1. **a.**

Passes	Fails
Addition	Addition
- Closure - Compatibility - Associativity - Commutativity	- Identity - For there to be an identity, there must be a positive integer b such that $b+a = a$ for all positive integers a. This means that b must be greater than all a. This is obviously not possible as there is no greatest positive integer. - Inverses - Since there is no identity, there cannot be an inverse. Furthermore, we do not have opposite numbers as all elements are positive integers.
Multiplication	Multiplication
- Closure - Compatibility - Associativity - Commutativity - Identity - $1 \times a$ will always equal a because all positive integers are $\geq$ than 1. - Distributivity over addition	- None

b.

Passes	Fails
Addition	Addition
- Closure - Compatibility - Associativity - Commutativity - Identity - $(0, 0)$ - Inverses - If an ordered pair (a, b) exists, it has an invers $(-(a, b))$ or $(-a, -b)$.	- None
Multiplication	Multiplication
- Closure - Compatibility - Associativity - Commutativity - Identity - $(1, 1)$ - Distributivity over addition	- None

2. By the reflexivity of equals $0 = 0$. By compatibility of addition with equality, we can add $(x+y) + (-(x+y)) + x + (-x) + y + (-y)$ to both sides.

$$0 = 0$$

$$0 + (x + y) + (-(x + y)) + x + (-x) + y + (-y) = 0 + (x + y) + (-(x + y)) + x + (-x) + y + (-y)$$

By the additive inverse

$$0 + (x + y) + (-(x + y)) + 0 + 0 = 0 + 0 + x + (-x) + y + (-y)$$

$$(x + y) + (-(x + y)) = +x + (-x) + y + (-y)$$

By compatibility, we can add $(-x) + (-y)$ to both sides.

$$(x + y) + (-(x + y)) + (-x) + (-y) = x + (-x) + y + (-y) + (-x) + (-y)$$

$$x + y + (-(x + y)) + (-x) + (-y) = x + (-x) + y + (-y)) + (-x) + (-y)$$

By much associativity

$$x + (-x) + y + (-y) + (-(x + y)) = x + (-x) + (-x) + y + (-y) + (-y)$$

Thus, by the additive inverse

$$0 + 0 + (-(x + y)) = 0 + (-x) + 0 + (-y)$$

$$(-(x + y)) = (-x) + (-y)$$

Thus, we have shown that $(-(x + y)) = (-x) + (-y)$.

3. By left distributivity

$$z \cdot (x + y) = z \cdot x + z \cdot y$$

Let $w = (x + y)$.

$$z \cdot w = z \cdot x + z \cdot y$$

By the commutativity of multiplication, $z \cdot w = w \cdot z$, so

$$w \cdot z = z \cdot x + z \cdot y$$

Substituting $(x + y)$ back in for w,

$$(x + y) \cdot z = z \cdot x + z \cdot y$$

Thus, we have proven right distributivity.

4. **a**. In class, we found

$$0 \cdot a = 0$$

By the additive inverse, $1 + (-1) = 0$, thus

$$(1 + (-1)) \cdot a = 0$$

By right distributivity,

$$1 \cdot a + (-1) \cdot a = 0$$

By the multiplicative identity, $1 \cdot a = a$, so

$$a + (-1) \cdot a = 0$$

By additive compatibility,

$$a + (-1) \cdot a + (-a) = 0 + (-a)$$

By additive associativity,

$$a + (-a) + (-1) \cdot a = (-a)$$

By the additive inverse,

$$0 + (-1) \cdot a = (-a)$$

$$(-1) \cdot a = (-a)$$

Thus, we have shown that $(-1) \cdot a = (-a)$.

b. Let us start with $(-a) \cdot b$. From part a, we know $a = (-1) \cdot a$. Thus,

$$(-a) \cdot b = ((-1) \cdot a) \cdot b$$

By multiplicative associativity,

$$= (-1) \cdot (a \cdot b)$$

By part a again, we can say that this equals $(-(a \cdot b))$. Therefore, we have shown that $(-a) \cdot b = -(a \cdot b)$.