Problem 1

Convert each number to the given base.

1. 37016 to base 7
2. 212.56 to base 5
3. $\frac{1}{4}$ to base 3
4. π to base 14 (up to 4 decimal places)

1.

q	r
37016	
5288	0
755	3
107	6
15	2
2	1
0	2

212630_7

2. $212.56 = 212 + \frac{14}{25}$

q	r
212	
42	2
8	2
1	3
0	1

$$\frac{14}{25} \cdot 5 = \frac{14}{5} = 2 + \frac{4}{5}$$
$$\frac{4}{5} \cdot 5 = 4$$

$212.56_{10} = 1322 + 0.24 = 1322.24_5$

3.

$$\frac{1}{4} \cdot 3 = 0 + \frac{3}{4}$$
$$\frac{3}{4} \cdot 3 = \frac{9}{4} = 2 + \frac{1}{4}$$
$$\frac{1}{4} \cdot 3 = 0 + \frac{3}{4}$$
$$\vdots$$

$0.\overline{02}_3$

4. $\pi = 3 + (0.14159\ldots)$. $3_{10} = 3_{14}$.

$$(\pi - 3) \cdot 14 = 1.98\ldots = 1 + 0.98\ldots$$
$$((\pi - 3) \cdot 14 - 1) \cdot 14 = 13.75\ldots = D + 0.75\ldots$$
$$(((\pi - 3) \cdot 14 - 1) \cdot 14 - 13) \cdot 14 = 10.53\ldots = A + 0.53\ldots$$
$$((((\pi - 3) \cdot 14 - 1) \cdot 14 - 13) \cdot 14 - 10) \cdot 14 = 7.42\ldots = 7 + 0.42\ldots$$

$\pi_{10} = 3.1DA7\ldots_{14}$.

Problem 2

Convert each number to base 10.

1. 212.56_8
2. $0.\overline{02}_6$
3. $0.\overline{4321}_7$

1. $2 \cdot 8^2 + 1 \cdot 8^1 + 2 \cdot 8^0 + 5 \cdot 8^{-1} + 6 \cdot 8^{-2} = 138 + \frac{5}{8} + \frac{6}{64} = 138\frac{23}{32}$
2. Let $x = 0.\overline{02}_6$.

$$\begin{aligned}
6^2x &= 2.\overline{02}_6 \\
6^2x - x &= 2.\overline{02}_6 - 0.\overline{02}_6 \\
35x &= 2_6 \\
35x &= 2_{10} \\
x &= \frac{2}{35}
\end{aligned}$$

3. Let $x = 0.\overline{4321}_7$

$$\begin{aligned}
7^4x &= 4321.\overline{4321}_7 \\
7^4x - x &= 4321.\overline{4321}_7 - 0.\overline{4321}_7 \\
2400x &= 4321_7 \\
2400x &= 4 \cdot 7^3 + 3 \cdot 7^2 + 2 \cdot 7^1 + 1 \cdot 7^0 \\
2400x &= 1534_{10} \\
x &= \frac{1534}{2400} \\
&= \frac{767}{1200}
\end{aligned}$$

Problem 3

Determine the length of the period and the length of the pre-period for each of the following numbers.

1. $\frac{1}{75}$
2. $\frac{13}{56}$
3. $\frac{7}{91}$

1. $75 = 3 \cdot 5^2$. $5|10$, so $T = 25$ and $U = 3$. $T|10^c$ for $c = 2$, so the pre-period length is 2. a is the smallest integer such that $10^a \equiv 1 \pmod 3$. $10^1 \equiv 1 \pmod 3$, so $a = 1$ and the period length is 1.
2. $56 = 7 \cdot 2^3$. $2|10$, so $T = 8$ and $U = 7$. $T|10^c$ for $c = 3$, so the pre-period length is 3. a is the smallest integer such that $10^a \equiv 1 \pmod 7$.

$$\begin{aligned}
10^1 &\equiv 3 \pmod 7 \\
10^2 &\equiv 3^2 \equiv 2 \pmod 7 \\
10^6 &\equiv 2^3 \equiv 1 \pmod 7
\end{aligned}$$

Thus, $a = 6$ and the period length is 6.

3. $\frac{7}{91} = \frac{1}{13}$. $13 = 13$. $T = 1$ and $U = 13$. $T|10^c$ for $c = 0$, so the pre-period length is 1. a is the smallest integer such that $10^a \equiv 1 \pmod{13}$.

$$\begin{aligned}
10^1 &\equiv 10 \pmod{13} \\
10^2 &\equiv 10^2 \equiv 9 \pmod{13} \\
10^4 &\equiv 9^2 \equiv 3 \pmod{13} \\
10^6 &\equiv 9 \cdot 3 \equiv 1 \pmod{13}
\end{aligned}$$

Thus, $a = 6$ and the period length is 6.

Problem 4

Find the period length of the pre-period length for the following numbers if they were written in base 12.

1. $\frac{1}{8}$

2. $\frac{1}{96}$

3. $\frac{1}{132}$

4. $\frac{11}{360}$

1. $8 = 2^3$. $T = 8$ and $U = 1$. $T|12^c$ for $c = 2$, so the pre-period length is 2. a is the smallest integer such that $12^a \equiv 1 \pmod{1}$. $a = 0$, so the period length is 0.

2. $96 = 2^5 \cdot 3$. $T = 96$ and $U = 1$. $T|12^c$ for $c = 3$, so the pre-period length is 3. a is the smallest integer such that $12^a \equiv 1 \pmod{1}$. $a = 0$, so the period length is 0.

3. $132 = 11 \cdot 12$. $T = 12$ and $U = 11$. $T|12^c$ for $c = 1$, so the pre-period length is 1. a is the smallest integer such that $12^a \equiv 1 \pmod{11}$. We easily see that $12^1 \equiv 1 \pmod{11}$, so $a = 1$ and the period length is 1.

4. $360 = 2^3 \cdot 3^2 \cdot 5$. $T = 72$ and $U = 5$. $T|12^c$ for $c = 2$, so the pre-period length is 2. a is the smallest integer such that $12^a \equiv 1 \pmod{5}$.

$$\begin{aligned}
12^1 &\equiv 2 \pmod{5} \\
12^2 &\equiv 2^2 \equiv 4 \pmod{5} \\
12^4 &\equiv 4^2 \equiv 1 \pmod{5}
\end{aligned}$$

So, $a = 4$ and the period length is 4.

Problem 5

Find the smallest positive integer n such that the base-10 expansion of $\frac{1}{n}$ is periodic with pre-period length 3 and length 5.

Let $n = T \cdot U$. Since the pre-period length is 3, $T|10^3$ and $T \not| 10^2$. Since each factor of T divides 10, T must be in the form $2^3 5^c$ or $2^c 5^3$ for $c \leq 3$. We want to minimize n so we choose the former. Since the period length is 5, $10^5 \equiv 1 \pmod{U}$. Thus, $Uj = 10^5 - 1 = 99999$ for some $j \in \mathbb{Z}$. $99999 = 3^2 \cdot 41 \cdot 271$. We want to minimize n, so we choose small U. We see that 3 and 3^2 aren't valid, since they result in a power of 10 being congruent to 1 too soon. So, let's try $U = 41$.

$$\begin{aligned}
10^1 &\equiv 10 \pmod{41} \\
10^2 &\equiv 100 \equiv 18 \pmod{41} \\
10^3 &\equiv 180 \equiv 16 \pmod{41} \\
10^4 &\equiv 160 \equiv 37 \pmod{41} \\
10^5 &\equiv 370 \equiv 1 \pmod{41}
\end{aligned}$$

We see that $U = 41$ works. Now, we need to find n. We have $T = 2^3 \cdot 5^c$ and $U = 41$. The smallest c is 0, so the smallest n is $n = 8 \cdot 41 = 328$.

Problem 6

For each of $n = 1, 2, 3, 4, 5, 6$, find the set of primes p such that the decimal expansion of $\frac{1}{p}$ is periodic with period length n.

$\boldsymbol{n = 1}$
$10^1 \equiv 1 \pmod{p}$, so $px = 9$ for some $x \in \mathbb{Z}$. $9 = 3^2$, so $p_{n=1} = \{3\}$.
$\boldsymbol{n = 2}$
$10^2 \equiv 1 \pmod{p}$, so $px = 99$ for some $x \in \mathbb{Z}$. $99 = 3^2 \cdot 11$, so we believe $p_{n=2} = \{3, 11\}$. However, $p = 3$ is not valid as $3 \in p_{n=1}$, so $p_{n=2} = \{11\}$.
$\boldsymbol{n = 3}$
$10^3 \equiv 1 \pmod{p}$, so $px = 999$ for some $x \in \mathbb{Z}$. $999 = 3^3 \cdot 37$, and taking out previous primes, we get $p_{n=3} = \{37\}$.
$\boldsymbol{n = 4}$
$10^4 \equiv 1 \pmod{p}$, so $px = 9999$ for some $x \in \mathbb{Z}$. $9999 = 3^2 \cdot 11 \cdot 101$, and taking out previous primes, we get $p_{n=4} = \{101\}$.
$\boldsymbol{n = 5}$
$10^5 \equiv 1 \pmod{p}$, so $px = 99999$ for some $x \in \mathbb{Z}$. $99999 = 3^2 \cdot 41 \cdot 271$, and taking out previous primes, we get $p_{n=5} = \{41, 271\}$.
$\boldsymbol{n = 6}$
$10^6 \equiv 1 \pmod{p}$, so $px = 999999$ for some $x \in \mathbb{Z}$. $999999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$, and taking out previous primes, we get $p_{n=6} = \{7, 13\}$.