I created a quick program (after completing problem #1) to help compute $\sigma(n)$ and $\tau(n)$; https://jsfiddle.net/xfthwe0b/6/.

Problem 1

Calculate $\tau(n)$ and $\sigma(n)$ for $n = 143, 144,$ and 145.

$143 = 11 \cdot 13144 = 2^4 \cdot 3^2 \qquad 145 = 5 \cdot 29$

$$\begin{aligned} \tau(143) &= (1+1)(1+1) \\ &= \mathbf{4} \\ \sigma(143) &= \sigma(11)\sigma(13) \\ &= (1+11)(1+13) \\ &= \mathbf{168} \end{aligned} \qquad \begin{aligned} \tau(144) &= (4+1)(2+1) \\ &= \mathbf{15} \\ \sigma(144) &= \sigma(2^4)\sigma(3^2) \\ &= (1+2+4+8+16)(1+3+9) \\ &= \mathbf{403} \end{aligned} \qquad \begin{aligned} \tau(145) &= (1+1)(1+1) \\ &= \mathbf{4} \\ \sigma(145) &= \sigma(5)\sigma(29) \\ &= (1+5)(1+29) \\ &= \mathbf{180} \end{aligned}$$

Problem 2

Find 3 numbers with $\tau(n) = 24$. Find two numbers with $\sigma(n) = 432$.

Let's start with the first part. Let $n = p_1^{a_1} p_2^{a_2} p_3^{a_3}$, such that all p are prime and for any $i, j \in \mathbb{Z}, p_i \neq p_j$. $\tau(n)$ only depends on a_i, so we can figure out some combination of $a_1, a_2,$ and a_3 and then change $p_1, p_2,$ and p_3 to generate some ns.

$$\begin{aligned} \tau(n) &= (a_1+1)(a_2+1)(a_3+1) \\ &= 24 \\ &= 2 \cdot 3 \cdot 4 \\ 2 \cdot 3 \cdot 4 &= (a_1+1)(a_2+1)(a_3+1) \\ a_1 = 1, a_2 &= 2, a_3 = 3 \end{aligned}$$

Thus, $\sigma(p_1^1 p_2^2 p_3^3) = 24$. Now, we can substitute, in the form (p_1, p_2, p_3), $(2, 3, 5), (5, 17, 23),$ and $(43, 417, 24043)$. This gives us $2250, 17\,581\,315,$ and $103\,921\,769\,945\,775\,943\,089$.

Now, for the second part. Finding a number n such that $\sigma(n) = 432$ does not seem easy, so we will take advantage of the fact that $\sigma(n)$ is multiplicative and find $m, o \in \mathbb{N}$ such that $\sigma(m)\sigma(o) = 432$. $432 = 18 * 24$, so we need to find m, n such that $\sigma(m) = 18$ and $\sigma(o) = 24$. We can then find that $\sigma(10) = \sigma(17) = 18$ and $\sigma(15) = \sigma(23) = 18$. So, two numbers with $\sigma(n) = 432$ are 230 and 255.

Problem 3

Define $\sigma_4(n)$ to be the sum of the fourth powers of the divisors of n. Show that $\sigma_4(n)$ is a multiplicative function.

From the definition, $\sigma_4(n) = \sum_{d|n} d^4$.

$$\begin{aligned}
\sigma_4(n) = \sigma_4(ab) &= \sum_{d|ab} d^4 \\
&= \sum_{\alpha|a,\beta|b} (\alpha\beta)^4 \\
&= \sum_{\alpha|a,\beta|b} \alpha^4\beta^4 \text{ (since } x^4 \text{ is obviously multiplicative)} \\
&= \sum_{\alpha|a} \sum_{\beta|b} \alpha^4\beta^4 \\
&= \sum_{\alpha|a} \alpha^4 (\sum_{\beta|b} \beta^4) \text{ (since } \alpha^4 \text{ is a constant)} \\
&= \sum_{\alpha|a} \alpha^4 \sigma_4(b) \text{ (by definition of } \sigma_4(n)) \\
&= \sigma_4(b) \sum_{\alpha|a} \alpha^4 \text{ (since } \sigma_4(b) \text{ is a constant)} \\
&= \sigma_4(b)\sigma_4(a) \text{ (by definition of } \sigma_4(n))
\end{aligned}$$

Thus, $\sigma_4(n)$ is multiplicative.

Problem 4

Consider the function $\alpha(n)$ which is the product of all factors of n. Prove or disprove: $\alpha(n)$ is multiplicative.

We will disprove by counterexample.

$$\begin{aligned}
\alpha(3) &= 1 \cdot 3 \\
&= 3 \\
\alpha(4) &= 1 \cdot 2 \cdot 4 \\
&= 8
\end{aligned}$$

If $\alpha(n)$ were multiplicative, we would expect $\alpha(3)\alpha(4) = \alpha(3 \cdot 4) =$

$\alpha(12) = 3 \cdot 8 = 24$. Let us check this.

$$\begin{aligned}\alpha(12) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 12 \\ &= 1728\end{aligned}$$

Obviously, $24 \neq 1728$. Thus, by counterexample, we have shown that $\alpha(n)$ is not multiplicative.

Problem 5

Prove that $\tau(n)$ is odd if and only [if] n is a perfect square.

Let $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$. So, $\tau(n) = (a_1+1)(a_2+1)(a_3+1)\dots(a_n+1)$. We see that this product is odd only when all the factors are themselves odd. Thus, $a_1, a_2, a_3 \dots a_n$ must be even. Let $a_1 = 2b_1, a_2 = 2b_2, a_3 = 2b_3, \dots, a_n = 2b_n$.

$$\begin{aligned}n &= p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k} \\ n &= p_1^{2b_1} p_2^{2b_2} p_3^{2b_3} \dots p_k^{2b_k} \\ n &= (p_1^{b_1} p_2^{b_2} p_3^{b_3} \dots p_k^{b_k})^2\end{aligned}$$

Thus, if $\tau(n)$ is odd, n is a perfect square. By performing a similar proof, just in reverse, we can prove the converse. Thus, we have shown that $\tau(n)$ is odd if and only if n is a perfect square.

Problem 6

Determine and prove a criterion that is equivalent to $\sigma(n)$ being odd.

I assert $\sigma(n)$ to be odd if $n = 2^k \cdot o^2$ for $k, o \in \mathbb{N}$ such that $k \geq 0$ and o is an odd integer. Since σ is multiplicative, $\sigma(n) = \sigma(a_1)\sigma(a_2)\sigma(a_3)\dots\sigma(a_n)$. $\sigma(n)$ is odd only when $\sigma(a_1), \sigma(a_2), \sigma(a_3), \dots, \sigma(a_n)$ are all odd. Since $(2, o) = 1$, $\sigma(2^k o^2) = \sigma(2^k)\sigma(o^2)$.

$$\begin{aligned}\sigma(2^k) &= 2^0 + 2^1 + 2^2 + \dots + 2^k \\ &= 2(2^0 + 2^1 + \dots + 2^{k-1}) + 1\end{aligned}$$

Thus, $\sigma(2^k)$ is odd. All factors of o^2 are odd since o is odd. So, $\sigma(o^2)$ is a sum of odd integers. Furthermore, as we have shown in problem

5, $\tau(n)$, or the number of divisors of n, is odd when n is a perfect square. Thus, $\sigma(o^2)$ is the sum of an odd number of odd integers. We can then see that this means $\sigma(o^2)$ itself is odd. Since $\sigma(2^k)$ and $\sigma(o^2)$ are both odd, so is $\sigma 2^k o^2$. Thus, we have shown that if n is in the form $2^k \cdot o^2$, $\sigma(n)$ is odd.