1. **a.** $q = 6, r = 2$

b. $q = 12, r = 0$

c. $q = -2, r = 10$

d. $q = -3, r = 7$

2. **a.** If we have an even integer a, then $2|a$ and we can say that $a = 2k$ for some $k \in \mathbb{Z}$. Furthermore, we can say that any even integer will be in a similar form. Multiplying $a \cdot b$, where a is an even integer and b is some integer we get

$$a \cdot b$$

$$= 2k \cdot b$$

$$= 2(kb)$$

Since it is in this form, ab is an even integer.

b. If an integer a is odd, it is not even, and thus $2 \not| v$. Thus, it can be represented as $2k + 1$ for some $k \in \mathbb{Z}$, since $2k$ is divisible by 2, but 1 isn't. We are looking for the product of two odd integers x and y. Since they are odd, they can be written as $2m + 1$ and $2n + 1$ respectively for some $m, n \in \mathbb{Z}$.

$$x \cdot y$$

$$(2m + 1)(2n + 1)$$

$$= 2m(2n + 1) + 1(2n + 1)$$

$$= 4mn + 2m + 2n + 1$$

$$= 2(2mn + m + n) + 1$$

Since it is in this form, the product is odd. Thus, we have proven that the product of two odd integers is odd.

3. We have proven that if $a|b$ and $c|d$ then $ab|cd$. So, if $a|b$ and $a|b$, then we can say $(a \cdot a)|(b \cdot b) \Rightarrow a^2|b^2$.

4. Let the base case be $n = 1$. $a|b$ as it's given, so the base case is true. For the inductive hypothesis, assume we have a $k \in \mathbb{Z}$ such that $a^k|b^k$. By the inductive hypothesis and since $a|b$, we can apply the theorem from problem 3 to get

$$(a^k \cdot a)|(b^k \cdot b)$$
$$a^{k+1}|b^{k+1}$$

This is the same formula for $k + 1$; thus, by induction we have shown that $a^n|b^n$ for any positive integer n.

5. If n is composite, it has at least 2, not necessarily distinct, prime factors. Let the prime factors of n be $d_1, \dots, d_k$. Thus, $n = d_1 \times \cdots \times d_k$. Now, let's assume that all the factors are $> \sqrt{n}$. Plugging this in we get $n > \sqrt{n} \times \cdots \times \sqrt{n}$. Since, there n has at least two prime factors, it is at minimum greater than $\sqrt{n} \cdot \sqrt{n} \Rightarrow n$. However, this is impossible as an integer n can never be greater than itself. Thus, by contradiction, at least one of the prime factors of n must be $\leq \sqrt{n}$.

6. Let p and q be consecutive odd primes such that $p < q$. Since they are odd, they can be represented as $2k+1$ and $2j+1$ respectively for some $j, k \in \mathbb{Z}$.

$$\begin{aligned} &p + q \\ &= (2k+1) + (2j+1) \\ &= 2k + 2j + 2 \\ &= 2(k+j+1) \end{aligned}$$

As we can see, $p+q$ is even, and thus $2|(p+q)$. From above, we defined $p < q$. Let's expand on this.

$$\begin{aligned} p &< q \\ p + p &< q + p \\ 2p &< p + q \\ 2p &< 2(k+j+1) \\ p &< k + j + 1 \end{aligned} \qquad\qquad \begin{aligned} p &< q \\ p + q &< q + q \\ 2(k+j+1) &< 2q \\ k + j + 1 &< q \end{aligned}$$

So, since $p < q$, $p < k + j + 1 < q$. Since p and q are consecutive primes, this means that all the integers between them are not prime and, thus, have 1 factor or at least 2 factors. For it to have 1 factor, $k + j + 1$ would have to be equal to 1 and thus $2(k + j + 1)$ would have to equal to 2. However, this is not possible as there are no two positive consecutive primes that sum to 2. Thus, $k+j+1$ has to have at least 2 factors. Considering the additional factor of 2 we discovered above, this means that $2(k + j + 1)$, and thus $p + q$, has at least 3 prime factors.