Summary

- Topics:

1. Bio-medical imaging
2. Image processing
3. Geometric modeling and computer graphics
4. Mesh generation
   - Marching Cubes/Dual Contouring
   - Tri/Tet Meshing
   - Quad/Hex Meshing
   - Quality Improvement
5. Computational mechanics
6. Bio-medical applications

Mechanical Engineering
Carnegie Mellon

Topic 5: Introduction to Finite Element Method

Jessica Zhang
Department of Mechanical Engineering
Courtesy Appointment in Biomedical Engineering
Carnegie Mellon University
jessicaz@andrew.cmu.edu
http://www.andrew.cmu.edu/user/jessicaz

Mechanical Engineering
Carnegie Mellon

Overview

- What is Finite Element Method?
- A One-dimensional Model Problem
  - Boundary value problem (BVP)
  - Weak/variational formulation
  - Galerkin approximations
  - Basis functions
  - Finite element calculations
  - Flow chart
  - Interpretation of the approximate solution
  - Accuracy of the finite element approximation

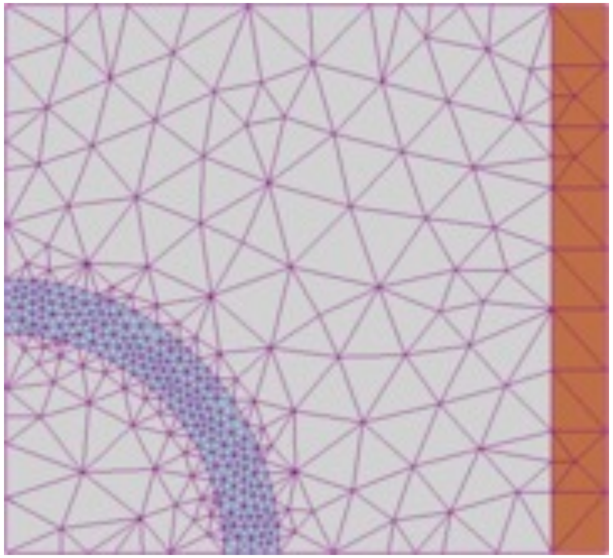
Finite Element Method

- The finite element method is a general technique for constructing approximate solutions to boundary-value problems (BVP).

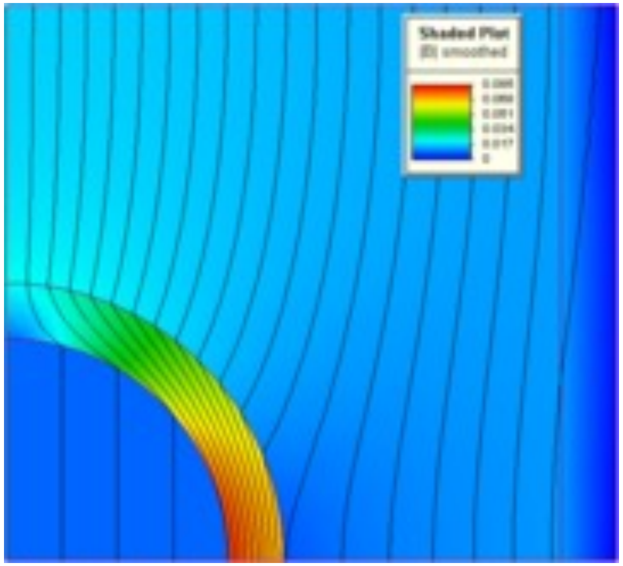
- This method involves dividing the domain of the solution into a finite number of simple subdomains, the finite elements, and using the variational concepts to construct an approximation of the solution over the collection of finite elements.

Finite Element Method

- Due to the generality and richness of the ideas underlying the method, it has been used with remarkable success in solving a wide range of problems in a lot of research fields.



2D mesh for the right image (mesh is denser around the object of interest)



2D FEM solution for a magnetostatic configuration (lines denote the direction of calculated flux density and color–its magnitude)

Mechanical Engineering
Carnegie Mellon

Overview

- What is Finite Element Method?
- A One-dimensional Model Problem
  - Boundary value problem (BVP)
  - Weak/variational formulation
  - Galerkin approximations
  - Basis functions
  - Finite element calculations
  - Flow chart
  - Interpretation of the approximate solution
  - Accuracy of the finite element approximation

Mechanical Engineering
Carnegie Mellon

The Model Problem

- A one-dimensional “two-point” boundary-value problem characterized by
  - a simple linear ordinary differential equation of second order
  - a pair of boundary conditions

Mechanical Engineering
Carnegie Mellon

The Statement of The Model Problem

- Consider the problem of finding a function $u = u(x)$, $0 \leq x \leq 1$, satisfying the following differential equation and boundary conditions:

$$\left.\begin{aligned} -u'' + u &= f(x), & 0 < x < 1 \\ u(0) &= 0, & u(1) = 0 \end{aligned}\right\}$$

where $u'' = d^2u/dx^2$, and $f(x) = x$.

- A problem such as this might be the study of the deflection of a string on an elastic foundation or of the temperature distribution in a rod.

Mechanical Engineering
Carnegie Mellon

Overview

- What is Finite Element Method?
- A One-dimensional Model Problem
  - Boundary value problem (BVP)
  - Weak/variational formulation
  - Galerkin approximations
  - Basis functions
  - Finite element calculations
  - Flow chart
  - Interpretation of the approximate solution
  - Accuracy of the finite element approximation

Weak/Variational Statement

- The requirement that a solution u to the BVP satisfies the differential equation at *every point* x, $0 < x < 1$, is too strong.

- To overcome this difficulty, we shall reformulate the BVP in a way that will admit weaker conditions on the solution and its derivatives. Such reformulation is called weak or variational formulation.

- Whenever a smooth "classical" solution to a problem exists, it is also the solution of the weak problem.

Weak/Variational Formulation

- Consider the same BVP problem:

$$\left.\begin{array}{ll} -u'' + u = x, & 0 < x < 1 \\ u(0) = 0. & u(1) = 0 \end{array}\right\} \qquad (1)$$

- The weak format is:

$$\int_0^1 \left(-u'' + u\right) v dx = \int_0^1 x v dx$$

for all members v of a suitable class of functions. v is the *weight function* or the *test function*.

Weak Statement

- The set of functions with zero values at $x = 0$ and $x = 1$ is denoted by the symbol H.
- The weak statement assumes the more compact form: find u such that:

$$\left.\begin{aligned} \int_0^1 (-u'' + u - x) v dx &= 0 \qquad \textit{for all } v \in H \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned}\right\} \qquad (2)$$

A Symmetric Variational Formulation

- If u and v are sufficiently smooth functions, then the standard integration-by-parts formula holds:

$$\int_0^1 -u''v\,dx = \int_0^1 u'v'\,dx - u'v\Big|_0^1.$$

- If we demand the test functions vanish at the endpoints, $v(0) = v(1) = 0$, then we have:

$$\int_0^1 -u''v\,dx = \int_0^1 u'v'\,dx$$

for all admissible test functions v.

Weak Statement

- The weak statement becomes: find $u \in H_0^1$ such that

$$\int_0^1 (u'v' + uv - xv)dx = 0 \qquad \text{for all } v \in H_0^1 \qquad (3)$$

- The set of admissible function $u \in H_0^1$ is defined as the set of all functions that vanish at the endpoints and whose 1st derivatives are *square-integrable*. Thus, a function w is a member of H_0^1 if

$$\int_0^1 (w')^2 dx < \infty \qquad \text{and} \qquad w(0) = 0 = w(1)$$

Properties of H_0^1

- **A linear space of functions:** if v_1 and v_2 are arbitrary test functions and α and β are arbitrary constants, then $\alpha v_1 + \beta v_2$ is also a test function.

- **Infinite-dimensional:** if we introduce the set of functions

$$\Psi_n(x) = \sqrt{2}\sin n\pi x, \qquad n = 1, 2, 3, \cdots$$

(trigonometric function) and v is a smooth test function in H_0^1, then it is easily verified that v can be represented in the form

$$v(x) = \sum_{n=1}^{m} a_n \Psi_n(x) \qquad a_n = \int_0^1 v(x)\Psi_n(x)dx$$

- An infinity of coefficients a_n must be specified in order to define any function $v \in H_0^1$; therefore the space H_0^1 of admissible function is infinite-dimensional.

Overview

- What is Finite Element Method?
- A One-dimensional Model Problem
  - Boundary value problem (BVP)
  - Weak/variational formulation
  - Galerkin approximations
  - Basis functions
  - Finite element calculations
  - Flow chart
  - Interpretation of the approximate solution
  - Accuracy of the finite element approximation

Mechanical Engineering
Carnegie Mellon

Galerkin Approximations

- Given an infinite set of functions in H_0^1

$$\{\phi_1(x), \phi_2(x), \phi_3(x), \cdots\}$$

Each test function v in H_0^1 can be represented as a linear combination of them

$$v(x) = \sum_{i=1}^{\infty} \beta_i \phi_i(x)$$

where β_i are constants, ϕ_i are called *basis functions*.

Mechanical Engineering
Carnegie Mellon

Approximation with Finite Number Basis Functions

- If we take only a finite number N of terms, then we will obtain only an approximation of v

$$v_N(x) = \sum_{i=1}^{N} \beta_i \phi_i(x). \qquad (4)$$

- The N basis functions

$$\{\phi_1(x), \phi_2(x), \cdots, \phi_N(x)\}$$

define an N-dimensional subspace H_0^N of H_0^1 .

Galerkin Approximations

- Galerkin's method consists of seeking an approximate solution to

$$\int_0^1 (u'v' + uv - xv)dx = 0 \qquad \textit{for all } v \in H_0^1$$

  in a finite-dimensional subspace H_0^N of the space of H_0^1 admissible functions rather than in the whole space H_0^1 .

- We seek an approximate solution in H_0^N,

$$u_N(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x). \qquad (5)$$

Mechanical Engineering
Carnegie Mellon

Galerkin Approximations

- The weak form becomes: find $u_N \in H_0^N$ such that

$$\int_0^1 \left(u'_N v'_N + u_N v_N\right) dx = \int_0^1 x v_N dx \qquad for\,all\; v_N \in H_0^N \qquad (6)$$

$$u_N(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x)$$

- Since the ϕ_i are known, u_N will be completely determined once the N coefficients α_i are determined. α_i are referred to as the *degrees of freedom* of the approximation.

Mechanical Engineering
Carnegie Mellon

Galerkin Approximations

- Plug the approximate solution u_N and the test function v_N into the weak statement (Eqn.6):

$$\int_0^1 \left(u'_N v'_N + u_N v_N\right)dx = \int_0^1 x v_N dx$$

$$u_N(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x) \longrightarrow \downarrow \longleftarrow v_N(x) = \sum_{i=1}^{N} \beta_i \phi_i(x)$$

$$\int_0^1 \left\{\frac{d}{dx}\left[\sum_{i=1}^{N} \beta_i \phi_i(x)\right] \frac{d}{dx}\left[\sum_{j=1}^{N} \alpha_j \phi_j(x)\right] + \left[\sum_{i=1}^{N} \beta_i \phi_i(x)\right]\left[\sum_{j=1}^{N} \alpha_j \phi_j(x)\right] - x \sum_{i=1}^{N} \beta_i \phi_i(x)\right\} dx = 0 \qquad \text{for all } \beta_i,\ i = 1, 2, \ldots, N$$

Expanding it and factoring the coefficients β_i gives

$$\sum_{i=1}^{N} \beta_i \left(\sum_{j=1}^{N} \left\{\int_0^1 [\phi'_i(x)\phi'_j(x) + \phi_i(x)\phi_j(x)]\, dx\right\} \alpha_j - \int_0^1 x\phi_i(x)\, dx\right) = 0$$

for all β_i, $i = 1, 2, \ldots, N$

$\phi'_i(x) = d\phi_i(x)/dx.$

Mechanical Engineering
Carnegie Mellon

Galerkin Approximations

- Rewrite it in the more compact form:

$$\sum_{i=1}^{N} \beta_i \left(\sum_{j=1}^{N} K_{ij}\alpha_j - F_i \right) = 0$$

for all choices of β_i, where

$$K_{ij} = \int_0^1 [\phi_i'(x)\phi_j'(x) + \phi_i(x)\phi_j(x)]\, dx$$

$$F_i = \int_0^1 x\phi_i\, dx \qquad i, j = 1, 2, \ldots, N \tag{7}$$

- The $N \times N$ matrix K=[K_{ij}] is referred to as the stiffness matrix, and the $N \times 1$ column vector F={F_i} is referred to as the load/force vector.

Galerkin Approximations

- Because β_i is arbitrary, we can obtain

$$\sum_{j=1}^{N} K_{ij}\alpha_j = F_i, \qquad i = 1, 2, \ldots, N$$

$$\alpha_j = \sum_{i=1}^{N} (K^{-1})_{ji} F_i$$

- The approximation solution is:

$$u_N(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x).$$

Galerkin Methods

- The stiffness matrix is symmetric. This symmetry provides us with an opportunity for reducing the computational effort.
- For a symmetric formulation, the Galerkin's method provides the best possible approximation of the solution.
- For a symmetric formulation, the spaces of trial functions and test functions coincide, only one set of basis functions ϕ_i need to be constructed.

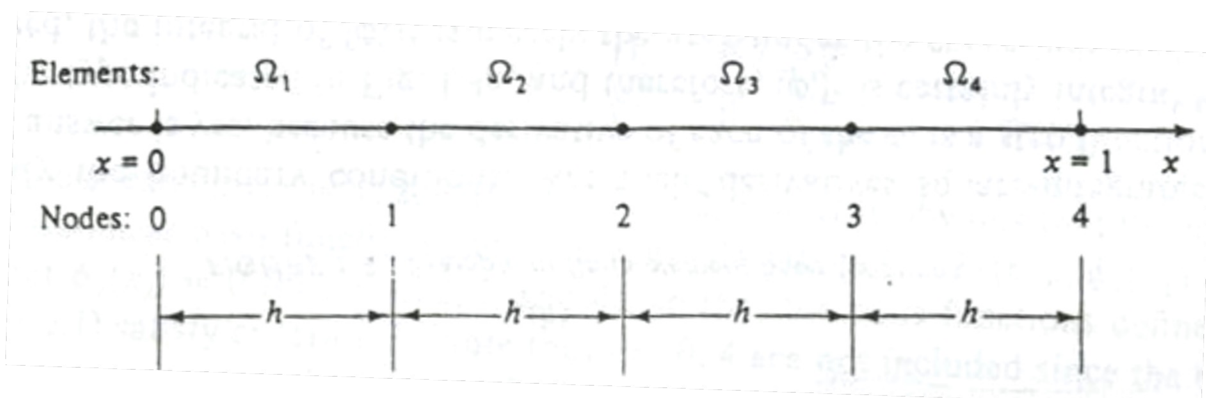
$$K_{ij} = \int_0^1 [\phi_i'(x)\phi_j'(x) + \phi_i(x)\phi_j(x)]\, dx$$

Overview

- What is Finite Element Method?
- A One-dimensional Model Problem
  - Boundary value problem (BVP)
  - Weak/variational formulation
  - Galerkin approximations
  - Basis functions
  - Finite element calculations
  - Flow chart
  - Interpretation of the approximate solution
  - Accuracy of the finite element approximation

Finite Element Basis Functions

- The basis function can be defined piecewise over subregions of the domain called finite elements, e.g., piecewise polynomials of low degree.
- First we partition the domain ($0 \leq x \leq 1$) into elements (mesh generation).

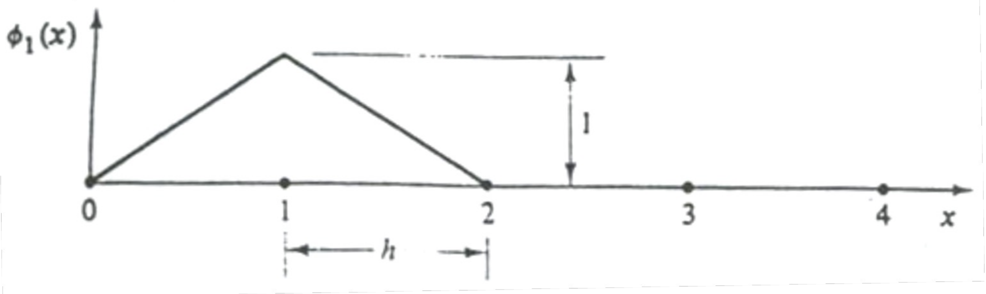


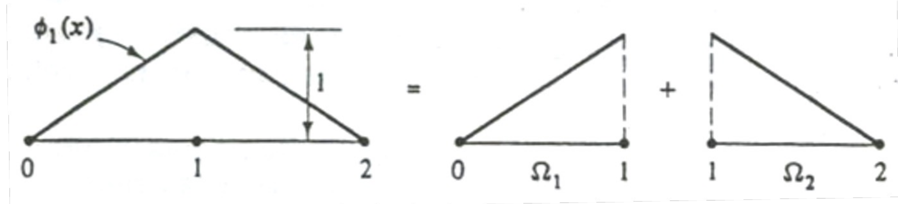
A finite element partition of the domain comprised of 4 elements with nodes at element endpoints.

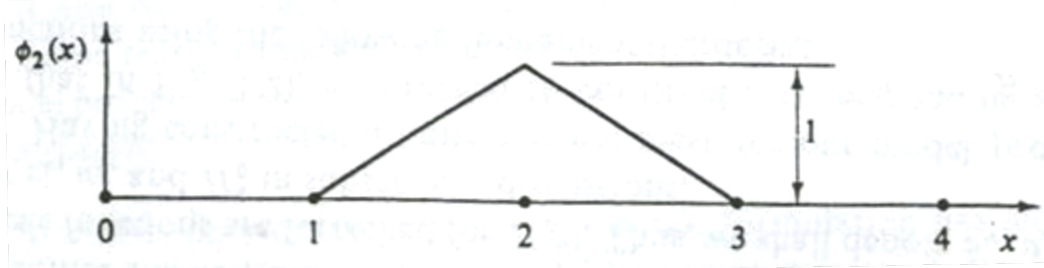
Finite Element Basis Functions

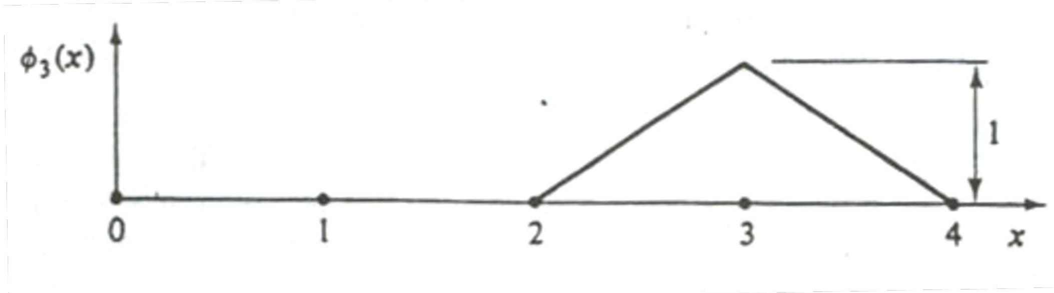
- Fundamental criteria to construct a set of basis functions:
  - The basis functions are generated by simple functions defined piecewise – element by element – over the finite element mesh.
  - The basis functions are smooth enough to be members of the class H_0^1 of test functions.
  - The basis functions are chosen in such a way that the parameters α_i define the approximation solution precisely at the nodal points.

Finite Element Basis Functions ("Hat"-Function)









Mechanical Engineering
Carnegie Mellon

Finite Element Basis Functions

- If the coordinates of the nodes are denoted x_i (i = 0, 1, 2, 3, 4), then the basis functions shown for i = 1, 2, and 3 are given by

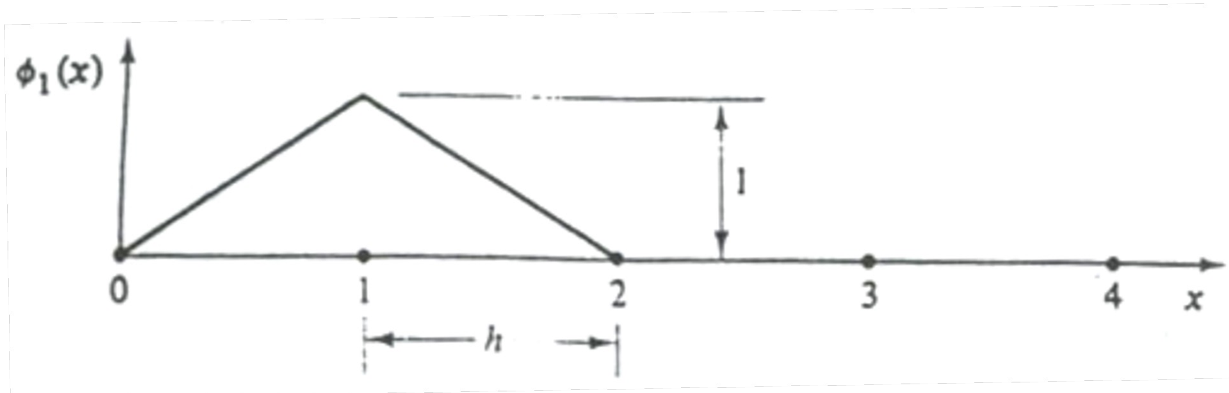
$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i} & \text{for } x_{i-1} \le x \le x_i \\[2ex] \dfrac{x_{i+1} - x}{h_{i+1}} & \text{for } x_i \le x \le x_{i+1} \\[2ex] 0 & \text{for } x \le x_{i-1} \quad \text{and} \quad x \ge x_{i+1} \end{cases}$$

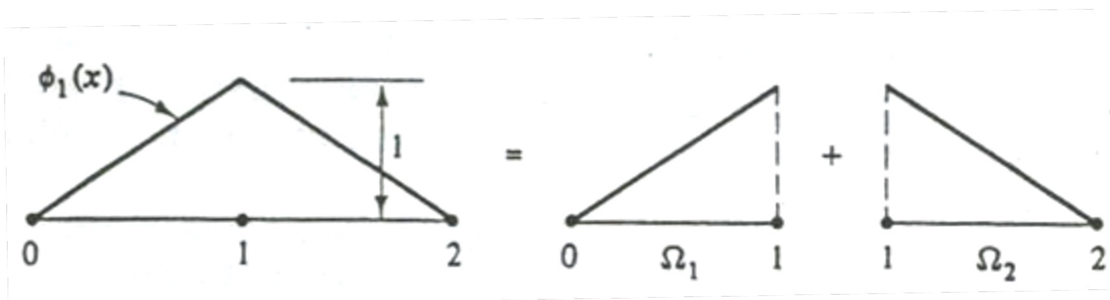
$$\phi_i'(x) = \begin{cases} \dfrac{1}{h_i} & \text{for } x_{i-1} < x < x_i \\[2ex] \dfrac{-1}{h_{i+1}} & \text{for } x_i < x < x_{i+1} \\[2ex] 0 & \text{for } x < x_{i-1} \quad \text{and} \quad x > x_{i+1} \end{cases}$$

where $h_i = x_i - x_{i-1}$ is the length of element Ω_i.

Check Criteria 1

- **Criteria 1:** The basis functions are generated by simple functions defined piecewise – element by element – over the finite element mesh.



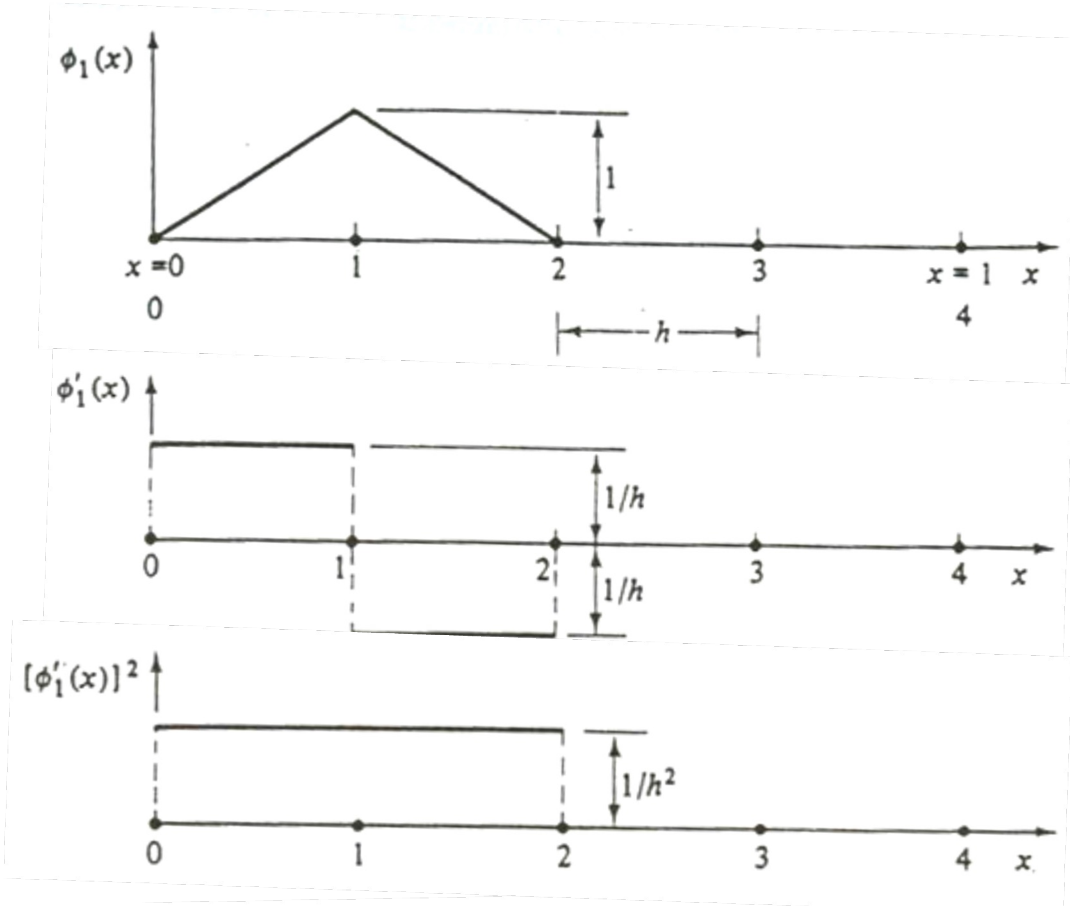


Check Criteria 2

- **Criteria 2:** The basis functions are smooth enough to be members of the class H_0^1 of test functions.
  - In order that $\phi_i \in H_0^1$, $i = 1, 2, 3$, each must have square-integrable 1st derivatives and must vanish at x=0 and x=1. It is obvious that the hat-function satisfies this condition.
  - Are their derivatives *square-integrable*? [Yes]

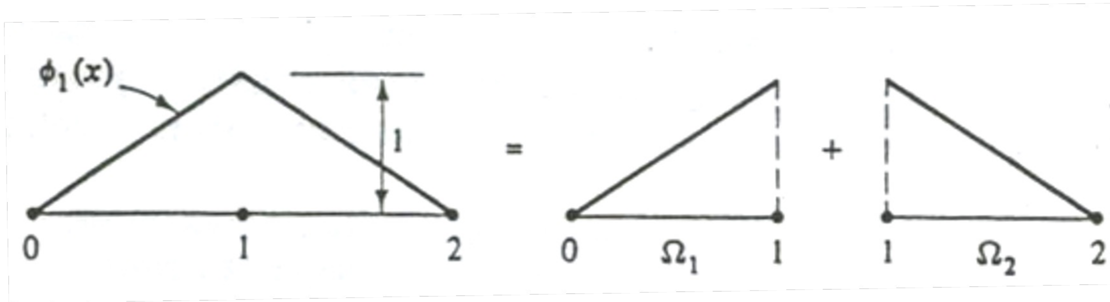
Check Criteria 2

– Are their derivatives square-integrable? [Yes]

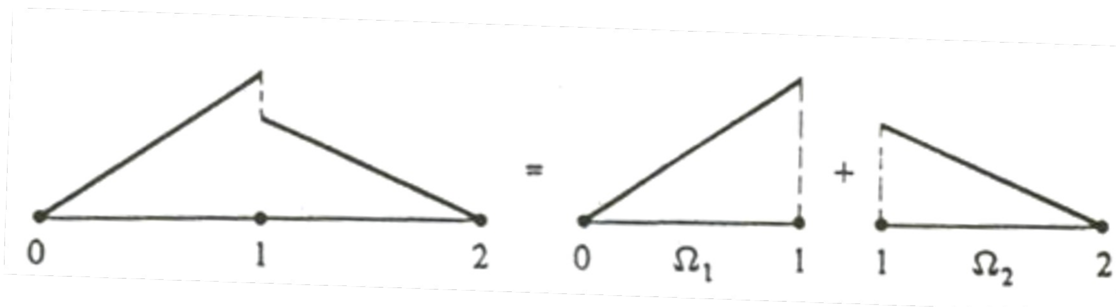


$$\int_0^1 [\phi_1'(x)]^2\, dx = \frac{1}{h^2} 2h = 2h^{-1} < \infty$$

Square-integrable vs Non-square-integrable



Square-integrable basis functions



Non-square-integrable basis functions

Check Criteria 3

- **Criteria 3:** The basis functions are chosen in such a way that the parameters α_i define the approximation solution precisely at the nodal points.
  - This criterion is not difficult to satisfy if each basis function is unity at one node and zero at all other nodes. If x_j is the x-coordinate of node j, then

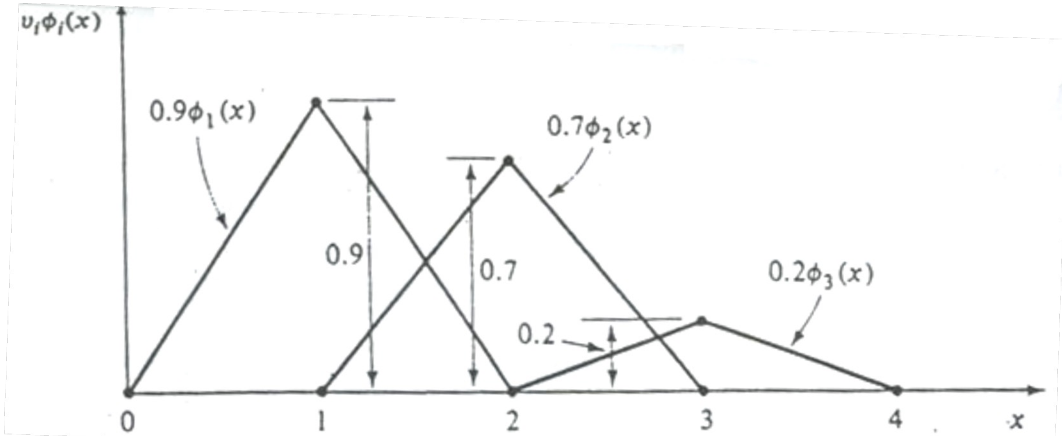
$$\phi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

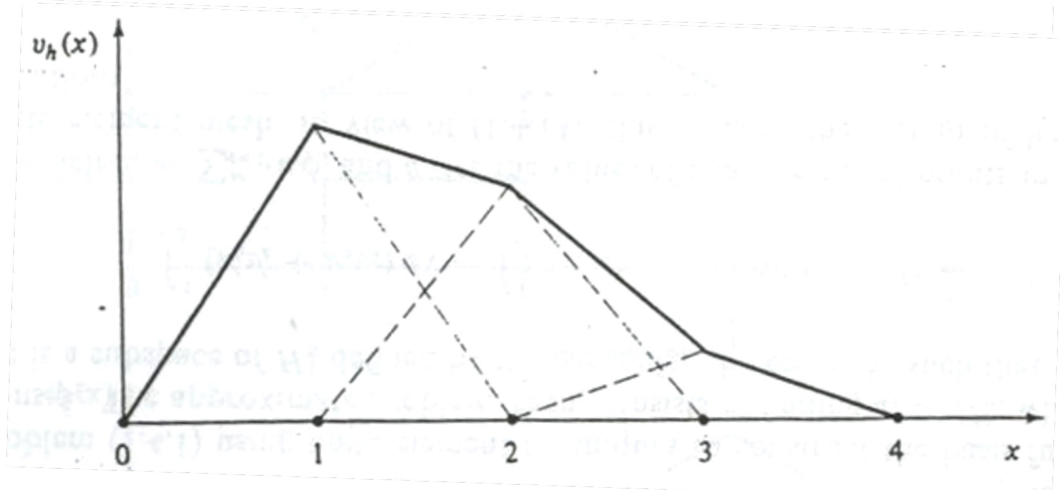
  - Note that $i = 0, 4$ are not included since the basis functions are required to satisfy the homogeneous end conditions.

Basis Functions

Suppose N = 3, β_1=0.9, β_2=0.7, and β_3=0.2.

$$v_h(x) = \sum_{i=1}^{N} \beta_i \phi_i(x). \rightarrow v_h(x) = 0.9\phi_1(x) + 0.7\phi_2(x) + 0.2\phi_3(x)$$





Mechanical Engineering
Carnegie Mellon

Overview

- What is Finite Element Method?
- A One-dimensional Model Problem
  - Boundary value problem (BVP)
  - Weak/variational formulation
  - Galerkin approximations
  - Basis functions
  - Finite element calculations
  - Flow chart
  - Interpretation of the approximate solution
  - Accuracy of the finite element approximation

Finite Element Calculations

- The weak form is: find $u_h \in H_0^h$ such that

$$\int_0^1 \left(u_h' v_h' + u_h v_h\right) dx = \int_0^1 x v_h \, dx \qquad \textit{for all } v_h \in H_0^h$$

$$u_h(x) = \sum_{i=1}^{N} u_i \phi_i(x).$$

u_i are the values of u_h at the nodal points in the finite element mesh. This leads to the system of linear equations

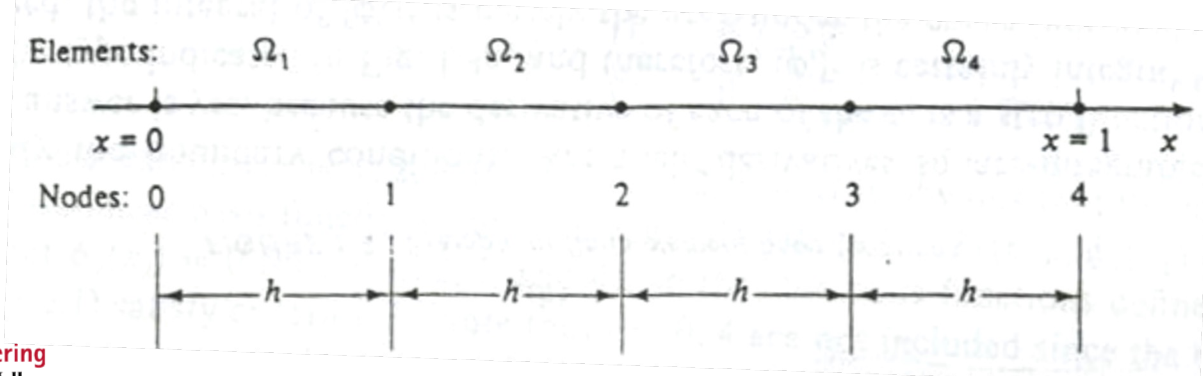
$$\sum_{j=1}^{N} K_{ij} \alpha_j = F_i, \qquad i = 1, 2, \ldots, N$$

Stiffness Matrix and Load Vector

- Summability of stiffness

$$
\begin{aligned}
K_{ij} &= \int_0^1 (\phi_i'\phi_j' + \phi_i\phi_j)\,dx \\
&= \int_0^h (\phi_i'\phi_j' + \phi_i\phi_j)\,dx + \int_h^{2h} (\phi_i'\phi_j' + \phi_i\phi_j)\,dx \\
&\quad + \int_{2h}^{3h} (\phi_i'\phi_j' + \phi_i\phi_j)\,dx + \int_{3h}^{1} (\phi_i'\phi_j' + \phi_i\phi_j)\,dx \\
&= \sum_{e=1}^{4} \int_{\Omega_e} (\phi_i'\phi_j' + \phi_i\phi_j)\,dx
\end{aligned}
$$

$\int_{\Omega_e}$ denotes integration over element Ω_e.



Stiffness Matrix and Load Vector

- Let the term

$$K_{ij}^{e} = \int_{\Omega_e} (\phi_i' \phi_j' + \phi_i \phi_j)\, dx$$

represent components of the element stiffness matrix for finite element Ω_e.

$$K_{ij} = \sum_{e=1}^{4} K_{ij}^{e}$$

Similarly

$$F_i = \sum_{e=1}^{4} F_i^{e}, \qquad F_i^{e} = \int_{\Omega_e} x \phi_i \, dx$$

Mechanical Engineering
Carnegie Mellon

Stiffness Matrix and Load Vector

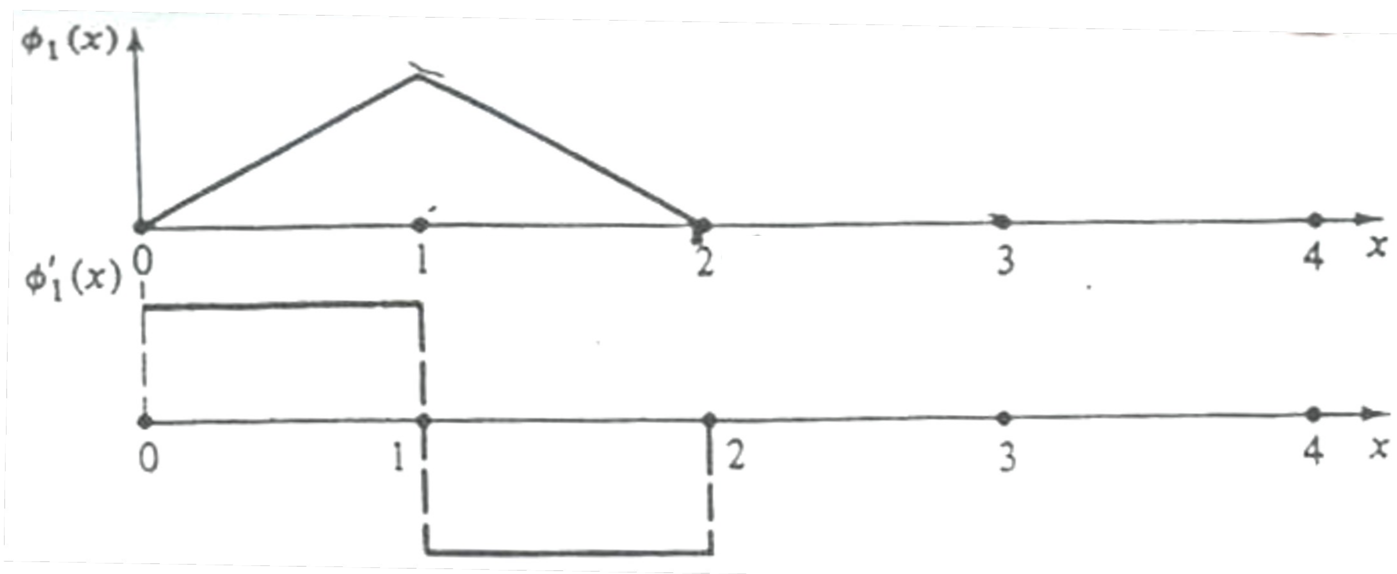
- The stiffness matrix and load vector can be computed as the sum of contributions from each element.

- First construct K^e and F^e for a typical element Ω_e, then construct K and F.

Mechanical Engineering
Carnegie Mellon

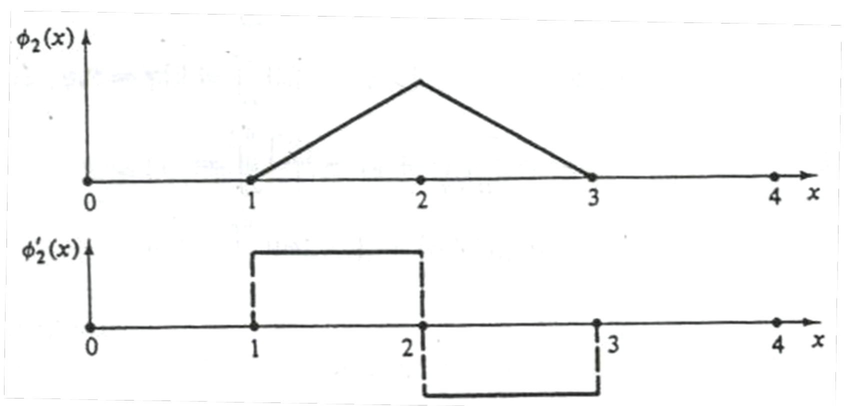
Sparseness of K

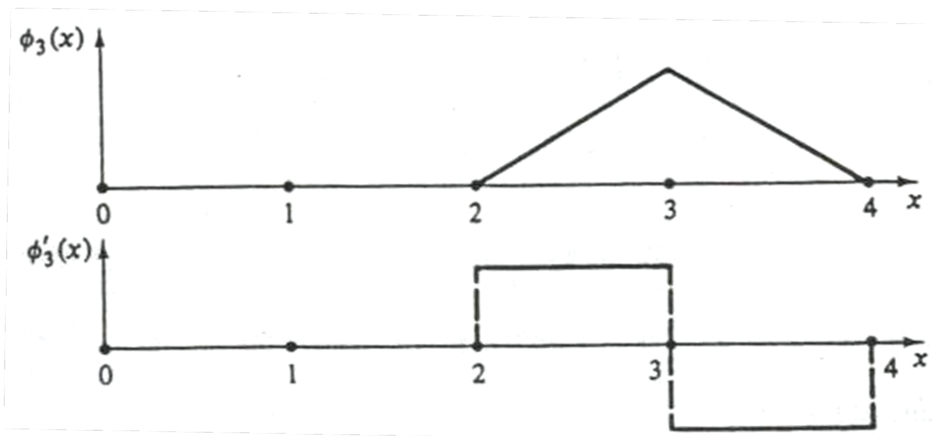
- Regarding the stiffness matrix K_{ij}, $i, j = 1, 2, 3$, we must compute 9 numbers.
- We found that ϕ_1 and ϕ_1' are not zero only in elements Ω_1 and Ω_2, which are adjacent to node 1. There K_{12}, K_{21} are nonzero, but K_{13}=K_{31}=0.



Sparseness of K

- Similarly, ϕ_2 and ϕ_2' are not zero only in elements Ω_2 and Ω_3, which are adjacent to node 2.
- ϕ_3 and ϕ_3' are not zero only in elements Ω_3 and Ω_4, which are adjacent to node 3.
- If we number nodes sequentially, the constructed stiffness matrix is to be banded.





Symmetry of K

- Interchanging i and j in the integral expression for K_{ij} does not change the value calculated, so that $K_{ij} = K_{ji}$ and the stiffness matrix is symmetric.

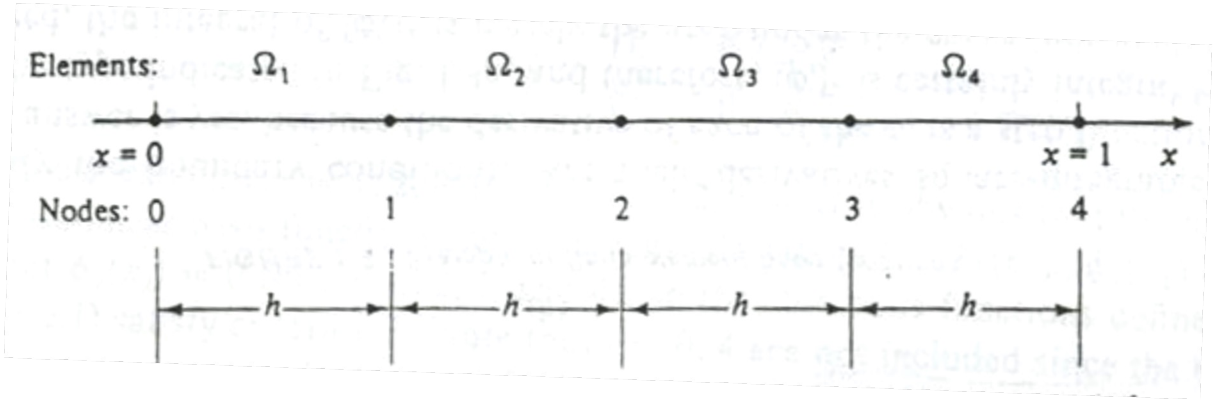
$$K_{ij} = \int_0^1 [\phi'_i(x)\phi'_j(x) + \phi_i(x)\phi_j(x)]\, dx$$

Mechanical Engineering
Carnegie Mellon

Solving The Model Problem

- The weak form is: find $u_N \in H_0^N$ such that

$$\int_0^1 \left(u'_N v'_N + u_N v_N\right) dx = \int_0^1 x v_N dx \qquad \textit{for all } v_N \in H_0^N$$

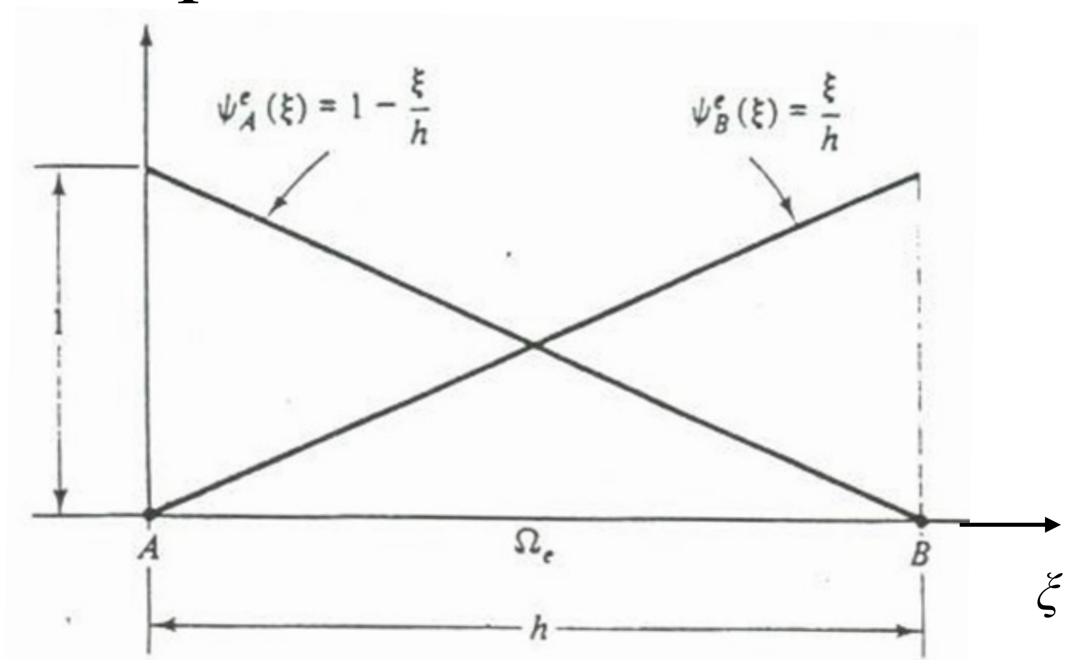


A finite element partition of the domain comprised of 4 elements with nodes at element endpoints.

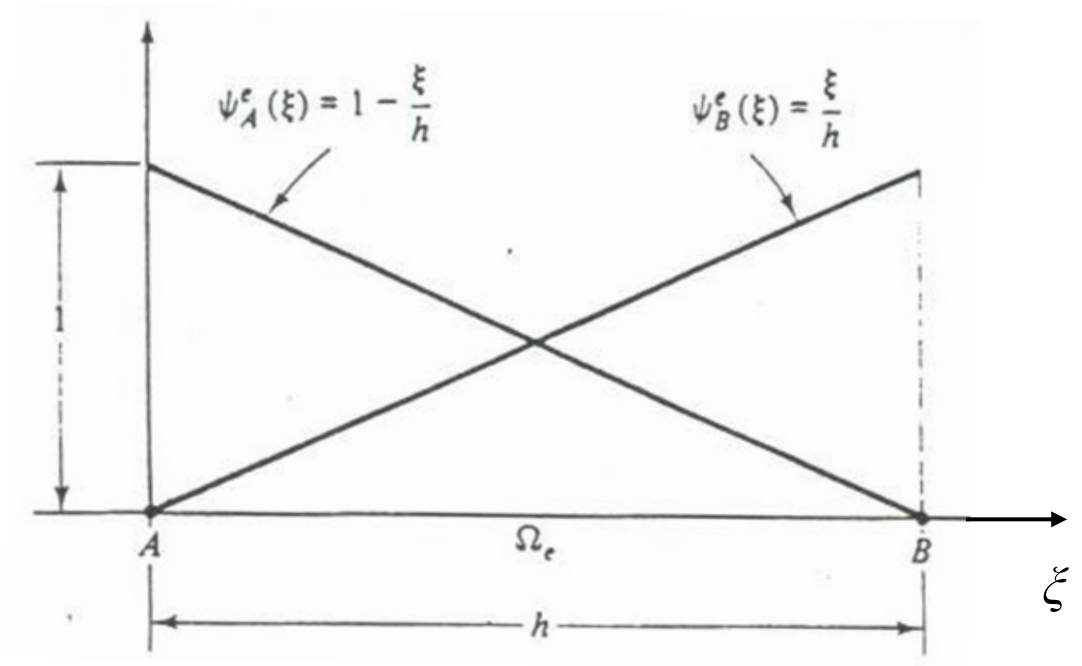
Mechanical Engineering
Carnegie Mellon

One Representative Element

- Let ξ be a local coordinate on this representative element with its origin at the left node A of Ω_e. As x traverses x_A to x_B, ξ goes from 0 to h, or $\xi = x - x_A$. $\Psi^e{}_A$ and $\Psi^e{}_B$ are the shape functions defined for Ω_e.



One Representative Element



$$\psi_A^e(\xi) = 1 - \frac{\xi}{h}, \qquad \psi_B^e(\xi) = \frac{\xi}{h}$$

$$\psi_A^{e\prime}(\xi) = -\frac{1}{h}, \qquad \psi_B^{e\prime}(\xi) = \frac{1}{h}$$

Mechanical Engineering
Carnegie Mellon

One Representative Element

Stiffness matrix:
$$k^e_{AA} = \int_0^h \{[\psi^{e\prime}_A(\xi)]^2 + [\psi^e_A(\xi)]^2\}\, d\xi$$
$$= \int_0^h \left[\frac{1}{h^2} + \left(1 - \frac{\xi}{h}\right)^2\right] d\xi = \frac{1}{h} + \frac{h}{3}$$
$$k^e_{AB} = k^e_{BA} = \int_0^h [\psi^{e\prime}_A(\xi)\psi^{e\prime}_B(\xi) + \psi^e_A(\xi)\psi^e_B(\xi)]\, d\xi$$
$$= \int_0^h \left[\left(\frac{-1}{h}\right)\frac{1}{h} + \left(1 - \frac{\xi}{h}\right)\frac{\xi}{h}\right] d\xi = -\frac{1}{h} + \frac{h}{6}$$
$$k^e_{BB} = \int_0^h \{[\psi^{e\prime}_B(\xi)]^2 + [\psi^e_B(\xi)]^2\}\, d\xi = \frac{1}{h} + \frac{h}{3}$$

Load vector:
$$F^e_A = \int_0^h (x_A + \xi)\left(1 - \frac{\xi}{h}\right) d\xi = \frac{h}{6}(2x_A + x_B)$$
$$F^e_B = \int_0^h (x_A + \xi)\left(\frac{\xi}{h}\right) d\xi = \frac{h}{6}(x_A + 2x_B)$$

Mechanical Engineering
Carnegie Mellon

One Representative Element

Construct local element stiffness matrix and load vector:

$$\mathbf{k}^e = \begin{bmatrix} \frac{1}{h} + \frac{h}{3} & -\frac{1}{h} + \frac{h}{6} \\ -\frac{1}{h} + \frac{h}{6} & \frac{1}{h} + \frac{h}{3} \end{bmatrix}, \qquad \mathbf{f}^e = \frac{h}{6}\begin{bmatrix} 2x_A + x_B \\ x_A + 2x_B \end{bmatrix}$$

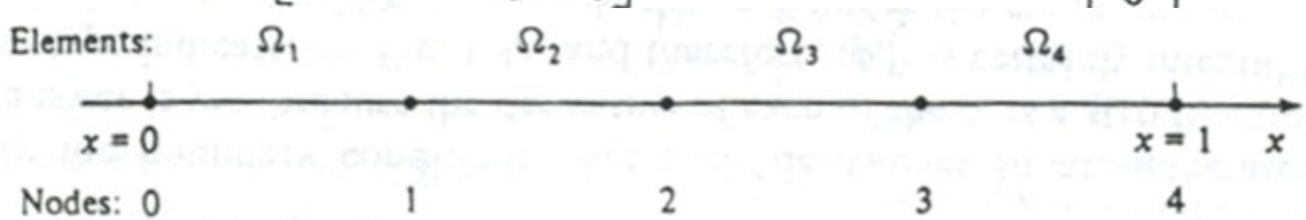
then we can calculate the element matrix for each element, $h = 1/4$.

Element Ω_1

$$\mathbf{K}^1 = [K_{ij}^1] = \frac{1}{24}\begin{bmatrix} 98 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{F}^1 = \{F_i^1\} = \frac{1}{96}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Element Ω_2

$$\mathbf{K}^2 = [K_{ij}^2] = \frac{1}{24}\begin{bmatrix} 98 & -95 & 0 \\ -95 & 98 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{F}^2 = \{F_i^2\} = \frac{1}{96}\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$$



Stiffness Matrix and Load Vector

Element Ω_3

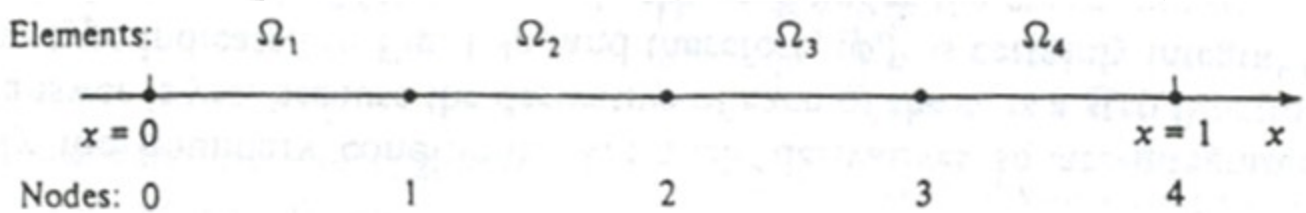
$$\mathbf{K}^3 = [K^3_{ij}] = \frac{1}{24}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 98 & -95 \\ 0 & -95 & 98 \end{bmatrix}, \qquad \mathbf{F}^3 = \{F^3_i\} = \frac{1}{96}\begin{bmatrix} 0 \\ 7 \\ 8 \end{bmatrix}$$

Element Ω_4

$$\mathbf{K}^4 = [K^4_{ij}] = \frac{1}{24}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 98 \end{bmatrix}, \qquad \mathbf{F}^4 = \{F^4_i\} = \frac{1}{96}\begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$\mathbf{K} = [K_{ij}] = \mathbf{K}^1 + \mathbf{K}^2 + \mathbf{K}^3 + \mathbf{K}^4 = \frac{1}{24}\begin{bmatrix} 196 & -95 & 0 \\ -95 & 196 & -95 \\ 0 & -95 & 196 \end{bmatrix}$$

$$\mathbf{F} = \{F_i\} = \mathbf{F}^1 + \mathbf{F}^2 + \mathbf{F}^3 + \mathbf{F}^4 = \frac{1}{96}\begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix}$$



Mechanical Engineering
Carnegie Mellon

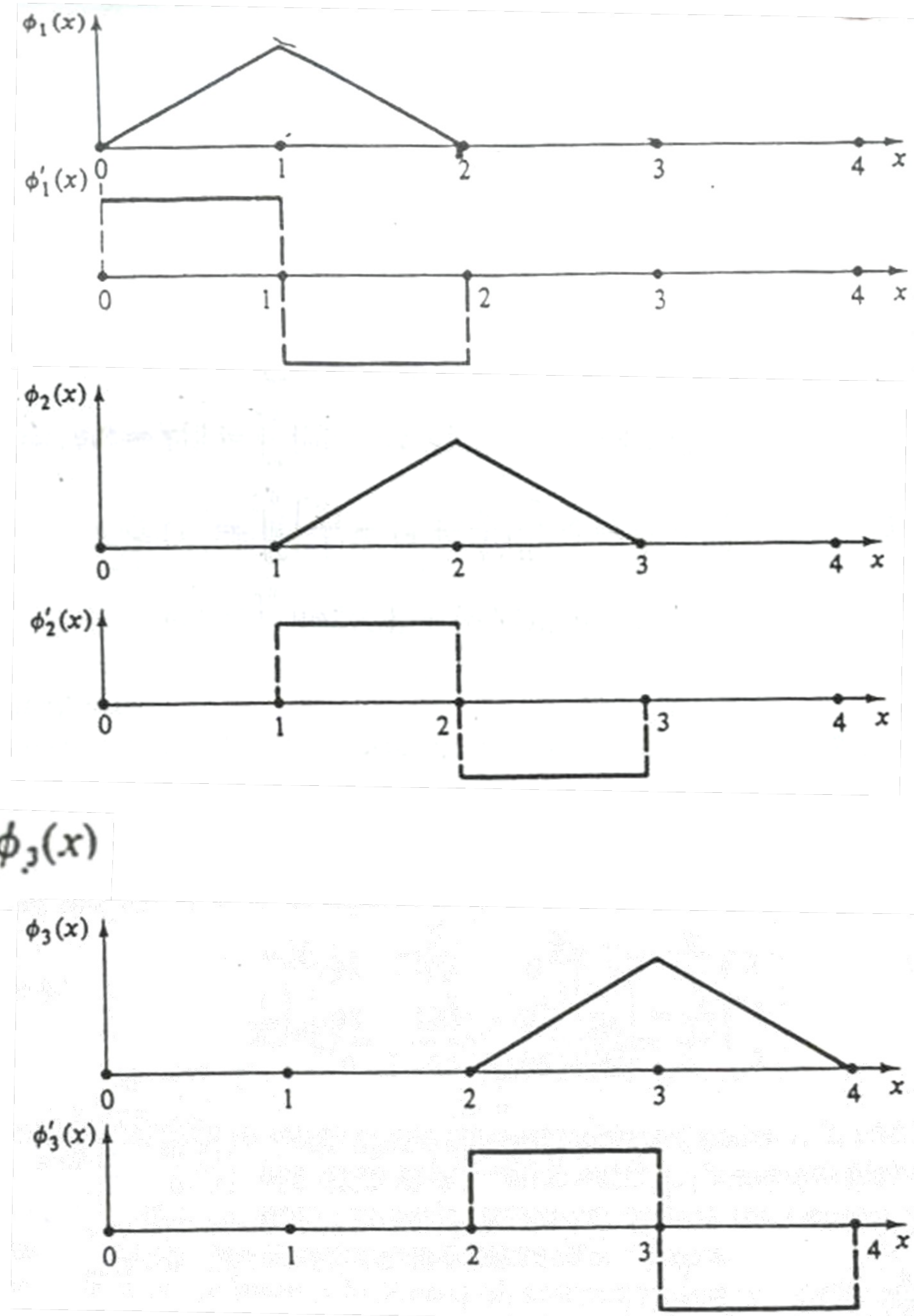
Final Linear System

$$\frac{1}{24}\begin{bmatrix} 196 & -95 & 0 \\ -95 & 196 & -95 \\ 0 & -95 & 196 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\mathbf{u} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.0353 \\ 0.0569 \\ 0.0505 \end{bmatrix}$$

$$u_h(x) = 0.0353\phi_1(x) + 0.0569\phi_2(x) + 0.0505\phi_3(x)$$

ϕ_i are “hat”-functions.



Overview

- What is Finite Element Method?
- A One-dimensional Model Problem
  - Boundary value problem (BVP)
  - Weak/variational formulation
  - Galerkin approximations
  - Basis functions
  - Finite element calculations
  - Flow chart
  - Interpretation of the approximate solution
  - Accuracy of the finite element approximation

Review: The Model Problem and Weak Form

- Consider the problem of finding a function $u = u(x)$, $0 \le x \le 1$, satisfying the following differential equation and boundary conditions:

$$\left.\begin{array}{ll} -u'' + u = f(x), & 0 < x < 1 \\ u(0) = 0, & u(1) = 0 \end{array}\right\}$$

where $u'' = d^2u/dx^2$, and $f(x) = x$.

- The weak statement becomes: find $u \in H_0^1$ such that

$$\int_0^1 \left(u'v' + uv - xv\right)dx = 0 \qquad \textit{for all } v \in H_0^1$$

Mechanical Engineering
Carnegie Mellon

Review: Galerkin Approximations

- Plug the approximate solution u_N and the test function v_N into the weak statement (finite-dimensional):

$$\int_0^1 \left(u_N' v_N' + u_N v_N\right) dx = \int_0^1 x v_N \, dx$$

$$u_N(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x) \qquad v_N(x) = \sum_{i=1}^{N} \beta_i \phi_i(x)$$

$$\int_0^1 \left\{ \frac{d}{dx}\left[\sum_{i=1}^{N} \beta_i \phi_i(x)\right] \frac{d}{dx}\left[\sum_{j=1}^{N} \alpha_j \phi_j(x)\right] + \left[\sum_{i=1}^{N} \beta_i \phi_i(x)\right]\left[\sum_{j=1}^{N} \alpha_j \phi_j(x)\right] - x \sum_{i=1}^{N} \beta_i \phi_i(x) \right\} dx = 0 \qquad \text{for all } \beta_i,\ i = 1, 2, \ldots, N$$

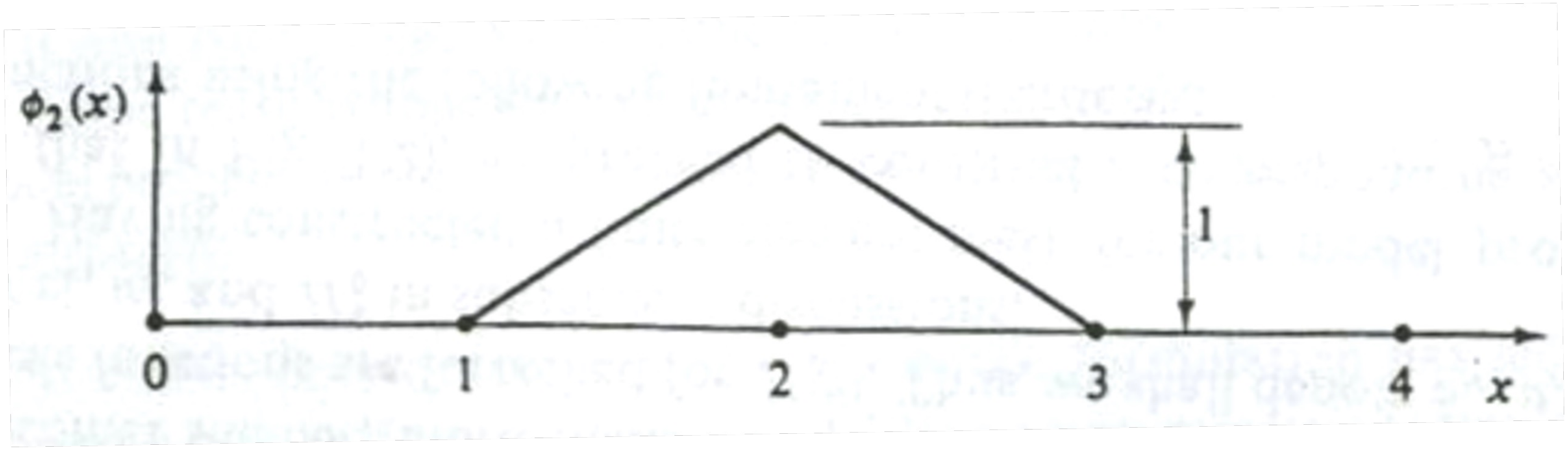
$$\sum_{j=1}^{N} K_{ij} \alpha_j = F_i, \qquad i = 1, 2, \ldots, N \quad \rightarrow \quad \alpha_j = \sum_{i=1}^{N} (K^{-1})_{ji} F_i$$

where $K_{ij} = \int_0^1 \left[\phi_i'(x)\phi_j'(x) + \phi_i(x)\phi_j(x)\right] dx$

$F_i = \int_0^1 x \phi_i \, dx \qquad i, j = 1, 2, \ldots, N$

Review: Three Criteria for Basis Functions

- The basis functions are generated by simple functions defined piecewise – element by element – over the finite element mesh.
- The basis functions are smooth enough to be members of the class H_0^1 of test functions.
- The basis functions are chosen in such a way that the parameters α_i defines the approximation solution precisely at the nodal points.



Mechanical Engineering
Carnegie Mellon

Review: Finite Element Calculations

- The weak form leads to the system of linear equations $\sum_{j=1}^{N} K_{ij}\alpha_j = F_i, \qquad i = 1, 2, \ldots, N$

- Let the term

$$K_{ij}^e = \int_{\Omega_e} (\phi_i'\phi_j' + \phi_i\phi_j)\,dx$$

represent components of the element stiffness matrix for finite element Ω_e.

$$K_{ij} = \sum_{e=1}^{4} K_{ij}^e$$

Similarly

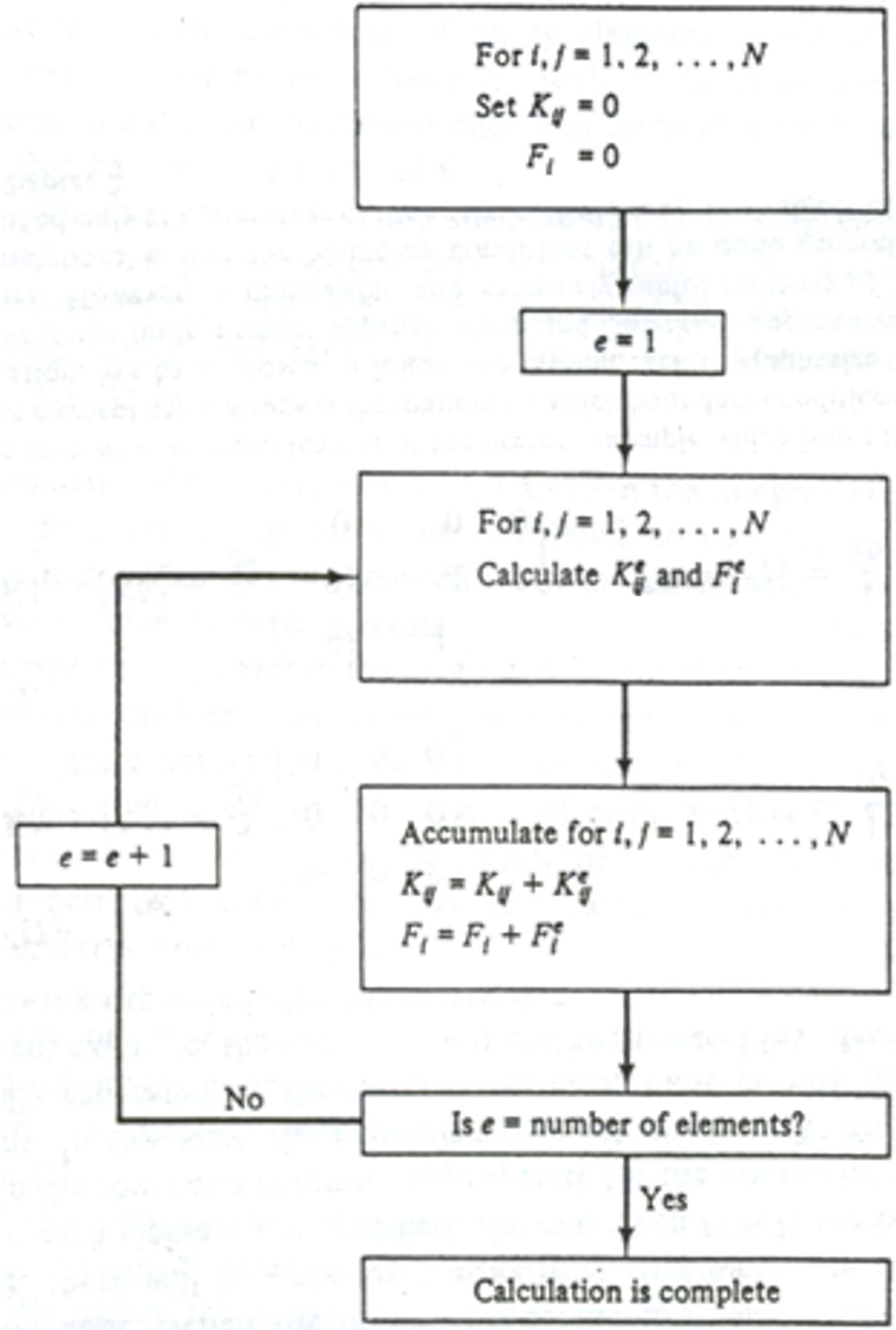
$$F_i = \sum_{e=1}^{4} F_i^e, \qquad F_i^e = \int_{\Omega_e} x\phi_i\,dx$$

Flow Chart

Flow chart shows element by element calculation and assembly of K and F.

$$K^e_{ij} = \int_{\Omega_e} (\phi'_i\phi'_j + \phi_i\phi_j)\, dx$$

$$F^e_i = \int_{\Omega_e} x\phi_i\, dx$$



Mechanical Engineering
Carnegie Mellon

Overview

- What is Finite Element Method?
- A One-dimensional Model Problem
  - Boundary value problem (BVP)
  - Weak/variational formulation
  - Galerkin approximations
  - Basis functions
  - Finite element calculations
  - Flow chart
  - Interpretation of the approximate solution
  - Accuracy of the finite element approximation

The Model Problem

- The weak form is: find $u_h \in H_0^h$ such that

$$\int_0^1 \left(u_h' v_h' + u_h v_h\right) dx = \int_0^1 x v_h \, dx \qquad \textit{for all } v_h \in H_0^h$$

- We have obtained the solution of the model problem. We can check graphs of the solution of its derivative.

$$\mathbf{u} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.0353 \\ 0.0569 \\ 0.0505 \end{bmatrix}$$

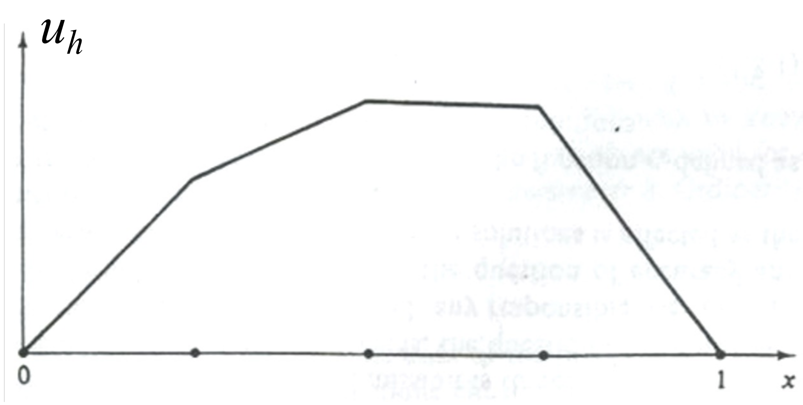
$$u_h(x) = 0.0353\phi_1(x) + 0.0569\phi_2(x) + 0.0505\phi_3(x)$$

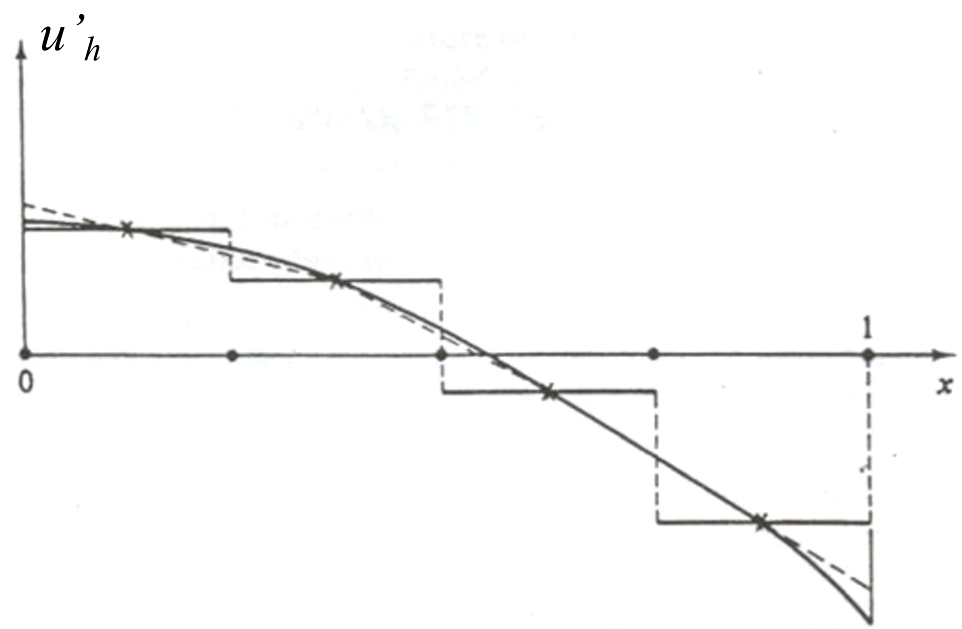
ϕ_i are "hat"-functions.

The Approximate Solution

$$u_h(x) = 0.0353\phi_1(x) + 0.0569\phi_2(x) + 0.0505\phi_3(x)$$

- The approximate solution is a fairly smooth function – there are no apparent oscillations or regions of very high gradients. (finer mesh gives better results)
- There is one extremum of the solution, a maximum value of 0.0569 at $x = 0.5$.
- The derivative of the solution is large near the endpoints, the largest absolute value occurring near $x = 1.0$.





Mechanical Engineering
Carnegie Mellon

Interpretation of The Approximate Solution

- Suppose the model problem has arisen in the analysis of a stretched string, supported on an elastic foundation and subjected to a transverse load whose distribution is given by $f(x) = x$.
- The solution u is the transverse deflection of the string and its derivative u' is proportional to the stress in the string.
- The total strain energy in the system (i.e., in the string and the elastic support) is given by

$$U = \frac{1}{2}\int_0^1 [(u')^2 + u^2]\, dx$$

Mechanical Engineering
Carnegie Mellon

Questions?

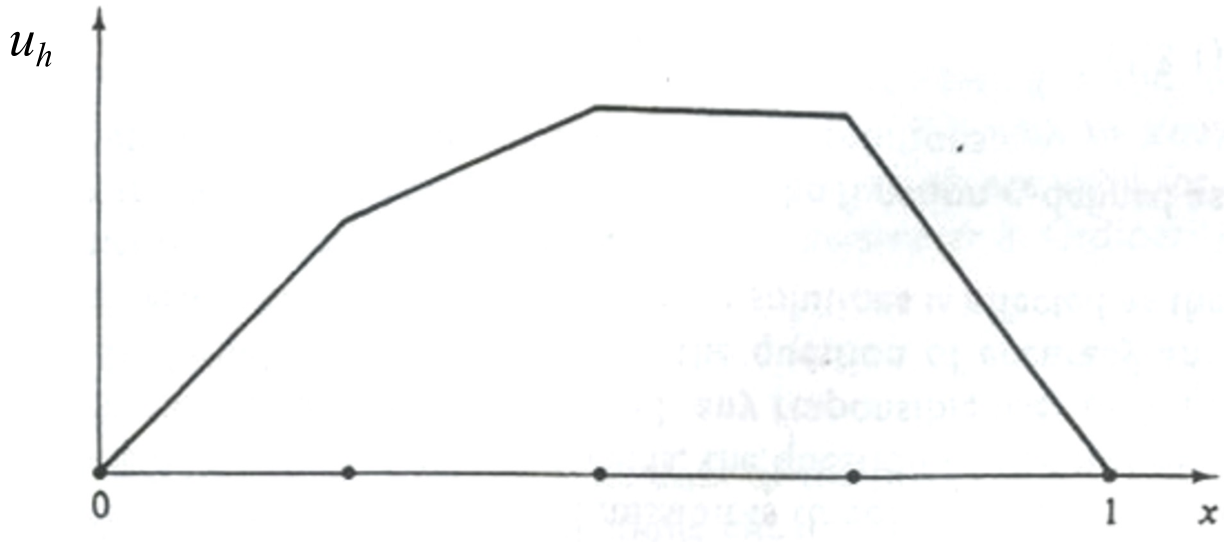
- What is the location and the value of the maximum deflection? (u)
- What is the location and the value of the maximum stress? (u')
- What is the value of total strain energy in the system? ($U = \frac{1}{2}\int_0^1 [(u')^2 + u^2]\,dx$)

Mechanical Engineering
Carnegie Mellon

Question 1

- *What is the location and the value of the maximum deflection?*

The maximal u_h is at node 2, $x = 0.5$, and $u_2 = 0.0569$.

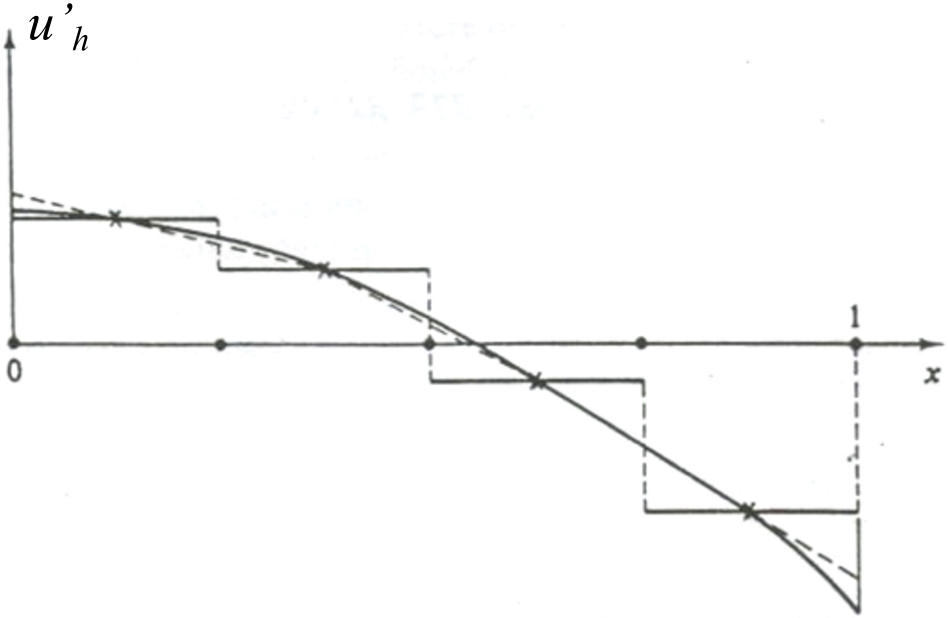


Question 2

- *What is the location and the value of the maximum stress?*

The maximum value of $|u'_h|$ is 0.202 and this value occurs throughout the element Ω_4. We assign it to the midpoint of the element (x = 0.875).

We expect the maximum stress to occur at the point x = 1, so we could extrapolate the values of stress to the boundary point at x = 1 (a smooth curve or piecewise straight lines)



Question 3

- *What is the value of total strain energy in the system?*

$$U = \frac{1}{2}\int_0^1 [(u')^2 + u^2]\, dx$$

Let ϕ denote the vector of basis functions and u the vector of nodal values of the solution,

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_2 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u_h(x) = \mathbf{u}^T \boldsymbol{\phi}(x)$$

$$\begin{aligned} U_h &= \frac{1}{2}\int_0^1 [(u_h')^2 + u_h^2]\, dx \\ &= \frac{1}{2}\mathbf{u}^T \int_0^1 (\boldsymbol{\phi}'^T \boldsymbol{\phi}' + \boldsymbol{\phi}^T \boldsymbol{\phi})\, dx\, \mathbf{u} \\ &= \frac{1}{2}\mathbf{u}^T \mathbf{K} \mathbf{u} \end{aligned}$$

$$\mathbf{K}\mathbf{u} = \mathbf{F}$$

$$\longrightarrow \quad \begin{aligned} U_h &= \frac{1}{2}\mathbf{u}^T \mathbf{F} \\ &= 0.0094 \end{aligned}$$

Interpretation of The Approximation Solution

- Calculations and observations indicate that a great deal of useful information can be extracted from a careful examination of properties of the solution.

Overview

- What is Finite Element Method?
- A One-dimensional Model Problem
  - Boundary value problem (BVP)
  - Weak/variational formulation
  - Galerkin approximations
  - Basis functions
  - Finite element calculations
  - Flow chart
  - Interpretation of the approximate solution
  - Accuracy of the finite element approximation

Approximation Accuracy

- How is the error in the approximate solutions effected as the number of elements in the mesh is increased?
- The error e is defined as the difference between the exact and the approximate solutions:

$$e(x) = u(x) - u_h(x).$$

energy norm:

$$||e||_E = \left\{\int_0^1 [(e')^2 + e^2]\, dx\right\}^{1/2}$$

mean-square (L^2) norm:

$$||e||_0 = \left(\int_0^1 e^2\, dx\right)^{1/2}$$

maximum norm:

$$||e||_\infty = \max_{0 \le x \le 1} |e(x)|$$

Mechanical Engineering
Carnegie Mellon

Asymptotic Estimates

- The error can be estimated although the exact solution is unknown.
- Suppose that the domain of the problem is discretized by a finite element mesh consisting of elements of equal length h. The error estimates will be of the form

$$\|e\| \leq Ch^{P}$$

where C is a constant depending upon the problem and generally p is a positive integer that depends on the chosen basis functions. When h tends to zero, then $||e||$ tends to zero.

a-*priori* Estimates and a-*posteriori* Estimates

$$\|e\| \leq Ch^{P}$$

- The above estimates require no information from the actual finite element solution, they are known prior to the construction of the solution, and are called a-*priori* estimates.

- More detailed estimates of accuracy can be based on information obtained from the finite element solution. Such estimates are called a-*posteriori* estimates.

a-*priori* Error Estimates

- In the case of our model problem, the following a-*priori* error estimates can be shown to hold for piecewise-linear basis functions:

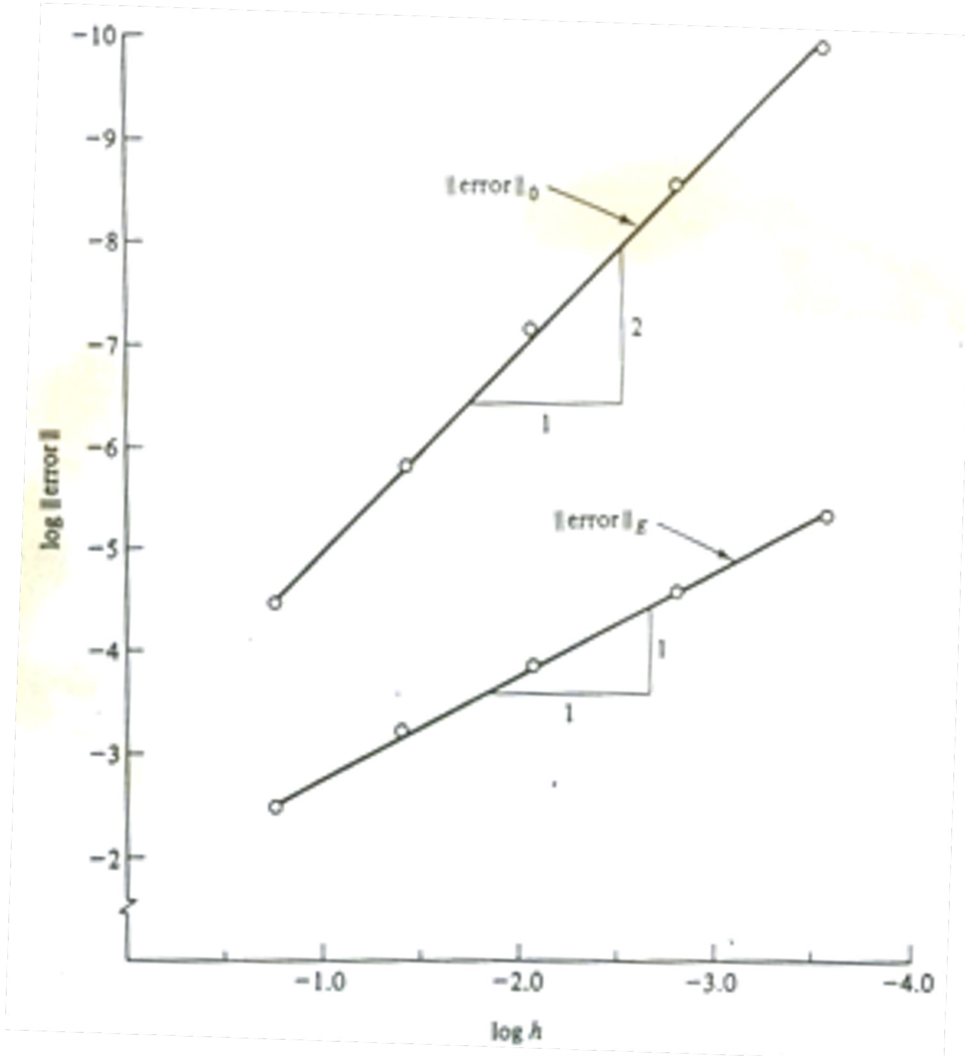
$$\left.\begin{aligned} \|e\|_E &\leq C_1 h \\ \|e\|_0 &\leq C_2 h^2 \\ \|e\|_\infty &\leq C_3 h^2 \end{aligned}\right\}$$

Log-log Plots of The Errors

- Considering the error estimates:

$$\|e\| \le E(h) = Ch^{P}$$

$$\log E(h) = p \log h + \log C$$



Summary

- What is Finite Element Method?
- A One-dimensional Model Problem
  - Boundary value problem (BVP)
  - Weak/variational formulation
  - Galerkin approximations
  - Basis functions
  - Finite element calculations
  - Flow chart
  - Interpretation of the approximate solution
  - Accuracy of the finite element approximation

Summary

- The same algorithm can be extended to 2D/3D.
- The finite element method has been used in a lot of research fields:
  - Mechanical Engineering
  - Civil Engineering and Architecture
  - Materials Science
  - Geological Engineering
  - Petroleum Engineering
  - Electrical Engineering and Computer Sciences
  - Biomedical Engineering
  - Biology, Chemistry, Physics, etc.

New Techniques of FEM

- XFEM: developed by Ted Belytschko in Northwestern University http://dilbert.engr.ucdavis.edu/~suku/xfem/
- Immersed FEM: developed by Wing Kam Liu in Northwestern University.
- Isogeometric Analysis: developed by Thomas J.R. Hughes in University of Texas at Austin. (Integration of CAD and FEM)

  http://users.ices.utexas.edu/~hughes/

Mechanical Engineering
Carnegie Mellon

References

- Y. J. Zhang. Geometric Modeling and Mesh Generation from Scanned Images. CRC Press, Taylor & Francis Group. 2016 (available free online through CMU Library)
- Finite Elements: A Second Course. J. T. Oden, G. F. Carey. Prentice Hall, Englewood Cliffs, 1983.
- The Finite Element Method – Linear Static and Dynamic Finite Element Analysis. T. J. R. Hughes. Dover Publishers, New York, 2000.
- http://en.wikipedia.org/wiki/Finite_element_method
- XFEM: developed by Ted Belytschko in Northwestern University http://dilbert.engr.ucdavis.edu/~suku/xfem/
- Immersed FEM: developed by Wing Kam Liu in Northwestern University.