1

Solve the system of linear congruences:

$$x \equiv 4 \pmod{11}$$
$$x \equiv 7 \pmod{12}$$
$$x \equiv -3 \pmod{13}$$

$N = 11 \cdot 12 \cdot 13 = 1716$. So, we must calculate m_n of the following

$$\frac{N}{11} = 156m_1 \equiv 2m_1 \equiv 1 \pmod{11}$$
$$\frac{N}{12} = 143m_2 \equiv 11m_2 \equiv 1 \pmod{12}$$
$$\frac{N}{13} = 132m_3 \equiv 2m_3 \equiv 1 \pmod{13}$$

q	r	m_1	y
	11	0	1
5	2	1	0
1	1	-5	1

q	r	m_2	y
	12	0	1
1	11	1	0
1	1	-1	1

q	r	m_3	y
	13	0	1
6	2	1	0
1	1	-6	1

Thus, we have $m_1 = 6$, $m_2 = 11$, and $m_3 = 7$. Now we can calculate the solution:

$$\begin{aligned} x &\equiv 4 \cdot 156 \cdot 6 + 7 \cdot 143 \cdot 11 + (-3) \cdot 132 \cdot 7 \pmod{1716} \\ &\equiv 3744 + 11011 - 2772 \pmod{1716} \\ &\equiv 11983 \pmod{1716} \\ &\equiv 1687 \pmod{1716} \end{aligned}$$

2

Solve the system of linear congruences:

$$2x \equiv 1 \pmod{3}$$
$$3x \equiv 2 \pmod{5}$$
$$4x \equiv 3 \pmod{7}$$
$$5x \equiv 4 \pmod{11}$$

$N = 3 \cdot 5 \cdot 7 \cdot 11 = 1155$. Now, let's find the inverses of the given congruences.

q	r	m_1	y
	3	0	1
1	2	1	0
1	1	-1	1

q	r	m_2	y
	5	0	1
1	3	1	0
2	1	-1	1

q	r	m_3	y
	7	0	1
1	4	1	0
3	1	-1	1

q	r	m_4	y
	11	0	1
2	5	1	0
1	1	-2	1

Thus, we found 2, 4, 6, and 9 respectively. Let's plug these in

$$x \equiv 1 \cdot 2 \equiv 2 \pmod{3}$$
$$x \equiv 2 \cdot 4 \equiv 3 \pmod{5}$$
$$x \equiv 3 \cdot 6 \equiv 4 \pmod{7}$$
$$x \equiv 4 \cdot 9 \equiv 36 \equiv 3 \pmod{11}$$

Now, we must solve for m_n in

$$\begin{aligned}
\frac{N}{3} &= 385m_1 \equiv m_1 \equiv 1 \pmod 3 \\
\frac{N}{5} &= 231m_2 \equiv m_2 \equiv 1 \pmod 5 \\
\frac{N}{7} &= 165m_3 \equiv 4m_3 \equiv 1 \pmod 7 \\
\frac{N}{11} &= 105m_4 \equiv 6m_4 \equiv 1 \pmod{11}
\end{aligned}$$

q	r	m_1	y
	3	0	1
1	2	1	0
1	1	-1	1
		1	-1

q	r	m_2	y
	5	0	1
1	3	1	0
1	2	-1	1
		1	-1

q	r	m_3	y
	7	0	1
1	4	1	0
1	3	-1	1
1	1	2	-1

q	r	m_4	y
	11	0	1
2	5	1	0
1	1	-2	1
		2	-2

Thus, we have $m_1 = 1$, $m_2 = 1$, $m_3 = 2$, and $m_4 = 2$. Now we can calculate the solution:

$$\begin{aligned}
x &\equiv 2 \cdot 385 \cdot 1 + 3 \cdot 231 \cdot 1 + 4 \cdot 165 \cdot 2 + 3 \cdot 105 \cdot 2 \pmod{1155} \\
&\equiv 770 + 693 + 1320 + 630 \pmod{1155} \\
&\equiv 3413 \pmod{1155} \\
&\equiv 839 \pmod{1155}
\end{aligned}$$

3

Solve the system of linear congruences:

$$\begin{aligned}
x &\equiv 2 \pmod 2 \\
x &\equiv 4 \pmod 6 \\
x &\equiv 2 \pmod{14} \\
x &\equiv 10 \pmod{15}
\end{aligned}$$

We see that for any pair of congruences in the system, the GCD of the moduli divides the difference of the residues. Thus, a solution exists. Let's start from the first congruence working our way down.

$$\begin{aligned}
x &\equiv 2 \pmod{2} \quad (1)\\
x &= 2 + 2k_1 \text{ for } k_1 \in \mathbb{Z}\\
2 + 2k_1 &\equiv 4 \pmod{6} \quad (2)\\
2k_1 &\equiv 2 \pmod{6}\\
k_1 &\equiv 1 \pmod{3}\\
k_1 &= 1 + 3k_2 \text{ for } k_2 \in \mathbb{Z}\\
x &= 2 + 2(1 + 3k_2)\\
x &= 4 + 6k_2\\
4 + 6k_2 &\equiv 2 \pmod{14} \quad (3)\\
6k_2 &\equiv -2 \pmod{14}\\
3k_2 &\equiv 6 \pmod{7}\\
3k_2 &= 6 + 7k_3 \text{ for } k_3 \in \mathbb{Z}\\
x &= 4 + 2(6 + 7k_3)\\
x &= 16 + 14k_3\\
16 + 14k_3 &\equiv 10 \pmod{15} \quad (4)\\
14k_3 &\equiv -6 \pmod{15}\\
14k_3 &\equiv 9 \pmod{15}\\
-k_3 &\equiv 9 \pmod{15}\\
k_3 &\equiv 6 \pmod{15}\\
k_3 &= 6 + 15k_4 \text{ for } k_4 \in \mathbb{Z}\\
x &= 16 + 14(6 + 15k_4)\\
x &= 100 + 210k_4\\
x &\equiv 100 \pmod{210} \quad (5)
\end{aligned}$$

4

Prove that there are arbitrarily long strings of consecutive integers in which every one of the numbers is divisible by a square (other than 1!).

First, let $P = \{p_1, p_2, \ldots, p_n\}$ be the set of all prime numbers where all p_i are unique. We have previously proven that there are infinite prime numbers, so $|P| = \infty$. Now, we want to find a string of n consecutive integers starting with x. Let's say that x is divisible by p_1^2, $x+1$ is divisible by $p_{1+1}^2 = p_2^2$, ..., and $x + n - 1$ is divisible by p_n^2. Now, let's rewrite in terms of congruences.

$$\begin{aligned}
x &\equiv 0 \pmod{p_1^2}\\
x + 1 &\equiv 0 \pmod{p_2^2}\\
&\vdots\\
x + n - 1 &\equiv 0 \pmod{p_n^2}
\end{aligned}$$

Rearranging gives us:

$$\begin{aligned}
x &\equiv 0 \pmod{p_1^2}\\
x &\equiv -1 \pmod{p_2^2}\\
&\vdots\\
x &\equiv -n + 1 \pmod{p_n^2}
\end{aligned}$$

Since $p_1, p_2, \ldots p_n$ are all unique primes, any pair of congruences has coprime moduli. Thus, we can use the Chinese Remainder Theorem. It tells us that there exists a solution to this system of congruences. Since we can choose n to be arbitrarily large, we can construct arbitrarily long sequences of consecutive integers, each divisible by a square.

5

Find the least positive residue of $3^{984} \pmod{31}$.

Luckily, 31 is prime, so we can simply use Fermat's Little Theorem. From it, we know $3^{31-1} = 3^{30} \equiv 1 \pmod{31}$.

$$\begin{aligned} 3^{984} &\equiv 3^{30\cdot 32+24} \pmod{31} \\ &\equiv (3^{30})^{32} \cdot 3^{24} \pmod{31} \\ &\equiv 1^{32} \cdot 3^{24} \pmod{31} \\ &\equiv 3^{24} \pmod{31} \end{aligned}$$

Now, let's compute $3^{24} \pmod{31}$ with successive squares.

$$\begin{aligned} 3^2 &\equiv 9 \pmod{31} \\ 3^4 &\equiv 81 \equiv 19 \pmod{31} \\ 3^8 &\equiv 361 \equiv 20 \pmod{31} \\ 3^{16} &\equiv 400 \equiv 28 \pmod{31} \end{aligned}$$

$$\begin{aligned} 3^{24} &\equiv 3^{16+8} \pmod{31} \\ &\equiv 3^{16} \cdot 3^8 \pmod{31} \\ &\equiv 28 \cdot 20 \pmod{31} \\ &\equiv 560 \pmod{31} \\ &\equiv 2 \pmod{31} \end{aligned}$$

Thus, the least possible residue of $3^{984} \pmod{31}$ is 2.

6

Find the least positive residue of $3^{984} \pmod{360}$.

$360 = 2^3 \cdot 3^2 \cot 5$. We can then split this into congruences of 8, 9, and 5 and combine with the Chinese Remainder Theorem. Let's calculate the congruences.

$$\begin{aligned} 3^{984} &= 9^{492} \\ &\equiv 1^{492} \pmod{8} \\ &\equiv 1 \pmod{8} \end{aligned}$$

$$\begin{aligned} 3^{984} &= 9^{492} \\ &\equiv 0^{492} \pmod{9} \\ &\equiv 0 \pmod{9} \end{aligned}$$

By Fermat's Little Theorem, we can say $3^{5-1} = 3^4 \equiv 1 \pmod{5}$.

$$\begin{aligned} 3^{984} &= 3^{4\cdot 246} \\ &= (3^4)^{246} \\ &\equiv 1^{246} \pmod{5} \\ &\equiv 1 \pmod{5} \end{aligned}$$

Thus, we have 3 congruences with coprime moduli.

$$\begin{aligned} x &\equiv 1 \pmod 8 \\ x &\equiv 0 \pmod 9 \\ x &\equiv 1 \pmod 5 \end{aligned}$$

Now, we can solve this using the Chinese remainder Theorem. $N = 8 \cdot 9 \cdot 5 = 360$. Then, we need to find all inverses m_n.

$$\begin{aligned} \frac{N}{8} &= 45m_1 \equiv 5m_1 \equiv 1 \pmod 8 \\ \frac{N}{9} &= 40m_2 \equiv 4m_2 \equiv 1 \pmod 9 \\ \frac{N}{5} &= 72m_3 \equiv 2m_3 \equiv 1 \pmod 5 \end{aligned}$$

q	r	m_1	y
	8	0	1
5	5	1	0
1	3	-1	1
1	2	2	-1
1	1	-3	2

q	r	m_2	y
	9	0	1
4	4	1	0
2	1	-2	1

q	r	m_3	y
	5	0	1
2	2	1	0
2	1	-2	1

Thus, we have $m_1 = 5$, $m_2 = 7$, and $m_3 = 3$. Now we can calculate the solution.

$$\begin{aligned} x &\equiv 1 \cdot 45 \cdot 5 + 0 \cdot 40 \cdot 7 + 1 \cdot 72 \cdot 3 \pmod{360} \\ &\equiv 225 + 0 + 216 \pmod{360} \\ &\equiv 441 \pmod{360} \\ &\equiv 81 \pmod{360} \end{aligned}$$

Thus, the least possible residue of $3^{984} \pmod{360}$ is 81.