1. **a**. If $a \equiv 3 \pmod 7$, then

$$a^2 \equiv 3^2 \pmod 7 \equiv 9 \pmod 7 \equiv 2 \pmod 7$$

$$5a \equiv 5 \cdot 3 \pmod 7 \equiv 15 \pmod 7 \equiv 1 (\text{mod } 7)$$

Adding the two, we get

$$a^2 + 5a \equiv (2+1) \pmod 7 \equiv 3 \pmod 7$$

Thus, it is <u>true</u> that, $\forall a \in \mathbb{Z}$, if $a \equiv 3 \pmod 7$, then $(a^2 + 5a) \equiv 3 \pmod 7$.

b. We can split modulo 7 into 7 equivalence classes for all $a \in \mathbb{Z}$, $[0], [1], [2], [3], [4], [5], [6], [7]$. Plugging all of these into $a^2 + 5a \pmod 7$, we get

$$[0] : 0^2 + 5 \cdot 0 \pmod 7 \equiv 0 \pmod 7$$

$$[1] : 1^2 + 5 \cdot 1 \pmod 7 \equiv 6 \pmod 7$$

$$[2] : 2^2 + 5 \cdot 2 \pmod 7 \equiv 14 \pmod 7 \equiv 0 \pmod 7$$

$$[3] : 3^2 + 5 \cdot 3 \pmod 7 \equiv 24 \pmod 7 \equiv 3 \pmod 7$$

$$[4] : 4^2 + 5 \cdot 4 \pmod 7 \equiv 36 \pmod 7 \equiv 1 \pmod 7$$

$$[5] : 5^2 + 5 \cdot 5 \pmod 7 \equiv 50 \pmod 7 \equiv 1 \pmod 7$$

$$[6] : 6^2 + 5 \cdot 6 \pmod 7 \equiv 66 \pmod 7 \equiv 3 \pmod 7$$

From this, we see that $a^2 + 5a \equiv \ 3 \pmod 7$ when $a \in [3] \lor [6]$. Thus, it is <u>false</u> that if $(a^2 + 5a) \equiv 3 \pmod 7$, then $a \equiv 3 \pmod 7$ as $a \equiv 6 \pmod 7$ is also possible.

2. Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{Z}$. Let $a \equiv a$ (mod n).

$$a \equiv a \text{ (mod n)}$$

$$a - a \equiv a - a \text{ (mod n)}$$

$$0 \equiv 0 \text{ (mod n)}$$

This is true, so $\sim$ is reflexive. Let $a \sim b$.

$$a \equiv b \text{ (mod n)}$$

$$a - a \equiv b - a \text{ (mod n)}$$

$$-1 \cdot 0 \equiv -1 \cdot (b - a) \text{ (mod n)}$$

$$0 \equiv a - b \text{ (mod n)}$$

$$b \equiv a \text{ (mod n)}$$

Thus, $\sim$ is symmetric. Let $a \sim b$ and $b \sim c$.

$$a \equiv b \text{ (mod n)}$$

$$b \equiv c \text{ (mod n)}$$

$$a + b \equiv b + c \text{ (mod n)}$$

$$a + b - b \equiv b + c - b \text{ (mod n)}$$

$$a \equiv c \text{ (mod n)}$$

Thus, $a \sim c$ and $\sim$ is transitive. Since $\sim$ is reflexive, symmetric, and transitive, $\sim$ is an equivalence relation.

3. Let $x \sim y$.

$$|x| + |y| = 10$$

If we reorganize, we get

$$|y| + |x| = 10$$

Thus, if $x \sim y$, then $y \sim x$, meaning that R is symmetric.

If R is reflexive, then $3 \sim 3$. Let us check this by plugging in:

$$|x| + |y| = 10$$

$$|3| + |3| = 10$$

$$3 + 3 = 10$$

$$6 = 10$$

This means $3 \not\sim 3$, thus R is not reflexive.

Plugging in, we see that $4 \sim 6$ and $6 \sim -4$ is true.

$$|4| + |6| = 10$$

$$4 + 6 = 10$$

$$10 = 10$$

$$|6| + |-4| = 10$$

$$6 + 4 = 10$$

$$10 = 10$$

If R is transitive, then $4 \sim -4$. Let's check this.

$$|4| + |-4| = 10$$

$$4 + 4 = 10$$

$$8 = 10$$

Obviously, $8 \neq 10$ meaning $4 \not\sim -4$, so R is not transitive, by contradiction.

4. **a**. Let $a, b, c \in \mathbb{R}$ such that $\sin(a) = \sin(b) = \sin(c)$. Obviously, $\sin(a) = \sin(a)$, so $f(a) = f(a)$, and $a \sim a$, meaning R is reflexive. Since $\sin(a) = \sin(b)$, $\sin(b) = \sin(a)$, so $f(a) = f(b)$ and $f(b) = f(a)$, thus $a \sim b$ and $b \sim a$, meaning R is symmetric. Since $\sin(a) = \sin(b) = \sin(c)$, $\sin(a) = \sin(b)$, $\sin(b) = \sin(c)$, and $\sin(a) = \sin(c)$, so $a \sim b$, $b \sim c$, and $a \sim c$, meaning R is transitive. Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

b. $[\pi] = \{k\pi | k \in \mathbb{Z}\}$

5. **a**. $[(3, 4)] = \{c, d \in \mathbb{R} | c^2 + d^2 = 25\}$

b. The equivalence class $[(a, b)]$, where $a, b \in \mathbb{Z}$, can, generally, be represented as a circle whose radius is $\sqrt{a^2 + b^2}$ and whose center is the origin.