q

1. Let the base case be $n = 1$.

$$\sum_{k=1}^{1} H_k = H_1 = \frac{1}{1} = 1 \tag{1}$$

$$(1+1)H_1 - 1 = (2)\frac{1}{1} - 1 = 2 \cdot 1 - 1 = 2 - 1 = 1 \tag{2}$$

Since $1 = 1$, the base case holds.

For the inductive step, assume we have a $j \in \mathbb{N}$, such that $j \geq 1$ and $\sum_{k=1}^{j} H_k = (j+1)H_j - j$. Additionally, notice that $H_n + \frac{1}{n+1} = H_{n+1}$, so $H_n = H_{n+1} - \frac{1}{n+1}$.

$$\sum_{k=1}^{j+1} H_k = (\sum_{k=1}^{j} H_k) + H_{j+1}$$
$$= (j+1)H_j - j + H_{j+1}$$
$$= (j+1)(H_{j+1} - \frac{1}{j+1}) - j + H_{j+1}$$
$$= (j+1)H_{j+1} - (j+1)\frac{1}{j+1} - j + H_{j+1}$$
$$= (j+1)H_{j+1} + 1 \cdot H_{j+1} - 1 - j$$
$$= (j+1+1)H_{j+1} - 1 - j$$
$$= ((j+1)+1)H_{j+1} - (j+1)$$

This is the same formula for $k + 1$; thus, by induction, we have shown that for any positive integer n, $\sum_{k=1}^{n} H_k = (n+1)H_n - n$.

2. Let the base cases be $n = 1$ and $n = 2$. $f(1) = 1$ and $f(2) = 5$.

$$n = 1 \qquad\qquad\qquad n = 2$$
$$f(1) = 1 \qquad\qquad\qquad f(2) = 5$$
$$2^1 + (-1)^1 = 2 - 1 = 1 \qquad 2^2 + (-1)^2 = 4 + 1 = 5$$

Since $1 = 1$ and $5 = 5$, the base cases hold.

For the inductive step, assume we have a $k \in \mathbb{N}$, such that $k \geq 2$, $f(k) = 2^k + (-1)^k$, and $f(k-1) = 2^{k-1} + (-1)^{k-1}$. By definition,

$$f(k+1) = f(k) + 2f(k-1)$$
$$= 2^k + (-1)^k + 2(2^{k-1} + (-1)^{k-1})$$
$$= 2^k + (-1)^k + 2 \cdot 2^{k-1} + 2 \cdot (-1)^{k-1}$$
$$= 2^k + (-1)^k + 2^k + 2 \cdot (-1)^k \cdot (-1)$$
$$= 2 \cdot 2^k + (-1)^k - 2(-1)^k$$
$$= 2^{k+1} - (-1)^k$$
$$= 2^{k+1} + (-1)^1(-1)^k$$
$$= 2^{k+1} + (-1)^{k+1}$$

This is the same formula for $k + 1$; thus, by induction, we have shown

p is some polynomial.

$$\begin{aligned}
x^{k+1} - y^{k+1} &= x^{k+1} - y^{k+1} + (-x^k y) + x^k y \\
&= x^{k+1} + (-x^k y) + x^k y - y^{k+1} \\
&= x^k \cdot x - x^k \cdot y + y \cdot x^k - y \cdot y^k \\
&= x^k(x - y) + y(x^k - y^k) \\
&= x^k(x - y) + y \cdot p \cdot (x - y) \\
&= (x - y)(x^k + y \cdot p)
\end{aligned}$$

Since $(x - y)|((x - y)(x^k + y \cdot p))$, $(x - y)|(x^{k+1} - y^{k+1})$. Thus, by induction, we have shown that for any positive integer n, $(x - y)|(x^n - y^n)$.

4. Let the base case be $n = 1$. 1 can be uniquely written as 2^0. So, the base case holds.

For the inductive step, we will use strong induction and assume that a $k \in \mathbb{N}$ and all $j \in \mathbb{N}$, such that $1 \leq j \leq k$, can be uniquely written as a sum of distinct powers of 2. Let $m \in \mathbb{N}$ such that 2^m is the greatest power of 2 less than $k + 1$, i.e. $2^m \leq k + 1 < 2^{m+1}$. Let $r \in \mathbb{N}$ such that $r = (k + 1) - 2^m$. Following this, $0 \leq r < 2^m$ and $r < k + 1$. We can represent $k + 1$ as $2^m + r$. There are 2 resulting cases, $r = 0$ and $r > 0$.

Case 1: If $r = 0$, then $k + 1 = 2^m$. Thus, $k + 1$ can be uniquely written as 2^m.

Case 2: If $r > 0$, then recall that $r < k + 1$, and thus $r \leq k$. By the strong inductive hypothesis, r can be uniquely written as a sum of distinct powers of 2. Moreover, because $r < 2^m$ and $1 + 2 + 4 + \cdots + 2^{m-1} < 2^m$, there will be no overlap between the powers used to represent r and 2^m. Therefore, $k + 1$ can be uniquely written as $2^m + r$, a sum of distinct powers of 2.

Thus, by strong induction, we have shown that every positive integer can be uniquely written as a sum of distinct powers of 2.