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Homework 4

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Problem 1: Given a triple of noncollinear points P_0, P_1, P_2 in the plane, any other point Q in the plane can be expressed uniquely as an affine combination of these three

$$Q = \omega_0 P_0 + \omega_1 P_1 + \omega_2 P_2,\ \omega_0 + \omega_1 + \omega_2 = 1 \tag{1}$$

The triplet $(\omega_0 + \omega_1 + \omega_2)$ of scalars is called the barycentric coordinates of Q relative to $\{P_i\}_{i=0}^{2}$.

(a) Give a drawing that illustrates the regions of the plane that are associated with each of the possible sign classes ($+$ or $-$) of the 3 barycentric coordinates. That is, label the region in which all 3 coordinates are positive with $(+,+,+)$, label the region in which only the first coordinate is negative with $(-,+,+)$, etc.

Answer: We label the sign classes I through VI and present the region of the plane in figure below.

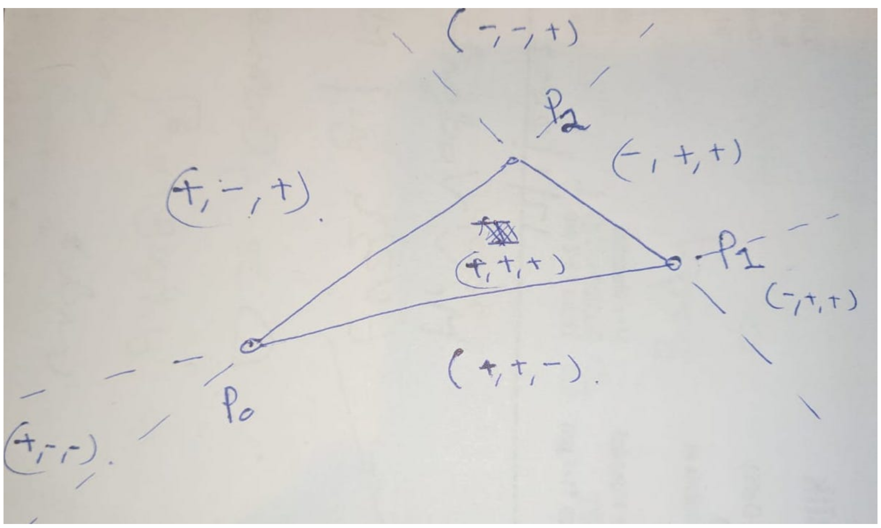

Figure 1: Affine regions

(b) Which sign class(es) are missing from the diagram? Why?

Answer: It is obvious that the sign class $(-,-,-)$ does not exist as $\omega_0 + \omega_1 + \omega_2 = 1$, as the sum of three negative numbers cannot be equal to a positive number.

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Problem 2: Answer the following two questions regarding the parametric continuity.

(a) Explain the differences between C^1 continuity and G^1 continuity for a parametric curve.

Answer: Continuity conditions for parametric curves represent how smoothly one curves blends or transitions into another.

We say that a function f is C^0 continuous at a point x, or over a region Ω, ie $f \in C^0(\Omega)$ meaning that at that point x, or for all points $x \in \Omega$, all partial derivatives up to first order are continuous (meaning that the gradient vector is continuous).

$$\{f \in C^1(\Omega) | \forall x \in \omega f,\, \partial_{x_i} f \text{ is continuous on } \Omega\} \tag{2}$$

For parametric curves, that means that the curve itself is continuous, and the first order parametric derivative is continuous (equal on all segments intersecting at that point). Such a curve is said to be first-order parametric continuous.

We say that a function f is first order geometric-continuous, $f \in G^1(\Omega)$, if the curve is continuous for all points x in Ω, as well as the parametric gradient vector is proportional (equal modulo a scaling factor).

(b) Suppose that we join two Bezier curves of degree 2 end-to-end, using the control points sequence $\langle P_1, P_2, P_3\rangle$, $\langle P_2, P_3, P_4\rangle$ respectively. Exactly what conditions must be satisfied by these five points for the combined curve to have C^1 parametric continuity at the point at which they are joined. How about G^1 parametric continuity? Prove your answer carefully by showing the continuity of the derivatives at this point.

Answer: The Bezier curve of degree 2 determined by points $\langle P_0, P_1, P_2\rangle$ is:

$$X(t) = (1 - t^2)P_0 + 2t(1 - t)P_1 + t^2 P_2,\, t \in [0, 1] \tag{3}$$

The parametric derivative is given by

$$X'(t) = 2(t - 1)P_0 + (2 - 4t)P_1 + 2tP_2,\, t \in [0, 1] \tag{4}$$

Let X_0 be the Bezier curve determined by $\langle P_0, P_1, P_2\rangle$, and X_1 determined by $\langle P2, P_3, P_4\rangle$.

$$\begin{aligned} X_0(t) &= (1 - t)^2 P_0 + 2t(1 - t)P_1 + t^2 P_2,\, t \in [0, 1] \\ X_1(t) &= (1 - t)^2 P_2 + 2t(1 - t)P_3 + t^2 P_4,\, t \in [0, 1] \end{aligned} \tag{5}$$

To enforce C^1 continuity between x_0, and x_1 at point P_2, we must have

(a) $X_0(1) = X_1(0)$

$$\begin{aligned} X_0(1) &= 0P_0 + 0P_1 + 1P_2 = P_2 \\ X_1(0) &= 1P_2 + 0P_3 + 0P_4 = P_2 \end{aligned} \tag{6}$$

This constraint is satisfied.

(b) $X_0'(1) = X_1'(0)$

$$\begin{aligned} X_0'(1) &= -2P_1 + 2P_2 \\ X_1'(0) &= -2P_2 + 2P_3 \end{aligned} \tag{7}$$

To satisfy this constraint,

$$\begin{aligned} -2P_1 + 4P_2 - 2P_3 &= 0 \\ -P_1 + 2P_2 - P_3 &= 0 \end{aligned} \tag{8}$$

This condition is not satisfied for arbitrary sets of points.

To enforce G^1 continuity between x_0, and x_1 at point P_2, we must have

(a) $X_0(1) = X_1(0)$

$$\begin{aligned} X_0(1) &= 0P_0 + 0P_1 + 1P_2 = P_2 \\ X_1(0) &= 1P_2 + 0P_3 + 0P_4 = P_2 \end{aligned} \tag{9}$$

This constraint is satisfied.

(b) $X_0'(1) = \lambda X_1'(0)$ for $\lambda \in \mathbb{R}$

$$\begin{aligned} X_0'(1) &= -2P_1 + 2P_2 \\ X_1'(0) &= -2P_2 + 2P_3 \end{aligned} \tag{10}$$

To satisfy this constraint,

$$\begin{aligned} -2P_1 + 2P_2 &= \lambda(-2P_2 + 2P_3) \\ -P_1 + P_2 &= \lambda(-P_2 + P_3) \end{aligned} \tag{11}$$

This condition is not satisfied for arbitrary sets of points.

Bezier curves are specifically constructed to enforce C^0 continuity, but may not satisfy C^1 continuity because computing the gradient would require information from outside the set of points the curve spans.