5 Vibration control

5.1 Total vibration and transients

In explaining the forced vibration response I have not been telling the whole story. Even for a single-degree-of-freedom model there can be a natural (free) vibration and a forced response at the same time forming the total vibration. For example, if a component vibrating in its steady-state response to a forcing vibration were hit by a hammer then it would also perform a natural vibration. In time, this natural vibration would be damped out to leave only the steady-state vibration.

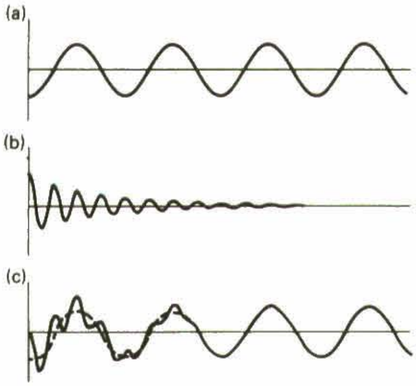

Figure 35

Consider a case in which the mounting is vibrating steadily, but the mass in our model is being held stationary. It is then released. What happens? If by sheer chance it was released having the appropriate speed and displacement and the correct phase to agree with its steady-state forced response then it would proceed to perform just that. However, normally the release conditions will not agree with the steady-state requirement. In Figure 35, say that it is released with zero speed and zero displacement at $t = 0$, when the steady-state response wanted it to be at $x = -A$, Figure 35(a). The result is that it performs a natural vibration of initial amplitude A, Figure 35(b), superimposed on the steady-state vibration, Figure 35(c). Because this natural vibration is damped and dies away it is called the *transient*. In due course there remains only the steady-state forced response. This is a particularly simple example of a transient, but it is intended to illustrate to you that a transient can arise whenever the responding object is disturbed from equilibrium or from a steady-state vibration. This can happen either because it has been acted on by additional forces, for example an impact, or because the forcing function changes in some way (amplitude, frequency or phase). A very common case is when the forcing vibration comes from a starting machine. However, even when the vibration source is running steadily it will not be perfectly regular. The response will always have minor transients affecting it, so you are reminded again that the theoretical steady-state solution is only an approximation to the true behaviour of a real system. The steady-state solution does, however, indicate the importance of resonance and its potential dangers.

transient

You should also note in this context that in certain circumstances the damping in a system can lead to an increasing transient vibration rather than one which dies away. This situation is more likely in systems with more than the one degree of freedom, to which I have restricted this introduction to vibration so far. An increasing transient is associated with *instability* in a vibrating system. Typical engineering examples are aircraft wing flutter and wheel shimmy. We do not, however, have the time to discuss instability further in this Unit.

instability

5.2 Vibration tolerance

Because of the dangers of vibration in general and resonance in particular, the mechanical engineering designer must always pay close attention to possible vibration problems. The acceptable level of vibration depends on the detailed application. A delicate instrument may need to be mounted in such a way as to prevent dangerous vibration reaching it. A structure may be damaged by the stresses associated with vibration – even if the yield stress is not reached, the cyclical stress variations can lead to fatigue failure as was mentioned in Unit 10. The *tolerance* of an instrument or structure to vibration is amenable to analysis, so in such cases the designer will be able to specify acceptable levels of vibration.

(vibration) tolerance

In the case of human beings such analysis is not practicable. In this case the limits are not so well defined because they vary considerably from one individual to another. Also the limits are not those of mechanical failure of the body so much as subjective factors such as concentration, headaches and deterioration of visual acuity. Extensive tests have given rise to standards for limits of human exposure to vibration. The acceptable levels depend upon the period for which they must be tolerated. Figure 36 shows a very simple vibration model of the human body, indicating the possibility of a number of resonance frequencies. Figure 37 shows the vibration limits for an 8-hour period – approximately a working day. By using logarithmic scales the limits conveniently form straight lines. The worst frequencies are those in the region 4 to 8 Hz, corresponding to a natural frequency of the internal organs. The three 'curves' represent the rather subjective variables of reduced comfort, decreased proficiency and exposure limit. The decreased proficiency limit is a suitable general guide to acceptable vibration for vehicle drivers or factory workers.

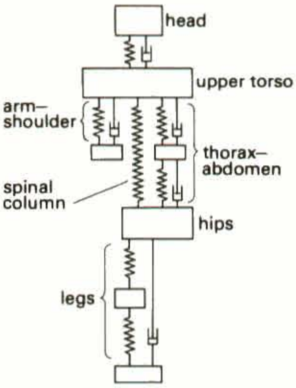

Figure 36

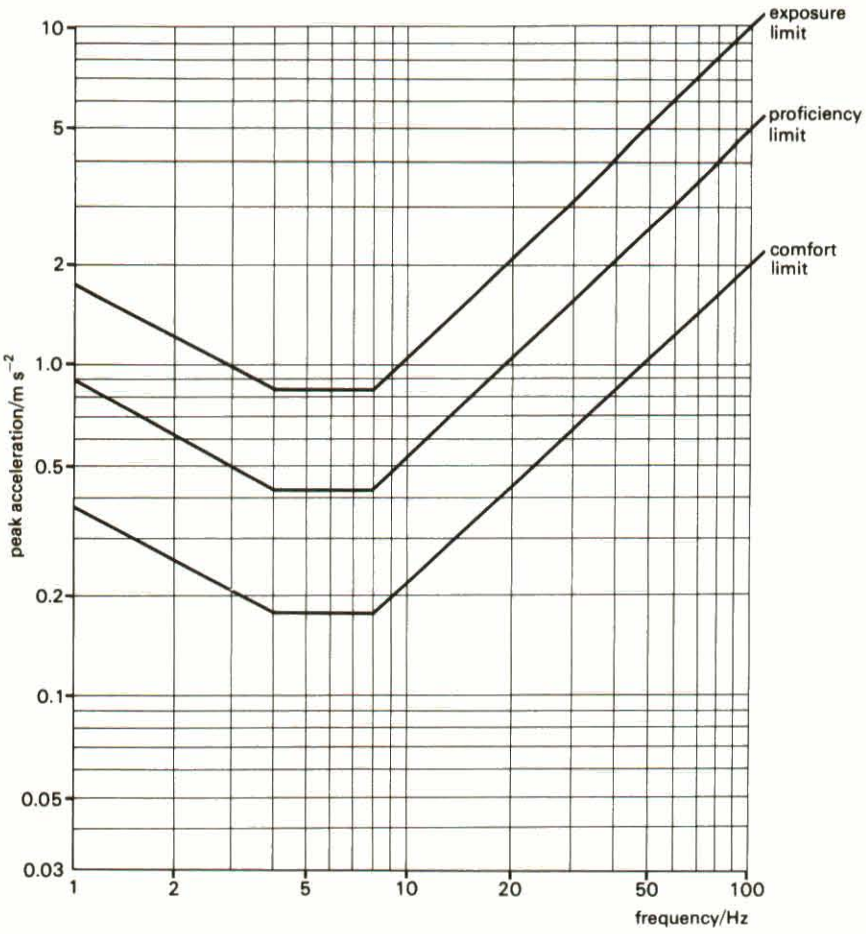

Figure 37 Human vibration limits (vertical whole body vibration, 8-hour period)

Exceeding the limit would be liable to lead to increased accident rates through loss of concentration. Exceeding the exposure limit can lead to a direct health risk from the vibration itself. Note that the vibration is expressed here in terms of the peak acceleration, giving a more convenient graph than displacement amplitude would. This amplitude can be found by dividing by the frequency (in rad s^{-1}) squared because for sinusoidal vibration $\ddot{x} = -\Omega^2 x$.

SAQ 20

If a bus driver is subject to an average of 40 Hz vibration from the engine for a full working day, and is not to have decreased proficiency, what is the maximum vibration to which the driver may be subjected, expressed as (a) peak acceleration, (b) amplitude?

SAQ 21

If the bus driver is mainly subject to 8 Hz vibration from the engine at tick-over, what is the maximum amplitude acceptable for (a) comfort, (b) proficiency, (c) exposure limit?

5.3 Control methods

The main message of the Unit so far has been that vibration can be damaging, especially if resonance occurs. What can the engineering designer do to minimize vibration problems? This area of study could be called 'vibration control', and you have now covered sufficient of the basic theory of vibration to look at the methods used to achieve this.

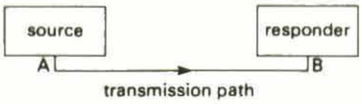

Figure 38

The first idea that you need to recognize is that there is always a source of vibration, and there is also an object which is subjected to that vibration. I shall call the latter the responder. The source and responder are not normally the same, so there is a transmission path between the two (Figure 38). Examples would be: (1) a lorry (source) shaking the windows of your house (responder); (2) an engine (source) shaking a carburettor (responder) mounted on it, causing fuel feed problems; (3) a rough road (source) shaking a car driver (responder). In practice one source may vibrate many responders, and equally one responder may be affected by many sources. Sometimes the source and responder are not readily distinguishable, for example a lathe tool vibrating because of the forces exerted on it by the workpiece – the source is really the cutting operation, a combination of tool and workpiece. Equally, an engine is its own source of vibration. However, in most cases the source and responder are readily identifiable. A good example typical of those met in practice would be a problem where a reciprocating engine mounted on a structure is liable to cause damage or other adverse effects to delicate instruments mounted elsewhere on the structure. The structure might be a building, motor car, a ship, or an aeroplane. (If your only experience of flying is in a modern jet aircraft, I can assure you that they are incomparably smoother than the older piston-engined aircraft used to be.)

In principle there are three possible methods of vibration control – we can modify the source, the transmission path or the responder.

Modification of the source of vibration

This is often the most effective and direct solution. Improving the balance of rotating machinery is one such procedure. A classic example is the wheel of a car. A badly unbalanced wheel can produce some very unpleasant effects on the occupants of the car. Sometimes the resonant response of the bonnet is clearly visible too. The sensible answer is to improve the wheel balance, which involves adding small pieces of lead to move the centre of mass to the centre of the axle, reducing the forces that cause the vibration to negligible amplitude. *Balancing* is, however, not always possible. Most reciprocating engine designs cannot be balanced perfectly even in principle, although the use of a larger number of cylinders generally reduces the residual unbalanceable forces. A rough road is another source of vibration that cannot normally be considered alterable. Within limits a driver can avoid pot-holes and choose a smooth route, but the vehicle design must take account of the fact that the road is by no means perfectly smooth.

balancing

Modification of the responder

This is often a satisfactory means to obviate vibration damage. If the forcing vibration has a substantially constant frequency then the natural frequency of the responder can be adjusted to ensure that resonance is avoided. This may be done by adjustment of stiffness or mass. However, this is more difficult if the vibration source covers a range of frequencies, for example an engine covering a range of speeds or a car being shaken by a road whose surface undulates in an irregular manner. In such cases adequate damping must be incorporated to minimize the effect of the unavoidable resonances (remember the effect of damping on resonance). Car body panels often have damping material such as felt applied to them. Another possible modification to the responder is to add an auxiliary mass and spring. If tuned to the appropriate frequency the motion of the auxiliary mass will be large, but the vibration of the original responder will be reduced. A device of this kind is often called a *vibration absorber*. Many car engine crankshafts are fitted with torsional vibration devices of this type.

vibration absorber

Modification of the vibration transmission path

By introducing a flexible member between the source and responder it may be possible to reduce the transmission of vibration. In practice this means mounting either the source or responder on a flexible support. This is called *vibration isolation*, because the machine is 'isolated' against vibration transmission. The springs of a car suspension are a good example, isolating the responder (the car body) from the source (the road). The flexibility of the tyres also plays an important role here. The passengers are further isolated from the car body by the flexibility of the seats. The remainder of this section is concerned with the performance of vibration isolating mountings.

vibration isolation

5.4 Vibration isolation

If attention to reducing the vibration at source and attention to the ability of the responder to withstand vibration prove inadequate, then the remaining possibility is to interrupt the transmission path in some way. The most convenient place to do this is normally at the source mounting or the responder mounting, A and B respectively in Figure 38. It turns out that these are very similar problems.

First consider a responder, shown in Figure 39 on a mounting modelled by a simple spring and damper in parallel. In the majority of practical cases a simple mounting such as the one shown is adequate. If the ground is vibrating with amplitude X_0 what is the amplitude A of the responder? You already know how to find that. It is $A = TX_0$ where T is the transmissibility of the mounting for the mass that it carries. What characteristics are required of the mounting to give vibration isolation? For a small A obviously we want a small T, and referring to Figure 34 it is clear that a high frequency ratio is the main thing to aim for. It is not so obvious what to do in the way of damping. If the frequency ratio is high then light damping (small ζ) would be better, but this might be dangerous if resonance occurs because of the high peak value of T for small ζ.

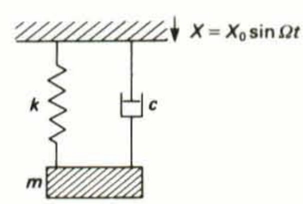

Figure 39

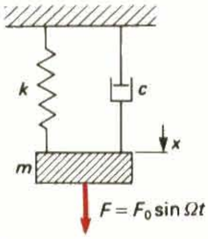

Figure 40

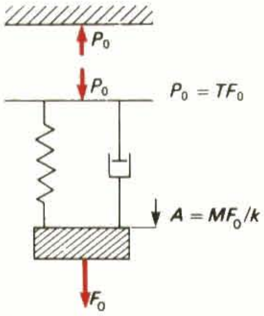

Figure 41 Amplitudes shown

Before saying any more about the practical implications of these curves let me turn to the problem of isolating a vibration source. Figure 40 shows a source with mass m again on a mounting that can be modelled by the spring of stiffness k and damper of coefficient c in parallel. There is a force $F = F_0 \sin \Omega t$ acting on it, usually as a result of the motion of internal components, where F_0 is the force amplitude. It is analogous to the displacement amplitude, the peak value of a sinusoidally varying displacement. Amplitude just means the peak value. Here I shall use amplitude to mean the peak value of a varying displacement and force amplitude to mean the peak value of a varying force.

We are trying to reduce the transmission of vibration from the source into the supporting structure. We need to know the force exerted by the spring and damper on the structure (Figure 41). If we neglect the displacement of the structure as small relative to that of the source, then the spring length change is just x, and the relative velocity of the two ends of the damper is just $\dot{x}$. The spring exerts a force kx and the damper a force $c\dot{x}$. We derived an expression for this total force $P = kx + c\dot{x}$ in Section 4.1. The total force is still sinusoidal with frequency Ω rad s^{-1}, as was the original vibrating force. The amplitude of the transmitted sinusoidal force is P_0, which is given by

$$P_0 = TF_0$$

where T is the transmissibility. Hence you can use Figure 34 for estimating the transmitted force, or else use the transmissibility formula

$$T = \left[\frac{1 + \{2\zeta(\Omega/\omega)\}^2}{\{1 - (\Omega/\omega)^2\}^2 + \{2\zeta(\Omega/\omega)\}^2}\right]^{1/2}$$

The numerical evaluation of T from this formula is quite straightforward with a scientific calculator, and is preferable. The graph permits a simple check of your calculation. If you decide to use the graph, then when ζ is not equal to one of the graph line values you can estimate a position in between the lines (i.e. interpolate). In using the graph, please remember that the scales are logarithmic. If the frequency range is beyond the graph then you will have to use the formula.

Example

A 150 kg car engine on rubber mountings of stiffness 60 kN m^{-1} and damping ratio 0.1 is subject to a sinusoidal force of amplitude 200 N and frequency 25 Hz. Estimate the force amplitude transmitted through the mountings.

Solution

$$\Omega = 2\pi \times 25 = 157 \text{ rad s}^{-1}$$

$$\omega = \sqrt{(k/m)} = (60 \times 10^3/150)^{1/2} = 20 \text{ rad s}^{-1}$$

$$\Omega/\omega = 7.85$$

$$T = 0.0307 \qquad \text{(by calculator)}$$

$$P_0 = TF_0 = 0.0307 \times 200 = 6.1 \text{ N}$$

My estimate of the transmitted force amplitude is 6.1 N.

SAQ 22

Estimate the transmitted force amplitude for the engine of the example if: (a) the damping is negligible, (b) $\zeta = 0.1$ and $k = 120$ kN m^{-1}, (c) $\zeta = 0.1$ and $k = 180$ kN m^{-1}, (d) $\zeta = 0.5$ and $k = 180$ kN m^{-1}.

Whether you are trying to isolate a structure from a vibration source, or a responder from a vibrating support structure, it is the transmissibility of the mounting that matters, as $A = TX_0$ and $P_0 = TF_0$. We need to make the transmissibility small. I have already said that this means that we want a large frequency ratio. We certainly want to keep away from resonance. The value of Ω is usually fixed by existing operating conditions, for example an engine speed, so for large Ω/ω we must make the natural frequency ω small. Now $\omega = \sqrt{(k/m)}$, so we want small k, that is a soft spring. Also you can see from Figure 34 that provided Ω/ω is beyond the 'crossover' point it is better to have small ζ.

In principle it is possible to achieve any required degree of isolation by making ω and ζ very small. However, there are practical limitations. The first one is that the 'spring' must support the weight of the object. A very soft spring will have a large deflection due to this ($e = mg/k$). Also if k is very small, any extra forces will cause further large deflections, for example, if someone leans on it. A frequency ratio of $\Omega/\omega = 10$ will, however, give a considerable degree of isolation especially for small ζ, so unless the requirements are very stringent it is usually possible to meet them with a simple isolating system of this type. More complicated systems could be used in extreme cases. Note that the more difficult cases tend to be when Ω is small, because this requires extremely small ω.

The other practical problem is that of 'start-up'. If say an engine is being isolated, the design may be quite satisfactory at the full operating speed, but during starting and stopping Ω is much smaller. In fact Ω must go from zero to greater than ω, so it must pass through the resonance condition when $\Omega = \omega$. This is often readily observable, for example on spin-driers, and also on diesel engines at very slow tick-over. If resonance is traversed rapidly then the amplitude does not have time to build up to really dangerous values. However, this does depend to some degree on the damping. More damping helps to prevent resonance problems (smaller peak T) at the cost of poorer isolation at full operating conditions. The optimum values obviously depend upon the exact application. In some cases, for example a car suspension, the manufacturers go to a good deal of trouble and expense to include special dampers. In many other cases the degree of damping needed is quite modest and not critical. This is one reason why rubber is popular for vibration isolation mountings. Not only does it provide the stiffness but also a useful small amount of damping resulting from its own internal friction. Hence it is especially useful for small machine mountings, engines in cars being a classic example. Figure 42 shows two typical rubber mountings in which the rubber is bonded to convenient metal mounting plates. The inherent damping in such mountings varies widely, depending upon the shape, rubber compound, frequency and amplitude but is usually in the range 0.01 to 0.1 with $\zeta = 0.05$ being a typical average value. This is, of course, not just a property of the material but also depends on the mass being supported.

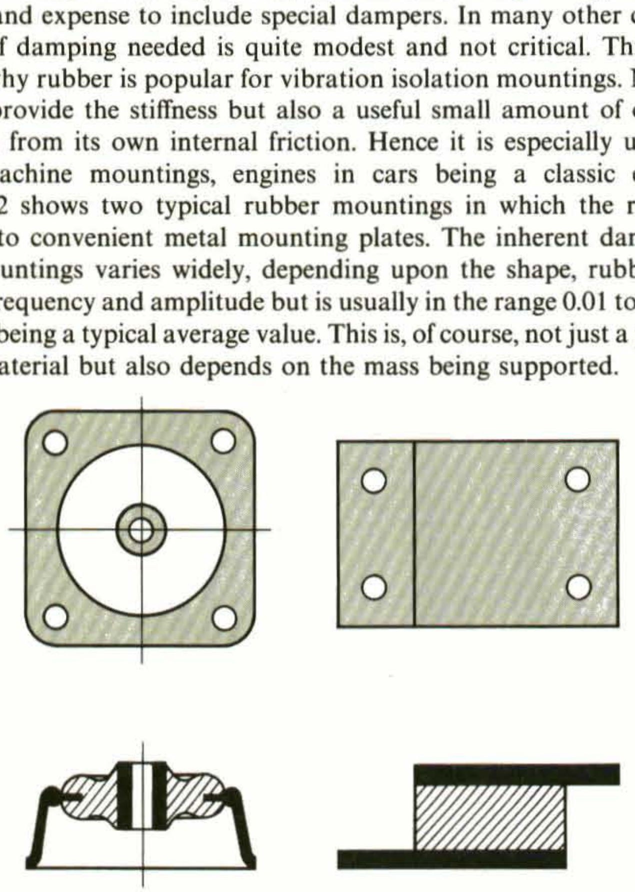

Figure 42 Vibration isolating rubber mountings

SAQ 23

A 20 kg single-cylinder motorcycle engine requires a vibration isolating mounting. As a result of the motion of its internal components at a typical running speed of 3000 rev min^{-1} the engine is subject to an approximately sinusoidal force of amplitude 400 N and frequency 50 Hz. The mounting proposed has stiffness 20 kN m^{-1} and negligible damping.

(a) Estimate the natural frequency.

(b) Estimate the transmissibility.

(c) Estimate the transmitted force.

SAQ 24

For the engine of the previous SAQ, if the specification for the transmitted force was that it be less than 10 N, estimate:

(a) The maximum acceptable transmissibility.

(b) The acceptable natural frequency.

(c) The acceptable mounting stiffness.

(d) If $\zeta = 0.1$ what would be the acceptable natural frequency?

(e) What would be the acceptable mounting stiffness?

In many practical cases of mechanical vibration, the force amplitude is proportional to the square of the running speed. This is typical of reciprocating engines and poorly balanced rotating machines.

SAQ 25

In these circumstances estimate the vibration force amplitude acting on the motorcycle engine at the following speeds.

(a) 750 rev min^{-1}, (b) 2000 rev min^{-1}, (c) 4000 rev min^{-1}, (d) 6000 rev min^{-1}.

SAQ 26

If the mountings were rubber blocks of stiffness 100 kN m^{-1} and damping ratio $\zeta = 0.1$, estimate (a) the resonance speed, (b) the transmissibility at resonance, (c) the transmitted force at resonance.

SAQ 27

Estimate the engine amplitude at resonance.

SAQ 28

If the resonance speed lies within the operating speed range of the engine then the results can be very unpleasant, and even disastrous. In practice the resonance speed is kept below the lowest speed, and if possible below about 0.4 times the lowest speed. The planned tick-over speed for the motorcycle engine is 600 rev min^{-1}.

(a) What undamped natural frequency should be aimed for?

(b) What mounting stiffness is required?

(c) The damping ratio is $\zeta = 0.1$. Estimate the transmitted force at tick-over.

(d) Estimate the transmitted force at 6000 rev min^{-1}.

SAQ 29

An air compressor of mass 260 kg normally runs at 900 rev min^{-1}, when it is subject to a vibratory force of amplitude 2 kN and frequency 15 Hz. It requires a vibration isolating mounting to reduce the force amplitude transmitted to the floor, which should not exceed 60 N. It is intended to use rubber mountings.

(a) Neglecting damping, estimate the necessary transmissibility, natural frequency and mounting stiffness.

(b) Taking $\zeta = 0.1$, estimate the necessary natural frequency and stiffness.

(c) Tests indicate that more damping is required to reduce the resonance amplitude, so extra dampers are added giving a total damping coefficient of 2 kN s m^{-1}, with the same stiffness as calculated in (b). Estimate the transmitted force in normal running.

(d) Estimate the amplitude and phase lag at resonance (the force is proportional to speed squared).

Summary

In this Unit you have been introduced to single-degree-of-freedom vibratory systems in translation with sinusoidal forces and displacements. This is not as restrictive as it might seem because it covers many practical cases, and is a necessary foundation for any further analysis covering rotation, multiple degrees of freedom and non-sinusoidal force variation. The vibration of a one-degree-of-freedom system, like the one in Figure 1, is in many cases analogous to the motion of a more complex system in one of its modes of vibration. The results of this Unit, therefore, at least give some insight into the vibratory behaviour of multiple-degree-of-freedom systems.

Vibration is divided into two main types – natural and forced. Natural vibration occurs at the damped natural frequency, usually with a decreasing amplitude. The undamped natural frequency is a theoretical notion, being the vibration frequency that would occur if there was zero damping. In many cases it is also a good approximation to the damped natural frequency.

The response of a system to forced harmonic vibration depends upon the frequency ratio – the forcing frequency divided by the undamped natural frequency. If this is close to 1 then very large response amplitudes can occur. The system is then in resonance.

In principle the total vibration of a system is a combination of natural and forced vibration. In practice, however, the natural vibration usually dies away leaving forced vibration as the dominant component, unless the system is unstable.

The effects of vibration can be very damaging, so vibration isolation is often necessary. The force amplitude transmitted from a vibration source to its supporting structure is given by $P_0 = TF_0$, where T is the transmissibility. The displacement amplitude transmitted to a responder from a vibrating support is $A = TX_0$. (The amplitude of vibration of a source due to a directly applied force is $A = MF_0/k$ where M is the magnification ratio.) To achieve vibration isolation a small value of T is needed. This requires a high frequency ratio. A small value of damping ratio improves isolation at high frequency ratios but permits a larger resonance amplitude.

Glossary

Term	Text reference	Explanation
Amplitude	1	A, magnitude of peak displacement from mean position during vibration
Balancing	5.3	Procedure for reducing vibration in rotating machinery by adjusting the mass distribution
Critical damping	2	Value of damping, given by a damping ratio $\zeta = 1$, below which damped natural vibration can occur. (See Figure 12)
Cycle	1	The motion of the system from any given position until the system returns to the same position and is moving in the same direction
Damped forced vibration	4	Forced vibration including the effects of damping
Damped natural frequency	2	Frequency of natural vibration when damping is allowed for: $\omega_d = \omega\sqrt{1-\zeta^2}$ (in rad s^{-1}) or $f_d = \omega_d/2\pi$ (in Hz)
Damped natural vibration	2	Natural vibration when damping is present
Damped period	2	Time for one cycle in damped natural vibration, $\tau_d = 1/f_d$ (in s)
Damping	2	Property of system, such as friction, which causes loss of energy and amplitude reduction
Damping coefficient	2	Coefficient c (in N s m^{-1}) in expression $c\dot{x}$ for damping force
Damping factor	2	Exponent α in exponential decay term in damped free vibration, defined as $\alpha = c/2m = \zeta\omega$ (in s^{-1})
Damping ratio	2	Ratio of damping to critical damping, defined by $\zeta = \dfrac{c}{2\sqrt{mk}}$
Dashpot	2	Device used to represent source of damping in a system. (See Figure 16)
Force amplitude	3.2	F_0 (or P_0), the peak value of an harmonic (sinusoidal) force
Forced vibration	1	Vibration in response to continuous external forcing
Forced harmonic vibration	3	Vibration of system due to external excitation arising from either ground motion or an applied force, which varies harmonically
Free vibration	1	See *natural vibration*
Frequency	1	Number of cycles of vibration performed in one second (f in Hz). Also quantified as ω (or Ω) in rad s^{-1}, where $\omega = 2\pi f$
Frequency ratio	3, 4	Ratio of frequency of applied force, or ground motion, to (undamped) natural frequency, Ω/ω
Ground motion	3.1, 4.2	Motion of previously steady mounting – sometimes referred to as displacement excitation
hertz (Hz)	1	Unit of frequency of one cycle per second
Instability	5.1	Condition in system when natural (free) vibration increases rather than decays, following a disturbance
Lightly damped systems	2	Systems with low damping, usually damping ratio $\zeta < 0.1$
Natural vibration	1	Vibration of a system after a disturbance with no further external forcing (also free vibration)
Natural (vibration) frequency	1	Frequency of natural, or free, vibration (ω in rad s^{-1})
Period	1	Time for one cycle of vibration (τ in s) for undamped system

Term	Text reference	Explanation
Phase lag	4.1	Factor ϕ which helps to specify forced harmonic vibration relative to the applied force, defined by $$\tan\phi = \frac{2\zeta(\Omega/\omega)}{1-(\Omega/\omega)^2}$$ ($-\phi$ is the phase angle). (See also Figure 30)
Resonance	3	Condition in forced harmonic vibration when amplitude of motion is large (when $\Omega = \omega$)
Resonance frequency	4	Frequency ($\Omega = \omega$) at which resonance occurs in forced harmonic vibration
Simple harmonic motion	1	Motion in which the acceleration is proportional to the displacement, and in the opposite direction
Transient	5.1	See *transient vibration*
Transient vibration	3	Natural (free) vibration of a system which dies away due to damping, sometimes called a transient. (See 5.1)
Transmissibility	3	Factor relating input on one side of a spring to the response on the other (either relates input force amplitude to transmitted force amplitude or input displacement amplitude to output motion amplitude). (See also 4.1 and 4.2)
Vibration absorber	5.3	Auxiliary mass and spring which is added to a vibrating system to reduce vibration
Vibration isolation	5.4	A technique for modifying the transmission path of a vibration to reduce the vibration output
(Vibration) tolerance	5.2	Acceptable level of vibration

Answers to Self-Assessment Questions

SAQ 1

(a) $\sqrt{(k/m)}$ has units of

$$\sqrt{(\text{N m}^{-1}/\text{kg})} = \sqrt{(\text{kg m s}^{-2}\ \text{m}^{-1}/\text{kg})}$$
$$= \sqrt{(\text{s}^{-2})}$$
$$= \text{s}^{-1}$$

ω has units of rad s^{-1}, which agrees because radians are dimensionless.

(b) $\omega = (k/m)^{1/2}$

$$= (10/0.035)^{1/2}$$
$$= 16.9 \text{ rad s}^{-1}$$

$f = \omega/2\pi$

$$= 2.7 \text{ Hz}$$

(c) The comparison is good.

(d) $\tau = 1/f$

$$= 0.37 \text{ s}$$

SAQ 2

(a) $\omega = (k/2m)^{1/2} = (k/m)^{1/2}/1.414$

(b) $\omega = (10/0.07)^{1/2} = 11.95 \text{ rad s}^{-1}$

$f = 1.9$ Hz

(c) The comparison is good.

SAQ 3

$$x = B \sin \omega t + C \cos \omega t$$

where, from SAQ 1,

$$\omega = 16.9 \text{ rad s}^{-1}$$

At $t = 0$,

$$x = C = 10 \text{ mm}$$

Differentiating the expression for x with respect to time,

$$\dot{x} = B\omega \cos \omega t - C\omega \sin \omega t$$

and at $t = 0$

$$\dot{x} = B\omega = 50 \text{ mm s}^{-1}$$

Therefore

$$C = 10 \text{ mm}$$

$$B = \frac{50}{16.9}\frac{\text{mm s}^{-1}}{\text{rad s}^{-1}} = 2.959 \text{ mm}$$

The amplitude

$$A = \sqrt{B^2 + C^2} = [(10)^2 + (2.959)^2]^{1/2}$$
$$= 10.43 \text{ mm}$$

SAQ 4

Damping force $F = c\dot{x}$, so $c = F/\dot{x}$.

Units are $\text{N/m s}^{-1} = \text{N s m}^{-1}$.

Substituting kg m s^{-2} for N gives kg s^{-1}. This is a more fundamental expression of the units, but it is more common practice to use N s m^{-1}.

SAQ 5

(a) rad s^{-1}

(b) rad s^{-1}

(c) N s m^{-1}, or kg s^{-1}

(d) s^{-1}

(e) ζ has no units.

SAQ 6

(a) $(1 - \zeta^2)^{1/2} = \frac{1}{2}$

$$1 - \zeta^2 = \tfrac{1}{4}$$
$$\zeta^2 = 0.75$$
$$\zeta = 0.866$$

(b) $(1 - \zeta^2)^{1/2} = 0.95$

$$\zeta = 0.31$$

SAQ 7

System 1

$$\omega = (k/m)^{1/2} = 120 \text{ rad s}^{-1}$$
$$f = \omega/2\pi = 19.1 \text{ Hz}$$
$$\alpha = c/2m = 15 \text{ s}^{-1}$$
$$\zeta = \alpha/\omega = 0.125$$
$$\omega_d = (\omega^2 - \alpha^2)^{1/2} = 119 \text{ rad s}^{-1}$$
$$f_d = \omega_d/2\pi = 18.95 \text{ Hz}$$
$$1/\alpha = 0.0667 \text{ s}$$
$$f_d/\alpha = 1.263 \text{ cycles}$$

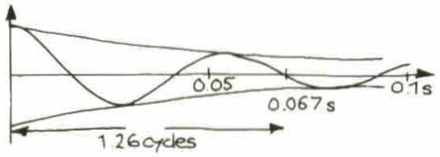

Figure 43

System 2

$\omega = 40 \text{ rad s}^{-1}$

$f = 6.37 \text{ Hz}$

$\alpha = 0.5 \text{ s}^{-1}$

$\zeta = 0.0125$

$\omega_d = 40 \text{ rad s}^{-1}$

$f_d = 6.37 \text{ Hz}$

$1/\alpha = 2 \text{ s}$

$f_d/\alpha = 12.73$ cycles

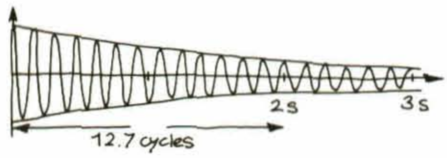

Figure 44

System 3

$\omega = 7.906 \text{ rad s}^{-1}$

$f = 1.26 \text{ Hz}$

$\alpha = 5.625 \text{ s}^{-1}$

$\zeta = 0.71$

$\omega_d = 5.56 \text{ rad s}^{-1}$

$f_d = 0.884 \text{ Hz}$

$1/\alpha = 0.178 \text{ s}$

$f_d/\alpha = 0.157$ cycles

In this case it is easiest to sketch the vibration of frequency ω_d with no damping, then sketch the declining amplitude, and then combine the two.

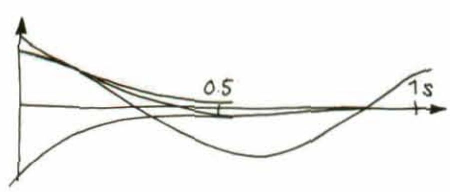

Figure 45

System 4

$\omega = 51.6 \text{ rad s}^{-1}$

$f = 8.2 \text{ Hz}$

$\alpha = 0.005 \text{ s}^{-1}$

$\zeta = 0.0001$

$\omega_d = 51.6 \text{ rad s}^{-1}$

$f_d = 8.2 \text{ Hz}$

$1/\alpha = 200 \text{ s}$

$f_d/\alpha = 1640$ cycles

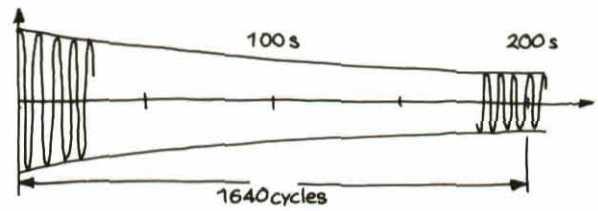

Figure 46

SAQ 8

In 1 second ten cycles are performed, so the damped natural frequency is $f_d = 10$ Hz ($\omega_d = 62.8 \text{ rad s}^{-1}$). The amplitude falls by the factor e in 2 s, so

$\alpha = 1/(2 \text{ s}) = 0.5 \text{ s}^{-1}$

$\omega = 62.8 \text{ rad s}^{-1}$

$\zeta = 0.008$

$\alpha = c/2m$, so

$c = 2\alpha m$

$= 3.4 \text{ N s m}^{-1}$

SAQ 9

One full cycle takes 2 s, so

$f_d = 0.5 \text{ Hz}, \qquad \omega_d = 3.14 \text{ rad s}^{-1}$

Over one cycle the amplitude ratio is 0.1.

$\alpha = -f_d \log_e R$

$= -0.5 \times \log_e 0.1$

$= 1.15 \text{ s}^{-1}$

$\omega = (\omega_d^2 + \alpha^2)^{1/2} = 3.344 \text{ rad s}^{-1}$

$\zeta = \alpha/\omega = 0.34$

$c = 2\alpha m$

$= 575 \times 10^3 \text{ N s m}^{-1}$

SAQ 10

(a) $\text{rad s}^{-1}/\text{rad s}^{-1} = 1$, so that Ω/ω has no units.

(b) $\text{m}/\text{m} = 1$, so that A/X_0 has no units.

(c) Both sides of the equation have numbers only – there are no m, s, kg or other units. Equations arranged in this way are very useful to engineers; they are often said to be non-dimensional.

SAQ 11

(a) $A/X_0 = 1$, 1.19, 2.78, 5.26, ∞, 4.76, 2.27, 1.04, 0.45, 0.33, 0.125, 0.01, 0.0001

(b)

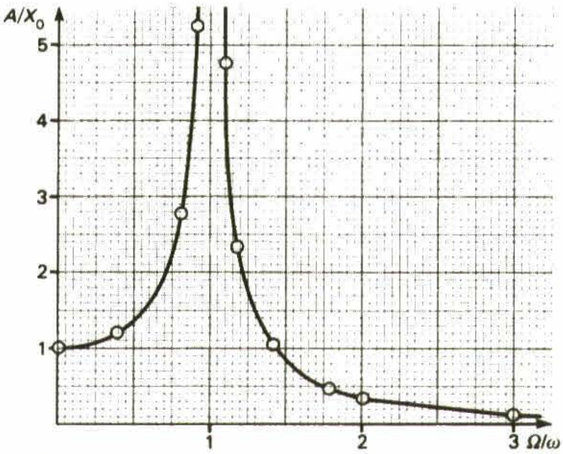

Figure 47

(c) A/X_0 is infinite at $\Omega/\omega = 1$.

(d) Trouble, possibly breakage

SAQ 12

(a) From $\omega^2 = k/m$ and $\omega \approx \omega_d$ (little damping)

$k = m\omega^2 = 500\ \text{N m}^{-1}$

(b) $\Omega/\omega = 30/25 = 1.2$

(c) $T = 2.27$

(d) $A = TX_0 = 2.27 \times 2\ \text{mm} = 4.54\ \text{mm}$

SAQ 13

(a) $X_0 = 2$ mm, $\quad A = 0.4$ mm maximum.

From $A = TX_0$, the maximum transmissibility is

$T = A/X_0 = 0.4/2 = 0.2.$

(b) The frequency ratio must be higher than the one giving $T = 0.2$. From the graph the minimum frequency ratio is $\Omega/\omega \approx 2.5$. If you calculate Ω/ω from T using the formula you will find $\Omega/\omega = 2.45$, allowing for the transmissibility formula giving a negative value when $\Omega/\omega > 1$ (remember you ignored the negative sign when plotting the amplitude in SAQ 11).

For $\Omega/\omega = 2.5$, and $\Omega = 2\pi \times 30 = 188.5\ \text{rad s}^{-1}$

$\omega = \Omega/2.5 = 75.4\ \text{rad s}^{-1}$

The natural frequency must be *less* than this, to give a frequency ratio *higher* than 2.5

(c) $\omega^2 = k/m$, so $k = m\omega^2 = 2.3\ \text{kN m}^{-1}$ is the maximum acceptable stiffness.

SAQ 14

(a) $\Omega = 2\pi \times 50 = 314\ \text{rad s}^{-1}$

$\omega = (k/m)^{1/2} = (6 \times 10^4/140)^{1/2} = 20.7\ \text{rad s}^{-1}$

$\Omega/\omega = 15.17$

Neglecting damping, $M = 0.0044$

$A = MF_0/k = 0.06$ mm

Note: $M = 0.0044$ even if an allowance is made for damping with $\zeta = 0.02$.

(b) $P_0 = kA = 3.6$ N amplitude

SAQ 15

(a) $1/2\zeta = 4.2$

(b) $1/2\zeta = 10$

(c) $1/2\zeta = 50$

(d) $1/2\zeta = 250$

SAQ 16

$$\omega = \sqrt{\frac{10 \times 1000}{35}} = 16.903\ \text{rad s}^{-1}$$

(a) $\Omega/\omega = 0.5916$

$T = 1.538$

$A = 7.7$ mm

(b) $\Omega/\omega = 1.0057$

$T = 80.06$

$A = 400$ mm

This amplitude could not be realised in practice. This theoretical amplitude is far beyond the possible limits of the spring extension.

(c) $\Omega/\omega = 5.9161$

$T = 0.0294$

$A = 0.15$ mm

In both cases (a) and (c), neglecting the damping would not have significantly affected the results.

SAQ 17

(a) $\Omega = 2\pi \times 50 = 314\ \text{rad s}^{-1}$

$\omega = (k/m)^{1/2} = 12.54\ \text{rad s}^{-1}$

$\Omega/\omega = 25$

$M = 0.0016$

$A = MF_0/k = 0.12$ mm

$\phi = 179.9^\circ$

as we would expect for a very high frequency ratio. Note that $\tan\phi = -0.0002$, so that $\phi = -0.01^\circ + 180^\circ = 179.9^\circ$.

(b) For $\Omega/\omega = 1$

$M = 1/2\zeta = 20$

(c) $A = MF_0/k = 2.7$ mm

$T = (1 + 4\zeta^2)^{1/2} M = 20.025$

$\therefore \quad P_0 = 20.025 \times 3 = 60.1$ N

(d) $F_0/k = 0.136$ mm

Maximum $M = 1/0.136 = 7.3$

(e) $M = 1/2\zeta, \qquad \zeta = 1/2M = 0.068$

(f) With $\Omega/\omega = 25$ and $\zeta = 0.068$

$M = 0.0016$

$A = 0.12$ mm (no significant change)

The frequency ratio is much too high for the increase in damping ratio to alter the magnification ratio.

SAQ 18

(a) $mg = 6 \times 9.81 = 58.86$ N

$F = ke$ (linear spring)

$$k = \frac{F}{e} = \frac{58.86}{0.03} = 1962 \text{ N m}^{-1}$$

(b) $\omega = \sqrt{(k/m)} = 18.08$ rad s^{-1}

$f = 2.88$ Hz

$\alpha = c/2m = 1$ s^{-1}

$\zeta = \alpha/\omega = 0.055$

$\omega_d = \omega\sqrt{1-\zeta^2} = 18.05$ rad s^{-1}

$f_d = 2.87$ Hz

(c) $1/\alpha = 1$ s

$f_d/\alpha = 2.87 \approx 2.9$ cycles

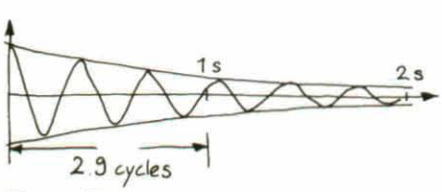

Figure 48

(d) $\Omega = 2\pi \times 12 = 75.4$ rad s^{-1}

$\Omega/\omega = 4.17$

$T = 0.0671$

$A = TX_0 = 0.134$ mm

(e) Maximum T at resonance $= 4$ mm/1 mm $= 4$

when $T = \dfrac{\sqrt{1+4\zeta^2}}{2\zeta}$,

so $\zeta = 0.129$

(f) $\zeta = \dfrac{c}{2\sqrt{(mk)}}$

$c = 2\zeta\sqrt{(mk)} = 28.0$ N s m^{-1}

(g) $\Omega = 2\pi \times 10 = 62.83$ rad s^{-1}

$\Omega/\omega = 3.475$

$T = 0.121$

$A = TX_0 = 0.145$ mm

SAQ 19

Remembering that ϕ is the phase lag and $-\phi$ is the phase angle,

$$\tan\phi = \frac{2\zeta(\Omega/\omega)}{1-(\Omega/\omega)^2}$$

(a) $$\frac{\tan 40^\circ}{\tan 155^\circ} = \frac{\Omega_1}{\Omega_2} \times \frac{1-(\Omega_2/\omega)^2}{1-(\Omega_1/\omega)^2}$$

where $\Omega_1 = 50$ Hz, $\Omega_2 = 55$ Hz; $\quad \Omega_1/\Omega_2 = 0.9091$

Hence

$$-1.7995\,[1-(\Omega_1/\omega)^2] = 0.9091[1-(\Omega_2/\omega)^2]$$

and

$$\omega^2[0.9091 + 1.7995] = 1.7995\,(50)^2 + 0.9091\,(55)^2$$

$\therefore \omega = 51.732$ Hz

(b) $\Omega_1/\omega = 0.9665$

$$\therefore \zeta = \frac{1-(0.9665)^2}{2(0.9665)}\tan 40^\circ$$

$\zeta = 0.0286$

(c) $$M_1 = \frac{1}{\{[1-(\Omega_1/\omega)^2]^2 + 4\zeta^2(\Omega_1/\omega)^2\}^{1/2}}$$

$\Omega_1/\omega = 0.9665$

$\therefore M_1 = 11.628$

At resonance, $\Omega/\omega = 1$, so that

$M_R = 1/2\zeta = 17.483$

$$\text{Amplitude at resonance} = \frac{M_R}{M_1} \times 15 \text{ mm} = 22.6 \text{ mm}$$

SAQ 20

(a) From the graph, the 8-hour proficiency limit is an acceleration amplitude of 2 m s^{-2} at 40 Hz.

(b) $\Omega = 2\pi \times 40 = 251$ rad s^{-1}

$= 2/(251)^2 = 0.03$ mm

SAQ 21

$\Omega = 2\pi \times 8 = 50.3$ rad s^{-1}

(a) Comfort: peak $\ddot{x} = 0.18$ m s^{-2}, $A = 0.07$ mm

(b) Proficiency: peak $\ddot{x} = 0.42$ m s^{-2}, $A = 0.17$ mm

(c) Exposure: peak $\ddot{x} = 0.83$ m s^{-2}, $A = 0.33$ mm

SAQ 22

(a) $\Omega/\omega = 7.85, \quad \zeta = 0$

$T = 0.0165$

$P_0 = TF_0 = 3.3$ N

(b) $\omega = \sqrt{k/m} = 28.3 \text{ rad s}^{-1}$

$\Omega/\omega = 157/28.3 = 5.55$

$T = 0.05$

$P_0 = TF_0 = 10$ N

(c) $\omega = \sqrt{k/m} = 34.64 \text{ rad s}^{-1}$

$\Omega/\omega = 4.53$

$T = 0.07$

$P_0 = TF_0 = 14$ N

(d) $\Omega/\omega = 4.53, \quad \zeta = 0.5$

$T = 0.23$

$P_0 = TF_0 = 46$ N

SAQ 23

(a) $\omega = \sqrt{k/m} = 31.62 \text{ rad s}^{-1}$

($f = 5.03$ Hz)

Damping is negligible, so

$\omega_d = 31.6 \text{ rad s}^{-1}, \quad f_d = 5.03$ Hz

(b) $\Omega = \dfrac{3000}{60} \times 2\pi = 314.2 \text{ rad s}^{-1}$

$\Omega/\omega = 9.94$

(or $50/5.03 = 9.94$)

$T = 0.0102$

(c) $P_0 = TF_0 = 4.1$ N

SAQ 24

(a) $F_0 = 400$ N, $\quad P_0 = 10$ N maximum

$T = 10/400 = 0.025$ maximum

(b) With $\zeta = 0$ (negligible)

$\Omega/\omega = 6.403$ minimum

$\Omega = 314.2 \text{ rad s}^{-1}$

$\omega = 314.2/6.403 = 49.07 \text{ rad s}^{-1}$

(Remember here that strictly $T = -0.025$ if $\Omega/\omega > 1$.)

(c) $k = m\omega^2 = 48.2 \text{ kN m}^{-1}$

(d) With $\zeta = 0.1$

$\Omega/\omega = 9.21$ minimum

$\omega = 314.2/9.2 = 34.12 \text{ rad s}^{-1}$

(e) $k = m\omega^2 = 23.3 \text{ kN m}^{-1}$

SAQ 25

(a) $400 \times (750/3000)^2 = 25$ N

(b) $400 \times (2000/3000)^2 = 178$ N

(c) $400 \times (4000/3000)^2 = 711$ N

(d) $400 \times (6000/3000)^2 = 1600$ N

SAQ 26

(a) The resonance speed is approximately the undamped natural frequency

$\omega = \sqrt{k/m} = 70.71 \text{ rad s}^{-1} \quad (11.25 \text{ Hz})$

(b) $\zeta = 0.1$, so at resonance

$$T = \frac{\sqrt{1 + 4\zeta^2}}{2\zeta} = 5.1$$

(c) $P_0 = TF_0$

$F_0 = 400(70.71/314.2)^2 = 20.26$ N

$P_0 = 103.3$ N

SAQ 27

$A = MF_0/k$

$M = 1/2\zeta = 5$

$$A = \frac{5 \times 20.26}{100 \times 10^3} = 1.013 \times 10^{-3} \text{ m} \approx 1 \text{ mm}$$

SAQ 28

(a) $600 \text{ rev min}^{-1} = 10 \text{ rev s}^{-1}$

$\Omega = 2\pi \times 10 = 62.83 \text{ rad s}^{-1}$

We require $f = 0.4 \times 10 = 4$ Hz

and $\omega = 2\pi f = 25.13 \text{ rad s}^{-1}$

(b) $k = m\omega^2 = 12.63 \text{ kN m}^{-1}$ maximum

(c) At tick-over $\Omega/\omega = 2.5$ (1/0.4 by design)

$\zeta = 0.1$

$T = 0.21$

$F_0 = 400 \times (600/3000)^2 = 16$ N

$P_0 = TF_0 = 3.4$ N

(d) $6000 \text{ rev min}^{-1}$

$F_0 = 400 \times (6000/3000)^2 = 1600$ N

$\Omega = 2\pi \times 6000/60 = 628.3 \text{ rad s}^{-1}$

$\Omega/\omega = 628.3/25.13 = 25$

[Alternatively $\Omega/\omega = 100/4 = 25$]

$\zeta = 0.1$

$T = 0.0082$

$P_0 = TF_0 = 13$ N

SAQ 29

(a) $T = 60/2000 = 0.03$ maximum

$\Omega/\omega = 5.86; \quad \Omega = 900/60 = 15 \text{ rev s}^{-1}$

$\omega = \Omega/5.86 = 2\pi \times 15/5.86 = 16.08 \text{ rad s}^{-1}$

$k = m\omega^2 = 67.3 \text{ kN m}^{-1}$

[Again remember here that strictly $T = -0.03$.]

(b) $T = 0.03, \quad \zeta = 0.1$

$\Omega/\omega = 7.99$

$\omega = \Omega/7.99 = 11.8 \text{ rad s}^{-1}$

$k = m\omega^2 = 36.2 \text{ kN m}^{-1}$

(c) $P_0 = TF_0$

$c = 2000 \text{ N s m}^{-1}$

$\alpha = c/2m = 3.846 \text{ s}^{-1}$

$\zeta = \alpha/\omega = 3.846/11.8 = 0.326$

$\Omega/\omega = 7.99$

$T = 0.084$

$P_0 = 168 \text{ N}$

(The extra damping has increased the transmitted force amplitude to beyond the specified value.)

(d) Resonance speed is approximately the undamped natural frequency:

$\omega = 11.8 \text{ rad s}^{-1}, \quad f = 1.88 \text{ Hz}$

$F_0 = 2000 \times (1.88/15)^2 = 31.4 \text{ N}$

$A = MF_0/k$

For $\zeta = 0.326, \quad M = 1.53$ at resonance

$\therefore A = 1.33 \text{ mm}$

Phase lag at resonance is given by $\tan \phi = \infty$, so $\phi = 90°$.

Index to Block 8