Problem 1

Let $g(x, y) = \sqrt[3]{xy}$.

a. Is g continuous at the origin?

b. Calculate $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ when $xy \neq 0$.

c. Show that $g_x(0, 0)$ and $g_y(0, 0)$ exist.

d. Is g differentiable at the origin?

Solution

a. We can check if g is continuous at the origin by simply checking if the limit exists and is equal to $g(0, 0)$.

$$g(0,0) = \sqrt[3]{0 \cdot 0} = 0$$
$$\lim_{(x,y)\to(0,0)} g(x, y) = \sqrt[3]{0 \cdot 0} = 0 = g(0, 0)$$

Thus, g is continuous at the origin.

b.

$$\frac{\partial g}{\partial x} = \frac{1}{3}x^{-\frac{2}{3}}\sqrt[3]{y} = \frac{\sqrt[3]{y}}{3\sqrt[3]{x^2}}$$
$$\frac{\partial g}{\partial y} = \frac{1}{3}y^{-\frac{2}{3}}\sqrt[3]{x} = \frac{\sqrt[3]{x}}{3\sqrt[3]{y^2}}$$

c. Using the limit definition of partial derivatives,

$$\begin{aligned} g_x(0,0) &= \lim_{h\to 0} \frac{g(0+h,0) - g(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt[3]{h \cdot 0} - 0}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= 0 \end{aligned}$$

$$\begin{aligned} g_y(0,0) &= \lim_{h\to 0} \frac{g(0,0+h) - g(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt[3]{0 \cdot h} - 0}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= 0 \end{aligned}$$

Thus, the partial derivatives exist.

d. For g to be differentiable at the origin, its partial derivatives must exist, and the following must hold:

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y) - L(x,y)}{||(x,y) - (0,0)||} = 0$$

where $L(x, y) = g(\vec{a}) + \nabla g(\vec{a}) \cdot ((x, y) - \vec{a})$, the linear approximation of g at $\vec{a}$. We know the partial derivatives exist, so we only need to check the last requirement.

$$L(0,0) = g(0,0) + g_x(0,0)(0-0) + g_y(0,0)(0-0) = 0$$

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{g(x,y)-L(x,y)}{||(x,y)-(0,0)||} &= \lim_{(x,y)\to(0,0)} \frac{g(x,y)-0}{||(x,y)||} \\
&= \lim_{(x,y)\to(0,0)} \frac{\sqrt[3]{xy}}{\sqrt{x^2+y^2}} \\
&= \lim_{r\to 0} \frac{\sqrt[3]{r\cos\theta\cdot r\sin\theta}}{\sqrt{r^2}} \\
&= \lim_{r\to 0} \frac{r^{\frac{2}{3}}\sqrt[3]{\cos\theta\cdot\sin\theta}}{|r|} \\
&= \lim_{r\to 0} |r|^{-\frac{1}{3}}\sqrt[3]{\cos\theta\cdot\sin\theta}
\end{aligned}
$$

There are two cases, $\cos\theta\cdot\sin\theta = 0$ and $\cos\theta\cdot\sin\theta \neq 0$. If $\cos\theta\cdot\sin\theta = 0$, then the limit is 0. If $\cos\theta\cdot\sin\theta \neq 0$, then the limit is not 0 as $|r|^{-\frac{1}{3}}$ goes to ∞ as $r \to 0$. Since this limit is not 0 from all directions, g is not differentiable at the origin.