# Topic 3: Geometric Modeling and Computer Graphics – Fundamentals

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# Spaces

- Computer graphics is concerned with the representation and manipulation of sets of geometric elements, such as points and line segments.
- We will review the rules governing three such spaces:
  - The (linear) vector space contains only two types of objects: scalars (such as real numbers) and vectors.
  - The affine space adds a third element: the point.
  - The Euclidean space adds the concept of distance.
- The vectors of interest in computer graphics are directed line segments and the *n*-tuples of numbers. Matrix algebra is a tool to manipulate *n*-tuples.

# Scalars

- Ordinary real numbers and the operations on them are one example of a scalar field.
- Let $S$ denote a set of elements called scalars, $\alpha, \beta, \cdots$ . Scalars have fundamental operations defined between pairs: addition and multiplication.

$$\forall \alpha, \beta \in S, \quad \alpha + \beta \in S, \quad \alpha \cdot \beta \in S$$

- These operations are commutative, associative, and distributive, $\forall \alpha, \beta, \gamma \in S$

$$\alpha + \beta = \beta + \alpha$$
$$\alpha \cdot \beta = \beta \cdot \alpha$$
$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$
$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$$
$$\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$$

If no ambiguity:

$$\alpha\beta \Leftrightarrow \alpha \cdot \beta$$

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# Scalars

- Two special scalars: the additive inverse (0) and the multiplicative inverse (1) – such that $\forall\, \alpha \in S$

$$\alpha + 0 = 0 + \alpha = \alpha$$
$$\alpha \cdot 1 = 1 \cdot \alpha = \alpha$$

- Each element has an additive inverse $-\alpha$ and a multiplicative inverse $\alpha^{-1}$, such that

$$\alpha + (-\alpha) = 0$$
$$\alpha \cdot \alpha^{-1} = 1$$

- The real numbers using ordinary addition and multiplication form a scalar field.

# Vector Spaces

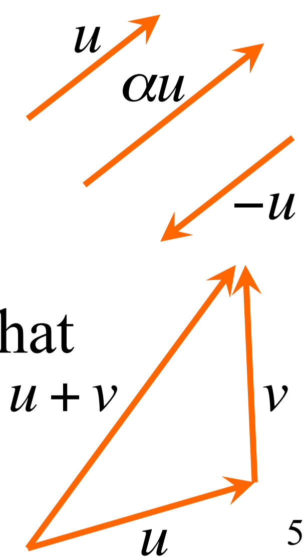
- A vector space, in addition to scalars, contains a 2nd type of entity: vectors.
- Vectors have two operations defined: vector-vector addition, scalar-vector multiplication.
- Let $u$, $v$, $w$ denote vectors in a vector space $V$. Vector addition is defined to be
  - Closed: $u + v \in V, \quad \forall u, v \in V$
  - Commutative: $u + v = v + u$
  - Associative: $u + (v + w) = (u + v) + w$

  A special vector 0 is defined such that $\forall u \in V$:

$$u + 0 = u$$

  Every vector $u$ has an additive inverse $-u$ such that

$$u + (-u) = 0$$



# Vector Spaces

- Scalar-vector multiplication is defined such that, for any scalar $\alpha$ and any vector $u$, $\alpha u$ is a vector in $V$. The scalar-vector operation is distributive:

$$\alpha(u+v) = \alpha u + \alpha v$$

$$(\alpha + \beta)u = \alpha u + \beta u$$

- $n$-tuples of scalars (real or complex numbers):

$$v = (v_1, v_2, \cdots, v_n)$$

Vector-vector addition and scalar-vector multiplication are

$$u + v = (u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n)$$
$$= (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$

$$\alpha v = (\alpha v_1, \alpha v_2, \cdots, \alpha v_n)$$

This space is denoted as $R^n$ and is the vector space in which we can manipulate vectors using matrix algebra.

# Vector Spaces

- A linear combination of $n$ vectors $u_1, u_2, \cdots, u_n$ is a vector of the form

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

- If the only set of scalars such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then the vectors are said to be linearly independent ($n$ is the dimension of the space) and they form a basis.

- If $v_1, v_2, \cdots, v_n$ is a basis for $V$, any vector $v$ can be expressed uniquely in terms of the basis vectors

$$v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$$

# Vector Spaces

- If $v'_1, v'_2, \cdots, v'_n$ is some other basis for $V$, there is a representation of $v$ in terms of the basis vectors

$$v = \beta'_1 v'_1 + \beta'_2 v'_2 + \cdots + \beta'_n v'_n$$

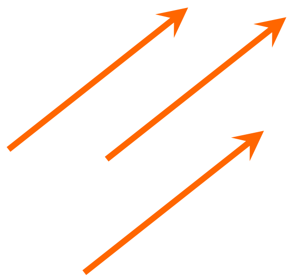
- There exists an $n$x$n$ matrix $M$ such that

$$\begin{bmatrix} \beta'_1 \\ \beta'_2 \\ \vdots \\ \beta'_n \end{bmatrix} = M \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

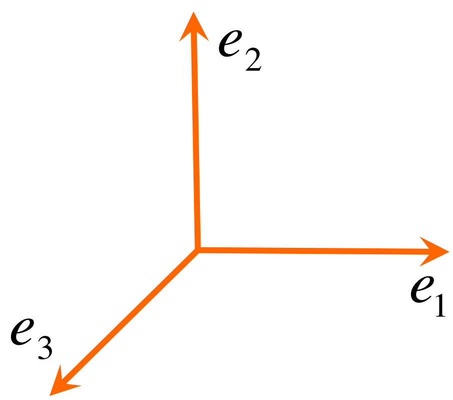
This matrix gives a way of changing representations through a simple linear transformation involving only scalar operations for carrying out matrix multiplication.

# Affine Spaces

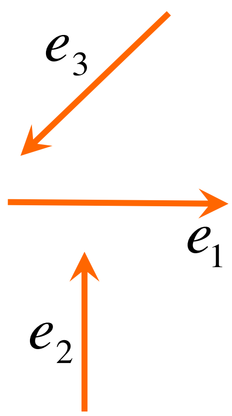
- A vector space lacks any geometric concepts such as location and distance. Vectors have only magnitude and direction, but have no position.
- A vector can be expressed in terms of a set of basis vectors that defines a coordinate system.



identical vectors




basis vectors located
at the origin




arbitrary placement
of basis vectors

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# Affine Spaces

- An affine space introduces a third type of entity to a vector space, points. ($P$, $Q$, $R$, …)
- Point-point subtraction yields a vector in $V$:

$$v = P - Q$$

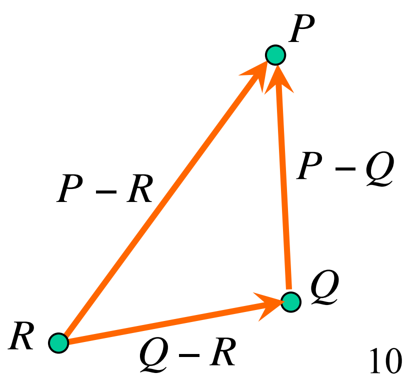
For every $v$ and $Q$, we can find a $P$ such that

$$P = v + Q$$

defining a vector-point addition.

- A consequence of the head-to-tail axiom is that for any three points $P$, $Q$, $R$,

$$(P - Q) + (Q - R) = P - R$$

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# Affine Spaces

- A frame consists of a point $P_0$, and a set of vector $v_1, v_2, \cdots, v_n$ that defines a basis for the vector space. Given a frame, an arbitrary vector can be written uniquely as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

and an arbitrary point can be written uniquely as

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$$

The two sets of scalars, $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\beta_1, \cdots, \beta_n\}$, give the representations of the vector and point. $P_0$ can be regarded as the origin of the frame, and all points are defined from this reference point.

# Euclidean Spaces

- Given scalars $(\alpha, \beta, \gamma, \cdots)$ and vectors $(u, v, w, \cdots)$, Euclidean spaces introduce the inner (dot) product, which combines two vectors to form a real.

$$u \cdot v = v \cdot u$$

$$(\alpha u + \beta v) \cdot w = \alpha u \cdot w + \beta v \cdot w$$

$$v \cdot v > 0 \quad if \quad v \neq 0 \qquad\qquad 0 \cdot 0 = 0$$

If $u \cdot v = 0$, then $u$ and $v$ are orthogonal. The magnitude (length) of a vector is usually measured as $|v| = \sqrt{v \cdot v}$

- For any two points $P$ and $Q$, $P - Q$ is a vector,

$$|P - Q| = \sqrt{(P - Q) \cdot (P - Q)}$$

- The angle between two vectors:

$$u \cdot v = |u||v| \cos\theta \qquad \cos\theta = \frac{u \cdot v}{|u||v|} = \begin{cases} 0 & u, v \ are \ orthogonal \\ 1 & u, v \ are \ parallel \end{cases}$$

# Projections

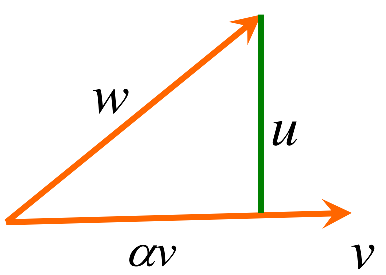
- The concept of projection arises from finding the shortest distance from a point to a line or plane. Given two vectors *v* and *w*, we can take one of them (*w*) and divide it into two parts: one parallel and one orthogonal to the other vector (*v*).

$$w = \alpha v + u$$

$$u \cdot v = 0$$

$$w \cdot v = (\alpha v + u) \cdot v = \alpha v \cdot v + u \cdot v = \alpha v \cdot v$$

$$\alpha = \frac{w \cdot v}{v \cdot v}$$




- The vector $\alpha v$ is the projection of *w* onto *v*,

$$u = w - \frac{w \cdot v}{v \cdot v} v$$

# Gram-Schmidt Orthogonalization

- Given a set of basis vectors, $a_1, a_2, \cdots, a_n$ , in a space of dimension *n*, it is relatively straightforward to create another basis $b_1, b_2, \cdots, b_n$ that is orthonormal, that is, a basis in which each vector has unit length and is orthogonal to each other in the basis,

$$b_i \cdot b_j = \begin{cases} 1 & if\ i = j \\ 0 & otherwise \end{cases}$$

- We process iteratively. We look for a vector in the form

$$b_2 = a_2 + \alpha b_1$$

which can make orthogonal to $b_1$ by choosing $\alpha$ properly.

$$b_2 \cdot b_1 = 0 = a_2 \cdot b_1 + \alpha b_1 \cdot b_1$$

$$\alpha = -\frac{a_2 \cdot b_1}{b_1 \cdot b_1} = -a_2 \cdot b_1 \qquad b_2 = a_2 - \frac{a_2 \cdot b_1}{b_1 \cdot b_1} b_1 = a_2 - (a_2 \cdot b_1) b_1$$

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# Gram-Schmidt Orthogonalization

- We have constructed the orthogonal vector by removing the part parallel to $b_1$, or the projection of $a_2$ onto $b_1$.
- The general iterative step is to find a

$$b_k = a_k + \sum_{i=1}^{k-1} \alpha_i b_i$$

that is orthogonal to $b_1, \ldots, b_{k-1}$. There are $k$-1 orthogonal conditions that allow us to find

$$\alpha_i = -\frac{a_k \cdot b_i}{b_i \cdot b_i}$$

- Each vector should be normalized by replacing $b_i$ by $b_i / | b_i |$.

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# Matrices

- A matrix is an $n \times m$ array of scalars, arranged conceptually as $n$ rows and $m$ columns. $n$ and $m$ are referred as the row and column dimensions of the matrix. If $m=n$, then the matrix is a square matrix of dimension $n$. We write the matrix A in terms of its elements:

$$A = \left[ a_{ij} \right] \qquad i = 1,\cdots,n \quad and \quad j = 1,\cdots,m$$

# Matrices

- The transpose of an $n \times m$ matrix $A$ is the $m \times n$ matrix that can be obtained by interchanging the rows and columns of $A$. We denote it as $A^{\mathrm{T}}$.

$$A^T = \left[ a_{ji} \right] \qquad i = 1, \cdots, n \quad and \quad j = 1, \cdots, m$$

- Matrix with one column ($n \times 1$) is column matrix, and matrix with one row ($1 \times m$) is row matrix.

Column matrix: $b = \left[ b_i \right]$

Row matrix: $b^T$

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# Matrix Operations

- Three basic matrix operations:
  - Scalar-matrix multiplication
  - Matrix-matrix addition
  - Matrix-matrix multiplication
- Scalar-matrix multiplication: defined for any size matrix $A$, and simply the element-by-element multiplication of the elements by a scalar $\alpha$.

$$\alpha A = \left[ \alpha a_{ij} \right]$$

# Matrix Operations

- **Matrix-matrix addition**: adding the corresponding elements of the two matrices, which have the same dimensions.

$$C = A + B = \left[ a_{ij} + b_{ij} \right]$$

- **Matrix-matrix multiplication**: the product of an $n \times l$ matrix $A$ by an $l \times m$ matrix $B$ is the $n \times m$ matrix. The number of columns of $A$ should be the same as the number of rows of $B$.

$$C = AB = \left[ c_{ij} \right] \qquad c_{ij} = \sum_{k=1}^{l} a_{ik} b_{kj}$$

# Matrix Operations

- Scalar-matrix multiplication obeys a number of simple rules that hold for any matrix A and scalars $\alpha$ and $\beta$ :

$$\alpha(\beta A) = (\alpha\beta)A$$

$$\alpha\beta A = \beta\alpha A$$

- Matrix-matrix addition has the commutative property. For any $n \times m$ matrices $A$ and $B$:

$$A + B = B + A$$

Associative property: for any three $n \times m$ matrices $A$, $B$ and $C$:

$$A + (B + C) = (A + B) + C$$

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# Matrix Operations

- Matrix-matrix multiplication is associative:

$$A(BC) = (AB)C$$

Not commutative:

$$AB \neq BA$$

In graphics application, matrices represent transformations such as translation and rotation. The order of transformation is important, and not commutative. A rotation followed by a translation is not the same as a translation followed by a rotation.

# Identity Matrix

- The identity matrix $I$ is a square matrix with 1s on the diagonal and 0s elsewhere:

$$I = \left[a_{ij}\right] \qquad a_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & elsewhere \end{cases}$$

$$AI = A$$

$$IB = B$$

# Row and Column Matrices

- A vector or a point in 3D can be represented as the column matrix, which is denoted by lowercase letters.

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- The transpose of $p$ is the row matrix:

$$p^T = \begin{bmatrix} x & y & z \end{bmatrix}$$

- The product of a square matrix of dimension $n$ and a column matrix of dimension $n$ is a new column matrix of dimension $n$. A square matrix is used to represent a transformation of the point (or vector):

$$p' = Ap$$

- $p' = ABCp$ describe sequences of transformations.

$$(AB)^T = B^T A^T \qquad\qquad p'^T = p^T C^T B^T A^T$$

# Rank

- For a matrix $A$, if a $B$ exists s.t. $BA=I$, then $B$ is the inverse of $A$ and $A$ is nonsingular. A noninvertible matrix is singular.

$$B = A^{-1}$$

- The inverse of a square matrix exists if and only if the determinant of the matrix is nonzero. The computational complexity of determinant calculation is $O(n^3)$ for an $n$-dimensional matrix, and the rank is $n$.
- For nonsquare matrices, the row (column) rank is the maximum number of linearly independent rows (columns).

# Change of Representation

- Suppose we have a vector space of dimension $n$. Let $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ be two bases for the vector space.
Hence a given vector $v$ can be expressed as either

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

or

$$v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$$

- We write the representation of $v$ as either

$$v = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}^T$$

or

$$v' = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix}^T$$

depending on which basis we use.

# Change of Representation

- Now we can address how to convert *v* to *v'*. The basis vectors $\{v_1, v_2, \cdots, v_n\}$ can be expressed as vectors in the basis $\{u_1, u_2, \cdots, u_n\}$. Thus, there exists a set of scalars $\gamma_{ij}$ such that

$$u_i = \gamma_{i1} v_1 + \gamma_{i2} v_2 + \cdots + \gamma_{in} v_n, \qquad i = 1, \cdots, n$$

We can write the expression in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = A \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

where $A$ is the $n \times n$ matrix:

$$A = \left[ \gamma_{ij} \right]$$

# Change of Representation

- We can use column matrices to express *v* and *v'*:

$$v = a^T \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad a = [\alpha_i]$$

$$v' = b^T \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad b = [\beta_i]$$

$$b^T = a^T A$$

- The matrix *A* is the matrix representation of the change between two bases.

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# The Cross Product

- Given two nonparallel vectors, $u$ and $v$, in a 3D space, the cross product gives a third vector, $w$, that is orthogonal to both.

$$w \cdot u = w \cdot v = 0$$

- Within a particular coordinate system, if $u$ has components $\alpha_1, \alpha_2, \alpha_3$ and $v$ has components $\beta_1, \beta_2, \beta_3$, then the cross product is

$$w = u \times v = \begin{bmatrix} \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ \alpha_3 \beta_1 - \alpha_1 \beta_3 \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 \end{bmatrix}$$

- Right-handed coordinate system: the cross product of $x$-axis and $y$-axis is along the $z$-axis.

# Eigenvalues and Eigenvectors

- $\lambda$ - eigenvalue, $u$ – eigenvector

$$Mu = \lambda u$$

$$Mu - \lambda u = Mu - \lambda I u = (M - \lambda I)u = 0$$

- The above equation can have a nontrivial solution if and only if the determinant

$$|M - \lambda I| = 0$$

# Eigenvalues and Eigenvectors

- If $M$ is $n \times n$, then the determinant yields a polynomial of degree $n$ , and there are $n$ roots. For each eigenvalue, we can find a corresponding eigenvector. If all the eigenvalues are distinct, then any set of corresponding eigenvectors form a basis for an $n$-dimensional vector space.
- Suppose $T$ is a nonsingular matrix, consider

$$Q = T^{-1}MT$$

$$Qv = T^{-1}MTv = \lambda v \longrightarrow MTv = \lambda Tv$$

The eigenvalues of $Q$ are the same as those of $M$, and the eigenvectors are the transformations of the eigenvectors of $M$. The matrices $M$ and $Q$ are said to be similar.

# Eigenvalues and Eigenvectors

- Eigenvalues and eigenvectors have a geometric interpretation. Consider an ellipsoid, centered at the origin, with its axes aligned with the coordinate axes.

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 1$$

or in matrix form ( $\lambda_1, \lambda_2, \lambda_3$ are positive):

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$$

- $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the diagonal matrix, they are also the inverse of the lengths of the major and minor axes of the ellipsoid. If we rotate the ellipsoid, then a new ellipsoid is created, but the length of axes of the ellipsoid stays the same.

# References


- Interactive Computer Graphics: A top-down approach using OpenGL. Edward Angel, 3rd Edition. Pearson Edition.