epsilon^{j)kl} S_k n^{\mathrm{C}}_l,} \tag{12.43}$$

where $r_{\mathrm{C}} = ||x^i - C^i||$ is the coordinate distance to the center of the black hole located at $x^i = C^i$, and $n^i = (x^i - C^i)/r_{\mathrm{C}}$ is the normal vector.