Summary

- Topics:

1. Bio-medical imaging
2. Image processing
3. Geometric modeling and computer graphics
4. Mesh generation
   - Marching Cubes/Dual Contouring
   - Tri/Tet Meshing
   - Quad/Hex Meshing
   - Quality Improvement
5. Computational mechanics
6. Bio-medical applications

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Topic 4: Mesh Generation – Surface Meshing and Quality Improvement

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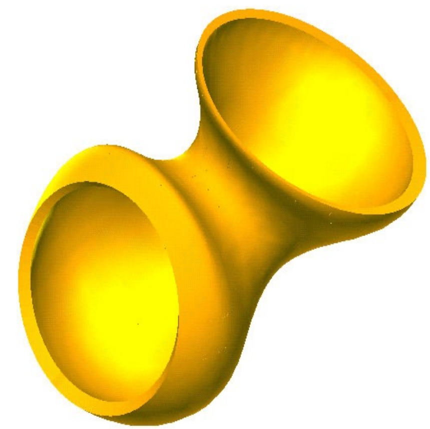
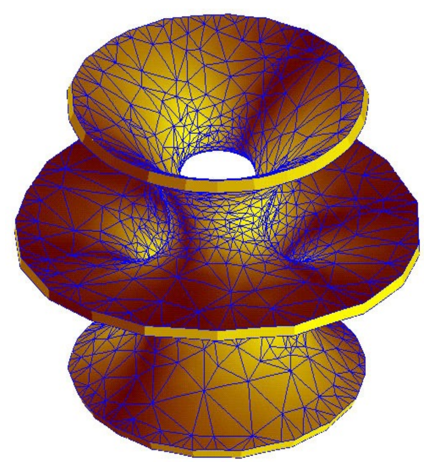
Surface Representation

- Surface meshing is to generate meshes (tri, quad) on 3D surfaces.
- These surfaces are typically represented by NURBS (Non-Uniform Rational B-Spline), which have been generated within a commercial CAD software package.
- The resulting surface elements can be used directly as structural shell elements, or used as input to a volumetric mesh generator.

Shell Finite Elements

- Airfoils
- Tin cans
- Shell canisters
- Sea shells
- Earth outer crust
- Human skin
- Skeletal Structures





Surface Mesh Generation

- Two categories:
  - Parametric space

$$x = x(u, v)$$
$$y = y(u, v)$$
$$z = z(u, v)$$

  - Direct 3D (implicit or explicit)

$$f(x, y, z) = 0$$

$$z = f(x, y)$$

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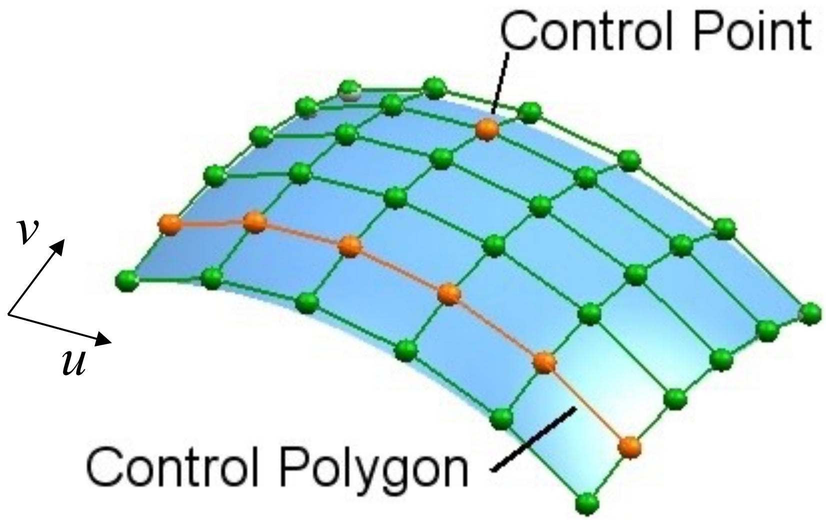
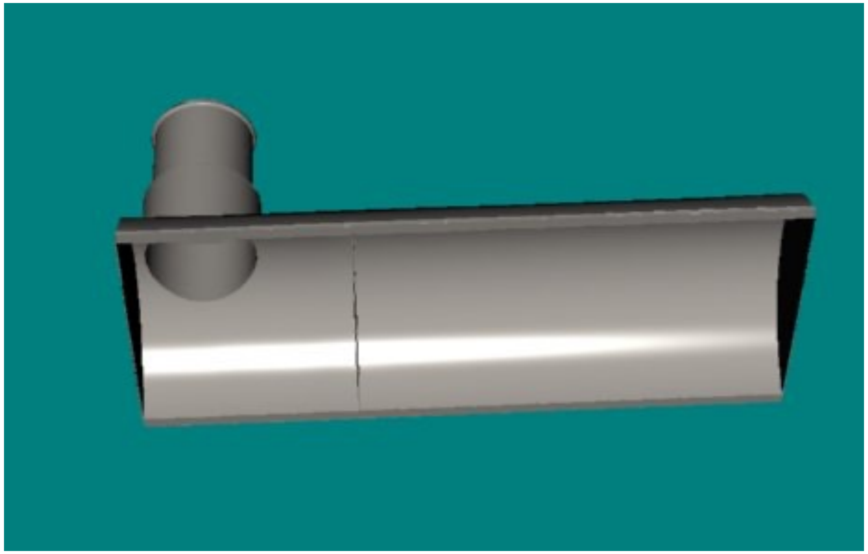
Parametric Space

- Parametric space algorithms will form elements in the 2D parametric space of the surface.
- Since all NURBS surfaces have an underlying *u*-*v* representation, it can often be efficient to mesh in 2D and as a final step, map the *u*-*v* coordinates back to physical space, *x*-*y*-*z* coordinates.

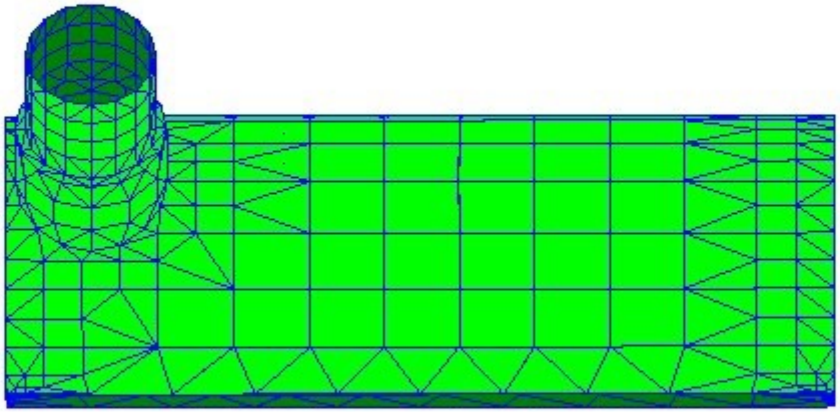


NURBS to Meshes

- The object consists of two orthogonal cylinders. The geometry was created inside Rhino.



NURBS model



Surface meshes

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Drawback

- The elements formed in the parametric space may not always form well-shaped elements in 3D once mapped back to the surface.
- Two methods to solve this problem:
  - Modify or re-parameterize the underlying parametric representation so there is a reasonable mapping from parametric space to physical space.
  - Modify the mesh generation algorithm so that stretched or anisotropic elements meshed in 2D will map back to well-shaped, isotropic elements in 3D.

Direct 3D

- Direct 3D surface mesh generators form elements directly on the geometry without regard to the parametric representation of the underlying geometry.
  - Triangular mesh
    - Octree
    - Delaunay triangulation
    - Advancing front approach
  - Quadrilateral mesh
    - Medial axis (domain decomposition)
    - Paving (advancing front approach for quad)
    - Octree

Mesh Post-Processing: Quality Improvement

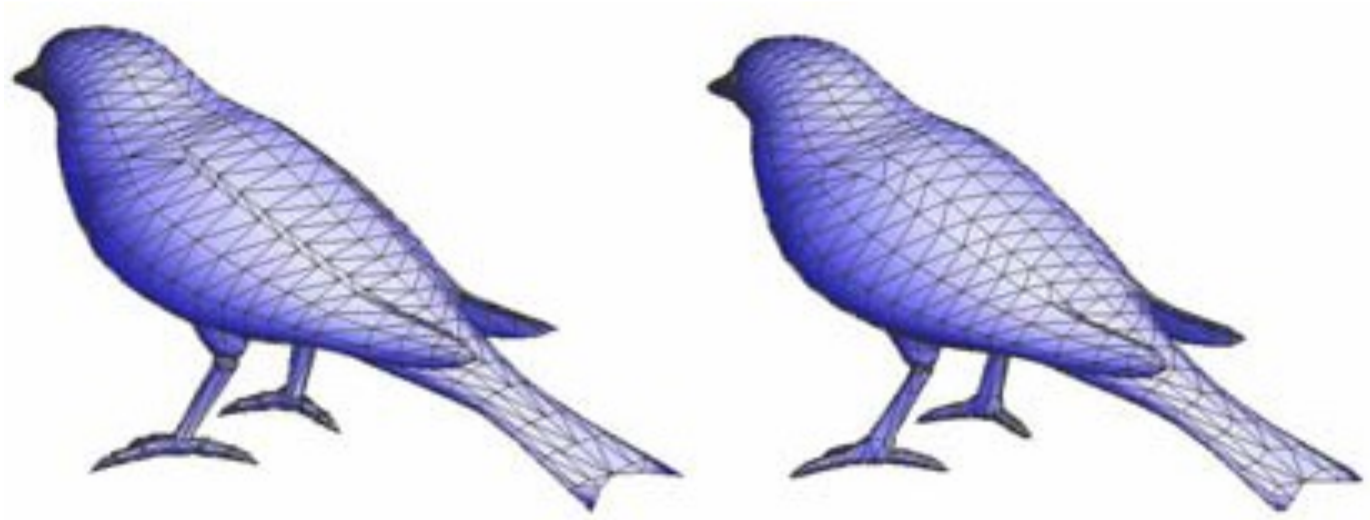
- The mesh quality needs to be optimized after mesh generation
  - Smoothing: adjust node locations while maintaining the element connectivity (topology).
  - Clean-up: change the element connectivity.

Smoothing

- Iteratively reposition individual nodes to improve the local quality of the elements.
  - Averaging methods
  - Optimization-based methods
  - Physically-based methods
  - Mid-node placement

Averaging Methods

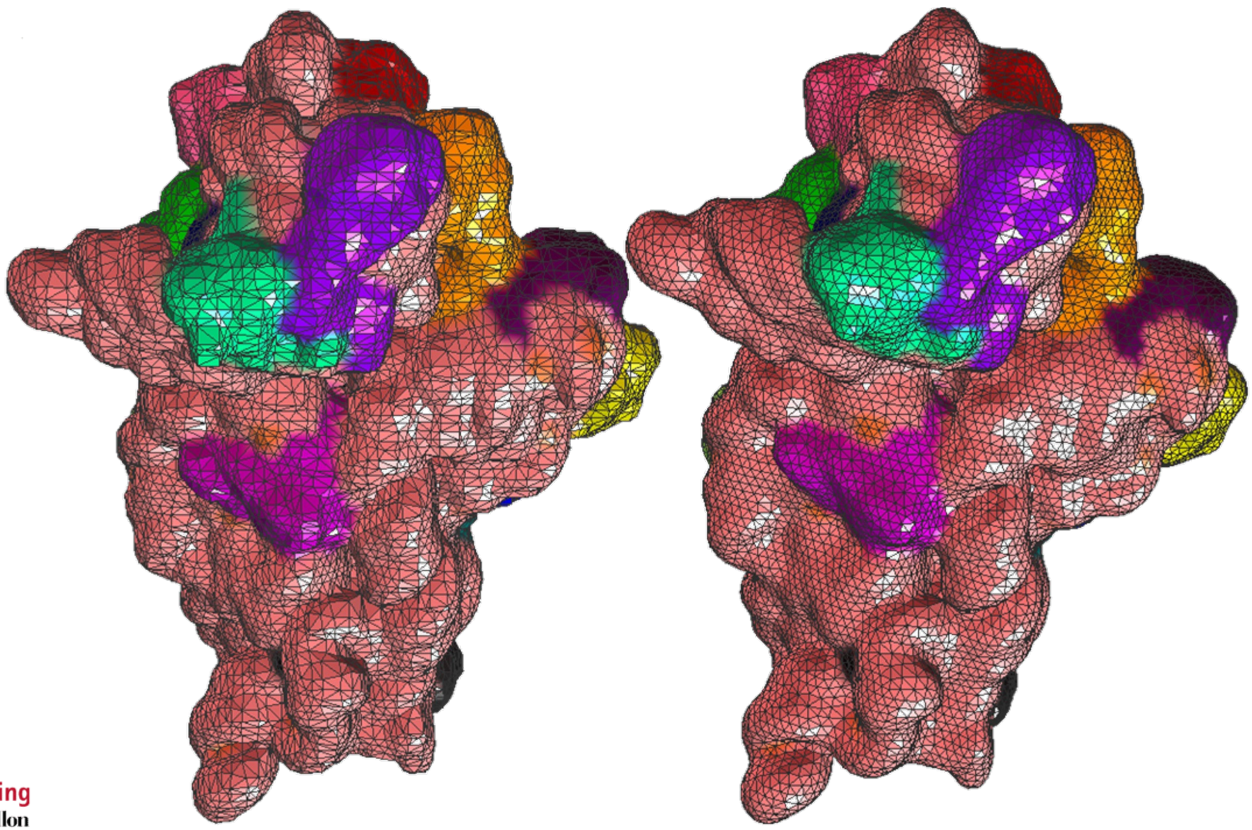
- The simplest and most straight-forward averaging method is Laplacian smoothing.
- An internal node in the mesh is placed at the average location of any node connected to it by an edge.
- This method works for tri/quad/tet/hex.



http://ieeexplore.ieee.org/iel5/10679/33713/01604646.pdf

Averaging Methods

- Most smoothing procedures will iterate through all the internal nodes in the mesh several times until any individual node has not moved more than a specified tolerance.
- This method is simple to implement and is in wide use.



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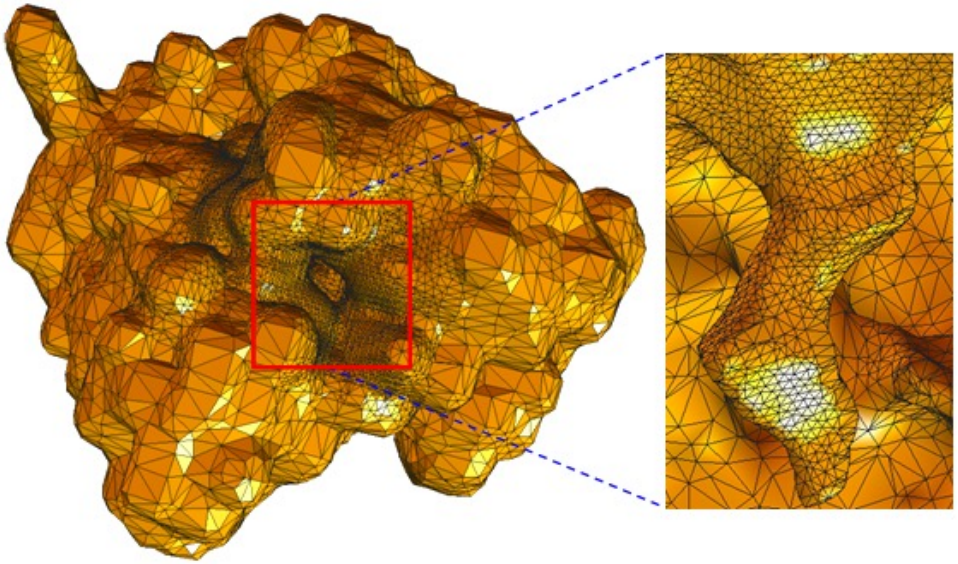
Weighted Averaging Methods

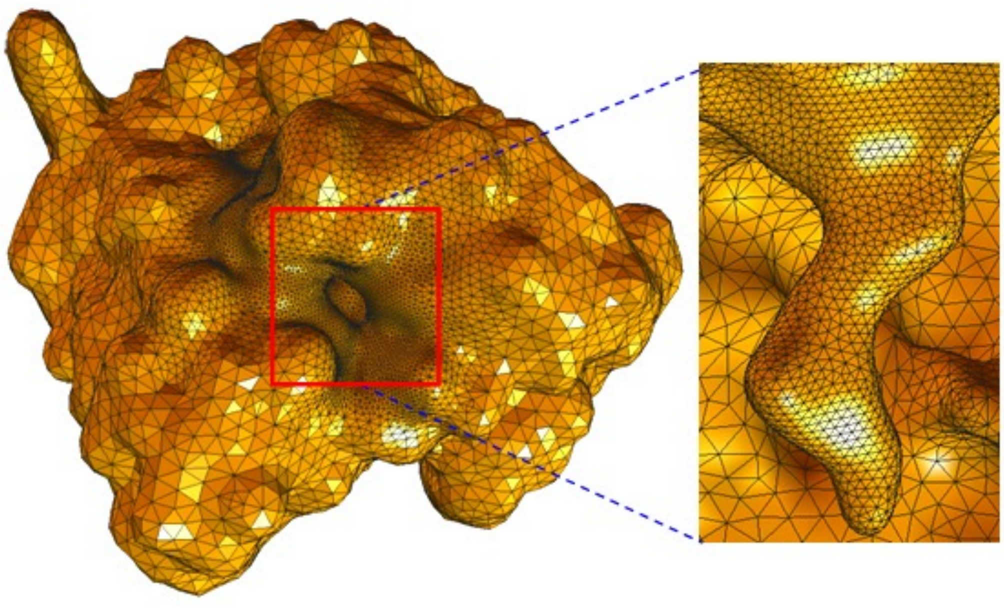
- Similar to Laplacian, there are a variety of other smoothing techniques, which iteratively reposition nodes based on a weighted average of the geometric properties of the surrounding nodes and elements.
  - Edge length
  - Area
  - Volume

Weighted Averaging Methods

Before quality improvement



After quality improvement
using area as the weight



Anisotropic Diffusion

- Isotropic Diffusion

$$\partial_t x(t) - \Delta_{M(t)} x(t) = 0$$

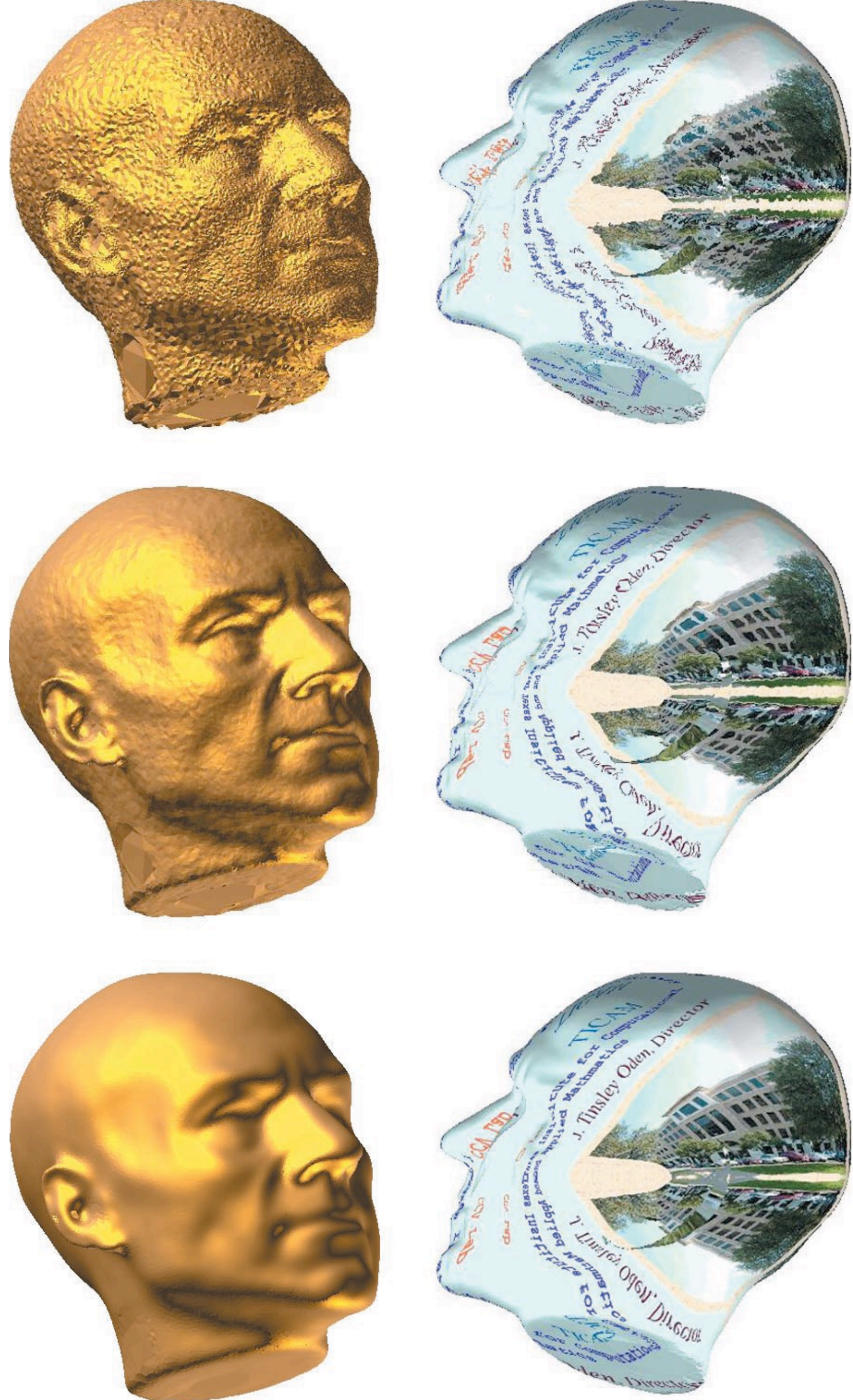
$\Delta_{M(t)} = \mathrm{div} \circ \nabla_{M(t)}$ is known as the Laplace–Beltrami operator on $M(t)$.

- Anisotropic Diffusion

$$\partial_t x(t) - \mathrm{div}_{M(t)}\left(D \nabla_{M(t)} x(t)\right) = 0$$

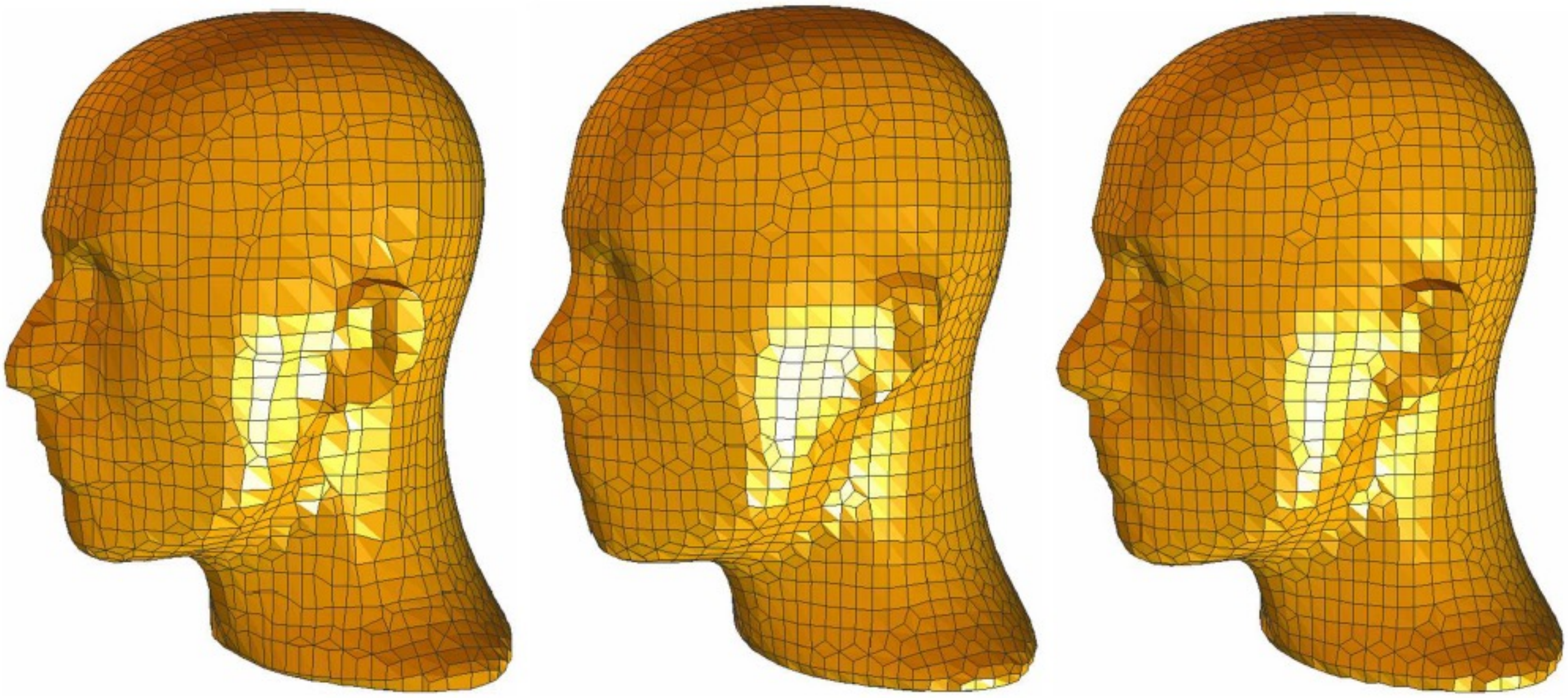
D is the diffusion tensor, defined by surface curvature.

C. Bajaj, G. Xu. Anisotropic Diffusion of Surfaces and Functions on Surfaces. ACM Transactions on Graphics, 22, 1, (2003), 4-32.



Feature-Preserved Smoothing

Geometric features are preserved by restricting the vertex movement on the tangent plane.



From left to the right: the original mesh, Laplacian smoothing, feature-preserved smoothing.

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Constrained Laplacian Smoothing

- Averaging methods also employ some form of additional constraint on the movement of a node.
- A comparison of local element quality is made before and after the proposed move and the node moved only if element quality is improved. This is often referred to as constrained Laplacian smoothing.

Optimization-Based Methods

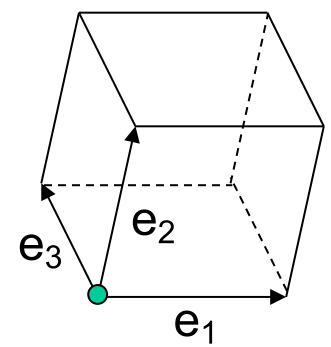
- Optimization-based smoothing techniques measure the quality of the surrounding elements to a node and attempt to optimize by computing the local gradient of the element quality w.r.t. the node location.

- The node is moved in the direction of the increasing gradient until an optimum is reached (conjugate gradient method).

Optimization Method

- Three error metrics [Kober et. al. 2000] [Knupp 2000] [Oddy et. al. 1988]
  Edge vector $e_i = x_i - x,\ i = 1,\ \ldots,\ m$
  Jacobian matrix $J = [e_1\, e_2\, _{\ldots}\, e_m]$

$$
\begin{aligned}
Jacobian(x) &= det(J) = \sqrt{det(J^T J)} && (1)\\
\kappa(x) &= \frac{1}{m}|J^{-1}||J| && (2)\\
Oddy(x) &= \frac{(|J^T J|^2 - \frac{1}{m}|J|^4)}{det(J)^{\frac{4}{m}}} && (3)
\end{aligned}
$$



- Quality metric in Finite Element Method [Finite Elements, Oden 1981]:

$$
x = \sum_{i=1}^{8} x_i \Phi_i \qquad y = \sum_{i=1}^{8} y_i \Phi_i \qquad z = \sum_{i=1}^{8} z_i \Phi_i \qquad \mathrm{J} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{vmatrix} \qquad |\mathrm{J}| = \det \mathrm{J}
$$

- Set an object function, use the conjugate gradient method to find an optimized position for a node with the worst condition number.

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Optimization-Based Methods

- Optimization-based smoothing techniques are time-consuming.
- A combined Laplacian/optimization-based approach is recommended: Laplacian smoothing is done for the majority of the time, reverting to optimization-based smoothing only when local element shape metrics drop below a certain threshold.

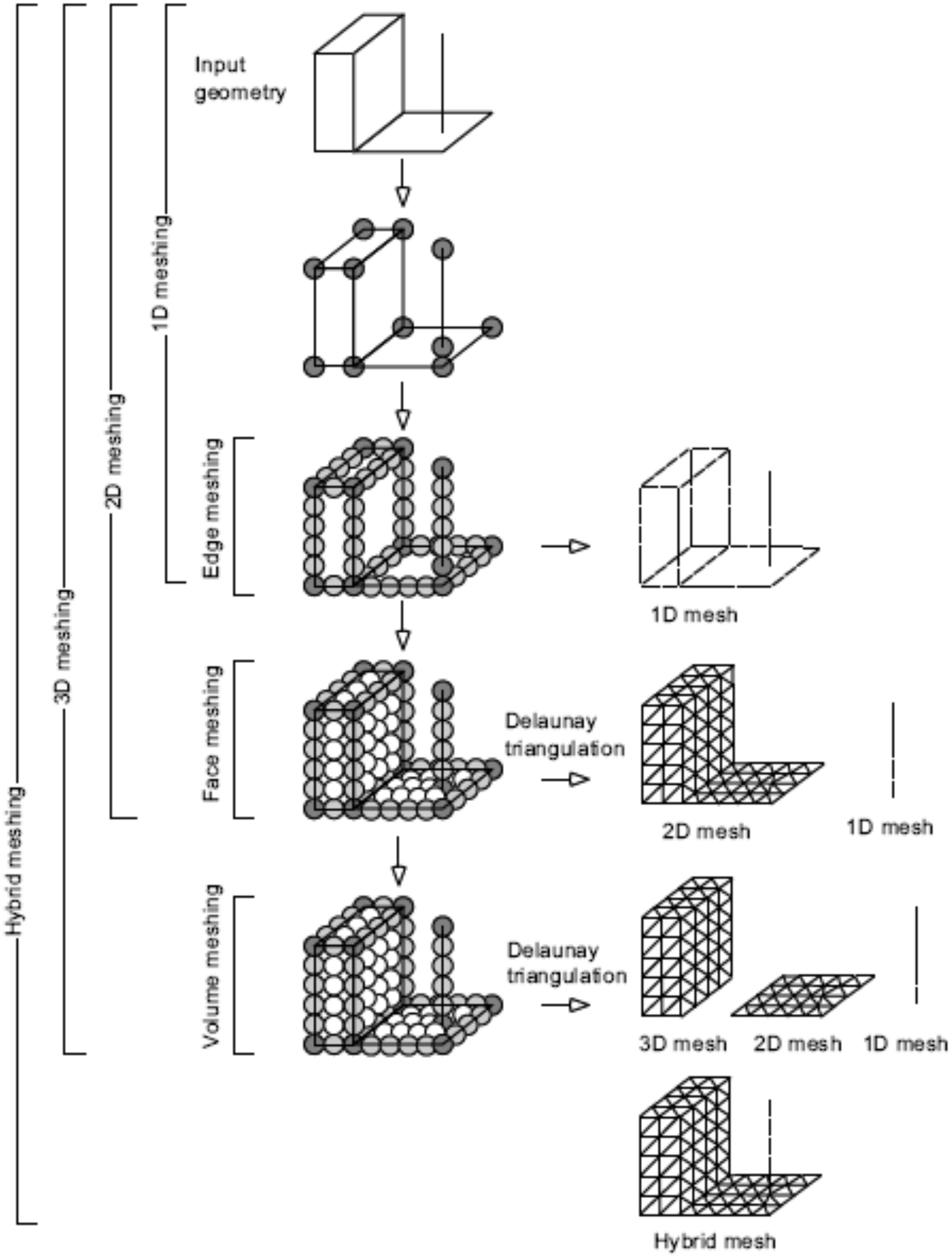
Physically-Based Methods

- Physically-based smoothing techniques are to reposition nodes based on a simulated physically based attraction or repulsion force.
- Lohner simulated the force between neighboring nodes as a system of springs interacting with each other.

R. Lohner, K. Morgan and O. C. Zienkiewicz, (1986) "Adaptive Grid Refinement for Compressible Euler Equations", *Accuracy Estimates and Adaptive Refinements in Finite Element Computations*, I. Babuska et al. eds., Wiley, pp. 281-297.

Physically-Based Methods

- Shimada viewed the nodes as the center of bubbles that are repositioned to attain equilibrium. With changes in the magnitude and direction of interparticle forces, different anisotropic characteristics and element sizes can be achieved.



Kenji Shimada, Atsushi Yamada and Takayuki Itoh, (1997) "Anisotropic Triangular Meshing of Parametric Surfaces via Close Packing of Ellipsoidal Bubbles", Proceedings, 6th International Meshing Roundtable, pp.375-390.

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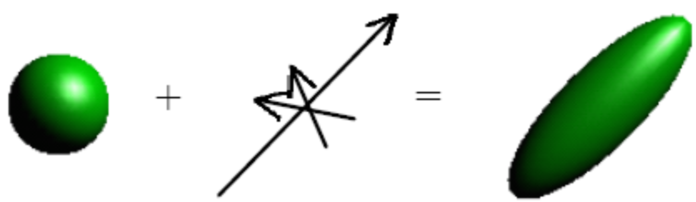
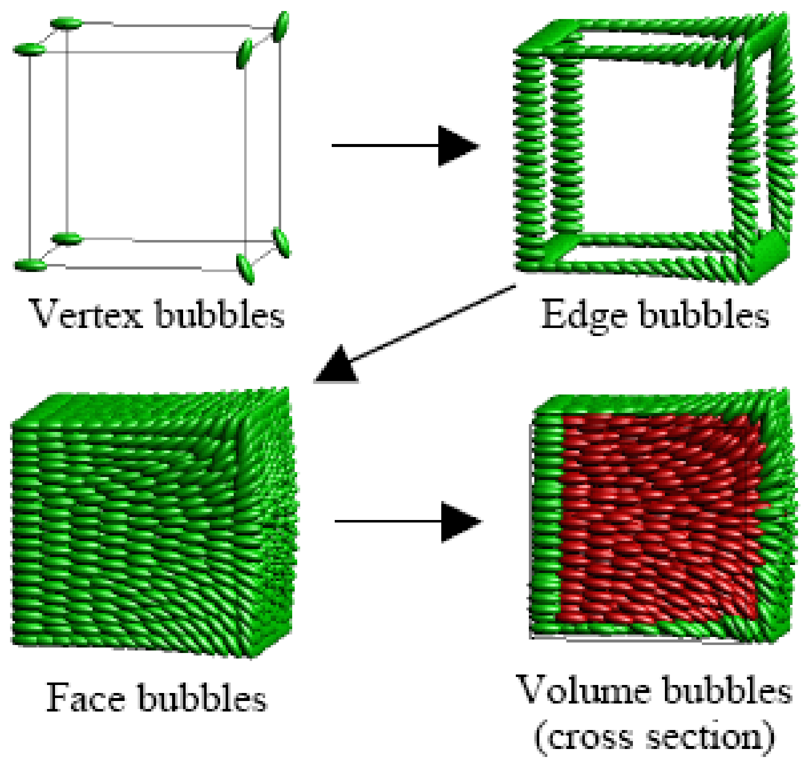
Anisotropic Tetrahedral Mesh Generation via Ellipsoidal Bubble Packing

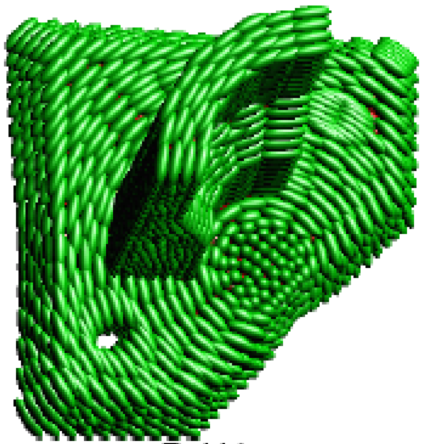
- Ellipsoidal bubbles are closely packed on the boundary and inside a geometric domain, and nodes are placed at the centers of the bubbles. This method then connects the nodes to create a tet mesh by the advancing front method.



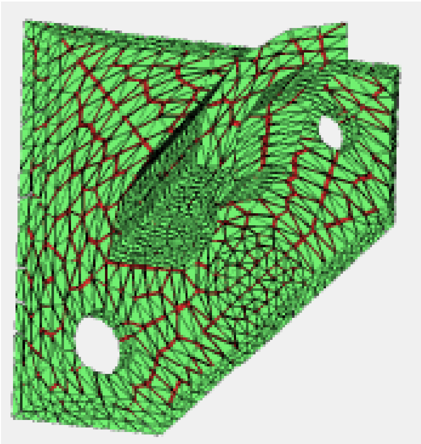
Yamakawa, S. and K. Shimada, "High Quality Anisotropic Tetrahedral Mesh Generation via Packing Ellipsoidal Bubbles," The 9th International Meshing Roundtable, pp.263-73, 2000.



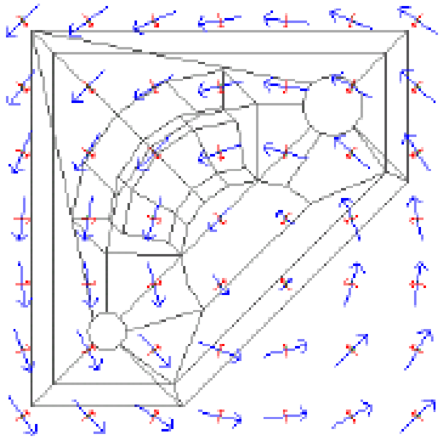
Anisotropic Tetrahedral Mesh Generation via Ellipsoidal Bubble Packing



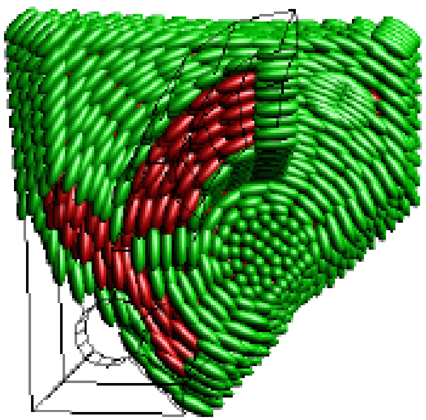
Bubbles



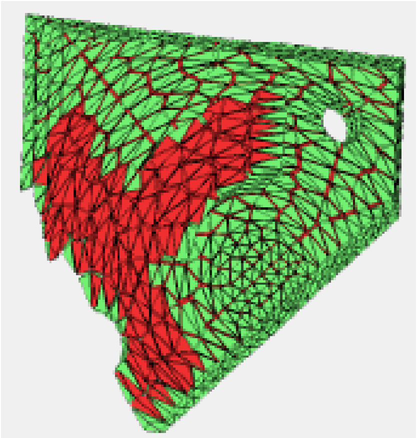
Tet Mesh



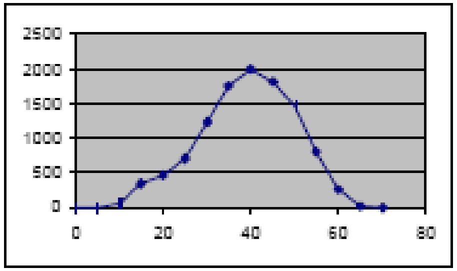
Anisotropy



Bubbles (Cross Section)



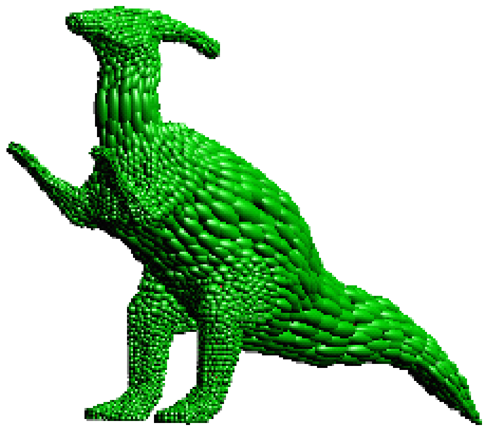
Tet Mesh (Cross Section)



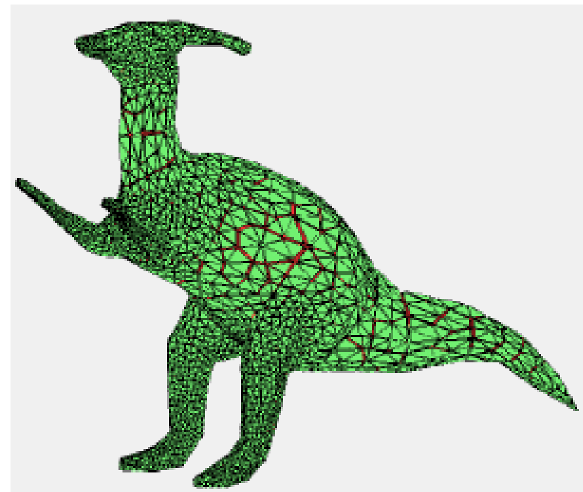
Minimum Dihedral Angle=8.0deg

Statistics and Quality Measurement

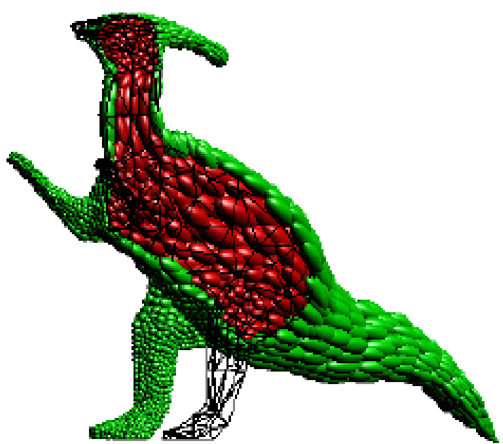
Anisotropic Tetrahedral Mesh Generation via Ellipsoidal Bubble Packing



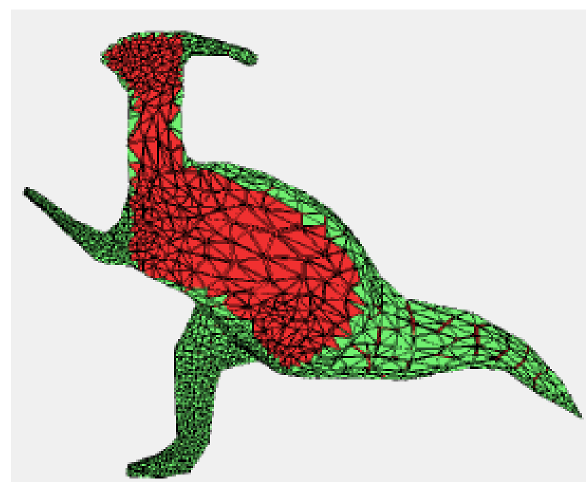
Bubbles



Tet Mesh



Bubbles (Cross Section)



Tet Mesh (Cross Section)

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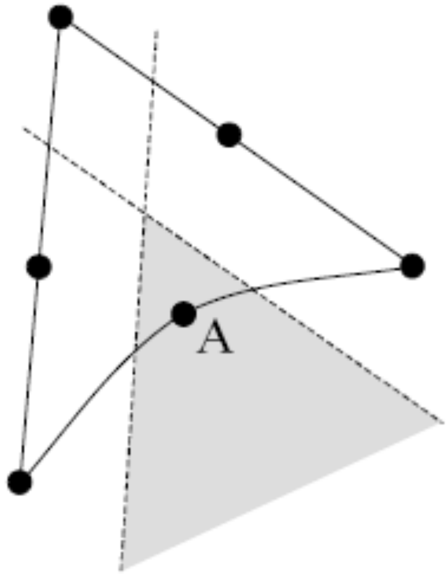
Mid-node Placement

- Salem introduced a method providing criteria for repositioning mid-nodes on quadratic elements to improve element quality.
- This method computes a region surrounding the mid-node known as the mid-node admissible space (MAS), where the mid-node can safely be moved and maintain or improve element quality.

Ahmed Z.I. Salem, Scott A. Canann, and Sunil Saigal, (1997) "Robust Distortion Metric for Quadratic Triangular 2D Finite Elements", AMD-Vol. 220 Trends in Unstructured Mesh Generation, pp.73-80.

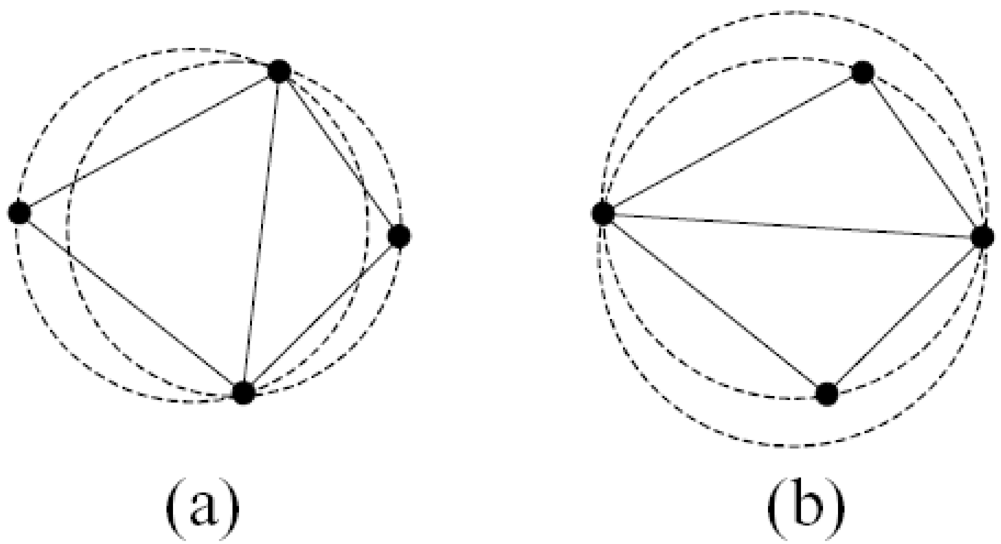


Clean-Up

- Cleanup methods improve mesh quality by making local changes to the mesh connectivity.
- Cleanup methods apply some criteria:
  - Shape improvement
  - Topological improvement
- Cleanup is used in conjunction with smoothing.

Shape Improvement

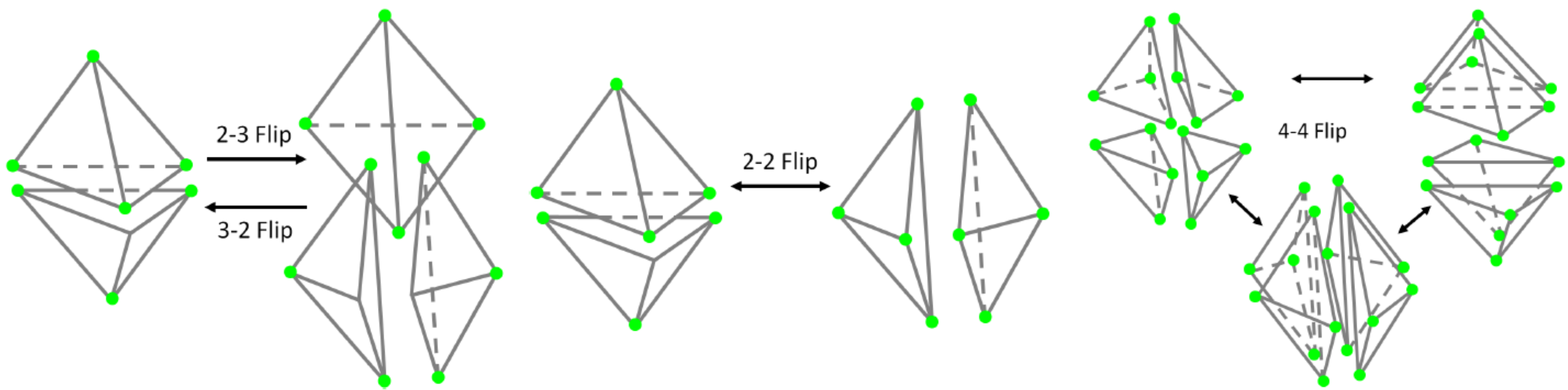
- For triangle meshes, simple diagonal swaps are often performed.
- For each interior edge, a check can be made to determine which edge would effectively improve the overall or minimum aspect ratio of its two adjacent triangles.
- The Delaunay criterion can be used: “empty circle”.



(a) Satisfies the Delaunay criterion, while (b) doesn’t.

Shape Improvement

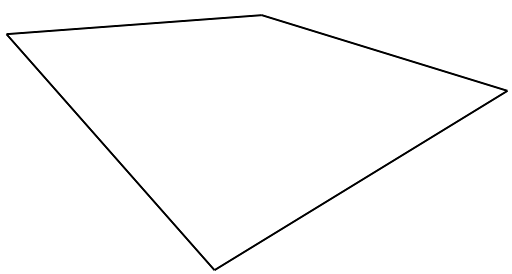
- For tetrahedral meshes, a series of local transformations are designed to improve the element quality.
  - Swapping interior edges.



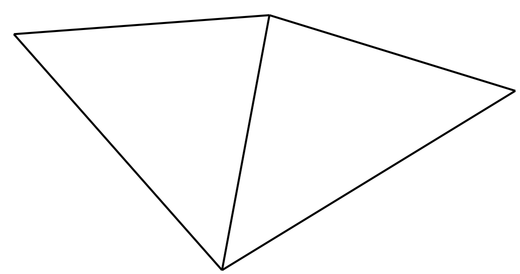
(a) (b) (c)

Shape Improvement

- In hybrid meshes (tri, quad), the element quality of two adjacent triangles may be preferable to a single poor quality quadrilateral. The quad may be split into two tris.



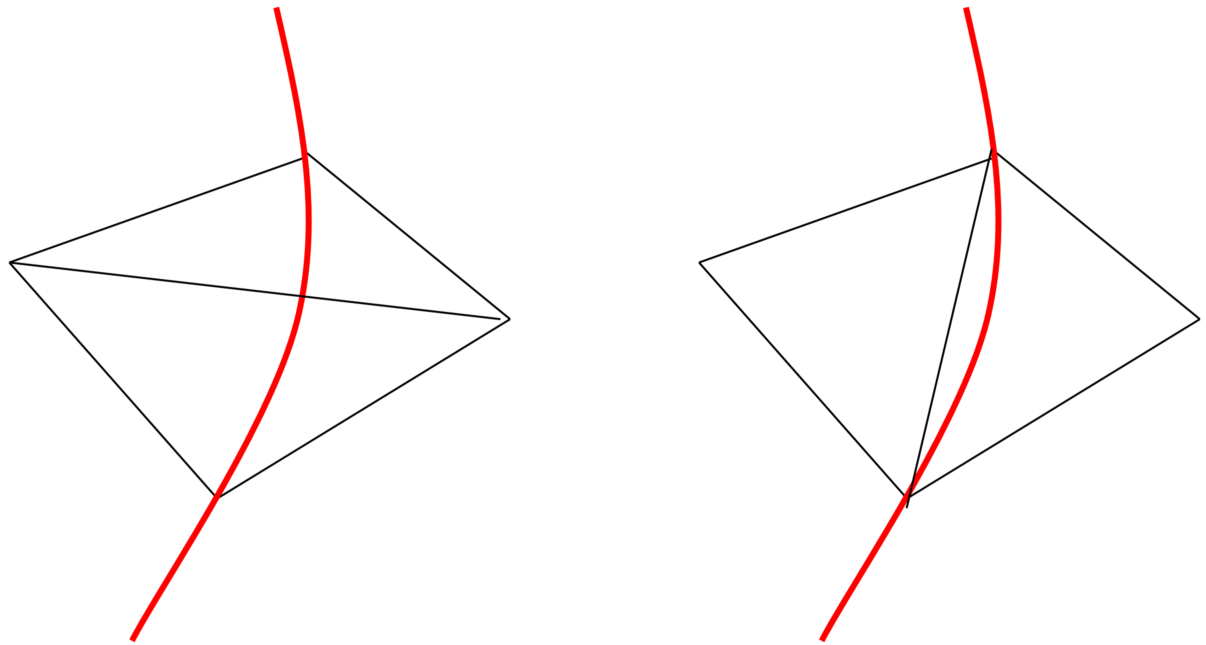
One quad



Two tris

Shape Improvement

- In some cases, particularly with curved surfaces, the elements resulting from the mesh generator may deviate significantly from the underlying geometry.
- Edge swaps can be performed based on which edge will deviate least from the surface.



Topology Improvement – Valence Optimization

- Attempt to optimize the number of edges sharing a single node (valence or degree), with the local element shapes improved at the same time.
- The ideal valence number is 6 for triangles and 4 for quadrilaterals.
- Performing local transformations to the elements can improve topology and enhance element quality.

Topology Improvement – Valence Optimization

- For volumetric meshes, valence optimization becomes more complex.
  - Optimizing the number of edges at a node.
  - Optimizing the number of faces at an edge.
- For tet meshes, this can involve a complex series of local transformations.
- For hex meshes, valence optimization is generally not considered tractable because the local modification to a hex mesh will propagate themselves to more than the immediate vicinity.

Mesh Refinement

- Refinement is defined as any operation performed on the mesh that effectively reduces the local mesh size.

- The reduction in size may be required in order to capture a local physical phenomenon, or improve the local element quality.

Mesh Refinement

- Some refinement methods in themselves can be considered mesh generation algorithms, adaptive mesh generation.
- Starting with a coarse mesh, a refinement procedure can be applied until the desired node density has been achieved.

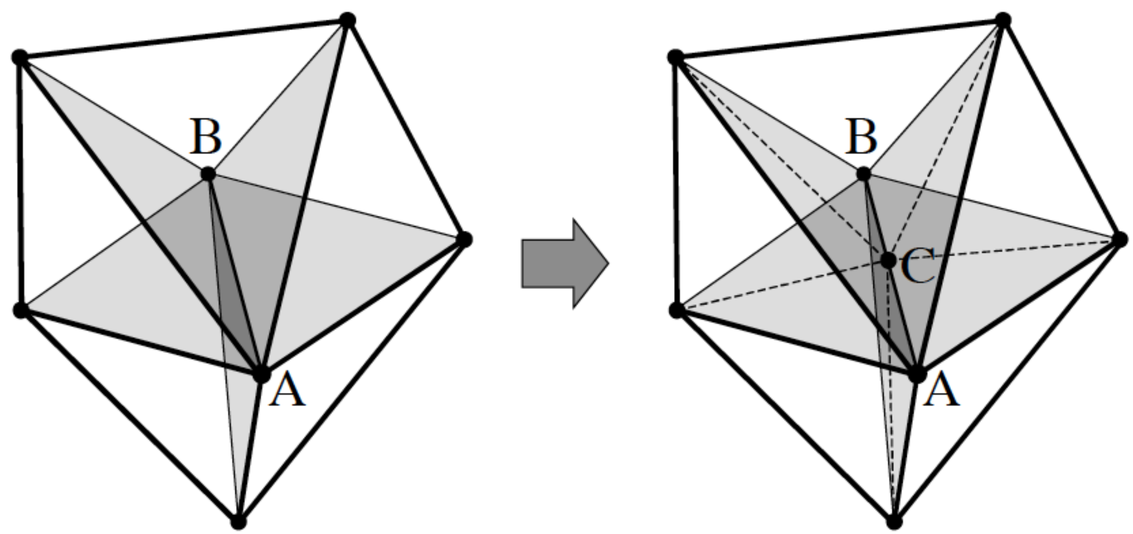
Triangle/Tetrahedral Refinement

Three principal methods for triangle and tetrahedral refinement:

- Edge bisection
- Point insertion
- Templates

Edge Bisection

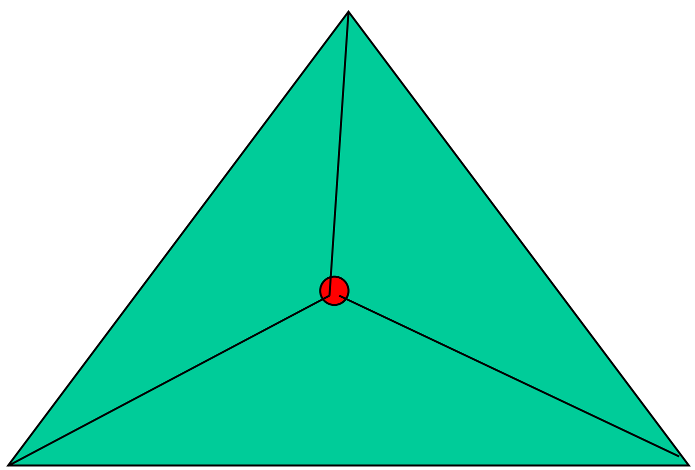
- Edge bisection is to split individual edges in the triangulation.
- As a result, the two triangles adjacent the edge are split into two. Extended to volumetric tet meshing, any tetrahedron sharing this edge should be split.

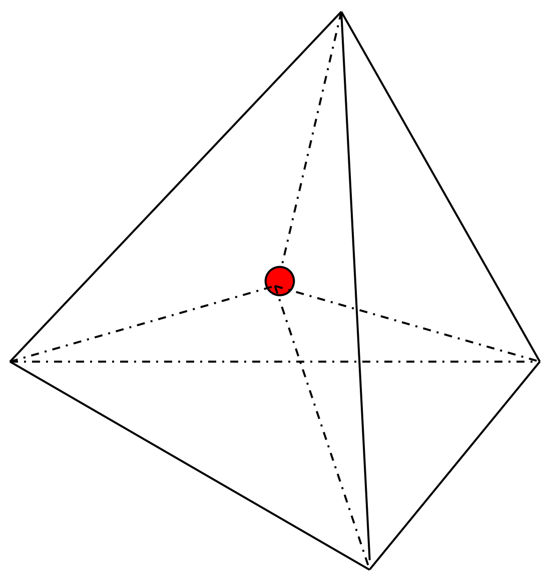


Edge A-B is split at point C, all its surrounding tetrahedra are split.

Point Insertion

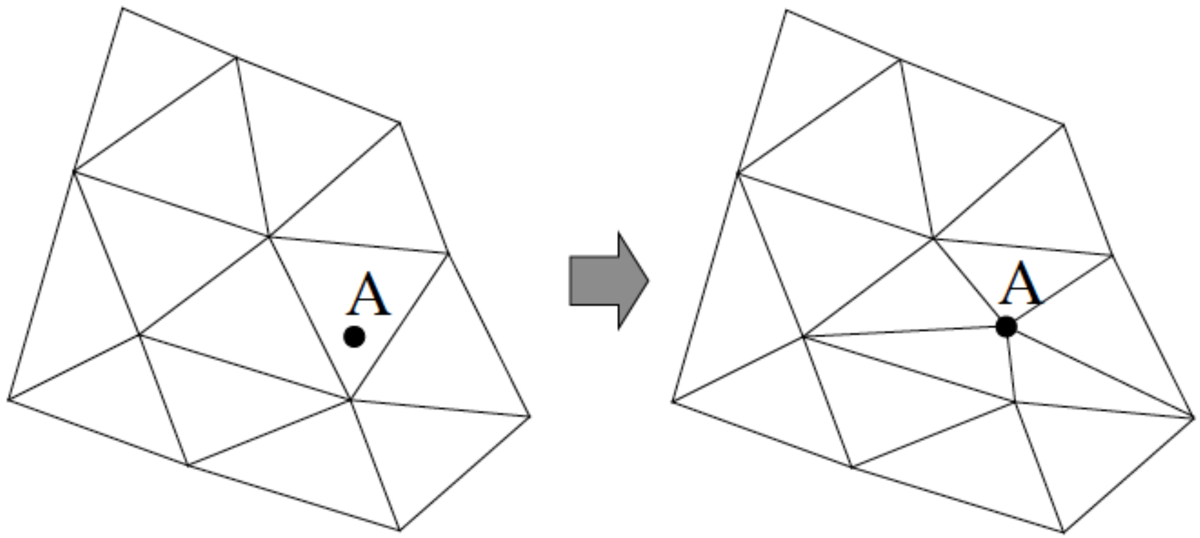
- Insert a single node at the centroid of an existing element, dividing the triangle into three or tetrahedron into four.





Point Insertion

- Poor quality elements may be introduced.
- A Delaunay approach can be used that will delete the local triangles or tetrahedra and connect the node to the triangulation maintaining the Delaunay criterion.



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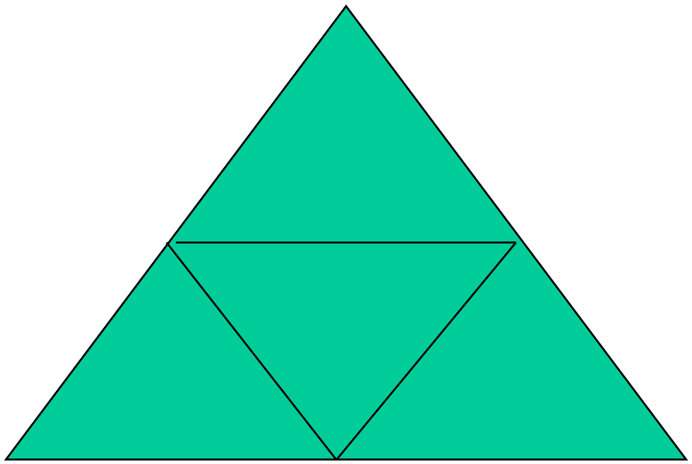
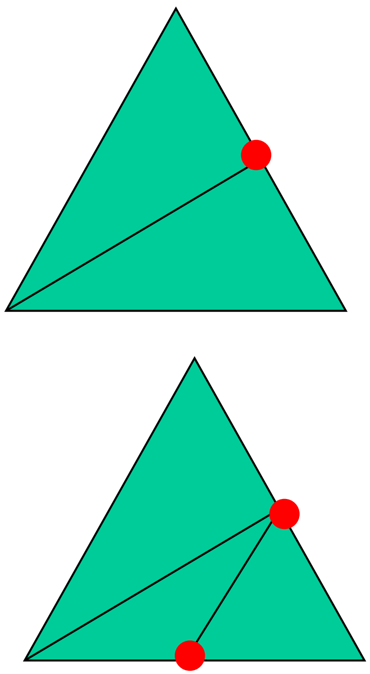
Templates

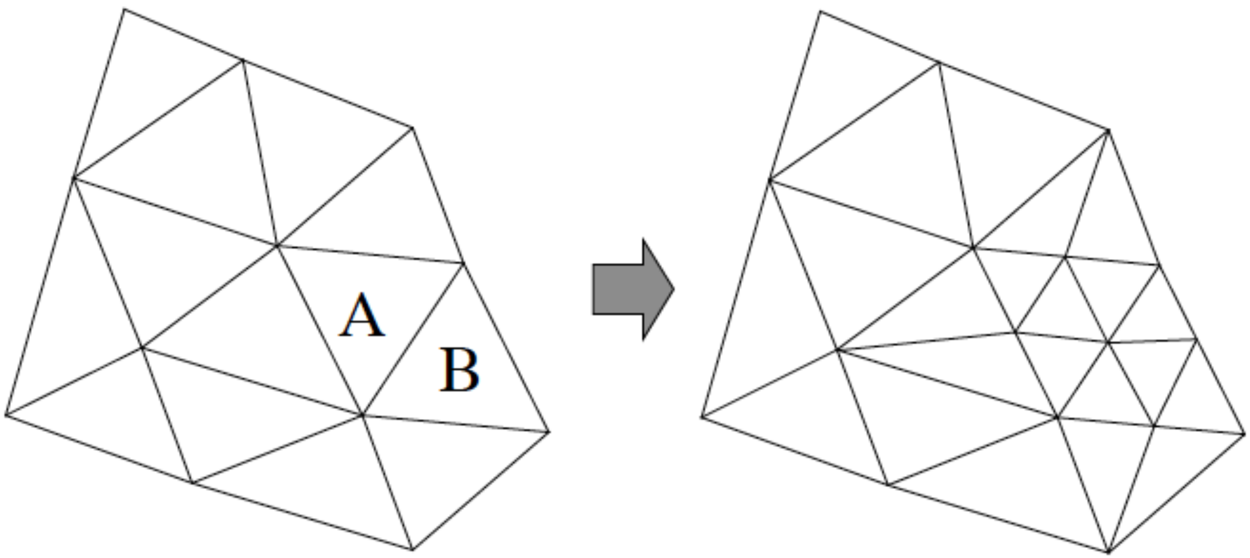
- A template refers to a specific decomposition of the triangle.
- One example is to decompose a single triangle into four similar triangles by inserting a new node at each of its edges.



Templates

- To maintain a conforming mesh, additional templates can also be defined based on the number of edges that have been split. (no hanging nodes)

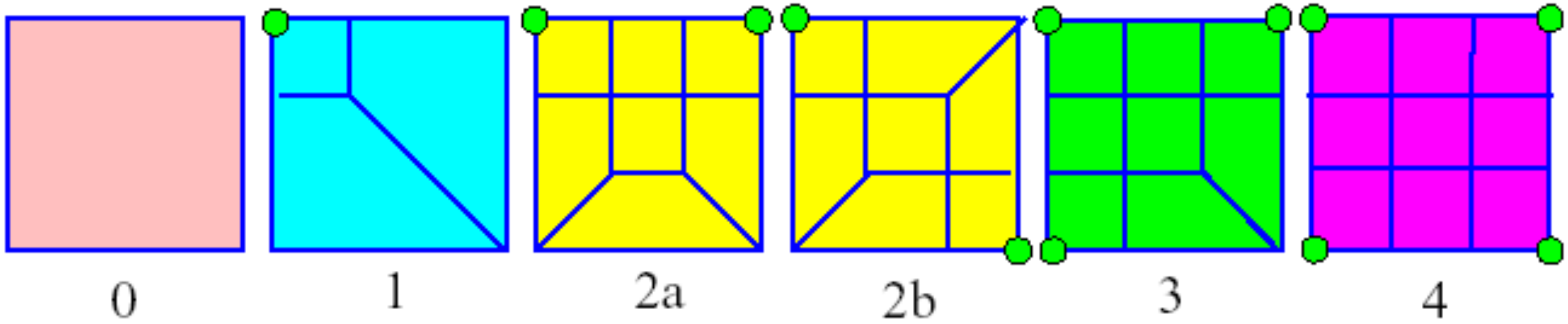


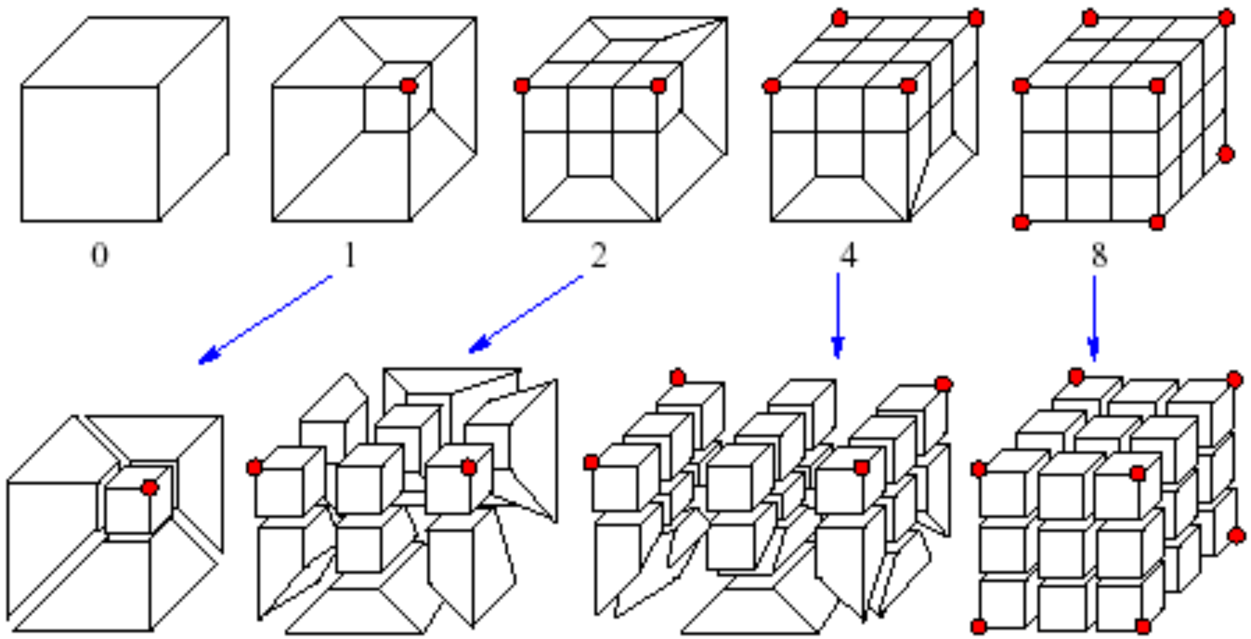


Quad/Hex Refinement

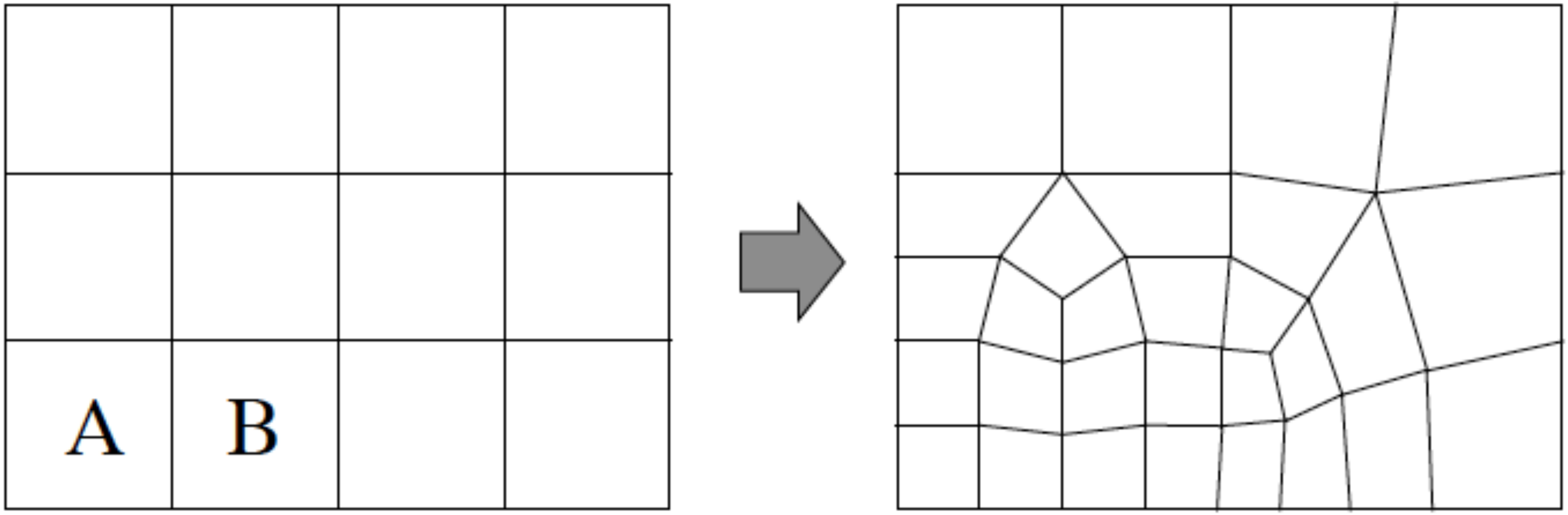
- Because of the structured nature of quad and hex meshes, the point insertion and edge bisection methods are generally not applicable. (no degenerated elements)
- The main methods used for quad/hex refinement involve decomposing the elements based on a set of predefined templates.

Quad/Hex Refinement Templates





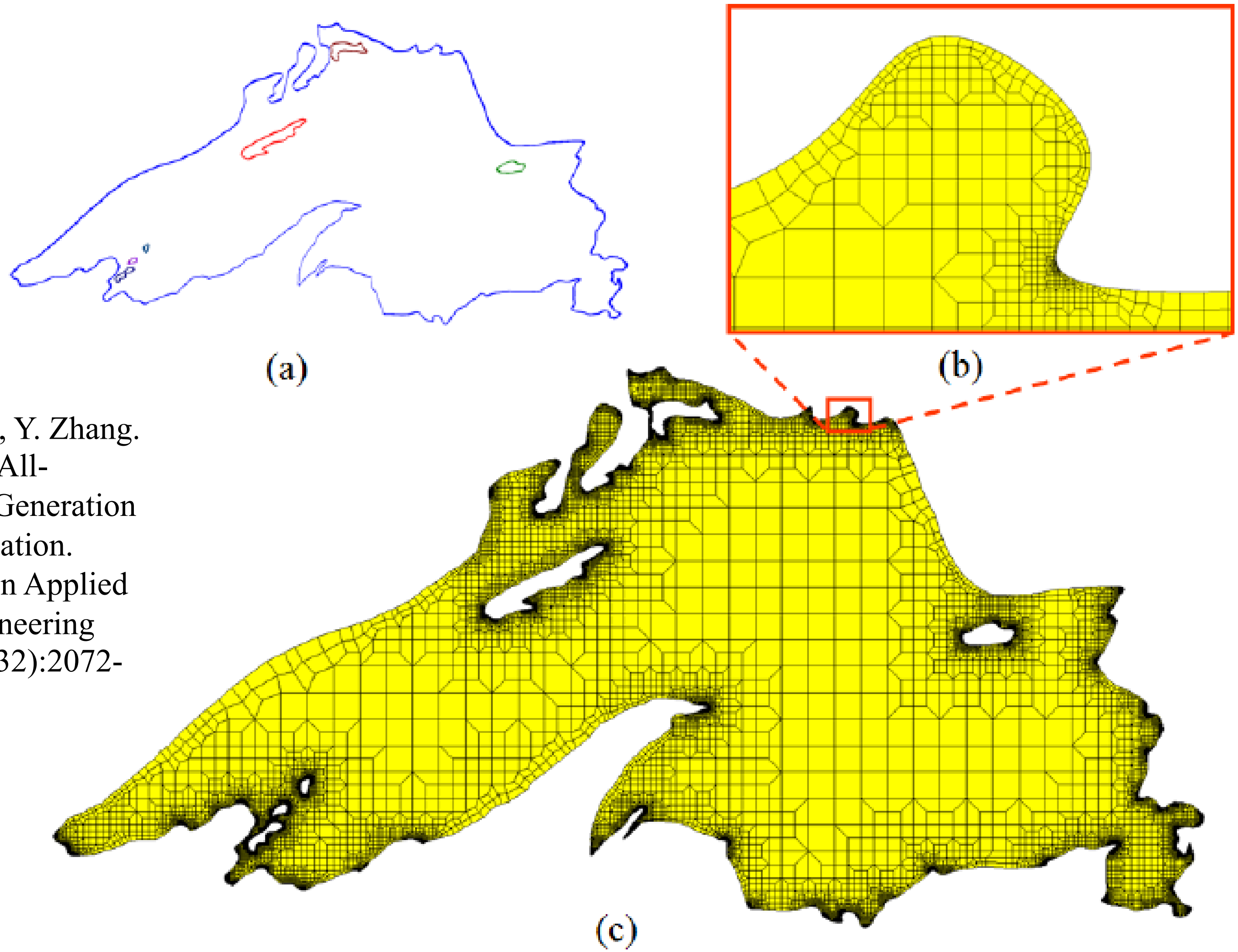
Quad/Hex Refinement



Example of local quad refinement where elements at A and B are refined by one half (2-refinement).

2-Refinement and Guaranteed-Quality Quad Meshing

X. Liang, M. Ebeida, Y. Zhang. Guaranteed-Quality All-Quadrilateral Mesh Generation with Feature Preservation. Computer Methods in Applied Mechanics and Engineering (CMAME). 199(29-32):2072-2083, 2010.

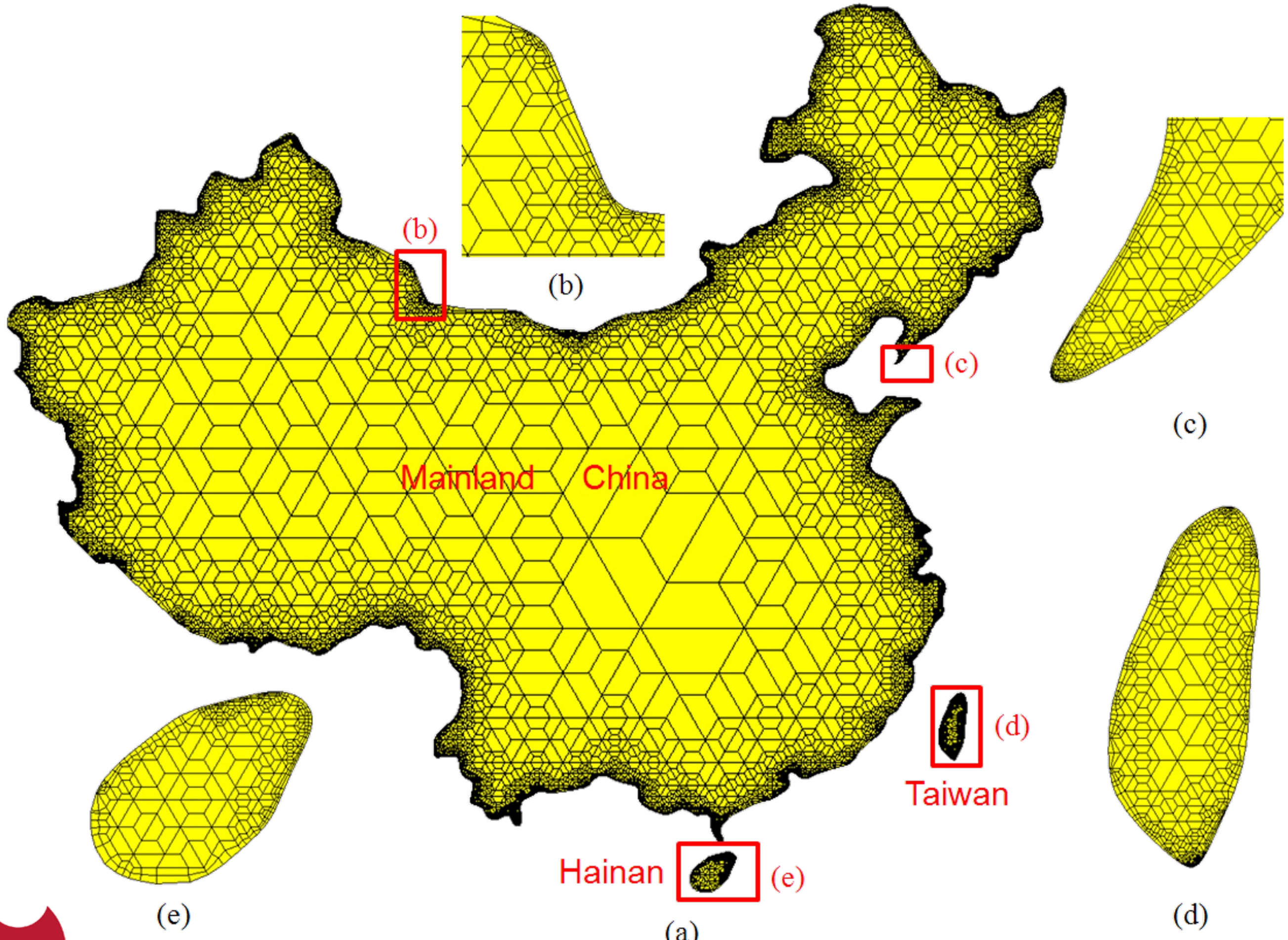


Mesh of the Superior Lake. (a) Curves as the contour of Super Continent. (b) Zoom-in picture. (c) Final guarantee-quality all-quad mesh, all the angles are within [43˚,135˚].

Hexagon-Based All-Quad Mesh Generation

To generate *guaranteed-quality all-quadrilateral (quad)* mesh for given smooth curves.

- Guaranteed-quality: all the angles in the mesh ∈ [$60^\circ - \varepsilon$, $120^\circ + \varepsilon$] ($\varepsilon \leq 5^\circ$)
- Preserving local geometric features and narrow regions



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Summary

- We reviewed unstructured triangular, tetrahedral, quadrilateral, and hexahedral mesh generation techniques.
- Some of them have been implemented in commercial software.
- Understand what have already been done, and try to use/improve them in your research.

SIAM International Meshing Roundtable Workshop (IMR)

- https://www.siam.org/conferences/cm/conference/imr23
- Conference webpage and proceedings are available online.
- The hybrid SIAM IMR will be held on March 6-9, 2023 in Amsterdam, The Netherlands.

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