1. Let the base case be $n = 1$.

$$\sum_{k=1}^{1} H_k = H_1 = \frac{1}{1} = 1 \tag{1}$$

$$(1+1)H_1 - 1 = (2)\frac{1}{1} - 1 = 2 \cdot 1 - 1 = 2 - 1 = 1 \tag{2}$$

Since $1 = 1$, the base case holds.
For the inductive step, assume we have a $j \in \mathbb{N}$, such that $j \geq 1$ and $\sum_{k=1}^{j} H_k = (j+1)H_j - j$. Additionally, notice that $H_n + \frac{1}{n+1} = H_{n+1}$, so $H_n = H_{n+1} - \frac{1}{n+1}$.

$$\begin{aligned}
\sum_{k=1}^{j+1} H_k &= (\sum_{k=1}^{j} H_k) + H_{j+1} \\
&= (j+1)H_j - 1 + H_{j+1} \\
&= (j+1)(H_{j+1} - \frac{1}{j+1}) - j + H_{j+1} \\
&= (j+1)H_{j+1} - (j+1)\frac{1}{j+1} - j + H_{j+1} \\
&= (j+1)H_{j+1} + 1 \cdot H_{j+1} - 1 - j \\
&= (j+1+1)H_{j+1} - 1 - j \\
&= ((j+1)+1)H_{j+1} - (j+1)
\end{aligned}$$

This is the same formula for $k+1$; thus, by induction, we have shown that for any positive integer n, $\sum_{k=1}^{n} H_k = (n+1)H_n - n$.

2. Let the base cases be $n = 1$ and $n = 2$. $f(1) = 1$ and $f(2) = 5$.

$$\begin{aligned}
n &= 1 & n &= 2 \\
f(1) &= 1 & f(2) &= 5 \\
2^1 + (-1)^1 = 2 - 1 &= 1 & 2^2 + (-1)^2 = 4 + 1 &= 5
\end{aligned}$$

Since $1 = 1$ and $5 = 5$, the base cases hold.
For the inductive step, assume we have a $k \in \mathbb{N}$, such that $k \geq 2$, $f(k) = 2^k + (-1)^k$, and $f(k-1) = 2^{k-1} + (-1)^{k-1}$. By definition,

$$\begin{aligned}
f(k+1) &= f(k) + 2f(k-1) \\
&= 2^k + (-1)^k + 2(2^{k-1} + (-1)^{k-1}) \\
&= 2^k + (-1)^k + 2 \cdot 2^{k-1} + 2 \cdot (-1)^{k-1} \\
&= 2^k + (-1)^k + 2^k + 2 \cdot (-1)^k \cdot (-1) \\
&= 2 \cdot 2^k + (-1)^k - 2(-1)^k \\
&= 2^{k+1} - (-1)^k \\
&= 2^{k+1} + (-1)^1(-1)^k \\
&= 2^{k+1} + (-1)^{k+1}
\end{aligned}$$

This is the same formula for $k+1$; thus, by induction, we have shown that for any positive integer n, $f(n) = 2^n + (-1)^n$.

3. Let the base case be $n = 1$. Obviously, $(x-y)|(x^1 - y^1)$, as $(x-y) = 1 \cdot (x^1 - y^1)$, so the base case holds.
For the inductive step, assume we have a $k \in \mathbb{N}$, such that $k \geq 1$ and $(x-y)|(x^k - y^k)$. Since

$(x-y)|(x^k - y^k)$, $(x^k - y^k) = p \cdot (x-y)$, where p is some polynomial.

$$\begin{aligned} x^{k+1} - y^{k+1} &= x^{k+1} - y^{k+1} + (-x^k y) + x^k y \\ &= x^{k+1} + (-x^k y) + x^k y - y^{k+1} \\ &= x^k \cdot x - x^k \cdot y + y \cdot x^k - y \cdot y^k \\ &= x^k (x-y) + y(x^k - y^k) \\ &= x^k (x-y) + y \cdot p \cdot (x-y) \\ &= (x-y)(x^k + y \cdot p) \end{aligned}$$

Since $(x-y)|((x-y)(x^k + y \cdot p))$, $(x-y)|(x^{k+1} - y^{k+1})$. Thus, by induction, we have shown that for any positive integer n, $(x-y)|(x^n - y^n)$.

4. Let the base case be $n = 1$. 1 can be uniquely written as 2^0. So, the base case holds.
For the inductive step, we will use strong induction and assume that a $k \in \mathbb{N}$ and all $j \in \mathbb{N}$, such that $1 \leq j \leq k$, can be uniquely written as a sum of distinct powers of 2. Let $m \in \mathbb{N}$ such that 2^m is the greatest power of 2 less than $k+1$, i.e. $2^m \leq k+1 < 2^{m+1}$. Let $r \in \mathbb{N}$ such that $r = (k+1) - 2^m$. Following this, $0 \leq r < 2^m$ and $r < k+1$. We can represent $k+1$ as $2^m + r$. There are 2 resulting cases, $r = 0$ and $r > 0$.
<u>Case 1:</u> If $r = 0$, then $k + 1 = 2^m$. Thus, $k+1$ can be uniquely written as 2^m.
<u>Case 2:</u> If $r > 0$, then recall that $r < k+1$, and thus $r \leq k$. By the strong inductive hypothesis, r can be uniquely written as a sum of distinct powers of 2. Moreover, because $r < 2^m$ and $1 + 2 + 4 + \cdots + 2^{m-1} < 2^m$, there will be no overlap between the powers used to represent r and 2^m. Therefore, $k+1$ can be uniquely written as $2^m + r$, a sum of distinct powers of 2.
Thus, by strong induction, we have shown that every positive integer can be uniquely written as a sum of distinct powers of 2.