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Misc Notes

October 15, 2021

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Chapter 1

Fluid Mechanics

1.1 Fluids

A fluid is a material that deforms continuously under action of a shear stress, however small. This contrasts with the behaviour of a solid which deforms only when stress values are in certain regimes. Solids are **elastic**: internal stresses in solids resist absolute deformation some original state; fluids are **viscous**: internal stresses in fluids resist the time rate of deformation. A corollary is that a fluid in static equilibrium supports only its weight and forces acting normal to its boundary. A material is said to be fluid if it seems to "flow" in the timescale of observation t_{obs}. That is, if the material is able to relax to a natural state in time less than t_{obs}. The relaxation time of a fluid is denoted λ_{relax}. The ratio of relaxation time to observation time is called Deborah Number. This dimensionless quantity is named after the prophet Deborah who, in the Book of Judges, proclaimed "The mountains flowed before the lord." For large enough t_{obs}, even mountains will behave like fluids. A material is fluid if De $<< 1$.

$$\text{De} = \frac{\lambda_{\text{relax}}}{t_{\text{obs}}} \tag{1.1}$$

We tend to neglect the discrete, molecular nature of matter and treat the fluid to be made of a "continuum". Macroscopic properties such as density and velocity are taken to be well defined for infinitesimal volume elements –small in comparison to system lengthscale but larger in comparison to molecular lengthscales. Fluid properties can vary continuously from one volume element to another and are average values of the molecular properties.

To derive the equations governing viscous fluid flow, we have to consider the time rate of change of quantities/properties in fluid control volumes. A control volume is an arbitrarily defined volume with a closed bounding surface. A material volume is a control volume that contains the same particles of matter at all times. A particular material volume may be defined by the closed bounding surface that envelops its material particles at a certain time. Hence, the velocity of the surface at every point is equal to the flow velocity at that point. The term "fluid element" is synonymous with a material volume.

We consider two perspectives to view motion in a continuum: the **Lagrangian** perspective expresses position, velocity and other state variables in terms of material points travelling with a fluid; the **Eulerian** perspective expresses fluid properties with respect to some reference coordinate system. The time evolution of the position of the material point $\boldsymbol{X}$ can be tracked on an Eulerian coordinate system with a vector $\boldsymbol{x}(\boldsymbol{X}, t)$. Since only one material point can be located at an Eulerian coordinate at a given time, there exists an inverse relation $\boldsymbol{X}(\boldsymbol{x}, t)$ mapping material points $\boldsymbol{X}$ to the Eulerian coordinate position they occupy at time t. To illustrate the difference, consider velocity. The velocity of a material point $\boldsymbol{X}(\boldsymbol{x_0}, t)$ is given by $\partial_t \boldsymbol{X}(\boldsymbol{x}, t)\big|_{\boldsymbol{x_0}} = \boldsymbol{v}(\boldsymbol{X}(\boldsymbol{x_0}, t), t)$

Table 1.1: Lagrangian and Eulerian Points of View

	Eulerian	Langrangian
Coordinate	$\boldsymbol{x}(\boldsymbol{X},t)$	$\boldsymbol{X}(\boldsymbol{x},t)$
Input		
	t (time)	t (time)
Output		
Velocity	$\partial_t \boldsymbol{x}(\boldsymbol{X},t)$	$\partial_t \boldsymbol{X}(\boldsymbol{x},t)$

Let $\boldsymbol{F}$ be some property of a fluid at some material point $\boldsymbol{X} \in \Omega$ at time $t > 0$. $\boldsymbol{F}$ depends on $\boldsymbol{X}$, i.e. which material point one chooses and time t as material points may interact with other points and lose/gain $\boldsymbol{F}$ over time. Hence the property $\boldsymbol{F}$ following a material point varies with the point's Eulerian coordinate and time, i.e. $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x},t)$. The rate of change of $\boldsymbol{F}$ following the material point is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{F} = \partial_t \boldsymbol{F} + \sum_{i=1}^{n} \partial_{x_i}\boldsymbol{F} \partial_t x_i$$
$$\boldsymbol{F} := \partial_t \boldsymbol{F} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{F}$$
(1.2)

Another way to think of the time rate of change of property $\boldsymbol{F}$ as one follows a material point is the following: $\partial_t \boldsymbol{F}$ is the time rate of change of $\boldsymbol{F}$ at fixed position $\boldsymbol{x}$, and $\partial_{x_i}\boldsymbol{F}\partial_t x_i$ is the rate of change of $\boldsymbol{F}$ as one would travel in the direction i multiplied by how fast the material point is travelling in said direction i.

Consider two points, $\boldsymbol{x_0}$, $\boldsymbol{x_0}+\boldsymbol{h}$, in a flow at time $t = 0$. We call the matrix J the Jacobian of $\boldsymbol{X}$ (Lagrangian coordinate) with respect to $\boldsymbol{x}$ (Eulerian coordinate).

$$J(x_0,t) = [J_{ij}] = \lim_{|h|\to 0}\frac{X_i(x_{0j}+h_j,t) - X_i(x_{0j},t)}{|h|} = \partial_{x_j}X_i(\boldsymbol{x},t)\Big|_{\boldsymbol{x}=\boldsymbol{x_0}}$$
(1.3)

An infinitesimal line element $\mathrm{d}V$ at time $t = 0$ is stretched to $\det(J)\mathrm{d}V$ at time t. Consider the material derivative of the Jacobian. <mark>todo:</mark>

$$\partial_t \det(J) = \partial_t \det\left(\partial_{x_j}X_i\right) = \sum_{i,j}\frac{\partial \det(J)}{\partial J_{ij}}\partial_t \frac{\partial X_i}{\partial x_j} = \sum_{i,j}\frac{\partial \det(J)}{\partial J_{ij}}\frac{\partial v_i}{\partial x_j}$$
$$= \sum_{i,j}\frac{\partial \det(J)}{\partial J_{ij}}\left(\sum_k \frac{\partial v_i}{\partial X_k}\frac{\partial X_k}{\partial x_j}\right) = \sum_{i,k}\frac{\partial v_i}{\partial X_k}\left(\sum_j \frac{\partial \det(J)}{\partial J_{ij}}\frac{\partial X_k}{\partial x_j}\right)$$
(1.4)

<mark>todo:</mark> flow visualization - streamlines, streaklines, pathlines

1.2 Stress

Stress is defined as a force across a "small" boundary ($\in \mathbb{R}^2$) per unit area of that boundary, for all orientations of the boundary. Stress is defined at a point with respect to a surface on which it would act. Consequently, stress depends on the orientation of the surface on which it acts. Hence, $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{x},t,\boldsymbol{n})$ where $\boldsymbol{n}$ is the outward pointing normal to the surface. In the limit $\delta A \to 0$, stress at a point is independent of the magnitude of the area.

$$\boldsymbol{\tau} = \lim_{\delta A \to 0}\frac{\delta \boldsymbol{F}}{\delta A}$$
(1.5)

We probe the dependence of stress on the normal the surface it acts on. We consider a cases that reveals stress at a point acting on two sides of a surface to be equal in magnitude and opposite in direction. Consider

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an infinitesimal disk-shaped fluid element with area δA and height δh. Without loss of generality, label the normal on one face $\boldsymbol{n}$ and the other $-\hat{\boldsymbol{n}}$. Let x_0 be the middle of the disk and let $\boldsymbol{g}$ be acceleration due to some applied body force. The equation of motion for the fluid element is:

$$\delta A \delta h \boldsymbol{\rho v} = \boldsymbol{\tau}(\boldsymbol{x}_0 + 0.5\delta h\hat{\boldsymbol{n}}, t, \hat{\boldsymbol{n}})\delta A + \boldsymbol{\tau}(\boldsymbol{x}_0 - 0.5\delta h\hat{\boldsymbol{n}}, t, -\hat{\boldsymbol{n}})\delta A + \rho\delta A\delta h\boldsymbol{g}$$
$$\delta h\boldsymbol{\rho v} = \boldsymbol{\tau}(\boldsymbol{x}_0 + 0.5\delta h\hat{\boldsymbol{n}}, t, \hat{\boldsymbol{n}}) + \boldsymbol{\tau}(\boldsymbol{x}_0 - 0.5\delta h\hat{\boldsymbol{n}}, t, -\hat{\boldsymbol{n}}) + \rho\delta h\boldsymbol{g} \tag{1.6}$$

In the limit $\delta h \to 0$, the rate of change of momentum of the disk falls off, and we get

$$\boldsymbol{\tau}(\boldsymbol{x}_0, t, -\hat{\boldsymbol{n}}) + \boldsymbol{\tau}(\boldsymbol{x}_0, t, \hat{\boldsymbol{n}}) = 0$$
$$\boldsymbol{\tau}(\boldsymbol{x}_0, t, -\hat{\boldsymbol{n}}) = -\boldsymbol{\tau}(\boldsymbol{x}_0, t, \hat{\boldsymbol{n}}) \tag{1.7}$$

Let stress at a point x_0 at time t_0 be a function of the normal to the surface it acts on i.e $\boldsymbol{\tau} = \boldsymbol{\tau}(\hat{\boldsymbol{n}})$. Let σ_{ij} be the component of stress in the $\hat{\boldsymbol{j}}$ direction on a surface with normal $\hat{\boldsymbol{i}}$. Similarly, we have

$$\boldsymbol{\tau}(\hat{\boldsymbol{i}}) = \sigma_{ii}\hat{\boldsymbol{i}} + \sigma_{ji}\hat{\boldsymbol{j}} + \sigma_{ki}\hat{\boldsymbol{k}}$$
$$\boldsymbol{\tau}(\hat{\boldsymbol{j}}) = \sigma_{ij}\hat{\boldsymbol{i}} + \sigma_{jj}\hat{\boldsymbol{j}} + \sigma_{kj}\hat{\boldsymbol{k}} \tag{1.8}$$
$$\boldsymbol{\tau}(\hat{\boldsymbol{k}}) = \sigma_{ik}\hat{\boldsymbol{i}} + \sigma_{jk}\hat{\boldsymbol{j}} + \sigma_{kk}\hat{\boldsymbol{k}}$$

where each σ_{**} is a scalar field varying in space and time. We now show that stress on a surface with an arbitrary normal can be represented as linear combinations of $\boldsymbol{\tau}(\hat{\boldsymbol{i}})$, $\boldsymbol{\tau}(\hat{\boldsymbol{j}})$, $\boldsymbol{\tau}(\hat{\boldsymbol{k}})$. Consider a fluid element in the shape of an infinitesimal tetrahedral with vertices $\boldsymbol{x_0}$, $\boldsymbol{x_0} + \delta x\hat{\boldsymbol{i}}$, $\boldsymbol{x_0} + \delta y\hat{\boldsymbol{j}}$, $\boldsymbol{x_0} + \delta z\hat{\boldsymbol{k}}$. This tetrahedral has a faces parallel to the xy, xz, yz planes with areas δA_z, δA_y, δA_x, and normals $-\hat{\boldsymbol{k}}$, $-\hat{\boldsymbol{j}}$, $-\hat{\boldsymbol{i}}$ respectively. Let the fourth face have area δA and some arbitrary normal $\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{i}} + n_y\hat{\boldsymbol{j}} + n_z\hat{\boldsymbol{k}}$. Let θ_x, θ_y, θ_z be the angle between $\hat{\boldsymbol{n}}$ and the coordinate axes. The areas are related by the following expressions:

$$\delta A_x = \cos\theta_x = n_x\delta A$$
$$\delta A_y = \cos\theta_y = n_y\delta A \tag{1.9}$$
$$\delta A_z = \cos\theta_z = n_z\delta A$$

We now apply Newton's laws to the fluid element in the limit δx, δy, $\delta z \to 0$. Since the volume of the fluid element, δV falls much faster than any of the surface areas, the mass time acceleration and body forces go to zero faster than the surface forces.

$$\delta V \propto \delta x\delta y\delta z$$
$$\delta A \propto \delta z\sqrt{\delta x^2 + \delta y^2}$$
$$\delta A_x \propto \delta y\delta z \tag{1.10}$$
$$\delta A_y \propto \delta x\delta z$$
$$\delta A_z \propto \delta x\delta y$$

Say δx, δy, δz go to zero as $\frac{1}{N}$. Then, for bounded constants c, c_v,

$$0 = \delta A\boldsymbol{\tau}(\hat{\boldsymbol{n}}) + \delta A_x\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + \delta A_y\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + \delta A_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}})$$
$$0 = \delta A\boldsymbol{\tau}(\hat{\boldsymbol{n}}) + n_x\delta A\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + n_y\delta A\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + n_z\delta A\boldsymbol{\tau}(-\hat{\boldsymbol{k}})$$
$$0 = \lim_{N\to\infty} \frac{c\boldsymbol{\tau}(\hat{\boldsymbol{n}})}{N^2} + \frac{cn_x\boldsymbol{\tau}(-\hat{\boldsymbol{i}})}{N^2} + \frac{cn_y\boldsymbol{\tau}(-\hat{\boldsymbol{j}})}{N^2} + \frac{cn_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}})}{N^2} + \frac{c_v}{N^3}(\rho\boldsymbol{g} - \rho\boldsymbol{v})$$
$$0 = \lim_{N\to\infty} c\boldsymbol{\tau}(\hat{\boldsymbol{n}}) + cn_x\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + cn_y\boldsymbol{\tau}(-\hat{\boldsymbol{j}}) + cn_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}}) + \frac{c_v}{N}(\rho\boldsymbol{g} - \rho\boldsymbol{v})$$
$$0 = \boldsymbol{\tau}(\hat{\boldsymbol{n}}) + n_x\boldsymbol{\tau}(-\hat{\boldsymbol{i}}) + n_y\boldsymbol{\tau}(-\hat{\boldsymbol{j}}) + n_z\boldsymbol{\tau}(-\hat{\boldsymbol{k}})$$
$$\boldsymbol{\tau}(\hat{\boldsymbol{n}}) = n_x\boldsymbol{\tau}(\hat{\boldsymbol{i}}) + n_y\boldsymbol{\tau}(\hat{\boldsymbol{j}}) + n_z\boldsymbol{\tau}(\hat{\boldsymbol{k}}) \tag{1.11}$$

Hence, forces due to surface stresses must balance each other in the limit of the tetrahedral shrinking to a point. Expressing the above expression in matrix notation,

$$\boldsymbol{\tau}(\hat{\boldsymbol{n}}) = \begin{bmatrix} \boldsymbol{\tau}(\hat{\boldsymbol{i}}) & \boldsymbol{\tau}(\hat{\boldsymbol{j}}) & \boldsymbol{\tau}(\hat{\boldsymbol{k}}) \end{bmatrix} \cdot \hat{\boldsymbol{n}}$$

$$\boldsymbol{\tau}(\hat{\boldsymbol{n}}) = \begin{bmatrix} \sigma_{ii} & \sigma_{ij} & \sigma_{ik} \\ \sigma_{ji} & \sigma_{jj} & \sigma_{jk} \\ \sigma_{ki} & \sigma_{kj} & \sigma_{kk} \end{bmatrix} \cdot \hat{\boldsymbol{n}} \tag{1.12}$$

We call $\boldsymbol{\tau}$ the traction vector and $\underline{\underline{\sigma}}$ the stress tensor.

$$\underline{\underline{\sigma}} = \begin{bmatrix} \boldsymbol{\tau}(\hat{\boldsymbol{i}}) & \boldsymbol{\tau}(\hat{\boldsymbol{j}}) & \boldsymbol{\tau}(\hat{\boldsymbol{k}}) \end{bmatrix} = \begin{bmatrix} \sigma_{ii} & \sigma_{ij} & \sigma_{ik} \\ \sigma_{ji} & \sigma_{jj} & \sigma_{jk} \\ \sigma_{ki} & \sigma_{kj} & \sigma_{kk} \end{bmatrix} \tag{1.13}$$

$$\boldsymbol{\tau}(\boldsymbol{x}, t, \hat{\boldsymbol{n}}) = \underline{\underline{\sigma}}(\boldsymbol{x}, t) \cdot \hat{\boldsymbol{n}} \tag{1.14}$$

<mark>todo:</mark> We show that the stress tensor is symmetric. Since a fluid continuously deforms under shear stress, a static, non-deforming fluid element in consequently under no shear stress. Hence for an arbitrary surface element in a fluid at rest, stress is in the normal direction. From equilibrium arguments, it can be proven that stress in a fluid at rest is isotropic.

$$\boldsymbol{\tau} = -p(\boldsymbol{x}, t)\boldsymbol{n}$$

$$\boldsymbol{\tau} = -p(\boldsymbol{x}, t)\underline{\underline{\delta}} \cdot \boldsymbol{n} \tag{1.15}$$

We still need to relate the stress tensor to the flow field. The *Newtonian Model* is based on the following assumptions:

1. shear stress is proportional to the rate of shear strain in a fluid particle;

2. shear stress is zero when the rate of shear strain is zero;

3. the stress to rate-of-strain relation is isotropic—that is, there is no preferred orientation in the fluid.

From the first assumption, we have $\sigma_{ij} = \mathrm{K}_{ijkl}e_{kl}$ where K is a fourth order tensor.

<mark>todo:</mark> derive

$$\underline{\underline{\sigma}} = -p\underline{\underline{\delta}} + \underline{\underline{\tau_v}}$$

$$= -p\underline{\underline{\delta}} + \mu\left(\nabla\boldsymbol{v} + \nabla\boldsymbol{v} - \frac{2}{3}(\nabla \cdot \boldsymbol{v})\underline{\underline{\delta}}\right) \tag{1.16}$$

1.3   Reynolds Transport Theorem and Consequences

We begin with the derivation for the Reynolds Transport Theorem which relates time rate of change of quantities in material volumes to the distribution of said properties in the volume. Consider an arbitrary, finite (finite, nonzero measure) control volume $\Omega(t) \subset \mathbb{R}^3$ bounded by some control surface $\partial\Omega(t) \subset \mathbb{R}^3$ in some time varying flow. For some property $\phi(\boldsymbol{x}, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} = \lim_{\Delta t \to 0} \frac{\int_{\Omega(t+\Delta t)} \phi(\boldsymbol{x}, t + \Delta t)\mathrm{d}\boldsymbol{x} - \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x}}{\Delta t} \tag{1.17}$$

For reasonably smooth $\phi(\boldsymbol{x}, t)$, use the taylor expansion of $\phi(\boldsymbol{x}, t)$ about $\phi(\boldsymbol{x}, t)$

$$\phi(\boldsymbol{x}, t + \Delta t) = \phi(\boldsymbol{x}, t) + \Delta t \partial_t \phi(\boldsymbol{x}, t) + \mathcal{O}(\Delta t^2) \tag{1.18}$$

Substituting,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = \lim_{\Delta t\to 0}\int_{\Omega(t+\Delta t)}\partial_t\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} + \frac{\int_{\Omega(t+\Delta t)}\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} - \int_{\Omega(t)}\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} + \mathcal{O}(\Delta t^2)}{\Delta t}$$
$$= \int_{\Omega(t)}\partial_t\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} + \lim_{\Delta t\to 0} + \frac{\int_{\Omega(t+\Delta t)}\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} - \int_{\Omega(t)}\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}}{\Delta t} \tag{1.19}$$

To explain the thought process, without loss of generality, let $\phi = 1$. We are looking for the difference in volume between the two integrals. That is equal to the net volume that the boundary has expanded into. For an infinitesimal patch $\mathrm{d}\boldsymbol{S_x}$ on $\partial\Omega$ (containing point $\boldsymbol{x}$, and with outward normal $\hat{\boldsymbol{n}}$), that is

$$(\boldsymbol{x}(t+\Delta t) - \boldsymbol{x}(t))\cdot\hat{\boldsymbol{n}}\mathrm{d}\boldsymbol{S_x} \tag{1.20}$$

Capturing the difference with a Taylor expansion and integrating over $\partial\Omega$, we get

$$\int_{\Omega(t+\Delta t)}\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} - \int_{\Omega(t)}\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} =$$
$$\int_{\partial\Omega(t)}\phi(\boldsymbol{x},t)\cdot(\Delta t\boldsymbol{v_c})\cdot\hat{\boldsymbol{n}}\mathrm{d}\boldsymbol{S_x} + \int_{\partial\Omega(t)}\phi(\boldsymbol{x},t)\cdot(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v_c}\frac{\Delta t^2}{2})\cdot\hat{\boldsymbol{n}}\mathrm{d}\boldsymbol{S_x} + \mathcal{O}(\Delta t^3) \tag{1.21}$$

where $\boldsymbol{v_c}(\boldsymbol{x},t)$ is the velocity of the control surface at time t at point $\boldsymbol{x}\in\partial\Omega(t)$ and $\hat{\boldsymbol{n}}$ is the outward pointing unit normal vector on $\partial\Omega(t)$. An infinitesimal area element $\mathrm{d}\boldsymbol{S_x}\subset\partial\Omega(t)$ moves a distance of $(\Delta t\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}} + \mathcal{O}(\Delta t^2))$ normal to the surface in time Δt. Another way to think of this approximation is the following: project the value of $\phi(\boldsymbol{x},t)$ on $\mathrm{d}\boldsymbol{S_x}\subset\partial\Omega(t)$ throughout the volume $\mathrm{d}\boldsymbol{S_x}(\Delta t\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}}) + \mathcal{O}(\Delta t^2)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = \int_{\Omega(t)}\partial_t\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} + \lim_{\Delta t\to 0}\frac{\int_{\partial\Omega(t)}\phi(\boldsymbol{x},t)(\Delta t\boldsymbol{v_c})\cdot\hat{\boldsymbol{n}}dA + \mathcal{O}(\Delta t^2)}{\Delta t}$$
$$= \int_{\Omega(t)}\partial_t\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega(t)}\phi(\boldsymbol{x},t)\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}}dA \tag{1.22}$$

$\partial_t\phi$ is the local the rate of production (or accumulation) of ϕ. It accounts for the effects of change in ϕ in the interior of the domain (say due to creation/dissipation or advection/diffusion). The divergence term accounts for the capture or loss of ϕ by the motion of the control surface. If $\Omega(t)$ is a material volume then, $\boldsymbol{v_c} = \boldsymbol{v}$, at all points in $\partial\Omega(t)$. Hence, the final form of the Reynolds Transport Theorem for material volumes $\Omega(t)$ is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = \int_{\Omega(t)}\partial_t\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega(t)}\phi(\boldsymbol{x},t)\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\mathrm{d}\boldsymbol{S_x} \tag{1.23}$$

We now apply the well known divergence theorem, also referred to as Gauss' Law,

$$\int_{\partial\Omega}\boldsymbol{\psi}\cdot\hat{\boldsymbol{n}}\mathrm{d}\boldsymbol{S_x} = \int_{\Omega}\boldsymbol{\nabla}\cdot\boldsymbol{\psi}\mathrm{d}\boldsymbol{x} \tag{1.24}$$

to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = \int_{\Omega(t)}\partial_t\phi + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\phi)\mathrm{d}\boldsymbol{x}$$
$$= \int_{\Omega(t)}\frac{\mathrm{D}}{\mathrm{D}t}\phi + \phi(\boldsymbol{\nabla}\cdot\boldsymbol{v})\mathrm{d}\boldsymbol{x} \tag{1.25}$$

Since a material volume always contains the same fluid elements, time rate of change of total mass in a material volume should be zero.

$$m(t) = \int_{\Omega}\rho\mathrm{d}V$$
$$0 = \frac{\mathrm{d}}{\mathrm{d}t}m = \int_{\Omega}\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})\mathrm{d}V \tag{1.26}$$

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Since the integral is zero for arbitrary material volumes, the integrand must be zero.

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0$$
$$D_t \rho = \partial_t \rho + (\boldsymbol{v} \cdot \nabla)\rho = -\rho(\nabla \cdot \boldsymbol{v}) \tag{1.27}$$

Let $\boldsymbol{p}(t)$ denote the total momentum of a fluid element.

$$\boldsymbol{p}(t) = \int_\Omega \rho \boldsymbol{v} \mathrm{d}V$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} = \int_\Omega \partial_t(\rho \boldsymbol{v}) + \nabla \cdot (\boldsymbol{v} \otimes \rho \boldsymbol{v})\mathrm{d}V$$
$$= \int_\Omega \partial_t(\rho \boldsymbol{v}) + \rho \boldsymbol{v}(\nabla \cdot \boldsymbol{v}) + (\boldsymbol{v} \cdot \nabla)(\rho \boldsymbol{v})\mathrm{d}V \tag{1.28}$$
$$= \int_\Omega D_t(\rho \boldsymbol{v}) + \rho \boldsymbol{v}(\nabla \cdot \boldsymbol{v})\mathrm{d}V$$

Consider the total force applied on the fluid element.

$$\boldsymbol{F}_{\text{ext}} = \int_\Omega \rho \boldsymbol{g}\mathrm{d}V + \int_{\partial\Omega} \underline{\underline{\sigma}} \cdot \boldsymbol{n}\mathrm{d}A$$
$$= \int_\Omega \rho \boldsymbol{g} + \nabla \cdot \underline{\underline{\sigma}}\mathrm{d}V$$
$$= \int_\Omega \rho \boldsymbol{g} + \nabla \cdot \left(-p\underline{\underline{\delta}} + \mu(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}) + \lambda(\nabla \cdot \boldsymbol{v})\underline{\underline{\delta}}\right)\mathrm{d}V \tag{1.29}$$
$$= \int_\Omega \rho \boldsymbol{g} - \nabla p + \mu(\nabla^2 \boldsymbol{v} + \nabla(\nabla \cdot \boldsymbol{v})) + \lambda\nabla(\nabla \cdot \boldsymbol{v})\mathrm{d}V$$
$$= \int_\Omega \rho \boldsymbol{g} - \nabla p + \mu\nabla^2 \boldsymbol{v} + (\mu + \lambda)\nabla(\nabla \cdot \boldsymbol{v})\mathrm{d}V$$

From Newton's second law, time rate of change of momentum is equal to the sum of all external forces applied.

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} - \boldsymbol{F}_{\text{ext}}$$
$$= \int_\Omega \partial_t(\rho \boldsymbol{v}) + \rho \boldsymbol{v}(\nabla \cdot \boldsymbol{v}) + (\boldsymbol{v} \cdot \nabla)(\rho \boldsymbol{v}) - \left(\rho \boldsymbol{g} - \nabla p + \mu\nabla^2 \boldsymbol{v} + (\mu + \lambda)\nabla(\nabla \cdot \boldsymbol{v})\right)\mathrm{d}V \tag{1.30}$$

Since the integral is zero for an arbitrary domain (fluid element) $\Omega(t)$, it follows that the integrand must be zero everywhere.

$$\partial_t(\rho \boldsymbol{v}) + \rho \boldsymbol{v}(\nabla \cdot \boldsymbol{v}) + (\boldsymbol{v} \cdot \nabla)(\rho \boldsymbol{v}) = \rho \boldsymbol{g} - \nabla p + \mu\nabla^2 \boldsymbol{v} + (\mu + \lambda)\nabla(\nabla \cdot \boldsymbol{v})$$
$$\boldsymbol{v}(\underbrace{\partial_t \rho + \nabla \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \rho}_{0\,(\text{continuity})}) + \rho D_t \boldsymbol{v} = \rho \boldsymbol{g} - \nabla p + \mu\nabla^2 \boldsymbol{v} + (\mu + \lambda)\nabla(\nabla \cdot \boldsymbol{v}) \tag{1.31}$$
$$\rho(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}) = \rho \boldsymbol{g} - \nabla p + \mu\nabla^2 \boldsymbol{v} + (\mu + \lambda)\nabla(\nabla \cdot \boldsymbol{v})$$

We have arrived at the Navier-Stokes equations. Take the divergence of the momentum equation.

$$\rho(\partial_t(\nabla \cdot \boldsymbol{v}) + \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})(\nabla \cdot \boldsymbol{v})) = \rho(\nabla \cdot \boldsymbol{g}) - \nabla^2 p + \mu\nabla^2(\nabla \cdot \boldsymbol{v}) + (\mu + \lambda)\nabla^2(\nabla \cdot \boldsymbol{v})$$
$$\rho D_t(\nabla \cdot \boldsymbol{v}) = \nabla^2(-p + (\lambda + 2\mu)(\nabla \cdot \boldsymbol{v})) + \rho(\nabla \cdot \boldsymbol{g} - \boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{v}) \tag{1.32}$$

where $\underline{\underline{A}} : \underline{\underline{B}} = A_{ij}B_{ji}$ is the double dot product. This equation simplifies into a laplace equation for pressure in case of incompressible flow.

$$-\nabla^2 p = \rho(\boldsymbol{\nabla}\boldsymbol{v} : \boldsymbol{\nabla}\boldsymbol{v} - \nabla \cdot \boldsymbol{g}) \tag{1.33}$$

We take the curl of the momentum equation. Define $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$

$$\nabla \times \rho(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}) = \nabla \times (\rho \boldsymbol{g} - \nabla p + \mu \nabla^2 \boldsymbol{v} + (\mu + \lambda)\nabla(\nabla \cdot \boldsymbol{v}))$$

$$\boldsymbol{\nabla}\rho \times \mathrm{D}_t \boldsymbol{v} + \rho \mathrm{D}_t \boldsymbol{\omega} + \epsilon_{ijk} v_{l,i} v_{j,l} = \nabla \times \boldsymbol{g} + \boldsymbol{\nabla}\rho \times \boldsymbol{g} + \mu \nabla^2 \boldsymbol{\omega} + \epsilon_{ijk}(-p + (\nabla \cdot \boldsymbol{v}))_{,ji} \qquad (1.34)$$

$$\boldsymbol{\nabla}\rho \times (\mathrm{D}_t \boldsymbol{v} - \boldsymbol{g}) + \rho \mathrm{D}_t \boldsymbol{\omega} + \epsilon_{ijk} v_{l,i} v_{j,l} = \nabla \times \boldsymbol{g} + \mu \nabla^2 \boldsymbol{\omega} + \epsilon_{ijk}(-p + (\nabla \cdot \boldsymbol{v}))_{,ji}$$

Informally, the Laplacian operator ∇^2 gives the difference between the average value of a function in the neighborhood of a point, and its value at that point. Thus, if u is the temperature, $\nabla^2 u$ tells whether (and by how much) the material surrounding each point is hotter or colder, on the average, than the material at that point.

We explore energy conservation properties of the incompressible Navier Stokes equations. Define the energy in a fluid element $\Omega(t)$ as $\mathcal{E} = \int_{\Omega(t)} \frac{1}{2}\rho|\boldsymbol{v}|^2 \mathrm{d}\boldsymbol{x}$. Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \frac{1}{2}\rho v_j v_j \mathrm{d}\boldsymbol{x} = \int_{\Omega(t)} \frac{1}{2}\rho \mathrm{D}_t(v_j v_j)\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega(t)} \rho \boldsymbol{v}_t^{\mathrm{D}}(\boldsymbol{v})\mathrm{d}\boldsymbol{x} = \int_{\Omega(t)} \boldsymbol{v}^{(-\boldsymbol{\nabla}p + \mu\nabla^2\boldsymbol{v})}\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega(t)} -\nabla \cdot (p\boldsymbol{v}) + \mu v_j v_{j,ii}\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega(t)} -\nabla \cdot (p\boldsymbol{v}) + \mu \left((v_j v_{j,i})_{,i} - v_{j,i} v_{j,i}\right)\mathrm{d}\boldsymbol{x} \\
&= \int_{\partial\Omega(t)} \boldsymbol{v} \cdot (-p\hat{\boldsymbol{n}} + \mu\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} - \int_{\Omega(t)} \mu v_{j,i} v_{j,i}\mathrm{d}\boldsymbol{x}
\end{aligned} \qquad (1.35)$$

1.4 Stokeslet and Stresslets

We discuss properties of fundamental solution to the Stokes equation. But first, review of potential theory..?

$$\begin{aligned}
\boldsymbol{\nabla}p - \mu\nabla^2\boldsymbol{v} &= \boldsymbol{f} \cdot \delta(\boldsymbol{x}) \\
\boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0
\end{aligned} \qquad (1.36)$$

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Chapter 2

Turbulence

2.1 Navier-Stokes Equations

A number of practical flows are governed by Eq. 2.1, i.e. the *unsteady incompressible Navier-Stokes* equations. The equations are coupled with appropriate initial conditions and boundary conditions on domain Ω to form a closed system. The Navier-Stokes system admits a form of fluid motion characterised by chaotic pressure and velocity fields called *turbulence*. Flows which are turbulent, or turbulent flows, are characterised by highly irregular pressure and velocity signals in space and time. Due to the chaotic nature of the Navier-Stokes system (Eq. 2.1), turbulence problems cannot be treated deterministically: miniscule perturbations in initial or boundary conditions magnify into completely different pressure and velocity fields. The trajectory to any particular state of the flow is virtually untraceable as a single bitwise shift caused due to roundoff in computation, or any minuscule perturbation to the experiment setup, would pollute the entire velocity and pressure fields. Turbulent problems are, therefore, treated statistically with the purpose of numerical simulations being the deduction of mean behaviour and high-order correlations.

$$\rho(\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}) = -\boldsymbol{\nabla}p + \mu\nabla^2\boldsymbol{v} + \rho\boldsymbol{f} \tag{2.1a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{2.1b}$$

Eq. 2.1 are defined on $(\boldsymbol{x}, t) \in \Omega \times (0, T)$ where Ω, the computational domain, is an open subset of $\mathbb{R}^d$, $d = 2, 3$, $\mu \left[\mathsf{M}^1\mathsf{L}^{-1}\mathsf{T}^{-1}\right]$ is the dynamic viscosity, $\rho \left[\mathsf{M}^1\mathsf{L}^{-3}\right]$ is the fluid density, and $\boldsymbol{f}$ some known forcing function. Eq. 2.1a is called the *momentum equation* as it arises from a balance of forces and rate of change of momentum on an infinitesimal fluid-element.

$$\frac{\partial}{\partial t}(\rho\boldsymbol{v}) + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho\boldsymbol{v} = \rho\boldsymbol{f} + \boldsymbol{\nabla} \cdot \underline{\underline{\tau}} \tag{2.2}$$

The operator $(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\nabla})$, called the material derivative, computes the time-rate of change of its argument inside a fluid-element, as represented on an Eulerian coordinate system $\boldsymbol{x} = (x, y, z)$.

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \tag{2.3}$$

$\rho\boldsymbol{f}$ is a vector forcing function that acts on the interior points of a fluid-element. Gravitational force, electromagnetic forces, and inertial forces are examples. $\underline{\underline{\tau}}$ is the Newtonian stress tensor that acts only on the bounding surface of a fluid-element. Two components of $\underline{\underline{\tau}}$ are $\underline{\underline{\delta}}p$, the pressure stress, and $\underline{\underline{\tau_v}} = \mu\left(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^T - \frac{2}{3}(\boldsymbol{\nabla} \cdot \boldsymbol{v})\underline{\underline{\delta}}\right)$, the viscous stress.

$$\underline{\underline{\tau}} = -\underline{\underline{\delta}}p + \mu\left(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^T - \frac{2}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\underline{\underline{\delta}}\right) \tag{2.4}$$

Eq. 2.1b is called the *continuity equation* as it ensures incompressibility of the flow by imposing the divergence-free constraint on the velocity field. Eq. 2.1b is obtained by applying mass conservation to a fluid-element for a constant density ρ

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0 \tag{2.5}$$

We obtain the momentum equations Eq. 2.1a and the continuity equation Eq. 2.1b by applying the *Newton's second law* and the *law of conservation of mass* respectively to arbitrary fluid-elements. We first prove the *Reynolds Transport Theorem* which relates time rate of change of quantities in material volumes to the distribution of said properties in the volume.

Consider an arbitrary, finite (finite, nonzero measure) control volume $\Omega(t) \subset \mathbb{R}^3$ bounded by some control surface $\partial\Omega(t) \subset \mathbb{R}^3$ in some time varying flow.

Theorem 2.1.1 (Reynolds Transport Theorem). For some $\phi(\boldsymbol{x}, t)$, and material volume $\Omega(t) \subset \mathbb{R}^d$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} = \int_{\Omega(t)}\partial_t\phi + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\phi)\mathrm{d}\boldsymbol{x} \tag{2.6}$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} = \lim_{\Delta t\to 0}\frac{\int_{\Omega(t+\Delta t)}\phi(\boldsymbol{x}, t+\Delta t)\mathrm{d}\boldsymbol{x} - \int_{\Omega(t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x}}{\Delta t} \tag{2.7}$$

For reasonably smooth $\phi(\boldsymbol{x}, t)$, use the taylor expansion of $\phi(\boldsymbol{x}, t)$ about $\phi(\boldsymbol{x}, t)$

$$\phi(\boldsymbol{x}, t+\Delta t) = \phi(\boldsymbol{x}, t) + \Delta t\partial_t\phi(\boldsymbol{x}, t) + \mathcal{O}(\Delta t^2) \tag{2.8}$$

Substituting,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} &= \lim_{\Delta t\to 0}\int_{\Omega(t+\Delta t)}\partial_t\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} + \frac{\int_{\Omega(t+\Delta t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} - \int_{\Omega(t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} + \mathcal{O}(\Delta t^2)}{\Delta t} \\
&= \int_{\Omega(t)}\partial_t\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} + \lim_{\Delta t\to 0}+\frac{\int_{\Omega(t+\Delta t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} - \int_{\Omega(t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x}}{\Delta t}
\end{aligned} \tag{2.9}$$

To explain the thought process, without loss of generality, let $\phi = 1$. We are looking for the difference in volume between the two integrals. That is equal to the net volume that the boundary has expanded into. For an infinitesimal patch $\mathrm{d}\boldsymbol{S}_{\boldsymbol{x}}$ on $\partial\Omega$ (containing point $\boldsymbol{x}$, and with outward normal $\hat{\boldsymbol{n}}$), that is

$$(\boldsymbol{x}(t+\Delta t) - \boldsymbol{x}(t))\cdot\hat{\boldsymbol{n}}\mathrm{d}\boldsymbol{S}_{\boldsymbol{x}} \tag{2.10}$$

Capturing the difference with a Taylor expansion and integrating over $\partial\Omega$, we get

$$\begin{aligned}
&\int_{\Omega(t+\Delta t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} - \int_{\Omega(t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} = \\
&\int_{\partial\Omega(t)}\phi(\boldsymbol{x}, t)\cdot(\Delta t\boldsymbol{v_c})\cdot\hat{\boldsymbol{n}}\mathrm{d}\boldsymbol{S}_{\boldsymbol{x}} + \int_{\partial\Omega(t)}\phi(\boldsymbol{x}, t)\cdot(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_c\frac{\Delta t^2}{2})\cdot\hat{\boldsymbol{n}}\mathrm{d}\boldsymbol{S}_{\boldsymbol{x}} + \mathcal{O}(\Delta t^3)
\end{aligned} \tag{2.11}$$

where $\boldsymbol{v_c}(\boldsymbol{x}, t)$ is the velocity of the control surface at time t at point $\boldsymbol{x} \in \partial\Omega(t)$ and $\hat{\boldsymbol{n}}$ is the outward pointing unit normal vector on $\partial\Omega(t)$. An infinitesimal area element $\mathrm{d}\boldsymbol{S}_{\boldsymbol{x}} \subset \partial\Omega(t)$ moves a distance of $(\Delta t\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}} + \mathcal{O}(\Delta t^2))$ normal to the surface in time Δt. Another way to think of this approximation is the following: project the value of $\phi(\boldsymbol{x}, t)$ on $\mathrm{d}\boldsymbol{S}_{\boldsymbol{x}} \subset \partial\Omega(t)$ throughout the volume $\mathrm{d}\boldsymbol{S}_{\boldsymbol{x}}(\Delta t\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}}) + \mathcal{O}(\Delta t^2)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} &= \int_{\Omega(t)}\partial_t\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} + \lim_{\Delta t\to 0}\frac{\int_{\partial\Omega(t)}\phi(\boldsymbol{x}, t)(\Delta t\boldsymbol{v_c})\cdot\hat{\boldsymbol{n}}dA + \mathcal{O}(\Delta t^2)}{\Delta t} \\
&= \int_{\Omega(t)}\partial_t\phi(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega(t)}\phi(\boldsymbol{x}, t)\boldsymbol{v_c}\cdot\hat{\boldsymbol{n}}dA
\end{aligned} \tag{2.12}$$

$\partial_t \phi$ is the local the rate of production (or accumulation) of ϕ. It accounts for the effects of change in ϕ in the interior of the domain (say due to creation/dissipation or advection/diffusion). The divergence term accounts for the capture or loss of ϕ by the motion of the control surface. If $\Omega(t)$ is a material volume then, $\boldsymbol{v_c} = \boldsymbol{v}$, at all points in $\partial\Omega(t)$. Hence, the final form of the Reynolds Transport Theorem for material volumes $\Omega(t)$ is:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = \int_{\Omega(t)} \partial_t\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega(t)} \phi(\boldsymbol{x},t)\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\mathrm{d}\boldsymbol{S_x} \tag{2.13}$$

We now apply the well known divergence theorem, also referred to as Gauss' Law,

$$\int_{\partial\Omega} \boldsymbol{\psi}\cdot\hat{\boldsymbol{n}}\mathrm{d}\boldsymbol{S_x} = \int_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}\mathrm{d}\boldsymbol{x} \tag{2.14}$$

to get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} &= \int_{\Omega(t)} \partial_t\phi + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\phi)\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega(t)} \frac{\mathrm{D}}{\mathrm{D}t}\phi + \phi(\boldsymbol{\nabla}\cdot\boldsymbol{v})\mathrm{d}\boldsymbol{x}
\end{aligned} \tag{2.15}$$

$\square$

Since a material volume always contains the same fluid elements, time rate of change of total mass in a material volume should be zero per law of conservation of mass.

$$\begin{aligned}
m(t) &= \int_{\Omega} \rho\mathrm{d}\boldsymbol{x} \\
0 = \frac{\mathrm{d}}{\mathrm{d}t}m &= \int_{\Omega} \partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\mathrm{d}\boldsymbol{x}
\end{aligned} \tag{2.16}$$

Now, we apply Newton's second law of motion Eq. 2.17a to an arbitrary fluid element. Let $\boldsymbol{p}(t)$ denote the total momentum of a fluid element.

$$\boldsymbol{F}_{\text{ext}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}(t) \tag{2.17a}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \rho\boldsymbol{v}\mathrm{d}\boldsymbol{x} \tag{2.17b}$$

$$= \int_{\Omega} \partial_t(\rho\boldsymbol{v}) + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\otimes\rho\boldsymbol{v})\mathrm{d}\boldsymbol{x} \tag{2.17c}$$

$$= \int_{\Omega} \partial_t(\rho\boldsymbol{v}) + \rho\boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\rho\boldsymbol{v})\mathrm{d}\boldsymbol{x} \tag{2.17d}$$

$$= \int_{\Omega} \frac{\mathrm{D}}{\mathrm{D}t}(\rho\boldsymbol{v}) + \rho\boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\mathrm{d}\boldsymbol{x} \tag{2.17e}$$

The net external force acting on a material element, $\boldsymbol{F}_{\text{ext}}$, is equation to the sum for body forces acting on the interior $\Omega(t)$ and traction forces acting on the boundary $\partial\Omega(t)$. The traction forces are modelled by the stress tensor $\underline{\underline{\tau}}$ given in Eq. 2.4

$$\begin{aligned}
\boldsymbol{F}_{\text{ext}} &= \int_{\Omega} \rho\boldsymbol{f}\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \underline{\underline{\tau}}\cdot\boldsymbol{n}\mathrm{d}\boldsymbol{S_x} \\
&= \int_{\Omega} \rho\boldsymbol{f} + \boldsymbol{\nabla}\cdot\underline{\underline{\tau}}\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega} \rho\boldsymbol{f} + \boldsymbol{\nabla}\cdot\left(-p\underline{\underline{\delta}} + \mu(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^{)} + \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{v})\underline{\underline{\delta}}\right)\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega} \rho\boldsymbol{f} - \boldsymbol{\nabla}p + \mu(\nabla^2\boldsymbol{v} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})) + \lambda\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega} \rho\boldsymbol{f} - \boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{v} + (\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\mathrm{d}\boldsymbol{x}
\end{aligned} \tag{2.18}$$

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From Newton's second law, time rate of change of momentum is equal to the sum of all external forces applied.

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} - \boldsymbol{F}_{\text{ext}} \\
&= \int_{\Omega} \partial_t(\rho\boldsymbol{v}) + \rho\boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\rho\boldsymbol{v}) - \left(\rho\boldsymbol{f} - \boldsymbol{\nabla}p + \mu\nabla^2\boldsymbol{v} + (\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\right)\mathrm{d}\boldsymbol{x}
\end{aligned}
\tag{2.19}
$$

Since the integral is zero for an arbitrary domain (fluid element) $\Omega(t)$, it follows that the integrand must be zero everywhere.

$$
\begin{aligned}
\partial_t(\rho\boldsymbol{v}) + \rho\boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\rho\boldsymbol{v}) &= \rho\boldsymbol{f} - \boldsymbol{\nabla}p + \mu\nabla^2\boldsymbol{v} + (\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) \\
\boldsymbol{v}(\underbrace{\partial_t\rho + \boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\rho}_{0}) + \rho\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{v} &= \rho\boldsymbol{f} - \boldsymbol{\nabla}p + \mu\nabla^2\boldsymbol{v} + \underbrace{(\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})}_{0} \\
\rho(\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}) &= \rho\boldsymbol{f} - \boldsymbol{\nabla}p + \mu\nabla^2\boldsymbol{v}
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) &= 0 \\
\frac{\mathrm{D}}{\mathrm{D}t}\rho = \partial_t\rho + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho &= -\rho(\boldsymbol{\nabla}\cdot\boldsymbol{v})
\end{aligned}
\tag{2.21}
$$

For incompressible (constant density) flows, the continuity equation boils down to

$$
\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0
\tag{2.22}
$$

We have arrived at the Navier-Stokes equations.

Before performing further analysis, we nondimensionalise velocity, time, and pressure with a canonical length L, and velocity U.

$$
\begin{aligned}
\boldsymbol{v}^* &= \frac{v}{U}, \ t^* = \frac{tU}{L}, \ p^* = \frac{p}{\rho U^2}, \ \boldsymbol{f}^* = \frac{\boldsymbol{f}}{U^2/L} \\
\frac{\partial}{\partial t^*} &= \frac{L}{U}\frac{\partial}{\partial t}, \ \boldsymbol{\nabla}^* = L\boldsymbol{\nabla}
\end{aligned}
\tag{2.23}
$$

We normalize pressure p with the stagnation pressure ρU^2. Substituting Eq. 2.23 into equation Eq. 2.1a and multiplying by $\frac{L}{\rho U^2}$, we obtain the Navier-Stokes equations in non-dimensional form

$$
\frac{\partial}{\partial t^*}\boldsymbol{v}^* + \boldsymbol{v}^*\cdot\boldsymbol{\nabla}\boldsymbol{v}^* = -\boldsymbol{\nabla}p^* + \frac{1}{\text{Re}}\nabla^2\boldsymbol{v}^* + \boldsymbol{f}^*
\tag{2.24a}
$$

$$
\boldsymbol{\nabla}^*\cdot\boldsymbol{v}^* = 0
\tag{2.24b}
$$

$Re = \dfrac{\rho U L}{\mu}$, the Reynolds number of the flow, is the ratio of viscous forces to inertial forces. Osborne Reynolds used the Reynolds number to predict turbulent transition of laminar flow in a pipe in his famous experiment in 1883 [**reynolds**]. He observed that low Reynolds number flows tend to be laminar, and high Reynolds number flows tend towards turbulent motion. Turbulent flows with higher Reynolds number contain more intricate, and chaotic motion.

Turbulent flows are characterised by a competition between viscous forces, which damp out velocity fluctuations by converting kinetic energy into heat, and inertial forces, which tend to generate and preserve velocity and pressure fluctuations[**wall-bounded-turb**]. In low Reynolds number flows, viscous forces dominate and the flow is near perfectly damped, meaning that any random fluctuation in the velocity field would be smoothed out in no time. The velocity fields adjusts almost instantaneously to any changes in the pressure gradient that drives the flow. At high Reynolds numbers, however, viscous forces may not be strong

enough to dampen out velocity fluctuations. As a result, even tiny disturbances may cause the entire flow to destabilise.

$$Re = \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{ma}{\tau A} = \frac{\rho V \cdot \frac{\mathrm{d}u}{\mathrm{d}t}}{\mu \frac{\mathrm{d}u}{\mathrm{d}y} \cdot A} \propto \frac{\rho L^3 \cdot \frac{\mathrm{d}u}{\mathrm{d}t}}{\mu \frac{\mathrm{d}u}{\mathrm{d}y} \cdot L^2} = \frac{\rho L \frac{\mathrm{d}y}{\mathrm{d}t}}{\mu} \propto \frac{\rho U L}{\mu} = \frac{UL}{\nu} \tag{2.25}$$

In Eq. 2.25, $\tau = \left| \underline{\underline{\tau}} \cdot \hat{\boldsymbol{n}} \right|$ is the magnitude of shear stress, t is time, y is the cross-sectional position, $u = \frac{\mathrm{d}}{\mathrm{d}t}x$ is the local flow streamwise velocity, and $\nu = \frac{\mu}{\rho} \left[\mathsf{L}^2 \mathsf{T}^{-1} \right]$ is the kinematic viscosity. In further discussions, we refer to kinematic viscosity, ν, as simply viscosity.

We drop the asterisks and restate the nondimensional Navier-Stokes system for further investigation.

$$\frac{\partial}{\partial t}\boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}p + \frac{1}{Re}\nabla^2\boldsymbol{v} + \boldsymbol{f} \tag{2.26a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{2.26b}$$

In low Reynolds number regimes, viscous effects dissipate energy from high-wavenumber modes into heat, making the velocity field rather smooth. Conversely, at high Reynolds numbers, the ability to dissipate energy from high wavenumber modes of the flow is decreased resulting in a large number of energy containing modes. The high Reynolds number regime, where the nonlinear advection term dominates, is characterised by thin boundary layers at the interface of fluid and a solid boundary as a consequence of the imposition of the no-slip boundary condition. At high Reynolds numbers, Eq. 2.26a become singularly perturbed, and give rise to thin boundary layers. [**wall-energy-cascade**] The resolution requirements are heightened as it is pertinent to resolve the boundary layers in order to get the main flow right.

2.2 Turbulence

We begin this section with a discussion Reynolds decomposition, an important mathematical technique in understanding turbulence, and then proceed to describing the energy cascade process, and the subtleties of wall-bounded turbulent flows. We derive the *Reynolds Stress Transport Equations* (RSTE) and discuss their role in describing mechanisms of turbulent energy transport, production, and dissipation.

2.2.1 Reynolds Decomposition

Reynolds decomposition is a mathematical technique of splitting a quantity into its mean or expected value, and a time or space varying fluctuation. Let a space, and time-varying quantity, $\phi(\boldsymbol{x}, t) \in C\left(\Omega \times (0, T)\right)$, equal to the sum of its expected value $\langle \phi \rangle$, and a fluctuation $\phi'(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, t) - \langle \phi \rangle$

$$\phi(\boldsymbol{x}, t) = \langle \phi \rangle + \phi'(\boldsymbol{x}, t) \tag{2.27}$$

In practice, expected values or ensemble averages are computed by averaging a quantity in time, or over in time and over a homogeneous direction.

$$\langle \phi \rangle = \frac{1}{T} \int_0^T \phi(\boldsymbol{x}, t)\mathrm{d}t \tag{2.28}$$

Some properties of Reynolds decomposition are below. Let ϕ and ψ be any quantities. Then,

1. Averaging and differentiation commute

   Space

   $$\langle \partial_j \phi(\boldsymbol{x}, t) \rangle = \frac{1}{T} \int_0^T \partial_j \phi(\boldsymbol{x}, t)\mathrm{d}t = \tag{2.29}$$

   Time

2. Ensemble of a fluctuation is zero.

$$\langle \phi' \rangle = \langle \phi - \langle \phi \rangle \rangle$$
$$\langle \phi \rangle = \langle \phi \rangle - \langle \phi' \rangle \tag{2.30}$$
$$\langle \phi' \rangle = 0$$

3. Ensemble of product is product of ensemble plus ensemble of product of fluctuations

$$
\begin{aligned}
\langle \phi\psi \rangle &= \langle (\langle \phi \rangle + \phi')(\langle \psi \rangle + \psi') \rangle \\
&= \langle \langle \phi \rangle \langle \psi \rangle + \langle \phi \rangle \psi' + \phi' \langle \psi \rangle + \phi'\psi' \rangle \\
&= \langle \phi \rangle \langle \psi \rangle + \langle \phi \rangle \underbrace{\langle \psi' \rangle}_{0} + \underbrace{\langle \phi' \rangle}_{0} \langle \psi \rangle + \langle \phi'\psi' \rangle \\
&= \langle \phi \rangle \langle \psi \rangle + \langle \phi'\psi' \rangle
\end{aligned}
\tag{2.31}
$$

2.2.2   Energy Cascade

Here, we briefly cover the energy cascade in turbulent flows. Energy cascade in turbulent flows refers to the transfer of energy between differing lengthscales or, equivalently, wavenumbers. We discuss the work on Richardson (1922) and Kolmogorov (1941) in describing energy transfer from larger eddies to smaller ones.

Turbulence is a multi-scale phenomenon involving transfer of energy between a spectrum of lengthscales and timescales. In turbulent flows, energy is said to reside in *eddies*, coherent swirls of fluid motion. Energy is transferred between scales of motion through the interaction, breakup, and formation of eddies. and cannot be dissipated until it is transferred to the smallest ones, in which viscosity acts[**wall-energy-cascade**].

Pertinent assumptions of the theories in this section are that energy transfer between eddies is local in scale, and restricted to eddies of similar sizes. § 2.2.3 discusses turbulent flows in presence of walls, which creates anisotropy and the presence of smaller lengthscales than predicted by Eq. 2.37.

Of note in the momentum equation (Eq. 2.1a) is the nonlienar advection term $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$ which is responsible for the entire velocity spectrum interacting with itself. To investigate the cascade process of the velocity field, we consider the Fourier expansion of the velocity field in free space

$$\boldsymbol{v}(\boldsymbol{x},t) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \hat{\boldsymbol{v}}_{\boldsymbol{k}}(t)\mathrm{e}^{\imath \boldsymbol{k}\cdot\boldsymbol{x}}, \quad \boldsymbol{x} \in \mathbb{R}^d, \, t \in \mathbb{R} \tag{2.32}$$

where $\hat{\boldsymbol{v}} \in \mathbb{C}^d$. We substitute the Fourier expansion in the advection equation

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = 0 \tag{2.33}$$

For the $\boldsymbol{k}^{\mathrm{th}}$ Fourier mode, we get

$$\frac{\partial}{\partial t}\hat{\boldsymbol{v}}_{\boldsymbol{k}} + \imath \boldsymbol{k} \cdot \sum_{\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}} \hat{\boldsymbol{v}}_{\boldsymbol{p}}\hat{\boldsymbol{v}}_{\boldsymbol{q}} = 0 \tag{2.34}$$

expressing the interaction of the wavenumber triad $\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q} \in \mathbb{Z}^d$. As the entire spectrum is present in the equation for time-evolution of every more, it is pertinent for numerical simulations to sufficiently resolve/or account for the energy containing component of the velocity spectrum in order to correctly represent flow-physics.

According to Richardson (1922), a flow can be thought of to be composed of eddies of differing sizes. The largest scales of turbulent motion, typically the size of the domain, are fed by an external source such as an imposed pressure gradient and contain the most kinetic energy. The moving fluid creates a negative-pressure gradient that gives rise to smaller eddies. Larger eddies, that are unstable, successively break-up and transfer their kinetic energy to smaller ones until energy in the smallest eddies is dissipated into heat by viscous effects.

Let $Re_l = \dfrac{u_l l}{\nu}$ be the Reynolds number for eddies of size l with velocity u_l. The energy cascade continues till eddies are sufficiently small that $Re_l \approx 1$ and molecular viscosity can effectively dissipate its kinetic energy. In the inertial subrange, where Re_l is large, the velocity cascades to higher and higher wavenumbers due to nonlinear interactions, mirroring the breakup of large, unstable eddies into smaller ones. As dissipation is the last step in the process, the rate of dissipation $\epsilon\left[\mathsf{L}^2\mathsf{T}^{-3}\right]$ is determined by the first process in the sequence—that is energy transfer from the largest eddies. This is because at every stable state, the rate of energy gained from larger eddies must equal the rate of energy lost to smaller eddies, which is then equal to the dissipation rate for the smallest eddies. From dimensional analysis, dissipation rate is found to be

$$\epsilon \propto \frac{U^3}{L} \tag{2.35}$$

independent of viscosity for high Reynolds number flows.

The theory of Kolmogorov (1941) answered questions regarding the lengthscale of the smallest eddies by hypothesising local isotropy, and statistic similarity at small enough scales for high Reynolds number flows. *Kolmogorov's hypothesis of local isotropy* states that the directional biases of large scales are lost in the chaotic scale-reduction process as energy is successively transferred to smaller eddies. The next hypothesis in Kolmogorov's theory, called *Kolmogorov's first similarity hypothesis* states that the statistics of small-scale turbulent motion have a universal form that is uniquely determined by viscosity ν and dissipation rate ϵ. From dimensional analysis, the lengthscale η and velocity u_η of the smallest eddies are

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$$
$$u_\eta = (\epsilon\nu)^{1/4} \tag{2.36}$$

The Reynolds number for the Kolmogorov scales $Re_\eta = \dfrac{\eta u_\eta}{\nu}$, charecterising the smallest dissipative eddies, is unity, consistent with the notion that the energy cascade proceeds to smaller scales until the Re_l is small enough for dissipation to be effective. Further, the ratio of the smallest to largest lengthscales is readily determined by the Reynolds number of the flow

$$\frac{\eta}{L} = Re^{-3/4} \tag{2.37}$$

2.2.3 Wall-Bounded Turbulence

The theories of Kolmogorov and Richardson are applicable to homogeneous isotropic turbulent flows—that is flows with properties independent of position and direction. However, the application of the *no-slip boundary condition* introduces a new set of physical processes and associated lenthscales near the boundary. In this section, we consider turbulent flows in the vicinity of smooth walls. This section is motivated in part by the review of wall-bounded turbulence [**wall-bounded-turb**].

We align the Cartesian coordinates x, y, z with the streamwise (in the direction of the flow), wall-normal, and spanwise directions. The associated velocity components are u, v, w respectively. For flows in a channel or a pipe, turbulent statistics become independent of downstream distance when measured far from the inlet if inlet conditions are steady. Then the flow is said to be fully developed. Viscous forces impose the no-slip boundary condition at the boundary-fluid interface dictating that

$$(\boldsymbol{v}_{\text{fluid}} - \boldsymbol{v}_{\text{wall}}) \cdot \hat{\boldsymbol{n}} = 0 \qquad \text{\textit{no-penetration condition}} \tag{2.38a}$$
$$(\boldsymbol{v}_{\text{fluid}} - \boldsymbol{v}_{\text{wall}}) \cdot \hat{\boldsymbol{n}} = 0 \qquad \text{\textit{no-slip condition}} \tag{2.38b}$$

todo: second no-slip condition evaluated at $y = 0$ where y is the distance from the wall. Far from the wall, the streamwise velocity is close U, the flow's characteristic velocity. For example, laminar flow between

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two walls obeys a parabolic profile for streamwise velocity with the maximum at the centerline. Turbulent flows have smaller velocity gradients in the bulk as turbulent eddies mix momentum effectively, and since the no-slip boundary condition has to be satisfied at the walls, velocity gradients are large near the wall. Viscous forces, characterised by the magnitude of shear stress τ, which are proportional to velocity gradients, are also large near the wall.

$$\tau = \left| \mu \left(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T - \frac{2}{3}(\boldsymbol{\nabla \cdot v})\underline{\underline{\delta}} \right) \cdot \hat{\boldsymbol{y}} \right| \propto \mu \frac{\mathrm{d}u}{\mathrm{d}y} \tag{2.39}$$

Prandtl introduced the concept of a viscous *boundary layer*, a region in the immediate vicinity of the wall where viscous effects dominate, and an inviscid region away from the wall following different scalings. Near the wall, velocity goes to zero, but its gradients become large, leading to viscous forces dominating over inertial forces. We define τ_w to be the magnitude of shear stress applied by the fluid on the wall, and u_τ to be the friction velocity at the wall.

$$\tau_w = \tau|_{y=0}$$
$$u_\tau = \sqrt{\frac{\tau_w}{\rho}} \tag{2.40}$$

Here, the relevant velocity and length scales are u_τ and ν/u_τ. Quantities nondimensionalised by these wall units, u_τ and ν, are labelled with a subscript $\cdot^+$. For example, the non-dimensional wall-distance y^+ and non-dimensional streamwise velocity u^+ are

$$u^+ = \frac{u}{u_\tau}$$
$$y^+ = \frac{y}{\nu/u_\tau} \tag{2.41}$$

respectively.

Turbulent eddies in the boundary layer are restricted in size by the boundary layer thickness. The assumption in § 2.2.2 on homogeneous, isotropic turbulence that the most energetic eddies are much larger than the least energetic ones breaks down in the near-wall region, where the most energetic eddies are relatively small, often on the order or η. Turbulent energy is typically supplied to wall-bounded flows by the anisotropy due to the presence of the wall, there can even exist a reverse energy cascade from smaller to larger scales[**wall-bounded-turb**].

Prandtl postulated that the mean streamwise velocity near the wall is independent of boundary layer thickness $\delta \left[\mathsf{L}^1 \right]$ leading to *the law of the wall*

$$\langle u^+ \rangle = F(y^+) \tag{2.42}$$

where $F(\cdot)$ is some universal function. In the inviscid region, the characteristic scales U and L remain valid while δ and u_τ are still relevant as the outer edge of the boundary layer sets up a no-slip condition for the outer flow. von Karman formulated the *velocity-defect law* in 1930 as <mark>todo:</mark> cite

$$\langle u^+ \rangle - u_e^+ = G(\eta) \tag{2.43}$$

where u_e is the streamwise velocity of the outer edge of the boundary layer, $\eta = \frac{y}{L}$ and $G(\cdot)$ is another universal function. The Reynolds number based on the friction velocity and the domain characteristic lengthscale is called the friction Reynolds number. Re_τ is experimentally measured, and gives an idea of the scale separation in wall-bounded flows.

$$Re_\tau = \frac{u_\tau L}{\nu} \tag{2.44}$$

The distance y^+ is the Reynolds number for structures reaching from y to the wall, and is never large in the boundary layer[**wall-energy-cascade**]. Between the inner and outer flow, lies a buffer layer (typically $10 \leq y^+ \leq 100$) where both inertial and viscous effects are important. Below $y^+ \approx 5$, fluid viscosity is dominant and the scaled mean velocity resembles a linear profile. In the buffer region, the streamwise velocity transitions to a logarithmic profile. Most of the turbulent energy is generated in the buffer region through a self-sustaining nonlinear *near-wall cycle* at least in moderate Reynolds number flows[**near-wall-cycle**].

<mark>todo:</mark> momentum fluxes, von Karman constant. few words on importance of wall-bounded flows

2.2.4 Boundary Curvature

todo: streamline curvature and pressure-differences

todo: turbulence - mean strain rate

2.2.5 Roughness

We expand on the discussion in § 2.2.3

todo: definition of roughness/rough wall

Flow features associated with roughness[**roughwalls**]

Roughness - [**nakayama2002**][**roughwalls**]

2.2.6 Reynolds Stress Transport Equations

todo: refer to TSFP1979 to improve this section

For brevity, we employ the Einstein summation notation.

We derive the Reynolds stress transport equations (RSTE) which describe the production, transport, and dissipation of energy in fluctuating modes of a flow. Examining the energy budgets of turbulent flows can lead to insights into factors driving turbulence and leading to its decay, spatial distribution, and how energy is transferred between mean and fluctuating modes. Reynolds stress budgets provide valuable information on the relative magnitudes of the terms and their possible scaling as well as reveal the production, dissipation, and spatial redistribution mechanisms of turbulent energy.

We repeat the non-dimensional, unsteady, incompressible Navier-Stokes equations in Einstein indicial notation.

$$\frac{\partial}{\partial t} v_j + v_i v_{j,i} = -p_{,j} + \frac{1}{Re} v_{j,ii} + f_j$$

$$v_{i,i} = 0$$

(2.45)

We take the expected value of the continuity equation to find that $\langle v_i \rangle$ is divergence free.

$$\langle 0 \rangle = \langle v_{i,i} \rangle$$

$$0 = \langle v_i \rangle_{,i}$$

(2.46)

Similarly, we expand the continuity equation in terms of expected values of fluctuations, to find that v_i' is also divergence free.

$$0 = v_{i,i} = \langle v_i \rangle_{,i}^{\;\;0} + v_{i,i}'$$

$$0 = v_{i,i}'$$

(2.47)

We take the expected value of the momentum equation.

$$\frac{\partial}{\partial t} \langle v_j \rangle + \langle v_i \rangle \langle v_{j,i} \rangle + \langle v_i' v_{j,i}' \rangle = -\langle p_{,j} \rangle + \frac{1}{Re} \langle v_j \rangle_{,ii} + \langle f_i \rangle$$

$$\frac{\partial}{\partial t} \langle v_j \rangle + \langle v_i \rangle \langle v_{j,i} \rangle = -\langle p_{,j} \rangle + \frac{1}{Re} \langle v_j \rangle_{,ii} + \langle f_i \rangle - \langle v_i' v_j' \rangle_{,i}$$

(2.48)

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$\eta_{ij} = \langle v'_i v'_j \rangle$ is called the Reynolds stress tensor. We apply Reynolds decomposition to the momentum equation.

$$\frac{\partial}{\partial t}(\langle v_j \rangle + v'_j) + (\langle v_i \rangle + v'_i)(\langle v_{j,i} \rangle + v'_{j,i}) = -(\langle p_{,j} \rangle + p'_j) + \frac{1}{Re}(\langle v_{j,ii} \rangle + v'_{j,ii}) + \langle f_j \rangle + \cancelto{0}{f'_j}$$

$$\frac{\partial}{\partial t}(\langle v_j \rangle + v'_j) + \langle v_i \rangle \langle v_{j,i} \rangle + \langle v_i \rangle v'_{j,i} + v'_i \langle v_{j,i} \rangle + v'_i v'_{j,i} = -(\langle p_{,j} \rangle + p'_j) + \frac{1}{Re}(\langle v_{j,ii} \rangle + v'_{j,ii}) + \langle f_j \rangle$$

(2.49)

Subtracting Eq. 2.48 from Eq. 2.49, we get

$$\frac{\partial}{\partial t}v'_j + \langle v_i \rangle v'_{j,i} + v'_i \langle v_{j,i} \rangle + v'_i v'_{j,i} - \langle v'_i v'_{j,i} \rangle = -p'_{,j} + \frac{1}{Re}v'_{j,ii}$$

(2.50)

Multiply both sides by v'_k and take the expected value.

$$\left\langle v'_k \frac{\partial}{\partial t}v'_j \right\rangle + \langle v_i \rangle \langle v'_k v'_{j,i} \rangle + \langle v'_k v'_i \rangle \langle v_{j,i} \rangle + \langle v'_k v'_i v'_{j,i} \rangle - \cancelto{0}{\langle v'_k \rangle}\langle v'_i v'_{j,i} \rangle = -\langle v'_k p'_{,j} \rangle + \frac{1}{Re}\langle v'_k v'_{j,ii} \rangle$$

(2.51)

Take the transpose and add.

$$\frac{\partial}{\partial t}\langle v'_j v'_k \rangle + \langle v_i \rangle \langle v'_j v'_k \rangle_{,i} + \langle v'_k v'_i \rangle \langle v_{j,i} \rangle + \langle v'_j v'_i \rangle \langle v_{k,i} \rangle + \langle v'_i v'_j v'_k \rangle_{,i}$$
$$= -\left(\langle v'_k p'_{,j} \rangle + \langle v'_j p'_{,k} \rangle\right) + \frac{1}{Re}\langle v'_j v'_k \rangle_{,ii} - \frac{2}{Re}\langle v'_{j,i} v'_{k,i} \rangle$$

(2.52)

We write the pressure transport term in terms of the pressure strain and pressure diffusion terms.

$$v'_k p'_{,j} + v'_j p'_{,k} = -p'(v'_{j,k} + v'_{k,j}) + (p' v'_j)_{,k} + (p' v'_k)_{,j}$$

(2.53)

We finally arrive at the tensor equation describing the behaviour of Reynolds Stresses over time.

$$\frac{\partial}{\partial t}\langle v'_j v'_k \rangle + \langle v_i \rangle \langle v'_j v'_k \rangle_{,i} + \langle v'_k v'_i \rangle \langle v_{j,i} \rangle + \langle v'_j v'_i \rangle \langle v_{k,i} \rangle + \langle v'_i v'_j v'_k \rangle_{,i}$$
$$= \langle p'(v'_{j,k} + v'_{k,j}) \rangle - \langle (p' v'_j)_{,k} + (p' v'_k)_{,j} \rangle + \frac{1}{Re}\langle v'_j v'_k \rangle_{,ii} - \frac{2}{Re}\langle v'_{j,i} v'_{k,i} \rangle$$

(2.54)

$$\frac{\partial}{\partial t}\eta_{jk} + \langle v_i \rangle \eta_{jk,i} + \eta_{ki} \langle v_{j,i} \rangle + \eta_{ji} \langle v_{k,i} \rangle + \langle v'_i v'_j v'_k \rangle_{,i}$$
$$= \langle p'(v'_{j,k} + v'_{k,j}) \rangle - \langle (p' v'_j)_{,k} + (p' v'_k)_{,j} \rangle + \frac{1}{Re}\eta_{jk,ii} - \frac{2}{Re}\langle v'_{j,i} v'_{k,i} \rangle$$

(2.55)

To obtain the equation for Turbulent Kinetic Energy, we consider one-half of the trace of Eq. 2.54 by multiplying with $\frac{1}{2}\delta_{ij}$.

$$\frac{\partial}{\partial t}k + \langle v_i \rangle k_{,i} + \langle v'_j v'_i \rangle \langle v_{j,i} \rangle + \langle v'_i v'_j v'_j \rangle_{,i} = \cancelto{0}{\langle p' v'_{i,i} \rangle} - \langle p' v'_j \rangle_{,j} + \frac{1}{Re}k_{,ii} - \frac{1}{Re}\langle v'_{j,i} v'_{j,i} \rangle$$

$$\frac{\partial}{\partial t}k + \langle v_i \rangle k_{,i} + \langle v'_j v'_i \rangle \langle v_{j,i} \rangle + \langle v'_i v'_j v'_j \rangle_{,i} = -\langle p'_{,j} v'_j \rangle - \cancelto{0}{\langle p' v'_{i,i} \rangle} + \frac{1}{Re}k_{,ii} - \frac{1}{Re}\langle v'_{j,i} v'_{j,i} \rangle$$

(2.56)

$$\frac{\partial}{\partial t}k + \langle v_i \rangle k_{,i} + \langle v'_j v'_i \rangle \langle v_{j,i} \rangle + \langle v'_i v'_j v'_j \rangle_{,i} = -\langle p'_{,j} v'_j \rangle + \frac{1}{Re}k_{,ii} - \frac{1}{Re}\langle v'_{j,i} v'_{j,i} \rangle$$

Eq. 2.54 and Eq. 2.56 describe the generation, transport and decay of Reynolds stresses in a flow. For each η_{ik} in the symmetric Reynolds stress tensor, and for k, we label the terms in Eq. 2.54 and Eq. 2.56.

Table 2.1: Budgets for Reynolds Stresses

Reynolds Stress Expression	Budget Term	TKE Expression
$\left\langle v_j' v_k' \right\rangle$	Reynold Stress	$k = \dfrac{1}{2} \left\langle v_j' v_j' \right\rangle$
$\left\langle v_i \right\rangle \left\langle v_j' v_k' \right\rangle_{,i}$	Convection	$\left\langle v_i \right\rangle k_{,i}$
$-\left\langle v_j' v_i' \right\rangle \left\langle v_{k,i} \right\rangle - \left\langle v_k' v_i' \right\rangle \left\langle v_{j,i} \right\rangle$	Production	$-\left\langle v_j' v_i' \right\rangle \left\langle v_{j,i} \right\rangle$
$-\left\langle v_i' v_j' v_k' \right\rangle_{,i}$	Turbulent Diffusion	$\left\langle v_i' v_j' v_j' \right\rangle_{,i}$
$-\left\langle v_k' p_{,j}' + v_j' p_{,k}' \right\rangle$	Pressure Transport	$-\left\langle v_j' p_j' \right\rangle$
$-\left\langle (p' v_j')_{,k} + (p' v_k')_{,j} \right\rangle$	Pressure Diffusion	$-\left\langle v_j' p_j' \right\rangle$
$\left\langle p'(v_{j,k}' + v_{k,j}') \right\rangle$	Pressure Strain	0
$\dfrac{1}{Re} \left\langle v_j' v_k' \right\rangle_{,ii}$	Viscous Diffusion	$\dfrac{1}{Re} k_{,ii}$
$-\dfrac{2}{Re} \left\langle v_{j,i}' v_{k,i}' \right\rangle$	Viscous Dissipation	$\dfrac{-1}{Re} \left\langle v_{j,i}' v_{j,i}' \right\rangle$

The convection term describes the transport of Reynolds stresses due to the motion of the mean flow $\left\langle v_i \right\rangle$. The production term arises from interactions between fluctuations and shearing of the mean flow, resulting in a net transfer of energy from mean to turbulent fluctuations. The viscous diffusion term, as the name suggests, describes the diffusion of Reynolds stresses in the flow domain.

2.3   Modelling and Simulation

todo: difficulties in meshing due to complicated geometries resulting in low-order methods.

todo: Inability to accurately represent boundary irregularities leading to differences between model and true setup.

todo: no existing statistical closure models

todo: onto LES, RANS

Numerical calculations in which energy containing lengthscale are sufficiently resolved are called Direct Numerical Simulations (DNS), which are an effective tool in turbulence research [**moin˙DNS˙research**] as DNS allow one to gain insight into the mechanisms pertinent to energy transfer between fluctuating and mean modes. Large Eddy Simulation (LES) is a technique to simulate turbulent flows whereby the Navier-Stokes equations are low-pass filtered, and only the small wavenumber modes are solved for. large eddy simulation (LES) of such flows, where the resolved/filtered scales of motion are evolved, and the unresolved scaled are modeled, is still intractable. Courtesy of the nonlinear advection term, the effect of the unresolved modes on the resolved ones has to be accounted for by means of an appropriate model. Wall models too. A common technique in the domain of subgrid stress modelling is to perform DNS of flows in simplified domains, filter the resulting velocity-pressure fields and find functional relations between the DNS results and filtered data. Despite recent advances in computing technology, wall-resolved LES still remain intractable for complex geometries.

todo: accurate/turbulent inlet-outlet, BC data generation

Flows of practical interest often involve modelling complicated geometries containing multiscale features. Due to infeasibility of modelling extremely complicated geometries, the effect of small-scale features on

the flow may be accounted for as roughness, though it is not clear what features must be modelled and what discarded[**nakayama2002**]. Larger undulations are considered as part of the boundary shape. If a boundary irregularity is not modelled, the motion associated with that detail is lost. Various studies of numerical simulation of smooth-surface wall flows have indicated that proper representation of the near-wall flow is essential for accurate reproduction of the main flow[**nakayama2009**]. Proper representation of the flow near rough and irregular boundaries can also be very important in computation of flows over rough surfaces. In order to study the effects of small boundary irregularities, a direct numerical simulation of a model flow with a "simplified" complex boundary has been conducted.

Chapter 3

Analysis Repository

3.1 $\mathbb{R}$

Relations, Sets, and Functions

Definition 3.1.1 (Relation). A relation $\sim$ (or R) on nonempty set X is a subset of the set $X \times X$. For $a, b \in X$, if $a \sim b$, then $(a, b) \in R$.

Properties of relations: $\forall a, b\, c \in X$,

1. Reflexive: $a \sim a$

2. Symmetric: $a \sim b \iff b \sim a$

3. Transitive:
$$\left. \begin{array}{l} a \sim b \\ b \sim c \end{array} \right\} \implies a \sim c \tag{3.1}$$

4. Antisymmetric:
$$\left. \begin{array}{l} a \sim b \\ b \sim a \end{array} \right\} \implies a = b \tag{3.2}$$

A relation that is reflexive, symmetric, and transitive is called equivalent relation. For $a \in X$, the equivalent class of a is
$$[a] := \{x \in X | a \sim x\} \tag{3.3}$$

Equivalent classes of elements in X form a partition of X. An ordered relation is a relation that is reflexive, transitive, and antisymmetric, usually denoted by $\leq$.

todo: functions (one-one, onto), fields, ordered fields, vector spaces

Theorem 3.1.1 (Pigeonhole Principle). If n items are put into m containers, with $n > m$, then there exists at least one container with more than one item.

Theorem 3.1.2. For $a \in \mathbb{R} \setminus \mathbb{Q}$, the set $[na] = \{[na]| \in \mathbb{N}\}$ of fractional parts is dense in $[0, 1]$.

Proof. Fix > 0. In this proof, $[\cdot]$ denotes the the fractional part of the argument. We aim to show that for every $x \in [0, 1] \exists n$ *such that* $|x - [na]| < \epsilon$. Starting with $x = 0$, fix M *such that* $\frac{1}{M} < \epsilon$. If $[0, 1]$ is partitioned into M panels of size $\frac{1}{M}$, then, by the Pigeonhole principle, $\exists n_1, n_2 \in \{1, \ldots, M + 1\}$ *such that* $[n_1 a], [n_2 a]$ lie in the same panel. $\implies [(n_2 - n_1)a] < \frac{1}{M} < \epsilon$. Hence 0 is a limit point of $[na]$.

Now consider $x \in (0, 1]$. Find n *such that* $[na] < \frac{1}{M} < \epsilon$. If $x \in [0, \frac{1}{M}]$, we are done. Else, let x belong to the jth panel: $x \in [\frac{j}{M}, \frac{j+1}{M}]$. Since $[na] < \frac{1}{M}, \exists p \in \mathbb{N}$ *such that* $p[na]$ belongs to the $(j - 1)^{\text{th}}$ panel. Setting $p = \sup\{s \in \mathbb{N} | s[na] < \frac{j}{M}\}$. $\implies |x - [(p + 1)na]| < \frac{1}{M} < \epsilon$. $\qquad\square$

todo: $\mathbb{R}$, the completion of $\mathbb{Q}$
todo: Construction of $\mathbb{R}$ using Dedekind's Cuts

Theorem 3.1.3 (Archimedean Property)**.** $\forall x, y \in \mathbb{R}^+ \exists N \in \mathbb{N} : y < nx$

Proof. Fix $x, y \in \mathbb{R}$. Let $A = \mathbb{N}x$. Consider, *ad absurdum,* $\forall n \in \mathbb{N}, y \geq nx$. That is, y is an upper bound for A. Let $\alpha = \sup A = n\alpha$ for some $n \in \mathbb{N}$. As $x > 0, \alpha < \alpha + x = (n + 1)x \in A$. Therefore α is not an upper bound for A. $\qquad\square$

Theorem 3.1.4. $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e. between any two reals, there exists a rational number.

Proof. Let $x, y \in \mathbb{R}, x < y$. Without loss of generality, consider the case where $x, y > 0$. Applying the Archimedean property, we get

$$\exists n \in \mathbb{N}, n(y - x) > 1$$
$$\implies x < x + \frac{1}{n} < y \tag{3.4}$$
$$\implies x < \frac{nx + 1}{n} < y$$

Since the numerator is not guaranteed to be rational, we seek $m \in \mathbb{N}$ such that $nx < m < nx + 1$. Apply the Archimedean property again to find positive integer m_1 such that $0 < nx < m_1$. Hence there exists $0 < m \leq m_1$ such that

$$m < nx < m + 1$$
$$nx < m < nx + 1 < ny \tag{3.5}$$
$$x < \frac{m}{n} < y$$

$\qquad\square$

Definition 3.1.2 (Dense subset)**.** A set $S \subset X$ is dense in X if every element in X is a limit point of S.

The real number system is an ordered field $\mathbb{R}$, a complete extension of $\mathbb{Q}$ that has the least upper bound property.

Definition 3.1.3 (Least Upper Bound Property)**.** For $S \subset \mathbb{R}$, an upper bound of S in $\mathbb{R}$ is an element $x \in \mathbb{R}$ such that $\forall s \in S, s < x$. The least upper bound of S in $\mathbb{R}$ is an element y such that

1. y is an upper bound for S

2. if x is an upper bound for S, then $y < x$.

The least upper bound property of $\mathbb{R}$ is that any nonempty set of real numbers that is bounded from above has a least upper bound.

An ordered set satisfies the completeness axiom if every subset S that is bounded above has a least upper bound denoted $\sup S \in \mathbb{R}$. If $-S = \{-s | s \in S\}$, then $\inf(S) = -\sup(-S)$

Theorem 3.1.5 (Knaster-Tarski Fixed Point Theorem). Consider a set $X \subset \mathbb{R}$, for which $a = \inf(X)$, $b = \sup(X) \in X$, then every increasing function $f : X \to X$ has at least one fixed point, i.e. $x_0 \in X$ *such that* $f(x_0) = x_0$.

Proof. Consider the set $S = \{x \in X | x \leq f(x)\}$. S is nonempty since $a \in S$. Hence $\beta = \sup(S) \in X$ must exist.

$$
\begin{aligned}
&\forall x \in S, x \leq \beta \\
&\implies x \leq f(x),\ f(x) \leq f(\beta) \qquad (f \text{ increasing function}) \\
&\implies \forall x \in S,\ x \leq f(\beta) \\
&\implies \beta \leq f(\beta) \qquad\qquad\qquad\ (\text{since } \beta = \sup(S)) \\
&\implies f(\beta) \leq f(f(\beta))
\end{aligned}
\tag{3.6}
$$

$\square$

We define the following for $x, y \in \mathbb{R}$

$$
|x| = \begin{cases} x, & x > 0 \\ -x, & x > 0 \end{cases}
$$

$$
\min(x) = \frac{x + y - |x - y|}{2}
\tag{3.7}
$$

$$
\max(x) = \frac{x + y + |x - y|}{2}
$$

Definition 3.1.4 (Extended Real Line). We define $\infty := \sup \mathbb{R}$ and $-\infty := \inf \mathbb{R}$, and call $\mathbb{R} \cup \{-\infty, \infty\}$ the extended real line.

Theorem 3.1.6 (Existence of nth root). $\forall a \in \mathbb{R}^+, \exists b \in \mathbb{R}+$ *such that* $b^n = a$

Proof. <u>Uniqueness:</u> Say $b^n = c^n = a$.

$$
\implies 0 = b^n - a^n = (b - c)\left(b^{n-1} + ab^{n-2} + \cdots + a^{n-1}\right)
\tag{3.8}
$$

Since a, b, $c > 0$, $b - c = 0$. <u>Existence:</u> Consider the function $f : [0, a + 1] \to \mathbb{R}$

$$
f(x) = x + \frac{a - x^n}{n(a + 1)^{n-1}}
\tag{3.9}
$$

Observe that $f(x)$ is increasing, and that all fixed points of f satisfy $x = a^n$. To prove that f is increasing, let $0 < x \leq y \leq a + 1$

$$
\begin{aligned}
f(y) - f(x) &= (y - x) - \frac{y^n - x^n}{n(a + 1)^{n-1}} \\
&= (y - x)\left(1 - \frac{\sum_{k=0}^{n-1} x^k y^{n-1-k}}{n(a + 1)^{n-1}}\right) \\
&\geq 0
\end{aligned}
\tag{3.10}
$$

Now we eliminate $x = 0$, $a + 1$ as fixed points, and apply the fixed point theorem.

$$
0 < f(0) = \frac{a}{n(a + 1)^{n-1}} < f(a + 1) = \frac{(n - 1)(a + 1)^n + a}{n(a + 1)^{n-1}} < \frac{(n - 1)(a + 1)^n + (a + 1)^n}{n(a + 1)^{n-1}} = a + 1
\tag{3.11}
$$

$\square$

3.2 Sequences

Definition 3.2.1 (Sequence). A sequence $(a_n)_{n \in \mathbb{N}}$ of elements in a set X is a function $a. : \mathbb{N} \to X$.

Definition 3.2.2 (Limit of a sequence). A sequence (x_n) converges to $x \in X$ in norm $\|\cdot\|$, abbreviated as $x_n \xrightarrow{n} x$, if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \; such \; that \; \forall n \geq N, \|x - x_n\| < \epsilon \tag{3.12}$$

The element $\lim_{n \to \infty} x_n = x$ is called the limit of sequence x_n.

We will consider sequences of real numbers under the absolute value norm. Below listed are some properties of convergent sequences.

1. The limit of a convergent sequence is unique.

   Proof. $\epsilon > 0$. Let $x_n \to a$, $x_n \to b \in \mathbb{R}$. Without loss of generality, let $b > a$, and fix $\epsilon = \dfrac{b - a}{3}$. Then $\exists N_a, N_b \in \mathbb{N}$ *such that*

   $$\forall n > N_a, |a - x_n| < \epsilon$$
   $$\forall n > N_b, |b - x_n| < \epsilon \tag{3.13}$$

   Then, $\forall n > \max(N_a, N_b)$,

   $$\implies |b - a| < |a - x_n| + |b - x_n| < (b - a)\frac{2}{3} \Rightarrow\!\Leftarrow \tag{3.14}$$

   $\square$

2. (a_n) convergent $\implies \{a_n\}_n$ bounded.

   Proof. Let $a_n \to a$. $\forall \epsilon, \exists N$ *such that* $\forall n > N, |a - a_n| < \epsilon$. Therefore,

   $$\forall n, |a_n| < \max \left(\{|a_m|\}_{m=1}^{N-1} \cup \{|a \pm \epsilon|\} \right) \tag{3.15}$$

   $\square$

3.
   $$\left. \begin{array}{c} a_n \to a \\ b_n \to b \end{array} \right\} \implies a_n b_n \to ab \tag{3.16}$$

   Proof. Fix $\epsilon > 0$. For $\epsilon > 0$, $\exists N_a, N_b$ *such that*

   $$\forall n > N_a |a - a_n| < \epsilon_a > 0$$
   $$\forall n > N_b |b - b_n| < \epsilon_b > 0 \tag{3.17}$$

   Since b_n is a convergent sequence, $\exists |b_n| < M_b \in \mathbb{R}$. Let $\epsilon_a = \dfrac{\epsilon}{2M_b}$, $\epsilon_b = \dfrac{\epsilon}{2a}$. For $n > \max(N_a, N_b)$,

   $$\begin{aligned} |ab - a_n b_n| &= |ab + ab_n - ab_n - a_n b_n| \\ &\leq a|b - b_n| + b_n|a - a_n| \\ &< a\frac{\epsilon}{2a} + b_n\frac{\epsilon}{2M_b} \\ &< \epsilon \end{aligned} \tag{3.18}$$

   $\square$

4. $a_n \to a \neq 0 \implies \dfrac{1}{a_n} \to \dfrac{1}{b}$.

Proof. Fix $0 < \epsilon < \frac{|b|}{2}$ (bounding b_n away from 0). $\exists N$ *such that* $\forall n > N |b - b_n| < \epsilon_b$.

$$
\begin{aligned}
\implies \left| \frac{1}{b} - \frac{1}{b_n} \right| &= \left| \frac{b - b_n}{bb_n} \right| \\
&< \frac{\epsilon_b}{|b||b_n|} \\
&< \frac{2\epsilon_b}{|b|^2}
\end{aligned}
\tag{3.19}
$$

Allowing $\epsilon_b = \frac{|b|^2 \epsilon}{2}$, we complete the proof. $\qquad \square$

5.

Theorem 3.2.1 (Squeeze Theorem). $\forall n > N, L \leftarrow a_n \leq b_n \leq c_n \to L \in \mathbb{R} \implies b_n \to L$.

Proof. $\forall n, |b_n - a_n| < c_n - a_n \to L - L = 0$. Fix $\epsilon > 0$. $\exists N > 0$ *such that* $\forall n > N, |L - a_n| < \frac{\epsilon}{2}, |c_n - a_n| < \frac{\epsilon}{2}$

$$
\begin{aligned}
\implies |L - b_n| &= |L - a_n + a_n - b_n| \\
&< |b_n - a_n| + |a_n - L| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
\tag{3.20}
$$

$\qquad \square$

6.

$$
\left.
\begin{array}{r}
a_n \to a \\
\forall n, a_n > 0
\end{array}
\right\} \implies \forall k \geq 1, \sqrt[k]{a_n} \to \sqrt[k]{a}
\tag{3.21}
$$

Proof. Fix $0 < \epsilon \leq \frac{|a|}{2}$. When $a = 0, \exists N$ *such that* $\forall n > N |a_n| < \epsilon^k$. $\implies \left| \sqrt[k]{a_n} \right| < \sqrt[k]{\epsilon^k} = \epsilon$. Consider when $a > 0$. Let $\epsilon_a > 0$. $\exists N$ *such that* $\forall n > N, |a - a_n| < \epsilon_a$

We use the identity

$$
A - B = \frac{A^k - B^k}{\sum_{i=0}^{k-1} A^i B^{k-1-i}}
\tag{3.22}
$$

When $A, B > 0$, we trivially have $|A - B| < \dfrac{\left| A^k - B^k \right|}{B^{k-1}}$

$$
\forall n > N, 0 \leq \left| \sqrt[k]{a} - \sqrt[k]{a_n} \right| < \frac{|a - a_n|}{\sqrt[k]{a^{k-1}}} < \frac{\epsilon_a}{\sqrt[k]{a^{k-1}}}
\tag{3.23}
$$

Allowing $\epsilon_a = \epsilon \sqrt[k]{a^{k-1}}$ completes the proof. $\qquad \square$

7. $|a_n| \to 0 \implies a_n \to 0$

Proof. Apply squeeze theorem to $0 \leftarrow (-|a_n|) \leq a_n \leq |a_n| \to 0$. $\qquad \square$

8.

Theorem 3.2.2 (Monotone Convergence Theorem). For a_n bounded and monotone, $a_n \nearrow \sup\{a_n\}_n$ or $a_n \searrow \inf\{a_n\}$. (Note on notation: $a_n \nearrow L \iff a_n$ is a monotone increasing sequence , $a_n \to L$.)

Proof. Consider the case when a_n is a monotone increasing sequence. The assumption is without loss of generality, for if a_n is monotone decreasing, we consider $-a_n$ and prove its convergence to $\sup\{-a_n\}_n = -\inf\{a_n\}_n$. By the least upper bound property, we have a unique $L := \sup\{a_n\}_n$. We aim to prove that $\forall \epsilon > 0 \exists N$ *such that* $\forall n > N, |L - a_n| < \epsilon$. If the statement is not true, then $\exists \epsilon_0 > 0$ *such that* $\forall k \exists n_k > k, |L - a_{n_k}| \geq \epsilon_0$. Since $a_k \nearrow$, we have $\forall k, a_k \leq a_{n_k} \leq L - \epsilon_0 \implies (L - \epsilon_0) < L$ is an upper bound for $\{a_n\}$. $\Rightarrow\!\Leftarrow$ $\qquad\qquad\qquad\square$

Example 3.2.1. $-1 < a < 1 \implies a^n \to 0$.

The example is trivial when $a = 0$. Consider the case when $0 < |a| < 1$. We consider the case when $0 < |a| < 1 \implies \frac{1}{|a|} > 1$. Let $\frac{1}{|a|} := 1 + h$.

$$
\begin{aligned}
\left(\frac{1}{|a|}\right)^n &= (1+h)^n = 1 + nh + \mathcal{O}(h^2) > 1 + nh \\
&\implies |a|^n < \frac{1}{1 + nh} \\
&\implies |a|^n \to 0
\end{aligned}
\tag{3.24}
$$

Since $a^n \leq |a^n| \leq |a|^n$ we conclude that $a^n \to 0$.

Example 3.2.2. $a_n = \sqrt[n]{n}$.

For $n > 1 \implies \sqrt[n]{n} > \sqrt[n]{1} = 1$. For $h_n > 0$, let $a_n = 1 + h_n$. We aim to show that $h_n \to 0$.

$$
\begin{aligned}
\implies n = a_n^n &= (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2}h_n^2 + + \frac{n(n-1)(n-2)}{3!}h_n^3 \cdots \\
&\geq 1 + \frac{n(n-1)}{2}h_n^2
\end{aligned}
\tag{3.25}
$$

We get $h_n^2 \leq \frac{2}{n} \implies h_n \to 0$.

Example 3.2.3. $a_n = \dfrac{a^n}{n!}$ for fixed $a > 0$.

We consider the ration $\dfrac{a_{n+1}}{a_n} = \dfrac{a}{n+1} < 1$ for large enough n.

$$
\left.\begin{aligned}
(a_n)_{n > n_0} \searrow \\
\forall n, a_n > 0
\end{aligned}\right\} \implies a_n \to L := \inf(a_n) \geq 0
\tag{3.26}
$$

If $L > 0$, $\dfrac{a_{n+1}}{a_n} \xrightarrow{n} \dfrac{L}{L} = 1 = 0 \xleftarrow{n} \dfrac{a}{n+1} \Rightarrow\!\Leftarrow$. Therefore, $L = 0$.

Example 3.2.4 (Decimal Expansion). Given $d_0 \in \mathbb{Z}, (d_n)_{n>0} \in \{0, 1, 2, \ldots, 9\}$, consider

$$
s_n = \sum_{i=0}^{n} \frac{d_i}{10^i}
\tag{3.27}
$$

The sequence is monotonically increasing, and

$$
d_0 \leq s_n \leq s_\infty = d_0 + \sum_{i=1}^{\infty} \frac{d_i}{10^i} \leq d_0 + 9 \sum_{i=1}^{\infty} \frac{1}{10^i} \leq d_0 + 9 \left(\frac{1}{1 - \frac{1}{10}} - 1\right) = d_0 + 1
\tag{3.28}
$$

$$
\left.\begin{aligned}
d_0 \leq s_n \leq d_0 + 1 \\
s_n \nearrow
\end{aligned}\right\} \implies s_n \text{convergent}
\tag{3.29}
$$

27

Hence, $s := \lim_n s_n = d_0.d_1 d_2 \ldots \in [d_0, d_0 + 1]$. This representation is not unique as

$$0.99\ldots = \sum_{i=1}^{\infty} \frac{9}{10^i} = 9\left(\frac{1}{1 - \frac{1}{10}} - 1\right) = 1.00\ldots \tag{3.30}$$

We can recapture the digits in s as follows: $d_0 = \lfloor s \rfloor$. Define $T(x) = \lfloor 10(s - d_0) \rfloor$. Now, $d_1 = T^1(s)$, $d_n = T^n(x)$

Definition 3.2.3 (Cauchy Sequence).

$$a_n \text{Cauchy sequence} \iff \forall \epsilon > 0 \exists N \text{ such that } \forall m, n > N, |a_m - a_n| < \epsilon \tag{3.31}$$

The elements in the tail of a cauchy sequence get arbitrarily close to each other. $\forall k, |a_n - a_{n+k}| \xrightarrow{n} 0$

Proposition 3.2.1. a_n convergent $\implies a_n$ cauchy

Proof. Suppose $a_n \to L$. Fix > 0, and sufficiently large m, n, we get $a_m, a_n \in B(L, \frac{\epsilon}{2}) \implies$ maximum distance between a_m, a_n is equal to the diameter of the ball. $\implies |a_m - a_n| <$. $\square$

Example 3.2.5 (Harmonic Series). $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Clearly $s_{n+k} - s_n = \sum_{i=1}^{k} \frac{1}{n+i}$.

$$\implies s_{2n} - s_n = \sum_{i=1}^{n} \frac{1}{n+i} \geq n\frac{1}{2n} = \frac{1}{2} \tag{3.32}$$

Since s_n is not a cauchy sequence, it is not convergent either. And since, $s_n \nearrow$, s_n diverges to infinity.

Example 3.2.6. $s_n = 1 + \frac{1}{n^2} + \cdots + \frac{1}{n^2}$

Clearly, $s_n \nearrow$. We try to find an upper bound for s_n

$$s_n < 1 + \sum_{i=1}^{n} \frac{1}{i(i-1)} = 1 + \sum_{i=1}^{n} \frac{1}{i-1} - \frac{1}{i}$$
$$= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n} \tag{3.33}$$

Hence, s_n convergent, and therefore cauchy.

Example 3.2.7. $a_n = \sqrt[n]{n!}$

We aim to show that the tail of a_n is larger than any arbitrarily chosen real. To do so, fix integer $M > 1$. Now, $\forall n > 2M$,
$$n! = 1 \cdot 2 \cdots M \cdot (M+1) \cdots (2M)$$
$$a_n \geq M^{\frac{n-M}{n}} = M\frac{1}{\sqrt[n]{M^M}} \tag{3.34}$$

As $\sqrt[n]{M^M} \to 1, \exists N > 2M$ such that $\forall n > N \sqrt[n]{M^M} > \frac{1}{2}$. Therefore, $\forall n > N, a_n > \frac{M}{2}$.

Example 3.2.8. $s_n = s_n(x) = 1 + \frac{x}{1} + \frac{x^2}{2} + \cdots \frac{x^n}{n!}$

$$s_{n+k} - s_n = \sum_{i=1}^{k} \frac{x^{n+i}}{(n+i)!} = \frac{x^{n+1}}{(n+1)!} \left(1 + \frac{x}{n+2} + \frac{x^2}{(n+2)(n+3)} + \cdots + \frac{x^{k-1}}{(n+2)\cdots(n+k)} \right)$$

$$|s_{n+k} - s_n| \leq \frac{|x|^{n+1}}{(n+1)!} \left(1 + \frac{|x|}{n+2} + \frac{|x|^2}{(n+2)^2} + \cdots + \frac{|x|^{k-1}}{(n+2)^{k-1}} \right)$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \sum_{j=0}^{\infty} \left(\frac{|x|}{n+2} \right)^j$$

$$= \frac{|x|^{n+1}}{(n+1)!} \frac{1}{1 - \frac{|x|}{n+2}}$$

(3.35)

We look at the upper bound in the limit $n \to \infty$.

$$\lim_n \frac{|x|^{n+1}}{(n+1)!} \frac{1}{1 - \frac{|x|}{n+2}} = \lim_n \frac{|x|^{n+1}}{(n+1)!} \lim_n \frac{1}{1 - \frac{|x|}{n+2}} = 0 \cdot 1 = 0$$

(3.36)

Hence s_n is cauchy and therefore convergent.

Theorem 3.2.3. Cauchy $\implies$ bounded.

Proof. Fix $\epsilon > 0$. For cauchy sequence a_n, $\exists N$ *such that* $\forall n > N, \forall k, |a_{n+k} - a_n| < \epsilon$. The set $\{a_n\}_{n \in \mathbb{N}}$ lies in the interval $[-M, M]$ where

$$M := \max \left(\{|a_n|\}_{n < N} \cup \{|a_N| + \epsilon\} \right)$$

(3.37)

$\square$

Definition 3.2.4 (Limit Inferior and Limit Superior)**.** Let a_n be a sequence.

$$\liminf_{x \to 0} = \lim_{n \to \infty} \sup_{k > n} a_k$$

$$\limsup_{x \to 0} = \lim_{n \to \infty} \inf_{k > n} a_k$$

(3.38)

Limit inferior, and limit superior, are the limits of the infimum/supremum of the tail of the sequence $(a_k)_{k > n}$ in the limit $n \to \infty$. For every n, it is always true that a finite number of elements lie in the discarded part of the sequence, $(k < n)$, and an infinite number of elements lie in the tail, $(k > n)$, and $\liminf$, $\limsup$ are the bounds of the infinitely long tail. Consider , $\forall k$, the truncated sequence $(a_n)_{n \geq k}$ and the set $A_k := \{a_n\}_{n \geq k}$. Define $u_k = \inf A_k$, $v_k = \sup A_k$. Because

$$A_1 \supset A_2 \supset A_3 \cdots A_{k-1} \supset A_k$$

(3.39)

we have,

$$u_1 \leq u_2 \cdots u_{k-1} \leq u_k \leq \cdots \leq v_k \leq v_{k+1} \cdots \leq v_2 \leq v_1$$

(3.40)

If (a_n) is bounded, $-\infty < u_1 \leq v_1 < \infty \implies u_n, v_n$ convergent (by monotone convergence theorem). Therefore,

$$\liminf_n a_n = \lim_k u_k = \sup_k u_k$$

$$\limsup_n a_n = \lim_k v_k = \inf_k v_k$$

(3.41)

Another way to look at $\liminf$, $\limsup$ is the following: $L_i = \liminf_n a_n$ if

1. $\forall x < L_i$, there exist infinite number of a_n greater than x

2. $\forall x > L_i$, there exist infinite number of a_n less than x

Some properties of $\liminf$, $\limsup$ are as follows:

1. $\limsup_n a_n = -\liminf_n(-a_n)$

2. $\limsup_n(a_n + b_n) \leq \limsup_n a_n + \limsup_n b_n$ (follows from $\sup(A + B) \leq \sup A + \sup B$)

Lemma 3.2.4 (Nested Interval Property)**.** Let $I_n = [a_n, b_n] \subset \mathbb{R}$ be a sequence of intervals such that I_n are nested as follows:

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots \tag{3.42}$$

Then, $\cap_n I_n \neq \varnothing$. Further, if length of the interval is going to zero with n, i.e., $(b_n - a_n) \to 0$, then $\cap_n I_n = \{x_0\}$ where $a_n, b_n \to x_0$.

Proof. todo: □

Definition 3.2.5 (Subsequence)**.** A subsequence of a sequence a_n is the sequence $(b_k)_k = (a_{n_k})_k$ where $n_. : \mathbb{N} \to \mathbb{N}$ with the restriction that $\forall k, k \leq n_k$.

Definition 3.2.6 (Limit Point)**.** (Also called accumulation point) A limit point a_n is a limit of convergence of a subsequence $(a_{n_k})_k$.

Theorem 3.2.5 (Bolzano-Weierstrass)**.**

Lemma 3.2.6 (Finite intersection property)**.**

todo: those silly tests

Theorem 3.2.7 (Rearrangement theorem)**.**

Theorem 3.2.8 (Fubini-Tonelli)**.** Summability of infinite matrices

Theorem 3.2.9 (Summability of infinite matrices)**.**

3.3 Sets, Spaces and Functions

For now, we restrict ourselves to sets in finite dimensional spaces.

Norm

Theorem 3.3.1 (Cauchy-Schwarz inequality)**.**

Open, closed/ closure, interior, etc.

DeMorgan's Laws (Unions and Intersections)

The Cantor set

Definition 3.3.1 (Open Ball)**.** The open ball of radius r around $x_0 \in X$ is defined as follows:

$$B(x_0, r) = \{x \in X | \|x - x_0\| < r\} \subset X \tag{3.43}$$

Definition 3.3.2 (Continuity)**.** We say a function $f : X \to Y$ is continuous at $x_0 \in X$

Definition 3.3.3 (Compact Set)**.** A set for which every open cover contains a finite subcover.

Theorem 3.3.2. Compact $\iff$ sequentially compact

Proof. The spirit of compactness is "any bounded sequence has a sequence converging in the set." □

Definition 3.3.4 (Precompact Set). A set in a normed vector space X is precompact (also called relatively compact set) if its closure is compact.

equivalance between different definitions of continuity.

==todo:== darboux integral, shuffling limits of integrals, prove everything from calculus

Definition 3.3.5 (Fixed Point Theorem). A **fixed point** of $T : X \to X$ is an element $x_0 \in X$ *such that* $T(x_0) = x_0$. A fixed point is **globally attracting** if $\forall x \in X, T^n(x) \xrightarrow{n} x_0$

We call T a **contraction** if $\forall x_1, x_2 \in X, d(Tx_1, Tx_2) \le \alpha d(x_1, x_2), \alpha \in [0, 1)$

Theorem 3.3.3 (Contraction Mapping Principle). For a contraction map $T : X \to X$ on a complete metric space X, there exists a unique fixed point $x_0 \in X$. The fixed point is globally attracting with geometric convergence rate.

Proof. Uniqueness: Suppose $x_1, x_2 \in X$ *such that* $x_1 = Tx_1$, $x_2 = Tx_2$. Then T is no longer a contraction since $d(Tx_1, Tx_2) = d(x_1, x_2)$.

$\forall x \in X$, we show that the sequence $T^n x$ is cauchy due to the contraction property.

$$d(T^{n+1}x, T^n x) \le \alpha d(T^n x, T^{n-1} x) \le \cdots \le \alpha^n d(Tx, x) \tag{3.44}$$

This implies, the distance between nonconsecutive iterates is bounded by

$$\begin{aligned} d(T^{n+m}x, T^n x) &\le d(T^{n+m}x, T^{n+m-1}x) + \cdots + d(T^{n+1}x, T^n x) \\ &\le (\alpha^{n+m-1} + \cdots + \alpha^n) d(Tx, x) \\ &\le \alpha^n \frac{1 - \alpha^m}{1 - \alpha} d(Tx, x) \\ &\le \frac{\alpha^n}{1 - \alpha} d(Tx, x) \end{aligned} \tag{3.45}$$

Let $x_0 = \lim_n T^n x$ for some x. $\implies Tx_0 = T \lim_n T^n x = \lim_n T^{n+1} x = x_0$. □

Theorem 3.3.4 (Contraction Power Mapping Principle). For $T : X \to X$, if T^m is a contraction, then T has a fixed point that is globally attracting.

Proof. Uniqueness is clear since any fixed point of T is a fixed point of T^m, and T^m has a unique fixed point that is globally attracting. To prove existence, we observe that

$$\begin{aligned} &\forall x, T^{mn} x \xrightarrow{n} x_0 \\ \implies & T^{mn+k} x \xrightarrow{n} x_0 \\ \implies & d\left(T^{mn+k}x, T^{mn}x\right) \le \alpha^n d\left(T^k x, x\right) \to 0 \end{aligned} \tag{3.46}$$

illustrating that for sufficiently large $N = mn$, $d\left(T^{N+k}x, T^N x\right) \to 0$. □

Application to picard iteration, fredhold and volterra integral equations are straightforward. Existence and uniqueness of solutions are obtained from fixed point theorems by establishing that the operators are contractions (provided we are working in Banach spaces). However contractivity is a rather strong property. We now prove the existence of a fixed point assuming compactness of a mapping.

Definition 3.3.6 (Compact Operator). An operator $T : X \to Y$ over normed vector spaces is compact if it is continuous, and for every bounded sequence $(x_n) \in X$, Tx_n has a convergent subsequence. In other words, every bounded set is mapped to a precompact set.

Chapter 4

Algebra

We discuss algebra...

4.1 Linear

An important object in linear algebra is a *vector*, a quantity described completely by a *magnitude* $\in \mathbb{R}$ and an abstract notion of a *direction*, properties which can be independently varied. The magnitude of a vector is obtained by applying a metric to the vector, whereas the direction is defined to be the vector itself, albeit of magnitude unity. Vectors live in *vector spaces*, sets on which operations are defined to manipulate these properties.

It is not the objective of this section to describe vector algebra. Instead, we focus on proving facts pertinent to developing a geometric intuition of vector operators.

Definition 4.1.1 (Vector Space). A vector space V over a field $\mathbb{F}$ is a set with operation of *vector addition* (+) and *scalar multiplication* defined so that $\forall \boldsymbol{x}, \boldsymbol{y} \in V, \forall \lambda \in \mathbb{F}$, there is a unique element $\boldsymbol{x} + \lambda \boldsymbol{y} \in V$ such that the following conditions hold.

- todo: add axioms of vector space operations

Relevant examples of vector spaces are

- $\mathbb{R}^n$ - represented as a column vector

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \tag{4.1}$$

- $M_{m \times n}(\mathbb{R})$ - space of $m \times n$ real matrices over field $\mathbb{R}$

$$A = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \dots & A_{mn} \end{bmatrix} \in M_{m \times n}(\mathbb{R}) \tag{4.2}$$

where $A_{ij} \in \mathbb{R}$. Scalar multiplication, and vector addition are defined element-vise as follows:

$$(A + B)_{ij} = A_{ij} + B_{ij} \qquad\qquad\qquad vector\ addition \qquad\qquad (4.3a)$$
$$(\lambda A)_{ij} = \lambda A_{ij} \qquad\qquad\qquad scalar\ multiplication \qquad (4.3b)$$

Every linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ has an associated $m \times n$ matrix A written as follows (where $\underline{e_j}$ are the n orthonormal basis of $\mathbb{R}^n$):

$$\underline{y} = A\underline{x} = \begin{bmatrix} L(\underline{e_1}) & \dots & L(\underline{e_n}) \end{bmatrix}_{m \times n} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1} = \sum_{j=1}^{n} x_j L(\underline{e_j}) = L(\sum_{j=1}^{n} x_j \underline{e_j}) = L(\underline{x}) \qquad (4.4)$$

- $\mathbb{P}^n$ - space of polynomials of degree up to n

$$p(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{P}^n \qquad\qquad (4.5)$$

Definition 4.1.2 (Vector Subspace). W subspace of V if W is a subset of V and is a vector space. Two trivial subspaces of V are $\{\mathbf{0}\}$ and V itself.

Definition 4.1.3. We denote $\mathrm{span}(\{v_i\}_i) = \{\sum_i a_i \boldsymbol{v_i} : \forall \{a_i\}_i \in \mathbb{R}\}$. We say $\{\boldsymbol{v_i}\}_i$ is **linearly independent** if $\sum_i a_i \boldsymbol{v_i} = 0 \iff \forall i, a_i = 0$. Simply put, $\{v_i\}_i$ is linearly independent if its elements can be combined to form $\mathbf{0}$ in only one way (the trivial way). If elements can be combined to form the additive identity, then it is obvious that at least one of the $\boldsymbol{v_j}$ is in the span of the others meaning that it is not adding anything new to the set $\{\boldsymbol{v_i}\}_i$ (which is boring). We call a (nonunique) minimum spanning (linearly independent) set of a vector space (V) its **basis** and call its size (number of items) the **dimension** of the space $\dim(V)$. Vectors in a space can be uniquely expressed as a linear combination of its basis vectors.

Theorem 4.1.1 (Replacement Theorem). Any set of vectors in V of size greater than $\dim(V)$ is linearly dependent.

Proof. Let $\{\boldsymbol{v_i}\}_{i=1}^{n}$ be a basis for V. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

todo: linear operations - null spaces, ranges, matrix meaning, rank, determinant, diagonalization, adjoint of an operator, canonical forms, matrix exponentials

4.2 Abstract

abstract algebra is boring but necessary. something about symetry

Chapter 5

Functional Analysis

5.1 Multi-Index Notation

A multi-index is a d-tuple of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_d)$. The degree of a multi-index α is $|\alpha| = \sum_{i=1}^{d} \alpha_i$. Consider a point in d-dimensional real Euclidean space $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. We denote by x^α the monomial $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. If $Dx_i = \partial_{x_i}$, denotes the partial differential operator with respect to variable x_i, then D^α denotes a differential operator of order $|\alpha|$.

$$D^\alpha = Dx_1^{\alpha_1} \cdots Dx_d^{\alpha_d} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \tag{5.1}$$

For two multi-indices β, α, we say $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for $1 \leq i \leq d$. In this case, $\alpha \pm \beta$ are also multi-indices with elements $\alpha_i \pm \beta_i$ and order $|\alpha \pm \beta| = |\alpha| \pm |\beta|$. We also denote $\alpha! = (\alpha_1!, \ldots, \alpha_d!)$ and if $\beta \leq \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = (\binom{\alpha_1}{\beta_1}, \ldots, \binom{\alpha_d}{\beta_d}) \tag{5.2}$$

Theorem 5.1.1. *Multinomial Theorem*

$$(x_1 + \cdots + x_d)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} x^\alpha \tag{5.3}$$

Theorem 5.1.2 (Leibniz Formula)**.**

$$D^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v \tag{5.4}$$

Proof. We prove the Leibniz formula by induction on $|\alpha|$. It is trivial to show that the equality holds for $|\alpha| = 0, 1$. Assuming the equality holds for for some $\alpha \neq 0$, we prove that the equality holds for $\alpha + e_i$ where is a multi-index with 1 in the ith position $\forall i \in \{1, \cdots, d\}$ and zeros elsewhere.

$$
\begin{aligned}
D^{\alpha+e_i}(uv) &= D^{e_i}\left(\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v \right) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{e_i}\left(D^\beta u D^{\alpha-\beta} v \right) \\
&= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{\beta+e_i} u D^{\alpha-\beta} v + \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta+e_i} v
\end{aligned}
\tag{5.5}
$$

$\square$

Theorem 5.1.3. *(Taylor's Formula)* For sufficiently smooth functions $f : \mathbb{R}^d \to \mathbb{R}$,

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \mathrm{D}^\alpha f(x_0)(x - x_0)^\alpha + \mathcal{O}(|x|^{k+1}) \tag{5.6}$$

5.2   Spaces and Norms

For the purposes of this text, a topological vector space (TVS) is a vector space for which the vector operations of addition and scalar multiplication are continuous. That is for $x, y \in X$, $c \in \mathbb{C}$ where X is some TVS, the maps $(x, y) \to x + y$ and $(c, x) \to cx$ are continuous.

A **norm** on a vector space X is a real-valued functional $\|\cdot\| : X \to \mathbb{R}$ such that for $x, y \in X$, $c \in \mathbb{C}$,

- $\|x\| \geq 0$ with equality holding iff $x = 0$.

- $\|cx\| = |c| \|x\|$

- $\|x + y\| \leq \|x\| + \|y\|$

A TVS is normable if its topology coincides with that induced by some norm. Two norms $\|\cdot\|_1$, $\|\cdot\|_2$ are equivalent if for all $x \in X$, there exists some $c \in \mathbb{R}$ such that

$$c\|x\|_1 \leq \|x\|_2 \leq \frac{1}{c}\|x\|_1 \tag{5.7}$$

If X is a normed space and all Cauchy sequences converge to a limit in X with respect to the norm, then X is called a **Banach space**. A vector space X is called **pre-Hilbert** if there exists a functional called the **scalar product** $(\cdot, \cdot) : X \times X \to \mathbb{R}$ such that $\forall u, v \in X$,

- $(u, u) \geq 0.\, (u, u) = 0 \iff u = 0$

- $(u, v) = (v, u)$

- $(u_1 + u_2, v) = (u_1, v) + (u_2, v),\ u_1, u_2 \in X$

- $(\lambda u, v) = \lambda(u, v),\ \lambda \in \mathbb{R}$.

The scalar product induces a norm $\|u\| = \sqrt{(u, u)}$. Another functional defined on a vector space X is the **inner product** $(\cdot, \cdot) : X \times X \to \mathbb{C}$ such that for $x, y, z \in X$, $a, b \in \mathbb{C}$,

- $(x, y) = \overline{(y, x)}$

- $(ax + by, z) = a(x, z) + b(y, z)$

- $(x, x) = 0 \iff x = 0$

If X is a Banach space under a norm that is induced by an inner product, then X is called a **Hilbert space**. Given an inner-product, a norm on X can be specified as follows: for $x \in X$,

$$\|x\| = x, x^{1/2} \tag{5.8}$$

We consider maps (or operators) between vector spaces X and Y, $L : X \to Y$. A map is linear if $\forall x_1, x_2 \in X, \forall \alpha \in \mathbb{R}, f(x_1 + \alpha x_2) = f(x_1) + \alpha f(x_2)$. For example, differentiation is a linear map from C^n to C^{n-1}, and integration from L^1_{loc} to $\mathbb{R}$. A linear operator is said to be bounded if $\exists C > 0$ *such that* $\|Lx\|_X \leq C\|x\|_X$

Theorem 5.2.1. For linear operators, the following are equivalent.

1. L is bounded

2. L is continuous at one point

3. L is continuous for all inputs

We denote with $\mathcal{L}(X, Y)$ the vector space of bounded linear operators from X to Y. We define the following induced norm on $\mathcal{L}(X, Y)$

$$\|L\| = \sup_{\|x\|_X = 1} \|Lx\|_Y \tag{5.9}$$

Theorem 5.2.2. If X and Y are normed vector spaces and Y is complete, then $\mathcal{L}(X, Y)$ is complete.

A map from X to its field, $f : X \to \mathbb{R}$ is called a **functional**. The set of linear functionals on X is called the **dual** of X and is denoted by X', which is also a vector space under pointwise addition and scalar multiplication. We represent the application of a functional on an element as follows:

$$f(x) = f, x \tag{5.10}$$

The operator norm on dual space X' is

$$\|x'\|_{X'} = \sup_{\|x\|_X = 1} |x'(x)| \tag{5.11}$$

Theorem 5.2.3. *(Riesz Representation Theorem)* For a Hilbert space X, a linear functional x' on X belongs in X' if and only if there exists an $y \in X$ such that for every $x \in X$,

$$x'(x) = (x, y)_X \tag{5.12}$$

We consider an open subset $\Omega \in \mathbb{R}^d$ as domain in d-dimensional Euclidean space. For some $S \subset \Omega$, we denote by $\overline{S}$ the closure of S. We write $S \subset\subset \Omega \subset \mathbb{R}^d$ if $\overline{S} \subset \Omega$ and $\overline{S}$ is compact. If u is a function defined on Ω, we define the support of u as

$$\text{supp}(u) = \overline{\{x \in \Omega | u(x) \neq 0\}} \tag{5.13}$$

We say $A \subset\subset B$ if and only if $\overline{A}$ is a compact subset of B. If, for some function u, $\text{supp}(u) \subset\subset \Omega$, we say that u has **compact support** in Ω. A measurable function u on Ω is said to be essentially bounded on Ω if there exists a constant K such that $|u(x)| \leq K$ a.e. on Ω. The greatest lower bound of such constants K is called the essential supremum of $|u|$ on Ω and is denoted by $\text{ess}\sup_{x \in \Omega} |u(x)|$. We denote by $L^\infty(\Omega)$ the vector space consisting of all equivalent classes of functions u (that are equal almost everywhere) which are essentially bounded on Ω.

$$\|u\|_\infty = \text{ess}\sup_{x \in \Omega} |u(x)| \tag{5.14}$$

We define the following family of vector spaces. For some integer m, the vector space $C^m(\Omega)$ consists of all functions defined on Ω which, along with their partial derivatives up to order m are continuous.

$$C^m(\Omega) = \{\phi : \Omega \to \mathbb{R} : \forall \alpha \text{ with } |\alpha| \leq m, \, \mathrm{D}^\alpha \phi \text{ is continuous}\}$$
$$C^\infty(\Omega) = \cap_{m=0}^\infty C^m(\Omega) \tag{5.15}$$

We abbreviate $C^0(\Omega)$ to $C(\Omega)$. The subspace $C_0^m(\Omega)$ consists of all functions in $C^m(\Omega)$ that have compact support in Ω. We wish to define equivalent spaces for $\overline{\Omega}$, $C^m(\overline{\Omega})$. However, since Ω is open, continuous functions on Ω need not be bounded. Therefore, not all functions in $C^m(\Omega)$ can be continuously extended to $\overline{\Omega}$. For example, the function $f(x) = 1/x$ is continuous on the open interval $(0, 1)$, but discontinuous

on $x = 0$. Functions that are uniformly continuous and bounded on Ω can be uniquely and continuously extended to $\overline{\Omega}$. We define $C^{(}\overline{\Omega})$ as follows:

$$C^m(\overline{\Omega}) = \{\phi \in C^m(\Omega) : \forall \alpha, |\alpha| \leq m, \mathrm{D}^\alpha \phi \text{ bounded and uniformly continuous on } \Omega\} \tag{5.16}$$

A function $f : \Omega \to \mathbb{R}$ is said to be **Hölder continuous** for exponent γ $(0 < \gamma \leq 1)$ if

$$\exists K \geq 0 \, \forall x, y \in \Omega, x \neq y, \frac{|f(x) - f(y)|}{\|x - y\|} \leq K \tag{5.17}$$

We define $C^{m,\gamma}(\overline{\Omega})$ to be the subspace of $C^m(\overline{\Omega})$ consisting of functions for which, for $|\alpha| = m$, $\mathrm{D}^\alpha \phi$ satisfies the Hölder condition. We define the following norms and seminorms:

$$\|u\|_{C^0(\overline{\Omega})} = \sup_{x \in \Omega} |\mathrm{D}^\alpha u| \qquad\qquad |u|_{C^{0,\gamma}(\overline{\Omega})} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{\|x - y\|_2} \tag{5.18}$$

$$\|u\|_{C^m(\overline{\Omega})} = \max_{0 \leq |\alpha| \leq m} \|u\|_{C^0(\overline{\Omega})} \qquad\qquad \|u\|_{C^{m,\gamma}(\overline{\Omega})} = \|u\|_{C^m(\overline{\Omega})} + \max_{|\alpha|=m} |\mathrm{D}^\alpha u|_{C^{0,\gamma}(\overline{\Omega})} \tag{5.19}$$

Theorem 5.2.4. $C^{m,\gamma}(\Omega)$ is complete for nonnegative integer m and $0 < \gamma \leq 1$ with respect to the norm $\|\cdot\|_{C^{m,\gamma}(\overline{\Omega})}$.

Proof. Observe that $\forall x \in \Omega$, $\{u_n(x)\} \subset \mathbb{R}$ form a Cauchy sequence. By completeness of $\mathbb{R}$, $\forall x, u_x(x) \to u(x)$. It is left to prove that $\lim_{n \to \infty} \|u - u_n\|_{C^{m,\gamma}(\Omega)} = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Let Ω be a domain in $\mathbb{R}^d$ and let $p \in \mathbb{R}^+$. We denote by $L^p(\Omega)$ the class of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |u(x)|^p x < \infty \tag{5.20}$$

The elements of $L^p(\Omega)$ are equivalent classes of functions that are equal almost everywhere on Ω that satisfy the above inequality. We show that $L^p(\Omega)$ is a vector space. We say $u = 0$ in $L^p(\Omega)$ if u is zero almost everywhere on Ω (i.e. $u(x) = 0$ a.e. in Ω). The restriction to Ω of any piecewise continuous, compactly supported function on $\mathbb{R}^d$ would belong to an equivalent class of functions in $L^p(\Omega)$. For $u, v \in L^p(\Omega)$, $x \in \Omega$, $c \in \mathbb{C}$,

$$|u(x) + v(x)|^p \leq (|u(x)| + |v(x)|)^p \leq 2^p(|u(x)|^p + |v(x)|^p) \tag{5.21}$$

Hence $u + v \in L^p(\Omega)$. Clearly $cu(x) \in L^p(\Omega)$. Therefore $L^p(\Omega)$ is a vector space. We claim that the function $\|\cdot\|_p$ defined below is a norm on $L^p(\Omega)$ for $1 \leq p < \infty$. $\|\cdot\|_p$ is not a norm for $0 < p < 1$.

$$\|u\|_p = \left\{ \int_\Omega |u(x)|^p x \right\}^{1/p} \tag{5.22}$$

For $u \in L^p(\Omega)$, it is clear that $\|u\|_p \geq 0$ with equality holding if and only if $u = 0$ in $L^p(\Omega)$. Moreover, $\|cu\|_p = |c| \|u\|_p$. It remains to be shown that $\|\cdot\|_p$ satisfies the triangle inequality, in $L^p(\Omega)$, which is known as *Minkowski's inequality*. The condition certainly holds for $p = 1$, since

$$\int_\Omega |u(x) + v(x)| x \leq \int_\Omega |u(x)| x + \int_\Omega |v(x)| x \tag{5.23}$$

For $p > 1$, we denote by p' the number $\frac{p}{p-1}$ so that $p' > 1$ and

$$\frac{1}{p} + \frac{1}{p'} = 1 \tag{5.24}$$

We call p' the exponential conjugate to p.

Theorem 5.2.5. *(Hölder's inequality)* For $p > 1$, $u \in L^p(\Omega)$ $v \in L^{p'}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)|x = \|uv\|_1 \le \|u\|_p \|v\|_{p'} \tag{5.25}$$

Proof. If either $\|u\|_p = 0$ or $\|v\|_{p'} = 0$ then $u(x)v(x) = 0$ almost everywhere and the inequality is satisfied. Otherwise, consider the function $f : \mathbb{R}^+ \to \mathbb{R}$, $f(t) = t^p/p + 1/p' - t$. The only critical point of the function is at $f(1) = 0$, which is a minimum. Hence for all t,

$$t \le \frac{t^p}{p} + \frac{1}{p'} \tag{5.26}$$

For $a, b \ge 0$, we substitute $t = ab^{-p'/p}$,

$$ab^{-p'/p} \le \frac{a^p b^{-p'}}{p} + \frac{1}{p'}$$
$$ab^{-p'/p+p'} \le \frac{a^p b^{-p'+p'}}{p} + \frac{b^{p'}}{p'} \tag{5.27}$$
$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

with equality occurring if and only if $a^p = b^{p'}$. Substitute $a = \frac{|u(x)|}{\|u\|_p}$, $b = \frac{|v(x)|}{\|v\|_{p'}}$ and integrate over Ω.

$$\int_\Omega \frac{|u(x)||v(x)|}{\|u\|_p \|v\|_{p'}} x \le \frac{1}{p} \int_\Omega \frac{|u(x)|^p}{\|u\|_p^p} x + \frac{1}{p'} \int_\Omega \frac{|v(x)|^{p'}}{\|v\|_{p'}^{p'}} x = \frac{1}{p} + \frac{1}{p'}$$
$$\int_\Omega |u(x)v(x)|x \le \int_\Omega |u(x)||v(x)|x \le \|u\|_p \|v\|_{p'} \tag{5.28}$$

Hence, $uv \in L^1(\Omega)$. $\qquad \square$

From Hölder's inequality, we get that each $v \in L^{p'}(\Omega)$ is associated with an element in the dual of $L^p(\Omega)$, $Fu = \int_\Omega uvx$. The case $p = p' = 2$ is special because every element in $L^2(\Omega)$ can be identified with an element in its dual space $L^2(\Omega)'$.

Theorem 5.2.6. *(Minkowski's Inequality)* Triangle Inequality for $p \ge 1$, $u, v \in L^p(\Omega)$.

$$\|u + v\|_p \le \|u\|_p + \|v\|_p \tag{5.29}$$

Proof. Since the case with $p = 1$ is trivial, consider $p > 1$. For $u, v \in L^p(\Omega)$, $u + v \in L^p(\Omega)$.

$$\int_\Omega |u(x) + v(x)|^{p'(p-1)} x = \int_\Omega |u(x) + v(x)|^p x < \infty \tag{5.30}$$

Hence, $(u+v)^{p-1} \in L^{p'}(\Omega)$.

$$\|u + v\|_p^p = \int_\Omega |u(x) + v(x)|^p x \le \int_\Omega |u(x) + v(x)|^{p-1} (|u(x)| + |v(x)|) x$$
$$\int_\Omega |u(x) + v(x)|^p x \le \left\{ \int_\Omega |u(x) + v(x)|^{p'(p-1)} x \right\}^{1/p'} \left\{ \int_\Omega \||u(x)| + |v(x)|\|^p x \right\}^{1/p}$$
$$\left\{ \int_\Omega |u(x) + v(x)|^p x \right\}^{1-1/p'} \le \left\{ \int_\Omega \||u(x)| + |v(x)|\|^p x \right\}^{1/p} \tag{5.31}$$
$$\left\{ \int_\Omega |u(x) + v(x)|^p x \right\}^{1/p} \le \left\{ \int_\Omega |u(x)|^p x \right\}^{1/p} + \left\{ \int_\Omega |v(x)|^p x \right\}^{1/p}$$
$$\|u + v\|_p \le \|u\|_p + \|v\|_p$$

Therefore, $\|\cdot\|_p$ is a norm on $L^p(\Omega)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 5.2.7 (Hahn-Banach Theorem). Let X be a Banach space and X_0 be a subspace of X. $\forall x_0' \in X_0', \exists\, x' \in X'$ *such that* x_0 and x' agree on X_0, and $\|x_0'\|_{X_0'} = \|x'\|_X$.

Theorem 5.2.8. If X is a normed vector space, then X', along with the induced operator norm, is a banach space (irrespective of whether X is complete).

Theorem 5.2.9 (Generalised Cauchy Identity). $|(x', x)| \le \|x\|_X \|x'\|_{X'}$

5.3 Adjoint Operator

We explain how a linear transformation between two normed vector spaces, $L \in \mathcal{L}(X, Y)$, is naturally associated with another transformation from Y' to X' by the **adjoint operator**, $L^* : Y' \to X'$, as follows: $\forall y' \in Y, \exists$ unique $x' \in X'$ *such that* $\forall x \in X, y' \circ L(x) = x'(x) \implies \forall y' \in Y', \forall x \in X,$

$$
\begin{aligned}
L^* y'(x) &= y' \circ L(x) \in X' \\
y' &\to Ly' \\
y', L(x) &= L^* y', x
\end{aligned}
\tag{5.32}
$$

The adjoint operator is also bounded since $|y' \circ L(x)| \le \|y'\|_{Y'} \|L(x)\|_Y \le \|y'\|_{Y'} \|L\| \|x\|_X$.

We show that for a map $L : \mathbb{R}^n \to \mathbb{R}^m$ with associated matrix A, the matrix of the adjoint is A. The action of the adjoint operator is one that for every $\underline{b} \in \mathbb{R}^m$ produces a unique $\underline{c} \in \mathbb{R}^n$ *such that*

$$
\begin{aligned}
\underline{c}^{\,\underline{x}=\underline{b}^A\underline{x}}, &\quad \forall \underline{x} \in \mathbb{R}^n \\
\implies \underline{x}^{(\underline{c}-A^{\underline{b}})=0} & \\
\implies \underline{c} = A^{\underline{b}} &
\end{aligned}
\tag{5.33}
$$

Most differential operators will be defined on a subset of $L^2(\Omega)$. For linear operators on space $L^2(\Omega)$, we have

$$
\begin{aligned}
v', L(u) = L^* v', u, &\qquad \forall u \in L^2(\Omega), v \in L^2(\Omega)' \\
\implies (v, Lu) = (L^* v, u), &\qquad\qquad \forall u, v \in L^2(\Omega)
\end{aligned}
\tag{5.34}
$$

Definition 5.3.1 (Adjoint Operator). For differential operator L, the adjoint is the differential operator L^* that satisfies

$$
(Lu, v) = (u, L^* v)
\tag{5.35}
$$

for sufficiently smooth u, v with compact support in Ω.

Theorem 5.3.1. For $L : X \to Y$, $\|L^*\| = \|L\|$

Proof.

$$
\begin{aligned}
|y' \circ L(x)| &\le \|y'\| \|L\| \|x\|, &\qquad \forall x \in X, \forall y' \in Y' \\
\frac{|y' \circ L(x)|}{\|y'\| \|x\|} &\le \|L\|, &\qquad \forall x \ne 0, \forall y' \ne 0 \\
\|L^*\| &\le \|L\|, &\qquad \text{by taking supremum over } x, y'
\end{aligned}
\tag{5.36}
$$

Now we prove the converse

$$
\begin{aligned}
|y' \circ L(x)| &\le \|L^*\| \|y'\| \|x\|, &\qquad \forall x, y' \\
\frac{|y'\,(L(x))|}{\|y'\| \|x\|} &\le \|L^*\|, &\qquad \forall x \ne 0, \forall y' \ne 0 \\
\|L\| &\le \|L^*\|, &\qquad \text{by taking supremum over } x, y'
\end{aligned}
\tag{5.37}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Adjoints are useful in understanding problems like the following: given normed vector spaces X, Y and mapping $\mathcal{L}(X, Y)$ and $b \in Y$, find $x \in X$ *such that*

$$Lx = b \tag{5.38}$$

Consider the $m \times n$ system $Ax = b$. We enlarge the system by adding the adjoint system $c = A^y$ as follows:

$$\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix} \tag{5.39}$$

<mark>todo:</mark> Consider the problem

$$\begin{cases} Lu = f, & x \in \Omega \\ \text{suitable b.c, i.c,} & x \in \partial\Omega \end{cases} \tag{5.40}$$

The Green's function $\phi(x, y)$ satisfies

$$\begin{cases} L^*\phi(x, y) = \delta(x - y), & x \in \Omega \\ \text{adjoint b.c, i.c,} & x \in \partial\Omega \end{cases} \tag{5.41}$$

We obtain the following representation formula:

$$\begin{aligned} u(y) &= (u(x), \delta(x - y)) = (u(x), L^*\phi(x, y)) \\ &= (Lu(x), \phi(x, y)) = (f(x), \phi(x, y)) \\ &= \int_\Omega f(x)\phi(x, y)x \end{aligned} \tag{5.42}$$

We enlarge the system as follows:

$$\begin{cases} Lu = f, & x \in \Omega \\ \text{suitable b.c, i.c,} & x \in \partial\Omega \\ L^*\phi(x, y) = \delta(x - y), & x \in \Omega \\ \text{adjoint b.c, i.c,} & x \in \partial\Omega \end{cases} \tag{5.43}$$

5.4   Some Imbeddings

We say that the normed space X is imbedded in the normed space Y and write $X \to Y$ if:

- X is a vector subspace of Y.

- the identity operator, I defined on X into Y is continuous. Since I is linear, it is equivalent to the existence of a constant M such that, for $x \in X$,

$$\|I(x)\|_Y \le M\|x\|_X \tag{5.44}$$

For nonnegative integer m and $0 < \nu \le \lambda \le 1$, we prove the following imbeddings:

- $C^{m+1}(\overline{\Omega}) \to C^m(\overline{\Omega})$
  It is clear that $C^{m+1}(\overline{\Omega}) \subset C^m(\overline{\Omega})$ and $\|\phi\|_{C^m(\overline{\Omega})} \le \|\phi\|_{C^{m+1}(\overline{\Omega})}$

- $C^{m,\lambda}(\overline{\Omega}) \to C^m(\overline{\Omega})$
  It is clear that $C^{m,\lambda}(\overline{\Omega}) \subset C^m(\overline{\Omega})$ and $\|\phi\|_{C^m(\overline{\Omega})} \le \|\phi\|_{C^{m,\lambda}(\overline{\Omega})}$

- For $0 < \nu < \lambda \leq 1$, $C^{m,\lambda}(\overline{\Omega}) \to C^{m,\nu}(\overline{\Omega})$

  Consider $\phi \in C^{m,\lambda}(\overline{\Omega})$. There exists K such that for all α with $|\alpha| \leq m$, for all $x, y \in \Omega$, $x \neq y$

$$
\begin{aligned}
\frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x-y|^\lambda} &< K \\
\frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x-y|^\nu} &< K|x-y|^{\lambda-\nu}
\end{aligned}
\tag{5.45}
$$

The term $|x-y|^{\lambda-\nu}$ is bounded for bounded Ω. Hence, $C^{m,\lambda}(\overline{\Omega}) \subset C^{m,\nu}(\overline{\Omega})$. Next, we compare the norms of $C^{m,\lambda}(\overline{\Omega})$ and $C^{m,\nu}(\overline{\Omega})$. For some $\phi \in C^{m,\lambda}(\overline{\Omega})$, we note that

$$
\begin{aligned}
\sup_{\substack{x,y\in\Omega \\ |x-y|<1}} \frac{|\mathrm{D}^\alpha\phi(x) - \mathrm{D}^\alpha\phi(y)|}{|x-y|^\nu} &\leq \sup_{\substack{x,y\in\Omega \\ |x-y|<1}} \frac{|\mathrm{D}^\alpha\phi(x) - \mathrm{D}^\alpha\phi(y)|}{|x-y|^\lambda} \\
\sup_{\substack{x,y\in\Omega \\ |x-y|\geq 1}} \frac{|\mathrm{D}^\alpha\phi(x) - \mathrm{D}^\alpha\phi(y)|}{|x-y|^\nu} &\leq 2\sup_{x\in\Omega} |\mathrm{D}^\alpha\phi(x)|
\end{aligned}
\tag{5.46}
$$

Hence, $\|\phi\|_{C^{m,\nu}(\overline{\Omega})} \leq 3\|\phi\|_{C^{m,\lambda}(\overline{\Omega})}$

Hölder's inequality <mark>clearly</mark> extends to cover this case with $p = \infty$, $p' = 1$. We now prove an imbedding theorem on $L^p(\Omega)$.

Theorem 5.4.1. Suppose $\mathrm{vol}\,\Omega = \int_\Omega x$, and $1 \leq p \leq q \leq \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and

$$
\|u\|_p \leq (\mathrm{vol}\,\Omega)^{1/p-1/q}\|u\|_q
\tag{5.47}
$$

Hence, $L^q(\Omega) \to L^p(\Omega)$. If $u \in L^\infty(\Omega)$, then

$$
\lim_{p\to\infty} \|u\|_p = \|u\|_\infty
\tag{5.48}
$$

Finally, if $u \in L^p(\Omega)$ for $1 \leq p < \infty$, and if there is a constant K such that for all p, $\|u\|_p \leq K$, then $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leq K$.

Proof. Consider $u \in L^q(\Omega)$. If $p = q$, the imbedding theorem is trivial. For $1 \leq p < q \leq \infty$, Hence $u^p \in L^{q/p}(\Omega)$. Then, Hölder's inequality gives us

$$
\int_\Omega |u(x)|^p x \leq \left\{ \int_\Omega |u(x)|^{p(q/p)} x \right\}^{p/q} \left\{ \int_\Omega 1x \right\}^{1-p/q}
\tag{5.49}
$$
$$
\|u\|_p \leq \mathrm{vol}(\Omega)^{1/p-1/q}\|u\|_q
$$

Hence, for $1 \leq p \leq q \leq \infty$, $L^q(\Omega) \to L^p(\Omega)$. If $u \in L^\infty(\Omega)$, then

$$
\lim_{p\to\infty} \|u\|_p \leq \|u\|_\infty
\tag{5.50}
$$

On the other hand, $\forall \varepsilon > 0 \, \exists A \subset \Omega$ of nonzero measure $\mu(A)$ such that $\forall x \in A$,

$$
\begin{aligned}
|u(x)| &\geq \|u\|_\infty - \varepsilon \\
\int_\Omega |u(x)|^p x &\geq \int_A |u(x)|^p x \geq \mu(A)(\|u\|_\infty - \varepsilon)^p \\
\|u\|_p &\geq (\mu(A))^{1/p}(\|u\|_\infty - \varepsilon) \\
\lim_{p\to\infty} \|u\|_p &\geq \|u\|_\infty
\end{aligned}
\tag{5.51}
$$

Therefore,

$$\lim_{p \to \infty} \|u\|_p = \|u\|_\infty \tag{5.52}$$

Finally, suppose $u \in L^p(\Omega)$ for $1 \le p < \infty$ and $\exists K$ such that for all such p, $\|u\|_p \le K$. Suppose, *ad absurdum*, u is essentially bounded. Then $\exists K_1 > K$, $A \subset \Omega$ with $\mu(A) > 0$ such that $\forall x \in A$, $|u(x)| \ge K_1$. Applying the same argument as before, we get

$$\lim_{p \to \infty} \|u\|_p \ge K_1 \tag{5.53}$$

which is a contradiction. $\qquad\square$

Theorem 5.4.2. If Ω is measurable, then $L^p(\Omega)$ is a Banach space for $1 \le p \le \infty$.

Theorem 5.4.3. For $f \in L^p(\Omega) \cap L^q(\Omega)$, $1 \le p \le q \le \infty$, then $\forall r \in [p, q]$, $f \in L^r(\Omega)$ and $\|f\|_r \le \|f\|_p^\alpha \|f\|_q^{1-\alpha}$ where $\dfrac{\alpha}{p} + \dfrac{1-\alpha}{q} = \dfrac{1}{r}$.

Proof. The proof is trivial for $r = p$ or $r = q$. If $p < r < q$, then $\dfrac{1}{p} > \dfrac{1}{r} > \dfrac{1}{q}$. Then, there exists α such that

$$\alpha \frac{1}{p} + (1 - \alpha) \frac{1}{q} = \frac{1}{r} \tag{5.54}$$

Then, for any $m, n \ge 1$ with $\dfrac{1}{m} + \dfrac{1}{n} = 1$, we have

$$\int_\Omega |f(x)|^r x = \int_\Omega |f(x)|^{\alpha r} |f(x)|^{(1-\alpha)r} \le \left\{ \int_\Omega |f(x)|^{r\alpha m} x \right\}^{1/m} \left\{ \int_\Omega |f(x)|^{r(1-\alpha)n} x \right\}^{1/n} \tag{5.55}$$

Choose $m = \dfrac{p}{r\alpha}$ and $m = \dfrac{q}{r(1-\alpha)}$ if $\alpha \ne 1$. If $\alpha = 1$, choose $n = \infty$.

$$\int_\Omega |f(x)|^r x \le \|f\|_p^\alpha \|f\|_q^{1-\alpha} \le \infty \tag{5.56}$$

Therefore $f \in L^r(\Omega)$. $\qquad\square$

todo: We now consider the normed dual of $L^p(\Omega)$.

5.5 Measure Theory Intro

The **measure** of a set refers to its size in some measure space. We say a set $S \subset \mathbb{R}^d$ has **measure zero** ($\mu(S) = 0$) if $\forall \varepsilon > 0$, S can be covered by open balls of total volume less than ε. $\mathbb{Q} \subset \mathbb{R}$, C^1 curves in $\mathbb{R}^2$, C^1 surfaces in $\mathbb{R}^3$ are examples of sets with measure zero. A countable union of sets of measure zero has measure zero. We say that a condition holds **almost everywhere (a.e.)** if the points where it does not hold form a set of zero measure. For example, the characteristic function for rationals on the set of real numbers is equal to zero almost everywhere.

Definition 5.5.1 (Measurable function). A function is measurable if it coincides a.e. with the limit of a sequence of piecewise continuous functions which is convergent almost everywhere.

We define the characteristic function of a set $A \subset \mathbb{R}^d$ as follows:

$$\chi_A(x) = \left\{ \begin{matrix} 1 & x \in A \\ 0 & x \notin A \end{matrix} \right. \tag{5.57}$$

We say that a set is measurable if its characteristic function is measurable. $\mathbb{Q}$ is measurable since its characteristic function coincides with zero almost everywhere. A countable union of measurable sets is also measurable.

Definition 5.5.2 (Piecewise Continuous). $g : \mathbb{R}^d \to \mathbb{R}$ is piecewise continuous if there exist disjoint, open and connected domains $\{D_i\}_{i \in I \subset \mathbb{N}}$ with piecewise C^1 boundary such that any sphere can be covered by finitely many $\overline{D_i}$. Further, g is continuous on each D_i and can be continuously extended to the boundary of D_i.

Consider a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, $f(x) \geq 0$. One can construct a nondecreasing sequence of piecewise continuous functions $\{g_k\}$ with compact support, convergent almost everywhere to f. We say that f is **Lebesgue integrable** if the sequence of Riemann integrals

$$\int_{\mathbb{R}^d} g_k(x)x \tag{5.58}$$

has an upper bound (hence a limit since g_k is a nondecreasing). One can show that $\int_{\mathbb{R}^d} g_k(x)x$ does not depend on the choice of sequence g_k.

$$\int_{\mathbb{R}^d} f(x)x = \lim_{k \to \infty} \int_{\mathbb{R}^d} g_k(x)x \tag{5.59}$$

For example, $\chi_{\mathbb{Q}}$ is not Riemann integrable since it discontinuous everywhere, but is Lebesgue integrable since it is equal to zero a.e. We define the Lebesgue integral of a measurable function f over a measurable subset Ω of $\mathbb{R}^d$ as follows:

$$\int_{\Omega} f(x)x = \int_{\mathbb{R}^d} f(x)\chi_{\Omega}(x)x \tag{5.60}$$

We say a measurable function $u : \mathbb{R}^d \to \Omega$ is **summable** if $\int_{\mathbb{R}^d} |u|x < \infty$ and define the set of summable functions

$$L^1(\mathbb{R}^d) = \{u : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^d} |u(x)|x < \infty\} \tag{5.61}$$

A measurable function is **locally integrable** on $\Omega \subset \mathbb{R}^d$ if it is integrable on any compact $K \subset \Omega$. The set of locally integrable functions on Ω is denoted $L^1_{\text{loc}}(\Omega)$.

$$L^1_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R} : \forall K \subset \Omega, \int_K u(x)x < \infty\} \tag{5.62}$$

Any piecewise continuous function is locally integrable in $\mathbb{R}^d$. An important fact is that summable functions are "approximately continuous" at every point.

Theorem 5.5.1 (Lebesgue's Differentiation Theorem). Let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then for a.e. point in $\mathbb{R}^d$,

$$\lim_{r \to 0} \int_{B_{x_0}(r)} u(x)x = u(x_0) \tag{5.63}$$

5.6 Distributions

We restate the divergence theorem: for $u \in C^1(\overline{\Omega})$ in scalar form (this can be derived from the vector form by considering a vector field with scalar u being the ith component, and all other components being zero),

$$\int_{\partial \Omega} u n_i \mathrm{d}S = \int_{\Omega} \partial_{x_i} u \mathrm{d}x \tag{5.64}$$

where n_i is the ith component of the unit normal vector defined on $\partial\Omega$. We derive the formula for integration by parts: for $u, v \in C^1(\Omega)$,

$$\int_{\partial\Omega} uvn_i \mathrm{d}S = \int_{\Omega} \partial_{x_i}(uv)\mathrm{d}x = \int_{\Omega} u\partial_{x_i}v + v\partial_{x_i}v\mathrm{d}x$$
$$\int_{\Omega} u\partial_{x_i}v\mathrm{d}x = \int_{\partial\Omega} uvn_i\mathrm{d}S - \int_{\Omega} v\partial_{x_i}u\mathrm{d}x \tag{5.65}$$

The space $C_0^\infty(\Omega)$ contains infinitely smooth functions with compact support on Ω. We call functions belonging to $C_0^\infty(\Omega)$ **test functions**. A key feature of test functions is that their extension to $\partial\Omega$ is the zero function as they are continuous and compactly supported in a subset of Ω. Let $u, v \in L^1_{\text{loc}}(\Omega)$ and α be a multi-index. We say that v is the αth **weak derivative** of u if

$$\forall\phi \in C_0^\infty(\Omega), \ \int_{\Omega} u\mathrm{D}^\alpha\phi\mathrm{d}x = (-1)^{|\alpha|}\int_{\Omega} v\phi\mathrm{d}x \tag{5.66}$$

It is straightforward to show that the weak derivative of u, if it exists, is uniquely defined up to a set of measure zero. Further, if $u \in C^{|\alpha|}(\Omega)$, its weak derivative is equal to its classical derivative.

A **mollifier**, $J(x)$, is a nonnegative, real-valued function belonging to $C_0^\infty(\mathbb{R}^d)$ used to create smooth functions approximating non-smooth functions via convolutions. Mollifiers satisfy the condition that $\int_{\mathbb{R}^d} J(x)\mathrm{d}x = 1$. We define the standard mollifier

$$J(x) = \begin{cases} C\exp\left\{\frac{-1}{1-|x|^2}\right\} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \tag{5.67}$$

where C is chosen such that the integral of J over $\mathbb{R}^d$ is equal to 1. For $\varepsilon > 0$, we define

$$J_\varepsilon(x) = \varepsilon^{-d}J(x/\varepsilon) \tag{5.68}$$

which is compactly supported in the ball of radius ε around the origin. If $u \in L^1_{\text{loc}}(\mathbb{R}^d)$, we define the mollification of u as the convolution

$$u_\varepsilon(x) = J_\varepsilon * u(x) = \int_{\mathbb{R}^d} J_\varepsilon(x-y)u(y)\mathrm{d}y \tag{5.69}$$

We define $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}$.

Theorem 5.6.1 (Properties of mollifiers). Let $u : \mathbb{R}^d \to \mathbb{R}$ and vanishes identically outside Ω.

1. If $u \in L^1_{\text{loc}}(\overline{\Omega})$, $u_\varepsilon \in C_0^\infty(\mathbb{R}^d)$

2. If also u has compact support in Ω, then $u_\varepsilon \in C_0^\infty(\Omega)$ provided $\varepsilon < \text{dist}(\text{supp}(u), \partial\Omega)$

3. $u_\varepsilon \to u$ as $\varepsilon \to 0$.

4. if $G \subset\subset \Omega$ and $u \in C(\Omega)$, then $u_\varepsilon \to u$ uniformly on G.

5. If $u \in C(\overline{\Omega})$, then $u_\varepsilon \to u$ uniformly on Ω.

Proof.

$$\mathrm{D}^\alpha(J_\varepsilon * u)(x) = \int_{\mathbb{R}^d} (\mathrm{D}^\alpha J_\varepsilon(y-x))u(y)\mathrm{d}y \tag{5.70}$$

$J_\varepsilon(y-x)$ is infinitely differentiable, and vanishes outside $B_x(\varepsilon)$ for every x for every multi-index α. If $\text{supp}(u) \subset\subset \Omega$, and $\varepsilon < \text{dist}(\text{supp}(u), \partial\Omega)$, then $\forall x \notin \Omega$, $u_\varepsilon(x) = 0$. Therefore, u_ε has compact support in Ω. From the Lebesgue differentiation theorem,

$$\lim_{r\to 0} \int_{\Omega} |u(y) - u(x)|\mathrm{d}y = 0 \tag{5.71}$$

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a.e. on Ω. Fix such a point x on Ω. Then,

$$|u_\varepsilon(x) - u(x)| = \left| \int_\Omega J_\varepsilon(y - x)u(y)\mathrm{vol}(\Omega)u(x)\mathrm{d}y \right| \tag{5.72}$$

$\square$

5.7   Sobolev Intro

We introduce Sobolev spaces of integer order and establish some of their basic properties. These are vector subspaces of various spaces $L^p(\Omega)$. We define the functional $\|\cdot\|_{m,p}$ where m is a nonnegative integer and $1 \le p \le \infty$ as follows:

$$\|u\|_{m,p} = \left\{ \sum_{0 \le |\alpha| \le m} \|\mathrm{D}^\alpha u\|_p^p \right\}^{1/p}$$
$$\|u\|_{m,\infty} = \max_{0 \le |\alpha| \le m} \|\mathrm{D}^\alpha u\|_\infty \tag{5.73}$$

for any function for which the right side makes sense. The above functional defines a norm on any vector space of functions where the right side takes finite values provided functions are identified in the space if they are equal almost everywhere in Ω. We consider three such spaces corresponding to any given values of m and p.

- $H^{m,p}(\Omega) \equiv$ the completion of $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}$
- $W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) : \mathrm{D}^\alpha u \in L^p(\Omega) \,\text{for}\, 0 \le |\alpha| \le m\}$
- $W_0^{m,p}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Any compactly supported function in $C^\infty(\Omega)$ belongs to $W^{m,p}(\Omega)$ for any m. If Ω is bounded, then $C^m(\Omega) \subset W^{m,p}(\Omega)$ for any $1 \le p \le \infty$.

Theorem 5.7.1. The space $W^{m,p}(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{m,p}$ where $m \in \mathbb{N}$, $1 \le p \le \infty$.2

Proof. <mark>todo</mark> Consider the sequence $\{u_n\}_n \subset W^{m,p}(\Omega)$ such that $\forall \varepsilon > 0 \exists N \,\text{s.t.}\, \forall m, n \ge N,\, \|u_m - u_n\|_{m,p} < \varepsilon$. $\square$

A set Ω is said to have the **segment property** if $\forall x_0 \in \partial\Omega$, there exists an open neighbourhood of x_0 called Ω_{x_0} and a nonzero direction $y_{x_0} \in \mathbb{R}^d$ such that $\forall t \in [0,1]$, $\overline{\Omega} \cap \Omega_{x_0} + ty_{x_0} \subset \Omega$. Sets with the segment property are only present on one side of their boundary. Balls, polytopes and open sets with C^1 boundary have the segment property. However, not all sets with piecewise C^1 boundary have the segment property.

Theorem 5.7.2. Let $\Omega \subset \mathbb{R}^d$ be an open set with a bounded, piecewise C^1 boundary. If Ω has the segment property, then for $1 \le p < \infty$, there exists the trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$. T is linear and continuous and $Tf = f|_{\partial\Omega}$

Proof. First we define T on $C_0^\infty(\mathbb{R}^d)$ and then extend it by density to $W^{1,p}(\Omega)$. For $f \in C_0^\infty(\mathbb{R}^d)$, $Tf = f|_{\partial\Omega}$. Clearly, T is linear. It remains to be shown that T is continuous with respect to the norm in $W^{1,p}(\Omega)$ and $L^p(\Omega)$, i.e. there exists constant K such that $\forall f \in C_0^\infty(\mathbb{R}^d)$,

$$\|Tf\|_{L^p(\partial\Omega)} \le K\|f\|_{W^{1,p}(\Omega)} \tag{5.74}$$

$\square$

<mark>todo:</mark> trace operator

Chapter 6

Numerics

We write out the spectral discretization for the Navier-Stokes equations in d dimensions. Consult appendix for full writeup.

$$\frac{\partial}{\partial t}\boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p + \frac{1}{Re}\nabla^2\boldsymbol{u} + \boldsymbol{f}, \quad (\boldsymbol{x},t) \in \Omega \times (0,T] \subset \mathbb{R}^{d+1}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \tag{6.1}$$

We select an appropriate n dimensional spaces $X^N, Y^N \subset \mathcal{L}^2(\Omega)$ for velocity and pressure respectively.

1d operators

$$\hat{B}_{ij} = \int_{-1}^1 \phi_i \phi_j x, \quad \hat{A}_{ij} = \int_{-1}^1 \nu \frac{\phi_i}{x} \frac{\phi_j}{x} x, \quad R = \texttt{eye}(n)(2:\texttt{end}-1,:), \quad \hat{J}_{vp} = \texttt{interp\_mat(zp,zv)} \tag{6.2}$$

R restriction matrix, ones at interior/neumann points, zeros at dirichlet points

diagonal jacobian matrix

$$J = \texttt{diag}(J(\boldsymbol{x}_i)), \tag{6.3}$$

2D operators

$$\overline{B} = \hat{B}_s \otimes \hat{B}_r, \quad \overline{A} = \hat{B}_s \otimes \hat{A}_r + \hat{A}_s \otimes \hat{B}_r, \quad \overline{H} = \beta_0\overline{B} + \nu\overline{AJ} = \hat{J}_s \otimes \hat{J}_r, \tag{6.4}$$

$$B = R\overline{B}R, \quad A = R\overline{A}R, \quad H = R\overline{H}R \tag{6.5}$$

2D vector operators

$$\mathbf{R} = \begin{bmatrix} R_{v_x} & \\ & R_{v_y} \end{bmatrix}, \quad \overline{\mathbf{B}} = \begin{bmatrix} \overline{B} & \\ & \overline{B} \end{bmatrix}, \quad \overline{\mathbf{A}} = \begin{bmatrix} \overline{A} & \\ & \overline{A} \end{bmatrix}, \quad \overline{\mathbf{H}} = \begin{bmatrix} \overline{H} & \\ & \overline{H} \end{bmatrix}, \tag{6.6}$$

$$(q, \boldsymbol{\nabla} \cdot \boldsymbol{u}) = \left(J_{vp}^{\frac{p}{v}}\right)^{B}{}^v \begin{bmatrix} D_x & D_y \end{bmatrix} \underline{\boldsymbol{u}}$$
$$= \boldsymbol{q}_{vp}^J \overline{B^v} \begin{bmatrix} D_x & D_y \end{bmatrix} \underline{\boldsymbol{u}} \tag{6.7}$$
$$= \boldsymbol{q}^{\overline{\mathbf{D}}\underline{\boldsymbol{u}}}$$

$$\underline{u} = \underline{u_b} + \mathbf{R}\underline{u_b}$$

(6.8)

Stokes system

$$\begin{bmatrix} \mathbf{R}\overline{\mathbf{H}}\mathbf{R} & -\mathbf{R}\overline{\mathbf{D}} \\ -\overline{\mathbf{D}}\mathbf{R} & \end{bmatrix} \begin{pmatrix} \underline{u}_h^n \\ \underline{\delta p}^n \end{pmatrix} = \begin{pmatrix} \mathbf{R}\left(\underline{f} + \overline{\mathbf{D}}\underline{p}^{n-1}\right) \\ \overline{\mathbf{D}}\overline{u}_b \end{pmatrix}$$

$$\begin{bmatrix} \mathbf{H} & -\mathbf{D} \\ -\mathbf{D} & \end{bmatrix} \begin{pmatrix} \underline{u}_h^n \\ \underline{\delta p}^n \end{pmatrix} = \begin{pmatrix} \underline{f}^* \\ \overline{\overline{\mathbf{D}}}\overline{u}_b \end{pmatrix}$$

$$\begin{bmatrix} \mathbf{H} & -\mathbf{H}\mathbf{Q}\mathbf{D} \\ -\mathbf{D} & \end{bmatrix} \begin{pmatrix} \underline{u}_h^n \\ \underline{\delta p}^n \end{pmatrix} = \begin{pmatrix} \underline{f}^* \\ \overline{\overline{\mathbf{D}}}\overline{u}_b \end{pmatrix}$$

$$\begin{bmatrix} \mathbf{H} & -\mathbf{H}\mathbf{Q}\mathbf{D} \\ & -\mathbf{D}\mathbf{Q}\mathbf{D} \end{bmatrix} \begin{pmatrix} \underline{u}_h^n \\ \underline{\delta p}^n \end{pmatrix} = \begin{pmatrix} \underline{f}^* \\ \overline{\mathbf{D}}(\overline{u}_b + \mathbf{H}^{-1}\underline{f}^*) \end{pmatrix}$$

(6.9)

$$\mathbf{H}^{-1} \approx \mathbf{Q} = \frac{1}{\beta_0}(\mathbf{R}\overline{\mathbf{B}}\mathbf{R}^{)-1} = \frac{1}{\beta_0}\mathbf{R}\overline{\mathbf{B}}^{-1}\mathbf{R}$$

(6.10)

Pressure solve

$$\mathbf{D}\mathbf{Q}\mathbf{D}\underline{\delta p} = -\overline{\mathbf{D}}\underline{u}^*$$

$$\overline{\mathbf{D}}\mathbf{R}^{\mathbf{R}}\overline{\mathbf{B}}^{-1}\mathbf{R}^{\mathbf{R}}\overline{\mathbf{D}}\underline{\delta p} = \underline{g}$$

$$J_{vp}\overline{B}\begin{bmatrix} D_x D_y \end{bmatrix} \begin{bmatrix} R^{v_x\,R_{v_x}}\overline{B}^{-1}R^{v_x\,R_{v_x}} & \\ & R^{v_y\,R_{v_y}}\overline{B}^{-1}R^{v_y\,R_{v_y}} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \end{bmatrix} \overline{B}J^{vp}\underline{\delta p} = \underline{g}$$

$$J_{vp}\overline{B}\begin{bmatrix} D_x D_y \end{bmatrix} \begin{bmatrix} M_{v_x}\overline{B}^{-1}M_{v_x} & \\ & M_{v_y}\overline{B}^{-1}M_{v_y} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \end{bmatrix} \overline{B}J^{vp}\underline{\delta p} = \underline{g}$$

(6.11)

Chapter 7

Discrete Differential Geometry

Based on Keenan Crane's youtube lecture series

7.1 Meshes

Definition 7.1.1 (Simplicial Complex)**.**

Chains and co-chains

7.2 Manifold

Definition 7.2.1 (Manifold)**.** An n-manifold is a space that locally looks like (is homeomorphic to) $\mathbb{R}^n$.

Definition 7.2.2 (Simplicial Manifold)**.** A simplicial k-complex is a manifold if the linik of every vertex looks like (i.e. is homeomorphic to) a $(k-1)$-dimensional sphere.

7.3 Exterior Algebra

A language for manipulating signed volumes.

Definition 7.3.1 (Wedge Product $\wedge$)**.** (Analogy: oriented span of vectors) For vectors u, v, the wedge product, $u \wedge v$ describe the oriented area vector. Wedging k vectors gives us a k-**vector**, or k-dimensional volumes. A k-vector encodes only two pieces of information: a direction and a magnitude. therefore, we can say that k-vectors (or k-vector) are the equivalance class of k-dimensional volumes with the same k-dimensional volume and orientation. For convenience, we say that a 0-vector has no orientation, just magnitude. i.e. just a scalar.

Since the wedge product is oriented, it is antisymmetric. Wedge product is also associative, i.e. $(u \wedge v) \wedge w = u \wedge (v \wedge w)$. Wedge product is also distributive over addition: i.e. $u \wedge v_1 + u \wedge v_2 = u \wedge (v_1 + v_2)$.

Definition 7.3.2 (Orthogonal Complement)**.** Let $U \subset V$ be linear subspace of vector space V with an inner product $\langle \cdot, \cdot \rangle$. The orthogonal complement of U is the collection of vectors

$$U^{\perp} := \{ v \in V | \langle u, v \rangle = 0 \, \forall u \in U \} \tag{7.1}$$

Definition 7.3.3 (Hodge Star $\star$)**.** (analogy: orthogonal complement to a wedge product) When you say,

$$\star(u \wedge v) = w \tag{7.2}$$

it means that the vector w is orthogonal to u, and v. More generally, in n-dimensional space, the hodge star of a k-vector v, $w = \star v = \star(\bigwedge_{i=1}^{k} v_i)$ is the $(n-k)$-vector $w = \bigwedge_{i=1}^{n-k} w_i$ the span of whose elements $\{w_i\}$ is the orthogonal complement of $\{v_i\}$ with some inner product. These two definitions are describing the same thing!

To fix orientation, we assert the convention that for k-vector z, $z \wedge \star z$ is a positively oriented n-vector.

For k-vector u, we have:
$$\begin{aligned} u \wedge \star u &> 0 \\ \star u \wedge \star \star u &> 0 \\ \implies \star \star u &= -u \end{aligned} \tag{7.3}$$

Summary: let V be an n-dimensional vector space consisting of 1-vectors. We can wedge together k vectors to get a k-vector (k dimensional signed volume). We can also apply the Hodge star to get the complementary $(n-k)$-vector.

The geometric meaning of k-vector and vectors has nothing to do with coordinates. A vector can be represented by a list of numbers, but it isn't fundamentally a list of numbers.

For any basis k-vector $\alpha = \bigwedge_{l=1}^{k} e_{i_l}$ where $i_{l l=1}^{k} \subset 1, \ldots, n$, we define the Hodge star such that $\det(\alpha \wedge \star \alpha) = 1$. In other words, if we start with a unit k-volume, wedging with its Hodge star must give a unit, positively oriented unit n-volume. For example, if $\alpha = e_2$, then $\star \alpha = e_3 \wedge e_1$ since $\det(e_2 \wedge e_3 \wedge e_1) = 1$, since its an even permutation of $e_1 \wedge e_2 \wedge e_3$.

7.3.1 k-Forms and Duality

A k-form is something that measures a k-vector. Something to remember is that measurement devices have the same dimension as the thing they're measuring, i.e. a ruler (1D) measures length, a container (3D) measures volume. A vector can be paired with another vector to get some measurement (inner product). Exterior calculus generalises this idea: a k-dimensional volume (k-form) is paired with a *dual* (k-form) k-dimensional volume to get a measurement.

Duality is a pervasive idea in mathematics. In this chapter, we consider duality between (primal) vectors (k-vectors) i.e. things that are, and (dual) covectors (k-forms) i.e. things that measure. Just as vectors can be wedged together to form k-vectors, covectors can be wedged together to form k-forms. Covectors are represented by row vectors (adjoints), and vectors by column vectors. Then they can be combined by inner products. We can think of row vectors as a linear map that sends column vectors to $\mathbb{R}$ via the inner product.

Definition 7.3.4 (Dual Space)**.** For any real vector space V, its dual space, V^* is the collection of linear functions $\alpha : V \to \mathbb{R}$ together with the operations of addition and scalar multiplication.

An element of the dual space is called a dual vector or a covector. Example: for $V = \mathcal{L}[0,1]$, $\delta : V \to \mathbb{R}$; $\delta(f) = f(0)$. i.e. the Dirac delta function is a member of the dual space of $\mathcal{L}[0,1]$.

Definition 7.3.5 (Sharp $\sharp$, and Flat $\flat$). (analogous to transpose operator in matrix algebra). Flat $\flat$ "flattens" a primal (column) vector and turns it into its dual (row) vector.

$$u, v \to^{\flat} u^{\flat}(v) \tag{7.4}$$

Likewise sharp $\sharp$ converts a covector (row vector) into its primal (column vector)

$$\alpha, \beta \to^{\sharp} \alpha(\beta^{\sharp}) \tag{7.5}$$

In general, the inner product involves a mass matrix. Eg. for vectors $u, v \in \mathbb{R}^n$, $\langle u, v \rangle = u^T M v$ where $M \in \mathbb{R}^{n \times n}$ is a positive-definite "mass" matrix (or a metric) that conveys that angle means in the space we're working in. The standard dot product is obtained when M is the identity matrix. Then,

$$\begin{aligned} u^{\flat}(v) &= (u^T M)v \\ \alpha(\beta^{\sharp}) &= \alpha(M^{-1}\beta^T) \end{aligned} \tag{7.6}$$

In other words,

$$\begin{aligned} u^{\flat}(\cdot) &= \langle u, \cdot \rangle \\ \langle \alpha^{\sharp}, \cdot \rangle &= \alpha(\cdot) \end{aligned} \tag{7.7}$$

Definition 7.3.6 (Determinant and Singed Volumes). Determinant is a signed volume. In 2D, the determinant of two vectors is the signed area of parallelogram spanned by the vectors. More generally, the determinant of vectors $\{v_i\}^n \subset \mathbb{R}^n$ is the signed volume of the parallelopipped spanned by the vectors.

Definition 7.3.7 (k-forms). What does it mean to take a multi-linear measurement of a 2-vector $v = v_1 \wedge v_2$ done by $\alpha = \alpha_1, \alpha_2$? It is the area of the projection of v onto the plane described by a, scaled by the magnitude of α.

$$\alpha(v) := \alpha_1(v_1)\alpha_2(v_2) - \alpha_1(v_2)\alpha_2(v_1) \tag{7.8}$$

In general a k-forms is a fully antisymmetric (i.e. exchanging any two arguments changes the sign), multi-linear measurement of a k-vector. Conceptually, you project $v_1, \ldots, v_k$ onto $\alpha_1, \ldots, \alpha_k$ and make the area measurement via the determinant.

$$(\bigwedge^k \alpha_i)(\bigwedge^k v_k) = \det \begin{bmatrix} \alpha_1(v_1) & \ldots & \alpha_1(v_k) \\ \vdots & \ddots & \vdots \\ \alpha_k(v_1) & \ldots & \alpha_k(v_k) \end{bmatrix} \tag{7.9}$$

$\alpha_i(v_j)$ can be thought of the component of v_j along the direction α_i

Moving the analogy ahead, a 0-form is something that takes no vector as input and gives a $\mathbb{R}$ output.

When working with coordinates, the **Dual basis** are written with superscripts $e^1, \ldots, e^n$, and primal basis are written with subscripts $e_1, \ldots, e_n$. Further, e^i is the dual basis for e_i have a special relationship:

$$e^i(e_j) = \delta^i_j \tag{7.10}$$

We also follow **Einstein summation notation**: if an index appears twice (once superscript, and once subscript), we consider an implicit summation of the quantity over the range of the index.

$$x^i y_i := \sum_{i=1}^{n} x^i y_i \tag{7.11}$$

7.4 Differential Forms

If a vector field is the assignment of a vector to each point in space, then a differential form is the assignment of a little volume measurement to each point in space. A differential 0-form is just a scalar field - it takes zero vectors as input and assigns a value at each point in space. A differential 1-form is a linear function mapping a vector to $\mathbb{R}$ at each point in space. We apply all exterior algebra operations pointwise to differential k-forms.

Basis vector fields: Just as we can pick a basis for vectors, we can pick a basis for vector fields: $\partial_{x^1}, \partial_{x^2}, \ldots, \partial_{x^n}$ are constant vector fields of unit magnitude along the coordinate axes in $\mathbb{R}^n$. They look like partial derivatives but don't think about what these symbols have to do with derivatives. These are just the basis for vector fields in $\mathbb{R}^n$. Vector fields can be formed by a linear combination of the basis vector fields, but the coefficient of the linear combination may vary across the domain. So a vector field on these basis in $\mathbb{R}^2$ would look like

$$v = a(x_1, x_2)\partial_{x^1} + b(x, y)\partial_{x^2} \tag{7.12}$$

For differential one forms, the dual basis are written as $\mathrm{d}x^i$ with the relationship $\mathrm{d}x^i(\partial_{x^j}) = \delta^i_j$. These are just the basis for 1-forms, no relation to derivatives. A differential 1-form is written as

$$\alpha = \alpha_1 \mathrm{d}x^1 + \alpha_2 \mathrm{d}x^2 \tag{7.13}$$

The explanation for the relation to the derivative is as follows: consider the scalar field $u = x_1$. The gradient is the constant field pointing to the right. The derivative of a coordinate function gives a constant field pointing in that direction.

7.5 Exterior Calculus

We discuss differentiation of k-forms via the exterior derivative, and integration of k-forms over k-dimensional regions of space. In calculus, integration and differentiation are linked by the fundamental theorem of calculus, which states

$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a) \tag{7.14}$$

In exterior calculus, the Stokes theorem says something similar: if we integrate a derivative, it gives out some values at the boundary:

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha \tag{7.15}$$

7.5.1 Differentiation

We think about derivatives of all kinds (partial, total, directional, gradient, curl, divergence) in terms of a 'pushforward': if a map (function) tells you how a domain is deformed, its derivative tells you how tangent

vectors on a domain get stretched out when the map is applied.

$$
\begin{aligned}
\text{Scalar Field} \quad & \phi & : \mathbb{R}^3 \to \mathbb{R} \\
\text{Vector Field} \quad & X & = u\partial_x + v\partial_y + w\partial_z \\
\text{Gradient} \quad & \boldsymbol{\nabla}\phi & = (\partial_x\phi)\partial_x + (\partial_y\phi)\partial_y + (\partial_z\phi)\partial_z \\
\text{Divergence} \quad & \boldsymbol{\nabla} \cdot X & = \partial_x u + \partial_y v + \partial_z w \\
\\
\text{Curl} \quad & \boldsymbol{\nabla} \times X & = \det \begin{bmatrix} \partial_x & \partial_y & \partial_z \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{bmatrix}
\end{aligned}
\tag{7.16}
$$

Definition 7.5.1 (Exterior Derivative)**.** The exterior derivative is a unique linear map $\mathrm{d} : \Omega^k \to \Omega^{k+1}$ (where Ω^k is the space of all differential k-forms) such that:

1. Differential: for $k = 0$, $\mathrm{d}\phi(X) = D_X\phi := \lim_{\epsilon \to 0} \dfrac{\phi(x + \epsilon X) - \phi(x)}{\epsilon}$, i.e. the differential of 0-form ϕ gives a 1-form which when applied to vector field X gives the directional derivative of ϕ along X. The exterior derivative of a 0-form behaves like the gradient, $\mathrm{d}\phi = (\partial_x\phi)\mathrm{d}_x + (\partial_y\phi)\mathrm{d}_y + (\partial_z\phi)\mathrm{d}_z$.

   In a coordinate free approach, we would say the gradient is the unique vector field $\boldsymbol{\nabla}\phi$ whose inner product with any vector field X yield the directional derivative $D_X\phi$ along X, $\langle \boldsymbol{\nabla}\phi, X \rangle = D_X\phi$ for all X, assuming ϕ is differentiable.

   We have, $\mathrm{d}\phi(\cdot) = \langle \boldsymbol{\nabla}\phi, \cdot \rangle = $ the directional derivative of ϕ. The gradient depends on the inner product, whereas the exterior derivative can be defined in the absence of an inner product and coordinates. $(\mathrm{d}\phi)^\sharp = \boldsymbol{\nabla}\phi$, and $(\boldsymbol{\nabla}\phi)^\flat = \mathrm{d}\phi$

2. Product rule: $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$ where $\alpha \in \Omega^k$, $\beta \in \Omega^l$

3. Exactness: $\mathrm{d} \circ \mathrm{d} = 0$. Consider exterior derivative of 1-form $u\mathrm{d}x$. $\mathrm{d}(u\mathrm{d}x) = \mathrm{d}(u \wedge \mathrm{d}x)$ (u is a 0-form), $= \mathrm{d}u \wedge \mathrm{d}x + (-1)^0 u \wedge \mathrm{d}\mathrm{d}x = \mathrm{d}u \wedge \mathrm{d}x = (\partial_x u\mathrm{d}x + \partial_y u\mathrm{d}y + \partial_z u\mathrm{d}z) \wedge \mathrm{d}x$

d increases the degree of differential forms.

Example: compute exterior derivative of 1-form $\mathrm{d}(u\mathrm{d}x + v\mathrm{d}y + w\mathrm{d}z)$

$$
\begin{aligned}
\mathrm{d}(u\mathrm{d}x + v\mathrm{d}y + w\mathrm{d}z) &= \mathrm{d}(u \wedge \mathrm{d}x) + \mathrm{d}(v \wedge \mathrm{d}y) + \mathrm{d}(w \wedge \mathrm{d}z) \\
&= (\mathrm{d}u \wedge \mathrm{d}x + u \wedge \mathrm{d}\mathrm{d}x) + (\mathrm{d}v \wedge \mathrm{d}y + v \wedge \mathrm{d}\mathrm{d}y) + (\mathrm{d}w \wedge \mathrm{d}z + w \wedge \mathrm{d}\mathrm{d}z) \\
&= (\mathrm{d}u \wedge \mathrm{d}x) + (\mathrm{d}v \wedge \mathrm{d}y) + (\mathrm{d}w \wedge \mathrm{d}z) \\
&= (\partial_x u\mathrm{d}x + \partial_y u + \partial_z u\mathrm{d}z) \wedge \mathrm{d}x + (\partial_x v\mathrm{d}x + \partial_y v + \partial_z v\mathrm{d}z) \wedge \mathrm{d}y + (\partial_x w\mathrm{d}x + \partial_y w + \partial_z w\mathrm{d}z) \wedge \mathrm{d}z
\end{aligned}
\tag{7.17}
$$

Simplifies to the curl. For vector field X, $\boldsymbol{\nabla} \times X = (\star \mathrm{d}X^\flat)^\sharp$. **Codifferential operator**: $\delta = \star\mathrm{d}\star$

7.5.2 Integration

Integration of 1-forms

7.6 Discrete Exterior Calculus

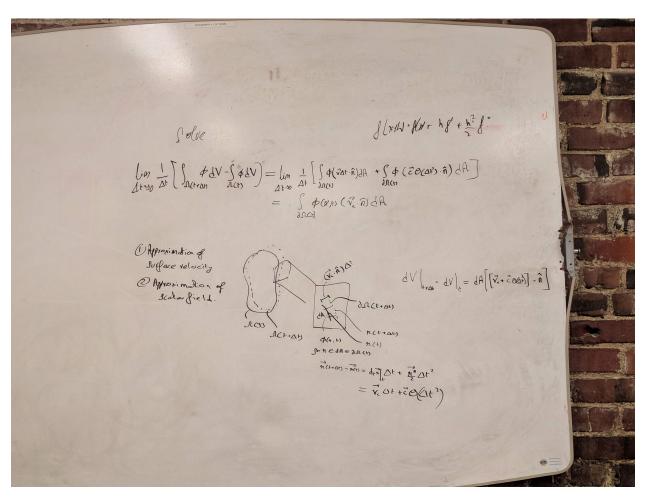### 7.6.1 Discrete Differential Forms

Appendix A

Fluids

Figure A.1: Evaluating Key Limit in RTT Proof
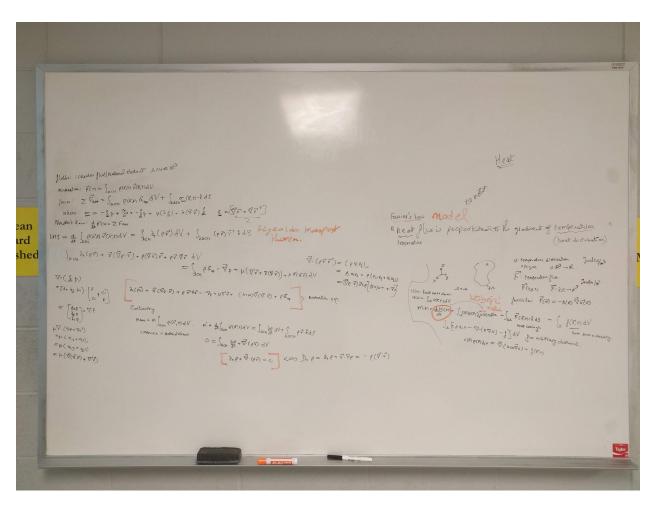
more stuff to come.

Figure A.2: Navier-Stokes for Compressible Flow
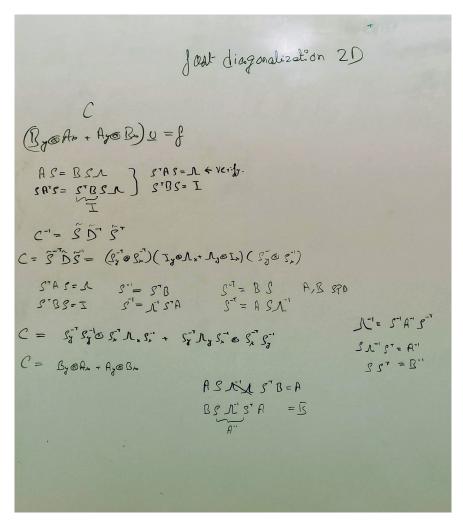
Appendix B

Numerics

Figure B.1: Fast Diagonalization Method

Figure B.2: Variational Method for Navier Stokes

Figure B.3: Navier-Stokes Pressure Solve