UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Misc Notes

September 27, 2019

Author:
Vedant Puri `vpuri3@illinois.edu` (347)-330-1343

Contents

Chapter 1

Fluid Mechanics

1.1 Fluids

A fluid is a material that deforms continuously under action of a shear stress, however small. This contrasts with the behaviour of a solid which deforms only when stress values are in certain regimes. Solids are **elastic**: internal stresses in solids resist absolute deformation some original state; fluids are **viscous**: internal stresses in fluids resist the time rate of deformation. A corollary is that a fluid in static equilibrium supports only its weight and forces acting normal to its boundary. A material is said to be fluid if it seems to "flow" in the timescale of observation t_{obs}. That is, if the material is able to relax to a natural state in time less than t_{obs}. The relaxation time of a fluid is denoted λ_{relax}. The ratio of relaxation time to observation time is called Deborah Number. This dimensionless quantity is named after the prophet Deborah who, in the Book of Judges, proclaimed "The mountains flowed before the lord." For large enough t_{obs}, even mountains will behave like fluids. A material is fluid if De << 1.

$$\mathrm{De} = \frac{\lambda_{\mathrm{relax}}}{t_{\mathrm{obs}}} \tag{1.1}$$

1.1.1 Continuum Assumption

We tend to neglect the discrete, molecular nature of matter and treat the fluid to be made of a "continuum". Macroscopic properties such as density and velocity are taken to be well defined for infinitesimal volume elements –small in comparison to system lengthscale but larger in comparison to molecular lengthscales. Fluid properties can vary continuously from one volume element to another and are average values of the molecular properties.

1.1.2 Control and Material Volumes

To derive the equations governing viscous fluid flow, we have to consider the time rate of change of quantities/properties in fluid control volumes. A control volume is an arbitrarily defined volume with a closed bounding surface. A material volume is a control volume that contains the same particles of matter at all times. A particular material volume may be defined by the closed bounding surface that envelops its material particles at a certain time. Hence, the velocity of the surface at every point is equal to the flow velocity at that point. The term "fluid element" is synonymous with a material volume.

We consider two disparate perspectives to view motion in a continuum: the **Lagrangian** perspective expresses position, velocity and other state variables in terms of material points travelling with a fluid; the **Eulerian** perspective expresses fluid properties with respect to some reference coordinate system. Consider the following example to illustrate the differences between the two perspectives. Let $\boldsymbol{F}$ be some property of a fluid at some material point $(\boldsymbol{X}, t) \in \Omega \times [0, T]$. $\boldsymbol{F}$ depends on $\boldsymbol{X}$, i.e. which material element one chooses and time t as material elements may interact with other elements and lose/gain the property over time. The time evolution of the position of the material element $\boldsymbol{X}$ can be tracked on an Eulerian coordinate system with a vector $\boldsymbol{x}(\boldsymbol{X}, t)$. Since only one material element can be located at an Eulerian coordinate at a given time, there exists an inverse relation $\boldsymbol{X}(\boldsymbol{x}, t)$ mapping material elements $\boldsymbol{X}$ to the Eulerian coordinate position they occupy at time t. Hence the property $\boldsymbol{F}$ following a material element varies with the element's Eulerian coordinate and time, i.e. $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x}, t)$. The rate of change of $\boldsymbol{F}$ following the material element is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{F} &= \partial_t \boldsymbol{F} + \sum_{i=1}^{n} \partial_{x_i} \boldsymbol{F} \partial_t x_i \\ \mathrm{D}_t \boldsymbol{F} &:= \partial_t \boldsymbol{F} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{F} \end{aligned} \tag{1.2}$$

Another way to think of the time rate of change of property $\boldsymbol{F}$ as one follows a material point is the following: $\partial_t \boldsymbol{F}$ is the time rate of change of $\boldsymbol{F}$ at fixed position $\boldsymbol{x}$, and $\partial_{x_i} \boldsymbol{F} \partial_t x_i$ is the rate of change of $\boldsymbol{F}$ as one would travel in the direction i multiplied by how fast the material point is travelling in said direction i.

1.2 Reynolds Transport Theorem (RTT)

RTT relates time rate of change of quantities in material volumes to the distribution of said properties in the volume. Consider an arbitrary, finite (finite, nonzero measure) control volume $\Omega(t) \subset \mathbb{R}^3$ bounded by some control surface $\partial\Omega(t) \subset \mathbb{R}^3$ in some time varying flow. For some property $\phi(\boldsymbol{x}, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\,\mathrm{d}V = \lim_{\Delta t \to 0} \frac{\int_{\Omega(t+\Delta t)} \phi(\boldsymbol{x}, t+\Delta t)\,\mathrm{d}V - \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\,\mathrm{d}V}{\Delta t} \tag{1.3}$$

For reasonably smooth $\phi(\boldsymbol{x}, t)$, use the taylor expansion of $\phi(\boldsymbol{x}, t)$ about $\phi(\boldsymbol{x}, t)$

$$\phi(\boldsymbol{x}, t+\Delta t) = \phi(\boldsymbol{x}, t) + \Delta t \partial_t \phi(\boldsymbol{x}, t) + \mathcal{O}(\Delta t^2) \tag{1.4}$$

Substituting,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\,\mathrm{d}V &= \lim_{\Delta t \to 0} \int_{\Omega(t+\Delta t)} \partial_t \phi(\boldsymbol{x}, t)\,\mathrm{d}V + \frac{\int_{\Omega(t+\Delta t)} \phi(\boldsymbol{x}, t)\,\mathrm{d}V - \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\,\mathrm{d}V + \mathcal{O}(\Delta t^2)}{\Delta t} \\ &= \int_{\Omega(t)} \partial_t \phi(\boldsymbol{x}, t)\,\mathrm{d}V + \lim_{\Delta t \to 0} + \frac{\int_{\Omega(t+\Delta t)} \phi(\boldsymbol{x}, t)\,\mathrm{d}V - \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\,\mathrm{d}V}{\Delta t} \end{aligned} \tag{1.5}$$

We can approximate the difference between the integrals using a Taylor approximation (see appendix for more details).

$$\int_{\Omega(t+\Delta t)} \phi(\boldsymbol{x}, t)\,\mathrm{d}V - \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\,\mathrm{d}V = \int_{\partial\Omega(t)} \phi(\boldsymbol{x}, t)(\Delta t \boldsymbol{v_c}) \cdot \hat{\boldsymbol{n}}\,\mathrm{d}A + \int_{\partial\Omega(t)} \phi(\boldsymbol{x}, t)(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_c \frac{\Delta t^2}{2}) \cdot \hat{\boldsymbol{n}}\,\mathrm{d}A + \mathcal{O}(\Delta t^3) \tag{1.6}$$

where $\boldsymbol{v_c}(\boldsymbol{x}, t)$ is the velocity of the control surface at time t at point $\boldsymbol{x} \in \partial\Omega(t)$ and $\hat{\boldsymbol{n}}$ is the outward pointing unit normal vector on $\partial\Omega(t)$. An infinitesimal area element $\mathrm{d}A \subset \partial\Omega(t)$ moves a distance of $(\Delta t \boldsymbol{v_c} \cdot \hat{\boldsymbol{n}} + \mathcal{O}(\Delta t^2))$ normal to the surface in time Δt. Another way to think of this approximation is the following: project the value of $\phi(\boldsymbol{x},t)$ on $\mathrm{d}A \subset \partial\Omega(t)$ throughout the volume $\mathrm{d}A(\Delta t\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}}) + \mathcal{O}(\Delta t^2)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x},t)\,\mathrm{d}V &= \int_{\Omega(t)}\partial_t\phi(\boldsymbol{x},t)\,\mathrm{d}V + \lim_{\Delta t\to 0}\frac{\int_{\partial\Omega(t)}\phi(\boldsymbol{x},t)(\Delta t\boldsymbol{v_c})\cdot\hat{\boldsymbol{n}}dA + \mathcal{O}(\Delta t^2)}{\Delta t}\\ &= \int_{\Omega(t)}\partial_t\phi(\boldsymbol{x},t)\,\mathrm{d}V + \int_{\partial\Omega(t)}\phi(\boldsymbol{x},t)\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}}dA\end{aligned} \tag{1.7}$$

$\partial_t\phi$ is the local the rate of production (or accumulation) of ϕ. It accounts for the effects of change in ϕ in the interior of the domain (say due to creation/dissipation or advection/diffusion). The divergence term accounts for the capture or loss of ϕ by the motion of the control surface. If $\Omega(t)$ is a material volume then, $\boldsymbol{v_c} = \boldsymbol{v}$, at all points in $\partial\Omega(t)$. Hence, the final form of the Reynolds Transport Theorem for material volumes $\Omega(t)$ is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x},t)\,\mathrm{d}V = \int_{\Omega(t)}\partial_t\phi(\boldsymbol{x},t)\,\mathrm{d}V + \int_{\partial\Omega(t)}\phi(\boldsymbol{x},t)\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}A \tag{1.8}$$

Applying Gauss' law,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x},t)\,\mathrm{d}V = \int_{\Omega(t)}\partial_t\phi + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\phi)\,\mathrm{d}V \tag{1.9}$$

1.3 Continuity/ Mass Conservation

Since a material volume always contains the same fluid elements, time rate of change of total mass in a material volume should be zero.

$$\begin{aligned}m(t) &= \int_\Omega \rho\,\mathrm{d}V\\ 0 = \frac{\mathrm{d}}{\mathrm{d}t}m &= \int_\Omega \partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\,\mathrm{d}V\end{aligned} \tag{1.10}$$

Since the integral is zero for arbitrary material volumes, the integrand must be zero.

$$\begin{aligned}\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) &= 0\\ \mathrm{D}_t\rho = \partial_t\rho + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho &= -\rho(\boldsymbol{\nabla}\cdot\boldsymbol{v})\end{aligned} \tag{1.11}$$

1.4 Stress

Stress is defined as a force across a "small" boundary ($\in \mathbb{R}^2$) per unit area of that boundary, for all orientations of the boundary. Stress is defined at a point with respect to a surface on which it would act. Consequently, stress depends on the orientation of the surface on which it acts. Hence, $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{x},t,\boldsymbol{n})$ where $\boldsymbol{n}$ is the outward pointing normal to the surface. In the limit $\delta A \to 0$, stress at a point is independent of the magnitude of the area.

$$\boldsymbol{\tau} = \lim_{\delta A\to 0}\frac{\delta\boldsymbol{F}}{\delta A} \tag{1.12}$$

We probe the dependence of stress on the normal the surface it acts on. We consider a cases that reveals stress at a point acting on two sides of a surface to be equal in magnitude and opposite in direction. Consider an infinitesimal disk-shaped fluid element with area δA and height δh. Without loss of generality, label the normal on one face $\boldsymbol{n}$ and the other $-\hat{\boldsymbol{n}}$. Let x_0 be the middle of the disk and let $\boldsymbol{g}$ be acceleration due to some applied body force. The equation of motion for the fluid element is:

$$\begin{aligned}\delta A\delta h\mathrm{D}_t\boldsymbol{\rho v} &= \boldsymbol{\tau}(\boldsymbol{x}_0 + 0.5\delta h\hat{\boldsymbol{n}},t,\hat{\boldsymbol{n}})\delta A + \boldsymbol{\tau}(\boldsymbol{x}_0 - 0.5\delta h\hat{\boldsymbol{n}},t,-\hat{\boldsymbol{n}})\delta A + \rho\delta A\delta h\boldsymbol{g}\\ \delta h\mathrm{D}_t\boldsymbol{\rho v} &= \boldsymbol{\tau}(\boldsymbol{x}_0 + 0.5\delta h\hat{\boldsymbol{n}},t,\hat{\boldsymbol{n}}) + \boldsymbol{\tau}(\boldsymbol{x}_0 - 0.5\delta h\hat{\boldsymbol{n}},t,-\hat{\boldsymbol{n}}) + \rho\delta h\boldsymbol{g}\end{aligned} \tag{1.13}$$

In the limit $\delta h \to 0$, the rate of change of momentum of the disk falls off, and we get

$$\begin{aligned}\boldsymbol{\tau}(\boldsymbol{x}_0, t, -\hat{\boldsymbol{n}}) + \boldsymbol{\tau}(\boldsymbol{x}_0, t, \hat{\boldsymbol{n}}) &= 0 \\ \boldsymbol{\tau}(\boldsymbol{x}_0, t, -\hat{\boldsymbol{n}}) &= -\boldsymbol{\tau}(\boldsymbol{x}_0, t, \hat{\boldsymbol{n}})\end{aligned} \tag{1.14}$$

Let stress at a point x_0 at time t_0 be a function of the normal to the surface it acts on i.e $\boldsymbol{\tau} = \boldsymbol{\tau}(\hat{\boldsymbol{n}})$. Let σ_{ij} be the component of stress in the $\hat{\boldsymbol{j}}$ direction on a surface with normal $\hat{\boldsymbol{i}}$. Similarly, we have

$$\begin{aligned}\boldsymbol{\tau}(\hat{\boldsymbol{i}}) &= \sigma_{ii}\hat{\boldsymbol{i}} + \sigma_{ji}\hat{\boldsymbol{j}} + \sigma_{ki}\hat{\boldsymbol{k}} \\ \boldsymbol{\tau}(\hat{\boldsymbol{j}}) &= \sigma_{ij}\hat{\boldsymbol{i}} + \sigma_{jj}\hat{\boldsymbol{j}} + \sigma_{kj}\hat{\boldsymbol{k}} \\ \boldsymbol{\tau}(\hat{\boldsymbol{k}}) &= \sigma_{ik}\hat{\boldsymbol{i}} + \sigma_{jk}\hat{\boldsymbol{j}} + \sigma_{kk}\hat{\boldsymbol{k}}\end{aligned} \tag{1.15}$$

where each σ_{**} is a scalar field varying in space and time. We now show that stress on a surface with an arbitrary normal can be represented as linear combinations of $\boldsymbol{\tau}(\hat{\boldsymbol{i}})$, $\boldsymbol{\tau}(\hat{\boldsymbol{j}})$, $\boldsymbol{\tau}(\hat{\boldsymbol{k}})$. Consider a fluid element in the shape of an infinitesimal tetrahedral with vertices $\boldsymbol{x_0}$, $\boldsymbol{x_0} + \delta x\hat{\boldsymbol{i}}$, $\boldsymbol{x_0} + \delta y\hat{\boldsymbol{j}}$, $\boldsymbol{x_0} + \delta z\hat{\boldsymbol{k}}$. This tetrahedral has a faces parallel to the xy, xz, yz planes with areas δA_z, δA_y, δA_x, and normals $-\hat{\boldsymbol{k}}$, $-\hat{\boldsymbol{j}}$, $-\hat{\boldsymbol{i}}$ respectively. Let the fourth face have area δA and some arbitrary normal $\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{i}} + n_y\hat{\boldsymbol{j}} + n_z\hat{\boldsymbol{k}}$. Let θ_x, θ_y, θ_z be the angle between $\hat{\boldsymbol{n}}$ and the coordinate axes. The areas are related by the following expressions:

$$\begin{aligned}\delta A_x &= \cos\theta_x = n_x\delta A \\ \delta A_y &= \cos\theta_y = n_y\delta A \\ \delta A_z &= \cos\theta_z = n_z\delta A\end{aligned} \tag{1.16}$$

We now apply Newton's laws to the fluid element in the limit δx, δy, $\delta z \to 0$. Since the volume of the fluid element, δV falls much faster than any of the surface areas, the mass time acceleration and body forces go to zero faster than the surface forces.

$$\begin{aligned}\delta V &\propto \delta x\delta y\delta z \\ \delta A &\propto \delta z\sqrt{\delta x^2 + \delta y^2} \\ \delta A_x &\propto \delta y\delta z \\ \delta A_y &\propto \delta x\delta z \\ \delta A_z &\propto \delta x\delta y\end{aligned} \tag{1.17}$$

Say δx, δy, δz go to zero as $\frac{1}{N}$. Then, for bounded constants c, c_v,

$$\begin{aligned}0 &= \delta A\boldsymbol{\tau}(\hat{\boldsymbol{n}}) + \delta A_x\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + \delta A_y\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + \delta A_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}}) \\ 0 &= \delta A\boldsymbol{\tau}(\hat{\boldsymbol{n}}) + n_x\delta A\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + n_y\delta A\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + n_z\delta A\boldsymbol{\tau}(-\hat{\boldsymbol{k}}) \\ 0 &= \lim_{N\to\infty} \frac{c\boldsymbol{\tau}(\hat{\boldsymbol{n}})}{N^2} + \frac{cn_x\boldsymbol{\tau}(-\hat{\boldsymbol{i}})}{N^2} + \frac{cn_y\boldsymbol{\tau}(-\hat{\boldsymbol{j}})}{N^2} + \frac{cn_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}})}{N^2} + \frac{c_v}{N^3}(\rho\boldsymbol{g} - \mathrm{D}_t\rho\boldsymbol{v}) \\ 0 &= \lim_{N\to\infty} c\boldsymbol{\tau}(\hat{\boldsymbol{n}}) + cn_x\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + cn_y\boldsymbol{\tau}(-\hat{\boldsymbol{j}}) + cn_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}}) + \frac{c_v}{N}(\rho\boldsymbol{g} - \mathrm{D}_t\rho\boldsymbol{v}) \\ 0 &= \boldsymbol{\tau}(\hat{\boldsymbol{n}}) + n_x\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + n_y\boldsymbol{\tau}(-\hat{\boldsymbol{j}}) + n_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}}) \\ \boldsymbol{\tau}(\hat{\boldsymbol{n}}) &= n_x\boldsymbol{\tau}(\hat{\boldsymbol{i}}) + n_y\boldsymbol{\tau}(\hat{\boldsymbol{j}}) + n_z\boldsymbol{\tau}(\hat{\boldsymbol{k}})\end{aligned} \tag{1.18}$$

Hence, forces due to surface stresses must balance each other in the limit of the tetrahedral shrinking to a point. Expressing the above expression in matrix notation,

$$\begin{aligned}\boldsymbol{\tau}(\hat{\boldsymbol{n}}) &= \begin{bmatrix}\boldsymbol{\tau}(\hat{\boldsymbol{i}}) & \boldsymbol{\tau}(\hat{\boldsymbol{j}}) & \boldsymbol{\tau}(\hat{\boldsymbol{k}})\end{bmatrix} \cdot \hat{\boldsymbol{n}} \\ \boldsymbol{\tau}(\hat{\boldsymbol{n}}) &= \begin{bmatrix}\sigma_{ii} & \sigma_{ij} & \sigma_{ik} \\ \sigma_{ji} & \sigma_{jj} & \sigma_{jk} \\ \sigma_{ki} & \sigma_{kj} & \sigma_{kk}\end{bmatrix} \cdot \hat{\boldsymbol{n}}\end{aligned} \tag{1.19}$$

We call $\boldsymbol{\tau}$ the traction vector and $\underline{\underline{\sigma}}$ the stress tensor.

$$\underline{\underline{\sigma}} = \begin{bmatrix}\boldsymbol{\tau}(\hat{\boldsymbol{i}}) & \boldsymbol{\tau}(\hat{\boldsymbol{j}}) & \boldsymbol{\tau}(\hat{\boldsymbol{k}})\end{bmatrix} = \begin{bmatrix}\sigma_{ii} & \sigma_{ij} & \sigma_{ik} \\ \sigma_{ji} & \sigma_{jj} & \sigma_{jk} \\ \sigma_{ki} & \sigma_{kj} & \sigma_{kk}\end{bmatrix} \tag{1.20}$$

$$\boldsymbol{\tau}(\boldsymbol{x}, t, \hat{\boldsymbol{n}}) = \underline{\underline{\sigma}}(\boldsymbol{x}, t) \cdot \hat{\boldsymbol{n}} \tag{1.21}$$

We now show that the stress tensor is symmetric.

1.4.1 Fluid at Rest

Since a fluid continuously deforms under shear stress, a static, non-deforming fluid element in consequently under no shear stress. Hence for an arbitrary surface element in a fluid at rest, stress is in the normal direction. From equilibrium arguments, it can be proven that stress in a fluid at rest is isotropic.

$$\begin{aligned} \boldsymbol{\tau} &= -p(\boldsymbol{x}, t)\boldsymbol{n} \\ \boldsymbol{\tau} &= -p(\boldsymbol{x}, t)\underline{\underline{\delta}} \cdot \boldsymbol{n} \end{aligned} \tag{1.22}$$

1.4.2 Stress Tensor for Newtonian Fluids

We still need to relate the stress tensor to the flow field. The *Newtonian Model* is based on the following assumptions:

1. shear stress is proportional to the rate of shear strain in a fluid particle;
2. shear stress is zero when the rate of shear strain is zero;
3. the stress to rate-of-strain relation is isotropicthat is, there is no preferred orientation in the fluid.

From the first assumption, we have $\sigma_{ij} = \mathrm{K}_{ijkl} e_{kl}$ where K is a fourth order tensor.

derive

$$\begin{aligned} \underline{\underline{\sigma}} &= -p\underline{\underline{\delta}} + \underline{\underline{\tau_v}} \\ &= -p\underline{\underline{\delta}} + \mu(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^{\mathsf{T}}) + \lambda(\boldsymbol{\nabla} \cdot \boldsymbol{v})\underline{\underline{\delta}} \end{aligned} \tag{1.23}$$

1.5 Momentum Conservation

Let $\boldsymbol{p}(t)$ denote the total momentum of a fluid element.

$$\begin{aligned} \boldsymbol{p}(t) &= \int_\Omega \rho \boldsymbol{v} \,\mathrm{d}V \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} &= \int_\Omega \partial_t(\rho\boldsymbol{v}) + \boldsymbol{\nabla} \cdot (\boldsymbol{v} \otimes \rho\boldsymbol{v}) \,\mathrm{d}V \\ &= \int_\Omega \partial_t(\rho\boldsymbol{v}) + \rho\boldsymbol{v}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\rho\boldsymbol{v}) \,\mathrm{d}V \\ &= \int_\Omega \mathrm{D}_t(\rho\boldsymbol{v}) + \rho\boldsymbol{v}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \,\mathrm{d}V \end{aligned} \tag{1.24}$$

Consider the total force applied on the fluid element.

$$
\begin{aligned}
\boldsymbol{F}_{\mathrm{ext}} &= \int_\Omega \rho \boldsymbol{g} \,\mathrm{d}V + \int_{\partial\Omega} \underline{\underline{\sigma}} \cdot \boldsymbol{n} \,\mathrm{d}A \\
&= \int_\Omega \rho \boldsymbol{g} + \boldsymbol{\nabla} \cdot \underline{\underline{\sigma}} \,\mathrm{d}V \\
&= \int_\Omega \rho \boldsymbol{g} + \boldsymbol{\nabla} \cdot \Big(- p\underline{\underline{\delta}} + \mu(\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^\intercal) + \lambda(\boldsymbol{\nabla} \cdot \boldsymbol{v})\underline{\underline{\delta}}\Big) \,\mathrm{d}V \\
&= \int_\Omega \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu(\boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v})) + \lambda \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \,\mathrm{d}V \\
&= \int_\Omega \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{v} + (\mu + \lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \,\mathrm{d}V
\end{aligned}
\tag{1.25}
$$

From Newton's second law, time rate of change of momentum is equal to the sum of all external forces applied.

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} - \boldsymbol{F}_{\mathrm{ext}} \\
&= \int_\Omega \partial_t(\rho \boldsymbol{v}) + \rho \boldsymbol{v}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\rho \boldsymbol{v}) - \Big(\rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{v} + (\mu + \lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v})\Big) \,\mathrm{d}V
\end{aligned}
\tag{1.26}
$$

Since the integral is zero for an arbitrary domain (fluid element) $\Omega(t)$, it follows that the integrand must be zero everywhere.

$$
\begin{aligned}
\partial_t(\rho \boldsymbol{v}) + \rho \boldsymbol{v}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\rho \boldsymbol{v}) &= \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{v} + (\mu + \lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \\
\boldsymbol{v}\cancelto{0\,(\text{continuity})}{(\partial_t \rho + \boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho)} + \rho \mathrm{D}_t \boldsymbol{v} &= \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{v} + (\mu + \lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \\
\rho(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}) &= \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{v} + (\mu + \lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v})
\end{aligned}
\tag{1.27}
$$

1.5.1 Divergence of Momentum Equation

Take the divergence of the momentum equation.

$$
\begin{aligned}
\rho(\partial_t(\boldsymbol{\nabla} \cdot \boldsymbol{v}) + \boldsymbol{\nabla v} : \boldsymbol{\nabla v}^\intercal + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\boldsymbol{\nabla} \cdot \boldsymbol{v})) &= \rho(\boldsymbol{\nabla} \cdot \boldsymbol{g}) - \boldsymbol{\nabla}^2 p + \mu \boldsymbol{\nabla}^2(\boldsymbol{\nabla} \cdot \boldsymbol{v}) + (\mu + \lambda)\boldsymbol{\nabla}^2(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \\
\rho \mathrm{D}_t(\boldsymbol{\nabla} \cdot \boldsymbol{v}) &= \boldsymbol{\nabla}^2(-p + (\lambda + 2\mu)(\boldsymbol{\nabla} \cdot \boldsymbol{v})) + \rho(\boldsymbol{\nabla} \cdot \boldsymbol{g} - \boldsymbol{\nabla v} : \boldsymbol{\nabla v}^\intercal)
\end{aligned}
\tag{1.28}
$$

where $\underline{\underline{A}} : \underline{\underline{B}} = A_{ij}B_{ji}$ is the double dot product. This equation simplifies into a laplace equation for pressure in case of incompressible flow.

$$
-\boldsymbol{\nabla}^2 p = \rho(\boldsymbol{\nabla v} : \boldsymbol{\nabla v}^\intercal - \boldsymbol{\nabla} \cdot \boldsymbol{g})
\tag{1.29}
$$

1.5.2 Vorticity Equation

We take the curl of the momentum equation. Define $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$

$$
\begin{aligned}
\boldsymbol{\nabla} \times \rho(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}) &= \boldsymbol{\nabla} \times (\rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{v} + (\mu + \lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v})) \\
\boldsymbol{\nabla}\rho \times \mathrm{D}_t \boldsymbol{v} + \rho \mathrm{D}_t \boldsymbol{\omega} + \epsilon_{ijk} v_{l,i} v_{j,l} &= \boldsymbol{\nabla} \times \boldsymbol{g} + \boldsymbol{\nabla}\rho \times \boldsymbol{g} + \mu \boldsymbol{\nabla}^2 \boldsymbol{\omega} + \epsilon_{ijk}(-p + (\boldsymbol{\nabla} \cdot \boldsymbol{v}))_{,ji} \\
\boldsymbol{\nabla}\rho \times (\mathrm{D}_t \boldsymbol{v} - \boldsymbol{g}) + \rho \mathrm{D}_t \boldsymbol{\omega} + \epsilon_{ijk} v_{l,i} v_{j,l} &= \boldsymbol{\nabla} \times \boldsymbol{g} + \mu \boldsymbol{\nabla}^2 \boldsymbol{\omega} + \epsilon_{ijk}(-p + (\boldsymbol{\nabla} \cdot \boldsymbol{v}))_{,ji}
\end{aligned}
\tag{1.30}
$$

Chapter 2

Reynolds Stress Transport Equations

We derive the Reynolds stress transport equations (RSTE) that describe the production, transport, and dissipation of energy in fluctuating modes in a flow. Examining the energy budgets of turbulent flows can lead to insights into factors driving turbulence and leading to its decay, spatial distribution, and how energy is transferred between mean and fluctuating modes. The detailed Reynolds stress budgets from DNS provide valuable information on the relative magnitudes of the terms and their possible scaling.

2.1 Navier-Stokes Equation

Consider the Navier-Stokes equations for incompressible fluid flow:

$$\begin{aligned} \rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}) &= -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{v} + \rho \boldsymbol{g} \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0 \end{aligned} \tag{2.1}$$

where $v_j(x_i, t)$, and $p(x_i, t)$ are time and space varying fields, and g_i is a constant body force. We nondimensionalise velocity, time, and pressure with some canonical length L and velocity U as follows:

$$\begin{aligned} v^* &= \frac{v}{U},\, t^* = \frac{tU}{L},\, p^* = \frac{p}{\rho U^2},\, g^* = \frac{g}{U^2/L} \\ \partial_{t^*} &= \frac{L}{U}\partial_t,\, \boldsymbol{\nabla}^* = L\boldsymbol{\nabla} \end{aligned} \tag{2.2}$$

Substituting Equation 2.2 into equation Equation 2.1 and multiplying by $\dfrac{L}{\rho U^2}$, we obtain the Navier-Stokes equations in non-dimensional form.

$$\begin{aligned} \partial_{t^*} \boldsymbol{v}^* + \boldsymbol{v}^* \cdot \boldsymbol{\nabla} \boldsymbol{v}^* &= -\boldsymbol{\nabla} p^* + \frac{1}{\mathrm{Re}} \boldsymbol{\nabla}^2 \boldsymbol{v}^* + \boldsymbol{g}^* \\ \boldsymbol{\nabla}^* \cdot \boldsymbol{v}^* &= 0 \end{aligned} \tag{2.3}$$

We drop the asterisks and write in Einstein indicial notation for convenience.

$$\begin{aligned} \partial_t v_j + v_i v_{j,i} &= -p_{,j} + \frac{1}{\mathrm{Re}} v_{j,ii} + g_j \\ v_{i,i} &= 0 \end{aligned} \tag{2.4}$$

2.2 Reynolds Decomposition

Splitting a quantity into its average value and a fluctuation from the average is called a Reynolds Decomposition. For a space and time varying quantity $\phi(x_i, t)$,

$$\phi = \langle \phi \rangle + \phi' \tag{2.5}$$

where $\langle \phi \rangle$ is the ensemble average of ϕ, and ϕ' is fluctuation or deviation of ϕ from $\langle \phi \rangle$. In practice, ensemble averages are obtained by averaging in time and over homogeneous directions.

$$\langle \phi \rangle = \frac{1}{\int \mathrm{d}t} \int \phi \, \mathrm{d}t \tag{2.6}$$

Some properties of Reynolds decomposition are below. Let ϕ and ψ be any quantities. Then,

- Averaging and differentiating commute since average of a derivative is the derivative of the average.

$$\begin{aligned} \langle \phi \rangle_{,i} &= \langle \phi_i \rangle \\ \partial_t \langle \phi \rangle &= \langle \partial_t \phi \rangle \end{aligned} \tag{2.7}$$

- Ensemble of a fluctuation is zero.

$$\begin{aligned} \langle \phi' \rangle &= \langle \phi - \langle \phi \rangle \rangle \\ \langle \phi \rangle &= \langle \phi \rangle - \langle \phi' \rangle \\ \langle \phi' \rangle &= 0 \end{aligned} \tag{2.8}$$

- Ensemble of product is product of ensemble plus ensemble of product of fluctuations

$$\begin{aligned} \langle \phi\psi \rangle &= \big\langle \big(\langle \phi \rangle + \phi' \big) \big(\langle \psi \rangle + \psi' \big) \big\rangle \\ &= \langle \langle \phi \rangle \langle \psi \rangle + \langle \phi \rangle \psi' + \phi' \langle \psi \rangle + \phi'\psi' \rangle \\ &= \langle \phi \rangle \langle v \rangle + \langle \phi \rangle \langle v' \rangle + \langle \phi' \rangle \langle \psi \rangle + \langle \phi'\psi' \rangle \\ &= \langle \phi \rangle \langle \psi \rangle + \langle \phi'\psi' \rangle \end{aligned} \tag{2.9}$$

2.3 Reynolds Stresses

We take the expected value of the continuity equation to find that $\langle v_i \rangle$ is divergence free.

$$\begin{aligned} \langle 0 \rangle &= \langle v_{i,i} \rangle \\ 0 &= \langle v_i \rangle_{,i} \end{aligned} \tag{2.10}$$

Similarly, we expand the continuity equation in terms of expected values of fluctuations,to find that v_i' is also divergence free.

$$\begin{aligned} 0 &= v_{i,i} = \cancelto{0}{\langle v_i \rangle_{,i}} + v'_{i,i} \\ 0 &= v'_{i,i} \end{aligned} \tag{2.11}$$

We take the expected value of the momentum equation.

$$\begin{aligned} \partial_t \langle v_j \rangle + \langle v_i \rangle \langle v_{j,i} \rangle + \langle v_i' v_{j,i}' \rangle &= -\langle p_{,j} \rangle + \frac{1}{\mathrm{Re}} \langle v_j \rangle_{,ii} + \langle g_i \rangle \\ \partial_t \langle v_j \rangle + \langle v_i \rangle \langle v_{j,i} \rangle &= -\langle p_{,j} \rangle + \frac{1}{\mathrm{Re}} \langle v_j \rangle_{,ii} + \langle g_i \rangle - \langle v_i' v_j' \rangle_{,i} \end{aligned} \tag{2.12}$$

$\eta_{ij} = \left\langle v'_i v'_j \right\rangle$ is called the Reynolds stress tensor. We apply Reynolds decomposition to the momentum equation.

$$\partial_t(\langle v_j \rangle + v'_j) + (\langle v_i \rangle + v'_i)(\langle v_{j,i} \rangle + v'_{j,i}) = -(\langle p_{,j} \rangle + p'_j) + \frac{1}{\mathrm{Re}}(\langle v_{j,ii} \rangle + v'_{j,ii}) + \langle g_j \rangle + \cancelto{0}{g'_j}$$

$$\partial_t(\langle v_j \rangle + v'_j) + \langle v_i \rangle \langle v_{j,i} \rangle + \langle v_i \rangle v'_{j,i} + v'_i \langle v_{j,i} \rangle + v'_i v'_{j,i} = -(\langle p_{,j} \rangle + p'_j) + \frac{1}{\mathrm{Re}}(\langle v_{j,ii} \rangle + v'_{j,ii}) + \langle g_j \rangle \tag{2.13}$$

Subtracting Equation 2.12 from Equation 2.13, we get

$$\partial_t v'_j + \langle v_i \rangle v'_{j,i} + v'_i \langle v_{j,i} \rangle + v'_i v'_{j,i} - \left\langle v'_i v'_{j,i} \right\rangle = -p'_{,j} + \frac{1}{\mathrm{Re}} v'_{j,ii} \tag{2.14}$$

Multiply both sides by v'_k and take the expected value.

$$\left\langle v'_k \partial_t v'_j \right\rangle + \langle v_i \rangle \left\langle v'_k v'_{j,i} \right\rangle + \langle v'_k v'_i \rangle \langle v_{j,i} \rangle + \left\langle v'_k v'_i v'_{j,i} \right\rangle - \cancelto{0}{\langle v'_k \rangle} \left\langle v'_i v'_{j,i} \right\rangle = -\left\langle v'_k p'_{,j} \right\rangle + \frac{1}{\mathrm{Re}} \left\langle v'_k v'_{j,ii} \right\rangle \tag{2.15}$$

Take the transpose and add.

$$\begin{aligned} \partial_t \left\langle v'_j v'_k \right\rangle + \langle v_i \rangle \left\langle v'_j v'_k \right\rangle_{,i} + \langle v'_k v'_i \rangle \langle v_{j,i} \rangle + \left\langle v'_j v'_i \right\rangle \langle v_{k,i} \rangle + \left\langle v'_i v'_j v'_k \right\rangle_{,i} \\ = -\left(\left\langle v'_k p'_{,j} \right\rangle + \left\langle v'_j p'_{,k} \right\rangle \right) + \frac{1}{\mathrm{Re}} \left\langle v'_j v'_k \right\rangle_{,ii} - \frac{2}{\mathrm{Re}} \left\langle v'_{j,i} v'_{k,i} \right\rangle \end{aligned} \tag{2.16}$$

We write the pressure transport term in terms of the pressure strain and pressure diffusion terms.

$$v'_k p'_{,j} + v'_j p'_{,k} = -p'(v'_{j,k} + v'_{k,j}) + (p' v'_j)_{,k} + (p' v'_k)_{,j} \tag{2.17}$$

We finally arrive at the tensor equation describing the behaviour of Reynolds Stresses over time.

$$\begin{aligned} \partial_t \left\langle v'_j v'_k \right\rangle + \langle v_i \rangle \left\langle v'_j v'_k \right\rangle_{,i} + \langle v'_k v'_i \rangle \langle v_{j,i} \rangle + \left\langle v'_j v'_i \right\rangle \langle v_{k,i} \rangle + \left\langle v'_i v'_j v'_k \right\rangle_{,i} \\ = \left\langle p'(v'_{j,k} + v'_{k,j}) \right\rangle - \left\langle (p' v'_j)_{,k} + (p' v'_k)_{,j} \right\rangle + \frac{1}{\mathrm{Re}} \left\langle v'_j v'_k \right\rangle_{,ii} - \frac{2}{\mathrm{Re}} \left\langle v'_{j,i} v'_{k,i} \right\rangle \end{aligned} \tag{2.18}$$

$$\begin{aligned} \partial_t \eta_{jk} + \langle v_i \rangle \eta_{jk,i} + \eta_{ki} \langle v_{j,i} \rangle + \eta_{ji} \langle v_{k,i} \rangle + \left\langle v'_i v'_j v'_k \right\rangle_{,i} \\ = \left\langle p'(v'_{j,k} + v'_{k,j}) \right\rangle - \left\langle (p' v'_j)_{,k} + (p' v'_k)_{,j} \right\rangle + \frac{1}{\mathrm{Re}} \eta_{jk,ii} - \frac{2}{\mathrm{Re}} \left\langle v'_{j,i} v'_{k,i} \right\rangle \end{aligned} \tag{2.19}$$

2.4 Turbulent Kinetic Energy

To obtain the equation for Turbulent Kinetic Energy, we consider one-half of the trace of Equation 2.18 by multiplying with $\frac{1}{2}\delta_{ij}$.

$$\begin{aligned} \partial_t k + \langle v_i \rangle k_{,i} + \left\langle v'_j v'_i \right\rangle \langle v_{j,i} \rangle + \left\langle v'_i v'_j v'_j \right\rangle_{,i} &= \cancelto{0}{\langle p' v'_{i,i} \rangle} - \left\langle p' v'_j \right\rangle_{,j} + \frac{1}{\mathrm{Re}} k_{,ii} - \frac{1}{\mathrm{Re}} \left\langle v'_{j,i} v'_{j,i} \right\rangle \\ \partial_t k + \langle v_i \rangle k_{,i} + \left\langle v'_j v'_i \right\rangle \langle v_{j,i} \rangle + \left\langle v'_i v'_j v'_j \right\rangle_{,i} &= -\left\langle p'_{,j} v'_j \right\rangle - \cancelto{0}{\langle p' v'_{i,i} \rangle} + \frac{1}{\mathrm{Re}} k_{,ii} - \frac{1}{\mathrm{Re}} \left\langle v'_{j,i} v'_{j,i} \right\rangle \\ \partial_t k + \langle v_i \rangle k_{,i} + \left\langle v'_j v'_i \right\rangle \langle v_{j,i} \rangle + \left\langle v'_i v'_j v'_j \right\rangle_{,i} &= -\left\langle p'_{,j} v'_j \right\rangle + \frac{1}{\mathrm{Re}} k_{,ii} - \frac{1}{\mathrm{Re}} \left\langle v'_{j,i} v'_{j,i} \right\rangle \end{aligned} \tag{2.20}$$

2.5 RSTE Budgets

Equation 2.18 and Equation 2.20 describe the generation, transport and decay of Reynolds stresses in a flow. For each η_{ik} in the symmetric Reynolds stress tensor, and for k, we label the terms in Equation 2.18 and Equation 2.20.

Table 2.1: Budgets for Reynolds Stresses

Reynolds Stress Expression	**Budget Term**	**TKE Expression**
$\left\langle v'_j v'_k \right\rangle$	Reynold Stress	$k = \frac{1}{2} \left\langle v'_j v'_j \right\rangle$
$\langle v_i \rangle \left\langle v'_j v'_k \right\rangle_{,i}$	Convection	$\langle v_i \rangle k_{,i}$
$-\left\langle v'_j v'_i \right\rangle \langle v_{k,i} \rangle - \langle v'_k v'_i \rangle \langle v_{j,i} \rangle$	Production	$-\left\langle v'_j v'_i \right\rangle \langle v_{j,i} \rangle$
$-\left\langle v'_i v'_j v'_k \right\rangle_{,i}$	Turbulent Diffusion	$\left\langle v'_i v'_j v'_j \right\rangle_{,i}$
$-\left\langle v'_k p'_{,j} + v'_j p'_{,k} \right\rangle$	Pressure Transport	$-\left\langle v'_j p'_j \right\rangle$
$-\left\langle (p' v'_j)_{,k} + (p' v'_k)_{,j} \right\rangle$	Pressure Diffusion	$-\left\langle v'_j p'_j \right\rangle$
$\left\langle p' (v'_{j,k} + v'_{k,j}) \right\rangle$	Pressure Strain	0
$\frac{1}{\mathrm{Re}} \left\langle v'_j v'_k \right\rangle_{,ii}$	Viscous Diffusion	$\frac{1}{\mathrm{Re}} k_{,ii}$
$-\frac{2}{\mathrm{Re}} \left\langle v'_{j,i} v'_{k,i} \right\rangle$	Viscous Dissipation	$\frac{-1}{\mathrm{Re}} \left\langle v'_{j,i} v'_{j,i} \right\rangle$

The convection term describes the transport of Reynolds stresses due to the motion of the mean flow $\langle v_i \rangle$. The production term arises from interactions between fluctuations and shearing of the mean flow, resulting in a net transfer of energy from mean to turbulent fluctuations. The viscous diffusion term, as the name suggests, describes the diffusion of Reynolds stresses in the flow domain.

Chapter 3

Analysis

3.1 basic

Relations, Sets, and Functions

equivalance relations, functions (one-one, onto), fields, order, vector spaces

pigeonhole principle

$\mathbb{Q}$

3.2 $\mathbb{R}$

3.2.1 Construction: Dedekind's Cuts

todo The completion of $\mathbb{Q}$

3.2.2 Properties

Theorem 3.2.1 (Archimedean Property)**.** $\forall x, y \in \mathbb{R}^+ \, \exists N \in \mathbb{N} : y < nx$

Proof. Fix x, $y \in \mathbb{R}$. Let $A = \mathbb{N}x$. Consider, *ad absurdum,* $\forall n \in \mathbb{N}$, $y \geq nx$. That is, y is an upper bound for A. Let $\alpha = \sup A = n\alpha$ for some $n \in \mathbb{N}$. As $x > 0$, $\alpha < \alpha + x = (n+1)x \in A$. Therefore α is not an upper bound for A. □

Theorem 3.2.2. $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e. between any two reals, there exists a rational number.

Proof. Let $x, y \in \mathbb{R}, x < y$. Without loss of generality, consider the case where $x, y > 0$. Applying the

Archimedean property, we get

$$\begin{aligned}
&\exists n \in \mathbb{N}, n(y-x) > 1 \\
&\implies x < x + \frac{1}{n} < y \\
&\implies x < \frac{nx+1}{n} < y
\end{aligned} \tag{3.1}$$

Since the numerator is not guaranteed to be rational, we seek $m \in \mathbb{N}$ such that $nx < m < nx + 1$. Apply the Archimedean property again to find positive integer m_1 such that $0 < nx < m_1$. Hence there exists $0 < m \leq m_1$ such that

$$\begin{aligned}
&m < nx < m + 1 \\
&nx < m < nx + 1 < ny \\
&x < \frac{m}{n} < y
\end{aligned} \tag{3.2}$$

□

Definition 3.2.1 (Dense subset)**.** A set $S \subset X$ is dense in X if every element in X is a limit point of S.

The real number system is an ordered field $\mathbb{R}$, a complete extension of $\mathbb{Q}$ that has the least upper bound property.

Definition 3.2.2 (Least Upper Bound Property)**.** For $S \subset \mathbb{R}$, an upper bound of S in $\mathbb{R}$ is an element $x \in \mathbb{R}$ such that $\forall s \in S, s < x$. The least upper bound of S in $\mathbb{R}$ is an element y such that

1. y is an upper bound for S
2. if x is an upper bound for S, then $y < x$.

The least upper bound property of $\mathbb{R}$ is that any nonempty set of real numbers that is bounded from above has a least upper bound.

An ordered set satisfies the completeness axiom if every subset S that is bounded above has a least upper bound denoted $\sup S \in \mathbb{R}$. If $-S = \{-s | s \in S\}$, then $\inf(S) = -\sup(-S)$

Theorem 3.2.3 (Knaster-Tarski Fixed Point Theorem)**.** Consider a set $X \subset \mathbb{R}$, for which $a = \inf(X)$, $b = \sup(X) \in X$, then every increasing function $f : X \to X$ has at least one fixed point, i.e. $x_0 \in X$ such that $f(x_0) = x_0$.

Proof. Consider the set $S = \{x \in X | x \leq f(x)\}$. S is nonempty since $a \in S$. Hence $\beta = \sup(S) \in X$ must exist.

$$\begin{aligned}
&\forall x \in S, x \leq \beta \\
\implies& x \leq f(x),\ f(x) \leq f(\beta) && (f \text{ increasing function}) \\
\implies& \forall x \in S,\ x \leq f(\beta) \\
\implies& \beta \leq f(\beta) && (\text{since } \beta = \sup(S)\ ) \\
\implies& f(\beta) \leq f\left(f(\beta)\right)
\end{aligned} \tag{3.3}$$

□

3.2.3 Extended Real Line: $\mathbb{R} \cup \{-\infty, \infty\}$

todo

3.3 Sequences

Definition 3.3.1 (Sequence)**.** A sequence $(a_n)_{n\in\mathbb{N}}$ of elements in a set X is a function $a. : \mathbb{N} \to X$.

Definition 3.3.2 (Limit of a sequence)**.** A sequence (x_n) converges to $x \in X$ in norm $\|\cdot\|$, abbreviated as $x_n \xrightarrow{n} x$, if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \|x - x_n\| < \epsilon \tag{3.4}$$

The element $\lim_{n\to\infty} x_n = x$ is called the limit of sequence x_n.

We will consider sequences of real numbers under the absolute value norm. Below listed are some properties of convergent sequences.

1. The limit of a convergent sequence is unique.

   Proof. $\epsilon > 0$. Let $x_n \to a$, $x_n \to b \in \mathbb{R}$. Without loss of generality, let $b > a$, and fix $\epsilon = \dfrac{b-a}{3}$. Then $\exists N_a, N_b \in \mathbb{N}$ such that

$$\begin{aligned} \forall n > N_a, |a - x_n| < \epsilon \\ \forall n > N_b, |b - x_n| < \epsilon \end{aligned} \tag{3.5}$$

   Then, $\forall n > \max(N_a, N_b)$,

$$\implies |b - a| < |a - x_n| + |b - x_n| < (b-a)\frac{2}{3} \Rightarrow\Leftarrow \tag{3.6}$$

   $\square$

2. (a_n) convergent $\implies$ $\{a_n\}_n$ bounded.

   Proof. Let $a_n \to a$. $\forall \epsilon, \exists N$ such that $\forall n > N, |a - a_n| < \epsilon$. Therefore,

$$\forall n, |a_n| < \max\left(\{|a_m|\}_{m=1}^{N-1} \cup \{|a \pm \epsilon|\}\right) \tag{3.7}$$

   $\square$

3.

$$\left.\begin{aligned} a_n \to a \\ b_n \to b \end{aligned}\right\} \implies a_n b_n \to ab \tag{3.8}$$

   Proof. Fix $\epsilon > 0$. For $\epsilon > 0$, $\exists N_a$, N_b such that

$$\begin{aligned} \forall n > N_a |a - a_n| < \epsilon_a > 0 \\ \forall n > N_b |b - b_n| < \epsilon_b > 0 \end{aligned} \tag{3.9}$$

   Since b_n is a convergent sequence, $\exists |b_n| < M_b \in \mathbb{R}$. Let $\epsilon_a = \dfrac{\epsilon}{2M_b}$, $\epsilon_b = \dfrac{\epsilon}{2a}$. For $n > \max(N_a, N_b)$,

$$\begin{aligned} |ab - a_n b_n| &= |ab + ab_n - ab_n - a_n b_n| \\ &\leq a|b - b_n| + b_n|a - a_n| \\ &< a\frac{\epsilon}{2a} + b_n \frac{\epsilon}{2M_b} \\ &< \epsilon \end{aligned} \tag{3.10}$$

   $\square$

4. $a_n \to a \neq 0 \implies \dfrac{1}{a_n} \to \dfrac{1}{b}$.

   Proof. Fix $0 < \epsilon < \frac{|b|}{2}$ (bounding b_n away from 0). $\exists N$ such that $\forall n > N |b - b_n| < \epsilon_b$.

$$\begin{aligned} \implies \left|\frac{1}{b} - \frac{1}{b_n}\right| &= \left|\frac{b - b_n}{bb_n}\right| \\ &< \frac{\epsilon_b}{|b||b_n|} \\ &< \frac{2\epsilon_b}{|b|^2} \end{aligned} \tag{3.11}$$

   Allowing $\epsilon_b = \frac{|b|^2\epsilon}{2}$, we complete the proof. □

5. **Theorem 3.3.1** (Squeeze Theorem)**.** $\forall n > N, L \leftarrow a_n \leq b_n \leq c_n \to L \in \mathbb{R} \implies b_n \to L$.

   Proof. $\forall n, |b_n - a_n| < c_n - a_n \to L - L = 0$. Fix $\epsilon > 0$. $\exists N > 0$ such that $\forall n > N, |L - a_n| < \dfrac{\epsilon}{2}, |c_n - a_n| < \dfrac{\epsilon}{2}$

$$\begin{aligned} \implies |L - b_n| &= |L - a_n + a_n - b_n| \\ &< |b_n - a_n| + |a_n - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned} \tag{3.12}$$

   □

6. $$\left.\begin{array}{r} a_n \to a \\ \forall n, a_n > 0 \end{array}\right\} \implies \forall k \geq 1, \sqrt[k]{a_n} \to \sqrt[k]{a} \tag{3.13}$$

   Proof. Fix $0 < \epsilon \leq \frac{|a|}{2}$. When $a = 0, \exists N$ such that $\forall n > N |a_n| < \epsilon^k$. $\implies \left|\sqrt[k]{a_n}\right| < \sqrt[k]{\epsilon^k} = \epsilon$. Consider when $a > 0$. Let $\epsilon_a > 0$. $\exists N$ such that $\forall n > N, |a - a_n| < \epsilon_a$

   We use the identity

$$A - B = \frac{A^k - B^k}{\sum_{i=0}^{k-1} A^i B^{k-1-i}} \tag{3.14}$$

   When A, $B > 0$, we trivially have $|A - B| < \dfrac{\left|A^k - B^k\right|}{B^{k-1}}$

$$\forall n > N, 0 \leq \left|\sqrt[k]{a} - \sqrt[k]{a_n}\right| < \frac{|a - a_n|}{\sqrt[k]{a^{k-1}}} < \frac{\epsilon_a}{\sqrt[k]{a^{k-1}}} \tag{3.15}$$

   Allowing $\epsilon_a = \epsilon\sqrt[k]{a^{k-1}}$ completes the proof. □

7. $|a_n| \to 0 \implies a_n \to 0$

   Proof. Apply squeeze theorem to $0 \leftarrow (-|a_n|) \leq a_n \leq |a_n| \to 0$. □

8. **Theorem 3.3.2** (Monotone Convergence Theorem)**.** For a_n bounded and monotone, $a_n \nearrow \sup\{a_n\}_n$ or $a_n \searrow \inf\{a_n\}$. (Note on notation: $a_n \nearrow L \iff a_n$ is a monotone sequence $, a_n \to L$.)

Proof. Consider the case when a_n is a monotone increasing sequence. The assumption is without loss of generality, for if a_n is monotone decreasing, we consider $-a_n$ and prove its convergence to $\sup\{-a_n\}_n = -\inf\{a_n\}_n$. By the least upper bound property, we have a unique $L := \sup\{a_n\}_n$. We aim to prove that $\forall \epsilon > 0 \exists N$ such that $\forall n > N, |L - a_n| < \epsilon$. If the statement is not true, then $\exists \epsilon_0 > 0$ such that $\forall k \exists n_k > k, |L - a_{n_k}| \geq \epsilon_0$. Since $a_k \nearrow$, we have $\forall k, a_k \leq a_{n_k} \leq L - \epsilon_0 \implies (L - \epsilon_0) < L$ is an upper bound for $\{a_n\}$. $\Rightarrow\Leftarrow$ $\square$

Example 3.3.1. $-1 < a < 1 \implies a^n \to 0$.

The example is trivial when $a = 0$. Consider the case when $0 < |a| < 1$. We consider the case when $0 < |a| < 1 \implies \frac{1}{|a|} > 1$. Let $\frac{1}{|a|} := 1 + h$.

$$\begin{aligned}
&\left(\frac{1}{|a|}\right)^n = (1+h)^n = 1 + nh + \mathcal{O}(h^2) > 1 + nh \\
&\implies |a|^n < \frac{1}{1+nh} \\
&\implies |a|^n \to 0
\end{aligned} \tag{3.16}$$

Since $a^n \leq |a^n| \leq |a|^n$ we conclude that $a^n \to 0$.

Example 3.3.2. $a_n = \sqrt[n]{n}$.

For $n > 1 \implies \sqrt[n]{n} > \sqrt[n]{1} = 1$. For $h_n > 0$, let $a_n = 1 + h_n$.

Example 3.3.3. $a_n = \dfrac{a^n}{n!}$

Example 3.3.4 (Decimal Expansion)**.**

Definition 3.3.3 (Cauchy Sequences)**.**

$$a_n \text{Cauchy sequence} \iff \forall \epsilon > 0 \exists N \text{ such that } \forall m, n > N, |a_m - a_n| < \epsilon \tag{3.17}$$

$$\liminf_{x\to 0}$$
$$\limsup_{x\to 0} \tag{3.18}$$

Properties of $\liminf, \limsup$

3.3.1 Subsequences

Definition 3.3.4 (Subsequence)**.** A subsequence of a sequence a_n is the sequence $(b_k)_k = (a_{n_k})_k$ where $n. : \mathbb{N} \to \mathbb{N}$ with the restriction that $\forall k, k \leq n_k$.

Definition 3.3.5 (Limit Point)**.**

Properties of subsequential limits

1.

Lemma 3.3.3 (Nested Interval Property)**.**

Theorem 3.3.4 (Bolzano-Weierstrass)**.**

Lemma 3.3.5 (Finite intersection property)**.**

3.4 Series

All those tests

Theorem 3.4.1 (Rearrangement theorem)**.**

Theorem 3.4.2 (Fubini-Tonelli)**.** Summability of infinite matrices

Theorem 3.4.3 (Summability of infinite matrices)**.**

3.5 Sets and Metric Spaces

We restrict ourselves to sets in finite dimensional spaces.

Norm

Theorem 3.5.1 (Cauchy-Schwarz inequality)**.**

Open, closed/ closue, interior, etc.

DeMorgan's Laws (Unions and Intersections)

The Cantor set

Definition 3.5.1 (Open Ball)**.** The open ball of radius r around $x_0 \in X$ is defined as follows:

$$B(x_0, r) = \{x \in X | \|x - x_0\| < r\} \subset X \tag{3.19}$$

Precompact, compact, bounded.

explain relation between precompact, compact, bounded, closed in finite dimensions.

3.6 Continuous Functions

We consider functions between metric spaces.

Definition 3.6.1 (Continuity)**.** We say a function $f : X \to Y$ is continuous at $x_0 \in X$

equivalance between different definitions of continuity.

Inverse images

darboux integral, shuffling limits of integrals, prove everything from calculus

Chapter 4

Sobolev Spaces

4.1 Multi-Index Notation

A multi-index is a d-tuple of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_d)$. The degree of a multi-index α is $|\alpha| = \sum_{i=1}^{d} \alpha_i$. Consider a point in d-dimensional real Euclidean space $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. We denote by x^α the monomial $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. If $\mathrm{D}_{x_i} = \partial_{x_i}$, denotes the partial differential operator with respect to variable x_i, then D^α denotes a differential operator of order $|\alpha|$.

$$\mathrm{D}^\alpha = \mathrm{D}_{x_1}^{\alpha_1} \cdots \mathrm{D}_{x_d}^{\alpha_d} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \tag{4.1}$$

For two multi-indices β, α, we say $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for $1 \leq i \leq d$. In this case, $\alpha \pm \beta$ are also multi-indices with elements $\alpha_i \pm \beta_i$ and order $|\alpha \pm \beta| = |\alpha| \pm |\beta|$. We also denote $\alpha! = (\alpha_1!, \ldots, \alpha_d!)$ and if $\beta \leq \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \left(\binom{\alpha_1}{\beta_1}, \ldots, \binom{\alpha_d}{\beta_d} \right) \tag{4.2}$$

Theorem 4.1.1. *Multinomial Theorem*

$$(x_1 + \cdots + x_d)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} x^\alpha \tag{4.3}$$

Proof. todo □

Theorem 4.1.2 (Leibniz Formula)**.**

$$\mathrm{D}^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathrm{D}^\beta u \, \mathrm{D}^{\alpha-\beta} v \tag{4.4}$$

Proof. We prove the Leibniz formula by induction on $|\alpha|$. It is trivial to show that the equality holds for $|\alpha| = 0, 1$. Assuming the equality holds for for some $\alpha \neq 0$, we prove that the equality holds for $\alpha + e_i$ where is a multi-index with 1 in the i^{th} position $\forall i \in \{1, \cdots, d\}$ and zeros elsewhere.

$$\begin{aligned} \mathrm{D}^{\alpha+e_i}(uv) &= \mathrm{D}^{e_i}\left(\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathrm{D}^\beta u \, \mathrm{D}^{\alpha-\beta} v \right) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathrm{D}^{e_i}\big(\mathrm{D}^\beta u \, \mathrm{D}^{\alpha-\beta} v \big) \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathrm{D}^{\beta+e_i} u \, \mathrm{D}^{\alpha-\beta} v + \binom{\alpha}{\beta} \mathrm{D}^\beta u \, \mathrm{D}^{\alpha-\beta+e_i} v \end{aligned} \tag{4.5}$$

□

Theorem 4.1.3. *(Taylor's Formula)* For sufficiently smooth functions $f : \mathbb{R}^d \to \mathbb{R}$,

$$f(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} \mathrm{D}^\alpha f(x_0)(x - x_0)^\alpha + \mathcal{O}(|x|^{k+1}) \tag{4.6}$$

Proof. todo □

4.2 Functionals, Dual Spaces, and Norms

For the purposes of this text, a topological vector space (TVS) is a vector space for which the vector space operations of addition and scalar multiplication are continuous. That is for $x, y \in X$, $c \in \mathbb{C}$ where X is some TVS, the maps $(x, y) \to x + y$ and $(c, x) \to cx$ are continuous.

A **functional** on a vector space X is a scalar valued function $f : X \to \mathbb{C}$. The functional is linear if for $x, y \in X$, $a, b \in \mathbb{C}$, $f(ax + by) = af(x) + bf(y)$. If X is a TVS, a functional on X is continuous if it is continuous from X into $\mathbb{C}$ where $\mathbb{C}$ has the usual topology induced by the Euclidean metric. The set of continuous, linear functionals on X is called the **dual** of X and is denoted by X', which is also a vector space under pointwise addition and scalar multiplication.

A **norm** on a vector space X is a real-valued functional $\|\cdot\| : X \to \mathbb{R}$ such that for $x, y \in X$, $c \in \mathbb{C}$,

- $\|x\| \ge 0$ with equality holding iff $x = 0$.
- $\|cx\| = |c|\|x\|$
- $\|x + y\| \le \|x\| + \|y\|$

A TVS is normable if its topology coincides with that induced by some norm. Two norms $\|\cdot\|_1$, $\|\cdot\|_2$ are equivalent if for all $x \in X$, there exists some $c \in \mathbb{R}$ such that

$$c\|x\|_1 \le \|x\|_2 \le \frac{1}{c}\|x\|_1 \tag{4.7}$$

If X is a normed space and all Cauchy sequences converge in that norm, then X is called a **Banach space**. A vector space X is called **pre-Hilbert** if there exists a functional called the **scalar product** $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ such that $\forall u, v \in X$,

- $\langle u, u \rangle \ge 0$. $\langle u, u \rangle = 0 \iff u = 0$
- $\langle u, v \rangle = \langle v, u \rangle$
- $\langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle$, $u_1, u_2 \in X$
- $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$, $\lambda \in \mathbb{R}$.

The scalar product induces a norm $\|u\| = \sqrt{\langle u, u \rangle}$. Another functional defined on a vector space X is the **inner product** $(\cdot, \cdot) : X \times X \to \mathbb{C}$ such that for $x, y, z \in X$, $a, b \in \mathbb{C}$,

- $(x, y) = \overline{(y, x)}$
- $(ax + by, z) = a(x, z) + b(y, z)$

- $(x,x) = 0 \iff x = 0$

If X is a Banach space under a norm that is induced by an inner product, then X is called a **Hilbert space**. Given an inner-product, a norm on X can be specified as follows: for $x \in X$,

$$\|x\| = (x,x)^{1/2} \tag{4.8}$$

We define an induced norm on the dual space X' of some vector space X that has norm $\|\cdot\|_X$. For some $x' \in X'$,

$$\|x'\|_{X'} = \sup_{x \in X,\, x \neq 0} \frac{|x'(x)|}{\|x\|_X} \tag{4.9}$$

Theorem 4.2.1. *(Riesz Representation Theorem)* For a Hilbert space X, a linear functional x' on X belongs in X' if and only if there exists an $y \in X$ such that for every $x \in X$,

$$x'(x) = (x,y)_X \tag{4.10}$$

Proof. todo □

We say that the normed space X is imbedded in the normed space Y and write $X \to Y$ if:

- X is a vector subspace of Y.
- the identity operator, I defined on X into Y is continuous. Since I is linear, it is equivalent to the existence of a constant M such that, for $x \in X$,

$$\|I(x)\|_Y \leq M\|x\|_X \tag{4.11}$$

We consider an open subset $\Omega \in \mathbb{R}^d$ as domain in d-dimensional Euclidean space. For some $S \subset \Omega$, we denote by $\overline{S}$ the closure of S. We write $S \subset\subset \Omega \subset \mathbb{R}^d$ if $\overline{S} \subset \Omega$ and $\overline{S}$ is compact. If u is a function defined on Ω, we define the support of u as

$$\text{supp}(u) = \overline{\{x \in \Omega | u(x) \neq 0\}} \tag{4.12}$$

We say $A \subset\subset B$ if and only if $\overline{A}$ is a compact subset of B. If, for some function u, $\text{supp}(u) \subset\subset \Omega$, we say that u has **compact support** in Ω.

4.3 Hölder Spaces

We define the following family of vector spaces. For some integer m, the vector space $C^m(\Omega)$ consists of all functions defined on Ω which, along with their partial derivatives up to order m are continuous.

$$\begin{aligned} &C^m(\Omega) = \{\phi : \Omega \to \mathbb{R} : \forall \alpha \text{ with } |\alpha| \leq m,\ \mathrm{D}^\alpha \phi \text{ is continuous}\} \\ &C^\infty(\Omega) = \cap_{m=0}^{\infty} C^m(\Omega) \end{aligned} \tag{4.13}$$

We abbreviate $C^0(\Omega)$ to $C(\Omega)$. The subspace $C_0^m(\Omega)$ consists of all functions in $C^m(\Omega)$ that have compact support in Ω. We wish to define equivalent spaces for $\overline{\Omega}$, $C^m(\overline{\Omega})$. However, since Ω is open, continuous functions on Ω need not be bounded. Therefore, not all functions in $C^m(\Omega)$ can be continuously extended to $\overline{\Omega}$. For example, the function $f(x) = 1/x$ is continuous on the open interval $(0,1)$, but discontinuous on $x = 0$. Functions that are uniformly continuous and bounded on Ω can be uniquely and continuously extended to $\overline{\Omega}$. We define $C^(\overline{\Omega})$ as follows:

$$C^m(\overline{\Omega}) = \{\phi \in C^m(\Omega) : \forall \alpha,\ |\alpha| \leq m,\ \mathrm{D}^\alpha \phi \text{ bounded and uniformly continuous on } \Omega\} \tag{4.14}$$

A function $f : \Omega \to \mathbb{R}$ is said to be **Hölder continuous** for exponent γ ($0 < \gamma \leq 1$) if

$$\exists K \geq 0 \, \forall x, y \in \Omega, x \neq y, \frac{|f(x) - f(y)|}{\|x - y\|} \leq K \tag{4.15}$$

We define $C^{m,\gamma}(\overline{\Omega})$ to be the subspace of $C^m(\overline{\Omega})$ consisting of functions for which, for $|\alpha| = m$, $\mathrm{D}^\alpha \phi$ satisfies the Hölder condition. We define the following norms and seminorms:

$$\|u\|_{C^0(\overline{\Omega})} = \sup_{x \in \Omega} |\mathrm{D}^\alpha u| \qquad |u|_{C^{0,\gamma}(\overline{\Omega})} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{\|x - y\|_2} \tag{4.16}$$

$$\|u\|_{C^m(\overline{\Omega})} = \max_{0 \leq |\alpha| \leq m} \|u\|_{C^0(\overline{\Omega})} \qquad \|u\|_{C^{m,\gamma}(\overline{\Omega})} = \|u\|_{C^m(\overline{\Omega})} + \max_{|\alpha| = m} |\mathrm{D}^\alpha u|_{C^{0,\gamma}(\overline{\Omega})} \tag{4.17}$$

Theorem 4.3.1. $C^{m,}(\Omega)$ is complete for nonnegative integer m and $0 < \gamma \leq 1$ with respect to the norm $\|\cdot\|_{C^{m,\gamma}(\overline{\Omega})}$.

Proof. Observe that $\forall x \in \Omega$, $\{u_n(x)\} \subset \mathbb{R}$ form a Cauchy sequence. By completeness of $\mathbb{R}, \forall x, u_x(x) \to u(x)$. It is left to prove that $\lim_{n \to \infty} \|u - u_n\|_{C^{m,\gamma}(\Omega)} = 0$. ☐

For nonnegative integer m and $0 < \nu \leq \lambda \leq 1$, we prove the following imbeddings:

- $C^{m+1}(\overline{\Omega}) \to C^m(\overline{\Omega})$
  It is clear that $C^{m+1}(\overline{\Omega}) \subset C^m(\overline{\Omega})$ and $\|\phi\|_{C^m(\overline{\Omega})} \leq \|\phi\|_{C^{m+1}(\overline{\Omega})}$
- $C^{m,\lambda}(\overline{\Omega}) \to C^m(\overline{\Omega})$
  It is clear that $C^{m,\lambda}(\overline{\Omega}) \subset C^m(\overline{\Omega})$ and $\|\phi\|_{C^m(\overline{\Omega})} \leq \|\phi\|_{C^{m,\lambda}(\overline{\Omega})}$
- For $0 < \nu < \lambda \leq 1$, $C^{m,\lambda}(\overline{\Omega}) \to C^{m,\nu}(\overline{\Omega})$
  Consider $\phi \in C^{m,\lambda}(\overline{\Omega})$. There exists K such that for all α with $|\alpha| \leq m$, for all $x, y \in \Omega$, $x \neq y$

$$\begin{aligned} \frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x - y|^\lambda} &< K \\ \frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x - y|^\nu} &< K|x - y|^{\lambda - \nu} \end{aligned} \tag{4.18}$$

  The term $|x - y|^{\lambda - \nu}$ is bounded for bounded Ω. Hence, $C^{m,\lambda}(\overline{\Omega}) \subset C^{m,\nu}(\overline{\Omega})$. Next, we compare the norms of $C^{m,\lambda}(\overline{\Omega})$ and $C^{m,\nu}(\overline{\Omega})$. For some $\phi \in C^{m,\lambda}(\overline{\Omega})$, we note that

$$\begin{aligned} \sup_{\substack{x,y \in \Omega \\ |x-y| < 1}} \frac{|\mathrm{D}^\alpha \phi(x) - \mathrm{D}^\alpha \phi(y)|}{|x - y|^\nu} &\leq \sup_{\substack{x,y \in \Omega \\ |x-y| < 1}} \frac{|\mathrm{D}^\alpha \phi(x) - \mathrm{D}^\alpha \phi(y)|}{|x - y|^\lambda} \\ \sup_{\substack{x,y \in \Omega \\ |x-y| \geq 1}} \frac{|\mathrm{D}^\alpha \phi(x) - \mathrm{D}^\alpha \phi(y)|}{|x - y|^\nu} &\leq 2 \sup_{x \in \Omega} |\mathrm{D}^\alpha \phi(x)| \end{aligned} \tag{4.19}$$

  Hence, $\|\phi\|_{C^{m,\nu}(\overline{\Omega})} \leq 3\|\phi\|_{C^{m,\lambda}(\overline{\Omega})}$

4.4 Measure Theory

The **measure** of a set refers to its size in some measure space. We say a set $S \subset \mathbb{R}^d$ has **measure zero** ($\mu(S) = 0$) if $\forall \varepsilon > 0$, S can be covered by open balls of total volume less than ε. $\mathbb{Q} \subset \mathbb{R}$, C^1 curves in $\mathbb{R}^2$,

C^1 surfaces in $\mathbb{R}^3$ are examples of sets with measure zero. A countable union of sets of measure zero has measure zero. We say that a condition holds **almost everywhere (a.e.)** if the points where it does not hold form a set of zero measure. For example, the characteristic function for rationals on the set of real numbers is equal to zero almost everywhere.

Definition 4.4.1 (Measurable function)**.** A function is measurable if it coincides a.e. with the limit of a sequence of piecewise continuous functions which is convergent almost everywhere.

We define the characteristic function of a set $A \subset \mathbb{R}^d$ as follows:

$$\chi_A(x) = \left\{ \begin{array}{ll} 1 & x \in A \\ 0 & x \notin A \end{array} \right. \tag{4.20}$$

We say that a set is measurable if its characteristic function is measurable. $\mathbb{Q}$ is measurable since its characteristic function coincides with zero almost everywhere. A countable union of measurable sets is also measurable.

Definition 4.4.2 (Piecewise Continuous)**.** $g : \mathbb{R}^d \to \mathbb{R}$ is piecewise continuous if there exist disjoint, open and connected domains $\{D_i\}_{i \in I \subset \mathbb{N}}$ with piecewise C^1 boundary such that any sphere can be covered by finitely many $\overline{D_i}$. Further, g is continuous on each D_i and can be continuously extended to the boundary of D_i.

4.4.1 Lebesgue Measure

Consider a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, $f(x) \geq 0$. One can construct a nondecreasing sequence of piecewise continuous functions $\{g_k\}$ with compact support, convergent almost everywhere to f. We say that f is **Lebesgue integrable** if the sequence of Riemann integrals

$$\int_{\mathbb{R}^d} g_k(x)\,\mathrm{d}x \tag{4.21}$$

has an upper bound (hence a limit since g_k is a nondecreasing). One can show that $\int_{\mathbb{R}^d} g_k(x)\,\mathrm{d}x$ does not depend on the choice of sequence g_k.

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_{\mathbb{R}^d} g_k(x)\,\mathrm{d}x \tag{4.22}$$

For example, $\chi_{\mathbb{Q}}$ is not Riemann integrable since it discontinuous everywhere, but is Lebesgue integrable since it is equal to zero a.e. We define the Lebesgue integral of a measurable function f over a measurable subset Ω of $\mathbb{R}^d$ as follows:

$$\int_{\Omega} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\chi_\Omega(x)\,\mathrm{d}x \tag{4.23}$$

We say a measurable function $u : \mathbb{R}^d \to \Omega$ is **summable** if $\int_{\mathbb{R}^d} |u|\,\mathrm{d}x < \infty$ and define the set of summable functions

$$L^1(\mathbb{R}^d) = \{u : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^d} |u(x)|\,\mathrm{d}x < \infty\} \tag{4.24}$$

A measurable function is **locally integrable** on $\Omega \subset \mathbb{R}^d$ if it is integrable on any compact $K \subset \Omega$. The set of locally integrable functions on Ω is denoted $L^1_{\text{loc}}(\Omega)$.

$$L^1_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R} : \forall K \subset \Omega, \int_K u(x)\,\mathrm{d}x < \infty\} \tag{4.25}$$

Any piecewise continuous function is locally integrable in $\mathbb{R}^d$. An important fact is that summable functions are "approximately continuous" at every point.

Theorem 4.4.1 (Lebesgue's Differentiation Theorem)**.** Let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then for a.e. point in $\mathbb{R}^d$,

$$\lim_{r\to 0} \int_{B_{x_0}(r)} u(x)\,\mathrm{d}x = u(x_0) \tag{4.26}$$

4.5 Distributions

We restate the divergence theorem: for $u \in C^1(\overline{\Omega})$ in scalar form (this can be derived from the vector form by considering a vector field with scalar u being the i^{th} component, and all other components being zero),

$$\int_{\partial\Omega} u n_i \,\mathrm{d}S = \int_\Omega \partial_{x_i} u \,\mathrm{d}x \tag{4.27}$$

where n_i is the i^{th} component of the unit normal vector defined on $\partial\Omega$. We derive the formula for integration by parts: for $u, v \in C^1(\Omega)$,

$$\begin{aligned} \int_{\partial\Omega} uvn_i \,\mathrm{d}S = \int_\Omega \partial_{x_i}(uv) \,\mathrm{d}x = \int_\Omega u\partial_{x_i} v + v\partial_{x_i} v \,\mathrm{d}x \\ \int_\Omega u\partial_{x_i} v \,\mathrm{d}x = \int_{\partial\Omega} uvn_i \,\mathrm{d}S - \int_\Omega v\partial_{x_i} v \,\mathrm{d}x \end{aligned} \tag{4.28}$$

The space $C_0^\infty(\Omega)$ contains infinitely smooth functions with compact support on Ω. We call functions belonging to $C_0^\infty(\Omega)$ **test functions**. A key feature of test functions is that their extension to $\partial\Omega$ is the zero function as they are continuous and compactly supported in a subset of Ω. Let u, $v \in L^1_{\text{loc}}(\Omega)$ and α be a multi-index. We say that v is the α^{th} **weak derivative** of u if

$$\forall \phi \in C_0^\infty(\Omega), \int_\Omega u \,\mathrm{D}^\alpha \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\phi \,\mathrm{d}x \tag{4.29}$$

It is straightforward to show that the weak derivative of u, if it exists, is uniquely defined up to a set of measure zero. Further, if $u \in C^{|\alpha|}(\Omega)$, its weak derivative is equal to its classical derivative.

4.6 Convolutions and Smoothing

A **mollifier**, $J(x)$, is a nonnegative, real-valued function belonging to $C_0^\infty(\mathbb{R}^d)$ used to create smooth functions approximating non-smooth functions via convolutions. Mollifiers satisfy the condition that $\int_{\mathbb{R}^d} J(x) \,\mathrm{d}x = 1$. We define the standard mollifier

$$J(x) = \begin{cases} C \exp\left\{\frac{-1}{1-|x|^2}\right\} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \tag{4.30}$$

where C is chosen such that the integral of J over $\mathbb{R}^d$ is equal to 1. For $\varepsilon > 0$, we define

$$J_\varepsilon(x) = \varepsilon^{-d} J(x/\varepsilon) \tag{4.31}$$

which is compactly supported in the ball of radius ε around the origin. If $u \in L^1_{\text{loc}}(\mathbb{R}^d)$, we define the mollification of u as the convolution

$$u_\varepsilon(x) = J_\varepsilon * u(x) = \int_{\mathbb{R}^d} J_\varepsilon(x-y)u(y) \,\mathrm{d}y \tag{4.32}$$

We define $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}$.

Theorem 4.6.1 (Properties of mollifiers)**.** Let $u : \mathbb{R}^d \to \mathbb{R}$ and vanishes identically outside Ω.

1. If $u \in L^1_{\text{loc}}(\overline{\Omega})$, $u_\varepsilon \in C_0^\infty(\mathbb{R}^d)$
2. If also u has compact support in Ω, then $u_\varepsilon \in C_0^\infty(\Omega)$ provided $\varepsilon < \text{dist}(\text{supp}(u), \partial\Omega)$

3. $u_\varepsilon \to u$ as $\varepsilon \to 0$.
4. if $G \subset\subset \Omega$ and $u \in C(\Omega)$, then $u_\varepsilon \to u$ uniformly on G.
5. If $u \in C(\overline{\Omega})$, then $u_\varepsilon \to u$ uniformly on Ω.

Proof.

$$\mathrm{D}^\alpha (J_\varepsilon * u)(x) = \int_{\mathbb{R}^d} (\,\mathrm{D}_x^\alpha J_\varepsilon(y-x))u(y)\,\mathrm{d}y \tag{4.33}$$

$J_\varepsilon(y-x)$ is infinitely differentiable, and vanishes outside $B_x(\varepsilon)$ for every x for every multi-index α. If $\mathrm{supp}(u) \subset\subset \Omega$, and $\varepsilon < \mathrm{dist}(\mathrm{supp}(u), \partial\Omega)$, then $\forall x \notin \Omega$, $u_\varepsilon(x) = 0$. Therefore, u_ε has compact support in Ω. From the Lebesgue differentiation theorem,

$$\lim_{r\to 0} \int_\Omega |u(y) - u(x)|\,\mathrm{d}y = 0 \tag{4.34}$$

a.e. on Ω. Fix such a point x on Ω. Then,

$$|u_\varepsilon(x) - u(x)| = \left| \int_\Omega J_\varepsilon(y-x)u(y)\mathrm{vol}(\Omega)u(x)\,\mathrm{d}y \right| \tag{4.35}$$

□

4.7 The Spaces $L^p(\Omega)$

Let Ω be a domain in $\mathbb{R}^d$ and let $p \in \mathbb{R}^+$. We denote by $L^p(\Omega)$ the class of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |u(x)|^p\,\mathrm{d}x < \infty \tag{4.36}$$

The elements of $L^p(\Omega)$ are equivalent classes of functions that are equal almost everywhere on Ω that satisfy the above inequality. We show that $L^p(\Omega)$ is a vector space. We say $u = 0$ in $L^p(\Omega)$ if u is zero almost everywhere on Ω (i.e. $u(x) = 0$ a.e. in Ω). The restriction to Ω of any piecewise continuous, compactly supported function on $\mathbb{R}^d$ would belong to an equivalent class of functions in $L^p(\Omega)$. For $u, v \in L^p(\Omega)$, $x \in \Omega$, $c \in \mathbb{C}$,

$$|u(x) + v(x)|^p \leq (|u(x)| + |v(x)|)^p \leq 2^p(|u(x)|^p + |v(x)|^p) \tag{4.37}$$

Hence $u + v \in L^p(\Omega)$. Clearly $cu(x) \in L^p(\Omega)$. Therefore $L^p(\Omega)$ is a vector space. We claim that the function $\|\cdot\|_p$ defined below is a norm on $L^p(\Omega)$ for $1 \leq p < \infty$. $\|\cdot\|_p$ is not a norm for $0 < p < 1$.

$$\|u\|_p = \left\{ \int_\Omega |u(x)|^p\,\mathrm{d}x \right\}^{1/p} \tag{4.38}$$

For $u \in L^p(\Omega)$, it is clear that $\|u\|_p \geq 0$ with equality holding if and only if $u = 0$ in $L^p(\Omega)$. Moreover, $\|cu\|_p = |c|\|u\|_p$. It remains to be shown that $\|\cdot\|_p$ satisfies the triangle inequality, in $L^p(\Omega)$, which is known as *Minkowski's inequality*. The condition certainly holds for $p = 1$, since

$$\int_\Omega |u(x) + v(x)|\,\mathrm{d}x \leq \int_\Omega |u(x)|\,\mathrm{d}x + \int_\Omega |v(x)|\,\mathrm{d}x \tag{4.39}$$

For $p > 1$, we denote by p' the number $\frac{p}{p-1}$ so that $p' > 1$ and

$$\frac{1}{p} + \frac{1}{p'} = 1 \tag{4.40}$$

We call p' the exponential conjugate to p.

Theorem 4.7.1. *(Hölder's inequality)* For $p > 1$, $u \in L^p(\Omega)\, v \in L^{p'}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \leq \|u\|_p \|v\|_{p'} \tag{4.41}$$

Proof. If either $\|u\|_p = 0$ or $\|v\|_{p'} = 0$ then $u(x)v(x) = 0$ almost everywhere and the inequality is satisfied. Otherwise, consider the function $f : \mathbb{R}^+ \to \mathbb{R}$, $f(t) = t^p/p + 1/p' - t$. The only critical point of the function is at $f(1) = 0$, which is a minimum. Hence for all t,

$$t \leq \frac{t^p}{p} + \frac{1}{p'} \tag{4.42}$$

For $a, b \geq 0$, we substitute $t = ab^{-p'/p}$,

$$\begin{aligned} ab^{-p'/p} &\leq \frac{a^p b^{-p'}}{p} + \frac{1}{p'} \\ ab^{-p'/p+p'} &\leq \frac{a^p b^{-p'+p'}}{p} + \frac{b^{p'}}{p'} \\ ab &\leq \frac{a^p}{p} + \frac{b^{p'}}{p'} \end{aligned} \tag{4.43}$$

with equality occurring if and only if $a^p = b^{p'}$. Substitute $a = \frac{|u(x)|}{\|u\|_p}$, $b = \frac{|v(x)|}{\|v\|_{p'}}$ and integrate over Ω.

$$\begin{aligned} \int_\Omega \frac{|u(x)||v(x)|}{\|u\|_p \|v\|_{p'}}\,\mathrm{d}x &\leq \frac{1}{p}\int_\Omega \frac{|u(x)|^p}{\|u\|_p^p}\,\mathrm{d}x + \frac{1}{p'}\int_\Omega \frac{|v(x)|^{p'}}{\|v\|_{p'}^{p'}}\,\mathrm{d}x = \frac{1}{p} + \frac{1}{p'} \\ \int_\Omega |u(x)v(x)|\,\mathrm{d}x &\leq \int_\Omega |u(x)||v(x)|\,\mathrm{d}x \leq \|u\|_p \|v\|_{p'} \end{aligned} \tag{4.44}$$

Hence, $uv \in L^1(\Omega)$. □

Theorem 4.7.2. *(Minkowski's Inequality)* Triangle Inequality for $p \geq 1$, $u, v \in L^p(\Omega)$.

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p \tag{4.45}$$

Proof. Since the case with $p = 1$ is trivial, consider $p > 1$. For $u, v \in L^p(\Omega)$, $u + v \in L^p(\Omega)$.

$$\int_\Omega |u(x) + v(x)|^{p'(p-1)}\,\mathrm{d}x = \int_\Omega |u(x) + v(x)|^p\,\mathrm{d}x < \infty \tag{4.46}$$

Hence, $(u + v)^{p-1} \in L^{p'}(\Omega)$.

$$\begin{aligned} \|u + v\|_p^p = \int_\Omega |u(x) + v(x)|^p\,\mathrm{d}x &\leq \int_\Omega |u(x) + v(x)|^{p-1}(|u(x)| + |v(x)|)\,\mathrm{d}x \\ \int_\Omega |u(x) + v(x)|^p\,\mathrm{d}x &\leq \left\{\int_\Omega |u(x) + v(x)|^{p'(p-1)}\,\mathrm{d}x\right\}^{1/p'} \left\{\int_\Omega ||u(x)| + |v(x)||^p\,\mathrm{d}x\right\}^{1/p} \\ \left\{\int_\Omega |u(x) + v(x)|^p\,\mathrm{d}x\right\}^{1-1/p'} &\leq \left\{\int_\Omega ||u(x)| + |v(x)||^p\,\mathrm{d}x\right\}^{1/p} \\ \left\{\int_\Omega |u(x) + v(x)|^p\,\mathrm{d}x\right\}^{1/p} &\leq \left\{\int_\Omega |u(x)|^p\,\mathrm{d}x\right\}^{1/p} + \left\{\int_\Omega |v(x)|^p\,\mathrm{d}x\right\}^{1/p} \\ \|u + v\|_p &\leq \|u\|_p + \|v\|_p \end{aligned} \tag{4.47}$$

□

Therefore, $\|\cdot\|_p$ is a norm on $L^p(\Omega)$.

A measurable function u on Ω is said to be essentially bounded on Ω if there exists a constant K such that $|u(x)| \leq K$ a.e. on Ω. The greatest lower bound of such constants K is called the essential supremum of $|u|$ on Ω and is denoted by $\operatorname{ess\,sup}_{x\in\Omega} |u(x)|$. We denote by $L^\infty(\Omega)$ the vector space consisting of all equivalent classes of functions u (that are equal almost everywhere) which are essentially bounded on Ω.

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \tag{4.48}$$

Hölder's inequality clearly extends to cover this case with $p = \infty$, $p' = 1$. We now prove an imbedding theorem on $L^p(\Omega)$.

Theorem 4.7.3. Suppose $\mathrm{vol}\Omega = \int_\Omega \mathrm{d}x$, and $1 \leq p \leq q \leq \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and

$$\|u\|_p \leq (\mathrm{vol}\Omega)^{1/p-1/q} \|u\|_q \tag{4.49}$$

Hence, $L^q(\Omega) \to L^p(\Omega)$. If $u \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|u\|_p = \|u\|_\infty \tag{4.50}$$

Finally, if $u \in L^p(\Omega)$ for $1 \leq p < \infty$, and if there is a constant K such that for all p, $\|u\|_p \leq K$, then $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leq K$.

Proof. Consider $u \in L^q(\Omega)$. If $p = q$, the imbedding theorem is trivial. For $1 \leq p < q \leq \infty$, Hence $u^p \in L^{q/p}(\Omega)$. Then, Hölder's inequality gives us

$$\begin{aligned} \int_\Omega |u(x)|^p \,\mathrm{d}x &\leq \left\{ \int_\Omega |u(x)|^{p(q/p)} \,\mathrm{d}x \right\}^{p/q} \left\{ \int_\Omega 1 \,\mathrm{d}x \right\}^{1-p/q} \\ \|u\|_p &\leq \mathrm{vol}(\Omega)^{1/p-1/q} \|u\|_q \end{aligned} \tag{4.51}$$

Hence, for $1 \leq p \leq q \leq \infty$, $L^q(\Omega) \to L^p(\Omega)$. If $u \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|u\|_p \leq \|u\|_\infty \tag{4.52}$$

On the other hand, $\forall \varepsilon > 0\, \exists A \subset \Omega$ of nonzero measure $\mu(A)$ such that $\forall x \in A$,

$$\begin{aligned} |u(x)| &\geq \|u\|_\infty - \varepsilon \\ \int_\Omega |u(x)|^p \,\mathrm{d}x &\geq \int_A |u(x)|^p \,\mathrm{d}x \geq \mu(A)(\|u\|_\infty - \varepsilon)^p \\ \|u\|_p &\geq (\mu(A))^{1/p}(\|u\|_\infty - \varepsilon) \\ \lim_{p\to\infty} \|u\|_p &\geq \|u\|_\infty \end{aligned} \tag{4.53}$$

Therefore,

$$\lim_{p\to\infty} \|u\|_p = \|u\|_\infty \tag{4.54}$$

Finally, suppose $u \in L^p(\Omega)$ for $1 \leq p < \infty$ and $\exists K$ such that for all such p, $\|u\|_p \leq K$. Suppose, *ad absurdum*, u is essentially bounded. Then $\exists K_1 > K$, $A \subset \Omega$ with $\mu(A) > 0$ such that $\forall x \in A$, $|u(x)| \geq K_1$. Applying the same argument as before, we get

$$\lim_{p\to\infty} \|u\|_p \geq K_1 \tag{4.55}$$

which is a contradiction. $\square$

Theorem 4.7.4. If Ω is measurable, then $L^p(\Omega)$ is a Banach space for $1 \leq p \leq \infty$.

Proof. Consider

todo ∎

4.7.1 Interpolation in $L^p(\Omega)$ Spaces

Theorem 4.7.5. For $f \in L^p(\Omega) \cap L^q(\Omega)$, $1 \le p \le q \le \infty$, then $\forall r \in [p, q]$, $f \in L^r(\Omega)$ and $\|f\|_r \le \|f\|_p^{\alpha} \|f\|_q^{1-\alpha}$ where $\dfrac{\alpha}{p} + \dfrac{1-\alpha}{q} = \dfrac{1}{r}$.

Proof. The proof is trivial for $r = p$ or $r = q$. If $p < r < q$, then $\dfrac{1}{p} > \dfrac{1}{r} > \dfrac{1}{q}$. Then, there exists α such that

$$\alpha \frac{1}{p} + (1-\alpha)\frac{1}{q} = \frac{1}{r} \tag{4.56}$$

Then, for any $m, n \ge 1$ with $\dfrac{1}{m} + \dfrac{1}{n} = 1$, we have

$$\int_{\Omega} |f(x)|^r \,\mathrm{d}x = \int_{\Omega} |f(x)|^{\alpha r} |f(x)|^{(1-\alpha) r} \le \left\{ \int_{\Omega} |f(x)|^{r\alpha m} \,\mathrm{d}x \right\}^{1/m} \left\{ \int_{\Omega} |f(x)|^{r(1-\alpha)n} \,\mathrm{d}x \right\}^{1/n} \tag{4.57}$$

Choose $m = \dfrac{p}{r\alpha}$ and $m = \dfrac{q}{r(1-\alpha)}$ if $\alpha \ne 1$. If $\alpha = 1$, choose $n = \infty$.

$$\int_{\Omega} |f(x)|^r \,\mathrm{d}x \le \|f\|_p^{\alpha} \|f\|_q^{1-\alpha} \le \infty \tag{4.58}$$

Therefore $f \in L^r(\Omega)$. ∎

We now consider the normed dual of $L^p(\Omega)$.

4.8 The Spaces $W^{m,p}(\Omega)$

We introduce Sobolev spaces of integer order and establish some of their basic properties. These are vector subspaces of various spaces $L^p(\Omega)$. We define the functional $\|\cdot\|_{m,p}$ where m is a nonnegative integer and $1 \le p \le \infty$ as follows:

$$\begin{aligned} \|u\|_{m,p} &= \left\{ \sum_{0 \le |\alpha| \le m} \|\mathrm{D}^{\alpha} u\|_p^p \right\}^{1/p} \\ \|u\|_{m,\infty} &= \max_{0 \le |\alpha| \le m} \|\mathrm{D}^{\alpha} u\|_{\infty} \end{aligned} \tag{4.59}$$

for any function for which the right side makes sense. The above functional defines a norm on any vector space of functions where the right side takes finite values provided functions are identified in the space if they are equal almost everywhere in Ω. We consider three such spaces corresponding to any given values of m and p.

- $H^{m,p}(\Omega) \equiv$ the completion of $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}$
- $W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) : \, \mathrm{D}^\alpha u \in L^p(\Omega) \, \text{for} \, 0 \leq |\alpha| \leq m\}$
- $W_0^{m,p}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Any compactly supported function in $C^\infty(\Omega)$ belongs to $W^{m,p}(\Omega)$ for any m. If Ω is bounded, then $C^m(\Omega) \subset W^{m,p}(\Omega)$ for any $1 \leq p \leq \infty$.

Theorem 4.8.1. The space $W^{m,p}(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{m,p}$ where $m \in \mathbb{N}$, $1 \leq p \leq \infty$.2

Proof. Consider the sequence $\{u_n\}_n \subset W^{m,p}(\Omega)$ such that $\forall \varepsilon > 0 \exists N \, \text{s.t.} \, \forall m, n \geq N$, $\|u_m - u_n\|_{m,p} < \varepsilon$.

todo □

4.8.1 Trace Operator

A set Ω is said to have the **segment property** if $\forall x_0 \in \partial\Omega$, there exists an open neighbourhood of x_0 called Ω_{x_0} and a nonzero direction $y_{x_0} \in \mathbb{R}^d$ such that $\forall t \in [0, 1]$, $\overline{\Omega} \cap \Omega_{x_0} + t y_{x_0} \subset \Omega$. Sets with the segment property are only present on one side of their boundary. Balls, polytopes and open sets with C^1 boundary have the segment property. However, not all sets with piecewise C^1 boundary have the segment property.

Theorem 4.8.2. Let $\Omega \subset \mathbb{R}^d$ be an open set with a bounded, piecewise C^1 boundary. If Ω has the segment property, then for $1 \leq p < \infty$, there exists the trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$. T is linear and continuous and $Tf = f|_{\partial\Omega}$

Proof. First we define T on $C_0^\infty(\mathbb{R}^d)$ and then extend it by density to $W^{1,p}(\Omega)$. For $f \in C_0^\infty(\mathbb{R}^d)$, $Tf = f|_{\partial\Omega}$. Clearly, T is linear. It remains to be shown that T is continuous with respect to the norm in $W^{1,p}(\Omega)$ and $L^p(\Omega)$, i.e. there exists constant K such that $\forall f \in C_0^\infty(\mathbb{R}^d)$,

$$\|Tf\|_{L^p(\partial\Omega)} \leq K \|f\|_{W^{1,p}(\Omega)} \tag{4.60}$$

□

Bibliography

[1] R.H. Ewoldt, *"Intermediate Fluid Mechanics"*, Department of Mechanical Science and Engineering, University of Illinois Urbana-Champaign 2018.

[2] A. A. Sonin, *Equations of Motion for Viscous Flow*, Department of Mechanical Engineering, Massachusetts Institute of Technology 2003.

[3] A. A. Sonin, *Fundamental Laws of Motion for Particles, Material Volumes, and Control Volumes*, Department of Mechanical Engineering, Massachusetts Institute of Technology 2003.

[4] S. B. Pope, *Turbulent Flows*, Cambridge University Press 2012.

[5] E. Kirr, *Reading Course on Sobolev Spaces*, Department of Mathematics, University of Illinois Urbana-Champaign 2018.

[6] R. Adams, *Sobolev Spaces*, Academic Press 1975.

[7] L.C. Evans, *Partial Differential Equations*, American Mathematical Society 1997.

Appendix A

Fluids

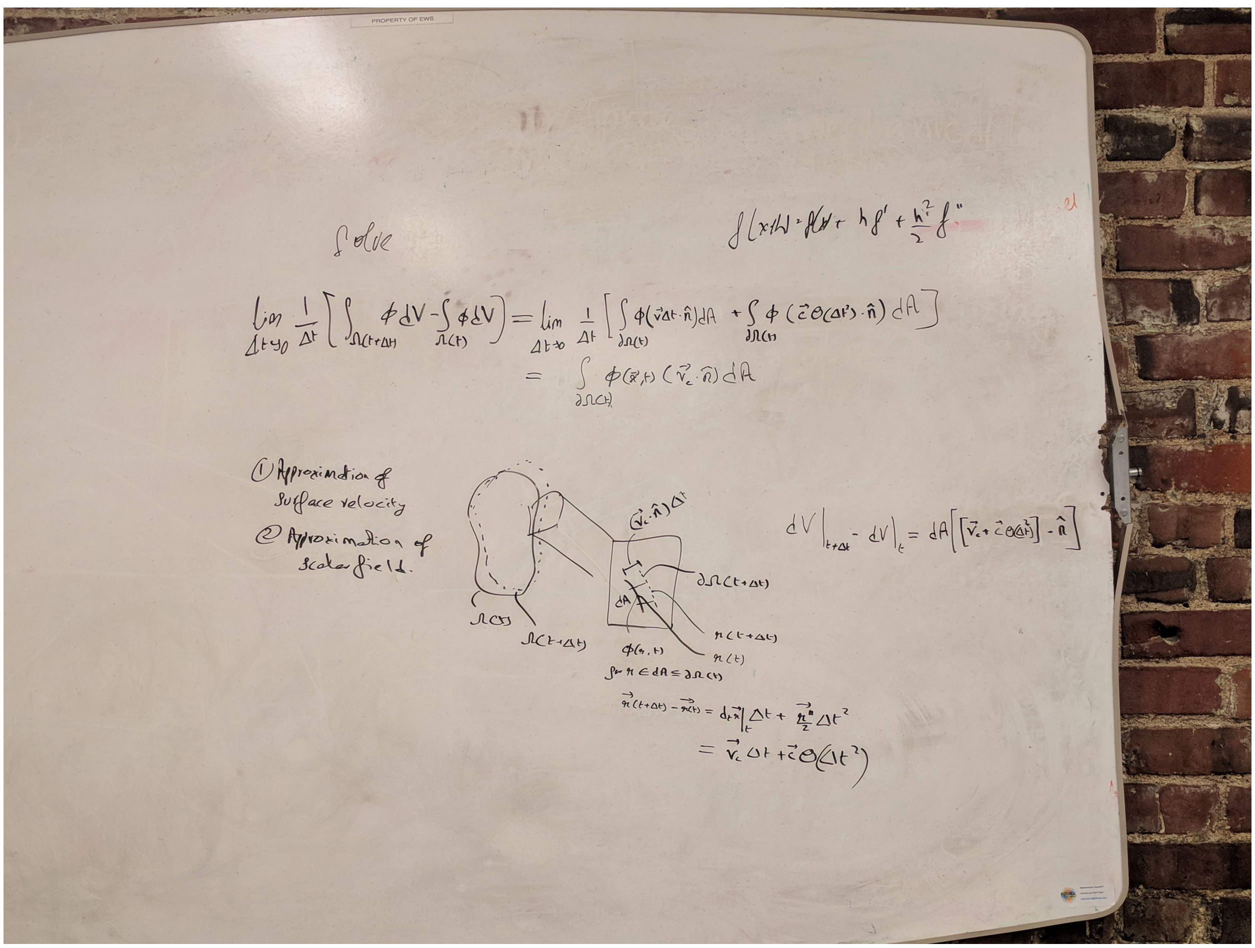

Figure A.1: Evaluating Key Limit in RTT Proof

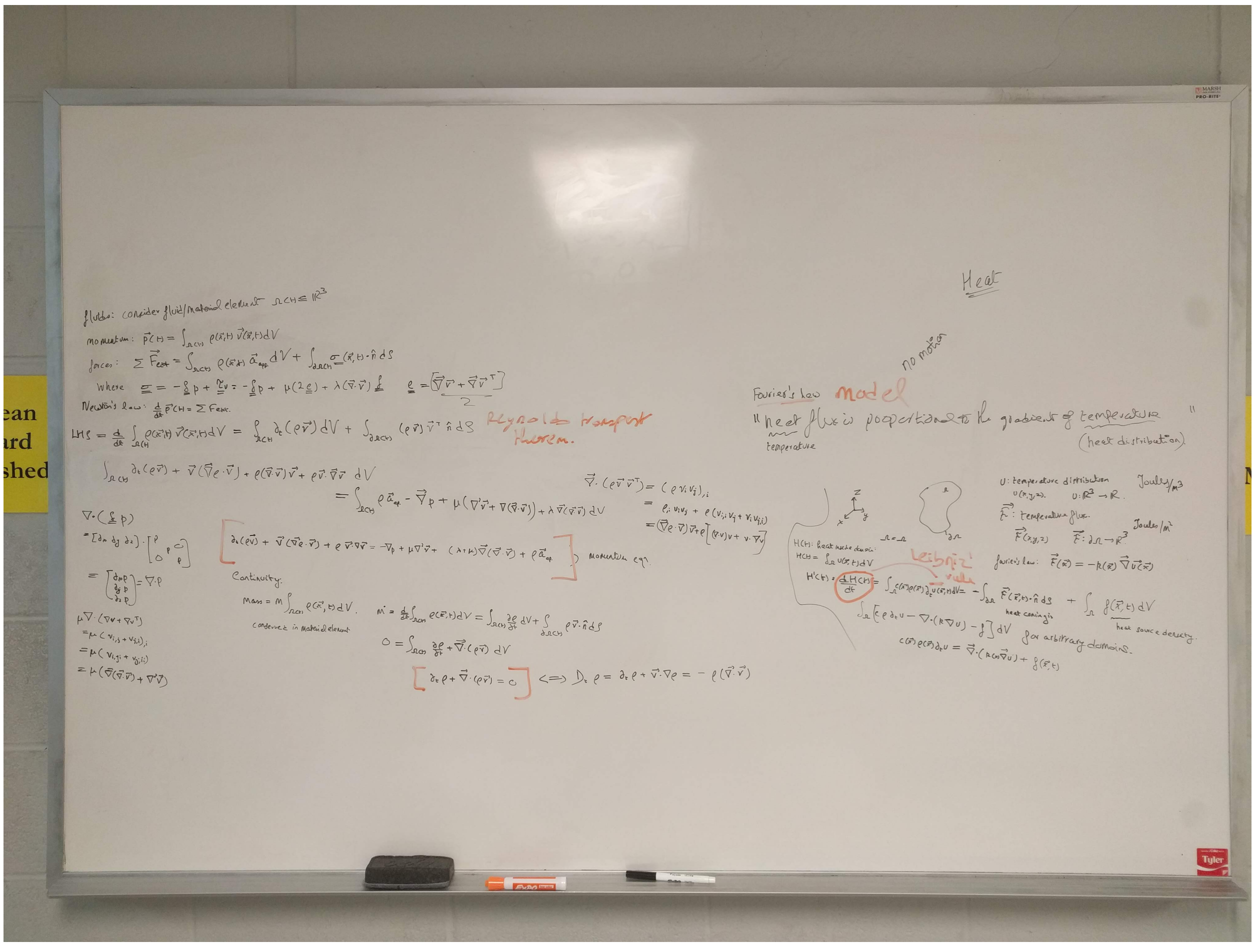

Figure A.2: Navier-Stokes for Compressible Flow

Appendix B

Sobolev Spaces

more stuff