Carl von Ossietzky University Oldenburg – Faculty V - Institute of Physics
Module Introductory Laboratory Course Physics – Part I

Forced Mechanical Oscillations

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HOOKE's law, harmonic oscillation, harmonic oscillator, eigenfrequency, damped harmonic oscillator, resonance, amplitude resonance, energy resonance, resonance curves.

Measuring program:
Measurement of the amplitude resonance curve and the phase curve for strong and weak damping.

References:

/1/ DEMTRÖDER, W.: „Experimentalphysik 1 – Mechanik und Wärme", Springer-Verlag, Berlin among others.
/2/ TIPLER, P.A.: „Physik", Spektrum Akademischer Verlag, Heidelberg among others.

1 Introduction

It is the object of this experiment to study the properties of a „harmonic oscillator" in a simple mechanical model. Such harmonic oscillators will be encountered again in different fields of physics, for example in electrodynamics (see experiment "*Electromagnetic resonant circuit*") and atomic physics. Therefore it is very important to understand this experiment, especially the importance of the amplitude resonance and phase curves.

2 Theory

2.1 Undamped Harmonic Oscillator

Let us observe a set-up according to Fig. 1, where a sphere of mass m_K is vertically suspended (x-direction) on a spring. Let us neglect the effects of friction for the moment. When the sphere is at rest, there is an equilibrium between the force of gravity, which points downwards, and the dragging resilience which points upwards; the centre of the sphere is then in the position $x = 0$. A deflection of the sphere from its equilibrium position by x causes a proportional dragging force F_R opposite to x:

(1) $$F_R \sim -x$$

The proportionality constant (*elastic* or *spring constant* or *directional quantity*) is denoted D, and Eq. (1) becomes the well-known HOOKE*'s law*[1]:

(2) $$F_R = -D\,x$$

Following deflection and release the dragging force causes an acceleration a of the sphere. According to *Newton's second law*

(3) $$F_R = m_K a$$

In combination with Eq. (2) we therefore obtain:

(4) $$m_K a = m_K \frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} = m_K\ \ddot{x} = -D\,x \qquad (t\text{: time})$$

the three terms on the left side merely representing different ways to write the relation force = mass × acceleration. Eq. (4) is the important differential equation (also called the *equation of motion*), by means of which all systems can be described which react with a dragging force on a deflection from their position of rest or equilibrium that is proportional to the degree of deflection. Such systems will be encountered very often in different fields of physics.
We are interested in learning which movement the sphere makes when it is deflected from its position at rest and then released, its initial velocity v at the moment of release being zero. So we look for the function

[1] ROBERT HOOKE (1635 – 1703)

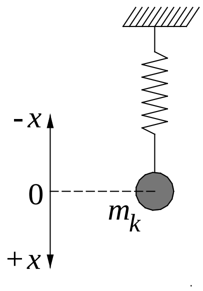

Fig. 1: Mass/spring system.

$x(t)$, which is a solution of the differential equation (4) under the condition $v(t = 0) = 0$. Note that apart from multiplicative factors, this function must be equal to its second time derivative. Hence, we attempt to solve the differential equation with a function $x(t)$, which describes a so-called *harmonic oscillation*:

(5) $$x(t) = x_0 \cos(\omega t + \varphi)$$

x_0 is the *amplitude*, $(\omega t + \varphi)$ the *phase*, φ the *initial phase* and ω the *angular eigenfrequency* of the oscillation (cf. Fig. 2). Inserting Eq. (5) into Eq. (4) and performing differentiation twice with respect to time t, we find:

(6) $$-m_K\, \omega^2\, x_0 \cos(\omega t + \varphi) = -D\, x_0 \cos(\omega t + \varphi)$$

From this follows the value of ω, for which Eq. (5) is a solution of Eq. (4):

(7) $$\omega = \sqrt{\frac{D}{m_K}} := \omega_0$$

Thus, the sphere performs oscillations with the angular eigenfrequency ω_0 when it is released. Since we assume that there is no friction, the amplitude x_0 of the oscillation remains constant. x_0 as well as the initial phase φ are free parameters which have to be chosen such that Eq. (5) is „adjusted“ to the process to be described, i.e. that Eq. (5) reflects the observed motion with the correct amplitude and initial phase.

Equation (7) is only valid if the mass of the spring, m_F , is negligible compared to the mass m_K of the sphere. If this is not true, we have to consider that the spring’s different elements of mass also oscillate following its deflection and release. The oscillation amplitudes of these elements of mass, however, are very different: They increase from zero at the point of suspension of the spring to a value x_0 at the end of the spring. An exact calculation[2] shows that the oscillation of the single elements of mass with different amplitudes equals the oscillation of one third of the whole spring mass with the amplitude x_0. Therefore, the correct equation for the angular eigenfrequency reads:

(8) $$\omega_0 = \sqrt{\frac{D}{m_K + \frac{1}{3} m_F}} := \sqrt{\frac{D}{m}} \quad \text{with } m := m_K + \frac{1}{3} m_F$$

In the experiment to be performed the sphere is not directly fixed to the spring but by a bar S_2, with an attached reflective plate R (Fig. 8). In that case, m_K in Eq. (8) has to be replaced by the total mass:

(9) $$m_G = m_K + m_S + m_R$$

m_S and m_R being the masses of S_2 and R.
An example illustrates the described relationships. According to Fig. 1 we observe a sphere of the mass $m_K = 0.11$ kg suspended by the bar and reflective plate ($m_S + m_R = 0.07$ kg) on a spring with the spring constant $D = 28$ kg/s^2 and the mass $m_F = 0.02$ kg. The sphere is deflected by $x_0 = 0.05$ m downwards from

[2] See for example ALONSO, M., FINN, E. J.: “Fundamental University Physics, Vol. 1: Mechanics“, Addison-Wesley Publishing Company, Reading (Mass.) among others.

its position at rest. Then we release the sphere and it performs oscillations with the amplitude x_0 and the eigenfrequency $f_0 = \omega_0/(2\pi) \approx 1.9$ Hz (Eq. (8)). If we start to record the motion $x(t)$ of the sphere exactly when it has achieved its maximum upward deflection, the cosine according to Eq. (5) „starts“ at an initial phase of $\varphi = \pi = 180°$ (mind the sign of x in Fig. 1!). This situation is represented in Fig. 2.

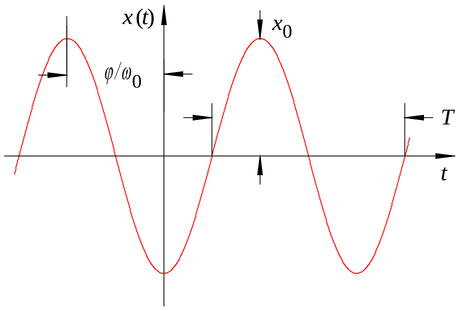

Fig. 2: Definition of the amplitude x_0, period length $T = 2\pi/\omega_0$ and initial phase φ of a harmonic oscillation. The phase φ must be divided by ω_0 for the presentation on the t-axis.

A system according to the arrangement considered here (also called *mass/spring system*) that performs harmonic oscillations is called a *harmonic oscillator*. The harmonic oscillator is characterized by a dragging force proportional to the deflection leading to a typical equation of motion in the form of (4) with a solution in the form of (5). Equally characteristic of the harmonic oscillator is the *parabolic* behaviour of its *potential energy* E_p as a function of the position (Fig. 3):

(10) $$E_p = \frac{1}{2} D\, x^2$$

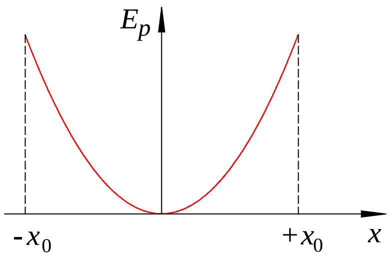

Fig. 3: Course of the potential energy E_p as a function of displacement x for the harmonic oscillator.

2.2 Damped Harmonic Oscillator

Now we observe the more realistic case of a mass/spring system under the influence of friction. We start from the simple case where, in addition to the restoring force $F_R = -Dx$, a frictional force F_b proportional to the velocity v is acting on the system. For F_b we can write:

(11) $$F_b = -bv = -b\frac{\mathrm{d}\,x}{\mathrm{d}\,t}$$

b being a *constant of friction*, which represents the magnitude of the friction.

Question 1:

- Which unit does b have? Why is there a minus sign in Eq. (11)?

In this case the equation of motion (4) takes on the form:

(12) $$m\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} = -D\,x - b\frac{\mathrm{d}\,x}{\mathrm{d}\,t}$$

Usually, this differential equation is written in the form:

(13) $$\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} + \frac{b}{m}\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \frac{D}{m}x = 0$$

Here again, it is interesting to know what type of motion the sphere performs after being deflected *once* from its position at rest and then released with an initial velocity of zero. Thus, we are once again searching for the function $x(t)$ which resolves the differential equation (13) under the condition $v(t = 0) = 0$. As a consequence of damping, we expect a decreasing amplitude of the oscillation and therefore try a solution with an exponentially decreasing amplitude (cf. Fig. 4):

(14) $$x = x_0\, e^{-\alpha t} \cos(\omega t + \varphi)$$ (α : *damping constant*)

We insert Eq. (14) into Eq. (13), perform the differentiations, and find that Eq. (14) represents a solution of Eq. (13) if the following is true for the parameters α and ω:

(15) $$\alpha = \frac{b}{2m}$$ and

(16) $$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

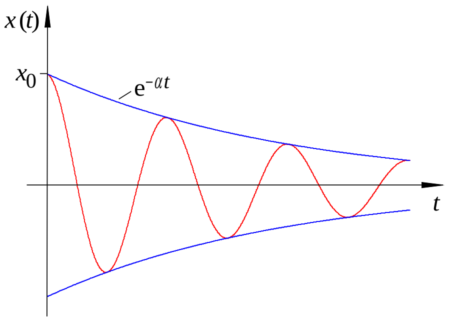

Fig. 4: Damped harmonic oscillation.

We will now interpret this result. First we note that the amplitude of the oscillation decreases more rapidly the larger the *damping constant* (the *damping coefficient*) α is. In the case of invariable mass this means according to Eq. (15) that the amplitude of the oscillation decreases more rapidly the larger the constant of friction b is - which is plausible.
From Eq. (16) we can read how the angular frequency ω of this damped harmonic oscillation changes with the constant of friction b. We study the following different cases:

(i) $b = 0$ $\rightarrow$ $\omega = \omega_0$

In the case of vanishing friction ($b = 0$) we have the case of the undamped harmonic oscillator as discussed in Chapter 2.1.The sphere performs a *periodic oscillation* at the angular eigenfrequency ω_0.

(ii) $(b/(2m))^2 = \omega_0^2$ $\rightarrow$ $\omega = 0$

This is the case of „critical damping“ in which the sphere does not perform a periodic oscillation any more. It is therefore called the case of critical damping. The sphere only returns to its starting position exponentially (cf. remarks).

(iii) $(b/(2m))^2 > \omega_0^2$ $\rightarrow$ ω imaginary

In the case of „supercritical damping” there is no periodic oscillation either. This case is called *aperiodical case* or *over damped case*. Here again, the sphere only returns to its starting position, however, with additional damping, i.e., more slowly (cf. remarks).

(iv) $0 < b < 2m\omega_0$ $\rightarrow$ $\omega < \omega_0$

This most general case, the *oscillation case,* leads to a periodic oscillation at a angular frequency ω (eq. (16)), which is slightly lower than the angular eigenfrequency ω_0 of the undamped harmonic oscillator.

Remarks:

Under the conditions discussed above ($v(t = 0) = 0$) there is no considerable difference between the case of *critical damping* and *supercritical damping*: In both cases the sphere returns to its starting position along an exponential path; in the case of supercritical damping there is only a stronger damping. We find a different situation in the case $v(t = 0) \neq 0$. If we do not only release the sphere, but push it thus giving it a certain starting velocity, it is possible in the case of *critical damping* that the sphere oscillates beyond its position at rest once, and only then returns to its starting position along an exponential path. In the case of *supercritical damping* such an oscillation beyond that position does not occur. The sphere always returns to its position at rest along an exponential path. Detailed calculations (solution of the differential equation (13) under the conditions (ii) and (iii)) confirm these relationships.

2.3 Forced Harmonic Oscillations

In Chapters 2.1 and 2.2 we have observed how the sphere oscillates if we deflect it *once* from its position at rest and then release it. Now we will investigate which oscillations the sphere performs if the system is subject to a *periodically* changing external force F_e (Fig. 5), for which the following is true:

(17) $$F_e = F_1 \sin\left(\omega_1 t\right)$$

F_1 is the amplitude of the external force and ω_1 its angular frequency. The sign is chosen such that the forces directed downwards are counted as positive and upward forces are counted as negative.

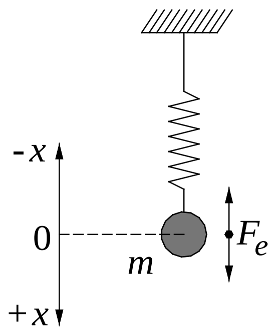

Fig. 5: Oscillation generation of a mass/spring system with an external force F_e, m being the mass according to Eqs. (8) and (9).

The external force F_e additionally acts on the spring. The equation of motion thus takes the form (cf. Eqs. (12) and (13)):

(18) $$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -D\,x - b\frac{\mathrm{d}x}{\mathrm{d}t} + F_e$$

and hence

(19) $$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{D}{m}x = \frac{1}{m}F_1 \sin\left(\omega_1 t\right)$$

It is expected that the motion of the sphere following a certain *transient time*, i.e., after the end of the *transient motion*, occurs at the same frequency as does the change of the external force. There would be no plausible explanation for another frequency. However, a *phase shift* ϕ between the stimulating force and the deflection of the sphere could be assumed. We may expect the oscillation amplitude to remain constant upon completion of the transient motion since the system is provided with new external energy again and again. Based on these considerations the following ansatz is suggested for the differential equation (19):

(20) $$x = x_0 \sin\left(\omega_1 t + \phi\right)$$

In this case ϕ is the phase shift between the deflection $x(t)$ and the external force F_e. For $\phi<0$ the deflection lags behind the stimulating force. By inserting Eq. (20) into Eq. (19) we find that Eq. (20) represents a solution of Eq. (19) if the following is true for the amplitude x_0 and the phase shift ϕ (derivation cf. Appendix chapter 4):

(21) $$x_0 = \frac{\frac{F_1}{m}}{\sqrt{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)^2 + \left(\frac{\omega_1 b}{m}\right)^2}}$$

(22) $$\phi = \arctan\left(\frac{\omega_0^2 - \omega_1^2}{\frac{\omega_1 b}{m}}\right) - \frac{\pi}{2}$$

Contrary to the cases discussed in Chapters 2.1 and 2.2, the amplitude x_0 and the phase ϕ are no longer freely selectable parameters, rather they are definitely determined by the quantities F_1, ω_1, m, b and $\omega_0^2 = D/m$.

Eq. (21) shows that the amplitude of the sphere's oscillation, the so called *resonance amplitude*, depends on the frequency of the stimulating force. Plotting x_0 over ω_1, we obtain the *amplitude resonance curve*. Fig. 6 (top) shows some typical amplitude resonance curves for different values of the friction constant b. In the stationary case, i.e. for $\omega_1 = 0$, we obtain the amplitude known from HOOKE's law from Eq.(21):

(23) $$x_0\left(\omega_1 = 0\right) := x_{00} = \frac{F_1}{D}$$

This is the value by which the sphere is deflected if it is affected by a constant force F_1.Substituting F_1 from Eq. (23) into Eq. (21), one obtains for the resonance amplitude x_0:

(24) $$x_0 = \frac{x_{00}\, D}{m\sqrt{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)^2 + \left(\frac{\omega_1 b}{m}\right)^2}}$$

The position of the maximum of x_0 as a function of ω_1 is found by means of the condition $dx_0/d\omega_1 = 0$. From Eq. (24) follows:

(25) $$x_0 = x_{0,\max} \qquad \text{for} \qquad \omega_1 = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}}$$

Except for the case $b = 0$, the maximum of the amplitude resonance curve is thus *not* found at the angular eigenfrequencies ω_0, but at a slightly lower angular frequencies $\omega_1 < \omega_0$.

The lower part of Fig. 6 shows the so called *phase curves* which determine the development of the phase shift ϕ as a function of the angular frequency ω_1. From Eq. (22) it follows that ϕ is always *negative*, i.e., the deflection of the sphere always *lags behind* the stimulating force except for the case $\omega_1 = 0$.

We will now discuss some special cases:

(i) In the case $\omega_1 << \omega_0$ the amplitude $x_0 \approx F_1/D$ is independent of b for „not too large“ b. The amplitude resonance curve is nearly horizontal for small excitation frequencies and the phase shift ϕ tends to 0: $\phi \approx 0°$. Thus the motion of the sphere almost directly follows the stimulating force.

(ii) In the *resonance case* (ω_1 according to Eq.(25)), the amplitude is maximal and given by

$$x_{0,\max} = \frac{F_1}{b\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}}$$

The smaller b is, the larger $x_{0,max}$ becomes; for $b \to 0$, $x_{0,max} \to \infty$. In this case the sphere's deflection lags behind the generating force by 90° ($\phi = -\pi/2$).

(iii) In the case $\omega_1 >> \omega_0$, $x_0 \approx F_1/(m\omega_1^2)$, i.e., the amplitude drops by $1/\omega_1^2$. The phase shift is $\phi = -\pi$ in this case, i.e., the sphere's deflection lags behind the generating force by 180°.

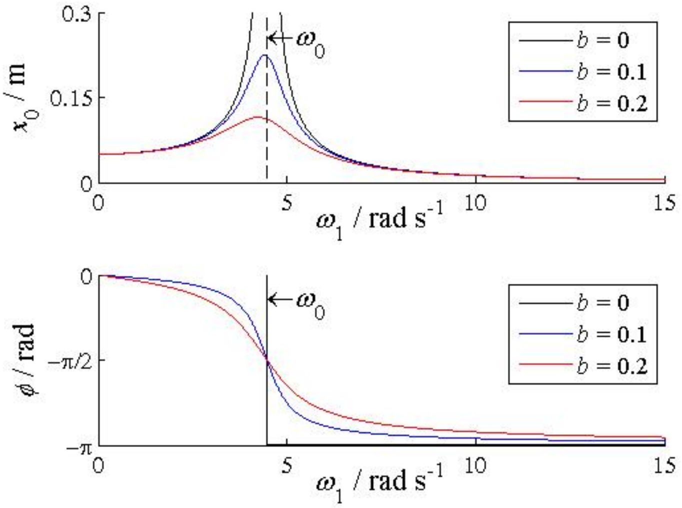

Fig. 6: Amplitude resonance curves (top) and phase curves (bottom) for a damped harmonic oscillator. (F_1 = 0.1 N, m = 0.1 kg, D = 2 kg/s^2, b in kg/s).

From the amplitude resonance curves and the special cases discussed in (i) - (iii) the *damping behaviour* of a *mass-spring-system* can be read, i.e. of a vibration isolating table, which is frequently used in optical precision metrology. The eigenfrequencies of such tables are in the range of about 1 Hz. If an external disturbance (e.g. building oscillation) has a very low frequency ($\omega_1 \to 0$), the amplitude of the perturbation is transferred onto the table without damping. Close to the angular eigenfrequency ($\omega_1 \approx \omega_0$) it is (unintentionally) amplified, whereas in the range of high frequencies ($\omega_1 >> \omega_0$) it is damped strongly.

The damping behaviour of such a system can be influenced by changing the mass m. Fig. 7 shows that a larger m reduces the angular eigenfrequency with the other parameters remaining unchanged and that the damping for frequencies above the angular eigenfrequency can be increased significantly. Thus, oscillation dampening tables often have large masses in the range of 10^3 kg.

Finally we will examine at which frequency the maximal *energy transfer* occurs from the generating system to the oscillating system. As we know that the maximal kinetic energy is equivalent to the maximum velocity, we first calculate the temporal course of the velocity v of the sphere using Eq. (20):

(26) $$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \omega_1 x_0 \cos\left(\omega_1 t + \phi\right) := v_0 \cos\left(\omega_1 t + \phi\right)$$

With Eq. (24) we thus obtain for the velocity v_0:

(27) $$v_0 = \omega_1 x_0 = \frac{\omega_1\, x_{00}\, D}{m\sqrt{\left(\omega_1^2 - \omega_0^2\right)^2 + \left(\frac{\omega_1 b}{m}\right)^2}}$$

and hence:

(28) $$v_0 = \frac{x_{00}\, D}{\sqrt{\left(m\omega_1 - \frac{D}{\omega_1}\right)^2 + b^2}}$$

v_0 becomes maximal when the denominator of Eq. (28) becomes minimal, i.e., if the following is true (for $b \neq 0$):

(29) $$m\omega_1 - \frac{D}{\omega_1} = 0 \quad \rightarrow \quad v_0 = v_{0,\max}$$

Hence it follows:

(30) $$\omega_1 = \sqrt{\frac{D}{m}} = \omega_0 \quad \rightarrow \quad v_0 = v_{0,\max}$$

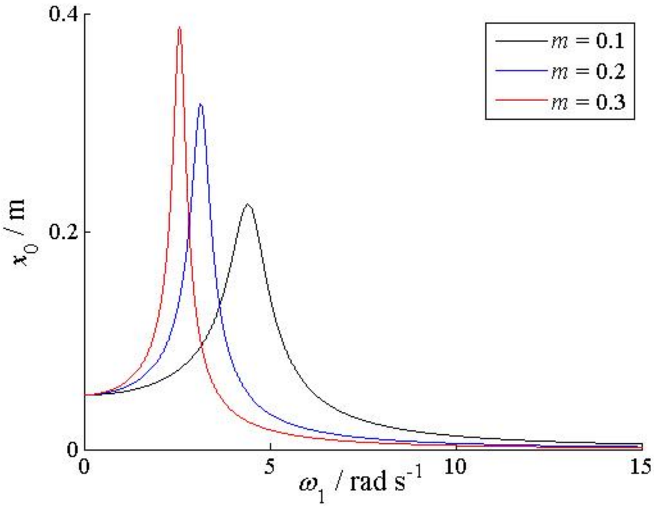

Fig. 7: Amplitude resonance curves for different masses m (in kg) with other parameters remaining unchanged (F = 0.1 N, D = 2 kg/s^2, b = 0.1 kg/s).

Thus the velocity and also the kinetic energy become maximal (in contrast to the resonance amplitude!) if the system is stimulated with its angular eigenfrequency ω_0. Therefore, this case is called *energy resonance*, a case in which the generating system can transfer the maximal energy to the oscillating system.

3 Experimental Procedure

Equipment:

Spring (D = (22.7 ± 0.5) kg/s^2, m_F = (0.0575 ±10^{-4}) kg), sphere on suspension bar with reflective plate (m_G needs to be weighed), excitation system on stand with motor and light barrier, electronic speed controller for motor, laser distance sensor (type BAUMER OADM 12U6460/S35, measuring range (16 – 120) mm), power supplies (PHYWE (0 – 15 / 0 – 30) V) for motor, light barrier and laser distance sensor, 2 glasses with different glycerine/water mixtures ($b \approx$ 0,7 kg/s for the more viscous mixture at T = 20 °C), desk for the glasses, digital oscilloscope TEKTRONIX TDS 1012 / 1012B / 2012C / TBS 1102B.

3.1 Description of Experimental Set-Up

The experiments are performed in a set-up according to Fig. 8. This allows for contact-free measurement of the *amplitude resonance curves* and *phase curves*. This set-up is described in the following, before presenting the actual measuring tasks in Chap. 3.2:
A sphere K of mass m_K is suspended on a spring by means of a bar S_2. The sphere is plunged into a glass B filled with a glycerine/water mixture to damp its oscillation. A reflective plate R is fixed on the bar. A laser beam from the laser distance sensor LDS (the operating principle was detailed in the experiment "*Sensors*...") is incident on the reflective plate. The sensor output is a voltage signal $U_{\mathrm{LDS}}(t)$, which varies linearly with the distance s between LDS and R.
The spring is connected to a piston rod P via a joint G_1 with a second bar S_1 which runs in a guide F. The piston rod P is fixed on a rotary disk D via a joint G_2. The disk can be rotated at an angular frequency ω_1 via a motor. Thus, the suspension point of the spring is set in a periodic vertical motion and a periodic driving force $F_e(t)$ is exerted on the spring. After the end of the transient motion, the sphere, together with S_2 and R, will also show a periodic vertical motion with amplitude x_0. This causes the laser distance sensor to produce a periodic voltage signal $U_{\mathrm{LDS}}(t)$ with an amplitude of $U_0 \sim x_0$ and an offset U_{DC} which depends on the distance s between LDS and R in the rest position of the sphere. The period T of U_{LDS} is given by

(31) $$T = \frac{2\pi}{\omega_1}$$

Thus the amplitude resonance curve $U_0(\omega_1)$ can be measured by varying ω_1. Using the calibration factor k of the laser distance sensor for voltage *differences*

(32) $$k = 0{,}0962 \text{ V/mm}$$

the *amplitude resonance curve* $x_0(\omega_1)$ can be determined. k may be taken as an error free quantity.

The measurement of the *phase curve*, i.e. the *phase shift* ϕ between the driving force $F_e(t)$ and the vertical displacement $x(t)$ of the sphere as a function of the angular frequency ω_1 can be carried out as follows:

With the aid of a marker M and the light barrier LS, which is interrupted by M, an electric pulse $U_{LS}(t)$ is generated every time the *suspension point of the spring* reaches its highest position (time t_1 in Fig. 9).
At this time, the *driving force* $F_e(t) = m d^2x/dt^2$ is at its minimum (keep in mind the sign according to Fig. 5). At a *later* time t_2, the *sphere* (not the suspension point of the spring!) reaches its highest position and thus the deflection $x(t)$ its minimum ($-x_0$; here too keep in mind the sign according to Fig. 5). In this position, the distance s between LDS and R and thus also $U_{LDS}(t)$ is minimal. The phase shift ϕ between $F_e(t)$ and $x(t)$ is then given by (cf. Fig. 9):

(33) $$\phi = -\frac{t_2 - t_1}{T} 2\pi := -\frac{\Delta t}{T} 2\pi = -\Delta t\, \omega_1$$

Therefore by variation of ω_1, the phase curve $\phi(\omega_1)$ can be measured.

In practice, the amplitude $U_0(\omega_1)$ and time difference $\Delta t(\omega_1)$ are measured simultaneously for each angular frequency ω_1 with the aid of an oscilloscope.
Finally one remark on the temporal course of the driving force $F_e(t)$: Except a constant phase shift, it corresponds to the temporal course of the vertical motion of the join G_1, i.e. the suspension point of the spring. This motion is described by the quantity $y(t)$ (cf. Fig. 10).

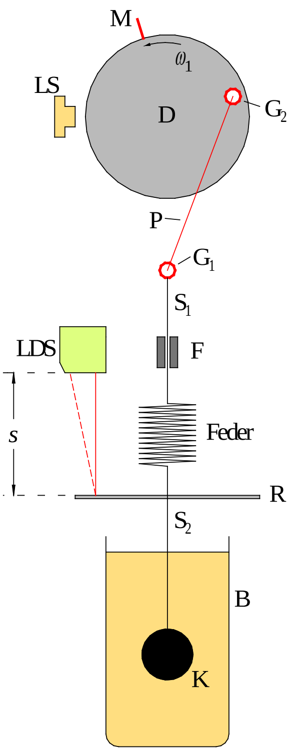

Fig. 8: Sketch of the experimental set-up.

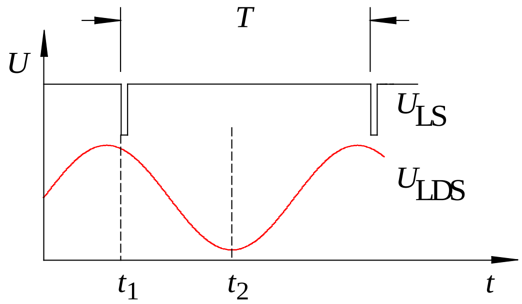

Fig. 9: Temporal course of the output voltages of the light barrier LS (U_{LS}) and the laser distance sensor LDS (U_{LDS}).Time t_1: suspension point of the spring at highest position, driving force $F_e(t)$ minimal. Time t_2: Sphere at highest position, $x(t)$ and U_{LDS} minimal.

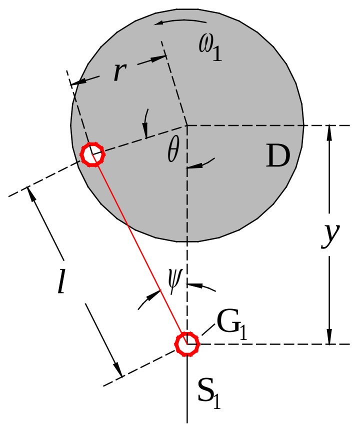

Fig. 10: Definition of quantities for calculating the movement of the join G_1 (cf. Fig. 8).

If the piston rod is mounted on the disk at a distance r from the axis of rotation, we obtain:

(34) $$y = r\cos\theta + l\cos\psi$$

and

(35) $$r\sin\theta = l\sin\psi \quad \rightarrow \quad \sin\psi = \frac{r}{l}\sin\theta$$

With

(36) $$\cos\psi = \sqrt{1-\sin^2\psi} = \sqrt{1-\frac{r^2}{l^2}\sin^2\theta}$$

and

(37) $$\theta = \omega_1 t$$

we finally obtain:

(38) $$y = r\cos\left(\omega_1 t\right) + \sqrt{l^2 - r^2\sin^2\left(\omega_1 t\right)}$$

The purely harmonic motion (r cos($\omega_1 t$)) is thus superimposed by a disturbance (square root term in Eq. (38)) which, unfortunately, is also time-dependent and therefore makes the motion anharmonic. Therefore, the driving force $F_e(t)$ is not completely harmonic either. If we choose $l >> r$, however, $l^2 >> r^2\sin^2(\omega_1 t)$ and hence $\sqrt{(...)} \approx l$. Instead of a time-dependent disturbance we then have to deal with a merely additive constant l which no longer disturbs the „harmony“.

3.2 Amplitude Resonance Curve and Phase Curve for Strong and Weak Damping

Using the setup according to Fig. 8, for a sphere with suspension bar S and reflective plate Rand a spring with known D and m_F (for data see *Equipment*) the amplitude resonance curve $x_0(\omega_1)$ and the phase curve $\phi(\omega_1)$ within the frequency range $f_1 = \omega_1/2\pi$ between 0 Hz and approx. 5 Hz are to be measured for two different dampings (glasses containing different glycerine/water mixtures).

The piston rod P of the excitation system is fixed to the disk in the second hole from the centre. The anharmonic disturbance according to Eq. (38) can be neglected in this case.
The output signals of the light barrier (U_{LS}) and the laser distance sensor (U_{LDS}) are displayed on a digital oscilloscope, which is triggered by the signal U_{LS}. The period time T of U_{LS} and the peak-peak value ($U_{SS} = 2\,U_0$) of U_{LDS} are determined by using the oscilloscope's MESSUNG / MEASURE function. From these quantities, the angular frequency $\omega_1$and the amplitudes U_0 and, respectively x_0 can be determined.

The time difference $\Delta t = t_2 - t_1$, from which the phase shift ϕ can be calculated according to Eq. (33) is measured by using the TIME-CURSOR (cf. Fig. 9).

Hint:

In order to achieve a mostly uniform motion of the disk, the disk must always be rotated *counter clockwise*. For the same reason, an electronic speed controller (operating voltage 12 V) must be used for adjusting the revolution number of the motor within the frequency range between 0 Hz and approx. 1.5 Hz, which is mounted between the power supply and the motor. For frequencies exceeding 1.5 Hz the motor can be directly connected with the power supply and the number of revolutions can be controlled via the operating voltage (increase voltage slowly from 0 V to max. 12 V).

For both glycerine/water mixtures , the amplitude $U_0(\omega_1)$ of the sphere motion, the period duration T, and the time difference Δt are measured for as many different values of ω_1 as possible (at least 20), especially near the eigenfrequency. The measurements are performed *after the end of the transient motion.*
For the case $\omega_1 \to 0$, the amplitude U_0 is determined by manually turning the axis of the motor (while the motor is switched off) to the positions “piston rod up”, “piston rod down“ and measuring the corresponding voltages U_{LDS}.
We plot x_0 over ω_1 for both mixtures in *one* diagram, and ϕ over ω_1 likewise in *one* diagram. The maximum errors of x_0 and ϕ are also entered in the form of error bars (estimate errors from the fluctuations of the measurements for U_{SS} and T at the oscilloscope). Then freehand regression curves are drawn through the measured values and their course is compared with the theoretical expectations.

Remarks:

In the vicinity of the angular eigenfrequency the measurement under weak damping may become difficult, because the amplitudes may be large and the spring (possibly even the mount) may get into uncontrollable motion or the sphere may even hit the bottom of the glass. In that case the spring system must be damped manually and rapidly proceeded to the next frequency value.

4 Appendix: Calculation of the Resonance Amplitude and the Phase Shift

We want to demonstrate that the resonance amplitude x_0 and the phase shift ϕ can be calculated with a few simple calculation steps, if we change over to complex representation. In complex representation Eq. (19) reads:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} + \frac{b}{m}\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \frac{D}{m}x = \frac{1}{m}F_1\,\mathrm{e}^{i\,\omega_1 t} \tag{39}$$

In analogy to Eq. (20) we choose as a complex approach:

$$x = x_0\,e^{i(\omega_1 t + \phi)} = x_0\,\mathrm{e}^{i\,\omega_1 t}\,\mathrm{e}^{i\,\phi} \tag{40}$$

Following differentiation and division by $e^{i\,\omega_1 t}$ insertion of Eq. (40) into Eq. (39) yields:

$$-\omega_1^2\,x_0\,\mathrm{e}^{i\,\phi} + i\,\omega_1\frac{b}{m}x_0\,\mathrm{e}^{i\,\phi} + \frac{D}{m}x_0\,\mathrm{e}^{i\,\phi} = \frac{F_1}{m} \tag{41}$$

Hence it follows with the definition of the angular eigenfrequency ω_0 according to Eq. (8):

$$(42) \qquad x_0\, e^{i\phi} = \frac{\frac{F_1}{m}}{\omega_0^2 - \omega_1^2 + i\omega_1 \frac{b}{m}} := z$$

As already demonstrated in the experiment "*measurement of capacities*...", Eq. (42) is *one* representation form of a complex number z, whose absolute value (modulus) $|z| = x_0$ is given by $\sqrt{zz^*}$, with z^* being the conjugate complex quantity of z. Hence it follows:

$$(43) \qquad x_0 = \sqrt{zz^*} = \sqrt{\left(\frac{\frac{F_1}{m}}{\omega_0^2 - \omega_1^2 + i\omega_1 \frac{b}{m}}\right)\left(\frac{\frac{F_1}{m}}{\omega_0^2 - \omega_1^2 - i\omega_1 \frac{b}{m}}\right)}$$

from which we obtain Eq. (21) by simple multiplication.

For calculating the phase angle we again use (cf. experiment "*measurement of capacities*...") the *second* representation of complex numbers, namely $z = \alpha + i\beta$, α being the real part and β the imaginary part of z. As is generally known, the phase angle ϕ can be calculated from these quantities as

$$(44) \qquad \phi = \arctan\left(\frac{\beta}{\alpha}\right)\begin{Bmatrix} +\pi & \text{for } \alpha < 0 \text{ and } \beta \geq 0 \\ -\pi & \text{for } \alpha < 0 \text{ and } \beta < 0 \end{Bmatrix}$$

In order to convert Eq. (42) into the form $\alpha + i\beta$, we extend the fraction in Eq. (42) with the conjugated complex denominator:

$$(45) \qquad x_0\, e^{i\phi} = \frac{\frac{F_1}{m}\left(\omega_0^2 - \omega_1^2 - i\omega_1 \frac{b}{m}\right)}{\left(\omega_0^2 - \omega_1^2 + i\omega_1 \frac{b}{m}\right)\left(\omega_0^2 - \omega_1^2 - i\omega_1 \frac{b}{m}\right)} = \frac{\frac{F_1}{m}\left(\omega_0^2 - \omega_1^2\right) - i\frac{F_1}{m}\omega_1 \frac{b}{m}}{\left(\omega_0^2 - \omega_1^2\right)^2 + \left(\frac{\omega_1 b}{m}\right)^2}$$

from which we can read off the quantities α and β:

$$(46) \qquad \alpha = \frac{\frac{F_1}{m}\left(\omega_0^2 - \omega_1^2\right)}{\left(\omega_0^2 - \omega_1^2\right)^2 + \left(\frac{\omega_1 b}{m}\right)^2} \qquad \text{and} \qquad \beta = -\frac{\frac{F_1}{m}\omega_1 \frac{b}{m}}{\left(\omega_0^2 - \omega_1^2\right)^2 + \left(\frac{\omega_1 b}{m}\right)^2}$$

which yields by insertion into Eq. (44):

$$(47) \qquad \phi = \arctan\left(-\frac{\frac{\omega_1 b}{m}}{\omega_0^2 - \omega_1^2}\right)\left\{-\pi \quad \Leftrightarrow \quad \omega_1 > \omega_0\right\}$$

With

$$(48) \qquad \arctan(-y) = \arctan\left(\frac{1}{y}\right) - \frac{\pi}{2}$$

it finally yields Eq. (22).