Summary

- Topics:

1. Bio-medical imaging
2. Image processing
3. Geometric modeling and computer graphics
4. Mesh generation
    - Marching Cubes/Dual Contouring
    - Tri/Tet Meshing
    - Quad/Hex Meshing
    - Quality Improvement
5. Computational mechanics
6. Bio-medical applications

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Topic 4: Mesh Generation – Delaunay Triangulation and Voronoi Diagram

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Grid and Mesh

- The terms grid and mesh are used interchangeably throughout with identical meaning.

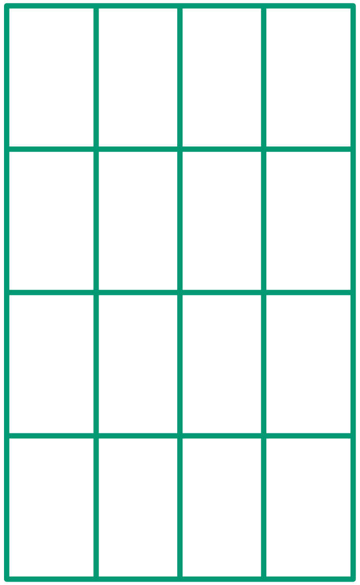
- The numerical solution of partial differential equations (PDEs) requires to replace the continuous differential equations with the discretization of the equations.
  - Finite Difference Method
  - Finite Element Method
  - Finite Volume Method

- The analysis domain is discretized into discrete points and cells/elements such as triangles, quadrilaterals, tetrahedra and hexahedra.

Structured Grids

- A structured grid of quads consists of a set of coordinates and connectivities that naturally map into elements of a matrix.
- In general, each interior vertex of a structured grid has the same valence number (e.g., 6 for triangles and 4 for quads), which is the number of elements sharing this point.

A 2D array $x(i, j)$ can be used to store coordinates of the structured grid.



$x(i, j)$

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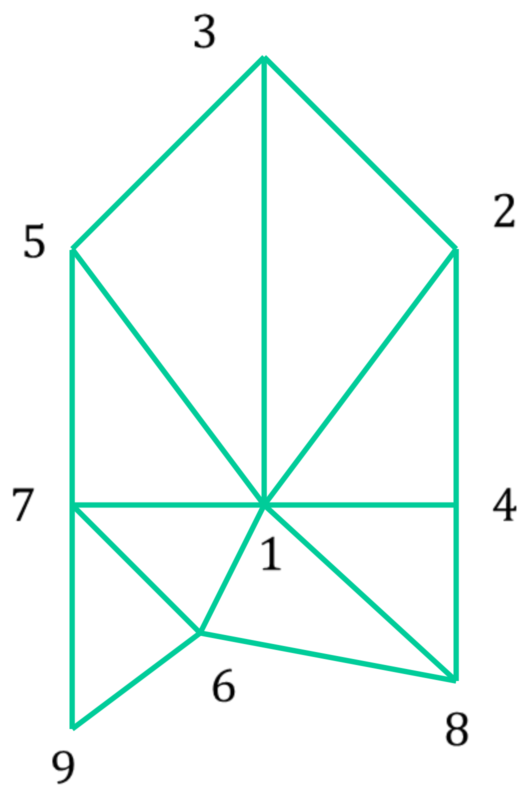
Unstructured Grids

- Vertices in unstructured grids don't have the same valence number everywhere.



Element	Nodes
1	1, 5, 7
2	1, 4, 2
3	1, 2, 3
4	1, 3, 5
5	1, 7, 6
6	1, 6, 8
7	1, 8, 4
8	7, 9, 6

Number of Points, Number of Elements

x_1, y_1

x_2, y_2

x_3, y_3

…

n_1, n_2, n_3

n_4, n_5, n_6

n_7, n_8, n_9

…

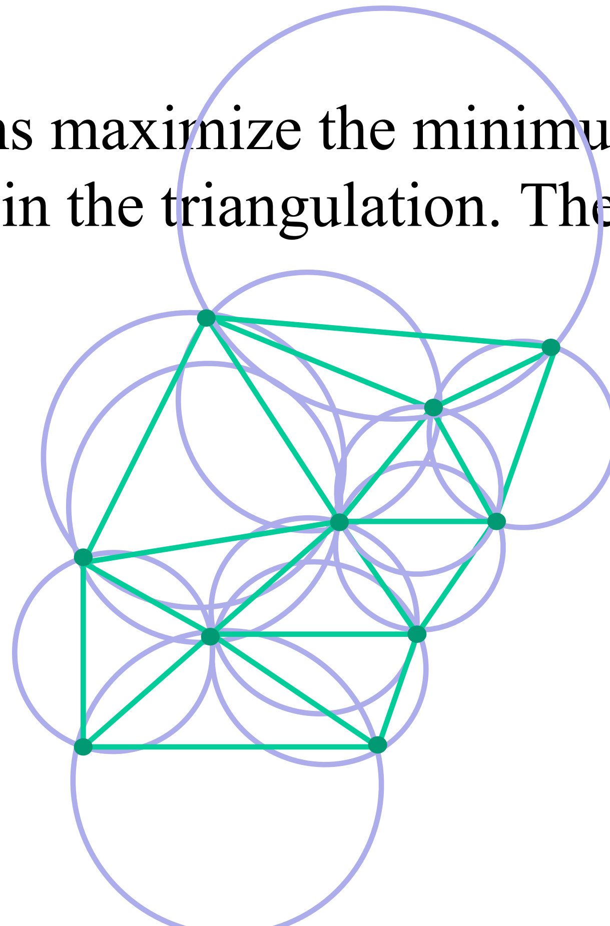
Grid Generation Approaches

- Triangular/Tetrahedral Mesh Generation
  - Delaunay Triangulation & Voronoi Diagram
  - Advancing Front Method
  - Octree-based Method

- Quadrilateral/Hexahedral Mesh Generation
  - Octree-based Method
  - Medial Surface Method
  - Plastering

Delaunay Triangulation

- Delaunay triangulation was invented by Boris Delaunay in 1934.
- A Delaunay triangulation for a set P of points in the plane is a triangulation DT(P) s.t. no point in P is inside the circumcircle of any triangle in DT(P).
- Delaunay triangulations maximize the minimum angle of all the angles of the triangles in the triangulation. They tend to avoid "sliver" triangles.



A Delaunay triangulation in the plane with circumcircles shown

Delaunay Triangulation

- Based on Delaunay definition, the circumcircle of a triangle formed by three points from the original point set is empty if it does not contain vertices other than the three that define it (Other points are permitted only on the very perimeter, not inside).

- A triangle net is a Delaunay triangulation if all the circumcircles of all the triangles in the net are empty. This Delaunay condition is also called "empty circle".

Special Cases

- For a set of points on the same line, there is no Delaunay triangulation (in fact, the notion of triangulation is undefined for this case).

- For 4 points on the same circle (e.g., the vertices of a rectangle), the Delaunay triangulation is not unique. Clearly, the two possible triangulations splitting the quadrangle into two triangles satisfy the Delaunay condition.

- Generalizations are possible to metrics other than Euclidean. However in these cases a Delaunay triangulation is not guaranteed to exist or be unique.

n-dimensional Delaunay

- For a set of points P in the (*d*-dimensional) Euclidean space, a Delaunay triangulation is a triangulation DT(P) s.t. no point in P is inside the circum-hypersphere of any simplex in DT(P).

- There exists a unique Delaunay triangulation for P, if P is a set of points in general position, that is, no three points are on the same line and no four are on the same circle for a 2D set of points. For an *n*-dimensional set of points, no *n*+1 points are on the same hyperplane and no *n*+2 points are on the same hypersphere.

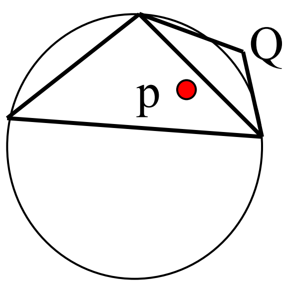
Properties

- Let n be the number of points and d the number of dimensions.
  1. The union of all simplices in the triangulation is the convex hull of the points.
  2. The Delaunay triangulation contains at most $O(n^{d/2})$ simplices.
  3. In the plane (2D), if there are b vertices on the convex hull, then any triangulation of the points has at most $2n-2-b$ triangles, plus one exterior face.
  4. The Delaunay triangulation maximizes the minimum angle.
  5. A circle circumscribing any Delaunay triangle doesn't contain any other input points in its interior.

Properties (continue)

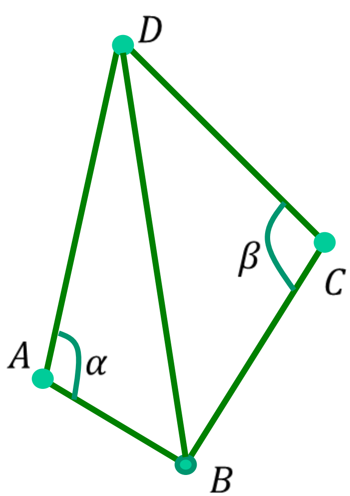
- Let *n* be the number of points and *d* the number of dimensions.

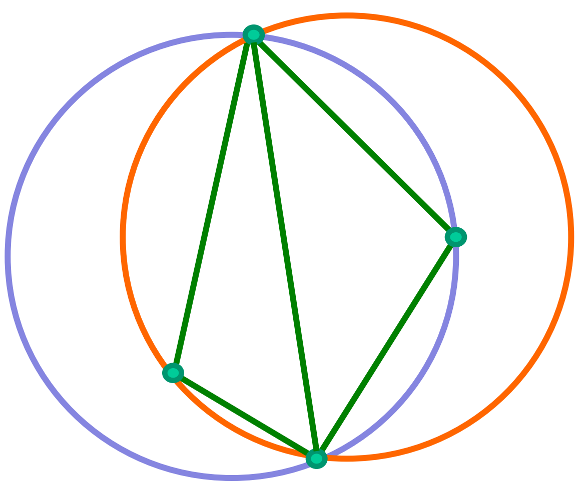
6. If a circle passing through two of the input points doesn't contain any other of them in its interior, then the segment connecting the two points is an edge of a Delaunay triangulation of the given points.
7. The Delaunay triangulation of a set of points in *d*-dimensional spaces is the projection of the convex hull of the projections of the points onto a $(d+1)$-dimensional paraboloid.
8. For a point *p* inside the convex hull of a Delaunay triangulation, the nearest vertex to *p* need not be one of the vertices of the triangle containing *p*.

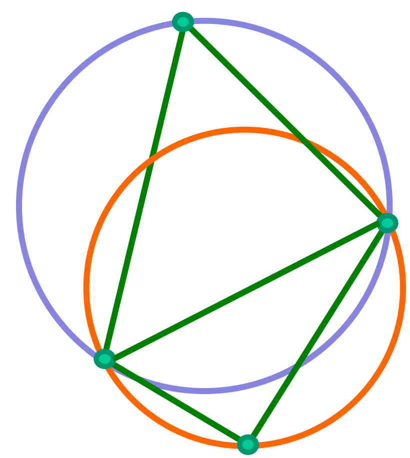


Visual Delaunay Definition: Flipping

- From the above properties an important feature arises: Looking at two triangles ABD and BCD with the common edge BD, if the sum of the angles α and β is less than or equal to 180° , the triangles meet the Delaunay condition.
- Flipping technique: If two triangles do not meet the Delaunay condition, switching the common edge BD for the common edge AC produces two triangles that do meet the Delaunay condition.







Algorithms

- All algorithms for computing Delaunay triangulations rely on fast operations for detecting when a point is within a triangle's circumcircle and an efficient data structure for storing triangles and edges.

- In 2D, one way to detect if point D lies in the circumcircle of A, B, C is to evaluate the determinant, assuming A, B, and C to lie counter-clockwise, this is positive if and only if D lies in the circumcircle.

$$\begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix} = \begin{vmatrix} A_x - D_x & A_y - D_y & (A_x - D_x)^2 + (A_y - D_y)^2 \\ B_x - D_x & B_y - D_y & (B_x - D_x)^2 + (B_y - D_y)^2 \\ C_x - D_x & C_y - D_y & (C_x - D_x)^2 + (C_y - D_y)^2 \end{vmatrix} > 0$$

Incremental

- The most straightforward way of computing the Delaunay triangulation is to repeatedly add one vertex at a time, retriangulating the affected parts of the graph.
- When a vertex is added, a search is done for all triangles' circumcircles containing the vertex. Then those triangles are removed and that part of the graph retriangulated. The time complexity is $O(n^2)$.

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Incremental

- A common way to speed up this method is sweepline, which involves sorting the vertices by one coordinate and adding them in that order. Then one only needs to keep track of circumcircles containing points of large enough first coordinate.
- The expected running time in 2D is $O(n^{3/2})$ although the worst case continues to be $O(n^2)$.
- Another efficient $O(n \log n)$ incremental algorithm keeps the whole history of the triangulation in the form of a tree. The elements replacing a conflict element in an insertion are called its children. This provides a fast way of getting the list of triangles to remove.

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Divide and Conquer

- A divide and conquer algorithm for triangulations in 2D is due to Lee and Schachter which was improved by Guibas and Stolfi and later by Dwyer.

- In this method, one recursively draws a line to split the vertices into two sets. The Delaunay triangulation is computed for each set, and then the two sets are merged along the splitting line.

- Using some clear tricks, the merge operation can be done in time $O(n)$, so the total running time is $O(n \log n)$.

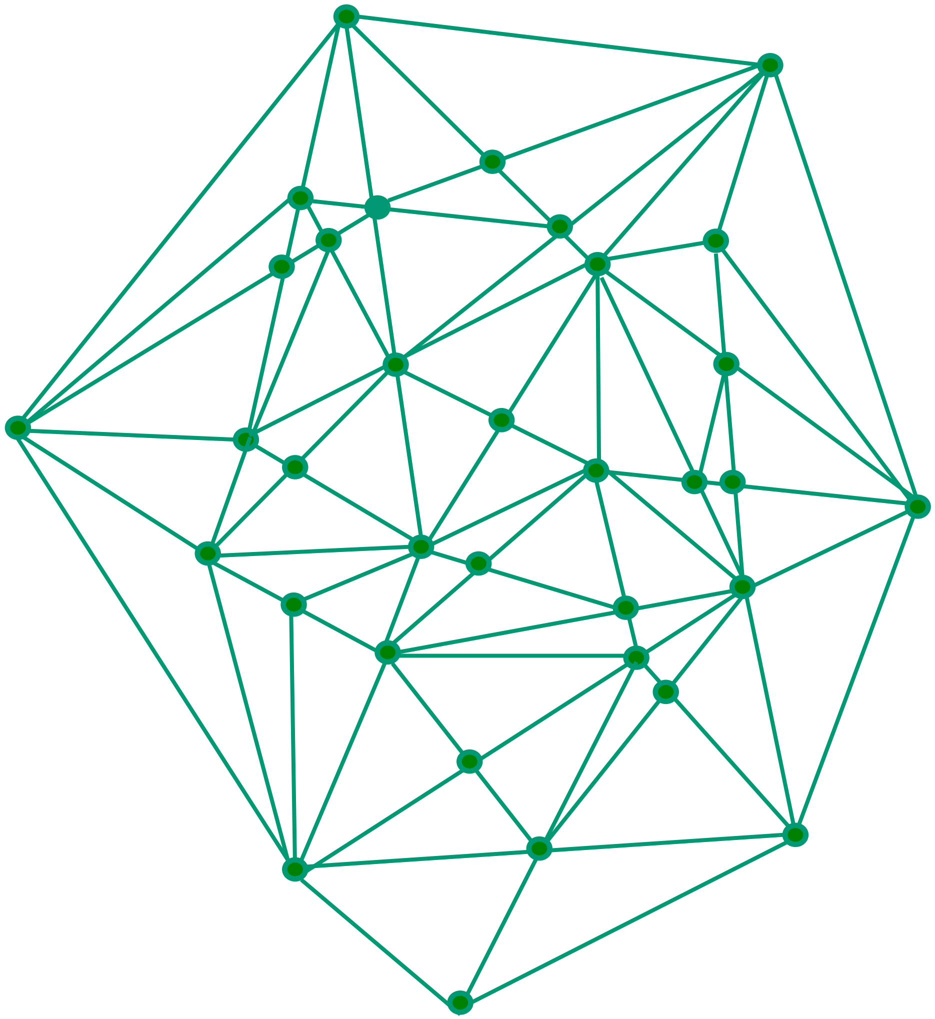
Applications

- For modeling terrain or other objects given a set of sample points, the Delaunay triangulation gives a nice set of triangles to use as polygons in the model. In particular, the Delaunay triangulation avoids narrow triangles.

- Delaunay triangulations are often used to build meshes for the finite element method, because of the angle guarantee and the fact that we know fast triangulation algorithms.

- Typically, the domain to be meshed is specified as a coarse simplicial complex. For the mesh to be numerically stable, it must be refined, for instance by using Ruppert's algorithm. This has been implemented by Jonathan Shewchuck in the freely available package.

Example



The Delaunay triangulation of a random set of 100 points in a plane.

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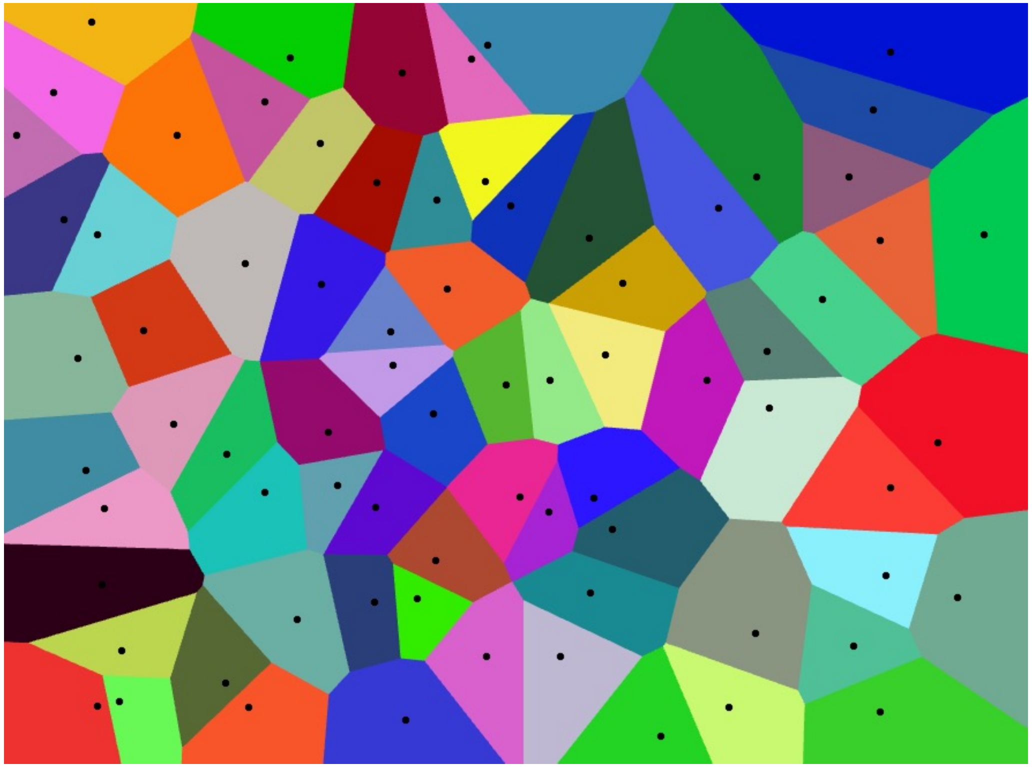
Voronoi Diagram

- The Delaunay triangulation of a discrete point set P corresponds to the dual graph of the Voronoi tessellation for P.

- A Voronoi diagram, named after George Voronoi, also called a Voronoi tessellation, a Voronoi decomposition, or a Dirichlet tessellation (after Lejeune Dirichlet), is a special kind of decomposition of a metric space determined by distances to a specified discrete set of points in the space.

Voronoi Diagram

- Given a set of points S, the Voronoi diagram for S is the partition of the plane which associates a region V(p) with each point p from S in such a way that all points in V(p) are closer to p than to any other point in S.



The Voronoi diagram of a random set of points in the plane (all points lie within the image).

Definition

- For any discrete set S of points in Euclidean space and for almost any point x, there is one point of S closest to x. The word "almost" is used to indicate exceptions where a point x may be equally close to two or more points of S.

- If S contains only two points, a and b, then the set of all points equidistant from a and b is a hyperplane, which is the boundary between the set of all points closer to a than to b, and the set of all points closer to b than to a. It is the perpendicular bisector of the line segment from a and b.

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Definition

- In general, the set of all points closer to a point c of S than to any other point of S is the interior of a convex polytope called the Dirichlet domain or Voronoi cell for c.
- The set of such polytopes tessellates the whole space, and is the Voronoi tessellation corresponding to the set S.
- If the dimension of the space is only 2, then it is easy to draw pictures of Voronoi tessellations, and in that case they are sometimes called Voronoi diagrams.

Properties

- The dual graph for a Voronoi diagram corresponds to the Delaunay triangulation for the same set of points *S*.

- The closest pair of points corresponds to two adjacent cells in the Voronoi diagram.

- Two points are adjacent on the convex hull if and only if their Voronoi cells share an infinitely long side.

History

- Informal use of Voronoi diagrams can be traced back to Descartes in 1644.
- Dirichlet used 2D and 3D Voronoi diagrams in his study of quadratic forms in 1850.
- British physician John Snow used a Voronoi diagram in 1854 to illustrate how the majority of people who died in the Soho Cholera epidemic lived closer to the infected Broad Street pump than to any other water pump.

History

- Voronoi diagrams are named after Russian mathematician Georgy F. Voronoi who defined and studied the general *n*-dimensional case in 1908.
- Voronoi diagrams that are used in geophysics and meteorology to analyze spatially distributed data (such as rainfall measurements) are called Thiessen polygons after American meteorologist Alfred H. Thiessen.
- Voronoi diagrams have been used in a lot of fields: In condensed matter physics, such tessellations are also known as Wigner-Seitz unit cells; In the case of general metric spaces, the cells are often called metric fundamental polygons; …

Examples

- Voronoi tessellations of regular lattices of points in 2D or 3D give rise to many familiar tessellations. For example,
  - A 2D lattice gives an irregular honeycomb tessellation, with equal hexagons with point symmetry; in the case of a regular triangular lattice it is regular; in the case of a rectangular lattice the hexagons reduce to rectangles in rows and columns; a square lattice gives the regular tessellation of squares.

Generalizations

- Voronoi cells can be defined for metric other than Euclidean. However in these cases the Voronoi tessellation is not guaranteed to exist.
- Voronoi cells can also be defined by measuring distances to objects that are not points. The Voronoi diagram with these cells is also called the medial axis, which is used in image segmentation, optical character recognition and other computational applications.
- In materials science, polycrystalline microstructures in metallic alloys are commonly represented using Voronoi tessellations.
- The Voronoi diagram of n points in d-dimensional space requires $O(n^{d/2})$ storage space. Therefore Voronoi diagrams are often not feasible for $d > 2$.

Application

- A point location data structure can be built on top of the Voronoi diagram in order to answer nearest neighbor queries, where you want to find the object that is closest to a given query point.
- Nearest neighbor queries have numerous applications. For example, when you want to find the nearest hospital, or the most similar object in a database.
- The Voronoi diagram is useful in polymer physics. It can be used to represent free volume of the polymer.
- It is also used in derivations of the capacity of a wireless network.
- In climatology, Voronoi diagrams are used to calculate the rainfall of an area, based on a series of point measurements. In this usage, they are generally referred to as Thiessen polygons.

Available Links and Source Codes

- Real time interactive Voronoi and Delaunay diagrams with source code http://www.cs.cornell.edu/Info/People/chew/Delaunay.html
- Delaunay triangulation and Voronoi diagram in CGAL, the Computational Geometry Algorithms Library
- Applet for calculation and visualization of convex hull, Delaunay triangulations and Voronoi diagrams in space
- Mathworld on Delaunay triangulation
  http://mathworld.wolfram.com/DelaunayTriangulation.html
- Qhull for computing Delaunay triangulations in 2-d, 3-d, etc.
  http://www.qhull.org/
- Triangle, a 2D Quality Mesh Generator and Delaunay Triangulator
  http://www.cs.cmu.edu/~quake/tripaper/triangle0.html
- Triangulate, an efficient algorithm for Terrain Modelling
  http://paulbourke.net/papers/triangulate/
- G. Leach: Improving Worst-Case Optimal Delaunay Triangulation Algorithms. (June 1992)

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