Topic 3: Geometric Modeling and Computer Graphics – Geometric Objects and Transformations

Jessica Zhang
Department of Mechanical Engineering
Courtesy Appointment in Biomedical Engineering
Carnegie Mellon University
jessicaz@andrew.cmu.edu
http://www.andrew.cmu.edu/user/jessicaz

Mechanical Engineering
Carnegie Mellon

Scalars, Points and Vectors

- In computer graphics, we work with sets of geometric objects, such as lines, polygons, and polyhedra.
- These basic geometric objects and the relationship among them can be described using three fundamental types: scalars, points and vectors.

Scalars, Points and Vectors

- A point is a location in the space. A mathematical point has neither a size nor a shape.
- A scalar is a number to specify quantities such as distance (real and complex numbers). Addition and multiplication are defined for scalars, and they are commutative and associative.
- A vector is a quantity with direction and magnitude, such as velocity and force. A vector does not have a fixed position in space. A vector is also called a directed line segment.
  - Scalar-vector multiplication
  - Vector-vector addition

The Geometric View

- Vector has two operations: the addition of two vectors (head-to-tail) and the multiplication of a vector by a scalar.
- Zero vector has a magnitude of zero, and the orientation is undefined.
- u is a vector, its inverse vector is $-u$, which has the same length but opposite directions.
- Point-vector addition: a directed line segment is moved from one point to another.
- Point-point subtraction forms a line segment or vector.

The Mathematical View

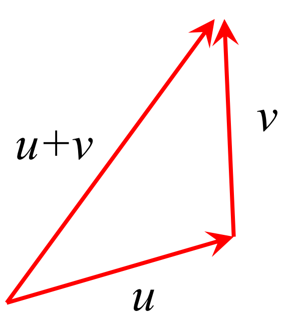
- Vector spaces have two distinct types of entities: vectors and scalars.
  - Scalar-vector multiplication
  - Vector-vector addition
- Euclidean spaces: introduce distance or a measure of size.
- Affine spaces: introduce the point.
  - Vector-point addition to produce a new point.
  - Point-point subtraction to produce a vector.

Geometric Abstract Data Types

- Scalar, point and vector are three Abstract Data Types (ADTs)
- Notation: Greek letters $\alpha, \beta, \gamma, \cdots$ denote scalars, upper-case letters P, Q, R, … define points, and lower-case letters u, v, w, … denote vectors.
- Vector-scalar multiplication: the direction of αv is the same as v if α is positive and the opposite direction if α is negative.

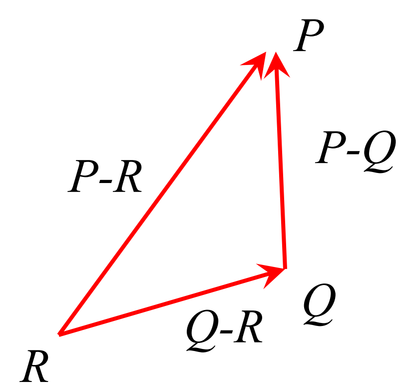
$$|\alpha v| = |\alpha||v|$$

- Point-point subtraction yields a vector: $v = P\text{-}Q$ and $P = v+Q$
- Vector-vector addition (head-to-tail rule):



$$u+v$$

$$(P\text{-}Q)+(Q\text{-}R)=P\text{-}R$$

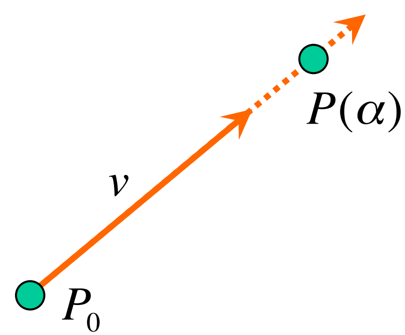


Lines

- The sum of a point and a vector leads to the notion of a line in an affine space.

$$P(\alpha) = P_0 + \alpha v$$

- Given a scalar value α, the function $P(\alpha)$ yields a point. This form is known as the parametric form of the line because we generate points on the line by varying the parameter α.



Mechanical Engineering
Carnegie Mellon

Affine Sums

- For any point Q, vector v, and positive scalar α ,

$$P = Q + \alpha v$$

describes all points on the line from Q in the direction of v. However, we can always find a point R such that

$$v = R - Q$$

thus:

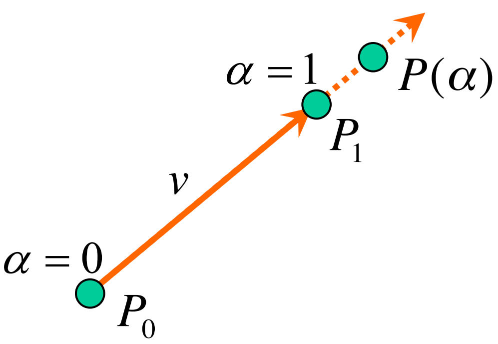
$$P = Q + \alpha(R - Q) = \alpha R + (1 - \alpha)Q$$

or:

$$P = \alpha_1 R + \alpha_2 Q$$

where

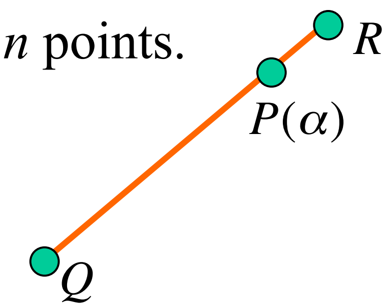
$$\alpha_1 + \alpha_2 = 1$$



Mechanical Engineering
Carnegie Mellon

Convexity

- A convex object is one for which any point lying on the line segment connecting any two points in the object is also in the object.
- We can use affine sums to help us gain a deeper understanding of convexity. For $0 \le \alpha \le 1$, the affine sum defines the line segment connecting *R* and *Q*, thus this line segment is a convex object.
- We can extend the affine sum to include objects defined by *n* points. Consider the form



$$P = \alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_n P_n$$

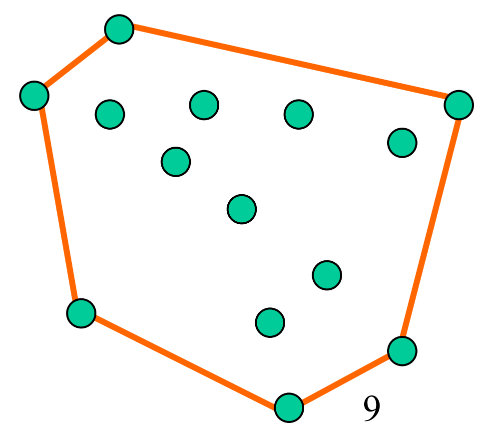
We can show, by induction, that this sum is defined iff

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

The set of points formed by the affine sum of *n* points, under the additional restriction

$$\alpha_i \ge 0, \qquad i = 1,2,\cdots,n$$

is called the convex hull of the set of points. It is the smallest convex object that includes the set of points.



Convex hull

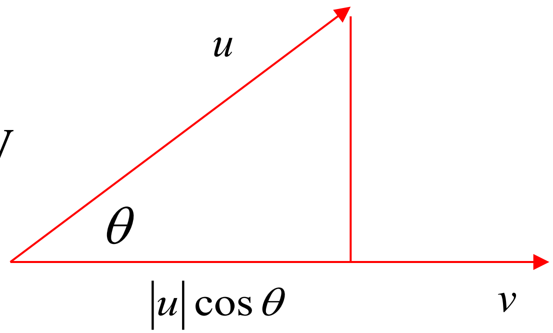
Dot Product

- The dot product of u and v is written $u \cdot v$. If $u \cdot v = 0$, u and v are said to be orthogonal. In an Euclidean space, the magnitude of a vector is defined. The square of the magnitude of a vector is given by the dot product

$$|u|^2 = u \cdot u$$

The angle between two vectors is given by

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$



In addition, $|u|\cos\theta = u \cdot v / |v|$ is the orthogonal projection of u onto v. The dot product expresses the geometric result that the shortest distance from the end point of u to the line segment v.

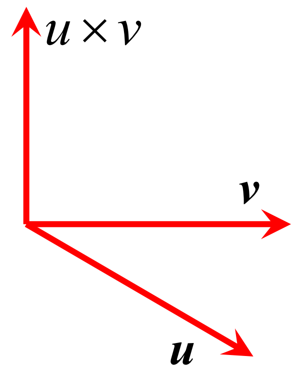
Cross Product

- In vector space, vectors are linearly independent if we can't write one in terms of the others using scalar-vector addition. The dimension of the vector space is the maximum number of linearly independent vectors in this space.
- Given two nonparallel vectors u and v, the cross product is to determine a vector w which is orthogonal to both of them.

$$w = u \times v$$

Then u, v, and w are mutually orthogonal.

$$|\sin\theta| = \frac{|u \times v|}{|u||v|}$$



Mechanical Engineering
Carnegie Mellon

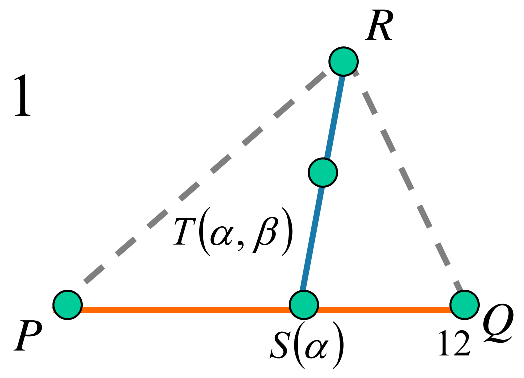
Planes

- A plane in an affine space can be defined as a direct extension of the parametric line.
- Three points, not on the same line, determine a unique plane. Suppose P, Q, and R are three such points, the line segment that joins P and Q is the set of points of the form

$$S(\alpha) = \alpha P + (1-\alpha)Q, \qquad 0 \le \alpha \le 1$$

Suppose that we take an arbitrary point on this line segment and form the line segment from this point to R. Using a second parameter β, we can describe points along this line segment as

$$T(\beta) = \beta S + (1-\beta)R, \qquad 0 \le \beta \le 1$$



Planes

- Combining the preceding two equations, we obtain one parametric form of the equation of a plane:

$$T(\alpha,\beta) = \beta[\alpha P + (1-\alpha)Q] + (1-\beta)R$$
$$= P + \beta(1-\alpha)(Q-P) + (1-\beta)(R-P)$$

Noting that *Q-P* and *R-P* are arbitrary vectors. We have shown that a plane can also be expressed in terms of points and two nonparallel vectors:

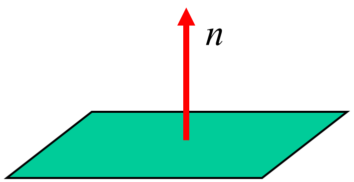
$$T(\alpha,\beta) = P_0 + \alpha u + \beta v$$

We can observe that, for $0 \le \alpha, \beta \le 1$, all the points $T(\alpha,\beta)$ lie in the triangle formed by *P*, *Q*, and *R*. If a point *P'* lies in the plane, then

$$P' - P_0 = \alpha u + \beta v$$

- The normal of the plane can be obtained by the cross product.

$$n = u \times v \qquad\qquad n \cdot (P - P_0) = 0$$



Coordinate Systems and Frames

- In a 3D vector space, we can represent any vector w uniquely in terms of any three linearly independent vectors, v_1, v_2, v_3 :

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

The scalars $\alpha_1, \alpha_2, \alpha_3$ are the components of w w.r.t the basis v_1, v_2, v_3 :

$$w = a^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \text{where} \qquad a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

- The basis vectors v_1, v_2, v_3 define a coordinate system.
- Both the origin and the basis vectors determine a frame. Within a given frame, every vector can be written uniquely as:

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

Every point can be written uniquely as: $P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$

Representations and N-tuples

- Suppose that vectors e_1, e_2, e_3 form a basis. A vector v can be represented by components $(\alpha_1, \alpha_2, \alpha_3)$.

$$v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$$

where $e_1 = (1,0,0)^T \qquad e_2 = (0,1,0)^T \qquad e_3 = (0,0,1)^T$

- We can write the representation of any vector v as a column matrix a or as the 3-tuple $(\alpha_1, \alpha_2, \alpha_3)$ where

$$v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$$

- These 3-tuples form a vector space known as the Euclidean R^3 that is equivalent (or homomorphic) to the vector space of our original geometric vectors.

Changes of Coordinate Systems

- Suppose that $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, u_3\}$ are two bases. Each basis vector in the 2nd set can be represented in terms of the 1st basis (and vice versa). Hence, there exist nine scalar components $\{\gamma_{ij}\}$ s.t.

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$
$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$
$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$

The 3 x 3 matrix:

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Changes of Coordinate Systems

- Suppose a vector w is represented in terms of two bases: $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$.

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$w = a^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{where} \quad a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$w = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3$$

$$w = b^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{where} \quad b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$w = b^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = b^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = a^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$a = M^T b \qquad\qquad b = \left(M^T\right)^{-1} a$$

Mechanical Engineering
Carnegie Mellon

Example of Change of Representation

- Suppose a vector w is represented in terms of $\{v_1, v_2, v_3\}$

$$w = a^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 + 2v_2 + 3v_3 \qquad a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- A new basis $\{u_1, u_2, u_3\}$

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= v_1 + v_2 \\ u_3 &= v_1 + v_2 + v_3 \end{aligned} \qquad M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A = \left(M^T\right)^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad b = Aa = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$w = -u_1 - u_2 + 3u_3$$

Homogeneous Coordinates

- A point P located at (x, y, z) is represented using a 3D frame defined by P_0, v_1, v_2, v_3

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{or} \quad P = P_0 + xv_1 + yv_2 + zv_3$$

- Homogeneous coordinates: use 4D column matrix to represent points and vectors in 3D. In the frame specified by $\{v_1, v_2, v_3, P_0\}$, any point P can be written uniquely as

$$P = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + P_0$$

$$P = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$0 \cdot P = 0$$

$$1 \cdot P = P$$

Homogeneous Coordinates

- P is represented by the column matrix

$$P = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix}$$

- In the same frame, any vector w can be written

$$w = \delta_1 v_1 + \delta_2 v_2 + \delta_3 v_3 = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

Mechanical Engineering
Carnegie Mellon

Homogeneous Coordinates

- If $\{v_1, v_2, v_3, P_0\}$ and $\{u_1, u_2, u_3, Q_0\}$ are two frames, then we can express the basis vectors and reference point of the 2nd frame in terms the 1st as:

$$u_1 = \gamma_{11} v_1 + \gamma_{12} v_2 + \gamma_{13} v_3$$

$$u_2 = \gamma_{21} v_1 + \gamma_{22} v_2 + \gamma_{23} v_3$$

$$u_3 = \gamma_{31} v_1 + \gamma_{32} v_2 + \gamma_{33} v_3$$

$$Q_0 = \gamma_{41} v_1 + \gamma_{42} v_2 + \gamma_{43} v_3 + P_0$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} \qquad M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

$$b^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = b^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = a^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$a = M^T b$$

M is called the matrix representation of the change of frames.

Example of Change in Frames

- If $\{v_1, v_2, v_3, P_0\}$ and $\{u_1, u_2, u_3, Q_0\}$ are two frames, then we can express the basis vectors and reference point of the 2nd frame in terms of the 1st as:

$$
\begin{aligned}
u_1 &= v_1 \\
u_2 &= v_1 + v_2 \\
u_3 &= v_1 + v_2 + v_3 \\
Q_0 &= P_0
\end{aligned}
$$

$$
\begin{aligned}
u_1 &= v_1 \\
u_2 &= v_1 + v_2 \\
u_3 &= v_1 + v_2 + v_3 \\
Q_0 &= v_1 + 2v_2 + 3v_3 + P_0
\end{aligned}
$$

$$
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

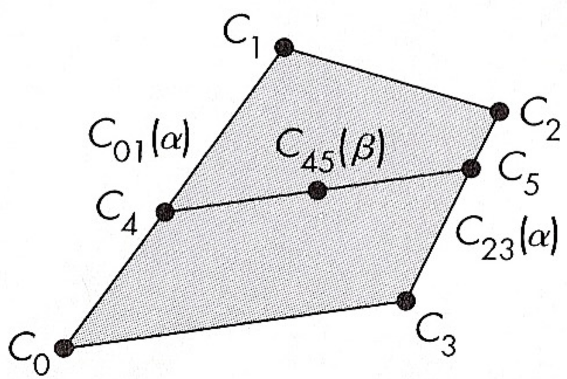
$$
M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix}
$$

Bilinear Interpolation

- The colors C_0, C_1, C_2, C_3 are the ones assigned to the vertices. We can use linear interpolation to interpolate color along the edges between vertices 0 and 1, and between 2 and 3.

$$C_{01}(\alpha) = (1-\alpha)C_0 + \alpha C_1,$$
$$C_{23}(\alpha) = (1-\alpha)C_2 + \alpha C_3.$$

$$C_{45}(\beta) = (1-\beta)C_4 + \beta C_5.$$



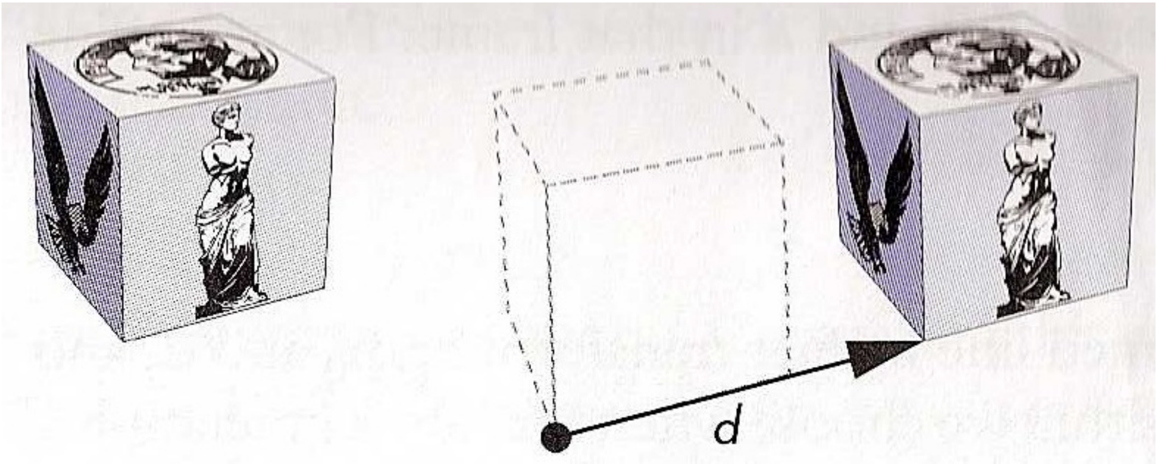
- For a flat quadrilateral, each color generated by this method corresponds to a point on the polygon. If the four vertices are not all in the same plane, then, although a color is generated, its location on a surface is not clearly defined.

Mechanical Engineering
Carnegie Mellon

Translation

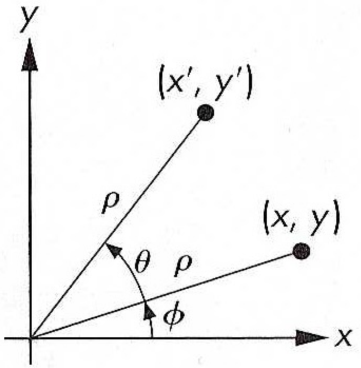
- Translation is an operation that displaces points by a fixed distance in a given direction.
- To specify a translation, we need only to specify a displacement vector *d*.
- This definition makes no reference to a frame or representation.
- Translation has 3 degrees of freedom.

$$P' = P + d$$



Rotation

- A particular point needs to be specified as the origin. A two-dimensional point at (x, y) in this frame is rotated about the origin by an angle θ to the position (x', y'). We can obtain the standard equations describing this rotation by representing (x, y) and (x', y') in polar form:



$$x = \rho \cos \phi,$$

$$y = \rho \sin \phi,$$

$$x' = \rho \cos(\theta + \phi),$$

$$y' = \rho \sin(\theta + \phi).$$

Expanding these terms using the trigonometric identities for the sine and cosine of the sum of two angles, we find

$$x' = \rho \cos \phi \cos \theta - \rho \sin \phi \sin \theta = x \cos \theta - y \sin \theta,$$

$$y' = \rho \cos \phi \sin \theta + \rho \sin \phi \cos \theta = x \sin \theta + y \cos \theta.$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation

- Three features of rotation:
  - Only the origin is unchanged by the rotation, it is called the fixed point
  - Rotation is in 3D (positive – counterclockwise)
  - 2D rotation in the plane is equivalent to 3D rotation about the *z*-axis.

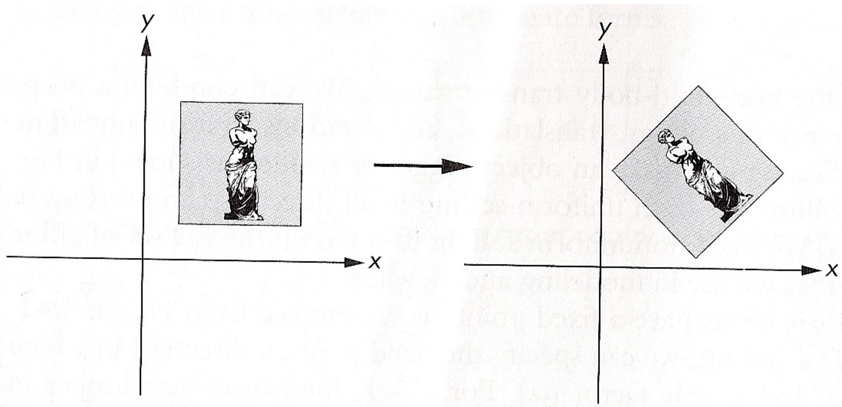


Figure 4.37 Rotation about a fixed point.

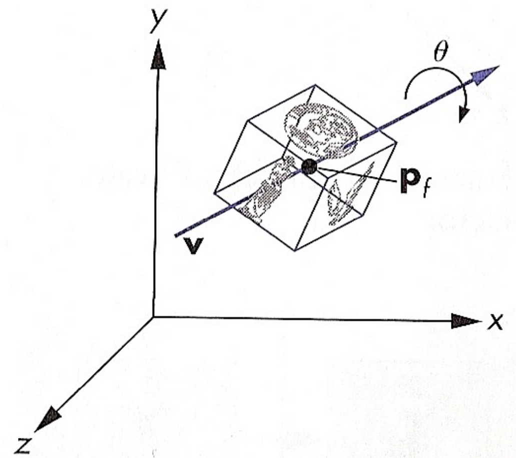
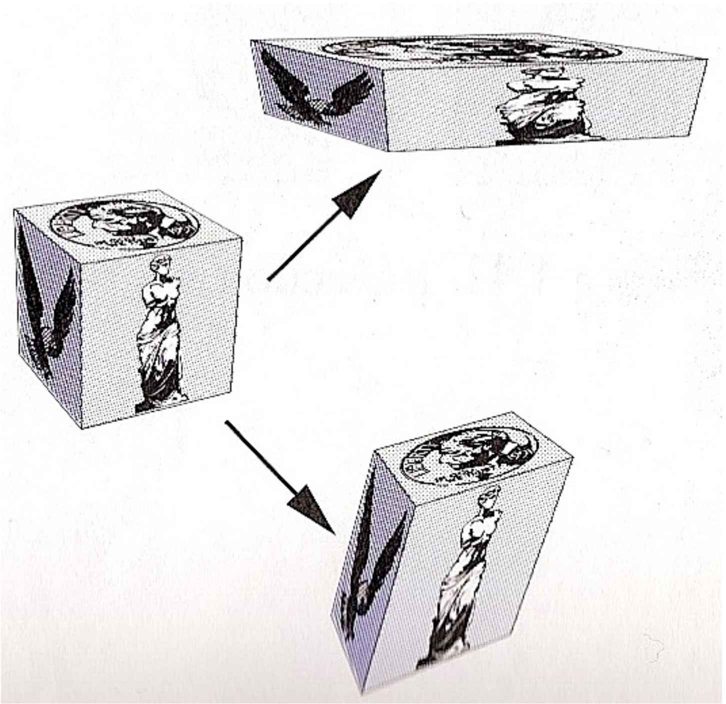


Figure 4.38 Three-dimensional rotation.

Mechanical Engineering
Carnegie Mellon

Rigid-Body Transformations

- Rotation and translation are rigid-body transformations. They can only change the object's location and orientation, not the shape of an object.
- Rotation and translation alone can't give us all possible affine transformations (may be not rigid-body transformation).



Scaling

- Scaling is an affine non-rigid-body transformation. (uniform or nonuniform)
- A properly chosen sequence of scaling, translation, rotation and shear forms any affine transformation.
- Scaling has a fixed point and a scalar factor *a*:
  - if $a > 1$, the objects get longer in the specific direction
  - if $0 <= a < 1$, the objects get smaller in that direction
  - if $a < 0$, the objects get reflection about the fixed point.

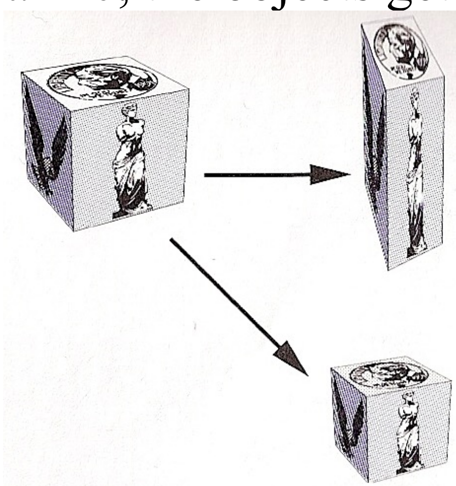


Figure 4.40 Uniform and nonuniform scaling.

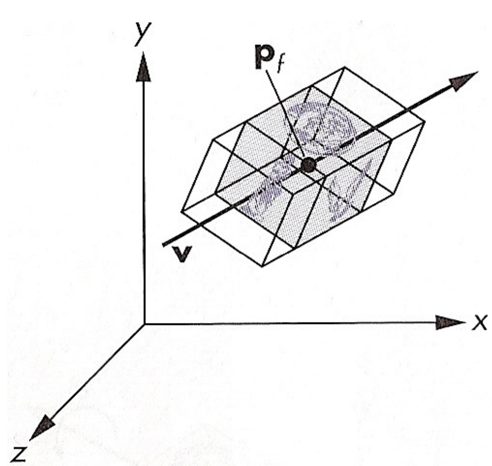


Figure 4.41 Effect of scale factor.

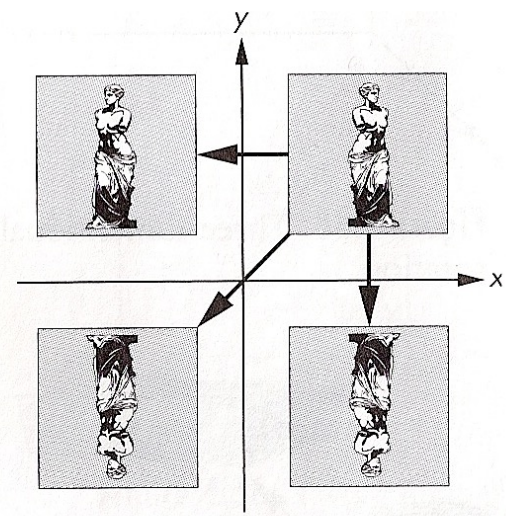


Figure 4.42 Reflection.

Transformations in Homogeneous Coordinates

- We work with representations in homogeneous coordinates:

$$\mathbf{q} = \mathbf{p} + \alpha \mathbf{v}$$

- Within a frame, each affine transformation is represented by a 4 x 4 matrix of the form

$$\mathbf{M} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation

Translation displaces points to new positions defined by a displacement vector. If we move the point $\mathbf{p}$ to $\mathbf{p}'$ by displacing by a distance $\mathbf{d}$, then

$$\mathbf{p}' = \mathbf{p} + \mathbf{d}.$$

Looking at their homogeneous-coordinate forms

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \qquad \mathbf{p}' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \\ 0 \end{bmatrix},$$

we see that these equations can be written component by component as

$$x' = x + \alpha_x,$$
$$y' = y + \alpha_y,$$
$$z' = z + \alpha_z.$$

Translation

- Use the matrix multiplication (with an additional column)

$$\mathbf{p}' = \mathbf{T}\mathbf{p}, \quad \text{where} \quad \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & \alpha_x \\ 0 & 1 & 0 & \alpha_y \\ 0 & 0 & 1 & \alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- T is called the translation matrix.
- If we displace a point by the vector *d*, we can return to the original position by a displacement of –*d*.

$$\mathbf{T}^{-1}(\alpha_x, \alpha_y, \alpha_z) = \mathbf{T}(-\alpha_x, -\alpha_y, -\alpha_z) = \begin{bmatrix} 1 & 0 & 0 & -\alpha_x \\ 0 & 1 & 0 & -\alpha_y \\ 0 & 0 & 1 & -\alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

- For both scaling and rotation, there is a fixed point which is unchanged by the transformation. We let the fixed point be the origin. A scaling matrix with a fixed point of the origin allows for independent scaling along the coordinate axes.

$$x' = \beta_x x,$$

$$y' = \beta_y y,$$

$$z' = \beta_z z.$$

$$\mathbf{p}' = \mathbf{S}\mathbf{p}, \quad \text{where} \quad \mathbf{S} = \mathbf{S}(\beta_x, \beta_y, \beta_z) = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{S}^{-1}(\beta_x, \beta_y, \beta_z) = \mathbf{S}\left(\frac{1}{\beta_x}, \frac{1}{\beta_y}, \frac{1}{\beta_z}\right)$$

Mechanical Engineering
Carnegie Mellon

Rotation

- We first look at rotation with a fixed point at the origin. There are 3 degrees of freedom corresponding independently rotation about the three coordinate axes.
- Matrix multiplication is not commutative.

rotation about the z-axis by an angle θ

$$x' = x\cos\theta - y\sin\theta,$$
$$y' = x\sin\theta + y\cos\theta,$$
$$z' = z;$$

$$\mathbf{p}' = \mathbf{R}_z\mathbf{p},$$

$$\mathbf{R}_z = \mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly:

$$\mathbf{R}_x = \mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_y = \mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta)$ and $\mathbf{R}^{-1}(\theta) = \mathbf{R}^{T}(\theta)$ $\mathbf{R} = \mathbf{R}_z\mathbf{R}_y\mathbf{R}_x$

Orthogonal matrix: $\mathbf{R}^{-1} = \mathbf{R}^{T}$

Shear

- Each shear is characterized by a single angle.

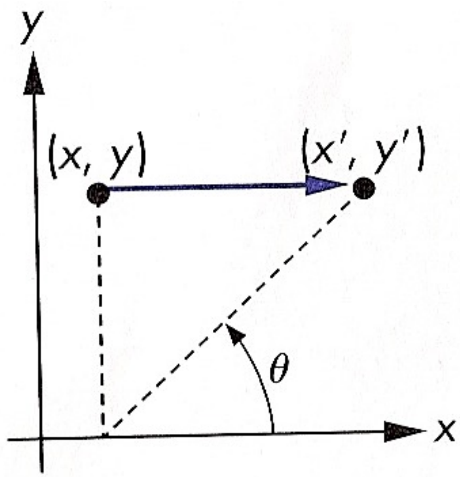


Figure 4.44 Computation of the shear matrix.

$$x' = x + y\cot\theta,$$

$$y' = y,$$

$$z' = z,$$

A shear matrix:

$$\mathbf{H}_x(\theta) = \begin{bmatrix} 1 & \cot\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_x^{-1}(\theta) = \mathbf{H}_x(-\theta)$$

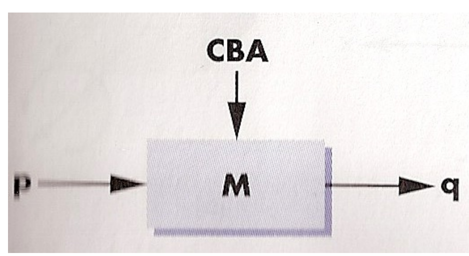
Concatenation of Transformations

- Affine transformations are obtained by multiplying or concatenating sequences of the basic transformations together.
- Suppose that we carry out three successive transformations on a point *p*, creating a new point *q*.

$$q = CBAp \qquad q = (C(B(Ap)))$$

Step 1: $M = CBA$

Step 2: $q = Mp$



Rotation About a Fixed Point

- The fixed point is not the origin

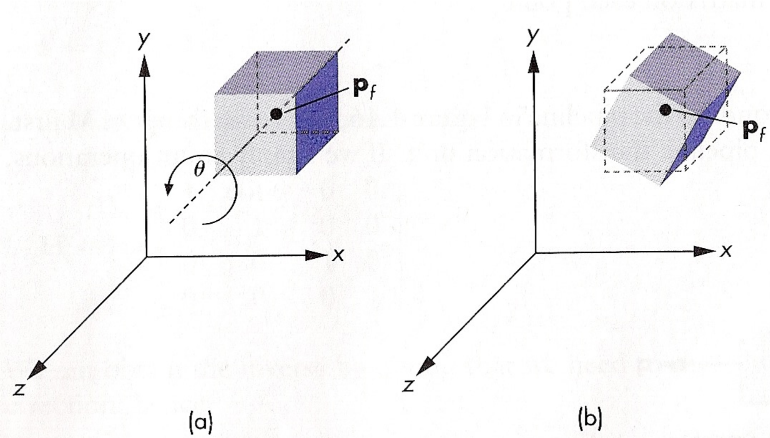


Figure 4.47 Rotation of a cube about its center.

$$\mathbf{M} = \mathbf{T}(\mathbf{p}_f)\mathbf{R}_z(\theta)\mathbf{T}(-\mathbf{p}_f)$$

$$\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & x_f - x_f\cos\theta + y_f\sin\theta \\ \sin\theta & \cos\theta & 0 & y_f - x_f\sin\theta - y_f\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

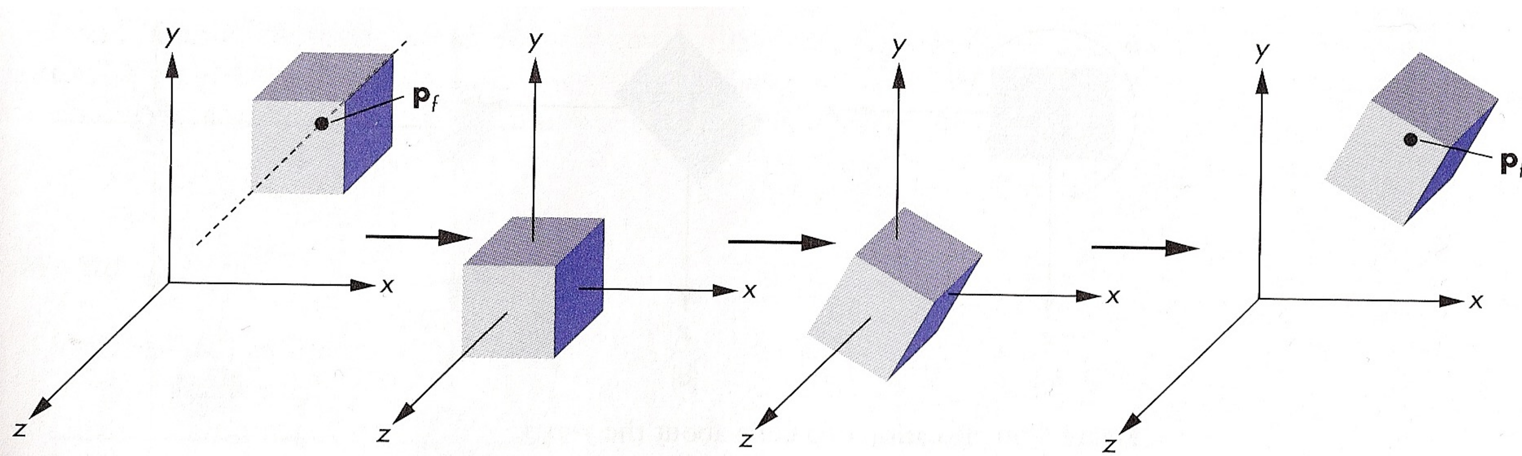


Figure 4.48 Sequence of transformations.

General Rotation

- An arbitrary rotation about the origin can be composed of three successive rotations about the three axes.
- We form the desired matrix by first doing a rotation about the *z*-axis, then doing a rotation about the *y*-axis, and concluding with a rotation about the *x*-axis.

$$\mathbf{R} = \mathbf{R}_x \mathbf{R}_y \mathbf{R}_z.$$

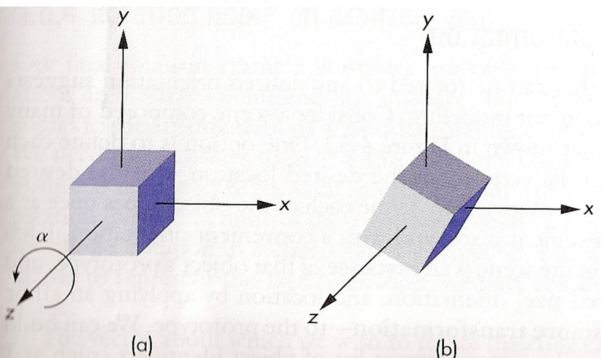


Figure 4.49 Rotation of a cube about the *z*-axis. The cube is shown (a) before rotation, and (b) after rotation.

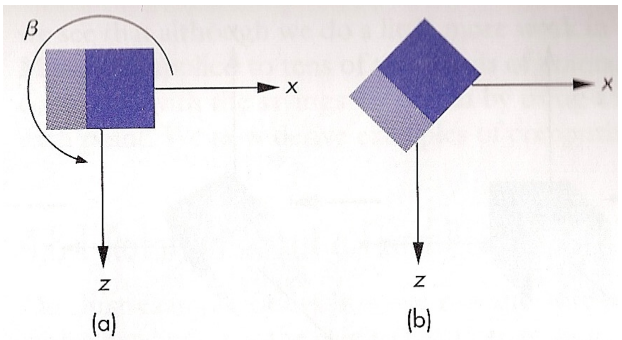


Figure 4.50 Rotation of a cube about the *y*-axis.

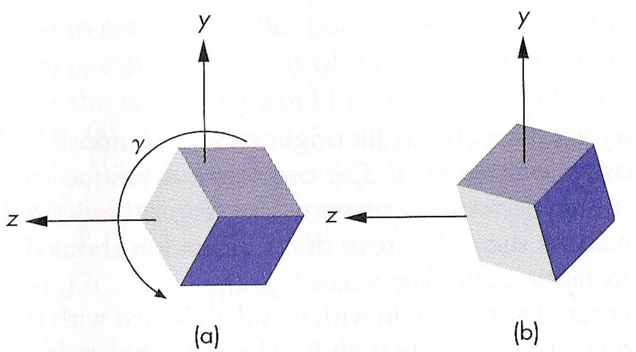


Figure 4.51 Rotation of a cube about the *x*-axis.

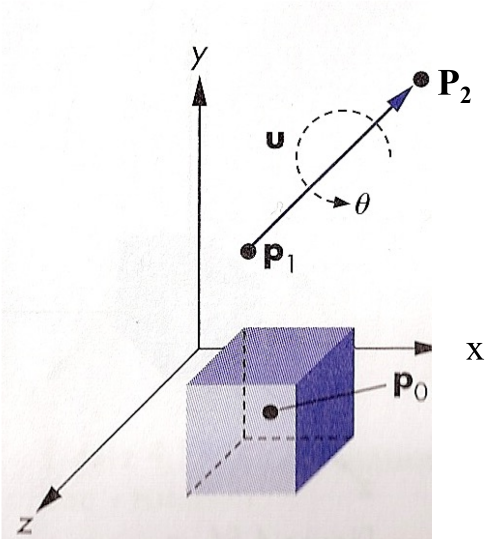
Rotation About an Arbitrary Axis

- The vector about which we wish to rotate is defined by two points:

$$\mathbf{u} = \mathbf{p}_2 - \mathbf{p}_1$$

$$\mathbf{v} = \frac{\mathbf{u}}{|\mathbf{u}|} = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$\alpha_x^2 + \alpha_y^2 + \alpha_z^2 = 1$$



- Our strategy is to carry out two rotations to align the axis of rotation, *v*, with the *z*-axis. Then, we can rotate about the *z*-axis. Finally undo the two rotations that did the aligning.

$$\mathbf{R} = \mathbf{R}_x(-\theta_x)\mathbf{R}_y(-\theta_y)\mathbf{R}_z(\theta)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x)$$

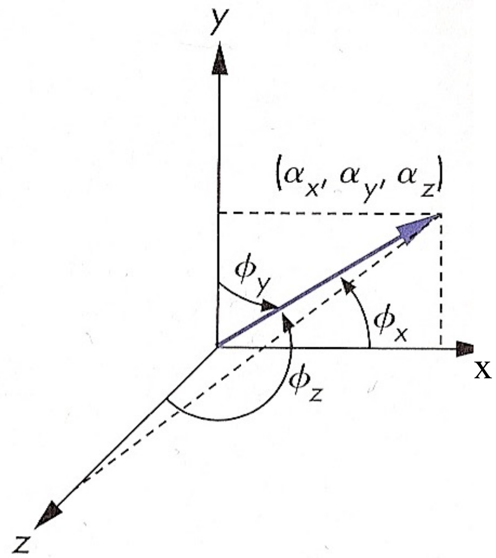
Mechanical Engineering
Carnegie Mellon

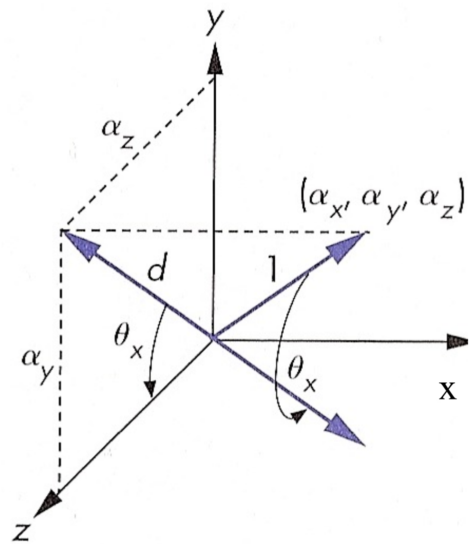
Rotation About an Arbitrary Axis

- The direction cosines:

$$\cos\phi_x = \alpha_x,$$
$$\cos\phi_y = \alpha_y,$$
$$\cos\phi_z = \alpha_z.$$

$$\cos^2\phi_x + \cos^2\phi_y + \cos^2\phi_z = 1$$





$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_z/d & -\alpha_y/d & 0 \\ 0 & \alpha_y/d & \alpha_z/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$

Rotation About an Arbitrary Axis

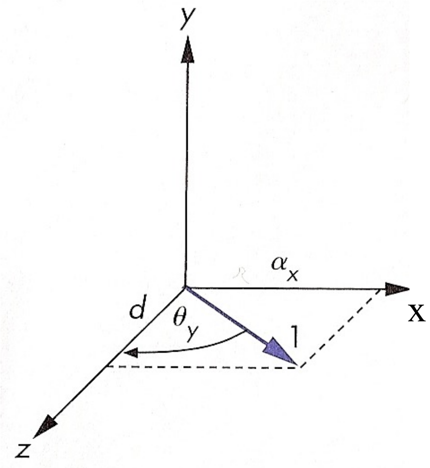


Figure 4.59 Computation the y rotation.

$$\mathbf{R}_y(\theta_y) = \begin{bmatrix} d & 0 & -\alpha_x & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M} = \mathbf{T}(\mathbf{p}_0)\mathbf{R}_x(-\theta_x)\mathbf{R}_y(-\theta_y)\mathbf{R}_z(\theta)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x)\mathbf{T}(-\mathbf{p}_0)$$

Suppose that we wish to rotate an object by 45 degrees about the line passing through the origin and the point (1, 2, 3).

$$\mathbf{R} = \mathbf{R}_x\left(-\cos^{-1}\frac{3}{\sqrt{13}}\right)\mathbf{R}_y\left(-\cos^{-1}\sqrt{\frac{13}{14}}\right)\mathbf{R}_z(45)\mathbf{R}_y\left(\cos^{-1}\sqrt{\frac{13}{14}}\right)$$
$$\mathbf{R}_x\left(\cos^{-1}\frac{3}{\sqrt{13}}\right)$$
$$= \begin{bmatrix} \frac{2+13\sqrt{2}}{28} & \frac{2-\sqrt{2}-3\sqrt{7}}{14} & \frac{6-3\sqrt{2}+4\sqrt{7}}{28} & 0 \\ \frac{2-\sqrt{2}+3\sqrt{7}}{14} & \frac{4+5\sqrt{2}}{14} & \frac{6-3\sqrt{2}-\sqrt{7}}{14} & 0 \\ \frac{6-3\sqrt{2}-4\sqrt{7}}{28} & \frac{6-3\sqrt{2}+\sqrt{7}}{14} & \frac{18+5\sqrt{2}}{28} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{M} = \mathbf{T}(\mathbf{p_f})\mathbf{R}\mathbf{T}(-\mathbf{p_f})$$

References

- Interactive Computer Graphics: A top-down approach using OpenGL. Edward Angel, 3rd Edition. Pearson Edition. Chapter 4.