Problem 1

Find the finite simple continued fraction $[a_0; a_1, \ldots, a_n]$ for the following rational numbers.

1. $\frac{4}{9}$
2. $\frac{372}{1001}$
3. $\frac{129}{51}$

1.

$$\begin{aligned}
\frac{4}{9} &= 0 + \frac{1}{\frac{9}{4}} \\
&= 0 + \frac{1}{2 + \frac{1}{\frac{4}{1}}} \\
&= 0 + \frac{1}{2 + \frac{1}{4}} \\
&= [0; 2, 4]
\end{aligned}$$

2.

$$\begin{aligned}
\frac{372}{1001} &= 0 + \frac{1}{\frac{1001}{372}} \\
&= 0 + \frac{1}{2 + \frac{372}{257}} \\
&= 0 + \frac{1}{2 + \frac{1}{1 + \frac{257}{115}}} \\
&= 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{115}{27}}}} \\
&= 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{4 + \frac{27}{7}}}}} \\
&= 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{4 + \frac{1}{3 + \frac{7}{6}}}}}} \\
&= 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{4 + \frac{1}{3 + \frac{1}{1 + \frac{1}{6}}}}}}} \\
&= [0; 2, 1, 2, 4, 3, 1, 6]
\end{aligned}$$

3.

$$\begin{aligned}
\frac{129}{51} &= 2 + \frac{27}{51} \\
&= 2 + \frac{1}{\frac{51}{27}} \\
&= 2 + \frac{1}{1 + \frac{24}{27}} \\
&= 2 + \frac{1}{1 + \frac{1}{1+\frac{3}{24}}} \\
&= 2 + \frac{1}{1 + \frac{1}{1+\frac{1}{\frac{24}{3}}}} \\
&= 2 + \frac{1}{1 + \frac{1}{1+\frac{1}{8+\frac{0}{3}}}} \\
&= 2 + \frac{1}{1 + \frac{1}{1+\frac{1}{8}}} \\
&= [2; 1, 1, 8]
\end{aligned}$$

Problem 2

For each finite simplified continued fraction, find the p_k, the q_k, and the convergents C_k.

1. $[2; 1, 5, 7, 3]$
2. $[2; 1, 1, 8]$

1.

$$\begin{aligned}
p_0 &= 2 & q_0 &= 1 & C_0 &= \frac{2}{1} = 2 \\
p_1 &= 1 \cdot 2 + 1 = 3 & q_1 &= 1 & C_1 &= \frac{3}{1} = 3 \\
p_2 &= 5 \cdot 3 + 2 = 17 & q_2 &= 5 \cdot 1 + 1 = 6 & C_2 &= \frac{17}{6} \\
p_3 &= 7 \cdot 17 + 3 = 122 & q_3 &= 7 \cdot 6 + 1 = 43 & C_3 &= \frac{122}{43} \\
p_4 &= 3 \cdot 122 + 17 = 383 & q_4 &= 3 \cdot 43 + 6 = 135 & C_4 &= \frac{383}{135}
\end{aligned}$$

2.

$$\begin{aligned}
p_0 &= 2 & q_0 &= 1 & C_0 &= \frac{2}{1} = 2 \\
p_1 &= 1 \cdot 2 + 1 = 3 & q_1 &= 1 & C_1 &= \frac{3}{1} = 3 \\
p_2 &= 1 \cdot 3 + 2 = 5 & q_2 &= 1 \cdot 1 + 1 = 2 & C_2 &= \frac{5}{2} \\
p_3 &= 8 \cdot 5 + 3 = 43 & q_3 &= 8 \cdot 2 + 1 = 17 & C_3 &= \frac{43}{17}
\end{aligned}$$

Problem 3

Explain why 2b should have given 1c, but did not.

Since the continued fractions are the same, we would have expected C_3 to be the same. However, this does not happen, because $\frac{129}{51}$ is not simplified. We see that in the final step that $\frac{3}{24} = \frac{1}{8+\frac{0}{3}}$. This last bit, $\frac{0}{3}$, "encodes" some information about the original rational, which is lost when we simplify it to 0. This simplification results in C_3 being a different rational.

Problem 4

Given a simple continued fraction $[a_0; a_1, \ldots, a_n]$, prove the two relations:

$$\frac{p_k}{p_{k-1}} = [a_k; a_{k-1}, \ldots, a_0] \tag{1}$$
$$\frac{q_k}{q_{k-1}} = [a_k; a_{k-1}, \ldots, a_1] \tag{2}$$

Relation 1:
We can prove this with induction. The base case is $k = 1$.

$$\begin{aligned}\frac{p_1}{p_0} &= \frac{a_1 a_0 + 1}{a_0} \\ &= a_1 + \frac{1}{a_0} \\ &= [a_1; a_0]\end{aligned}$$

Thus, the base case holds. Now, let's assume that for $k = j$ such that $j \in \mathbb{N}$, the relation holds. Now for $k = j + 1$:

$$\begin{aligned}\frac{p_{j+2}}{p_{j+1}} &= \frac{a_{j+2}p_{j+1} + p_j}{p_{j+1}} \\ &= a_{j+2} + \frac{p_j}{p_{j+1}} \\ &= a_{j+2} + \frac{1}{\frac{p_{j+1}}{p_j}} \\ &= a_{j+2} + \frac{1}{[a_{j+1}; a_j, \ldots, a_0]} \\ &= [a_{j+2}; a_{j+1}, \ldots, a_0]\end{aligned}$$

Thus, since the inductive step holds, relation 1 holds for all $k \in \mathbb{N}$.
Relation 2:
We can prove this with induction. The base case is $k = 1$.

$$\begin{aligned}\frac{q_1}{q_0} &= \frac{a_1}{1} \\ &= a_1 \\ &= [a_1]\end{aligned}$$

Thus, the base case holds. Now, let's assume that for $k = j$ such that $j \in \mathbb{N}$, the relation holds. Now for $k = j + 1$:

$$\begin{aligned}\frac{q_{j+2}}{q_{j+1}} &= \frac{a_{j+2}q_{j+1} + q_j}{q_{j+1}} \\ &= a_{j+2} + \frac{q_j}{q_{j+1}} \\ &= a_{j+2} + \frac{1}{\frac{q_{j+1}}{q_j}} \\ &= a_{j+2} + \frac{1}{[a_{j+1}; a_j, \ldots, a_1]} \\ &= [a_{j+2}; a_{j+1}, \ldots, a_1]\end{aligned}$$

Thus, since the inductive step holds, relation 2 holds for all $k \in \mathbb{N}$.