Problem 1

Carefully demonstrate that the distributive law holds for rational numbers.

Let $a, b, c, d, e,$ and $f \in \mathbb{Q}$.

$$\begin{aligned}
\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right) &= \frac{a}{b}\left(\frac{cf+de}{df}\right) \\
&= \frac{a(cf+de)}{bdf} \\
&= \frac{acf+ade}{bdf} \\
&= \frac{ac\cancel{f}}{bd\cancel{f}}+\frac{a\cancel{d}e}{b\cancel{d}f} \\
\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right) &= \frac{ac}{bd}+\frac{ae}{bf}
\end{aligned}$$

Thus, the distributive law holds for the rational numbers.

Problem 2

Given rational numbers $0 < x < y$, show that $x^{-1} > y^{-1}$.

Since $y > x$, we can say that $y = x + \epsilon$ for some $\epsilon > 0$. Thus:

$$\begin{aligned}
\frac{\epsilon}{x(x+\epsilon)} &> 0 \quad (x \text{ and } \epsilon \text{ are both positive}) \\
\frac{\epsilon+x-x}{x(x+\epsilon)} &> 0 \\
\frac{(x+\epsilon)-x}{x(x+\epsilon)} &> 0 \\
\frac{1}{x}-\frac{1}{x+\epsilon} &> 0 \\
\frac{1}{x}-\frac{1}{x+\epsilon}+\frac{1}{x+\epsilon} &> 0+\frac{1}{x+\epsilon} \\
\frac{1}{x} &> \frac{1}{x+\epsilon} \\
x^{-1} &> (x+\epsilon)^{-1} \\
x^{-1} &> y^{-1}
\end{aligned}$$

Thus, we have shown that $x^{-1} > y^{-1}$ when $0 < x < y$.

Problem 3

Show that the rational numbers have the *Archimedean property*: for any positive rational numbers x and y, you can find a positive integer n so that $nx > y$.

Let $a, b, c, d \in \mathbb{N}$, such that $a, b, c, d \geq 1$, $x = \frac{a}{b}$, and $y = \frac{c}{d}$.

$$\begin{aligned} nx &> y \\ n \cdot \frac{a}{b} &> \frac{c}{d} \\ n \cdot \frac{a}{b} \cdot \frac{d}{c} &> \frac{c}{d} \cdot \frac{d}{c} \\ n \cdot \frac{ad}{bc} &> 1 \end{aligned}$$

For simplicity, let $n = 2bc$. Then:

$$\begin{aligned} 2bc \cdot \frac{ad}{bc} &> 1 \\ 2ad &> 1 \\ ad &> \frac{1}{2} \end{aligned}$$

Since $a, d \geq 1$, $ad \geq 1 \cdot 1 \Rightarrow ad \geq 1$. $1 > \frac{1}{2}$, so $ad > \frac{1}{2}$ is true. Thus, the rational numbers have the Archimedean property.