Problem 1

Find the general solution to the linear diophantine equation $412x + 600y = -24$.

We fill first find one solution with the Extended Euclidean Algorithm.

q	r	x	y
	600	0	1
1	412	1	0
2	188	-1	1
5	36	3	-2
4	8	-16	11
2	4	67	-46

So, $(412, 600) = 4$ and $(67, -46)$ is a solution to the equation $412x + 600y = (412, 600) = 4$. $-24 \div 4 = -6$, so we can scale $(67, -46)$ by -6 to get a solution to the given equation. Scaling, we get $(-402, 276)$. We can then plug these values into the general solution equations to get

$$x = -402 + 150t$$

$$y = 276 - 103t$$

Problem 2

True or False, and why: $192x + 250y + 405z = A$ has an integer solution for every integer A.

Let's find the solutions for $192x + 250y = (192, 250)$ with the Extended Euclidean Algorithm.

q	r	x	y
	250	0	1
1	192	1	0
3	58	-1	1
3	18	4	-3
4	4	-13	10
2	2	56	-43

Thus, $(56, -43)$ is a solution to $192x + 250y = 2$. Now, let's scale by 203, so it's equal to $2 * 203 = 406$. Doing do, we get $(11368, -8729)$. Now, let's substitute into our original equation. For $(11368, -8729, z)$, $192x + 250y + 405z = 406 + 405z$. We can see that substituting

$z = -1$, we get $406 + 405z = 1$. So, for the values $(11368, -8729, -1)$, $192x + 250y + 405z = 1$. Since, $\forall A \in \mathbb{Z}$, $1 \cdot A = A$, we can simply scale this ordered pair by A to get any A. Thus, it is true that $192x + 250y + 405z = A$ has an integer solution for every integer A.

Problem 3

Find a non-negative solution to $34s + 76t = 754$. Obtain it methodically, not just by magically naming an answer.

Let us first find a solution for $34s + 76t = (34, 76)$.

q	r	s	t
	76	0	1
2	34	1	0
4	8	-2	1
4	2	9	-4

Thus, $(34, 76) = 2$ and $(9, -4)$ is a solution to $34s + 76t = 2$. We scale by $\frac{754}{2} = 377$ to get $(3393, -1508)$. We can get all solutions to $34s + 76t = 754$ with the formulas

$$s = 3393 + 38d$$

$$t = -1508 - 17d$$

For a non-negative solution, $s > 0$ and $t > 0$.

$$\begin{aligned} s &> 0 \\ 3393 + 38d &> 0 \\ 38d &> -3393 \\ d &\geq -89 \end{aligned} \qquad\qquad \begin{aligned} t &> 0 \\ -1508 - 17d &> 0 \\ -17d &> 1508 \\ d &\leq -89 \end{aligned}$$

Thus, the only such solution is when $d = -89$. Plugging this into the general solution formulas, we get $(11, 5)$ as the only non-negative solution to $34s + 76t = 754$.

Problem 4

What combinations of nickels, dimes, and quarters can have 16 coins totaling exactly \$2.00?

Let n be the number of nickels, d the number of dimes, and q the number of quarters. Thus, we can create two equations:

$$n + d + q = 16$$

$$5n + 10d + 25q = 200$$

Obviously, $(5, 10, 25, 200) = 5$, so we can divide the second equation by 5 giving us

$$n + 2d + 5q = 40$$

Let's rearrange and plug in the first equation.

$$n + d + q = 16$$

$$n = 16 - d - q$$

$$16 - d - q + 2d + 5q = 40$$

$$d + 4q = 24$$

This is a standard linear diophantine equation, so we can follow similar steps as previous problems. However, it is obvious that $(1, 4) = 1$ and we can easily find $(0, 6)$ as one solution. We can then find the equations for the general solutions.

$$d = 4t$$

$$q = 6 - t$$

From our rearranged first equation, we can say

$$n = 16 - d - q$$

$$n = 16 - 4t - 6 + t$$

$$n = 10 - 3t$$

Thus, for any $t \in \mathbb{Z}$, we will satisfy our original two equations. However, in the real world, we can not have negative coins, so let's account for that.

$$\begin{aligned} n &\geq 0 & d &\geq 0 & q &\geq 0 \\ 10 - 3t &\geq 0 & 4t &\geq 0 & 6 - t &\geq 0 \\ 3t &\leq 10 & t &\geq 0 & t &\leq 6 \\ t &\leq \frac{10}{3} \end{aligned}$$

Combining all 3 inequalities, we get $0 \leq t \leq \frac{10}{3}$. The integer solutions to this are $t = 0, 1, 2, 3$. Thus, the only combination of 16 nickels, dimes, or quarters that equal \$2.00 are (in the form (n, d, q)) $(10, 0, 6)$, $(7, 4, 5)$, $(4, 8, 4)$, and $(1, 12, 3)$.