

1.
 - a. $\forall x \exists y P(x, y)$
 - b. $\exists x \exists y \neg P(x, y)$
 - c. $\exists! x P(x, \text{Alaska})$
 - d. $\exists x \forall y (\neg(P(x, y) \oplus y = \text{Illinois}))$
2.
 - a. $\neg(\forall x \forall y (x + y = y + x)) \Rightarrow \exists x \exists y (x + y \neq y + x)$
 - b. $\neg(\forall x \exists y ((x \neq y) \wedge (x + y = 3))) \Rightarrow \exists x \forall y ((x = y) \vee (x + y \neq 3))$
3. Let $x \in \mathbb{Z}$.

$$\begin{aligned} & (x+1)^2 - x^2 \\ & x^2 + 2x + 1 - x^2 \\ & 2x + 1 \end{aligned}$$

Remember that every odd number can be written in the form $2k + 1$ where $k \in \mathbb{Z}$. Since the difference of the squares of some x and some $x + 1$ can be written in this form, it is odd. Therefore, since x can be substituted for any integer, we can get any and all odd integers from the difference of two squares.

4. Let $x_1, x_2, \dots, x_n \in \mathbb{R}$. We can prove with contradiction by assuming the opposite where all x_n are less than the average.

$$x_n < \frac{\sum_{k=1}^n x_n}{n}$$

We can then add all the terms of x_n together. This gives us

$$x_1 + x_2 + \dots + x_n < \frac{\sum_{k=1}^n x_n}{n} + \frac{\sum_{k=1}^n x_n}{n} + \dots + \frac{\sum_{k=1}^n x_n}{n}$$

$$\sum_{k=1}^n x_n < \sum_{k=1}^n x_n$$

Obviously, a number can not be less than itself, proving that, by contradiction, at least one of the real numbers x_1, x_2, \dots, x_n is greater than or equal to the average of all x_n .

5.
 - a. Let $a, n, j, k \in \mathbb{Z}$. Since $a|(8n + 7)$ and $a|(4n + 1)$, we can rewrite in the form:

$$\begin{aligned} 8n + 7 &= aj & 4n + 1 &= ak \\ \frac{aj - 7}{8} &= n & \frac{ak - 1}{4} &= n \end{aligned}$$

$$\begin{aligned} \frac{aj - 7}{8} &= \frac{ak - 1}{4} \\ \frac{aj - 7}{8} * 8 &= \frac{ak - 1}{4} * 8 \\ aj - 7 &= 2ak - 2 \\ aj - 2ak &= 5 \\ a(j - 2k) &= 5 \end{aligned}$$

Since $j - 2k$ is an integer, this means that a divides 5.

- b. Let $a, n, j, k \in \mathbb{Z}$. Since $a|(9n + 5)$ and $a|(6n + 1)$, we can rewrite in the form:

$$\begin{aligned} 9n + 5 &= aj & 6n + 1 &= ak \\ \frac{aj - 5}{9} &= n & \frac{ak - 1}{6} &= n \end{aligned}$$

$$\begin{aligned} \frac{aj - 5}{9} &= \frac{ak - 1}{6} \\ \frac{aj - 5}{9} * 18 &= \frac{ak - 1}{6} * 18 \end{aligned}$$

$$2aj - 10 = 3ak - 3$$

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$$2aj - 3ak = 7$$

$$a(2j - 3k) = 7$$

Since $2j - 3k$ is an integer, this means that a divides 7.

- c. Let $n \in \mathbb{Z}$ and be odd. This means it can be represented in the form $2k + 1$ where $k \in \mathbb{Z}$. Let us compute $n^4 + 4n^2 + 11$ and simplify.

$$n^4 + 4n^2 + 11$$

$$(2k + 1)^4 + 4(2k + 1)^2 + 11$$

$$16k^4 + 32k^3 + 24k^2 + 8k + 1 + 16k^2 + 16k + 4 + 11$$

$$16k^4 + 32k^3 + 40k^2 + 24k + 16$$

$$8(2k^4 + 4k^3 + 5k^2 + 3k + 2)$$

Since $2k^4 + 4k^3 + 5k^2 + 3k + 2$ is an integer, we have proved that $n^4 + 4n^2 + 11$ is equal to 8 times some integer when n is odd, meaning that $8|(n^4 + 4n^2 + 11)$.

- d. We can prove this false by letting $n = 1$ substituting:

$$8|(n^4 + n^2 + 2n)$$

$$8|(1^4 + 1^2 + 2 * 1) \Rightarrow 8|(1 + 1 + 2) \Rightarrow 8|4$$

Obviously, 4 is not divisible by 8 meaning that the proposition is false.

6. We will prove by contradiction. Let $m, n, p \in \mathbb{Z}$ and let p be odd. The equation $x^2 + 2mx + 2p = 0$, which we assume has integer solutions, can be written as

$$x^2 + 2mx = -2p$$

We can then do casework on the value of x depending on whether it is even or odd.

Case 1: Let n be an even integer and let us substitute for x . Since n is even, it will be in the form $2k$ such that $k \in \mathbb{Z}$.

$$(2k)^2 + 2m(2k) = -2p$$

$$4k^2 + 4mk = -2p$$

$$-2k^2 - 2mk = p$$

$$-2(k^2 + mk)$$

p is in the form $2k$, meaning that p is even in this case. This contradicts the assumption that p is odd.

Case 2: Let n be an odd integer and let us substitute for x . Since it is odd, it will be in the form $2k + 1$ such that $k \in \mathbb{Z}$.

$$(2k + 1)^2 + 2m(2k + 1) = -2p$$

$$4k^2 + 4k + 1 + 4mk + 2m = -2p$$

$$-2k^2 - 2k - \frac{1}{2} - 2mk - m = p$$

Since $-2k^2 - 2k - 2mk - m \in \mathbb{Z}$ and it is subtracted by $\frac{1}{2}$, p is not an integer, which contradicts the assumption that p is an integer.

As we can see in the above cases, when the solution is an even integer, p must be an even integer, and when the solution is an odd integer, p can not be an integer. This proves that, by contradiction, if p is odd, then there is no integer solution of $x^2 + 2mx + 2p = 0$.