University of Illinois at Urbana-Champiagn

Misc Notes

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Chapter 1

Fluid Mechanics

1.1 Fluids

A fluid is a material that deforms continuously under action of a shear stress, however small. This contrasts with the behaviour of a solid which deforms only when stress values are in certain regimes. Solids are **elastic**: internal stresses in solids resist absolute deformation some original state; fluids are **viscous**: internal stresses in fluids resist the time rate of deformation. A corollary is that a fluid in static equilibrium supports only its weight and forces acting normal to its boundary. A material is said to be fluid if it seems to "flow" in the timescale of observation t_{obs}. That is, if the material is able to relax to a natural state in time less than t_{obs}. The relaxation time of a fluid is denoted λ_{relax}. The ratio of relaxation time to observation time is called Deborah Number. This dimensionless quantity is named after the prophet Deborah who, in the Book of Judges, proclaimed "The mountains flowed before the lord." For large enough t_{obs}, even mountains will behave like fluids. A material is fluid if $\mathrm{De} << 1$.

$$\mathrm{De} = \frac{\lambda_{\mathrm{relax}}}{t_{\mathrm{obs}}} \tag{1.1}$$

We tend to neglect the discrete, molecular nature of matter and treat the fluid to be made of a "continuum". Macroscopic properties such as density and velocity are taken to be well defined for infinitesimal volume elements –small in comparison to system lengthscale but larger in comparison to molecular lengthscales. Fluid properties can vary continuously from one volume element to another and are average values of the molecular properties.

To derive the equations governing viscous fluid flow, we have to consider the time rate of change of quantities/properties in fluid control volumes. A control volume is an arbitrarily defined volume with a closed bounding surface. A material volume is a control volume that contains the same particles of matter at all times. A particular material volume may be defined by the closed bounding surface that envelops its material particles at a certain time. Hence, the velocity of the surface at every point is equal to the flow velocity at that point. The term "fluid element" is synonymous with a material volume.

We consider two perspectives to view motion in a continuum: the **Lagrangian** perspective expresses position, velocity and other state variables in terms of material points travelling with a fluid; the **Eulerian** perspective expresses fluid properties with respect to some reference coordinate system. The time evolution of the position of the material point $\boldsymbol{X}$ can be tracked on an Eulerian coordinate system with a vector $\boldsymbol{x}(\boldsymbol{X}, t)$. Since only one material point can be located at an Eulerian coordinate at a given time, there exists an inverse relation $\boldsymbol{X}(\boldsymbol{x}, t)$ mapping material points $\boldsymbol{X}$ to the Eulerian coordinate position they occupy at time t. To illustrate the difference, consider velocity. The velocity of a material point $\boldsymbol{X}(\boldsymbol{x_0}, t)$ is given by $\partial_t \boldsymbol{X}(\boldsymbol{x}, t)\big|_{\boldsymbol{x_0}} = \boldsymbol{v}(\boldsymbol{X}(\boldsymbol{x_0}, t), t)$

Let $\boldsymbol{F}$ be some property of a fluid at some material point $\boldsymbol{X} \in \Omega$ at time $t > 0$. $\boldsymbol{F}$ depends on $\boldsymbol{X}$, i.e. which material point one chooses and time t as material point may interact with other points and lose/gain the property over time. Hence the property $\boldsymbol{F}$ following a material point varies with the point's Eulerian coordinate and time, i.e. $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x}, t)$. The rate of change of $\boldsymbol{F}$ following the material point is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{F} &= \partial_t \boldsymbol{F} + \sum_{i=1}^{n} \partial_{x_i} \boldsymbol{F} \partial_t x_i \\ \mathrm{D}_t \boldsymbol{F} &:= \partial_t \boldsymbol{F} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{F} \end{aligned} \tag{1.2}$$

Another way to think of the time rate of change of property $\boldsymbol{F}$ as one follows a material point is the following: $\partial_t \boldsymbol{F}$ is the time rate of change of $\boldsymbol{F}$ at fixed position $\boldsymbol{x}$, and $\partial_{x_i} \boldsymbol{F} \partial_t x_i$ is the rate of change of $\boldsymbol{F}$ as one would travel in the direction i multiplied by how fast the material point is travelling in said direction i.

Consider two points, $\boldsymbol{x_0}$, $\boldsymbol{x_0} + \boldsymbol{h}$, in a flow at time $t = 0$. We call the matrix J the Jacobian of $\boldsymbol{X}$ (Lagrangian coordinate) with respect to $\boldsymbol{x}$ (Eulerian coordinate).

$$J(x_0, t) = [J_{ij}] = \lim_{|h| \to 0} \frac{X_i(x_{0j} + h_j, t) - X_i(x_{0j}, t)}{|h|} = \partial_{x_j} X_i(\boldsymbol{x}, t)\Big|_{\boldsymbol{x} = \boldsymbol{x_0}} \tag{1.3}$$

An infinitesimal line element $\mathrm{d}V$ at time $t = 0$ is stretched to $\det(J)\,\mathrm{d}V$ at time t. Consider the material derivative of the Jacobian. todo:

$$\begin{aligned} \partial_t \det(J) =& \partial_t \det\left(\partial_{x_j} X_i\right) = \sum_{i,j} \frac{\partial \det(J)}{\partial J_{ij}} \partial_t \frac{\partial X_i}{\partial x_j} = \sum_{i,j} \frac{\partial \det(J)}{\partial J_{ij}} \frac{\partial v_i}{\partial x_j} \\ =& \sum_{i,j} \frac{\partial \det(J)}{\partial J_{ij}} \left(\sum_k \frac{\partial v_i}{\partial X_k} \frac{\partial X_k}{\partial x_j} \right) = \sum_{i,k} \frac{\partial v_i}{\partial X_k} \left(\sum_j \frac{\partial \det(J)}{\partial J_{ij}} \frac{\partial X_k}{\partial x_j} \right) \end{aligned} \tag{1.4}$$

1.2 Stress

Stress is defined as a force across a "small" boundary ($\in \mathbb{R}^2$) per unit area of that boundary, for all orientations of the boundary. Stress is defined at a point with respect to a surface on which it would act. Consequently, stress depends on the orientation of the surface on which it acts. Hence, $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{x}, t, \boldsymbol{n})$ where $\boldsymbol{n}$ is the outward pointing normal to the surface. In the limit $\delta A \to 0$, stress at a point is independent of the magnitude of the area.

$$\boldsymbol{\tau} = \lim_{\delta A \to 0} \frac{\delta \boldsymbol{F}}{\delta A} \tag{1.5}$$

We probe the dependence of stress on the normal the surface it acts on. We consider a cases that reveals stress at a point acting on two sides of a surface to be equal in magnitude and opposite in direction. Consider an infinitesimal disk-shaped fluid element with area δA and height δh. Without loss of generality, label the normal on one face $\boldsymbol{n}$ and the other $-\hat{\boldsymbol{n}}$. Let x_0 be the middle of the disk and let $\boldsymbol{g}$ be acceleration due to some applied body force. The equation of motion for the fluid element is:

$$\begin{aligned} \delta A \delta h \mathrm{D}_t \boldsymbol{\rho v} &= \boldsymbol{\tau}(\boldsymbol{x}_0 + 0.5\delta h \hat{\boldsymbol{n}}, t, \hat{\boldsymbol{n}})\delta A + \boldsymbol{\tau}(\boldsymbol{x}_0 - 0.5\delta h \hat{\boldsymbol{n}}, t, -\hat{\boldsymbol{n}})\delta A + \rho \delta A \delta h \boldsymbol{g} \\ \delta h \mathrm{D}_t \boldsymbol{\rho v} &= \boldsymbol{\tau}(\boldsymbol{x}_0 + 0.5\delta h \hat{\boldsymbol{n}}, t, \hat{\boldsymbol{n}}) + \boldsymbol{\tau}(\boldsymbol{x}_0 - 0.5\delta h \hat{\boldsymbol{n}}, t, -\hat{\boldsymbol{n}}) + \rho \delta h \boldsymbol{g} \end{aligned} \tag{1.6}$$

In the limit $\delta h \to 0$, the rate of change of momentum of the disk falls off, and we get

$$\begin{aligned} \boldsymbol{\tau}(\boldsymbol{x}_0, t, -\hat{\boldsymbol{n}}) + \boldsymbol{\tau}(\boldsymbol{x}_0, t, \hat{\boldsymbol{n}}) &= 0 \\ \boldsymbol{\tau}(\boldsymbol{x}_0, t, -\hat{\boldsymbol{n}}) &= -\boldsymbol{\tau}(\boldsymbol{x}_0, t, \hat{\boldsymbol{n}}) \end{aligned} \tag{1.7}$$

Let stress at a point x_0 at time t_0 be a function of the normal to the surface it acts on i.e $\boldsymbol{\tau} = \boldsymbol{\tau}(\hat{\boldsymbol{n}})$. Let σ_{ij} be the component of stress in the $\hat{\boldsymbol{j}}$ direction on a surface with normal $\hat{\boldsymbol{\imath}}$. Similarly, we have

$$\begin{aligned}\boldsymbol{\tau}(\hat{\boldsymbol{\imath}}) &= \sigma_{ii}\hat{\boldsymbol{\imath}} + \sigma_{ji}\hat{\boldsymbol{j}} + \sigma_{ki}\hat{\boldsymbol{k}}\\ \boldsymbol{\tau}(\hat{\boldsymbol{j}}) &= \sigma_{ij}\hat{\boldsymbol{\imath}} + \sigma_{jj}\hat{\boldsymbol{j}} + \sigma_{kj}\hat{\boldsymbol{k}}\\ \boldsymbol{\tau}(\hat{\boldsymbol{k}}) &= \sigma_{ik}\hat{\boldsymbol{\imath}} + \sigma_{jk}\hat{\boldsymbol{j}} + \sigma_{kk}\hat{\boldsymbol{k}}\end{aligned} \tag{1.8}$$

where each σ_{**} is a scalar field varying in space and time. We now show that stress on a surface with an arbitrary normal can be represented as linear combinations of $\boldsymbol{\tau}(\hat{\boldsymbol{\imath}})$, $\boldsymbol{\tau}(\hat{\boldsymbol{j}})$, $\boldsymbol{\tau}(\hat{\boldsymbol{k}})$. Consider a fluid element in the shape of an infinitesimal tetrahedral with vertices $\boldsymbol{x_0}$, $\boldsymbol{x_0} + \delta x\hat{\boldsymbol{\imath}}$, $\boldsymbol{x_0} + \delta y\hat{\boldsymbol{j}}$, $\boldsymbol{x_0} + \delta z\hat{\boldsymbol{k}}$. This tetrahedral has a faces parallel to the xy, xz, yz planes with areas δA_z, δA_y, δA_x, and normals $-\hat{\boldsymbol{k}}$, $-\hat{\boldsymbol{j}}$, $-\hat{\boldsymbol{\imath}}$ respectively. Let the fourth face have area δA and some arbitrary normal $\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{j}} + n_z\hat{\boldsymbol{k}}$. Let θ_x, θ_y, θ_z be the angle between $\hat{\boldsymbol{n}}$ and the coordinate axes. The areas are related by the following expressions:

$$\begin{aligned}\delta A_x &= \cos\theta_x = n_x\delta A\\ \delta A_y &= \cos\theta_y = n_y\delta A\\ \delta A_z &= \cos\theta_z = n_z\delta A\end{aligned} \tag{1.9}$$

We now apply Newton's laws to the fluid element in the limit δx, δy, $\delta z \to 0$. Since the volume of the fluid element, δV falls much faster than any of the surface areas, the mass time acceleration and body forces go to zero faster than the surface forces.

$$\begin{aligned}\delta V &\propto \delta x\delta y\delta z\\ \delta A &\propto \delta z\sqrt{\delta x^2 + \delta y^2}\\ \delta A_x &\propto \delta y\delta z\\ \delta A_y &\propto \delta x\delta z\\ \delta A_z &\propto \delta x\delta y\end{aligned} \tag{1.10}$$

Say δx, δy, δz go to zero as $\frac{1}{N}$. Then, for bounded constants c, c_v,

$$\begin{aligned}0 &= \delta A\boldsymbol{\tau}(\hat{\boldsymbol{n}}) + \delta A_x\boldsymbol{\tau}(-\hat{\boldsymbol{\imath}}) + \delta A_y\boldsymbol{\tau}(-\hat{\boldsymbol{\imath}}) + \delta A_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}})\\ 0 &= \delta A\boldsymbol{\tau}(\hat{\boldsymbol{n}}) + n_x\delta A\boldsymbol{\tau}(-\hat{\boldsymbol{\imath}}) + n_y\delta A\boldsymbol{\tau}(-\hat{\boldsymbol{\imath}}) + n_z\delta A\boldsymbol{\tau}(-\hat{\boldsymbol{k}})\\ 0 &= \lim_{N\to\infty} \frac{c\boldsymbol{\tau}(\hat{\boldsymbol{n}})}{N^2} + \frac{cn_x\boldsymbol{\tau}(-\hat{\boldsymbol{\imath}})}{N^2} + \frac{cn_y\boldsymbol{\tau}(-\hat{\boldsymbol{j}})}{N^2} + \frac{cn_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}})}{N^2} + \frac{c_v}{N^3}(\rho\boldsymbol{g} - \mathrm{D}_t\rho\boldsymbol{v})\\ 0 &= \lim_{N\to\infty} c\boldsymbol{\tau}(\hat{\boldsymbol{n}}) + cn_x\boldsymbol{\tau}(-\hat{\boldsymbol{\imath}}) + cn_y\boldsymbol{\tau}(-\hat{\boldsymbol{j}}) + cn_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}}) + \frac{c_v}{N}(\rho\boldsymbol{g} - \mathrm{D}_t\rho\boldsymbol{v})\\ 0 &= \boldsymbol{\tau}(\hat{\boldsymbol{n}}) + n_x\boldsymbol{\tau}(-\hat{\boldsymbol{\imath}}) + n_y\boldsymbol{\tau}(-\hat{\boldsymbol{j}}) + n_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}})\\ \boldsymbol{\tau}(\hat{\boldsymbol{n}}) &= n_x\boldsymbol{\tau}(\hat{\boldsymbol{\imath}}) + n_y\boldsymbol{\tau}(\hat{\boldsymbol{j}}) + n_z\boldsymbol{\tau}(\hat{\boldsymbol{k}})\end{aligned} \tag{1.11}$$

Hence, forces due to surface stresses must balance each other in the limit of the tetrahedral shrinking to a point. Expressing the above expression in matrix notation,

$$\begin{aligned}\boldsymbol{\tau}(\hat{\boldsymbol{n}}) &= \begin{bmatrix}\boldsymbol{\tau}(\hat{\boldsymbol{\imath}}) & \boldsymbol{\tau}(\hat{\boldsymbol{j}}) & \boldsymbol{\tau}(\hat{\boldsymbol{k}})\end{bmatrix} \cdot \hat{\boldsymbol{n}}\\ \boldsymbol{\tau}(\hat{\boldsymbol{n}}) &= \begin{bmatrix}\sigma_{ii} & \sigma_{ij} & \sigma_{ik}\\ \sigma_{ji} & \sigma_{jj} & \sigma_{jk}\\ \sigma_{ki} & \sigma_{kj} & \sigma_{kk}\end{bmatrix} \cdot \hat{\boldsymbol{n}}\end{aligned} \tag{1.12}$$

We call $\boldsymbol{\tau}$ the traction vector and $\underline{\underline{\sigma}}$ the stress tensor.

$$\underline{\underline{\sigma}} = \begin{bmatrix}\boldsymbol{\tau}(\hat{\boldsymbol{\imath}}) & \boldsymbol{\tau}(\hat{\boldsymbol{j}}) & \boldsymbol{\tau}(\hat{\boldsymbol{k}})\end{bmatrix} = \begin{bmatrix}\sigma_{ii} & \sigma_{ij} & \sigma_{ik}\\ \sigma_{ji} & \sigma_{jj} & \sigma_{jk}\\ \sigma_{ki} & \sigma_{kj} & \sigma_{kk}\end{bmatrix} \tag{1.13}$$

$$\boldsymbol{\tau}(\boldsymbol{x}, t, \hat{\boldsymbol{n}}) = \underline{\underline{\sigma}}(\boldsymbol{x}, t) \cdot \hat{\boldsymbol{n}} \tag{1.14}$$

todo: We show that the stress tensor is symmetric. Since a fluid continuously deforms under shear stress, a static, non-deforming fluid element in consequently under no shear stress. Hence for an arbitrary surface element in a fluid at rest, stress is in the normal direction. From equilibrium arguments, it can be proven that stress in a fluid at rest is isotropic.

$$\begin{aligned}\boldsymbol{\tau} &= -p(\boldsymbol{x},t)\boldsymbol{n} \\ \boldsymbol{\tau} &= -p(\boldsymbol{x},t)\underline{\underline{\delta}} \cdot \boldsymbol{n}\end{aligned} \tag{1.15}$$

We still need to relate the stress tensor to the flow field. The *Newtonian Model* is based on the following assumptions:

1. shear stress is proportional to the rate of shear strain in a fluid particle;
2. shear stress is zero when the rate of shear strain is zero;
3. the stress to rate-of-strain relation is isotropic—that is, there is no preferred orientation in the fluid.

From the first assumption, we have $\sigma_{ij} = \mathrm{K}_{ijkl} e_{kl}$ where K is a fourth order tensor.

todo: derive

$$\begin{aligned}\underline{\underline{\sigma}} &= -p\underline{\underline{\delta}} + \underline{\underline{\tau_v}} \\ &= -p\underline{\underline{\delta}} + \mu(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^{\intercal}) + \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{v})\underline{\underline{\delta}}\end{aligned} \tag{1.16}$$

1.3 Reynolds Transport Theorem and Consequences

Reynolds Transport Theorem (RTT) relates time rate of change of quantities in material volumes to the distribution of said properties in the volume. Consider an arbitrary, finite (finite, nonzero measure) control volume $\Omega(t) \subset \mathbb{R}^3$ bounded by some control surface $\partial\Omega(t) \subset \mathbb{R}^3$ in some time varying flow. For some property $\phi(\boldsymbol{x}, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V = \lim_{\Delta t\to 0} \frac{\int_{\Omega(t+\Delta t)} \phi(\boldsymbol{x},t+\Delta t)\,\mathrm{d}V - \int_{\Omega(t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V}{\Delta t} \tag{1.17}$$

For reasonably smooth $\phi(\boldsymbol{x},t)$, use the taylor expansion of $\phi(\boldsymbol{x},t)$ about $\phi(\boldsymbol{x},t)$

$$\phi(\boldsymbol{x},t+\Delta t) = \phi(\boldsymbol{x},t) + \Delta t\partial_t\phi(\boldsymbol{x},t) + \mathcal{O}(\Delta t^2) \tag{1.18}$$

Substituting,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V &= \lim_{\Delta t\to 0}\int_{\Omega(t+\Delta t)} \partial_t\phi(\boldsymbol{x},t)\,\mathrm{d}V + \frac{\int_{\Omega(t+\Delta t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V - \int_{\Omega(t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V + \mathcal{O}(\Delta t^2)}{\Delta t} \\ &= \int_{\Omega(t)} \partial_t\phi(\boldsymbol{x},t)\,\mathrm{d}V + \lim_{\Delta t\to 0} + \frac{\int_{\Omega(t+\Delta t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V - \int_{\Omega(t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V}{\Delta t}\end{aligned} \tag{1.19}$$

To explain the thought process, WLOG let $\phi = 1$. We are looking for the difference in volume between the two integrals. That is equal to the net volume that the boundary has expanded into. For an infinitesimal patch $\mathrm{d}A$ on $\partial\Omega$ (containing point $\boldsymbol{x}$, and with outward normal $\hat{\boldsymbol{n}}$), that is

$$(\boldsymbol{x}(t+\Delta t) - \boldsymbol{x}(t)) \cdot \hat{\boldsymbol{n}}\,\mathrm{d}A \tag{1.20}$$

Capturing the difference with a Taylor expansion and integrating over $\partial\Omega$, we get

$$\int_{\Omega(t+\Delta t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V - \int_{\Omega(t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V = \int_{\partial\Omega(t)} \phi(\boldsymbol{x},t)\cdot(\Delta t\boldsymbol{v_c})\cdot\hat{\boldsymbol{n}}\,\mathrm{d}A + \int_{\partial\Omega(t)} \phi(\boldsymbol{x},t)\cdot(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_c\frac{\Delta t^2}{2})\cdot\hat{\boldsymbol{n}}\,\mathrm{d}A + \mathcal{O}(\Delta t^3) \tag{1.21}$$

where $\boldsymbol{v_c}(\boldsymbol{x},t)$ is the velocity of the control surface at time t at point $\boldsymbol{x} \in \partial\Omega(t)$ and $\hat{\boldsymbol{n}}$ is the outward pointing unit normal vector on $\partial\Omega(t)$. An infinitesimal area element $\mathrm{d}A \subset \partial\Omega(t)$ moves a distance of $(\Delta t\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}} + \mathcal{O}(\Delta t^2))$ normal to the surface in time Δt. Another way to think of this approximation is the following: project the value of $\phi(\boldsymbol{x},t)$ on $\mathrm{d}A \subset \partial\Omega(t)$ throughout the volume $\mathrm{d}A(\Delta t\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}}) + \mathcal{O}(\Delta t^2)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V &= \int_{\Omega(t)} \partial_t\phi(\boldsymbol{x},t)\,\mathrm{d}V + \lim_{\Delta t\to 0}\frac{\int_{\partial\Omega(t)}\phi(\boldsymbol{x},t)(\Delta t\boldsymbol{v_c})\cdot\hat{\boldsymbol{n}}dA + \mathcal{O}(\Delta t^2)}{\Delta t} \\ &= \int_{\Omega(t)} \partial_t\phi(\boldsymbol{x},t)\,\mathrm{d}V + \int_{\partial\Omega(t)} \phi(\boldsymbol{x},t)\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}}dA\end{aligned} \tag{1.22}$$

$\partial_t\phi$ is the local the rate of production (or accumulation) of ϕ. It accounts for the effects of change in ϕ in the interior of the domain (say due to creation/dissipation or advection/diffusion). The divergence term accounts for the capture or loss of ϕ by the motion of the control surface. If $\Omega(t)$ is a material volume then, $\boldsymbol{v_c} = \boldsymbol{v}$, at all points in $\partial\Omega(t)$. Hence, the final form of the Reynolds Transport Theorem for material volumes $\Omega(t)$ is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V = \int_{\Omega(t)} \partial_t\phi(\boldsymbol{x},t)\,\mathrm{d}V + \int_{\partial\Omega(t)} \phi(\boldsymbol{x},t)\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}A \tag{1.23}$$

Applying Gauss' law,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \phi(\boldsymbol{x},t)\,\mathrm{d}V &= \int_{\Omega(t)} \partial_t\phi + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\phi)\,\mathrm{d}V \\ &= \int_{\Omega(t)} \mathrm{D}_t\phi + \phi(\boldsymbol{\nabla}\cdot\boldsymbol{v})\,\mathrm{d}V\end{aligned} \tag{1.24}$$

Since a material volume always contains the same fluid elements, time rate of change of total mass in a material volume should be zero.

$$\begin{aligned}m(t) &= \int_\Omega \rho\,\mathrm{d}V \\ 0 = \frac{\mathrm{d}}{\mathrm{d}t}m &= \int_\Omega \partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\,\mathrm{d}V\end{aligned} \tag{1.25}$$

Since the integral is zero for arbitrary material volumes, the integrand must be zero.

$$\begin{aligned}\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0 \\ \mathrm{D}_t\rho = \partial_t\rho + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho = -\rho(\boldsymbol{\nabla}\cdot\boldsymbol{v})\end{aligned} \tag{1.26}$$

Let $\boldsymbol{p}(t)$ denote the total momentum of a fluid element.

$$\begin{aligned}\boldsymbol{p}(t) &= \int_\Omega \rho\boldsymbol{v}\,\mathrm{d}V \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} &= \int_\Omega \partial_t(\rho\boldsymbol{v}) + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\otimes\rho\boldsymbol{v})\,\mathrm{d}V \\ &= \int_\Omega \partial_t(\rho\boldsymbol{v}) + \rho\boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\rho\boldsymbol{v})\,\mathrm{d}V \\ &= \int_\Omega \mathrm{D}_t(\rho\boldsymbol{v}) + \rho\boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\,\mathrm{d}V\end{aligned} \tag{1.27}$$

Consider the total force applied on the fluid element.

$$
\begin{aligned}
\boldsymbol{F}_{\text{ext}} &= \int_\Omega \rho \boldsymbol{g}\,\mathrm{d}V + \int_{\partial\Omega} \underline{\underline{\sigma}} \cdot \boldsymbol{n}\,\mathrm{d}A \\
&= \int_\Omega \rho \boldsymbol{g} + \boldsymbol{\nabla} \cdot \underline{\underline{\sigma}}\,\mathrm{d}V \\
&= \int_\Omega \rho \boldsymbol{g} + \boldsymbol{\nabla} \cdot \Big(-p\underline{\underline{\delta}} + \mu(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^\intercal) + \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{v})\underline{\underline{\delta}}\Big)\,\mathrm{d}V \\
&= \int_\Omega \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu(\boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})) + \lambda\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\,\mathrm{d}V \\
&= \int_\Omega \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu\boldsymbol{\nabla}^2 \boldsymbol{v} + (\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\,\mathrm{d}V
\end{aligned}
\tag{1.28}
$$

From Newton's second law, time rate of change of momentum is equal to the sum of all external forces applied.

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} - \boldsymbol{F}_{\text{ext}} \\
&= \int_\Omega \partial_t(\rho\boldsymbol{v}) + \rho\boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\rho\boldsymbol{v}) - \Big(\rho\boldsymbol{g} - \boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{v} + (\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\Big)\,\mathrm{d}V
\end{aligned}
\tag{1.29}
$$

Since the integral is zero for an arbitrary domain (fluid element) $\Omega(t)$, it follows that the integrand must be zero everywhere.

$$
\begin{aligned}
\partial_t(\rho\boldsymbol{v}) + \rho\boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\rho\boldsymbol{v}) &= \rho\boldsymbol{g} - \boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{v} + (\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) \\
\boldsymbol{v}\cancelto{0\,\text{(continuity)}}{(\partial_t\rho + \boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\rho)} + \rho\mathrm{D}_t\boldsymbol{v} &= \rho\boldsymbol{g} - \boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{v} + (\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) \\
\rho(\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}) &= \rho\boldsymbol{g} - \boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{v} + (\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})
\end{aligned}
\tag{1.30}
$$

We have arrived at the Navier-Stokes equations. Take the divergence of the momentum equation.

$$
\begin{aligned}
\rho(\partial_t(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{v}^\intercal + (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\cdot\boldsymbol{v})) &= \rho(\boldsymbol{\nabla}\cdot\boldsymbol{g}) - \boldsymbol{\nabla}^2 p + \mu\boldsymbol{\nabla}^2(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\mu+\lambda)\boldsymbol{\nabla}^2(\boldsymbol{\nabla}\cdot\boldsymbol{v}) \\
\rho\mathrm{D}_t(\boldsymbol{\nabla}\cdot\boldsymbol{v}) &= \boldsymbol{\nabla}^2(-p + (\lambda+2\mu)(\boldsymbol{\nabla}\cdot\boldsymbol{v})) + \rho(\boldsymbol{\nabla}\cdot\boldsymbol{g} - \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{v}^\intercal)
\end{aligned}
\tag{1.31}
$$

where $\underline{\underline{A}} : \underline{\underline{B}} = A_{ij}B_{ji}$ is the double dot product. This equation simplifies into a laplace equation for pressure in case of incompressible flow.

$$
-\boldsymbol{\nabla}^2 p = \rho(\boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{v}^\intercal - \boldsymbol{\nabla}\cdot\boldsymbol{g})
\tag{1.32}
$$

We take the curl of the momentum equation. Define $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$

$$
\begin{aligned}
\boldsymbol{\nabla}\times\rho(\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}) &= \boldsymbol{\nabla}\times(\rho\boldsymbol{g} - \boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{v} + (\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})) \\
\boldsymbol{\nabla}\rho\times\mathrm{D}_t\boldsymbol{v} + \rho\mathrm{D}_t\boldsymbol{\omega} + \epsilon_{ijk}v_{l,i}v_{j,l} &= \boldsymbol{\nabla}\times\boldsymbol{g} + \boldsymbol{\nabla}\rho\times\boldsymbol{g} + \mu\boldsymbol{\nabla}^2\boldsymbol{\omega} + \epsilon_{ijk}(-p + (\boldsymbol{\nabla}\cdot\boldsymbol{v}))_{,ji} \\
\boldsymbol{\nabla}\rho\times(\mathrm{D}_t\boldsymbol{v} - \boldsymbol{g}) + \rho\mathrm{D}_t\boldsymbol{\omega} + \epsilon_{ijk}v_{l,i}v_{j,l} &= \boldsymbol{\nabla}\times\boldsymbol{g} + \mu\boldsymbol{\nabla}^2\boldsymbol{\omega} + \epsilon_{ijk}(-p + (\boldsymbol{\nabla}\cdot\boldsymbol{v}))_{,ji}
\end{aligned}
\tag{1.33}
$$

Informally, the Laplacian operator $\boldsymbol{\nabla}^2$ gives the difference between the average value of a function in the neighborhood of a point, and its value at that point. Thus, if u is the temperature, $\boldsymbol{\nabla}^2 u$ tells whether (and by how much) the material surrounding each point is hotter or colder, on the average, than the material at that point.

We explore energy conservation properties of the incompressible Navier Stokes equations. Define the energy in a fluid element $\Omega(t)$ as $\mathcal{E} = \int_{\Omega(t)} \frac{1}{2}\rho|\boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x}$. Then,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \frac{1}{2}\rho v_j v_j \,\mathrm{d}\boldsymbol{x} = \int_{\Omega(t)} \frac{1}{2}\rho \mathrm{D}_t(v_j v_j)\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega(t)} \rho \boldsymbol{v}^\mathsf{T} \mathrm{D}_t(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \int_{\Omega(t)} \boldsymbol{v}^\mathsf{T}\left(-\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{v}\right)\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega(t)} -\boldsymbol{\nabla}\cdot(p\boldsymbol{v}) + \mu v_j v_{j,ii}\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega(t)} -\boldsymbol{\nabla}\cdot(p\boldsymbol{v}) + \mu\left((v_j v_{j,i})_{,i} - v_{j,i}v_{j,i}\right)\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\partial\Omega(t)} \boldsymbol{v}\cdot\left(-p\hat{\boldsymbol{n}} + \mu\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\boldsymbol{v}\right)\,\mathrm{d}\boldsymbol{x} - \int_{\Omega(t)} \mu v_{j,i}v_{j,i}\,\mathrm{d}\boldsymbol{x}
\end{aligned}
\tag{1.34}
$$

1.4 Stokeslet and Stresslets

We discuss properties of fundamental solution to the Stokes equation.

$$
\begin{gathered}
\boldsymbol{\nabla} p - \mu \boldsymbol{\nabla}^2 \boldsymbol{v} = \boldsymbol{f}\cdot\delta(\boldsymbol{x}) \\
\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0
\end{gathered}
\tag{1.35}
$$

Chapter 2

Reynolds Stress Transport Equations

We derive the Reynolds stress transport equations (RSTE) that describe the production, transport, and dissipation of energy in fluctuating modes in a flow. Examining the energy budgets of turbulent flows can lead to insights into factors driving turbulence and leading to its decay, spatial distribution, and how energy is transferred between mean and fluctuating modes. The detailed Reynolds stress budgets from DNS provide valuable information on the relative magnitudes of the terms and their possible scaling.

Consider the Navier-Stokes equations for incompressible fluid flow:

$$\begin{aligned} \rho(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}) &= -\boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{v} + \rho\boldsymbol{g} \\ \boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0 \end{aligned} \tag{2.1}$$

where $v_j(x_i,t)$, and $p(x_i,t)$ are time and space varying fields, and g_i is a constant body force. We nondimensionalise velocity, time, and pressure with some canonical length L and velocity U as follows:

$$\begin{aligned} &v^* = \frac{v}{U},\ t^* = \frac{tU}{L},\ p^* = \frac{p}{\rho U^2},\ g^* = \frac{g}{U^2/L} \\ &\partial_{t^*} = \frac{L}{U}\partial_t,\ \boldsymbol{\nabla}^* = L\boldsymbol{\nabla} \end{aligned} \tag{2.2}$$

Substituting Equation 2.2 into equation Equation 2.1 and multiplying by $\dfrac{L}{\rho U^2}$, we obtain the Navier-Stokes equations in non-dimensional form.

$$\begin{aligned} \partial_{t^*}\boldsymbol{v}^* + \boldsymbol{v}^*\cdot\boldsymbol{\nabla}\boldsymbol{v}^* &= -\boldsymbol{\nabla}p^* + \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\boldsymbol{v}^* + \boldsymbol{g}^* \\ \boldsymbol{\nabla}^*\cdot\boldsymbol{v}^* &= 0 \end{aligned} \tag{2.3}$$

We drop the asterisks and write in Einstein indicial notation for convenience.

$$\begin{aligned} \partial_t v_j + v_i v_{j,i} &= -p_{,j} + \frac{1}{\mathrm{Re}} v_{j,ii} + g_j \\ v_{i,i} &= 0 \end{aligned} \tag{2.4}$$

2.1 Reynolds Decomposition

Splitting a quantity into its average value and a fluctuation from the average is called a Reynolds Decomposition. For a space and time varying quantity $\phi(x_i,t)$,

$$\phi = \langle\phi\rangle + \phi' \tag{2.5}$$

where $\langle \phi \rangle$ is the ensemble average of ϕ, and ϕ' is fluctuation or deviation of ϕ from $\langle \phi \rangle$. In practice, ensemble averages are obtained by averaging in time and over homogeneous directions.

$$\langle \phi \rangle = \frac{1}{\int \mathrm{d}t} \int \phi \, \mathrm{d}t \tag{2.6}$$

Some properties of Reynolds decomposition are below. Let ϕ and ψ be any quantities. Then,

- Averaging and differentiating commute since average of a derivative is the derivative of the average.

$$\begin{aligned} \langle \phi \rangle_{,i} &= \langle \phi_i \rangle \\ \partial_t \langle \phi \rangle &= \langle \partial_t \phi \rangle \end{aligned} \tag{2.7}$$

- Ensemble of a fluctuation is zero.

$$\begin{aligned} \langle \phi' \rangle &= \langle \phi - \langle \phi \rangle \rangle \\ \langle \phi \rangle &= \langle \phi \rangle - \langle \phi' \rangle \\ \langle \phi' \rangle &= 0 \end{aligned} \tag{2.8}$$

- Ensemble of product is product of ensemble plus ensemble of product of fluctuations

$$\begin{aligned} \langle \phi\psi \rangle &= \big\langle \big(\langle \phi \rangle + \phi' \big) \big(\langle \psi \rangle + \psi' \big) \big\rangle \\ &= \langle \langle \phi \rangle \langle \psi \rangle + \langle \phi \rangle \psi' + \phi' \langle \psi \rangle + \phi' \psi' \rangle \\ &= \langle \phi \rangle \langle v \rangle + \langle \phi \rangle \langle v' \rangle + \langle \phi' \rangle \langle \psi \rangle + \langle \phi' \psi' \rangle \\ &= \langle \phi \rangle \langle \psi \rangle + \langle \phi' \psi' \rangle \end{aligned} \tag{2.9}$$

2.2 Reynolds Stresses and Turbulence Budgets

We take the expected value of the continuity equation to find that $\langle v_i \rangle$ is divergence free.

$$\begin{aligned} \langle 0 \rangle &= \langle v_{i,i} \rangle \\ 0 &= \langle v_i \rangle_{,i} \end{aligned} \tag{2.10}$$

Similarly, we expand the continuity equation in terms of expected values of fluctuations,to find that v_i' is also divergence free.

$$\begin{aligned} 0 &= v_{i,i} = \cancelto{0}{\langle v_i \rangle_{,i}} + v'_{i,i} \\ 0 &= v'_{i,i} \end{aligned} \tag{2.11}$$

We take the expected value of the momentum equation.

$$\begin{aligned} \partial_t \langle v_j \rangle + \langle v_i \rangle \langle v_{j,i} \rangle + \langle v_i' v'_{j,i} \rangle &= - \langle p_{,j} \rangle + \frac{1}{\mathrm{Re}} \langle v_j \rangle_{,ii} + \langle g_i \rangle \\ \partial_t \langle v_j \rangle + \langle v_i \rangle \langle v_{j,i} \rangle &= - \langle p_{,j} \rangle + \frac{1}{\mathrm{Re}} \langle v_j \rangle_{,ii} + \langle g_i \rangle - \langle v_i' v_j' \rangle_{,i} \end{aligned} \tag{2.12}$$

$\eta_{ij} = \langle v_i' v_j' \rangle$ is called the Reynolds stress tensor. We apply Reynolds decomposition to the momentum equation.

$$\begin{aligned} \partial_t(\langle v_j \rangle + v_j') + (\langle v_i \rangle + v_i')(\langle v_{j,i} \rangle + v'_{j,i}) &= -(\langle p_{,j} \rangle + p_j') + \frac{1}{\mathrm{Re}}(\langle v_{j,ii} \rangle + v'_{j,ii}) + \langle g_j \rangle + \cancelto{0}{g_j'} \\ \partial_t(\langle v_j \rangle + v_j') + \langle v_i \rangle \langle v_{j,i} \rangle + \langle v_i \rangle v'_{j,i} + v_i' \langle v_{j,i} \rangle + v_i' v'_{j,i} &= -(\langle p_{,j} \rangle + p_j') + \frac{1}{\mathrm{Re}}(\langle v_{j,ii} \rangle + v'_{j,ii}) + \langle g_j \rangle \end{aligned} \tag{2.13}$$

Subtracting Equation 2.12 from Equation 2.13, we get

$$\partial_t v'_j + \langle v_i \rangle v'_{j,i} + v'_i \langle v_{j,i} \rangle + v'_i v'_{j,i} - \langle v'_i v'_{j,i} \rangle = -p'_{,j} + \frac{1}{\mathrm{Re}} v'_{j,ii} \tag{2.14}$$

Multiply both sides by v'_k and take the expected value.

$$\langle v'_k \partial_t v'_j \rangle + \langle v_i \rangle \langle v'_k v'_{j,i} \rangle + \langle v'_k v'_i \rangle \langle v_{j,i} \rangle + \langle v'_k v'_i v'_{j,i} \rangle - \cancelto{0}{\langle v'_k \rangle} \langle v'_i v'_{j,i} \rangle = - \langle v'_k p'_{,j} \rangle + \frac{1}{\mathrm{Re}} \langle v'_k v'_{j,ii} \rangle \tag{2.15}$$

Take the transpose and add.

$$\begin{aligned} \partial_t \langle v'_j v'_k \rangle + \langle v_i \rangle \langle v'_j v'_k \rangle_{,i} + \langle v'_k v'_i \rangle \langle v_{j,i} \rangle + \langle v'_j v'_i \rangle \langle v_{k,i} \rangle + \langle v'_i v'_j v'_k \rangle_{,i} \\ = -\big(\langle v'_k p'_{,j} \rangle + \langle v'_j p'_{,k} \rangle \big) + \frac{1}{\mathrm{Re}} \langle v'_j v'_k \rangle_{,ii} - \frac{2}{\mathrm{Re}} \langle v'_{j,i} v'_{k,i} \rangle \end{aligned} \tag{2.16}$$

We write the pressure transport term in terms of the pressure strain and pressure diffusion terms.

$$v'_k p'_{,j} + v'_j p'_{,k} = -p'(v'_{j,k} + v'_{k,j}) + (p' v'_j)_{,k} + (p' v'_k)_{,j} \tag{2.17}$$

We finally arrive at the tensor equation describing the behaviour of Reynolds Stresses over time.

$$\begin{aligned} \partial_t \langle v'_j v'_k \rangle + \langle v_i \rangle \langle v'_j v'_k \rangle_{,i} + \langle v'_k v'_i \rangle \langle v_{j,i} \rangle + \langle v'_j v'_i \rangle \langle v_{k,i} \rangle + \langle v'_i v'_j v'_k \rangle_{,i} \\ = \langle p'(v'_{j,k} + v'_{k,j}) \rangle - \langle (p' v'_j)_{,k} + (p' v'_k)_{,j} \rangle + \frac{1}{\mathrm{Re}} \langle v'_j v'_k \rangle_{,ii} - \frac{2}{\mathrm{Re}} \langle v'_{j,i} v'_{k,i} \rangle \end{aligned} \tag{2.18}$$

$$\begin{aligned} \partial_t \eta_{jk} + \langle v_i \rangle \eta_{jk,i} + \eta_{ki} \langle v_{j,i} \rangle + \eta_{ji} \langle v_{k,i} \rangle + \langle v'_i v'_j v'_k \rangle_{,i} \\ = \langle p'(v'_{j,k} + v'_{k,j}) \rangle - \langle (p' v'_j)_{,k} + (p' v'_k)_{,j} \rangle + \frac{1}{\mathrm{Re}} \eta_{jk,ii} - \frac{2}{\mathrm{Re}} \langle v'_{j,i} v'_{k,i} \rangle \end{aligned} \tag{2.19}$$

To obtain the equation for Turbulent Kinetic Energy, we consider one-half of the trace of Equation 2.18 by multiplying with $\frac{1}{2}\delta_{ij}$.

$$\begin{aligned} \partial_t k + \langle v_i \rangle k_{,i} + \langle v'_j v'_i \rangle \langle v_{j,i} \rangle + \langle v'_i v'_j v'_j \rangle_{,i} &= \cancelto{0}{\langle p' v'_{i,i} \rangle} - \langle p' v'_j \rangle_{,j} + \frac{1}{\mathrm{Re}} k_{,ii} - \frac{1}{\mathrm{Re}} \langle v'_{j,i} v'_{j,i} \rangle \\ \partial_t k + \langle v_i \rangle k_{,i} + \langle v'_j v'_i \rangle \langle v_{j,i} \rangle + \langle v'_i v'_j v'_j \rangle_{,i} &= - \langle p'_{,j} v'_j \rangle - \cancelto{0}{\langle p' v'_{i,i} \rangle} + \frac{1}{\mathrm{Re}} k_{,ii} - \frac{1}{\mathrm{Re}} \langle v'_{j,i} v'_{j,i} \rangle \\ \partial_t k + \langle v_i \rangle k_{,i} + \langle v'_j v'_i \rangle \langle v_{j,i} \rangle + \langle v'_i v'_j v'_j \rangle_{,i} &= - \langle p'_{,j} v'_j \rangle + \frac{1}{\mathrm{Re}} k_{,ii} - \frac{1}{\mathrm{Re}} \langle v'_{j,i} v'_{j,i} \rangle \end{aligned} \tag{2.20}$$

Equation 2.18 and Equation 2.20 describe the generation, transport and decay of Reynolds stresses in a flow. For each η_{ik} in the symmetric Reynolds stress tensor, and for k, we label the terms in Equation 2.18 and Equation 2.20.

Table 2.1: Budgets for Reynolds Stresses

Reynolds Stress Expression	**Budget Term**	**TKE Expression**
$\langle v'_j v'_k \rangle$	Reynold Stress	$k = \frac{1}{2} \langle v'_j v'_j \rangle$
$\langle v_i \rangle \langle v'_j v'_k \rangle_{,i}$	Convection	$\langle v_i \rangle k_{,i}$
$- \langle v'_j v'_i \rangle \langle v_{k,i} \rangle - \langle v'_k v'_i \rangle \langle v_{j,i} \rangle$	Production	$- \langle v'_j v'_i \rangle \langle v_{j,i} \rangle$
$- \langle v'_i v'_j v'_k \rangle_{,i}$	Turbulent Diffusion	$\langle v'_i v'_j v'_j \rangle_{,i}$
$- \left\langle v'_k p'_{,j} + v'_j p'_{,k} \right\rangle$	Pressure Transport	$- \langle v'_j p'_j \rangle$
$- \langle (p' v'_j)_{,k} + (p' v'_k)_{,j} \rangle$	Pressure Diffusion	$- \langle v'_j p'_j \rangle$
$\left\langle p'(v'_{j,k} + v'_{k,j}) \right\rangle$	Pressure Strain	0
$\frac{1}{\mathrm{Re}} \langle v'_j v'_k \rangle_{,ii}$	Viscous Diffusion	$\frac{1}{\mathrm{Re}} k_{,ii}$
$-\frac{2}{\mathrm{Re}} \left\langle v'_{j,i} v'_{k,i} \right\rangle$	Viscous Dissipation	$\frac{-1}{\mathrm{Re}} \langle v'_{j,i} v'_{j,i} \rangle$

The convection term describes the transport of Reynolds stresses due to the motion of the mean flow $\langle v_i \rangle$. The production term arises from interactions between fluctuations and shearing of the mean flow, resulting in a net transfer of energy from mean to turbulent fluctuations. The viscous diffusion term, as the name suggests, describes the diffusion of Reynolds stresses in the flow domain.

Chapter 3

Analysis Repository

3.1 $\mathbb{R}$

Relations, Sets, and Functions

Definition 3.1.1 (Relation)**.** A relation $\sim$ (or R) on nonempty set X is a subset of the set $X \times X$. For a, $b \in X$, if $a \sim b$, then $(a, b) \in R$.

Properties of relations: $\forall a$, $b\, c \in X$,

1. Reflexive: $a \sim a$
2. Symmetric: $a \sim b \iff b \sim a$
3. Transitive:
$$\left.\begin{aligned} a \sim b \\ b \sim c \end{aligned}\right\} \implies a \sim c \tag{3.1}$$
4. Antisymmetric:
$$\left.\begin{aligned} a \sim b \\ b \sim a \end{aligned}\right\} \implies a = b \tag{3.2}$$

A relation that is reflexive, symmetric, and transitive is called equivalent relation. For $a \in X$, the equivalent class of a is
$$[a] := \{x \in X | a \sim x\} \tag{3.3}$$
Equivalent classes of elements in X form a partition of X. An ordered relation is a relation that is reflexive, transitive, and antisymmetric, usually denoted by $\leq$.

todo: functions (one-one, onto), fields, ordered fields, vector spaces

Theorem 3.1.1 (Pigeonhole Principle)**.** If n items are put into m containers, with $n > m$, then there exists at least one container with more than one item.

Theorem 3.1.2. For $a \in \mathbb{R} \setminus \mathbb{Q}$, the set $[na] = \{[na] | \in \mathbb{N}\}$ of fractional parts is dense in $[0, 1]$.

Proof. Fix > 0. In this proof, $[\cdot]$ denotes the the fractional part of the argument. We aim to show that for every $x \in [0,1] \exists n$ such that $|x - [na]| < \epsilon$. Starting with $x = 0$, fix M such that $\frac{1}{M} < \epsilon$. If $[0,1]$ is partitioned into M panels of size $\frac{1}{M}$, then, by the Pigeonhole principle, $\exists n_1, n_2 \in \{1, \ldots, M+1\}$ such that $[n_1 a]$, $[n_2 a]$ lie in the same panel. $\implies [(n_2 - n_1)a] < \frac{1}{M} < \epsilon$. Hence 0 is a limit point of $[na]$.

Now consider $x \in (0,1]$. Find n such that $[na] < \frac{1}{M} < \epsilon$. If $x \in [0, \frac{1}{M}]$, we are done. Else, let x belong to the j^{th} panel: $x \in [\frac{j}{M}, \frac{j+1}{M}]$. Since $[na] < \frac{1}{M}, \exists p \in \mathbb{N}$ such that $p[na]$ belongs to the $(j-1)^{\text{th}}$ panel. Setting $p = \sup\{s \in \mathbb{N} | s[na] < \frac{j}{M}\}$. $\implies |x - [(p+1)na]| < \frac{1}{M} < \epsilon$. ☐

todo: $\mathbb{Q}$ todo: Construction: Dedekind's Cuts todo: The completion of $\mathbb{Q}$

Theorem 3.1.3 (Archimedean Property)**.** $\forall x, y \in \mathbb{R}^+ \exists N \in \mathbb{N} : y < nx$

Proof. Fix x, $y \in \mathbb{R}$. Let $A = \mathbb{N}x$. Consider, *ad absurdum,* $\forall n \in \mathbb{N}$, $y \geq nx$. That is, y is an upper bound for A. Let $\alpha = \sup A = n\alpha$ for some $n \in \mathbb{N}$. As $x > 0$, $\alpha < \alpha + x = (n+1)x \in A$. Therefore α is not an upper bound for A. ☐

Theorem 3.1.4. $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e. between any two reals, there exists a rational number.

Proof. Let $x, y \in \mathbb{R}, x < y$. Without loss of generality, consider the case where $x, y > 0$. Applying the Archimedean property, we get

$$\begin{aligned} &\exists n \in \mathbb{N}, n(y-x) > 1 \\ &\implies x < x + \frac{1}{n} < y \\ &\implies x < \frac{nx+1}{n} < y \end{aligned} \tag{3.4}$$

Since the numerator is not guaranteed to be rational, we seek $m \in \mathbb{N}$ such that $nx < m < nx + 1$. Apply the Archimedean property again to find positive integer m_1 such that $0 < nx < m_1$. Hence there exists $0 < m \leq m_1$ such that

$$\begin{aligned} &m < nx < m+1 \\ &nx < m < nx + 1 < ny \\ &x < \frac{m}{n} < y \end{aligned} \tag{3.5}$$

☐

Definition 3.1.2 (Dense subset)**.** A set $S \subset X$ is dense in X if every element in X is a limit point of S.

The real number system is an ordered field $\mathbb{R}$, a complete extension of $\mathbb{Q}$ that has the least upper bound property.

Definition 3.1.3 (Least Upper Bound Property)**.** For $S \subset \mathbb{R}$, an upper bound of S in $\mathbb{R}$ is an element $x \in \mathbb{R}$ such that $\forall s \in S, s < x$. The least upper bound of S in $\mathbb{R}$ is an element y such that

1. y is an upper bound for S
2. if x is an upper bound for S, then $y < x$.

The least upper bound property of $\mathbb{R}$ is that any nonempty set of real numbers that is bounded from above has a least upper bound.

An ordered set satisfies the completeness axiom if every subset S that is bounded above has a least upper bound denoted sup$S \in \mathbb{R}$. If $-S = \{-s | s \in S\}$, then $\inf(S) = -\sup(-S)$

Theorem 3.1.5 (Knaster-Tarski Fixed Point Theorem)**.** Consider a set $X \subset \mathbb{R}$, for which $a = \inf(X)$, $b = \sup(X) \in X$, then every increasing function $f : X \to X$ has at least one fixed point, i.e. $x_0 \in X$ such that $f(x_0) = x_0$.

Proof. Consider the set $S = \{x \in X | x \leq f(x)\}$. S is nonempty since $a \in S$. Hence $\beta = \sup(S) \in X$ must exist.

$$\begin{aligned}
&\forall x \in S, x \leq \beta \\
\implies &x \leq f(x),\ f(x) \leq f(\beta) && (f \text{ increasing function}) \\
\implies &\forall x \in S,\ x \leq f(\beta) \\
\implies &\beta \leq f(\beta) && (\text{since } \beta = \sup(S)\ ) \\
\implies &f(\beta) \leq f\left(f(\beta)\right)
\end{aligned} \tag{3.6}$$

∎

We define the following for $x, y \in \mathbb{R}$

$$\begin{aligned}
|x| &= \begin{cases} x, & x > 0 \\ -x, & x > 0 \end{cases} \\
\min(x) &= \frac{x + y - |x - y|}{2} \\
\max(x) &= \frac{x + y + |x - y|}{2}
\end{aligned} \tag{3.7}$$

Definition 3.1.4 (Extended Real Line)**.** We define $\infty := \sup \mathbb{R}$ and $-\infty := \inf \mathbb{R}$, and call $\mathbb{R} \cup \{-\infty, \infty\}$ the extended real line.

Theorem 3.1.6 (Existence of n^{th} root)**.** $\forall a \in \mathbb{R}^+, \exists b \in \mathbb{R}+$ such that $b^n = a$

Proof. <u>Uniqueness:</u> Say $b^n = c^n = a$.

$$\implies\ 0 = b^n - a^n = (b - c)\left(b^{n-1} + ab^{n-2} + \cdots + a^{n-1}\right) \tag{3.8}$$

Since a, b, $c > 0$, $b - c = 0$. <u>Existence:</u> Consider the function $f : [0, a + 1] \to \mathbb{R}$

$$f(x) = x + \frac{a - x^n}{n(a + 1)^{n-1}} \tag{3.9}$$

Observe that $f(x)$ is increasing, and that all fixed points of f satisfy $x = a^n$. To prove that f is increasing, let $0 < x \leq y \leq a + 1$

$$\begin{aligned}
f(y) - f(x) &= (y - x) - \frac{y^n - x^n}{n(a + 1)^{n-1}} \\
&= (y - x)\left(1 - \frac{\sum_{k=0}^{n-1} x^k y^{n-1-k}}{n(a + 1)^{n-1}}\right) \\
&\geq 0
\end{aligned} \tag{3.10}$$

Now we eliminate $x = 0$, $a + 1$ as fixed points, and apply the fixed point theorem.

$$0 < f(0) = \frac{a}{n(a + 1)^{n-1}} < f(a + 1) = \frac{(n - 1)(a + 1)^n + a}{n(a + 1)^{n-1}} < \frac{(n - 1)(a + 1)^n + (a + 1)^n}{n(a + 1)^{n-1}} = a + 1 \tag{3.11}$$

∎

3.2 Sequences

Definition 3.2.1 (Sequence)**.** A sequence $(a_n)_{n\in\mathbb{N}}$ of elements in a set X is a function $a.: \mathbb{N} \to X$.

Definition 3.2.2 (Limit of a sequence)**.** A sequence (x_n) converges to $x \in X$ in norm $\|\cdot\|$, abbreviated as $x_n \xrightarrow{n} x$, if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \|x - x_n\| < \epsilon \tag{3.12}$$

The element $\lim_{n\to\infty} x_n = x$ is called the limit of sequence x_n.

We will consider sequences of real numbers under the absolute value norm. Below listed are some properties of convergent sequences.

1. The limit of a convergent sequence is unique.

   Proof. $\epsilon > 0$. Let $x_n \to a$, $x_n \to b \in \mathbb{R}$. Without loss of generality, let $b > a$, and fix $\epsilon = \dfrac{b-a}{3}$. Then $\exists N_a, N_b \in \mathbb{N}$ such that

$$\begin{aligned} &\forall n > N_a, |a - x_n| < \epsilon \\ &\forall n > N_b, |b - x_n| < \epsilon \end{aligned} \tag{3.13}$$

   Then, $\forall n > \max(N_a, N_b)$,

$$\implies |b - a| < |a - x_n| + |b - x_n| < (b-a)\frac{2}{3} \Rightarrow\Leftarrow \tag{3.14}$$

   $\square$

2. (a_n) convergent $\implies$ $\{a_n\}_n$ bounded.

   Proof. Let $a_n \to a$. $\forall \epsilon, \exists N$ such that $\forall n > N, |a - a_n| < \epsilon$. Therefore,

$$\forall n, |a_n| < \max\left(\{|a_m|\}_{m=1}^{N-1} \cup \{|a \pm \epsilon|\}\right) \tag{3.15}$$

   $\square$

3. $$\left.\begin{aligned} a_n \to a \\ b_n \to b \end{aligned}\right\} \implies a_n b_n \to ab \tag{3.16}$$

   Proof. Fix $\epsilon > 0$. For $\epsilon > 0$, $\exists N_a$, N_b such that

$$\begin{aligned} &\forall n > N_a |a - a_n| < \epsilon_a > 0 \\ &\forall n > N_b |b - b_n| < \epsilon_b > 0 \end{aligned} \tag{3.17}$$

   Since b_n is a convergent sequence, $\exists |b_n| < M_b \in \mathbb{R}$. Let $\epsilon_a = \dfrac{\epsilon}{2M_b}$, $\epsilon_b = \dfrac{\epsilon}{2a}$. For $n > \max(N_a, N_b)$,

$$\begin{aligned} |ab - a_n b_n| &= |ab + ab_n - ab_n - a_n b_n| \\ &\leq a|b - b_n| + b_n|a - a_n| \\ &< a\frac{\epsilon}{2a} + b_n \frac{\epsilon}{2M_b} \\ &< \epsilon \end{aligned} \tag{3.18}$$

   $\square$

4. $a_n \to a \neq 0 \implies \dfrac{1}{a_n} \to \dfrac{1}{b}$.

Proof. Fix $0 < \epsilon < \frac{|b|}{2}$ (bounding b_n away from 0). $\exists N$ such that $\forall n > N |b - b_n| < \epsilon_b$.

$$\begin{aligned} \implies \left| \frac{1}{b} - \frac{1}{b_n} \right| &= \left| \frac{b - b_n}{bb_n} \right| \\ &< \frac{\epsilon_b}{|b||b_n|} \\ &< \frac{2\epsilon_b}{|b|^2} \end{aligned} \tag{3.19}$$

Allowing $\epsilon_b = \frac{|b|^2 \epsilon}{2}$, we complete the proof. □

5.

Theorem 3.2.1 (Squeeze Theorem)**.** $\forall n > N, L \leftarrow a_n \leq b_n \leq c_n \to L \in \mathbb{R} \implies b_n \to L$.

Proof. $\forall n, |b_n - a_n| < c_n - a_n \to L - L = 0$. Fix $\epsilon > 0$. $\exists N > 0$ such that $\forall n > N, |L - a_n| < \dfrac{\epsilon}{2}$, $|c_n - a_n| < \dfrac{\epsilon}{2}$

$$\begin{aligned} \implies |L - b_n| &= |L - a_n + a_n - b_n| \\ &< |b_n - a_n| + |a_n - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned} \tag{3.20}$$

□

6.

$$\left.\begin{aligned} a_n \to a \\ \forall n, a_n > 0 \end{aligned}\right\} \implies \forall k \geq 1, \sqrt[k]{a_n} \to \sqrt[k]{a} \tag{3.21}$$

Proof. Fix $0 < \epsilon \leq \frac{|a|}{2}$. When $a = 0, \exists N$ such that $\forall n > N |a_n| < \epsilon^k$. $\implies \left| \sqrt[k]{a_n} \right| < \sqrt[k]{\epsilon^k} = \epsilon$. Consider when $a > 0$. Let $\epsilon_a > 0$. $\exists N$ such that $\forall n > N, |a - a_n| < \epsilon_a$

We use the identity

$$A - B = \frac{A^k - B^k}{\sum_{i=0}^{k-1} A^i B^{k-1-i}} \tag{3.22}$$

When A, $B > 0$, we trivially have $|A - B| < \dfrac{\left|A^k - B^k\right|}{B^{k-1}}$

$$\forall n > N, 0 \leq \left| \sqrt[k]{a} - \sqrt[k]{a_n} \right| < \frac{|a - a_n|}{\sqrt[k]{a^{k-1}}} < \frac{\epsilon_a}{\sqrt[k]{a^{k-1}}} \tag{3.23}$$

Allowing $\epsilon_a = \epsilon \sqrt[k]{a^{k-1}}$ completes the proof. □

7. $|a_n| \to 0 \implies a_n \to 0$

Proof. Apply squeeze theorem to $0 \leftarrow (-|a_n|) \leq a_n \leq |a_n| \to 0$. □

8.

Theorem 3.2.2 (Monotone Convergence Theorem)**.** For a_n bounded and monotone, $a_n \nearrow \sup\{a_n\}_n$ or $a_n \searrow \inf\{a_n\}$. (Note on notation: $a_n \nearrow L \iff a_n$ is a monotone increasing sequence , $a_n \to L$.)

Proof. Consider the case when a_n is a monotone increasing sequence. The assumption is without loss of generality, for if a_n is monotone decreasing, we consider $-a_n$ and prove its convergence to $\sup\{-a_n\}_n = -\inf\{a_n\}_n$. By the least upper bound property, we have a unique $L := \sup\{a_n\}_n$. We aim to prove that $\forall \epsilon > 0 \exists N$ such that $\forall n > N, |L - a_n| < \epsilon$. If the statement is not true, then $\exists \epsilon_0 > 0$ such that $\forall k \exists n_k > k, |L - a_{n_k}| \geq \epsilon_0$. Since $a_k \nearrow$, we have $\forall k, a_k \leq a_{n_k} \leq L - \epsilon_0 \implies (L - \epsilon_0) < L$ is an upper bound for $\{a_n\}$. $\Rightarrow\Leftarrow$ ☐

Example 3.2.1. $-1 < a < 1 \implies a^n \to 0$.

The example is trivial when $a = 0$. Consider the case when $0 < |a| < 1$. We consider the case when $0 < |a| < 1 \implies \frac{1}{|a|} > 1$. Let $\frac{1}{|a|} := 1 + h$.

$$\begin{aligned}
\left(\frac{1}{|a|}\right)^n &= (1+h)^n = 1 + nh + \mathcal{O}(h^2) > 1 + nh \\
&\implies |a|^n < \frac{1}{1+nh} \\
&\implies |a|^n \to 0
\end{aligned} \tag{3.24}$$

Since $a^n \leq |a^n| \leq |a|^n$ we conclude that $a^n \to 0$.

Example 3.2.2. $a_n = \sqrt[n]{n}$.

For $n > 1 \implies \sqrt[n]{n} > \sqrt[n]{1} = 1$. For $h_n > 0$, let $a_n = 1 + h_n$. We aim to show that $h_n \to 0$.

$$\begin{aligned}
\implies n = a_n^n &= (1+h_n)^n = 1 + nh_n + \frac{n(n-1)}{2}h_n^2 + + \frac{n(n-1)(n-2)}{3!}h_n^3 \cdots \\
&\geq 1 + \frac{n(n-1)}{2}h_n^2
\end{aligned} \tag{3.25}$$

We get $h_n^2 \leq \frac{2}{n} \implies h_n \to 0$.

Example 3.2.3. $a_n = \dfrac{a^n}{n!}$ for fixed $a > 0$.

We consider the ration $\dfrac{a_{n+1}}{a_n} = \dfrac{a}{n+1} < 1$ for large enough n.

$$\left.\begin{aligned} &(a_n)_{n>n_0} \searrow \\ &\forall n, a_n > 0 \end{aligned}\right\} \implies a_n \to L := \inf(a_n) \geq 0 \tag{3.26}$$

If $L > 0$, $\dfrac{a_{n+1}}{a_n} \xrightarrow{n} \dfrac{L}{L} = 1 = 0 \xleftarrow{n} \dfrac{a}{n+1}$ $\Rightarrow\Leftarrow$. Therefore, $L = 0$.

Example 3.2.4 (Decimal Expansion)**.** Given $d_0 \in \mathbb{Z}, (d_n)_{n>0} \in \{0, 1, 2, \ldots, 9\}$, consider

$$s_n = \sum_{i=0}^{n} \frac{d_i}{10^i} \tag{3.27}$$

The sequence is monotonically increasing, and

$$d_0 \leq s_n \leq s_\infty = d_0 + \sum_{i=1}^{\infty} \frac{d_i}{10^i} \leq d_0 + 9\sum_{i=1}^{\infty} \frac{1}{10^i} \leq d_0 + 9\left(\frac{1}{1 - \frac{1}{10}} - 1\right) = d_0 + 1 \tag{3.28}$$

$$\left.\begin{aligned} d_0 \leq s_n \leq d_0 + 1 \\ s_n \nearrow \end{aligned}\right\} \implies s_n \text{convergent} \tag{3.29}$$

Hence, $s := \lim_n s_n = d_0.d_1d_2\ldots \in [d_0, d_0+1]$. This representation is not unique as

$$0.99\ldots = \sum_{i=1}^{\infty} \frac{9}{10^i} = 9\left(\frac{1}{1-\frac{1}{10}} - 1\right) = 1.00\ldots \tag{3.30}$$

We can recapture the digits in s as follows: $d_0 = \lfloor s \rfloor$. Define $T(x) = \lfloor 10(s-d_0) \rfloor$. Now, $d_1 = T^1(s)$, $d_n = T^n(x)$

Definition 3.2.3 (Cauchy Sequence)**.**

$$a_n \text{Cauchy sequence} \iff \forall \epsilon > 0 \exists N \text{ such that } \forall m,\, n > N, |a_m - a_n| < \epsilon \tag{3.31}$$

The elements in the tail of a cauchy sequence get arbitrarily close to each other. $\forall k, |a_n - a_{n+k}| \xrightarrow{n} 0$

Proposition 3.2.1. a_n convergent $\implies$ a_n cauchy

Proof. Suppose $a_n \to L$. Fix > 0, and sufficiently large m, n, we get $a_m, a_n \in B(L, \frac{\epsilon}{2}) \implies$ maximum distance between a_m, a_n is equal to the diameter of the ball. $\implies |a_m - a_n| <$. ☐

Example 3.2.5 (Harmonic Series)**.** $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Clearly $s_{n+k} - s_n = \sum_{i=1}^{k} \frac{1}{n+i}$.

$$\implies s_{2n} - s_n = \sum_{i=1}^{n} \frac{1}{n+i} \geq n\frac{1}{2n} = \frac{1}{2} \tag{3.32}$$

Since s_n is not a cauchy sequence, it is not convergent either. And since, $s_n \nearrow$, s_n diverges to infinity.

Example 3.2.6. $s_n = 1 + \frac{1}{n^2} + \cdots + \frac{1}{n^2}$

Clearly, $s_n \nearrow$. We try to find an upper bound for s_n

$$\begin{aligned} s_n &< 1 + \sum_{i=1}^{n} \frac{1}{i(i-1)} = 1 + \sum_{i=1}^{n} \frac{1}{i-1} - \frac{1}{i} \\ &= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) \quad = 2 - \frac{1}{n} \end{aligned} \tag{3.33}$$

Hence, s_n convergent, and therefore cauchy.

Example 3.2.7. $a_n = \sqrt[n]{n!}$

We aim to show that the tail of a_n is larger than any arbitrarily chosen real. To do so, fix integer $M > 1$. Now, $\forall n > 2M$,

$$\begin{aligned} n! &= 1 \cdot 2 \cdots M \cdot (M+1) \cdots (2M) \\ a_n &\geq M^{\frac{n-M}{n}} = M\frac{1}{\sqrt[n]{M^M}} \end{aligned} \tag{3.34}$$

As $\sqrt[n]{M^M} \to 1, \exists N > 2M$ such that $\forall n > N \sqrt[n]{M^M} > \frac{1}{2}$. Therefore, $\forall n > N, a_n > \dfrac{M}{2}$.

Example 3.2.8. $s_n = s_n(x) = 1 + \dfrac{x}{1} + \dfrac{x^2}{2} + \cdots \dfrac{x^n}{n!}$

$$
\begin{aligned}
s_{n+k} - s_n &= \sum_{i=1}^{k} \frac{x^{n+i}}{(n+i)!} = \frac{x^{n+1}}{(n+1)!}\left(1 + \frac{x}{n+2} + \frac{x^2}{(n+2)(n+3)} + \cdots + \frac{x^{k-1}}{(n+2)\cdots(n+k)}\right) \\
|s_{n+k} - s_n| &\leq \frac{|x|^{n+1}}{(n+1)!}\left(1 + \frac{|x|}{n+2} + \frac{|x|^2}{(n+2)^2} + \cdots + \frac{|x|^{k-1}}{(n+2)^{k-1}}\right) \\
&\leq \frac{|x|^{n+1}}{(n+1)!}\sum_{j=0}^{\infty}\left(\frac{|x|}{n+2}\right)^j \\
&= \frac{|x|^{n+1}}{(n+1)!}\frac{1}{1-\frac{|x|}{n+2}}
\end{aligned} \tag{3.35}
$$

We look at the upper bound in the limit $n \to \infty$.

$$
\lim_n \frac{|x|^{n+1}}{(n+1)!}\frac{1}{1-\frac{|x|}{n+2}} = \lim_n \frac{|x|^{n+1}}{(n+1)!}\lim_n \frac{1}{1-\frac{|x|}{n+2}} = 0 \cdot 1 = 0 \tag{3.36}
$$

Hence s_n is cauchy and therefore convergent.

Theorem 3.2.3. Cauchy $\implies$ bounded.

Proof. Fix $\epsilon > 0$. For cauchy sequence $a_n, \exists N$ such that $\forall n > N, \forall k, |a_{n+k} - a_n| < \epsilon$. The set $\{a_n\}_{n\in\mathbb{N}}$ lies in the interval $[-M, M]$ where

$$
M := \max\left(\{|a_n|\}_{n<N} \cup \{|a_N| + \epsilon\}\right) \tag{3.37}
$$

□

Definition 3.2.4 (Limit Inferior and Limit Superior)**.** Let a_n be a sequence.

$$
\begin{aligned}
\liminf_{x\to 0} &= \lim_{n\to\infty} \sup_{k>n} a_k \\
\limsup_{x\to 0} &= \lim_{n\to\infty} \inf_{k>n} a_k
\end{aligned} \tag{3.38}
$$

Limit inferior, and limit superior, are the limits of the infimum/supremum of the tail of the sequence $(a_k)_{k>n}$ in the limit $n \to \infty$. For every n, it is always true that a finite number of elements lie in the discarded part of the sequence, $(k < n)$, and an infinite number of elements lie in the tail, $(k > n)$, and $\liminf$, $\limsup$ are the bounds of the infinitely long tail. Consider ,$\forall k$, the truncated sequence $(a_n)_{n\geq k}$ and the set $A_k := \{a_n\}_{n\geq k}$. Define $u_k = \inf A_k$, $v_k = \sup A_k$. Because

$$
A_1 \supset A_2 \supset A_3 \cdots A_{k-1} \supset A_k \tag{3.39}
$$

we have,

$$
u_1 \leq u_2 \cdots u_{k-1} \leq u_k \leq \cdots \leq v_k \leq v_{k+1} \cdots \leq v_2 \leq v_1 \tag{3.40}
$$

If (a_n) is bounded, $-\infty < u_1 \leq v_1 < \infty \implies u_n, v_n$ convergent (by monotone convergence theorem). Therefore,

$$
\begin{aligned}
\liminf_n a_n &= \lim_k u_k = \sup_k u_k \\
\limsup_n a_n &= \lim_k v_k = \inf_k v_k
\end{aligned} \tag{3.41}
$$

Another way to look at $\liminf$, $\limsup$ is the following: $L_i = \liminf_n a_n$ if

1. $\forall x < L_i$, there exist infinite number of a_n greater than x
2. $\forall x > L_i$, there exist infinite number of a_n less than x

Some properties of $\liminf, \limsup$ are as follows:

1. $\limsup_n a_n = -\liminf_n(-a_n)$
2. $\limsup_n(a_n + b_n) \leq \limsup_n a_n + \limsup_n b_n$ (follows from $\sup(A+B) \leq \sup A + \sup B$)

Lemma 3.2.4 (Nested Interval Property)**.** Let $I_n = [a_n, b_n] \subset \mathbb{R}$ be a sequence of intervals such that I_n are nested as follows:

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots \tag{3.42}$$

Then, $\cap_n I_n \neq \varnothing$. Further, if length of the interval is going to zero with n, i.e., $(b_n - a_n) \to 0$, then $\cap_n I_n = \{x_0\}$ where $a_n, b_n \to x_0$.

Proof. todo: □

Definition 3.2.5 (Subsequence)**.** A subsequence of a sequence a_n is the sequence $(b_k)_k = (a_{n_k})_k$ where $n_\cdot : \mathbb{N} \to \mathbb{N}$ with the restriction that $\forall k, k \leq n_k$.

Definition 3.2.6 (Limit Point)**.** (Also called accumulation point) A limit point a_n is a limit of convergence of a subsequence $(a_{n_k})_k$.

Theorem 3.2.5 (Bolzano-Weierstrass)**.**

Lemma 3.2.6 (Finite intersection property)**.**

todo: those silly tests

Theorem 3.2.7 (Rearrangement theorem)**.**

Theorem 3.2.8 (Fubini-Tonelli)**.** Summability of infinite matrices

Theorem 3.2.9 (Summability of infinite matrices)**.**

3.3 Sets, Spaces and Functions

For now, we restrict ourselves to sets in finite dimensional spaces.

Norm

Theorem 3.3.1 (Cauchy-Schwarz inequality)**.**

Open, closed/ closure, interior, etc.

DeMorgan's Laws (Unions and Intersections)

The Cantor set

Definition 3.3.1 (Open Ball)**.** The open ball of radius r around $x_0 \in X$ is defined as follows:

$$B(x_0, r) = \{x \in X | \|x - x_0\| < r\} \subset X \tag{3.43}$$

The spirit of compactness is "any bounded sequence has a convergent subsequence."

Definition 3.3.2 (Continuity)**.** We say a function $f : X \to Y$ is continuous at $x_0 \in X$

Definition 3.3.3 (Compact Set)**.** A set

Definition 3.3.4 (Precompact Set)**.** A set in a normed vector space X is precompact (also called relatively compact set) if its closure is compact.

equivalance between different definitions of continuity.

todo: darboux integral, shuffling limits of integrals, prove everything from calculus

Definition 3.3.5 (Fixed Point Theorem)**.** A **fixed point** of $T : X \to X$ is an element $x_0 \in X$ such that $T(x_0) = x_0$. A fixed point is **globally attracting** if $\forall x \in X, T^n(x) \xrightarrow{n} x_0$

We call T a **contraction** if $\forall x_1, x_2 \in X, d(Tx_1, Tx_2) \le \alpha d(x_1, x_2), \alpha \in [0, 1)$

Theorem 3.3.2 (Contraction Mapping Principle)**.** For a contraction map $T : X \to X$ on a complete metric space X, there exists a unique fixed point $x_0 \in X$. The fixed point is globally attracting with geometric convergence rate.

Proof. Uniqueness: Suppose $x_1, x_2 \in X$ such that $x_1 = Tx_1$, $x_2 = Tx_2$. Then T is no longer a contraction since $d(Tx_1, Tx_2) = d(x_1, x_2)$.

$\forall x \in X$, we show that the sequence $T^n x$ is cauchy due to the contraction property.

$$d(T^{n+1}x, T^n x) \le \alpha d(T^n x, T^{n-1}x) \le \cdots \le \alpha^n d(Tx, x) \tag{3.44}$$

This implies, the distance between nonconsecutive iterates is bounded by

$$\begin{aligned} d(T^{n+m}x, T^n x) &\le d(T^{n+m}x, T^{n+m-1}x) + \cdots + d(T^{n+1}x, T^n x) \\ &\le (\alpha^{n+m-1} + \cdots + \alpha^n) d(Tx, x) \\ &\le \alpha^n \frac{1-\alpha^m}{1-\alpha} d(Tx, x) \\ &\le \frac{\alpha^n}{1-\alpha} d(Tx, x) \end{aligned} \tag{3.45}$$

Let $x_0 = \lim_n T^n x$ for some x. $\implies Tx_0 = T \lim_n T^n x = \lim_n T^{n+1} x = x_0$. □

Theorem 3.3.3 (Contraction Power Mapping Principle)**.** For $T : X \to X$, if T^m is a contraction, then T has a fixed point that is globally attracting.

Proof. Uniqueness is clear since any fixed point of T is a fixed point of T^m, and T^m has a unique fixed point that is globally attracting. To prove existence, we observe that

$$\begin{aligned} &\forall x, T^{mn} x \xrightarrow{n} x_0 \\ \implies &T^{mn+k} x \xrightarrow{n} x_0 \\ \implies &d\left(T^{mn+k}x, T^{mn}x\right) \le \alpha^n d\left(T^k x, x\right) \to 0 \end{aligned} \tag{3.46}$$

illustrating that for sufficiently large $N = mn$, $d\left(T^{N+k}x, T^N x\right) \to 0$. □

Application to picard iteration, fredhold and volterra integral equations are straightforward. Existence and uniqueness of solutions are obtained from fixed point theorems by establishing that the operators are contractions (provided we are working in Banach spaces). However contractivity is a rather strong property. We now prove the existence of a fixed point assuming compactness of a mapping.

Definition 3.3.6 (Compact Operator)**.** An operator $T : X \to Y$ over normed vector spaces is compact if it is continuous, and for every bounded sequence $(x_n) \in X$, Tx_n has a convergent subsequence. In other words, every bounded set is mapped to a precompact set.

Chapter 4

Functional Analysis

4.1 Multi-Index Notation

A multi-index is a d-tuple of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_d)$. The degree of a multi-index α is $|\alpha| = \sum_{i=1}^{d} \alpha_i$. Consider a point in d-dimensional real Euclidean space $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. We denote by x^α the monomial $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. If $\mathrm{D}_{x_i} = \partial_{x_i}$, denotes the partial differential operator with respect to variable x_i, then D^α denotes a differential operator of order $|\alpha|$.

$$\mathrm{D}^\alpha = \mathrm{D}_{x_1}^{\alpha_1} \cdots \mathrm{D}_{x_d}^{\alpha_d} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \tag{4.1}$$

For two multi-indices β, α, we say $\beta \le \alpha$ if $\beta_i \le \alpha_i$ for $1 \le i \le d$. In this case, $\alpha \pm \beta$ are also multi-indices with elements $\alpha_i \pm \beta_i$ and order $|\alpha \pm \beta| = |\alpha| \pm |\beta|$. We also denote $\alpha! = (\alpha_1!, \ldots, \alpha_d!)$ and if $\beta \le \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = (\binom{\alpha_1}{\beta_1}, \ldots, \binom{\alpha_d}{\beta_d}) \tag{4.2}$$

Theorem 4.1.1. *Multinomial Theorem*

$$(x_1 + \cdots + x_d)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} x^\alpha \tag{4.3}$$

Theorem 4.1.2 (Leibniz Formula)**.**

$$\mathrm{D}^\alpha(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \mathrm{D}^\beta u \, \mathrm{D}^{\alpha-\beta} v \tag{4.4}$$

Proof. We prove the Leibniz formula by induction on $|\alpha|$. It is trivial to show that the equality holds for $|\alpha| = 0, 1$. Assuming the equality holds for for some $\alpha \ne 0$, we prove that the equality holds for $\alpha + e_i$ where is a multi-index with 1 in the i^{th} position $\forall i \in \{1, \cdots, d\}$ and zeros elsewhere.

$$\begin{aligned} \mathrm{D}^{\alpha+e_i}(uv) &= \mathrm{D}^{e_i}\left(\sum_{\beta \le \alpha} \binom{\alpha}{\beta} \mathrm{D}^\beta u \, \mathrm{D}^{\alpha-\beta} v\right) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \mathrm{D}^{e_i}\big(\mathrm{D}^\beta u \, \mathrm{D}^{\alpha-\beta} v\big) \\ &= \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \mathrm{D}^{\beta+e_i} u \, \mathrm{D}^{\alpha-\beta} v + \binom{\alpha}{\beta} \mathrm{D}^\beta u \, \mathrm{D}^{\alpha-\beta+e_i} v \end{aligned} \tag{4.5}$$

□

Theorem 4.1.3. *(Taylor's Formula)* For sufficiently smooth functions $f : \mathbb{R}^d \to \mathbb{R}$,

$$f(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} \mathrm{D}^\alpha f(x_0)(x - x_0)^\alpha + \mathcal{O}(|x|^{k+1}) \tag{4.6}$$

4.2 Spaces and Norms

For the purposes of this text, a topological vector space (TVS) is a vector space for which the vector operations of addition and scalar multiplication are continuous. That is for $x, y \in X$, $c \in \mathbb{C}$ where X is some TVS, the maps $(x, y) \to x + y$ and $(c, x) \to cx$ are continuous.

A **norm** on a vector space X is a real-valued functional $\|\cdot\| : X \to \mathbb{R}$ such that for $x, y \in X$, $c \in \mathbb{C}$,

- $\|x\| \ge 0$ with equality holding iff $x = 0$.
- $\|cx\| = |c|\|x\|$
- $\|x + y\| \le \|x\| + \|y\|$

A TVS is normable if its topology coincides with that induced by some norm. Two norms $\|\cdot\|_1$, $\|\cdot\|_2$ are equivalent if for all $x \in X$, there exists some $c \in \mathbb{R}$ such that

$$c\|x\|_1 \le \|x\|_2 \le \frac{1}{c}\|x\|_1 \tag{4.7}$$

If X is a normed space and all Cauchy sequences converge to a limit in X with respect to the norm, then X is called a **Banach space**. A vector space X is called **pre-Hilbert** if there exists a functional called the **scalar product** $(\cdot, \cdot) : X \times X \to \mathbb{R}$ such that $\forall u, v \in X$,

- $(u, u) \ge 0. \, (u, u) = 0 \iff u = 0$
- $(u, v) = (v, u)$
- $(u_1 + u_2, v) = (u_1, v) + (u_2, v)$, $u_1, u_2 \in X$
- $(\lambda u, v) = \lambda(u, v)$, $\lambda \in \mathbb{R}$.

The scalar product induces a norm $\|u\| = \sqrt{(u, u)}$. Another functional defined on a vector space X is the **inner product** $(\cdot, \cdot) : X \times X \to \mathbb{C}$ such that for $x, y, z \in X$, $a, b \in \mathbb{C}$,

- $(x, y) = \overline{(y, x)}$
- $(ax + by, z) = a(x, z) + b(y, z)$
- $(x, x) = 0 \iff x = 0$

If X is a Banach space under a norm that is induced by an inner product, then X is called a **Hilbert space**. Given an inner-product, a norm on X can be specified as follows: for $x \in X$,

$$\|x\| = (x, x)^{1/2} \tag{4.8}$$

We consider maps (or operators) between vector spaces X and Y, $L : X \to Y$. A map is linear if $\forall x_1, x_2 \in X, \forall \alpha \in \mathbb{R}, f(x_1 + \alpha x_2) = f(x_1) + \alpha f(x_2)$. For example, differentiation is a linear map from C^n to C^{n-1}, and integration from L^1_{loc} to $\mathbb{R}$. Every linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ has an associated $m \times n$ matrix A written as follows (where $\underline{e_j}$ are unit vectors in the j^{th} cardinal direction):

$$\underline{y} = A\underline{x} = \begin{bmatrix} L(\underline{e_1}) & \dots & L(\underline{e_n}) \end{bmatrix}_{m\times n} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} = \sum_{j=1}^{n} x_j L(\underline{e_j}) = L\left(\sum_{j=1}^{n} x_j \underline{e_j}\right) = L(\underline{x}) \tag{4.9}$$

A linear operator is said to be bounded if $\exists C > 0$ such that $\|Lx\|_X \leq C\|x\|_X$

Theorem 4.2.1. For linear operators, the following are equivalent.

1. L is bounded
2. L is continuous at one point
3. L is continuous for all inputs

We denote with $\mathcal{L}(X,Y)$ the vector space of bounded linear operators from X to Y. We define the following induced norm on $\mathcal{L}(X,Y)$

$$\|L\| = \sup_{\|x\|_X=1} \|Lx\|_Y \tag{4.10}$$

Theorem 4.2.2. If X and Y are normed vector spaces and Y is complete, then $\mathcal{L}(X,Y)$ is complete.

A map from X to its field, $f : X \to \mathbb{R}$ is called a **functional**. The set of linear functionals on X is called the **dual** of X and is denoted by X', which is also a vector space under pointwise addition and scalar multiplication. We represent the application of a functional on an element as follows:

$$f(x) = \langle f, x\rangle \tag{4.11}$$

The operator norm on dual space X' is

$$\|x'\|_{X'} = \sup_{\|x\|_X=1} |x'(x)| \tag{4.12}$$

Theorem 4.2.3. *(Riesz Representation Theorem)* For a Hilbert space X, a linear functional x' on X belongs in X' if and only if there exists an $y \in X$ such that for every $x \in X$,

$$x'(x) = (x, y)_X \tag{4.13}$$

We consider an open subset $\Omega \in \mathbb{R}^d$ as domain in d-dimensional Euclidean space. For some $S \subset \Omega$, we denote by $\overline{S}$ the closure of S. We write $S \subset\subset \Omega \subset \mathbb{R}^d$ if $\overline{S} \subset \Omega$ and $\overline{S}$ is compact. If u is a function defined on Ω, we define the support of u as

$$\text{supp}(u) = \overline{\{x \in \Omega | u(x) \neq 0\}} \tag{4.14}$$

We say $A \subset\subset B$ if and only if $\overline{A}$ is a compact subset of B. If, for some function u, $\text{supp}(u) \subset\subset \Omega$, we say that u has **compact support** in Ω. A measurable function u on Ω is said to be essentially bounded on Ω if there exists a constant K such that $|u(x)| \leq K$ a.e. on Ω. The greatest lower bound of such constants K is called the essential supremum of $|u|$ on Ω and is denoted by $\operatorname{ess\,sup}_{x\in\Omega} |u(x)|$. We denote by $L^\infty(\Omega)$ the vector space consisting of all equivalent classes of functions u (that are equal almost everywhere) which are essentially bounded on Ω.

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \tag{4.15}$$

We define the following family of vector spaces. For some integer m, the vector space $C^m(\Omega)$ consists of all functions defined on Ω which, along with their partial derivatives up to order m are continuous.

$$\begin{aligned} C^m(\Omega) &= \{\phi : \Omega \to \mathbb{R} : \forall \alpha \text{ with } |\alpha| \le m,\ \mathrm{D}^\alpha \phi \text{ is continuous}\} \\ C^\infty(\Omega) &= \cap_{m=0}^{\infty} C^m(\Omega) \end{aligned} \tag{4.16}$$

We abbreviate $C^0(\Omega)$ to $C(\Omega)$. The subspace $C_0^m(\Omega)$ consists of all functions in $C^m(\Omega)$ that have compact support in Ω. We wish to define equivalent spaces for $\overline{\Omega}$, $C^m(\overline{\Omega})$. However, since Ω is open, continuous functions on Ω need not be bounded. Therefore, not all functions in $C^m(\Omega)$ can be continuously extended to $\overline{\Omega}$. For example, the function $f(x) = 1/x$ is continuous on the open interval $(0, 1)$, but discontinuous on $x = 0$. Functions that are uniformly continuous and bounded on Ω can be uniquely and continuously extended to $\overline{\Omega}$. We define $C^(\overline{\Omega})$ as follows:

$$C^m(\overline{\Omega}) = \{\phi \in C^m(\Omega) : \forall \alpha,\ |\alpha| \le m,\ \mathrm{D}^\alpha \phi \text{ bounded and uniformly continuous on } \Omega\} \tag{4.17}$$

A function $f : \Omega \to \mathbb{R}$ is said to be **Hölder continuous** for exponent γ $(0 < \gamma \le 1)$ if

$$\exists K \ge 0\, \forall x, y \in \Omega, x \ne y, \frac{|f(x) - f(y)|}{\|x - y\|} \le K \tag{4.18}$$

We define $C^{m,\gamma}(\overline{\Omega})$ to be the subspace of $C^m(\overline{\Omega})$ consisting of functions for which, for $|\alpha| = m$, $\mathrm{D}^\alpha \phi$ satisfies the Hölder condition. We define the following norms and seminorms:

$$\|u\|_{C^0(\overline{\Omega})} = \sup_{x \in \Omega} |\,\mathrm{D}^\alpha u| \qquad |u|_{C^{0,\gamma}(\overline{\Omega})} = \sup_{\substack{x,y \in \Omega \\ x \ne y}} \frac{|u(x) - u(y)|}{\|x - y\|_2} \tag{4.19}$$

$$\|u\|_{C^m(\overline{\Omega})} = \max_{0 \le |\alpha| \le m} \|u\|_{C^0(\overline{\Omega})} \qquad \|u\|_{C^{m,\gamma}(\overline{\Omega})} = \|u\|_{C^m(\overline{\Omega})} + \max_{|\alpha| = m} |\,\mathrm{D}^\alpha u|_{C^{0,\gamma}(\overline{\Omega})} \tag{4.20}$$

Theorem 4.2.4. $C^{m,\gamma}(\Omega)$ is complete for nonnegative integer m and $0 < \gamma \le 1$ with respect to the norm $\|\cdot\|_{C^{m,\gamma}(\overline{\Omega})}$.

Proof. Observe that $\forall x \in \Omega$, $\{u_n(x)\} \subset \mathbb{R}$ form a Cauchy sequence. By completeness of $\mathbb{R}, \forall x, u_x(x) \to u(x)$. It is left to prove that $\lim_{n\to\infty} \|u - u_n\|_{C^{m,\gamma}(\Omega)} = 0$. □

Let Ω be a domain in $\mathbb{R}^d$ and let $p \in \mathbb{R}^+$. We denote by $L^p(\Omega)$ the class of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |u(x)|^p \,\mathrm{d}x < \infty \tag{4.21}$$

The elements of $L^p(\Omega)$ are equivalent classes of functions that are equal almost everywhere on Ω that satisfy the above inequality. We show that $L^p(\Omega)$ is a vector space. We say $u = 0$ in $L^p(\Omega)$ if u is zero almost everywhere on Ω (i.e. $u(x) = 0$ a.e. in Ω). The restriction to Ω of any piecewise continuous, compactly supported function on $\mathbb{R}^d$ would belong to an equivalent class of functions in $L^p(\Omega)$. For $u, v \in L^p(\Omega)$, $x \in \Omega$, $c \in \mathbb{C}$,

$$|u(x) + v(x)|^p \le (|u(x)| + |v(x)|)^p \le 2^p(|u(x)|^p + |v(x)|^p) \tag{4.22}$$

Hence $u + v \in L^p(\Omega)$. Clearly $cu(x) \in L^p(\Omega)$. Therefore $L^p(\Omega)$ is a vector space. We claim that the function $\|\cdot\|_p$ defined below is a norm on $L^p(\Omega)$ for $1 \le p < \infty$. $\|\cdot\|_p$ is not a norm for $0 < p < 1$.

$$\|u\|_p = \left\{ \int_\Omega |u(x)|^p \,\mathrm{d}x \right\}^{1/p} \tag{4.23}$$

For $u \in L^p(\Omega)$, it is clear that $\|u\|_p \ge 0$ with equality holding if and only if $u = 0$ in $L^p(\Omega)$. Moreover, $\|cu\|_p = |c|\|u\|_p$. It remains to be shown that $\|\cdot\|_p$ satisfies the triangle inequality, in $L^p(\Omega)$, which is known as *Minkowski's inequality*. The condition certainly holds for $p = 1$, since

$$\int_\Omega |u(x) + v(x)| \,\mathrm{d}x \le \int_\Omega |u(x)| \,\mathrm{d}x + \int_\Omega |v(x)| \,\mathrm{d}x \tag{4.24}$$

For $p > 1$, we denote by p' the number $\frac{p}{p-1}$ so that $p' > 1$ and

$$\frac{1}{p} + \frac{1}{p'} = 1 \tag{4.25}$$

We call p' the exponential conjugate to p.

Theorem 4.2.5. *(Hölder's inequality)* For $p > 1$, $u \in L^p(\Omega)$ $v \in L^{p'}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x = \|uv\|_1 \leq \|u\|_p \|v\|_{p'} \tag{4.26}$$

Proof. If either $\|u\|_p = 0$ or $\|v\|_{p'} = 0$ then $u(x)v(x) = 0$ almost everywhere and the inequality is satisfied. Otherwise, consider the function $f : \mathbb{R}^+ \to \mathbb{R}$, $f(t) = t^p/p + 1/p' - t$. The only critical point of the function is at $f(1) = 0$, which is a minimum. Hence for all t,

$$t \leq \frac{t^p}{p} + \frac{1}{p'} \tag{4.27}$$

For $a, b \geq 0$, we substitute $t = ab^{-p'/p}$,

$$\begin{aligned} ab^{-p'/p} &\leq \frac{a^p b^{-p'}}{p} + \frac{1}{p'} \\ ab^{-p'/p+p'} &\leq \frac{a^p b^{-p'+p'}}{p} + \frac{b^{p'}}{p'} \\ ab &\leq \frac{a^p}{p} + \frac{b^{p'}}{p'} \end{aligned} \tag{4.28}$$

with equality occurring if and only if $a^p = b^{p'}$. Substitute $a = \frac{|u(x)|}{\|u\|_p}$, $b = \frac{|v(x)|}{\|v\|_{p'}}$ and integrate over Ω.

$$\begin{aligned} \int_\Omega \frac{|u(x)||v(x)|}{\|u\|_p \|v\|_{p'}}\,\mathrm{d}x \leq \frac{1}{p}\int_\Omega \frac{|u(x)|^p}{\|u\|_p^p}\,\mathrm{d}x + \frac{1}{p'}\int_\Omega \frac{|v(x)|^{p'}}{\|v\|_{p'}^{p'}}\,\mathrm{d}x = \frac{1}{p} + \frac{1}{p'} \\ \int_\Omega |u(x)v(x)|\,\mathrm{d}x \leq \int_\Omega |u(x)||v(x)|\,\mathrm{d}x \leq \|u\|_p \|v\|_{p'} \end{aligned} \tag{4.29}$$

Hence, $uv \in L^1(\Omega)$. □

From Hölder's inequality, we get that each $v \in L^{p'}(\Omega)$ is associated with an element in the dual of $L^p(\Omega)$, $Fu = \int_\Omega uv\,\mathrm{d}x$. The case $p = p' = 2$ is special because every element in $L^2(\Omega)$ can be identified with an element in its dual space $L^2(\Omega)'$.

Theorem 4.2.6. *(Minkowski's Inequality)* Triangle Inequality for $p \geq 1$, $u, v \in L^p(\Omega)$.

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p \tag{4.30}$$

Proof. Since the case with $p = 1$ is trivial, consider $p > 1$. For $u, v \in L^p(\Omega)$, $u + v \in L^p(\Omega)$.

$$\int_\Omega |u(x) + v(x)|^{p'(p-1)}\,\mathrm{d}x = \int_\Omega |u(x) + v(x)|^p\,\mathrm{d}x < \infty \tag{4.31}$$

Hence, $(u+v)^{p-1} \in L^{p'}(\Omega)$.

$$
\begin{aligned}
\|u+v\|_p^p = \int_\Omega |u(x)+v(x)|^p \,\mathrm{d}x &\leq \int_\Omega |u(x)+v(x)|^{p-1}(|u(x)|+|v(x)|)\,\mathrm{d}x \\
\int_\Omega |u(x)+v(x)|^p \,\mathrm{d}x &\leq \left\{\int_\Omega |u(x)+v(x)|^{p'(p-1)}\,\mathrm{d}x\right\}^{1/p'} \left\{\int_\Omega ||u(x)|+|v(x)||^p\,\mathrm{d}x\right\}^{1/p} \\
\left\{\int_\Omega |u(x)+v(x)|^p \,\mathrm{d}x\right\}^{1-1/p'} &\leq \left\{\int_\Omega ||u(x)|+|v(x)||^p\,\mathrm{d}x\right\}^{1/p} \\
\left\{\int_\Omega |u(x)+v(x)|^p \,\mathrm{d}x\right\}^{1/p} &\leq \left\{\int_\Omega |u(x)|^p\,\mathrm{d}x\right\}^{1/p} + \left\{\int_\Omega |v(x)|^p\,\mathrm{d}x\right\}^{1/p} \\
\|u+v\|_p &\leq \|u\|_p + \|v\|_p
\end{aligned} \tag{4.32}
$$

Therefore, $\|\cdot\|_p$ is a norm on $L^p(\Omega)$. □

Theorem 4.2.7 (Hahn-Banach Theorem)**.** Let X be a Banach space and X_0 be a subspace of X. $\forall x_0' \in X_0', \exists\, x' \in X'$ such that x_0 and x' agree on X_0, and $\|x_0'\|_{X_0'} = \|x'\|_X$.

Theorem 4.2.8. If X is a normed vector space, then X', along with the induced operator norm, is a banach space (irrespective of whether X is complete).

Theorem 4.2.9 (Generalised Cauchy Identity)**.** $|(x',x)| \leq \|x\|_X \|x'\|_{X'}$

4.3 Adjoint Operator

We explain how a linear transformation between two normed vector spaces, $L \in \mathcal{L}(X,Y)$, is naturally associated with another transformation from Y' to X' by the **adjoint operator**, $L^* : Y' \to X'$, as follows: $\forall y' \in Y, \exists$ unique $x' \in X'$ such that $\forall x \in X, y' \circ L(x) = x'(x) \implies \forall y' \in Y', \forall x \in X$,

$$
\begin{aligned}
L^* y'(x) &= y' \circ L(x) \in X' \\
y' &\to Ly' \\
\langle y', L(x)\rangle &= \langle L^* y', x\rangle
\end{aligned} \tag{4.33}
$$

The adjoint operator is also bounded since $|y' \circ L(x)| \leq \|y'\|_{Y'} \|L(x)\|_Y \leq \|y'\|_{Y'} \|L\| \|x\|_X$.

We show that for a map $L : \mathbb{R}^n \to \mathbb{R}^m$ with associated matrix A, the matrix of the adjoint is $A^\intercal$. The action of the adjoint operator is one that for every $\underline{b} \in \mathbb{R}^m$ produces a unique $\underline{c} \in \mathbb{R}^n$ such that

$$
\begin{aligned}
&\underline{c}^\intercal \underline{x} = \underline{b}^\intercal A \underline{x}, \quad \forall \underline{x} \in \mathbb{R}^n \\
\Longrightarrow\; &\underline{x}^\intercal (\underline{c} - A^\intercal \underline{b}) = 0 \\
\Longrightarrow\; &\underline{c} = A^\intercal \underline{b}
\end{aligned} \tag{4.34}
$$

Most differential operators will be defined on a subset of $L^2(\Omega)$. For linear operators on space $L^2(\Omega)$, we have

$$
\begin{aligned}
&\langle v', L(u)\rangle = \langle L^* v', u\rangle, \qquad \forall u \in L^2(\Omega), v \in L^2(\Omega)' \\
\Longrightarrow\;\; &(v, Lu) = (L^* v, u), \qquad\qquad \forall u, v \in L^2(\Omega)
\end{aligned} \tag{4.35}
$$

Definition 4.3.1 (Adjoint Operator)**.** For differential operator L, the adjoint is the differential operator L^* that satisfies

$$
(Lu, v) = (u, L^* v) \tag{4.36}
$$

for sufficiently smooth u, v with compact support in Ω.

Theorem 4.3.1. For $L : X \to Y$, $\|L^*\| = \|L\|$

Proof.

$$
\begin{aligned}
|y' \circ L(x)| &\leq \|y'\|\|L\|\|x\|, && \forall x \in X, \forall y' \in Y' \\
\frac{|y' \circ L(x)|}{\|y'\|\|x\|} &\leq \|L\|, && \forall x \neq 0, \forall y' \neq 0 \\
\|L^*\| &\leq \|L\|, && \text{by taking supremum over } x, y'
\end{aligned} \tag{4.37}
$$

Now we prove the converse

$$
\begin{aligned}
|y' \circ L(x)| &\leq \|L^*\|\|y'\|\|x\|, && \forall x, y' \\
\frac{|y'\left(L(x)\right)|}{\|y'\|\|x\|} &\leq \|L^*\|, && \forall x \neq 0, \forall y' \neq 0 \\
\|L\| &\leq \|L^*\|, && \text{by taking supremum over } x, y'
\end{aligned} \tag{4.38}
$$

□

Adjoints are useful in understanding problems like the following: given normed vector spaces X, Y and mapping $\mathcal{L}(X, Y)$ and $b \in Y$, find $x \in X$ such that

$$
Lx = b \tag{4.39}
$$

Consider the $m \times n$ system $Ax = b$. We enlarge the system by adding the adjoint system $c = A^\intercal y$ as follows:

$$
\begin{bmatrix} 0 & A \\ A^\intercal & 0 \end{bmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix} \tag{4.40}
$$

todo: Consider the problem

$$
\begin{cases} Lu = f, & x \in \Omega \\ \text{suitable b.c, i.c,} & x \in \partial\Omega \end{cases} \tag{4.41}
$$

The Green's function $\phi(x, y)$ satisfies

$$
\begin{cases} L^*\phi(x, y) = \delta(x - y), & x \in \Omega \\ \text{adjoint b.c, i.c,} & x \in \partial\Omega \end{cases} \tag{4.42}
$$

We obtain the following representation formula:

$$
\begin{aligned}
u(y) &= (u(x), \delta(x - y)) = (u(x), L^*\phi(x, y)) \\
&= (Lu(x), \phi(x, y)) = (f(x), \phi(x, y)) \\
&= \int_\Omega f(x)\phi(x, y)\,\mathrm{d}x
\end{aligned} \tag{4.43}
$$

We enlarge the system as follows:

$$
\begin{cases} Lu = f, & x \in \Omega \\ \text{suitable b.c, i.c,} & x \in \partial\Omega \\ L^*\phi(x, y) = \delta(x - y), & x \in \Omega \\ \text{adjoint b.c, i.c,} & x \in \partial\Omega \end{cases} \tag{4.44}
$$

4.4 Some Imbeddings

We say that the normed space X is imbedded in the normed space Y and write $X \to Y$ if:

- X is a vector subspace of Y.
- the identity operator, I defined on X into Y is continuous. Since I is linear, it is equivalent to the existence of a constant M such that, for $x \in X$,

$$\|I(x)\|_Y \leq M\|x\|_X \tag{4.45}$$

For nonnegative integer m and $0 < \nu \leq \lambda \leq 1$, we prove the following imbeddings:

- $C^{m+1}(\overline{\Omega}) \to C^m(\overline{\Omega})$
  It is clear that $C^{m+1}(\overline{\Omega}) \subset C^m(\overline{\Omega})$ and $\|\phi\|_{C^m(\overline{\Omega})} \leq \|\phi\|_{C^{m+1}(\overline{\Omega})}$
- $C^{m,\lambda}(\overline{\Omega}) \to C^m(\overline{\Omega})$
  It is clear that $C^{m,\lambda}(\overline{\Omega}) \subset C^m(\overline{\Omega})$ and $\|\phi\|_{C^m(\overline{\Omega})} \leq \|\phi\|_{C^{m,\lambda}(\overline{\Omega})}$
- For $0 < \nu < \lambda \leq 1$, $C^{m,\lambda}(\overline{\Omega}) \to C^{m,\nu}(\overline{\Omega})$
  Consider $\phi \in C^{m,\lambda}(\overline{\Omega})$. There exists K such that for all α with $|\alpha| \leq m$, for all $x, y \in \Omega$, $x \neq y$

$$\begin{aligned} \frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x-y|^\lambda} &< K \\ \frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x-y|^\nu} &< K|x-y|^{\lambda-\nu} \end{aligned} \tag{4.46}$$

  The term $|x-y|^{\lambda-\nu}$ is bounded for bounded Ω. Hence, $C^{m,\lambda}(\overline{\Omega}) \subset C^{m,\nu}(\overline{\Omega})$. Next, we compare the norms of $C^{m,\lambda}(\overline{\Omega})$ and $C^{m,\nu}(\overline{\Omega})$. For some $\phi \in C^{m,\lambda}(\overline{\Omega})$, we note that

$$\begin{aligned} \sup_{\substack{x,y\in\Omega \\ |x-y|<1}} \frac{|\mathrm{D}^\alpha \phi(x) - \mathrm{D}^\alpha \phi(y)|}{|x-y|^\nu} &\leq \sup_{\substack{x,y\in\Omega \\ |x-y|<1}} \frac{|\mathrm{D}^\alpha \phi(x) - \mathrm{D}^\alpha \phi(y)|}{|x-y|^\lambda} \\ \sup_{\substack{x,y\in\Omega \\ |x-y|\geq 1}} \frac{|\mathrm{D}^\alpha \phi(x) - \mathrm{D}^\alpha \phi(y)|}{|x-y|^\nu} &\leq 2 \sup_{x\in\Omega} |\mathrm{D}^\alpha \phi(x)| \end{aligned} \tag{4.47}$$

  Hence, $\|\phi\|_{C^{m,\nu}(\overline{\Omega})} \leq 3\|\phi\|_{C^{m,\lambda}(\overline{\Omega})}$

Hölder's inequality clearly extends to cover this case with $p = \infty$, $p' = 1$. We now prove an imbedding theorem on $L^p(\Omega)$.

Theorem 4.4.1. Suppose $\mathrm{vol}\Omega = \int_\Omega \mathrm{d}x$, and $1 \leq p \leq q \leq \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and

$$\|u\|_p \leq (\mathrm{vol}\Omega)^{1/p-1/q}\|u\|_q \tag{4.48}$$

Hence, $L^q(\Omega) \to L^p(\Omega)$. If $u \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|u\|_p = \|u\|_\infty \tag{4.49}$$

Finally, if $u \in L^p(\Omega)$ for $1 \leq p < \infty$, and if there is a constant K such that for all p, $\|u\|_p \leq K$, then $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leq K$.

Proof. Consider $u \in L^q(\Omega)$. If $p = q$, the imbedding theorem is trivial. For $1 \leq p < q \leq \infty$, Hence $u^p \in L^{q/p}(\Omega)$. Then, Hölder's inequality gives us

$$\begin{aligned} \int_\Omega |u(x)|^p \,\mathrm{d}x &\leq \left\{ \int_\Omega |u(x)|^{p(q/p)} \,\mathrm{d}x \right\}^{p/q} \left\{ \int_\Omega 1 \,\mathrm{d}x \right\}^{1-p/q} \\ \|u\|_p &\leq \mathrm{vol}(\Omega)^{1/p-1/q}\|u\|_q \end{aligned} \tag{4.50}$$

Hence, for $1 \le p \le q \le \infty$, $L^q(\Omega) \to L^p(\Omega)$. If $u \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|u\|_p \le \|u\|_\infty \tag{4.51}$$

On the other hand, $\forall \varepsilon > 0\, \exists A \subset \Omega$ of nonzero measure $\mu(A)$ such that $\forall x \in A$,

$$\begin{aligned} |u(x)| &\ge \|u\|_\infty - \varepsilon \\ \int_\Omega |u(x)|^p \, \mathrm{d}x &\ge \int_A |u(x)|^p \, \mathrm{d}x \ge \mu(A)(\|u\|_\infty - \varepsilon)^p \\ \|u\|_p &\ge (\mu(A))^{1/p}(\|u\|_\infty - \varepsilon) \\ \lim_{p\to\infty} \|u\|_p &\ge \|u\|_\infty \end{aligned} \tag{4.52}$$

Therefore,

$$\lim_{p\to\infty} \|u\|_p = \|u\|_\infty \tag{4.53}$$

Finally, suppose $u \in L^p(\Omega)$ for $1 \le p < \infty$ and $\exists K$ such that for all such p, $\|u\|_p \le K$. Suppose, *ad absurdum*, u is essentially bounded. Then $\exists K_1 > K$, $A \subset \Omega$ with $\mu(A) > 0$ such that $\forall x \in A$, $|u(x)| \ge K_1$. Applying the same argument as before, we get

$$\lim_{p\to\infty} \|u\|_p \ge K_1 \tag{4.54}$$

which is a contradiction. □

Theorem 4.4.2. If Ω is measurable, then $L^p(\Omega)$ is a Banach space for $1 \le p \le \infty$.

Theorem 4.4.3. For $f \in L^p(\Omega) \cap L^q(\Omega)$, $1 \le p \le q \le \infty$, then $\forall r \in [p, q]$, $f \in L^r(\Omega)$ and $\|f\|_r \le \|f\|_p^\alpha \|f\|_q^{1-\alpha}$ where $\dfrac{\alpha}{p} + \dfrac{1-\alpha}{q} = \dfrac{1}{r}$.

Proof. The proof is trivial for $r = p$ or $r = q$. If $p < r < q$, then $\dfrac{1}{p} > \dfrac{1}{r} > \dfrac{1}{q}$. Then, there exists α such that

$$\alpha\frac{1}{p} + (1-\alpha)\frac{1}{q} = \frac{1}{r} \tag{4.55}$$

Then, for any $m, n \ge 1$ with $\dfrac{1}{m} + \dfrac{1}{n} = 1$, we have

$$\int_\Omega |f(x)|^r \, \mathrm{d}x = \int_\Omega |f(x)|^{\alpha r} |f(x)|^{(1-\alpha)r} \le \left\{ \int_\Omega |f(x)|^{r\alpha m} \, \mathrm{d}x \right\}^{1/m} \left\{ \int_\Omega |f(x)|^{r(1-\alpha)n} \, \mathrm{d}x \right\}^{1/n} \tag{4.56}$$

Choose $m = \dfrac{p}{r\alpha}$ and $m = \dfrac{q}{r(1-\alpha)}$ if $\alpha \ne 1$. If $\alpha = 1$, choose $n = \infty$.

$$\int_\Omega |f(x)|^r \, \mathrm{d}x \le \|f\|_p^\alpha \|f\|_q^{1-\alpha} \le \infty \tag{4.57}$$

Therefore $f \in L^r(\Omega)$. □

todo: We now consider the normed dual of $L^p(\Omega)$.

4.5 Measure Theory

The **measure** of a set refers to its size in some measure space. We say a set $S \subset \mathbb{R}^d$ has **measure zero** ($\mu(S) = 0$) if $\forall \varepsilon > 0$, S can be covered by open balls of total volume less than ε. $\mathbb{Q} \subset \mathbb{R}$, C^1 curves in $\mathbb{R}^2$, C^1 surfaces in $\mathbb{R}^3$ are examples of sets with measure zero. A countable union of sets of measure zero has measure zero. We say that a condition holds **almost everywhere (a.e.)** if the points where it does not hold form a set of zero measure. For example, the characteristic function for rationals on the set of real numbers is equal to zero almost everywhere.

Definition 4.5.1 (Measurable function)**.** A function is measurable if it coincides a.e. with the limit of a sequence of piecewise continuous functions which is convergent almost everywhere.

We define the characteristic function of a set $A \subset \mathbb{R}^d$ as follows:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{4.58}$$

We say that a set is measurable if its characteristic function is measurable. $\mathbb{Q}$ is measurable since its characteristic function coincides with zero almost everywhere. A countable union of measurable sets is also measurable.

Definition 4.5.2 (Piecewise Continuous)**.** $g : \mathbb{R}^d \to \mathbb{R}$ is piecewise continuous if there exist disjoint, open and connected domains $\{D_i\}_{i \in I \subset \mathbb{N}}$ with piecewise C^1 boundary such that any sphere can be covered by finitely many $\overline{D_i}$. Further, g is continuous on each D_i and can be continuously extended to the boundary of D_i.

Consider a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, $f(x) \geq 0$. One can construct a nondecreasing sequence of piecewise continuous functions $\{g_k\}$ with compact support, convergent almost everywhere to f. We say that f is **Lebesgue integrable** if the sequence of Riemann integrals

$$\int_{\mathbb{R}^d} g_k(x)\,\mathrm{d}x \tag{4.59}$$

has an upper bound (hence a limit since g_k is a nondecreasing). One can show that $\int_{\mathbb{R}^d} g_k(x)\,\mathrm{d}x$ does not depend on the choice of sequence g_k.

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_{\mathbb{R}^d} g_k(x)\,\mathrm{d}x \tag{4.60}$$

For example, $\chi_{\mathbb{Q}}$ is not Riemann integrable since it discontinuous everywhere, but is Lebesgue integrable since it is equal to zero a.e. We define the Lebesgue integral of a measurable function f over a measurable subset Ω of $\mathbb{R}^d$ as follows:

$$\int_{\Omega} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\chi_\Omega(x)\,\mathrm{d}x \tag{4.61}$$

We say a measurable function $u : \mathbb{R}^d \to \Omega$ is **summable** if $\int_{\mathbb{R}^d} |u|\,\mathrm{d}x < \infty$ and define the set of summable functions

$$L^1(\mathbb{R}^d) = \{u : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^d} |u(x)|\,\mathrm{d}x < \infty\} \tag{4.62}$$

A measurable function is **locally integrable** on $\Omega \subset \mathbb{R}^d$ if it is integrable on any compact $K \subset \Omega$. The set of locally integrable functions on Ω is denoted $L^1_{\text{loc}}(\Omega)$.

$$L^1_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R} : \forall K \subset \Omega, \int_K u(x)\,\mathrm{d}x < \infty\} \tag{4.63}$$

Any piecewise continuous function is locally integrable in $\mathbb{R}^d$. An important fact is that summable functions are “approximately continuous” at every point.

Theorem 4.5.1 (Lebesgue's Differentiation Theorem)**.** Let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then for a.e. point in $\mathbb{R}^d$,

$$\lim_{r\to 0} \int_{B_{x_0}(r)} u(x)\,\mathrm{d}x = u(x_0) \tag{4.64}$$

4.6 Distributions

We restate the divergence theorem: for $u \in C^1(\overline{\Omega})$ in scalar form (this can be derived from the vector form by considering a vector field with scalar u being the i^{th} component, and all other components being zero),

$$\int_{\partial\Omega} u n_i \,\mathrm{d}S = \int_{\Omega} \partial_{x_i} u \,\mathrm{d}x \tag{4.65}$$

where n_i is the i^{th} component of the unit normal vector defined on $\partial\Omega$. We derive the formula for integration by parts: for $u, v \in C^1(\Omega)$,

$$\begin{aligned}\int_{\partial\Omega} uvn_i \,\mathrm{d}S &= \int_{\Omega} \partial_{x_i}(uv)\,\mathrm{d}x = \int_{\Omega} u\partial_{x_i} v + v\partial_{x_i} v \,\mathrm{d}x \\ \int_{\Omega} u\partial_{x_i} v \,\mathrm{d}x &= \int_{\partial\Omega} uvn_i \,\mathrm{d}S - \int_{\Omega} v\partial_{x_i} v \,\mathrm{d}x\end{aligned} \tag{4.66}$$

The space $C_0^\infty(\Omega)$ contains infinitely smooth functions with compact support on Ω. We call functions belonging to $C_0^\infty(\Omega)$ **test functions**. A key feature of test functions is that their extension to $\partial\Omega$ is the zero function as they are continuous and compactly supported in a subset of Ω. Let u, $v \in L^1_{\text{loc}}(\Omega)$ and α be a multi-index. We say that v is the α^{th} **weak derivative** of u if

$$\forall \phi \in C_0^\infty(\Omega), \int_{\Omega} u \,\mathrm{D}^\alpha \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_{\Omega} v\phi \,\mathrm{d}x \tag{4.67}$$

It is straightforward to show that the weak derivative of u, if it exists, is uniquely defined up to a set of measure zero. Further, if $u \in C^{|\alpha|}(\Omega)$, its weak derivative is equal to its classical derivative.

A **mollifier**, $J(x)$, is a nonnegative, real-valued function belonging to $C_0^\infty(\mathbb{R}^d)$ used to create smooth functions approximating non-smooth functions via convolutions. Mollifiers satisfy the condition that $\int_{\mathbb{R}^d} J(x)\,\mathrm{d}x = 1$. We define the standard mollifier

$$J(x) = \begin{cases} C\exp\left\{\frac{-1}{1-|x|^2}\right\} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \tag{4.68}$$

where C is chosen such that the integral of J over $\mathbb{R}^d$ is equal to 1. For $\varepsilon > 0$, we define

$$J_\varepsilon(x) = \varepsilon^{-d} J(x/\varepsilon) \tag{4.69}$$

which is compactly supported in the ball of radius ε around the origin. If $u \in L^1_{\text{loc}}(\mathbb{R}^d)$, we define the mollification of u as the convolution

$$u_\varepsilon(x) = J_\varepsilon * u(x) = \int_{\mathbb{R}^d} J_\varepsilon(x-y)u(y)\,\mathrm{d}y \tag{4.70}$$

We define $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$.

Theorem 4.6.1 (Properties of mollifiers)**.** Let $u : \mathbb{R}^d \to \mathbb{R}$ and vanishes identically outside Ω.

1. If $u \in L^1_{\text{loc}}(\overline{\Omega})$, $u_\varepsilon \in C_0^\infty(\mathbb{R}^d)$

2. If also u has compact support in Ω, then $u_\varepsilon \in C_0^\infty(\Omega)$ provided $\varepsilon < \operatorname{dist}(\operatorname{supp}(u), \partial\Omega)$

3. $u_\varepsilon \to u$ as $\varepsilon \to 0$.

4. if $G \subset\subset \Omega$ and $u \in C(\Omega)$, then $u_\varepsilon \to u$ uniformly on G.

5. If $u \in C(\overline{\Omega})$, then $u_\varepsilon \to u$ uniformly on Ω.

Proof.

$$\mathrm{D}^\alpha(J_\varepsilon * u)(x) = \int_{\mathbb{R}^d} (\mathrm{D}_x^\alpha J_\varepsilon(y-x))u(y)\,\mathrm{d}y \tag{4.71}$$

$J_\varepsilon(y-x)$ is infinitely differentiable, and vanishes outside $B_x(\varepsilon)$ for every x for every multi-index α. If $\operatorname{supp}(u) \subset\subset \Omega$, and $\varepsilon < \operatorname{dist}(\operatorname{supp}(u), \partial\Omega)$, then $\forall x \notin \Omega$, $u_\varepsilon(x) = 0$. Therefore, u_ε has compact support in Ω. From the Lebesgue differentiation theorem,

$$\lim_{r \to 0} \int_\Omega |u(y) - u(x)|\,\mathrm{d}y = 0 \tag{4.72}$$

a.e. on Ω. Fix such a point x on Ω. Then,

$$|u_\varepsilon(x) - u(x)| = \left| \int_\Omega J_\varepsilon(y-x)u(y)\operatorname{vol}(\Omega)u(x)\,\mathrm{d}y \right| \tag{4.73}$$

□

4.7 Sobolev Intro

We introduce Sobolev spaces of integer order and establish some of their basic properties. These are vector subspaces of various spaces $L^p(\Omega)$. We define the functional $\|\cdot\|_{m,p}$ where m is a nonnegative integer and $1 \le p \le \infty$ as follows:

$$\begin{aligned} \|u\|_{m,p} &= \left\{ \sum_{0 \le |\alpha| \le m} \|\mathrm{D}^\alpha u\|_p^p \right\}^{1/p} \\ \|u\|_{m,\infty} &= \max_{0 \le |\alpha| \le m} \|\mathrm{D}^\alpha u\|_\infty \end{aligned} \tag{4.74}$$

for any function for which the right side makes sense. The above functional defines a norm on any vector space of functions where the right side takes finite values provided functions are identified in the space if they are equal almost everywhere in Ω. We consider three such spaces corresponding to any given values of m and p.

- $H^{m,p}(\Omega) \equiv$ the completion of $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}$
- $W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) : \mathrm{D}^\alpha u \in L^p(\Omega) \,\text{for}\, 0 \le |\alpha| \le m\}$
- $W_0^{m,p}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Any compactly supported function in $C^\infty(\Omega)$ belongs to $W^{m,p}(\Omega)$ for any m. If Ω is bounded, then $C^m(\Omega) \subset W^{m,p}(\Omega)$ for any $1 \le p \le \infty$.

Theorem 4.7.1. The space $W^{m,p}(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{m,p}$ where $m \in \mathbb{N}$, $1 \le p \le \infty$.2

Proof. todo Consider the sequence $\{u_n\}_n \subset W^{m,p}(\Omega)$ such that $\forall \varepsilon > 0 \exists N \,\text{s.t.}\, \forall m, n \ge N$, $\|u_m - u_n\|_{m,p} < \varepsilon$. □

A set Ω is said to have the **segment property** if $\forall x_0 \in \partial\Omega$, there exists an open neighbourhood of x_0 called Ω_{x_0} and a nonzero direction $y_{x_0} \in \mathbb{R}^d$ such that $\forall t \in [0,1]$, $\overline{\Omega} \cap \Omega_{x_0} + ty_{x_0} \subset \Omega$. Sets with the segment property are only present on one side of their boundary. Balls, polytopes and open sets with C^1 boundary have the segment property. However, not all sets with piecewise C^1 boundary have the segment property.

Theorem 4.7.2. Let $\Omega \subset \mathbb{R}^d$ be an open set with a bounded, piecewise C^1 boundary. If Ω has the segment property, then for $1 \leq p < \infty$, there exists the trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$. T is linear and continuous and $Tf = f|_{\partial\Omega}$

Proof. First we define T on $C_0^\infty(\mathbb{R}^d)$ and then extend it by density to $W^{1,p}(\Omega)$. For $f \in C_0^\infty(\mathbb{R}^d)$, $Tf = f|_{\partial\Omega}$. Clearly, T is linear. It remains to be shown that T is continuous with respect to the norm in $W^{1,p}(\Omega)$ and $L^p(\Omega)$, i.e. there exists constant K such that $\forall f \in C_0^\infty(\mathbb{R}^d)$,

$$\|Tf\|_{L^p(\partial\Omega)} \leq K\|f\|_{W^{1,p}(\Omega)} \tag{4.75}$$

□

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Appendix A

Fluids

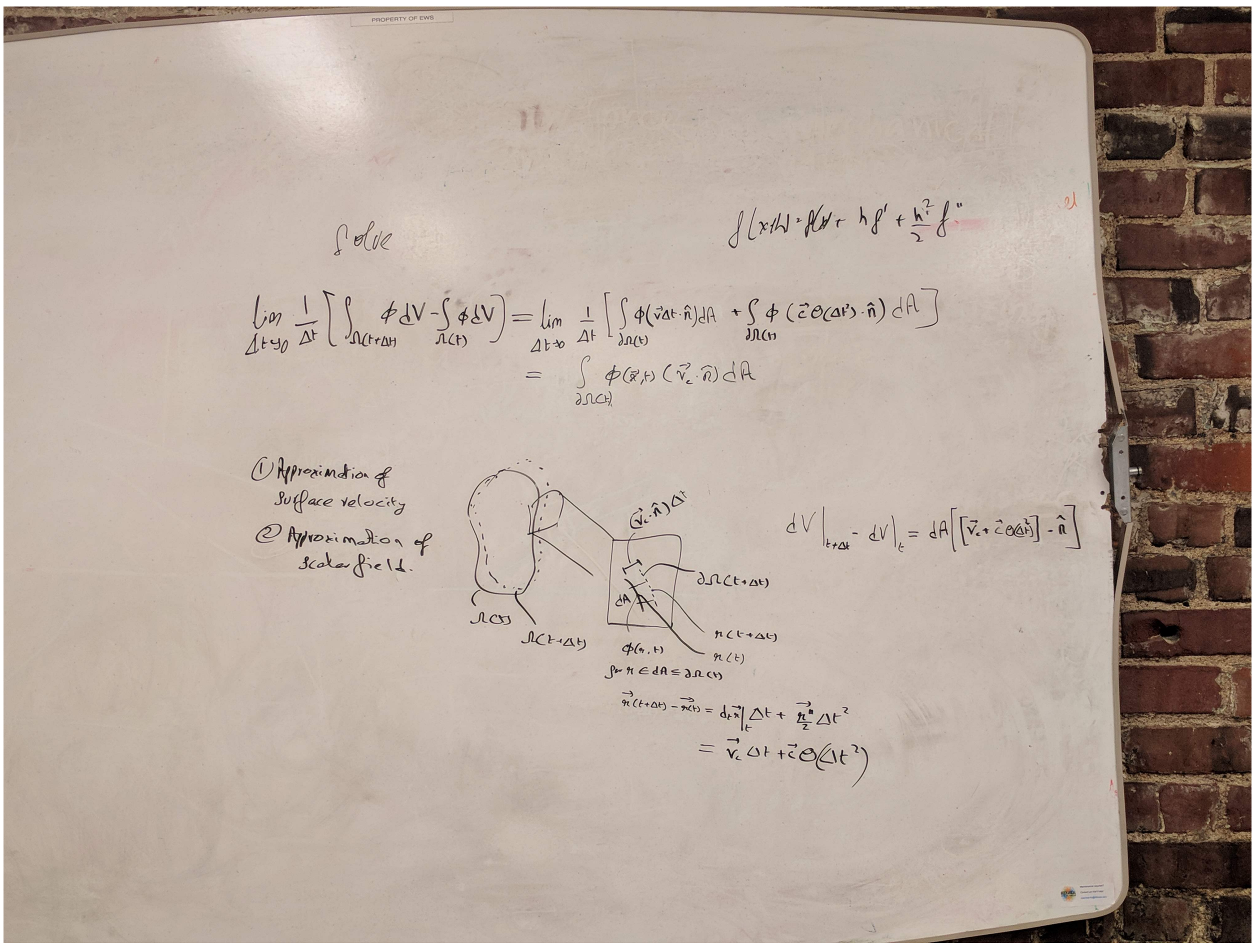

Figure A.1: Evaluating Key Limit in RTT Proof

more stuff to come.

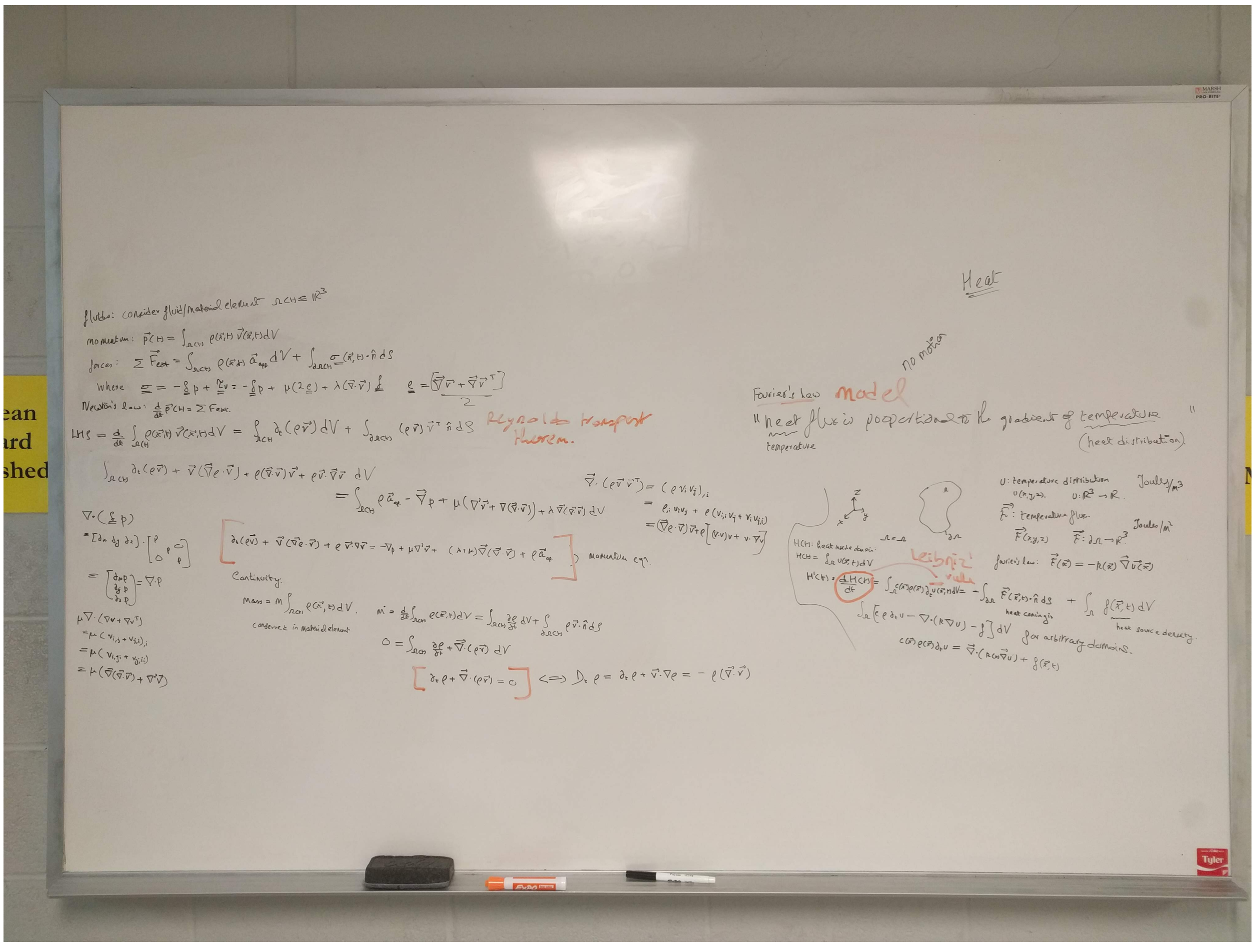

Figure A.2: Navier-Stokes for Compressible Flow