1. **a**. $\forall x \exists y\ P(x,y)$

b. $\exists x \exists y\ \neg P(x,y)$

c. $\exists! x\ P(x, Alaska)$

d. $\exists x \forall y\ (\neg(P(x,y) \oplus y = Illinois)$

2. **a**. $\neg(\forall x \forall y\ (x+y=y+x)) \Rightarrow \exists x \exists y\ (x+y \neq y+x)$

b. $\neg(\forall x \exists y\ ((x \neq y) \wedge (x+y=3)) \Rightarrow \exists x \forall y\ ((x=y) \vee (x+y \neq 3)$

3. Let $x \in \mathbb{Z}$.

$$(x+1)^2 - x^2$$
$$x^2 + 2x + 1 - x^2$$
$$2x + 1$$

Remember that every odd number can be written in the form $2k+1$ where $k \in \mathbb{Z}$. Since the difference of the squares of some x and some $x+1$ can be written in this form, it is odd. Therefore, since x can be substituted for any integer, we can get any and all odd integers from the difference of two squares.

4. Let $x_1, x_2, ..., x_n \in \mathbb{R}$. We can prove with contradiction by assuming the opposite where all x_n are less than the average.

$$x_n < \frac{\sum_{k=1}^{n} x_n}{n}$$

We can then add all the terms of x_n together. This gives us

$$x_1 + x_2 + ... + x_n < \frac{\sum_{k=1}^{n} x_n}{n} + \frac{\sum_{k=1}^{n} x_n}{n} + ... + \frac{\sum_{k=1}^{n} x_n}{n}$$

$$\sum_{k=1}^{n} x_n < \sum_{k=1}^{n} x_n$$

Obviously, a number can not be less than itself, proving that, by contradiction, at least one of the real numbers $x_1, x_2, ..., x_n$ is greater than or equal to the average of all x_n.

5. **a**. Let $a, n, j, k \in \mathbb{Z}$. Since $a|(8n+7)$ and $a|(4n+1)$, we can rewrite in the form:

$$8n + 7 = aj \qquad 4n + 1 = ak$$
$$\frac{aj-7}{8} = n \qquad \frac{ak-1}{4} = n$$

$$\frac{aj-7}{8} = \frac{ak-1}{4}$$
$$\frac{aj-7}{8} * 8 = \frac{ak-1}{4} * 8$$
$$aj - 7 = 2ak - 2$$
$$aj - 2ak = 5$$
$$a(j-2k) = 5$$

Since $j - 2k$ is an integer, this means that a divides 5.

b. Let $a, n, j, k \in \mathbb{Z}$. Since $a|(9n+5)$ and $a|(6n+1)$, we can rewrite in the form:

$$9n + 5 = aj \qquad 6n + 1 = ak$$
$$\frac{aj-5}{9} = n \qquad \frac{ak-1}{6} = n$$

$$\frac{aj-5}{9} = \frac{ak-1}{6}$$
$$\frac{aj-5}{9} * 18 = \frac{ak-1}{6} * 18$$

$$2aj - 10 = 3ak - 3$$
$$2aj - 3ak = 7$$
$$a(2j - 3k) = 7$$

Since $2j - 3k$ is an integer, this means that a divides 7.

c. Let $n \in \mathbb{Z}$ and be odd. This means it can be represented in the form $2k + 1$ where $k \in \mathbb{Z}$. Let us compute $n^4 + 4n^2 + 11$ and simplify.

$$n^4 + 4n^2 + 11$$

$$(2k + 1)^4 + 4(2k + 1)^2 + 11$$
$$16k^4 + 32k^3 + 24k^2 + 8k + 1 + 16k^2 + 16k + 4 + 11$$
$$16k^4 + 32k^3 + 40k^2 + 24k + 16$$
$$8(2k^4 + 4k^3 + 5k^2 + 3k + 2)$$

Since $2k^4 + 4k^3 + 5k^2 + 3k + 2$ is an integer, we have proved that $n^4 + 4n^2 + 11$ is equal to 8 times some integer when n is odd, meaning that $8|(n^4 + 4n^2 + 11)$.

d. We can prove this false by letting $n = 1$ substituting:

$$8|(n^4 + n^2 + 2n)$$

$$8|(1^4 + 1^2 + 2 * 1) \Rightarrow 8|(1 + 1 + 2) \Rightarrow 8|4$$

Obviously, 4 is not divisible by 8 meaning that the proposition is false.

6. We will prove by contradiction. Let $m, n, p \in \mathbb{Z}$ and let p be odd. The equation $x^2 + 2mx + 2p = 0$, which we assume has integer solutions, can be written as

$$x^2 + 2mx = -2p$$

We can then do casework on the value of x depending on whether it is even or odd.

Case 1: Let n be an even integer and let us substitute for x. Since n is even, it will be in the form $2k$ such that $k \in \mathbb{Z}$.

$$(2k)^2 + 2m(2k) = -2p$$
$$4k^2 + 4mk = -2p$$
$$-2k^2 - 2mk = p$$
$$-2(k^2 + mk)$$

p is in the form $2k$, meaning that p is even in this case. This contradicts the assumption that p is odd.

Case 2: Let n be an odd integer and let us substitute for x. Since it is odd, it will be in the form $2k + 1$ such that $k \in \mathbb{Z}$.

$$(2k + 1)^2 + 2m(2k + 1) = -2p$$
$$4k^2 + 4k + 1 + 4mk + 2m = -2p$$
$$-2k^2 - 2k - \frac{1}{2} - 2mk - m = p$$

Since $-2k^2 - 2k - 2mk - m \in \mathbb{Z}$ and it is subtracted by $\frac{1}{2}$, p is not an integer, which contradicts the assumption that p is an integer.

As we can see in the above cases, when the solution is an even integer, p must be an even integer, and when the solution is an odd integer, p can not be an integer. This proves that, by contradiction, if p is odd, then there is no integer solution of $x^2 + 2mx + 2p = 0$.