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Challenges and Solutions to Realisability Semantics for Intersection Types with Expansion Variables

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Abstract. Expansion is a crucial operation for calculating principal typings in intersection type systems. Because the early definitions of expansion were complicated, *E-variables* were introduced in order to make the calculations easier to mechanise and reason about. Recently, E-variables have been further simplified and generalised to also allow calculating other type operators than just intersection. There has been much work on semantics for type systems with intersection types, but none whatsoever before our work, on type systems with E-variables. In this paper we expose the challenges of building a semantics for E-variables and we provide a novel solution. Because it is unclear how to devise a space of meanings for E-variables, we develop instead a space of meanings for types that is hierarchical. First, we index each type with a natural number and show that although this intuitively captures the use of E-variables, it is difficult to index the universal type ω with this hierarchy and it is not possible to obtain completeness of the semantics if more than one E-variable is used. We then move to a more complex semantics where each type is associated with a list of natural numbers and establish that both ω and an arbitrary number of E-variables can be represented without losing any of the desirable properties of a realisability semantics.

Keywords: Realisability semantics, expansion variables, intersection types, completeness

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1. Introduction

Intersection types were developed in the late 1970s to type λ-terms that are untypable with simple types; they do this by providing a kind of finitary type polymorphism where the usage of types is listed rather than quantified over. They have been useful in reasoning about the semantics of the λ-calculus, and have been investigated for use in static program analysis. *Expansion* was introduced at the end of the 1970s as a crucial procedure for calculating *principal typings* for λ-terms in type systems with intersection types, enabling support for compositional type inference. Coppo, Dezani, and Venneri [4] introduced the operation of *expansion* on *typings* (pairs of a type environment and a result type) for calculating the possible typings of a term when using intersection types. As a simple example, the λ-term $M = (\lambda x.x(\lambda y.yz))$ can be assigned the typing $\Phi_1 = \langle (z : a) \vdash (((a \to b) \to b) \to c) \to c \rangle$, which happens to be its principal typing. The term M can also be assigned the typing $\Phi_2 = \langle (z : a_1 \sqcap a_2) \vdash (((a_1 \to b_1) \to b_1) \sqcap ((a_2 \to b_2) \to b_2) \to c) \to c \rangle$, and an expansion operation can obtain Φ_2 from Φ_1.

Because the early definitions of expansion were complicated, *E-variables* were introduced in order to make the calculations easier to mechanize and reason about. For example, in System E [2], the typing Φ_1 from above is replaced by $\Phi_3 = \langle (z : ea) \vdash (e((a \to b) \to b) \to c) \to c \rangle$, which differs from Φ_1 by the insertion of the E-variable e at two places, and Φ_2 can be obtained from Φ_3 by substituting for e the *expansion term* $E = (a := a_1, b := b_1) \sqcap (a := a_2, b := b_2)$. Carlier and Wells [3] have surveyed the history of expansion and also E-variables.

In many kinds of semantics, the meaning of a type T is calculated by an expression $[T]_\nu$ that takes two parameters, the type T and also a valuation ν that assigns to type variables the same kind of meanings that are assigned to types. To extend this idea to types with E-variables, we would need to devise some space of possible meanings for E-variables. Given that a type $e\,T$ can be turned by expansion into a new type $S_1(T) \sqcap S_2(T)$, where S_1 and S_2 are arbitrary substitutions (they can be arbitrary further expansions), and that this can introduce an unbounded number of new variables (both E-variables and regular type variables), the situation is complicated. Because it is unclear how to devise a space of meanings for expansions and E-variables, we instead develop a space of meanings for types that is hierarchical in the sense of having many degrees. We specifically avoid trying to give a semantics to the operation of expansion, and instead treat only the E-variables. Although this idea is not perfect, it seems to go quite far in giving an intuition for E-variables, namely that each E-variable acts as a kind of capsule that isolates parts of the λ-term being analyzed by the typing.

In the open problems published in the proceedings of the Lecture Notes in Computer Science symposium held in 1975 [7], it is suggested that an arrow type expresses functionality. Following this idea, a type's semantics is given as a set of closed λ-terms with behaviour related to the specification given by the type. Hence, the semantic approach we use is realisability semantics. Atomic types (e.g., type variables) are interpreted as sets of λ-terms that are *saturated*, meaning that they are closed under β-expansion (i.e., β-reduction in reverse). Arrow and intersection types are interpreted naturally by function spaces and set intersection. Realisability allows showing *soundness* in the sense that the meaning of a type T contains all closed λ-terms that can be assigned T as their result type. This has been shown useful for characterising the behaviour of typed λ-terms [14]. One also wants to show the converse of soundness which is called *completeness*, i.e., that every closed λ-term in the meaning of T can be assigned T as its result type.

Hindley [9, 10, 11] was the first to study this notion of completeness for a simple type system and he showed that all the types of that system have the completeness property. Then, he generalised his completeness proof for an intersection type system [8]. Using his completeness theorem for the realisability semantics based on the sets of λ-terms saturated by $\beta\eta$-equivalence, Hindley has shown that simple types are uniquely realised by the λ-terms which are typable by these types. However, Hindley's result does not hold for his intersection type system and the completeness theorems were established with the sets of λ-terms saturated by $\beta\eta$-equivalence. In this paper, our completeness result depends only on the weaker requirement of β-equivalence, and we have managed to make simpler proofs that avoid needing η-reduction, Church-Rosser (a.k.a. confluence), or strong normalisation (SN) (although we do establish both confluence and SN for both β and $\beta\eta$).

Other work on realizability we have consulted includes that by Labib-Sami [15], Farkh and Nour [6], and Coquand [5], although none of this work deals with intersection types or E-variables. Related work on realisability that deals with intersection types includes that by Kamareddine and Nour [12], which gives a realisability semantics with soundness and completeness for an intersection type system. This system is quite different from the three hierarchical systems we present in this paper. The main difference being the hierarchies which did not exist in [12].

Initially, we aimed to give a realisability semantics for the system of expansions proposed by Carlier and Wells in [3]. In order to simplify our study, we considered the system with the expansion variables but without the expansion rewriting rules. In essence, this meant that the syntax of terms is: $M ::= x \mid (M\,N) \mid (\lambda x.M)$ where x ranges over a countably infinite set of variables $\mathcal{V}$, that the syntax of types is: $T ::= a \mid \omega \mid T_1 \to T_2 \mid T_1 \sqcap T_2 \mid eT$ where a is a basic type ranging over a countably infinite set of type variables $\mathcal{A}$ and e is an expansion variable ranging over a countably infinite set of expansion variables $\mathcal{E}$, and that the typing rules are:

$$\frac{}{x : \langle (x : T) \vdash T \rangle}\ \text{var}$$

$$\frac{}{M : \langle () \vdash \omega \rangle}\ \omega$$

$$\frac{M : \langle \Gamma, (x : T_1) \vdash T_2 \rangle}{\lambda x.M : \langle \Gamma \vdash T_1 \to T_2 \rangle}\ \text{abs}$$

$$\frac{M_1 : \langle \Gamma_1 \vdash T_1 \to T_2 \rangle \quad M_2 : \langle \Gamma_2 \vdash T_1 \rangle}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T_2 \rangle}\ \text{app}$$

$$\frac{M : \langle \Gamma_1 \vdash T_1 \rangle \quad M : \langle \Gamma_2 \vdash T_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T_1 \sqcap T_2 \rangle}\ \sqcap$$

$$\frac{M : \langle \Gamma \vdash T \rangle}{M : \langle e\Gamma \vdash eT \rangle}\ \text{e-app}$$

In order to give a realisability semantics for this system, we needed to define the interpretation of a type to be a set of terms having this type. We were obviously forced to distinguish between the interpretation of T and eT. However, in the typing rule e-app, the term M is unchanged and this poses difficulties. For this reason, we modified slightly the above type system by indexing the terms of the λ-calculus giving us the syntax of terms as: $M ::= x^i \mid (M\,N) \mid (\lambda x^i.M)$ (where i are natural numbers and where M and N need to satisfy a certain condition before $(M\ N)$ is allowed as a term) and by slightly changing our type rules and in particular the rule e-app:

$$\frac{M : \langle \Gamma \vdash_i U \rangle}{M^+ : \langle e\Gamma \vdash_i eU \rangle}\ (exp)$$

In this rule, M^+ is M where all the indices are increased by 1. Obviously these indices needed a revision of the β-reduction and of the typing rules in order to preserve the desirable properties of the type system and the realisability semantics. For this, we defined the good terms and the good types and showed that these notions go hand in hand (e.g., a good type contains only good terms). We developed a realisability semantics where each use of an E-variable in a type corresponds to an index at which evaluation occurs in the λ-term that is assigned the type. This is an elegant solution that captures the intuition behind E-variables. However, in order for this new type system to function well, it was necessary to consider λI-terms only (removing a subterm from M also removes important information about M) and to drop ω completely. This led us to the introduction of $\lambda I^{\mathbb{N}}$-calculus and our first type system $\vdash_1$ for which we developed a sound realisability semantics for E-variables. However, although the first type system $\vdash_1$ is crucial to understand the intuition behind the indexing we propose, the realisability semantics for $\vdash_1$ does not satisfy completeness (and neither subject reduction). For this reason, we modified our system $\vdash_1$ by considering a smaller set of types (where intersections and expansions cannot occur directly to the right of an arrow), and by adding subtyping rules. This new system $\vdash_2$ has both soundness and subject reduction. As for completeness, we needed to limit the list of expansion variables to a single element list. This problem of completeness for $\vdash_2$ comes from the fact that the indexes (the natural numbers) do not permit us to differentiate between the types e_1T and e_2T for two different expansion variables e_1 and e_2. So, again, we were forced to revise our type system. For this, we decided to limit our λ-terms by indexing them by lists of natural numbers (where the natural number i represents the expansion variable e_i). This way the rule exp above will allow us to distinguish the interpretations of the types e_iT and e_jT when $e_i \neq e_j$. Furthermore, this way, our λ-terms are constructed in such a way that K-reductions do not limit the information on the starting terms (in fact, β-reduction is not always allowed). In order to obtain completeness with the ω-rule, we should also consider ω indexed by lists. This means that the new calculus becomes rather heavy but this is unavoidable. It is needed to obtain a complete realisability semantics where an arbitrary (possibly infinite) number of expansion variables is allowed and where the universal type ω is present. The use of lists complicates matters and hence, needs to be understood in the context of the first semantics where indices are natural numbers rather than lists of natural numbers. In addition to the above, we have considered three notions of saturations (in line with the literature) illustrating that these notions behave well in our complete realisability semantics.

Section 2 gives the syntax of the indexed calculi we consider in this paper: the $\lambda I^{\mathbb{N}}$-calculus, which is the λI-calculus with each variable marked by a natural number *degree*, and the full λ-calculus $\lambda^{\mathcal{L}_{\mathbb{N}}}$-calculus indexed with finite sequences of natural numbers. We show the confluence of β, $\beta\eta$ and weak head reduction h on our indexed λ-calculi. Section 3 introduces the syntax and terminology for types used in both indexed calculi. Section 4 introduces our three intersection type systems with E-variables $\vdash_i$ for $i \in \{1, 2, 3\}$, where in one, the syntax of types is not restricted (and hence subject reduction fails) but in the other two it is restricted but then extended with a subtyping relation. In Section 5 we study the type theoretical properties of our three type systems including subject reduction and expansion with respect to our various reduction relations $(\beta, \beta\eta, h)$. In Section 6, we introduce our realisability semantics and show its soundness for all the three type systems we consider (and for all the reduction relations). In

Section 7 we establish the challenges of showing completeness for the realisability semantics of the first two systems. We show that completeness does not hold for the first system and that it also does not hold for the second system if more than one expansion variable is used, but does hold for a restriction of this system to one single E-variable. This is an important study in the semantics of intersection type systems with expansion variables since a unique expansion variable can be used many times and can occur nested. In Section 8 we establish the completeness of $\vdash_3$ by introducing a special interpretation. We conclude in Section 9. Due to space limitations, we omit the details of the proofs. Full proofs however can be found in the expanded version of this article (currently at [13]) which will always be available at the authors' web pages.

2. The syntax of the indexed λ-calculi

We assume that if a metavariable v ranges over a set $\mathcal{S}$ then v_i for $i \geq 0$ and v', v'', etc. also range over $\mathcal{S}$. A binary relation is a set of pairs. Let *rel* range over binary relations. Let $\mathrm{dom}(rel) = \{x \mid \langle x, y\rangle \in rel\}$ and $\mathrm{ran}(rel) = \{y \mid \langle x, y\rangle \in rel\}$. A function is a binary relation *fun* such that if $\{\langle x, y\rangle, \langle x, z\rangle\} \subseteq fun$ then $y = z$. Let *fun* range over functions. Let $s \to s' = \{fun \mid \mathrm{dom}(fun) \subseteq s \wedge \mathrm{ran}(fun) \subseteq s'\}$. We sometimes write $x : s$ instead of $x \in s$.

Definition 2.1. (Indices)
We have two kinds of indices: natural numbers for our first semantics (clause 1) and lists of natural numbers for our second semantics (clauses 2..5). We let I, J, range over indices.

1. Let n, m range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$.
2. Let L, K, R range over $\mathcal{L}_\mathbb{N}$ the set of finite sequences of natural numbers $(n_i)_{1\leq i\leq l}$. We denote $\oslash$ the empty sequence of natural numbers.
3. If $L = (n_i)_{1\leq i\leq l}$, we use $m :: L$ to denote the sequence $(r_i)_{1\leq i\leq l+1}$ where $r_1 = m$ and for all $i \in \{2, \dots, l+1\}$, $r_i = n_{i-1}$. In particular, $k :: \oslash = (k)$.
4. If $L = (n_i)_{1\leq i\leq n}$ and $K = (m_i)_{1\leq i\leq m}$, we use $L :: K$ to denote the sequence $(r_i)_{1\leq i\leq n+m}$ where for all $i \in \{1, \dots, n\}$, $r_i = n_i$ and for all $i \in \{n+1, \dots, n+m\}$, $r_i = m_{i-n}$. In particular, $L :: \oslash = \oslash :: L = L$.
5. We define on $\mathcal{L}_\mathbb{N}$ a binary relation $\preceq$ by:

   $L_1 \preceq L_2$ (or $L_2 \succeq L_1$) if there exists $L_3 \in \mathcal{L}_\mathbb{N}$ such that $L_2 = L_1 :: L_3$.

Lemma 2.1. $\preceq$ is an order relation on $\mathcal{L}_\mathbb{N}$.

We assume that x, y, z range over a countably infinite set of variables $\mathcal{V}$.

We will define two indexed calculi: the $\lambda I^\mathbb{N}$-calculus (whose set of terms is called $\mathcal{M}_1$ as well as $\mathcal{M}_2$ for notational reasons) and the $\lambda^{\mathcal{L}_\mathbb{N}}$-calculus (whose set of terms is $\mathcal{M}_3$). As obvious, indices in $\lambda I^\mathbb{N}$ are simple but only allow the I-part of the calculus.

We let M, N, P range over $\mathcal{M}_1 = \mathcal{M}_2$ (resp. $\mathcal{M}_3$) and use $=$ for syntactic equality. We assume the usual definition of subterms and the usual convention for parentheses and their omission (see Barendregt [1] and Krivine [14]).

The joinability $M \diamond N$ of terms M and N ensures that in any term that contains M and N, each variable has a unique index (note that it is more accurate to include this as part of the simultaneous inductions in Definitions 2.3 and 2.5, but for clarity, we took it apart here).

Definition 2.2. (Joinability $\diamond$)
Let $i \in \{1, 2, 3\}$.

- Let M, N be terms of $\lambda I^{\mathbb{N}}$ (resp. $\lambda^{\mathcal{L}_{\mathbb{N}}}$) and let $\mathrm{FV}(M)$ and $\mathrm{FV}(N)$ be the corresponding free variables. We say that M and N are joinable and write $M \diamond N$ iff for all $x \in \mathcal{V}$, if $x^I \in \mathrm{FV}(M)$ and $x^J \in \mathrm{FV}(N)$ (where I, J are indices in $\mathbb{N}$ (resp. $\mathcal{L}_{\mathbb{N}}$)), then $I = J$.
- If $\mathcal{X} \subseteq \mathcal{M}_i$ such that $\forall M, N \in \mathcal{X}, M \diamond N$, we write, $\diamond \mathcal{X}$.
- If $\mathcal{X} \subseteq \mathcal{M}_i$ and $M \in \mathcal{M}_i$ such that $\forall N \in \mathcal{X}, M \diamond N$, we write, $M \diamond \mathcal{X}$.

Now we give the syntax of $\lambda I^{\mathbb{N}}$, an indexed version of the λI-calculus where indices (which range over the set of natural numbers $\mathbb{N}$) help categorise the *good terms* where the degree of a function is never larger than that of its argument. This amounts to having the full λI-calculus at each index and creating new λI-terms through a mixing recipe.

Definition 2.3. (The set of terms $\mathcal{M}_1$ (also called $\mathcal{M}_2$))
The set of terms $\mathcal{M}_1 = \mathcal{M}_2$, the set of free variables $\mathrm{FV}(M)$ of $M \in \mathcal{M}_2$ and the degree $\mathrm{d}(M)$ of a term M, are defined by simultaneous induction:

- If $x \in \mathcal{V}$, $n \in \mathbb{N}$, then $x^n \in \mathcal{M}_2$, $\mathrm{FV}(x^n) = \{x^n\}$, and $\mathrm{d}(x^n) = n$.
- If $M, N \in \mathcal{M}_2$ such that $M \diamond N$ (see Definition 2.2), then $M\,N \in \mathcal{M}_2$, $\mathrm{FV}(M\,N) = \mathrm{FV}(M) \cup \mathrm{FV}(N)$ and $\mathrm{d}(M\ N) = \min(\mathrm{d}(M), \mathrm{d}(N))$ (where $\min$ is the minimum)
- If $M \in \mathcal{M}_2$ and $x^n \in \mathrm{FV}(M)$, then $\lambda x^n.M \in \mathcal{M}_2$, $\mathrm{FV}(\lambda x^n.M) = \mathrm{FV}(M) \setminus \{x^n\}$, and $\mathrm{d}((\lambda x^n.M_1)) = \mathrm{d}(M_1)$.

Note that a subterm of $M \in \mathcal{M}_2$ is also in $\mathcal{M}_2$.

Here is now the syntax of good terms in the $\lambda I^{\mathbb{N}}$-calculus.

Definition 2.4. (The set of good terms $\mathbb{M} \subset \mathcal{M}_2$)

1. The set of good terms $\mathbb{M} \subset \mathcal{M}_2$ is defined by:
   - If $x \in \mathcal{V}$, $n \in \mathbb{N}$, then $x^n \in \mathbb{M}$,
   - If $M, N \in \mathbb{M}$, $M \diamond N$ and $\mathrm{d}(M) \leq \mathrm{d}(N)$ then $M\,N \in \mathbb{M}$.
   - If $M \in \mathbb{M}$ and $x^n \in \mathrm{FV}(M)$, then $\lambda x^n.M \in \mathbb{M}$.

   Note that a subterm of $M \in \mathbb{M}$ is also in $\mathbb{M}$.
2. For each $n \in \mathbb{N}$, we let: • $\mathbb{M}^n = \mathbb{M} \cap \mathcal{M}_2^n$

Lemma 2.2. 1. (M is good and $x^n \in FV(M)$) iff $\lambda x^n.M$ is good.

2. (M_1 and M_2 are good, $M_1 \diamond M_2$ and $\mathrm{d}(M_1) \leq \mathrm{d}(M_2)$) iff $M_1 M_2$ is good.

Now, we give the syntax of $\lambda^{\mathcal{L}_{\mathbb{N}}}$. Note that in $\mathcal{M}_3$, an application $M\ N$ is only allowed when $\mathrm{d}(M) \preceq \mathrm{d}(N)$. This restriction was not made in $\lambda I^{\mathbb{N}}$. Furthermore, we only allow the abstraction $\lambda x^L.M$ in $\lambda^{\mathcal{L}_{\mathbb{N}}}$ $L \succeq \mathrm{d}(M)$ which is also the case in $\lambda I^{\mathbb{N}}$ since there, we only consider the I-calculus. The elegance of $\lambda I^{\mathbb{N}}$ is the ability to give the syntax of good terms, which is not obvious in $\lambda^{\mathcal{L}_{\mathbb{N}}}$.

Definition 2.5. (The set of terms $\mathcal{M}_3$)
The set of terms $\mathcal{M}_3$, the set of free variables $\mathrm{FV}(M)$ of $M \in \mathcal{M}_3$ and the degree function $\mathrm{d} : \mathcal{M}_3 \to \mathcal{L}_{\mathbb{N}}$ are defined by simultaneous induction:

- If $x \in \mathcal{V}$ and $L \in \mathcal{L}_{\mathbb{N}}$, then $x^L \in \mathcal{M}_3$, $\mathrm{FV}(x^L) = \{x^L\}$ and $\mathrm{d}(x^L) = L$.
- If $M, N \in \mathcal{M}_3$, $\mathrm{d}(M) \preceq \mathrm{d}(N)$ and $M \diamond N$ (see Definition 2.2), then $M\ N \in \mathcal{M}_3$, $\mathrm{FV}(MN) = \mathrm{FV}(M) \cup \mathrm{FV}(N)$ and $\mathrm{d}(M\ N) = \mathrm{d}(M)$.
- If $x \in \mathcal{V}$, $M \in \mathcal{M}_3$ and $L \succeq \mathrm{d}(M)$, then $\lambda x^L.M \in \mathcal{M}_3$, $\mathrm{FV}(\lambda x^L.M) = \mathrm{FV}(M) \setminus \{x^L\}$ and $\mathrm{d}(\lambda x^L.M) = \mathrm{d}(M)$.

Note that every subterm of $M \in \mathcal{M}_3$ is also in $\mathcal{M}_3$.

As expansions change the degree of a term, indexes in a term need to increase/decrease. The next definitions turn terms of degree n into terms of higher degrees and also, if $n > 0$, they can be turned into terms of lower degrees. Note that $^+$ and $^-$ are well behaved operations with respect to all that matters (free variables, reduction, joinability, substitution, etc.).

Definition 2.6. 1. For each $n \in \mathbb{N}$, we let: • $\mathcal{M}_2^n = \{M \in \mathcal{M}_2 \mid \mathrm{d}(M) = n\}$
• $\mathcal{M}_2^{>n} = \mathcal{M}_2^{\geq n+1}$ • $\mathcal{M}_2^{\geq n} = \{M \in \mathcal{M}_2 \mid \mathrm{d}(M) \geq n\}$

2. We define $^+ : \mathcal{M}_2 \to \mathcal{M}_2$ and $^- : \mathcal{M}_2^{>0} \to \mathcal{M}_2$ by:
• $(x^n)^+ = x^{n+1}$ • $(M_1\ M_2)^+ = M_1^+\ M_2^+$ • $(\lambda x^n.M)^+ = \lambda x^{n+1}.M^+$
• $(x^n)^- = x^{n-1}$ • $(M_1\ M_2)^- = M_1^-\ M_2^-$ • $(\lambda x^n.M)^- = \lambda x^{n-1}.M^-$
3. Let $\mathcal{X} \subseteq \mathcal{M}_2$. If $\forall M \in \mathcal{X}$, $\mathrm{d}(M) > 0$, we write $\mathrm{d}(\mathcal{X}) > 0$. We define:
• $\mathcal{X}^+ = \{M^+ \mid M \in \mathcal{X}\}$ • If $\mathrm{d}(\mathcal{X}) > 0$, $\mathcal{X}^- = \{M^- \mid M \in \mathcal{X}\}$.
4. We define M^{-n} by induction on $\mathrm{d}(M) \geq n \geq 0$. If $n = 0$ then $M^{-n} = M$ and if $n \geq 0$ then $M^{-(n+1)} = (M^{-n})^-$.

Definition 2.7. Let $i \in \mathbb{N}$ and $M \in \mathcal{M}_3$.

1. For each $L \in \mathcal{L}_{\mathbb{N}}$, we let: • $\mathcal{M}_3^L = \{M \in \mathcal{M}_3 \mid \mathrm{d}(M) = L\}$
• $\mathcal{M}_3^{\succeq L} = \{M \in \mathcal{M}_3 \mid \mathrm{d}(M) \succeq L\}$
2. We define M^{+i} by:
•$(x^L)^{+i} = x^{i::L}$ •$(M_1\ M_2)^{+i} = M_1^{+i}\ M_2^{+i}$ •$(\lambda x^L.M)^{+i} = \lambda x^{i::L}.M^{+i}$
3. If $\mathrm{d}(M) = i :: L$, we define M^{-i} by: •$(x^{i::K})^{-i} = x^K$
•$(M_1\ M_2)^{-i} = M_1^{-i}\ M_2^{-i}$ •$(\lambda x^{i::K}.M)^{-i} = \lambda x^K.M^{-i}$

4. Let $\mathcal{X} \subseteq \mathcal{M}_3$. We define $\mathcal{X}^{+i} = \{M^{+i} \mid M \in \mathcal{X}\}$.
   Note that $(\mathcal{X} \cap \mathcal{Y})^{+i} = \mathcal{X}^{+i} \cap \mathcal{Y}^{+i}$.

Definition 2.8. (Substitution, alpha conversion, compatibility, reduction)

- Let $m \geq 0$, $1 \leq i \leq m$, M, N_i be terms of $\lambda I^{\mathbb{N}}$ (resp. $\lambda^{\mathcal{L}_{\mathbb{N}}}$) and $I_i \in \mathbb{N}$ (resp. $\mathcal{L}_{\mathbb{N}}$). $M[(x_i^{I_i} := N_i)_{1\leq i\leq m}]$ (or simply $M[(x_i^{I_i} := N_i)_m]$), the simultaneous substitution of N_i for all free occurrences of $x_i^{I_i}$ in M only matters when:
  - $\diamond\mathcal{X}$ where $\mathcal{X} = \{M\} \cup \{N_i \mid 1 \leq i \leq m\}$.
  - $\forall i$ such that $1 \leq i \leq m$, we have $d(N_i) = I_i$.

  We restrict substitution to incorporate these conditions. With $\mathcal{X}$ as above, $M[(x_i^{I_i} := N_i)_m]$ is only defined when $\diamond\mathcal{X}$ and when $d(N_i) = I_i$ for every i.[1] We may write $x_1^{I_1} := N_1, \ldots, x_m^{I_m} := N_m$ instead of $(x_i^{I_i} := N_i)_m$. We also write $M[(x_i^{I_i} := N_i)_{1\leq i\leq 1}]$ as $M[x_1^{I_1} := N_1]$.

- In $\lambda I^{\mathbb{N}}$ (resp. $\lambda^{\mathcal{L}_{\mathbb{N}}}$), we take terms modulo *α-conversion* given by: $\lambda x^I.M = \lambda y^I.(M[x^I := y^I])$ where $\forall J, y^J \notin \mathrm{FV}(M)$ (where $I, J \in \mathbb{N}$ (resp. $\mathcal{L}_{\mathbb{N}}$)). We use the Barendregt convention (BC) where the names of bound variables differ from the free ones and where we rewrite terms so that not both λx^I and λx^J co-occur when $I \neq J$.

- Let $i \in \{1, 2, 3\}$. A relation R on $\mathcal{M}_i$ is *compatible* iff for all $M, N, P \in \mathcal{M}_i$:
  - If $\langle M, N\rangle \in R$ and $\lambda x^I.M, \lambda x^I.N \in \mathcal{M}_i$ then $\langle \lambda x^I.M, \lambda x^I.N\rangle \in R$.
  - If $\langle M, N\rangle \in R$ and $MP, NP \in \mathcal{M}_i$ then $\langle MP, NP\rangle \in R$.
  - If $\langle M, N\rangle \in R$, and $PM, PN \in \mathcal{M}_i$ then $\langle PM, PN\rangle \in R$.

- Let $i \in \{1, 2, 3\}$. The reduction relation $\rhd_\beta$ on $\mathcal{M}_i$ is defined as the least compatible relation closed under the rule: $(\lambda x^I.M)N \rhd_\beta M[x^I := N]$ if $\mathrm{d}(N) = I$.

- Let $i \in \{1, 2, 3\}$. The reduction relation $\rhd_\eta$ on $\mathcal{M}_i$ is defined as the least compatible relation closed under the rule: $\lambda x^I.(M\ x^I) \rhd_\eta M$ if $x^I \notin \mathrm{FV}(M)$

- Let $i \in \{1, 2, 3\}$. The weak head reduction $\rhd_h$ on $\mathcal{M}_i$ is defined by:
  $(\lambda x^I.M)NN_1 \ldots N_n \rhd_h M[x^I := N]N_1 \ldots N_n$ where $n \geq 0$

- We let $\rhd_{\beta\eta} = \rhd_\beta \cup \rhd_\eta$.

- For a reduction relation $\rhd_r$, we denote by $\rhd_r^*$ the reflexive and transitive closure of $\rhd_r$. We denote by $\simeq_r$ the equivalence relation induced by $\rhd_r^*$.

The next theorem states that reductions preserve the free variables and the degree of a term.

Theorem 2.1. Let $i \in \{1, 2, 3\}$. Let $M \in \mathcal{M}_i$ and $r \in \{\beta, \beta\eta, h\}$.

1. If $M \rhd_\eta^* N$, then $\mathrm{FV}(N) = \mathrm{FV}(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$.

[1] We can prove the following lemma: Let $\mathcal{X} = \{M\} \cup \{N_j \mid 1 \leq j \leq m\}$. We have: $\diamond\mathcal{X}$ and $\forall 1 \leq j \leq m, d(N_j) = I_j$ iff $M[(x_j^{I_j} := N_j)_m] \in \mathcal{M}_i$ where $i \in \{1, 2, 3\}$.

2. If $i = 3$ and $M \rhd_r^* N$, then $\mathrm{FV}(N) \subseteq \mathrm{FV}(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$.

3. If $i \neq 3$ and $M \rhd_\beta^* N$ then $\mathrm{FV}(M) = \mathrm{FV}(N)$, $\mathrm{d}(M) = \mathrm{d}(N)$ and M is good iff N is good.

Proof:

1. By induction on $M \rhd_\eta^* N$.

2. Case $r = \beta$. By induction on $M \rhd_\beta^* N$.
   Case $r = \beta\eta$, by the β and η cases.
   Case $r = h$, by the β case.

3. By induction on $M \rhd_\beta^* N$.

□

Normal forms are defined as usual.

Definition 2.9. Let $i \in \{1, 2, 3\}$.

1. $M \in \mathcal{M}_i$ is in β- (resp. $\beta\eta$-, h-) normal form if there is no $N \in \mathcal{M}_i$ such that $M \rhd_\beta N$ (resp. $M \rhd_{\beta\eta} N$, $M \rhd_h N$).

2. $M \in \mathcal{M}_i$ is β-normalising (resp. $\beta\eta$-normalising, h-normalising) if there is an $N \in \mathcal{M}_i$ such that $M \rhd_\beta^* N$ (resp. $M \rhd_{\beta\eta} N$, $M \rhd_h N$) and N is in β-normal form (resp. $\beta\eta$-normal form, h-normal form).

Finally, β, $\beta\eta$ and h reductions are confluent on the indexed lambda calculi:

Theorem 2.2. (Confluence)
Let $i \in \{1, 2, 3\}$. Let $M, M_1, M_2 \in \mathcal{M}_i$ and $r \in \{\beta, \beta\eta, h\}$.

1. If $M \rhd_r^* M_1$ and $M \rhd_r^* M_2$, then there is M' such that $M_1 \rhd_r^* M'$ and $M_2 \rhd_r^* M'$.

2. $M_1 \simeq_r M_2$ iff there is a term M such that $M_1 \rhd_r^* M$ and $M_2 \rhd_r^* M$.

Proof:
We establish the confluence using the standard parallel reduction method. Full details can be found in [13]. □

3. The types of the indexed calculi

We assume that a, b range over a countably infinite set of type variables $\mathcal{A}$, and that e ranges over a countably infinite set of expansion variables $\mathcal{E} = \{\overline{e}_0, \overline{e}_1, \dots\}$. We denote $\overline{e}_{i_1} \dots \overline{e}_{i_n}$ by $\vec{e}_{i(1:n)}$ or alternatively by $\vec{e}_K$, where $K = (i_1, \dots, i_n)$. In all our type systems, we quotient types by taking $\sqcap$ to be commutative (i.e. $U_1 \sqcap U_2 = U_2 \sqcap U_1$), associative (i.e. $U_1 \sqcap (U_2 \sqcap U_3) = (U_1 \sqcap U_2) \sqcap U_3$) and idempotent (i.e. $U \sqcap U = U$), by assuming the distributivity of expansion variables over $\sqcap$ (i.e. $e_i(U_1 \sqcap U_2) = e_iU_1 \sqcap e_iU_2$). We denote $U_n \sqcap U_{n+1} \cdots \sqcap U_m$ by $\sqcap_{i=n}^m U_i$ (when $n \leq m$).

For $\lambda I^{\mathbb{N}}$, we study two type systems (none of which has the ω-type). In the first, there are no restrictions on where the arrow occurs. In the second, intersections and expansions cannot occur directly to the right of an arrow.

Definition 3.1. (Types, good types and degree of a type for $\lambda I^{\mathbb{N}}$)

1. The sets of types $\mathbb{T}_2 \subseteq \mathbb{U}_2 \subseteq \mathbb{U}_1$ are defined by $\mathbb{U}_1 ::= \mathcal{A} \mid \mathbb{U}_1 \to \mathbb{U}_1 \mid \mathbb{U}_1 \sqcap \mathbb{U}_1 \mid \mathcal{E}\mathbb{U}_1$ and $\mathbb{U}_2 ::= \mathbb{U}_2 \sqcap \mathbb{U}_2 \mid \mathcal{E}\mathbb{U}_2 \mid \mathbb{T}_2$ where $\mathbb{T}_2 ::= \mathcal{A} \mid \mathbb{U}_2 \to \mathbb{T}_2$. We let T, U, V, W (resp. T, resp. U, V, W) range over $\mathbb{U}_1$ (resp. $\mathbb{T}_2$, resp. $\mathbb{U}_2$).

2. We define a function $\mathrm{d} : \mathbb{U}_1 \to \mathbb{N}$ by (hence d is also defined on $\mathbb{U}_2$):
   • $\mathrm{d}(a) = 0$ • $\mathrm{d}(U \to T) = \min(\mathrm{d}(U), \mathrm{d}(T))$
   • $\mathrm{d}(eU) = \mathrm{d}(U) + 1$ • $\mathrm{d}(U \sqcap V) = \min(\mathrm{d}(U), \mathrm{d}(V))$.

3. We define the good types on $\mathbb{U}_1$ by (this also defines good types on $\mathbb{U}_2$):
   • If $a \in \mathcal{A}$, then a is good • If U is good and $e \in \mathcal{E}$, then eU is good
   • If U, T are good and $\mathrm{d}(U) \geq \mathrm{d}(T)$, then $U \to T$ is good
   • If U, V are good and $\mathrm{d}(U) = \mathrm{d}(V)$, then $U \sqcap V$ is good

The next lemma states when arrow, intersection and expansion types are good.

Lemma 3.1.

1. On $\mathbb{U}_1$ (hence on $\mathbb{U}_2$), we have the following:
   (a) (U, T are good and $\mathrm{d}(U) \geq \mathrm{d}(T)$) iff $U \to T$ is good.
   (b) (U, V are good and $\mathrm{d}(U) = \mathrm{d}(V)$) iff $U \sqcap V$ is good.
   (c) U is good iff eU is good.

2. On $\mathbb{U}_2$, we have in addition the following:
   (a) If $T \in \mathbb{T}_2$, then $\mathrm{d}(T) = 0$.
   (b) If $\mathrm{d}(U) = n$ then $U = \sqcap_{i=1}^{k} \vec{e}_{i(1:n)} V_i$ where $k \geq 1$ and $\exists i. V_i \in \mathbb{T}_2$.
   (c) If U is good and $\mathrm{d}(U) = n$, then $U = \sqcap_{i=1}^{k} \vec{e}_{i(1:n)} T_i$ where $k \geq 1$ and $\forall\ 1 \leq i \leq k$, $T_i \in \mathbb{T}_2$.
   (d) U and T are good iff $U \to T$ is good.

For $\lambda^{\mathcal{L}_{\mathbb{N}}}$, we study a type system (with the universal type ω). In this type system, in order to get subject reduction and hence completeness, intersections and expansions cannot occur directly to the right of an arrow (see $\mathbb{U}_3$ below). Note that our sets $\mathbb{U}_3$ and $\mathbb{T}_3$ are far more restricted here than for the $\lambda I^{\mathbb{N}}$-calculus and that we do not have the luxury of giving a syntax for a so-called good types. Note also that the definitions of degrees and types are simultaneous (unlike for $\mathbb{U}_2$ and $\mathbb{T}_2$ where types were defined without any reference to degrees).

Definition 3.2. (Types and degrees for $\lambda^{\mathcal{L}_{\mathbb{N}}}$)

1. We define sets of types $\mathbb{T}_3 \subseteq \mathbb{U}_3$, and a function $\mathrm{d} : \mathbb{U}_3 \to \mathcal{L}_{\mathbb{N}}$ by simultaneous induction as follows:

   • If $a \in \mathcal{A}$, then $a \in \mathbb{T}_3$ and $\mathrm{d}(a) = \oslash$.
   • If $U \in \mathbb{U}_3$ and $T \in \mathbb{T}_3$, then $U \to T \in \mathbb{T}_3$ and $\mathrm{d}(U \to T) = \oslash$.

- If $L \in \mathcal{L}_{\mathbb{N}}$, then $\omega^L \in \mathbb{U}_3$ and $\mathrm{d}(\omega^L) = L$.
- If $U_1, U_2 \in \mathbb{U}_3$ and $\mathrm{d}(U_1) = \mathrm{d}(U_2)$, then $U_1 \sqcap U_2 \in \mathbb{U}_3$ and $\mathrm{d}(U_1 \sqcap U_2) = \mathrm{d}(U_1) = \mathrm{d}(U_2)$.
- $U \in \mathbb{U}_3$ and $\overline{e}_i \in \mathcal{E}$, then $\overline{e}_i U \in \mathbb{U}_3$ and $\mathrm{d}(\overline{e}_i U) = i :: \mathrm{d}(U)$.

Note that d remembers the number of the expansion variables $\overline{e}_i$ in order to keep a trace of them.

2. We let T range over $\mathbb{T}_3$, and U, V, W range over $\mathbb{U}_3$. We quotient types further by having ω^L as a neutral (i.e. $\omega^L \sqcap U = U$). We also assume that for all $i \geq 0$ and $K \in \mathcal{L}_{\mathbb{N}}$, $\overline{e}_i \omega^K = \omega^{i::K}$.

All our type systems use the following definition (of course within the corresponding calculus, with the corresponding indices and types):

Definition 3.3. (Environments)

1. Let $k \in \{1, 2, 3\}$. A type environment for $\mathbb{U}_k$ is a set $\{x_1^{I_1} : U_1, \ldots, x_n^{I_n} : U_n \mid n \geq 0, \forall 1 \leq i, j \leq n, U_i \in \mathbb{U}_k$, and if $x_i^{I_i} = x_j^{I_j}$ then $U_i = U_j\}$. We denote such environment (call it Γ) by $x_1^{I_1} : U_1, \ldots, x_n^{I_n} : U_n$ or simply by $(x_i^{I_i} : U_i)_n$ and define $\mathrm{dom}(\Gamma) = \{x_i^{I_i} \mid 1 \leq i \leq n\}$. We let $Env_{\mathbb{U}_k}$ be the set of type environments for $\mathbb{U}_k$. If $\mathrm{dom}(\Gamma_1) \cap \mathrm{dom}(\Gamma_2) = \emptyset$, we write Γ_1, Γ_2 for $\Gamma_1 \cup \Gamma_2$. Let Γ, Δ range over environments and let $()$ be the empty environment.
2. If $\Gamma = (x_i^{I_i} : U_i)_n$ and $x^J \notin \mathrm{dom}(\Gamma)$, then we write $\Gamma, x^J : U$ for the type environment $x_1^{I_1} : U_1, \ldots, x_n^{I_n} : U_n, x^J : U$.
3. We say that Γ_1 is joinable with Γ_2 and write $\Gamma_1 \diamond \Gamma_2$ iff for all $x \in \mathcal{V}$, if $x^I \in \mathrm{dom}(\Gamma_1)$ and $x^J \in \mathrm{dom}(\Gamma_2)$, then $I = J$.
4. We say that a type environment Γ is OK (and write $\mathrm{OK}(\Gamma)$) iff for all $x^I : U \in \Gamma$, $\mathrm{d}(U) = I$.
5. Let $\Gamma_1 = (x_i^{I_i} : U_i)_n, \Gamma'_1$, $\Gamma_2 = (x_i^{I_i} : U'_i)_n, \Gamma'_2$ where $\mathrm{dom}(\Gamma'_1) \cap \mathrm{dom}(\Gamma'_2) = \emptyset$ and $\forall 1 \leq i \leq n$, $d(U_i) = d(U'_i)$. We denote $\Gamma_1 \sqcap \Gamma_2$ the type environment $(x_i^{I_i} : U_i \sqcap U'_i)_n, \Gamma'_1, \Gamma'_2$. Note that $\mathrm{dom}(\Gamma_1 \sqcap \Gamma_2) = \mathrm{dom}(\Gamma_1) \cup \mathrm{dom}(\Gamma_2)$ and that, on environments, $\sqcap$ is commutative, associative and idempotent.
6. In $\lambda I^{\mathbb{N}}$ (i.e., on $Env_{\mathbb{U}_1}$ and $Env_{\mathbb{U}_2}$), we define for $\Gamma = (x_i^{n_i} : U_i)_n$:
   - Γ is good iff , for every $1 \leq i \leq n$, U_i is good.
   - $\mathrm{d}(\Gamma) > 0$ iff for every $1 \leq i \leq n$, $\mathrm{d}(U_i) > 0$ and $n_i > 0$.
   - $e\Gamma = (x_i^{n_i+1} : eU_i)_n$. So $e(\Gamma_1 \sqcap \Gamma_2) = e\Gamma_1 \sqcap e\Gamma_2$.
7. In $\lambda^{\mathcal{L}_{\mathbb{N}}}$ (i.e., on $Env_{\mathbb{U}_3}$), we define:
   - If $M \in \mathcal{M}_3$ and $\mathrm{FV}(M) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$, let env_M^{ω} be the type environment $(x_i^{L_i} : \omega^{L_i})_n$.
   - Let $\Gamma = (x_i^{L_i} : U_i)_n$ and $\overline{e}_j \in \mathcal{E}$.
     - We denote $\overline{e}_j \Gamma = (x_i^{j::L_i} : \overline{e}_j U_i)_n$. Note that $e(\Gamma_1 \sqcap \Gamma_2) = e\Gamma_1 \sqcap e\Gamma_2$.
     - $\mathrm{d}(\Gamma) \succeq L$ if and only if for all $i \in \{1, \ldots, n\}$, $\mathrm{d}(U_i) \succeq L$.

As we did for terms, we decrease the indexes of types and environments.

Definition 3.4. (Degree decreasing in $\lambda I^{\mathbb{N}}$)

1. If $\mathrm{d}(U) > 0$, we inductively define the type U^- by:
   $\bullet\ (U_1 \sqcap U_2)^- = U_1^- \sqcap U_2^-$ $\qquad \bullet (eU)^- = U$
   If $\mathrm{d}(U) \geq n \geq 0$, we inductively define the type U^{-n} by:
   if $n = 0$ then $U^{-n} = U$ and if $n \geq 0$ then $U^{-(n+1)} = (U^{-n})^-$.

2. If $\Gamma = (x_i^{n_i} : U_i)_k$ and $\mathrm{d}(\Gamma) > 0$, then we let $\Gamma^- = (x_i^{n_i - 1} : U_i^-)_k$.
   If $\mathrm{d}(\Gamma) \geq n \geq 0$, then,
   if $n = 0$ then $\Gamma^{-n} = \Gamma$ and if $n \geq 0$ then $\Gamma^{-(n+1)} = (\Gamma^{-n})^-$.

Definition 3.5. (Degree decreasing in $\lambda^{\mathcal{L}_{\mathbb{N}}}$)

1. If $\mathrm{d}(U) \succeq L$, then if $L = \oslash$ then $U^{-L} = U$ else $L = i :: K$ and we inductively define the type U^{-L} as follows:
   $(U_1 \sqcap U_2)^{-i::K} = U_1^{-i::K} \sqcap U_2^{-i::K}$ $\qquad (\overline{e}_i U)^{-i::K} = U^{-K}$
   We write U^{-i} instead of $U^{-(i)}$.

2. If $\Gamma = (x_i^{L_i} : U_i)_k$ and $\mathrm{d}(\Gamma) \succeq L$, then for all $i \in \{1, \ldots, k\}$, $L_i = L :: L'_i$ and Γ^{-L} denote $(x^{L'_i} : U_i^{-L})_k$.
   We write Γ^{-i} instead of $\Gamma^{-(i)}$.

4. The type systems $\vdash_1$ and $\vdash_2$ for $\lambda I^{\mathbb{N}}$ and $\vdash_3$ for $\lambda^{\mathcal{L}_{\mathbb{N}}}$

In this section we introduce our three type systems $\vdash_i$ for $i \in \{1, 2, 3\}$, our intersection type systems with expansion variables. The systems $\vdash_1$ (which uses types in $\mathbb{U}_i$) and $\vdash_2$ (which uses types in $\mathbb{U}_2$) are for $\lambda I^{\mathbb{N}}$, $\vdash_3$ (which uses types in $\mathbb{U}_3$) is for $\lambda^{\mathcal{L}_{\mathbb{N}}}$. In $\vdash_1$, types are not restricted and Subject Reduction (SR) fails. In $\vdash_2$, the syntax of types is restricted (see $\mathbb{U}_2$), and in order to guarantee SR for this type system (and hence completeness later on), we introduce a subtyping relation which will allow intersection type elimination (something not available in the first type system). In $\vdash_3$, the syntax of types is restricted further (see $\mathbb{U}_3$) so that completeness will hold with an arbitrary number of expansion variables.

We follow [3] and write type judgements as $M : \langle \Gamma \vdash U \rangle$ instead of $\Gamma \vdash M : U$.

Definition 4.1. (The type systems)
Let $i \in \{1, 2, 3\}$. The type system $\vdash_i$ uses the set $\mathbb{U}_i$ of definitions 3.1 and 3.2. The typing rules of $\vdash_1$ and $\vdash_2$ are given on the left of Figure 1 (recall that when used for $\vdash_1$, U and T range over all of $\mathbb{U}_1$, and when used for $\vdash_2$, U ranges over $\mathbb{U}_2$ and T ranges only over $\mathbb{T}_2$). The typing rules of $\vdash_3$ are given on the left of Figure 2. In the last clause, the binary relation $\sqsubseteq$ is defined on $\mathbb{U}_2$ and $\mathbb{U}_3$ by the rules on the right hand side of Figures 1 and 2 respectively. For $j \in \{2, 3\}$, we let Φ denote types in $\mathbb{U}_j$, or environments Γ or j-typings $\langle \Gamma \vdash_j U \rangle$. When $\Phi \sqsubseteq \Phi'$, then Φ and Φ' belong to the same set ($\mathbb{U}_j$/$Env_{\mathbb{U}_j}$/j-typings).
- We say that Γ is $\vdash_i$-legal iff there are M, U such that $M : \langle \Gamma \vdash_i U \rangle$.
- Let $k \in \{1, 2\}$. We say that
  - $\langle \Gamma \vdash_k U \rangle$ is good iff Γ and U are good.
  - $\mathrm{d}(\langle \Gamma \vdash_k U \rangle) > 0$ iff $\mathrm{d}(\Gamma) > 0$ and $\mathrm{d}(U) > 0$.
- We say that $\mathrm{d}(\langle \Gamma \vdash_3 U \rangle) \succeq L$ if and only if $\mathrm{d}(\Gamma) \succeq L$ and $\mathrm{d}(U) \succeq L$.

To illustrate how our indexed type system works, we give an example:

Let $i \in \{1,2\}$
In $\vdash_1$, U and T range over all of $\mathbb{U}_1$.
In $\vdash_2$, U ranges over $\mathbb{U}_2$ and T ranges only over $\mathbb{T}_2$

$$\frac{T \text{ good} \quad \mathrm{d}(T) = n}{x^n : \langle (x^n : T) \vdash_1 T \rangle}\ (ax)$$

$$\frac{T \text{ good}}{x^0 : \langle (x^0 : T) \vdash_2 T \rangle}\ (ax)$$

$$\frac{M : \langle \Gamma, (x^n : U) \vdash_i T \rangle}{\lambda x^n.M : \langle \Gamma \vdash_i U \to T \rangle}\ (\to_I)$$

$$\frac{M_1 : \langle \Gamma_1 \vdash_i U \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash_i U \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_i T \rangle}\ (\to_E)$$

$$\frac{M : \langle \Gamma_1 \vdash_i U_1 \rangle \quad M : \langle \Gamma_2 \vdash_i U_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_i U_1 \sqcap U_2 \rangle}\ (\sqcap_I)$$

$$\frac{M : \langle \Gamma \vdash_i U \rangle}{M^+ : \langle e\Gamma \vdash_i eU \rangle}\ (exp)$$

$$\frac{M : \langle \Gamma \vdash_2 U \rangle \quad \langle \Gamma \vdash_2 U \rangle \sqsubseteq \langle \Gamma' \vdash_2 U' \rangle}{M : \langle \Gamma' \vdash_2 U' \rangle}\ (\sqsubseteq)$$

$\sqsubseteq$ is defined on:
$\mathbb{U}_2$, $Env_{\mathbb{U}_2}$ and 2-typings.

$$\frac{}{\Phi \sqsubseteq \Phi}\ (ref)$$

$$\frac{\Phi_1 \sqsubseteq \Phi_2 \quad \Phi_2 \sqsubseteq \Phi_3}{\Phi_1 \sqsubseteq \Phi_3}\ (tr)$$

$$\frac{U_2 \text{ good} \quad \mathrm{d}(U_1) = \mathrm{d}(U_2)}{U_1 \sqcap U_2 \sqsubseteq U_1}\ (\sqcap_E)$$

$$\frac{U_1 \sqsubseteq V_1 \quad U_2 \sqsubseteq V_2}{U_1 \sqcap U_2 \sqsubseteq V_1 \sqcap V_2}\ (\sqcap)$$

$$\frac{U_2 \sqsubseteq U_1 \quad T_1 \sqsubseteq T_2}{U_1 \to T_1 \sqsubseteq U_2 \to T_2}\ (\to)$$

$$\frac{U_1 \sqsubseteq U_2}{eU_1 \sqsubseteq eU_2}\ (\sqsubseteq_{exp})$$

$$\frac{U_1 \sqsubseteq U_2 \quad y^n \notin \mathrm{dom}(\Gamma)}{\Gamma, (y^n : U_1) \sqsubseteq \Gamma, (y^n : U_2)}\ (\sqsubseteq_c)$$

$$\frac{U_1 \sqsubseteq U_2 \quad \Gamma_2 \sqsubseteq \Gamma_1}{\langle \Gamma_1 \vdash_2 U_1 \rangle \sqsubseteq \langle \Gamma_2 \vdash_2 U_2 \rangle}\ (\sqsubseteq_{\langle\rangle})$$

Figure 1. Typing rules / Subtyping rules for $\vdash_1$ and $\vdash_2$

Example 4.1. Let $L_1 = 3 :: \oslash \preceq L_2 = 3 :: 2 :: \oslash \preceq L_3 = 3 :: 2 :: 1 :: \oslash \preceq L_4 = 3 :: 2 :: 1 :: 0 :: \oslash$ and let $a, b, c, d \in \mathcal{A}$. Consider M, M', U as follows:
$M = \lambda x^{L_2}.\lambda y^{L_1}.(y^{L_1}\,(x^{L_2}\,\lambda u^{L_3}.\lambda v^{L_4}.(u^{L_3}\,(v^{L_4}\,v^{L_4})))) \in \mathcal{M}_3$.
$M' = \lambda x^2.\lambda y^1.(y^1\,(x^2\,\lambda u^3.\lambda v^4.(u^3\,(v^4\,v^4)))) \in \mathcal{M}_2$.
$U = \overline{e}_3(\overline{e}_2(\overline{e}_1((\overline{e}_0 b \to c) \to (\overline{e}_0(a \sqcap (a \to b)) \to c)) \to d) \to (((\overline{e}_2 d \to a) \sqcap b) \to a))$.

We invite the reader to check that $M : \langle () \vdash_3 U \rangle$ and $M' : \langle () \vdash_2 U \rangle$. We simply give some steps in the derivation of $M : \langle () \vdash_3 U \rangle$ (note that the derivation of $M' : \langle () \vdash_2 U \rangle$ only differs from the derivation of $M : \langle () \vdash_3 U \rangle$ by replacing everywhere $\vdash_3$ by $\vdash_2$ and any list $n_1 :: n_2 \cdots :: n_k :: \oslash$ by k for any $k \geq 0$):

- $v^{\oslash} v^{\oslash} :< v^{\oslash} : a \sqcap (a \to b) \vdash_3 b >$
- $v^{0::\oslash} v^{0::\oslash} :< v^{0::\oslash} : \overline{e}_0(a \sqcap (a \to b)) \vdash_3 \overline{e}_0 b >$
- $u^{\oslash} :< u^{\oslash} : \overline{e}_0 b \to c \vdash_3 \overline{e}_0 b \to c >$
- $u^{\oslash}(v^{0::\oslash} v^{0::\oslash}) :< u^{\oslash} : \overline{e}_0 b \to c, v^{0::\oslash} : \overline{e}_0(a \sqcap (a \to b)) \vdash_3 c >$
- $\lambda v^{0::\oslash}.u^{\oslash}(v^{0::\oslash} v^{0::\oslash}) :< u^{\oslash} : \overline{e}_0 b \to c \vdash_3 \overline{e}_0(a \sqcap (a \to b)) \to c >$

U ranges over $\mathbb{U}_3$ and T ranges only over $\mathbb{T}_3$	$\sqsubseteq$ is defined on: $\mathbb{U}_3$, $Env_{\mathbb{U}_3}$ and 3-typing.
$\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash_3 T \rangle}\ (ax)$	$\dfrac{}{\Phi \sqsubseteq \Phi}\ (ref)$
$\dfrac{}{M : \langle env^{\omega}_{M} \vdash_3 \omega^{\mathrm{d}(M)} \rangle}\ (\omega)$	$\dfrac{\Phi_1 \sqsubseteq \Phi_2 \quad \Phi_2 \sqsubseteq \Phi_3}{\Phi_1 \sqsubseteq \Phi_3}\ (tr)$
$\dfrac{M : \langle \Gamma, (x^L : U) \vdash_3 T \rangle}{\lambda x^L.M : \langle \Gamma \vdash_3 U \to T \rangle}\ (\to_I)$	$\dfrac{d(U_1) = d(U_2)}{U_1 \sqcap U_2 \sqsubseteq U_1}\ (\sqcap_E)$
$\dfrac{M : \langle \Gamma \vdash_3 T \rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash_3 \omega^L \to T \rangle}\ (\to'_I)$	$\dfrac{U_1 \sqsubseteq V_1 \quad U_2 \sqsubseteq V_2 \quad d(U_1) = d(U_2)}{U_1 \sqcap U_2 \sqsubseteq V_1 \sqcap V_2}\ (\sqcap)$
$\dfrac{M_1 : \langle \Gamma_1 \vdash_3 U \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash_3 U \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_3 T \rangle}\ (\to_E)$	$\dfrac{U_2 \sqsubseteq U_1 \quad T_1 \sqsubseteq T_2}{U_1 \to T_1 \sqsubseteq U_2 \to T_2}\ (\to)$
$\dfrac{M : \langle \Gamma \vdash_3 U_1 \rangle \qquad M : \langle \Gamma \vdash_3 U_2 \rangle}{M : \langle \Gamma \vdash_3 U_1 \sqcap U_2 \rangle}\ (\sqcap_I)$	$\dfrac{U_1 \sqsubseteq U_2}{eU_1 \sqsubseteq eU_2}\ (\sqsubseteq_e)$
$\dfrac{M : \langle \Gamma \vdash_3 U \rangle}{M^{+j} : \langle \bar{e}_j \Gamma \vdash_3 \bar{e}_j U \rangle}\ (e)$	$\dfrac{U_1 \sqsubseteq U_2 \quad y^L \notin \mathrm{dom}(\Gamma)}{\Gamma, y^L : U_1 \sqsubseteq \Gamma, y^L : U_2}\ (\sqsubseteq_c)$
$\dfrac{M : \langle \Gamma \vdash_3 U \rangle \qquad \langle \Gamma \vdash_3 U \rangle \sqsubseteq \langle \Gamma' \vdash_3 U' \rangle}{M : \langle \Gamma' \vdash_3 U' \rangle}\ (\sqsubseteq)$	$\dfrac{U_1 \sqsubseteq U_2 \quad \Gamma_2 \sqsubseteq \Gamma_1}{\langle \Gamma_1 \vdash_3 U_1 \rangle \sqsubseteq \langle \Gamma_2 \vdash_3 U_2 \rangle}\ (\sqsubseteq_{\langle\rangle})$

Figure 2. Typing rules / Subtyping rules for $\vdash_3$

- $\lambda u^{\oslash}.\lambda v^{0::\oslash}.u^{\oslash}(v^{0::\oslash} v^{0::\oslash}) :< () \vdash_3 (\bar{e}_0 b \to c) \to (\bar{e}_0(a \sqcap (a \to b)) \to c) >$
- $\lambda u^{1::\oslash}.\lambda v^{1::0::\oslash}.u^{1::\oslash}(v^{1::0::\oslash} v^{1::0::\oslash}) :$
  $< () \vdash_3 \bar{e}_1((\bar{e}_0 b \to c) \to (\bar{e}_0(a \sqcap (a \to b)) \to c)) >$
- $x^{\oslash} :< x^{\oslash} : \bar{e}_1((\bar{e}_0 b \to c) \to (\bar{e}_0(a \sqcap (a \to b)) \to c)) \to d$
  $\vdash_3 \bar{e}_1((\bar{e}_0 b \to c) \to (\bar{e}_0(a \sqcap (a \to b)) \to c)) \to d >$
- $x^{\oslash}(\lambda u^{1::\oslash}.\lambda v^{1::0::\oslash}.u^{1::\oslash}(v^{1::0::\oslash} v^{1::0::\oslash})) :$
  $< x^{\oslash} : \bar{e}_1((\bar{e}_0 b \to c) \to (\bar{e}_0(a \sqcap (a \to b)) \to c)) \to d \vdash_3 d >$
- $x^{2::\oslash}(\lambda u^{2::1::\oslash}.\lambda v^{2::1::0::\oslash}.u^{2::1::\oslash}(v^{2::1::0::\oslash} v^{2::1::0::\oslash})) :$
  $< x^{2::\oslash} : \bar{e}_2(\bar{e}_1((\bar{e}_0 b \to c) \to (\bar{e}_0(a \sqcap (a \to b)) \to c)) \to d) \vdash_3 \bar{e}_2 d >$
- $y^{\oslash}(x^{2::\oslash}(\lambda u^{2::1::\oslash}.\lambda v^{2::1::0::\oslash}.u^{2::1::\oslash}(v^{2::1::0::\oslash} v^{2::1::0::\oslash}))) :$
  $< x^{2::\oslash} : \bar{e}_2(\bar{e}_1((\bar{e}_0 b \to c) \to (\bar{e}_0(a \sqcap (a \to b)) \to c)) \to d),$
  $y^{\oslash} : (\bar{e}_2 d \to a) \sqcap b \vdash_3 a >$
- $\lambda y^{\oslash}.(y^{\oslash}(x^{2::\oslash}(\lambda u^{2::1::\oslash}.\lambda v^{2::1::0::\oslash}.u^{2::1::\oslash}(v^{2::1::0::\oslash} v^{2::1::0::\oslash})))) :$
  $< x^{2::\oslash} : \bar{e}_2(\bar{e}_1((\bar{e}_0 b \to c) \to (\bar{e}_0(a \sqcap (a \to b)) \to c)) \to d),$

$\vdash_3 ((\overline{e}_2 d \to a) \sqcap b) \to a >$

- $\lambda x^{2::\oslash}.\lambda y^{\oslash}.(y^{\oslash}(x^{2::\oslash}(\lambda u^{2::1::\oslash}.\lambda v^{2::1::0::\oslash}.u^{2::1::\oslash}(v^{2::1::0::\oslash}v^{2::1::0::\oslash})))) :$
  $< () \vdash_3 \overline{e}_2(\overline{e}_1((\overline{e}_0 b \to c) \to (\overline{e}_0(a \sqcap (a \to b)) \to c)) \to d),$
  $\to (((\overline{e}_2 d \to a) \sqcap b) \to a) >$

- $\lambda x^{L_2}.\lambda y^{L_1}.(y^{L_1}(x^{L_2}(\lambda u^{L_3}.\lambda v^{L_4}.u^{L_3}(v^{L_4}v^{L_4})))) :$
  $< () \vdash_3 \overline{e}_3(\overline{e}_2(\overline{e}_1((\overline{e}_0 b \to c) \to (\overline{e}_0(a \sqcap (a \to b)) \to c)) \to d),$
  $\to (((\overline{e}_2 d \to a) \sqcap b) \to a)) >$

Definition 4.2. 1. In $\lambda I^{\mathbb{N}}$, if $U \in \mathbb{U}_2$ and $\Gamma \in Env_{\mathbb{U}_2}$ such that $\mathrm{d}(\Gamma) > 0$ and $\mathrm{d}(U) > 0$, then we let $(\langle \Gamma \vdash_2 U \rangle)^- = (\langle \Gamma^- \vdash_2 U^- \rangle)$.

2. In $\lambda^{\mathcal{L}_{\mathbb{N}}}$, if $U \in \mathbb{U}_3$ and $\Gamma \in Env_{\mathbb{U}_3}$ such that $\mathrm{d}(\Gamma) \succeq K$ and $\mathrm{d}(U) \succeq K$, then we denote $(\langle \Gamma \vdash_3 U \rangle)^{-K} = \langle \Gamma^{-K} \vdash_3 U^{-K} \rangle$.

Next we show how ordering propagates to environments and relates degrees:

Lemma 4.1. 1. If $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $x^I \notin \mathrm{dom}(\Gamma)$ then $\Gamma, (x^I : U) \sqsubseteq \Gamma', (x^I : U')$.

2. $\Gamma \sqsubseteq \Gamma'$ iff $\Gamma = (x_i^{I_i} : U_i)_n$, $\Gamma' = (x_i^{I_i} : U_i')_n$ and for all $i \in \{1, \ldots, n\}$, $U_i \sqsubseteq U_i'$.

3. Let $j \in \{2, 3\}$. $\langle \Gamma \vdash_j U \rangle \sqsubseteq \langle \Gamma' \vdash_j U' \rangle$ iff $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$.

4. $\sqsubseteq$ is well defined on $\mathbb{U}_j$, $Env_{\mathbb{U}_j}$ and on j-typings, for $j \in \{2, 3, \}$.

5. If $U_1 \sqsubseteq U_2$ then $d(U_1) = d(U_2)$ and U_1 is good iff U_2 is good.

6. If $\Gamma_1 \sqsubseteq \Gamma_2$ then $d(\Gamma_1) \succeq L$ iff $d(\Gamma_2) \succeq L$.

Proof:
1. and 2. By induction on the derivation $\Gamma \sqsubseteq \Gamma'$.
3. By induction on the derivation $\langle \Gamma \vdash_j U \rangle \sqsubseteq \langle \Gamma' \vdash_j U' \rangle$.
4. By induction on the derivation $\Phi_1 \sqsubseteq \Phi_2$
5. By induction on the derivation $U_1 \sqsubseteq U_2$.
6. By induction on the derivation $\Gamma_1 \sqsubseteq \Gamma_2$. □

The next theorem states that typings are well defined, that within a typing, degrees are well behaved and that we do not allow weakening.

Theorem 4.1. Let $j \in \{1, 2, 3\}$. We have:

1. $\vdash_j$ is well defined on $\mathcal{M}_j \times Env_{\mathbb{U}_j} \times \mathbb{U}_j$.

2. Let $\Gamma = (x_i^{I_i} : U_i)_n$ and $M : \langle \Gamma \vdash_j U \rangle$. Then:

   (a) $d(M) = d(U)$ and $\forall 1 \leq i \leq n$, $d(U_i) = I_i$.

   (b) If $j = 3$ then $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U)$.

(c) If $j \neq 3$ then U and M are good and $\forall\, 1 \leq i \leq n$, $d(U_i) \geq d(M)$ and U_i is good.

3. Let $M : \langle \Gamma \vdash_j U \rangle$. Then:
   (a) $\mathrm{dom}(\Gamma) = \mathrm{FV}(M)$.
   (b) If $j \neq 3$ and $\mathrm{d}(U) \geq k$ then $M^{-k} : \langle \Gamma^{-k} \vdash_j U^{-k} \rangle$.
   (c) If $j = 3$ and $\mathrm{d}(U) \succeq K$ then $M^{-K} : \langle \Gamma^{-K} \vdash_3 U^{-K} \rangle$.

Proof:
We prove 1 and 2 simultaneously by induction on the derivation $M : \langle \Gamma \vdash_j U \rangle$ using Lemma 4.1. We prove 3 by induction on the derivation $M : \langle \Gamma \vdash_j U \rangle$. $\square$

Here are some derivable typing rules.

Remark 4.1. Let $j \in \{2, 3\}$.

1. The rule $\dfrac{M : \langle \Gamma_1 \vdash_j U_1 \rangle \qquad M : \langle \Gamma_2 \vdash_j U_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_j U_1 \sqcap U_2 \rangle}\ \sqcap'_I$ is derivable.
2. The rule $\dfrac{U \text{ is good} \qquad \mathrm{d}(U) = n}{x^n : \langle (x^n : U) \vdash_2 U \rangle}\ ax'$ is derivable.
3. The rule $\dfrac{}{x^{\mathrm{d}(U)} : \langle (x^{\mathrm{d}(U)} : U) \vdash_3 U \rangle}\ ax''$ is derivable.
4. The rule $\dfrac{}{U \sqsubseteq \omega^{\mathrm{d}(U)}}\ \omega$ is derivable.

Lemma 4.2. Let $i \in \{1, 2, 3\}$.

1. If $M : \langle \Gamma \vdash_3 U \rangle$ then $\Gamma \sqsubseteq env^{\omega}_M$
2. If $\mathrm{dom}(\Gamma) = \mathrm{FV}(M)$, and $\forall x^L : U \in \Gamma$, $d(U) = L$ then $M : \langle \Gamma \vdash_3 \omega^{d(M)} \rangle$.
3. If $M_1 : \langle \Gamma_1 \vdash_i U \rangle$ and $M_2 : \langle \Gamma_2 \vdash_i U \rangle$ then $\Gamma_1 \diamond \Gamma_2$ iff $M_1 \diamond M_2$.

Proof:

1. Let $\Gamma = (x_i^{L_i} : U_i)_n$ where $\mathrm{FV}(M) = \{x_1^{L_1}, x_2^{L_2}, \ldots, x_n^{L_n}\}$ by Theorem 4.1.3a. Since by Remark 4.1.4 resp. Theorem 4.1.2, $\forall 1 \leq i \leq n$, $U_i \sqsubseteq \omega^{d(U_i)}$ resp. $d(U_i) = L_i$, then by Lemma 4.1.2, $\Gamma \sqsubseteq env^{\omega}_M$.

2. Let $\Gamma = (x_i^{L_i} : U_i)_n$ where $\mathrm{FV}(M) = \{x_1^{L_1}, x_2^{L_2}, \ldots, x_n^{L_n}\}$ and $\forall 1 \leq i \leq n$, $d(U_i) = L_i$. By Remark 4.1.4, $U_i \sqsubseteq \omega^{L_i}$. By Lemma 4.1.1, $\Gamma \sqsubseteq env^{\omega}_M = (x^{L_i} : \omega^{L_i})_n$. Since by ω, $M : \langle env^{\omega}_M \vdash_3 \omega^{d(M)} \rangle$, we have by $\sqsubseteq$ and $\sqsubseteq_{\langle\rangle}$, $M : \langle \Gamma \vdash_3 \omega^{d(M)} \rangle$.

3. If) Let $x^I \in \mathrm{dom}(\Gamma_1)$ and $x^J \in \mathrm{dom}(\Gamma_2)$ then by Theorem 4.1.3a, $x^I \in \mathrm{FV}(M_1)$ and $x^J \in \mathrm{FV}(M_2)$ so $\Gamma_1 \diamond \Gamma_2$. Only if) Let $x^I \in \mathrm{FV}(M_1)$ and $x^J \in \mathrm{FV}(M_2)$ then by Theorem 4.1.3a, $x^I \in \mathrm{dom}(\Gamma_1)$ and $x^J \in \mathrm{dom}(\Gamma_2)$ so $M_1 \diamond M_2$.

$\square$

5. Subject reduction and expansion properties

Now we list the generation lemmas for the three type systems (for proofs see [13]).

Lemma 5.1. (Generation for $\vdash_1$)

1. If $x^n : \langle \Gamma \vdash_1 T\rangle$, then $\Gamma = (x^n : T)$.
2. If $\lambda x^n.M : \langle \Gamma \vdash_1 T_1 \to T_2\rangle$, then $M : \langle \Gamma, x^n : T_1 \vdash_1 T_2\rangle$.
3. If $MN : \langle \Gamma \vdash_1 T\rangle$ then $\Gamma = \Gamma_1 \sqcap \Gamma_2$, $T = \sqcap_{i=1}^n \vec{e}_{i(1:m_i)} T_i$, $n \geq 1, m_i \geq 0$, $M : \langle \Gamma_1 \vdash_1 \sqcap_{i=1}^n \vec{e}_{i(1:m_i)}(T_i' \to T_i)\rangle$ and $N : \langle \Gamma_2 \vdash_1 \sqcap_{i=1}^n \vec{e}_{i(1:m_i)} T_i'\rangle$.

Lemma 5.2. (Generation for $\vdash_2$)

1. If $x^n : \langle \Gamma \vdash_2 U\rangle$, then $\Gamma = (x^n : V)$ where $V \sqsubseteq U$.
2. If $\lambda x^n.M : \langle \Gamma \vdash_2 U\rangle$ and $\mathrm{d}(U) = m$, then $U = \sqcap_{i=1}^k \vec{e}_{i(1:m)}(V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma, x^n : \vec{e}_{i(1:m)} V_i \vdash_2 \vec{e}_{i(1:m)} T_i\rangle$.

Lemma 5.3. (Generation for $\vdash_3$)

1. If $x^L : \langle \Gamma \vdash_3 U\rangle$, then $\Gamma = (x^L : V)$ and $V \sqsubseteq U$.
2. If $\lambda x^L.M : \langle \Gamma \vdash_3 U\rangle$, $x^L \in \mathrm{FV}(M)$ and $\mathrm{d}(U) = K$, then $U = \omega^K$ or $U = \sqcap_{i=1}^p \vec{e}_K(V_i \to T_i)$ where $p \geq 1$ and for all $i \in \{1, \ldots, p\}$, $M : \langle \Gamma, x^L : \vec{e}_K V_i \vdash_3 \vec{e}_K T_i\rangle$.
3. If $\lambda x^L.M : \langle \Gamma \vdash_3 U\rangle$, $x^L \notin \mathrm{FV}(M)$ and $\mathrm{d}(U) = K$, then $U = \omega^K$ or $U = \sqcap_{i=1}^p \vec{e}_K(V_i \to T_i)$ where $p \geq 1$ and for all $i \in \{1, \ldots, p\}$, $M : \langle \Gamma \vdash_3 \vec{e}_K T_i\rangle$.
4. If $M\ x^L : \langle \Gamma, (x^L : U) \vdash_3 T\rangle$ and $x^L \notin \mathrm{FV}(M)$, then $M : \langle \Gamma \vdash_3 U \to T\rangle$.

Proof:
1. By induction on the derivation $x^L : \langle \Gamma \vdash_3 U\rangle$. 2. By induction on the derivation $\lambda x^L.M : \langle \Gamma \vdash_3 U\rangle$. 3. Same proof as that of 2. 4. By induction on the derivation $M\,x^L : \langle \Gamma, x^L : U \vdash_3 T\rangle$. □

We also show that no β-redexes are blocked in a typable term.

Lemma 5.4. (No β-redexes are blocked in typable terms)
Let $i \in \{1, 2\}$ and $M : \langle \Gamma \vdash_i U\rangle$. If $(\lambda x^n.M_1)M_2$ is a subterm of M, then $\mathrm{d}(M_2) = n$ and hence $(\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2]$.

Lemma 5.5. (Substitution for $\vdash_2$ and $\vdash_3$)
Let $i \in \{2, 3\}$. If $M : \langle \Gamma, x^I : U \vdash_i V\rangle$, $N : \langle \Delta \vdash_i U\rangle$ and $M \diamond N$ then $M[x^I := N] : \langle \Gamma \sqcap \Delta \vdash_i V\rangle$.

Proof:
By induction on the derivation $M : \langle \Gamma, x^I : U \vdash_i V\rangle$. □

Lemma 5.6. (Substitution and Subject β-reduction fails for $\vdash_1$)
Let a, b, c be different elements of $\mathcal{A}$. We have:

1. $(\lambda x^0.x^0x^0)(y^0z^0) \rhd_\beta (y^0z^0)(y^0z^0)$

2. $(\lambda x^0.x^0x^0)(y^0z^0) : \langle y^0 : b \to ((a \to c) \sqcap a), z^0 : b \vdash_1 c\rangle$.

3. $x^0x^0 : \langle x^0 : (a \to c) \sqcap a \vdash_1 c\rangle$.

4. It is not possible that
$(y^0z^0)(y^0z^0) : \langle y^0 : b \to ((a \to c) \sqcap a), z^0 : b \vdash_1 c\rangle$.
Hence, the substitution and subject β-reduction lemmas fail for $\vdash_1$.

Proof:
1..3 are easy. For 4, assume $(y^0z^0)(y^0z^0) : \langle y^0 : b \to ((a \to c) \sqcap a), z^0 : b \vdash_1 c\rangle$. By lemma 5.1.3 twice using lemmas 4.1 and 5.1.1:

- $y^0z^0 : \langle y^0 : b \to ((a \to c) \sqcap a), z^0 : b \vdash_1 \sqcap_{i=1}^n (T_i \to c)\rangle$.
- $y^0 : \langle y^0 : b \to ((a \to c) \sqcap a) \vdash_1 b \to (a \to c) \sqcap a\rangle$.
- $z^0 : \langle z^0 : b \vdash_1 b\rangle$.
- $\sqcap_{i=1}^n (T_i \to c) = (a \to c) \sqcap a$.

Hence $a = T_i \to c$ for some T_i. Absurd. □

Nevertheless, we show that SR and subject expansion for β using $\vdash_2$ holds. This will be used in the proof of completeness (more specifically in lemma 7.2 which is basic for the completeness theorem 7.1).

Lemma 5.7. (Subject reduction and expansion for β and $\vdash_2$)

1. If $M : \langle \Gamma \vdash_2 U\rangle$ and $M \rhd_\beta^* N$, then $N : \langle \Gamma \vdash_2 U\rangle$.

2. If $N : \langle \Gamma \vdash_2 U\rangle$ and $M \rhd_\beta^* N$ then $M : \langle \Gamma \vdash_2 U\rangle$.

Since $\vdash_3$ does not allow weakening, we need the next definition since when a term is reduced, it may lose some of its free variables and hence will need to be typed in a smaller environment.

Definition 5.1. If Γ is a type environment and $\mathcal{U} \subseteq \mathrm{dom}(\Gamma)$, then we write $\Gamma \upharpoonright_{\mathcal{U}}$ for the restriction of Γ on the variables of $\mathcal{U}$. If $\mathcal{U} = \mathrm{FV}(M)$ for a term M, we write $\Gamma \upharpoonright_M$ instead of $\Gamma \upharpoonright_{\mathrm{FV}(M)}$.

Now we are ready to prove the main result of this section:

Theorem 5.1. (Subject reduction for $\vdash_3$)
If $M : \langle \Gamma \vdash_3 U\rangle$ and $M \rhd_{\beta\eta}^* N$, then $N : \langle \Gamma \upharpoonright_N \vdash_3 U\rangle$.

Proof:
By induction on the derivation $M : \langle \Gamma \vdash_3 U\rangle$. □

Corollary 5.1.

1. If $M : \langle \Gamma \vdash_3 U\rangle$ and $M \rhd_\beta^* N$, then $N : \langle \Gamma \upharpoonright_N \vdash_3 U\rangle$.

2. If $M : \langle \Gamma \vdash_3 U\rangle$ and $M \rhd_h^* N$, then $N : \langle \Gamma \upharpoonright_N \vdash_3 U\rangle$.

The next lemma is needed for expansion.

Lemma 5.8. If $M[x^L := N] : \langle \Gamma \vdash_3 U\rangle$, $\mathrm{d}(N) = L$ and $x^L \in \mathrm{FV}(M)$ then there exist a type V and two type environments Γ_1, Γ_2 such that $\mathrm{d}(V) = L$ and:
$M : \langle \Gamma_1, x^L : V \vdash_3 U\rangle \qquad N : \langle \Gamma_2 \vdash_3 V\rangle \qquad \Gamma = \Gamma_1 \sqcap \Gamma_2$

Proof:
By induction on the derivation $M[x^L := N] : \langle \Gamma \vdash_3 U\rangle$. □

Since more free variables might appear in the β-expansion of a term, the next definition gives a possible enlargement of an environment.

Definition 5.2. Let $m \geq n$, $\Gamma = (x_i^{L_i} : U_i)_n$ and $\mathcal{U} = \{x_1^{L_1}, \ldots, x_m^{L_m}\}$. We write $\Gamma\uparrow^{\mathcal{U}}$ for $x_1^{L_1} : U_1, \ldots, x_n^{L_n} : U_n, x_{n+1}^{L_{n+1}} : \omega^{L_{n+1}}, \ldots, x_m^{L_m} : \omega^{L_m}$. If $\mathrm{dom}(\Gamma) \subseteq \mathrm{FV}(M)$, we write $\Gamma\uparrow^M$ instead of $\Gamma\uparrow^{\mathrm{FV}(M)}$.

We are now ready to establish that subject expansion holds for β (next theorem) and that it fails for η (Lemma 5.9).

Theorem 5.2. (Subject expansion for β)
If $N : \langle \Gamma \vdash_3 U\rangle$ and $M \rhd_\beta^* N$, then $M : \langle \Gamma\uparrow^M \vdash_3 U\rangle$.

Proof:
By induction on the length of the derivation $M \rhd_\beta^* N$ using the fact that if $\mathrm{FV}(P) \subseteq \mathrm{FV}(Q)$, then $(\Gamma\uparrow^P)\uparrow^Q = \Gamma\uparrow^Q$. □

Corollary 5.2. If $N : \langle \Gamma \vdash_3 U\rangle$ and $M \rhd_h^* N$, then $M : \langle \Gamma\uparrow^M \vdash_3 U\rangle$.

Lemma 5.9. (Subject expansion fails for η)
Let a be an element of $\mathcal{A}$. We have:

1. $\lambda y^\oslash.\lambda x^\oslash.y^\oslash x^\oslash \rhd_\eta \lambda y^\oslash.y^\oslash$
2. $\lambda y^\oslash.y^\oslash : \langle () \vdash_3 a \to a\rangle$.
3. It is not possible that: $\lambda y^\oslash.\lambda x^\oslash.y^\oslash x^\oslash : \langle () \vdash_3 a \to a\rangle$.
   Hence, the subject η-expansion lemmas fail for $\vdash_3$.

Proof:
1. and 2. are easy. For 3., assume $\lambda y^\oslash.\lambda x^\oslash.y^\oslash x^\oslash : \langle () \vdash_3 a \to a\rangle$.
By Lemma 5.3.2, $\lambda x^\oslash.y^\oslash x^\oslash : \langle (y : a) \vdash_3 \to a\rangle$. Again, by Lemma 5.3.2, $a = \omega^\oslash$ or there exists $n \geq 1$ such that $a = \sqcap_{i=1}^n (U_i \to T_i)$, absurd. □

6. Realisability

Crucial to a realisability semantics is the notion of a saturated set:

Definition 6.1. (Saturated sets)
Let $i \in \{1, 2, 3\}$ and $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}_i$.

1. We use $\mathcal{P}(\mathcal{X})$ to denote the powerset of $\mathcal{X}$, i.e. $\{\mathcal{Y} \mid \mathcal{Y} \subseteq \mathcal{X}\}$.
2. We define $\mathcal{X} \rightsquigarrow \mathcal{Y} = \{M \in \mathcal{M}_i \mid \forall N \in \mathcal{X}.\ M \diamond N \Rightarrow MN \in \mathcal{Y}\}$.
3. We say that $\mathcal{X} \wr \mathcal{Y}$ iff for all $M \in \mathcal{X} \rightsquigarrow \mathcal{Y}$, there exist $N \in \mathcal{X}$ such that $M \diamond N$.
4. For $r \in \{\beta, \beta\eta, h\}$, we say that $\mathcal{X}$ is r-saturated if whenever $M \rhd_r^* N$ and $N \in \mathcal{X}$, then $M \in \mathcal{X}$.

Saturation is closed under intersection, lifting and arrows:

Lemma 6.1. 1. If $\mathcal{X}, \mathcal{Y}$ are r-saturated sets, then $\mathcal{X} \cap \mathcal{Y}$ is r-saturated.

2. If $\mathcal{X} \subseteq \mathcal{M}_3$ is r-saturated, then $\mathcal{X}^{+i}$ is r-saturated.
3. If $\mathcal{X} \subseteq \mathcal{M}_2$ is r-saturated, then $\mathcal{X}^{+}$ is r-saturated.
4. Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}_2$ (resp. $\mathcal{M}_3$). If $\mathcal{Y}$ is r-saturated, then, for every set $\mathcal{X}$, $\mathcal{X} \rightsquigarrow \mathcal{Y}$ is r-saturated.
5. If $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}_2$ then $(\mathcal{X} \rightsquigarrow \mathcal{Y})^+ \subseteq \mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+$.
6. If $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}_3$ then $(\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i} \subseteq \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$.
7. Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}_2$. If $\mathcal{X}^+ \wr \mathcal{Y}^+$, then $\mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+ \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})^+$.
8. Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}_3$. If $\mathcal{X}^{+i} \wr \mathcal{Y}^{+i}$, then $\mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i} \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i}$.
9. For every $n \in \mathbb{N}$, the set $\mathbb{M}^n$ is saturated.

The interpretations and meanings of types are crucial to a realisability semantics:

Definition 6.2. (Interpretations and meaning of types)
Let $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and $\mathcal{V}_1, \mathcal{V}_2$ are both countably infinite. Let $i \in \{1, 2, 3\}$.

1. Let $x \in \mathcal{V}_1$ and I an index. We define $\mathcal{N}_x^I = \{x^I\ N_1 ... N_k \in \mathcal{M}_i \mid k \geq 0\}$.
2. In $\lambda I^{\mathbb{N}}$, let $r = \beta$ and $I_0 = 0$. In $\lambda^{\mathcal{L}_{\mathbb{N}}}$, let $r \in \{\beta, \beta\eta, h\}$ and $I_0 = \oslash$.
   (a) An r_i-interpretation $\mathcal{I} : \mathcal{A} \to \mathcal{P}(\mathcal{M}_i^{I_0})$ is a function such that for all $a \in \mathcal{A}$:
   • $\mathcal{I}(a)$ is r-saturated • In $\lambda I^{\mathbb{N}}$, $\mathcal{I}(a) \subseteq \mathbb{M}^0$ • $\forall x \in \mathcal{V}_1,\ \mathcal{N}_x^{I_0} \subseteq \mathcal{I}(a)$.
   (b) Let an r_i-interpretation $\mathcal{I} : \mathcal{A} \to \mathcal{P}(\mathcal{M}_i^{I_0})$. We extend $\mathcal{I}$ (to $\mathbb{U}_1$ in case of $\lambda I^{\mathbb{N}}$ and to $\mathbb{U}_3$ in case of $\lambda^{\mathcal{L}_{\mathbb{N}}}$) as follows:
   • $\mathcal{I}(U_1 \sqcap U_2) = \mathcal{I}(U_1) \cap \mathcal{I}(U_2)$ • $\mathcal{I}(U \to T) = \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$
   In $\lambda I^{\mathbb{N}}$: • $\mathcal{I}(eU) = \mathcal{I}(U)^+$
   In $\lambda^{\mathcal{L}_{\mathbb{N}}}$: • $\mathcal{I}(\omega^L) = \mathcal{M}_3^L$ • $\mathcal{I}(\overline{e}_i U) = \mathcal{I}(U)^{+i}$

   Because $\cap$ is commutative, associative, idempotent, $(\mathcal{X} \cap \mathcal{Y})^+ = \mathcal{X}^+ \cap \mathcal{Y}^+$ and $(\mathcal{X} \cap \mathcal{Y})^{+i} = \mathcal{X}^{+i} \cap \mathcal{Y}^{+i}$, $\mathcal{I}$ is well defined.
   Let r_i-int = $\{\mathcal{I} \mid \mathcal{I}$ is an r_i-interpretation$\}$.
   (c) Let $U \in \mathbb{U}_i$. Let $r \in \{\beta, \beta\eta, h\}$. Define $[U]_{r_i}$, the r_i-interpretation of U by:
   $[U]_{r_i} = \{M \in \mathcal{M}_i \mid M$ is closed and $M \in \bigcap_{\mathcal{I} \in r_i\text{-int}} \mathcal{I}(U)\}$

It is easy to show that in $\lambda I^{\mathbb{N}}$, if $x^n\ N_1...N_k \in \mathcal{N}_x^n$ then $\forall\, 1 \leq i \leq k$, $\mathrm{d}(N_i) \geq n$. Hence, in $\lambda I^{\mathbb{N}}$, we have $\mathcal{N}_x^n = \{x^n\ N_1...N_k \in \mathbb{M} \mid k \geq 0\}$.

Type interpretations are saturated and interpretations of good types contain only good terms.

Lemma 6.2. Let $r \in \{\beta, \beta\eta, h\}$. Let $i \in \{2, 3\}$.

1. (a) For any $U \in \mathbb{U}_i$ and $\mathcal{I} \in r_i$-int, we have $\mathcal{I}(U)$ is r-saturated.
   (b) If $\mathrm{d}(U) = L$ and $\mathcal{I} \in r_3$-int, then for all $x \in \mathcal{V}_1, \mathcal{N}_x^L \subseteq \mathcal{I}(U) \subseteq \mathcal{M}_3^L$.
   (c) If U is a good type such that $\mathrm{d}(U) = n$ and $\mathcal{I}$ is an r_2-interpretation, then $\forall x \in \mathcal{V}_1$, $x^n \in \mathcal{N}_x^n \subseteq \mathcal{I}(U) \subseteq \mathbb{M}^n$.
2. Let $r \in \{\beta, \beta\eta, h\}$. If $\mathcal{I} \in r_i$-int and $U \sqsubseteq V$, then $\mathcal{I}(U) \subseteq \mathcal{I}(V)$.

Proof:
1a . By induction on U using lemma 6.1.
1b. We prove $\forall x \in \mathcal{V}_1.\, \mathcal{N}_x^L \subseteq \mathcal{I}(U) \subseteq \mathcal{M}_3^L$ by induction on U.
1c. Obviously, $x^n \in \mathcal{N}_x^n$. We prove $\mathcal{N}_x^n \subseteq \mathcal{I}(U) \subseteq \mathbb{M}^n$ by induction on U good.
2. By induction of the derivation $U \sqsubseteq V$. □

Corollary 6.1. (Meanings of good types consist of good terms)
On $\mathbb{U}_1$ (hence also on $\mathbb{U}_2$) we have: If U is a good type such that $\mathrm{d}(U) = n$ then $[U]_{\beta_2} \subseteq \mathbb{M}^n$.

Proof:
Simply note that by lemma 6.2, for any interpretation $\mathcal{I}, \mathcal{I}(U) \subseteq \mathbb{M}^n$. □

Lemma 6.3. (Soundness of $\vdash_1$/$\vdash_2$/$\vdash_3$)
Let $i \in \{1, 2, 3\}$, $r \in \{\beta, \beta\eta, h\}$, $\mathcal{I} \in r_i$-int. Let $M : \langle (x_j^{I_j} : U_j)_n \vdash_i U\rangle$ and $\forall 1 \leq j \leq n$, $N_j \in \mathcal{I}(U_j)$. If $\diamond\{M, N_1, N_2, \ldots, N_n\}$, then $M[(x_j^{I_j} := N_j)_n] \in \mathcal{I}(U)$.

Proof:
By induction on the derivation $M : \langle (x_j^{I_j} : U_j)_n \vdash_i U\rangle$. □

Corollary 6.2. Let $r \in \{\beta, \beta\eta, h\}$ and $i \in \{1, 2, 3\}$. If $M : \langle () \vdash_i U\rangle$, then $M \in [U]_{r_i}$.

Proof:
By Lemma 6.3, $M \in \mathcal{I}(U)$ for any r_i-interpretation $\mathcal{I}$. By Lemma 4.2, $\mathrm{FV}(M) = \mathrm{dom}(()) = \emptyset$ and hence M is closed. Therefore, $M \in [U]_{r_i}$. □

Lemma 6.4. (The meaning of types is closed under type operations)
Let $r \in \{\beta, \beta\eta, h\}$ and $j \in \{1, 2, 3\}$. The following hold:

1. $[\overline{e}_i U]_{r_3} = [U]_{r_3}^{+i}$ and if $k \in \{1, 2\}$ then $[eU]_{r_k} = [U]_{r_k}^{+}$
2. $[U \sqcap V]_{r_j} = [U]_{r_j} \cap [V]_{r_j}$
3. If $U \to T \in \mathbb{U}_3$ then for any interpretation $\mathcal{I}, \mathcal{I}(U) \wr \mathcal{I}(T)$.

4. If $U \to T$ is good then for any interpretation $\mathcal{I}$, $\mathcal{I}(U) \wr \mathcal{I}(T)$.
5. On $\mathbb{U}_1$ only (since $eU \to eT \notin \mathbb{U}_2$), we have:
   If $U \to T$ is good, then $[e(U \to T)]_{\beta_2} = [eU \to eT]_{\beta_2}$.

Proof:
1. and 2. are easy.

3. Let $\mathrm{d}(U) = L$, $M \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$ and $x \in \mathcal{V}_1$ such that for all K, $x^K \notin \mathrm{FV}(M)$, hence $M \diamond x^L$ and by lemma 6.2, $x^L \in \mathcal{I}(U)$.
4. Let $\mathrm{d}(U) = n$ and $M \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$. Take $x \in \mathcal{V}_1$ such that $\forall p \in \mathbb{N}$, $x^p \notin FV(M)$. Hence, $M \diamond x^n$. By lemma 3.1, U is good and by lemma 6.2, $x^n \in \mathcal{I}(U)$.
5. Since $U \to T$ is good, then, by lemma 3.1, U, T are good and $\mathrm{d}(U) \geq \mathrm{d}(T)$. Again by lemma 3.1, eU, eT are good, $\mathrm{d}(eU) \geq \mathrm{d}(eT)$ and $eU \to eT$ is good. Hence by 3. above, $\mathcal{I}(U)^+ \wr \mathcal{I}(T)^+$. Thus, by lemma 6.1.5, for any interpretation $\mathcal{I}$ we have $\mathcal{I}(e(U \to T)) = \mathcal{I}(eU \to eT)$.

$\square$

The next definition and lemma put the realisability semantics in use.

Definition 6.3. (Examples)
Let $a, b \in \mathcal{A}$ where $a \neq b$. We define:

- $Id_0 = a \to a$, $Id_1 = e_1(a \to a)$ and $Id_1' = e_1 a \to e_1 a$.
- $D = (a \sqcap (a \to b)) \to b$.
- $Nat_0 = (a \to a) \to (a \to a)$, $Nat_1 = e_1((a \to a) \to (a \to a))$, and $Nat_0' = (e_1 a \to a) \to (e_1 a \to a)$.

Moreover, if M, N are terms and $n \in \mathbb{N}$, we define $(M)^n\ N$ by induction on n: $(M)^0\ N = N$ and $(M)^{m+1}\ N = M\ ((M)^m\ N)$.

Lemma 6.5. 1. $[(a \sqcap b) \to a]_{\beta_1} = \{M \in \mathbb{M}^0 \mid M \rhd_\beta^* \lambda y^0.y^0\}$.

2. It is not possible that $\lambda y^0.y^0 : \langle () \vdash_1 (a \sqcap b) \to a\rangle$.
3. $\lambda y^0.y^0 : \langle () \vdash_2 (a \sqcap b) \to a\rangle$.
4. $[Id_0]_{\beta_3} = \{M \in \mathcal{M}_3^\oslash \mid M \rhd_\beta^* \lambda y^\oslash y^\oslash\}$.
5. $[Id_1]_{\beta_3} = [Id_1']_\beta = \{M \in \mathcal{M}_3^{(1)} \mid M \rhd_\beta^* \lambda y^{(1)}.y^{(1)}\}$. (Note that $Id_1' \notin \mathbb{U}_3$.)
6. $[D]_{\beta_3} = \{M \in \mathcal{M}_3^\oslash \mid M \rhd_\beta^* \lambda y^\oslash.y^\oslash y^\oslash\}$.
7. $[Nat_0]_{\beta_3} = \{M \in \mathcal{M}_3^\oslash \mid M \rhd_\beta^* \lambda f^\oslash.f^\oslash$ or $M \rhd_\beta^* \lambda f^\oslash.\lambda y^\oslash.(f^\oslash)^n y^\oslash$ where $n \geq 1\}$.
8. $[Nat_1]_{\beta_3} = \{M \in \mathcal{M}_3^{(1)} \mid M \rhd_\beta^* \lambda f^{(1)}.f^{(1)}$ or $M \rhd_\beta^* \lambda f^{(1)}.\lambda x^{(1)}.(f^{(1)})^n y^{(1)}$ where $n \geq 1\}$.
9. $[Nat_0']_{\beta_3} = \{M \in \mathcal{M}_3^\oslash \mid M \rhd_\beta^* \lambda f^\oslash.f^\oslash$ or $M \rhd_\beta^* \lambda f^\oslash.\lambda y^{(1)}.f^\oslash y^{(1)}\}$.

7. The challenges of completeness in $\lambda I^{\mathbb{N}}$

In this paper we are concerned with two realisability semantics of E-variables. These semantics are based on a hierarchy of types and terms. Considering how expansions can introduce new substitutions, new expansions and an unbounded number of new variables (even E-variables), it became clear that to give meanings to expansions, we needed to use a hierarchy on types and terms. At first, one thinks of labeling types and terms with a natural number and this is the hierarchy we used in $\lambda I^{\mathbb{N}}$. When assigning meanings to types, we ensured that each use of E-variables simply changes the labels and that each E-variable acted as a kind of capsule that isolates parts of the λ-term being analyzed by the typing. This captured accurately the intuition behind E-variables. However, this indexing poses two problems: it imposes that the type ω should have all possible indexes (which is impossible and hence we eliminated ω from the type systems for $\mathcal{M}_2$) and it implies that the realisability semantics can only be complete when a unique E-variable is used (as we will see in this section). In order to understand the challenges of the semantics of E-variables with ω and to understand the idea behind the hierarchy, we first studied the λI-calculus typed with two representative intersection type systems. The restriction to λI (where in every $(\lambda x.M)$ the variable x must appear free in M) was motivated by not knowing how to support the ω type while preserving the intuitive levels made of single natural numbers. For $\vdash_1$, the first of these type systems (the most natural), we showed that subject reduction and hence completeness do not hold.

Remark 7.1. (Failure of completeness for $\vdash_1$)
Items 1, 2 and 3 of Lemma 6.5 show that we can not have a completeness result (a converse of lemma 6.3 for closed terms) for $\vdash_1$. To type the term $\lambda y^0.y^0$ by the type $(a \sqcap b) \to a$, we need an elimination rule for $\sqcap$ which we do not have in $\vdash_1$.

Note that failure of completeness for $\vdash_1$ is related to the failure of its subject reduction. So, one might think that since $\vdash_2$, the second type system for $\lambda I^{\mathbb{N}}$, has subject reduction, its semantics is complete. This is not the case.

Remark 7.2. (Failure of completeness of $\vdash_2$ if more than one E-variable is used)
Let $a \in \mathcal{A}$, $e_1, e_2 \in \mathcal{E}$, $e_1 \neq e_2$ and $Nat''_0 = (e_1 a \to a) \to (e_2 a \to a)$. Then:
1) $\lambda f^0.f^0 \in [Nat''_0]$ and 2) It is not possible that $\lambda f^0.f^0 : \langle () \vdash_2 Nat''_0 \rangle$.

Hence $\lambda f^0.f^0 \in [Nat''_0]$ but $\lambda f^0.f^0$ is not typable by Nat''_0 and we do not have completeness in the presence of more than one expansion variable.

However, we will see that we have completeness for $\vdash_2$ if only one expansion variable is used.

7.1. Completeness of $\vdash_2$ with one expansion variable

The problem shown in remark 7.2 comes from the fact that for the realisability semantics that we considered for $\vdash_2$, we identify all expansion variables. In order to give a completeness theorem for $\vdash_2$, we will in what follows restrict our system to only one expansion variable. In the rest of this section, we assume that the set $\mathcal{E}$ contains only one expansion variable $\overline{e}_1$.

The need of one single expansion variable is clear in part 2) of lemma 7.1 which would fail if we use more than one expansion variable. For example, if $e_1 \neq e_2$ then $(e_1 a)^- = a = (e_2 a)^-$ but $e_1 a \neq e_2 a$. This lemma is crucial for the rest of this section and hence, a single expansion variable is also crucial.

Lemma 7.1. Let $U, V \in \mathbb{U}_2$ and $\mathrm{d}(U) = \mathrm{d}(V) > 0$. 1) $\overline{e}_1 U^- = U$ and 2) If $U^- = V^-$, then $U = V$.

Despite the difference of the number of expansion variables used in this completeness proof and that of the next section, there are a number of similarities of both proofs. We still write these two proofs independently to illustrate the method and especially since the proof for this section is far simpler. Furthermore, we only show the semantics in this section for β-reduction (although the semantics works for all our notions of reductions as we show in the next section).

The first step of the proof is to divide $\{y^n \mid y \in \mathcal{V}_2\}$ disjointly amongst types of order n.

Definition 7.1. Let $U \in \mathbb{U}_2$. We define the set of variables $\mathbb{V}_U$ by induction on $\mathrm{d}(U)$. If $\mathrm{d}(U) = 0$, then: $\mathbb{V}_U$ is an infinite set of variables of degree 0; if $y^0 \in \mathbb{V}_U$, then $y \in \mathcal{V}_2$; and if $U \neq V$ and $\mathrm{d}(U) = \mathrm{d}(V) = 0$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$. If $\mathrm{d}(U) = n + 1$, then we put $\mathbb{V}_U = \{y^{n+1} \mid y^n \in \mathbb{V}_{U^-}\}$.

Our partition of $\mathcal{V}_2$ allows useful infinite sets which contain type environments that will play a crucial role in one particular type interpretation. These sets and environments are given in the next definition.

Definition 7.2.

1. Let $n \in \mathbb{N}$. We let $\mathbb{G}^n = \{(y^n : U) \mid U \in \mathbb{U}_2, \mathrm{d}(U) = n \text{ and } y^n \in \mathbb{V}_U\}$ and $\mathbb{H}^n = \bigcup_{m \geq n} \mathbb{G}^m$. Note that $\mathbb{G}^n$ and $\mathbb{H}^n$ are not type environments because they are infinite sets.
2. Let $n \in \mathbb{N}$, $M \in \mathcal{M}_2$ and $U \in \mathbb{U}_2$, we write $M : \langle \mathbb{H}^n \vdash_2 U \rangle$ iff there is a type environment $\Gamma \subset \mathbb{H}^n$ where $M : \langle \Gamma \vdash_2 U \rangle$

Now, for every n, we define the set of the good terms of order n which contain some free variable x^i where $x \in \mathcal{V}_1$ and $i \geq n$.

Definition 7.3. Let $n \in \mathbb{N}$ and $\mathcal{O}^n = \{M \in \mathbb{M}^n \mid x^i \in \mathrm{FV}(M) \text{ where } x \in \mathcal{V}_1 \text{ and } i \geq n\}$. Obviously, if $n \in \mathbb{N}$ and $x \in \mathcal{V}_1$, then $\mathcal{N}_x^n \subseteq \mathcal{O}^n$.

Here is the crucial β_2-interpretation $\mathbb{I}$ for the proof of completeness:

Definition 7.4. Let $\mathbb{I}$ be the β_2-interpretation defined by:
for all type variables a, $\mathbb{I}(a) = \mathcal{O}^0 \cup \{M \in \mathcal{M}_2^0 \mid M : \langle \mathbb{H}^0 \vdash_2 a \rangle\}$.

$\mathbb{I}$ is indeed a β_2-interpretation and the interpretation of a type of order n contains the good terms of order n which are typable in the special environments which are parts of the infinite sets of definition 7.2:

Lemma 7.2.

1. $\mathbb{I}$ is a β_2-interpretation. I.e., $\forall a \in \mathcal{A}$, $\mathbb{I}(a)$ is β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^0 \subseteq \mathbb{I}(a) \subseteq \mathbb{M}^0$.
2. If $U \in \mathbb{U}_2$ is good and $\mathrm{d}(U) = n$, then $\mathbb{I}(U) = \mathcal{O}^n \cup \{M \in \mathbb{M}^n \mid M : \langle \mathbb{H}^n \vdash_2 U \rangle\}$.

$\mathbb{I}$ is used to prove completeness (the proof is on the authors web pages).

Theorem 7.1. (Completeness)
Let $U \in \mathbb{U}_2$ be good such that $\mathrm{d}(U) = n$. The following hold:

1. $[U]_{\beta_2} = \{M \in \mathbb{M}^n \mid M : \langle () \vdash_2 U \rangle\}$.
2. $[U]_{\beta_2}$ is stable by reduction: i.e., if $M \in [U]_{\beta_2}$ and $M \rhd_\beta^* N$, then $N \in [U]_{\beta_2}$.
3. $[U]_{\beta_2}$ is stable by expansion: i.e., if $N \in [U]_{\beta_2}$ and $M \rhd_\beta^* N$, then $M \in [U]_{\beta_2}$.

8. Completeness in $\lambda^{\mathcal{L}_{\mathbb{N}}}$

Having understood the challenges of E-variables and the difficulty of representing the type ω using natural numbers as indices for the hierarchy, we moved to the presentation of indices as sequences of natural numbers and we provided our third type system $\vdash_3$. We developed a realizability semantics where we allow the full λ-calculus, an arbitrary (possibly infinite) number of expansion variables and where ω is present, and we showed its soundness. Now, we show its completeness.

We need the following partition of the set of variables $\{y^L \mid y \in \mathcal{V}_2\}$.

Definition 8.1. 1. Let $L \in \mathcal{L}_{\mathbb{N}}$. We define $\mathbb{U}_3^L = \{U \in \mathbb{U}_3 \mid \mathrm{d}(U) = L\}$ and $\mathcal{V}^L = \{x^L \mid x \in \mathcal{V}_2\}$.

2. Let $U \in \mathbb{U}_3$. We inductively define a set of variables $\mathbb{V}_U$ as follows:
   - If $\mathrm{d}(U) = \oslash$ then:
     - $\mathbb{V}_U$ is an infinite set of variables of degree $\oslash$.
     - If $U \neq V$ and $\mathrm{d}(U) = \mathrm{d}(V) = \oslash$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.
     - $\bigcup_{U \in \mathbb{U}_3^{\oslash}} \mathbb{V}_U = \mathcal{V}^{\oslash}$.
   - If $\mathrm{d}(U) = L$, then we put $\mathbb{V}_U = \{y^L \mid y^{\oslash} \in \mathbb{V}_{U^{-L}}\}$.

Lemma 8.1. 1. If $\mathrm{d}(U), \mathrm{d}(V) \succeq L$ and $U^{-L} = V^{-L}$, then $U = V$.

2. If $\mathrm{d}(U) = L$, then $\mathbb{V}_U$ is an infinite subset of $\mathcal{V}^L$.

3. If $U \neq V$ and $\mathrm{d}(U) = \mathrm{d}(V) = L$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.

4. $\bigcup_{U \in \mathbb{U}_3^L} \mathbb{V}_U = \mathcal{V}^L$.

5. If $y^L \in \mathbb{V}_U$, then $y^{i::L} \in \mathbb{V}_{\overline{e}_i U}$.

6. If $y^{i::L} \in \mathbb{V}_U$, then $y^L \in \mathbb{V}_{U^{-i}}$.

Proof:
1. If $L = (n_i)_m$, we have $U = \overline{e}_{n_1} \ldots \overline{e}_{n_m} U'$ and $V = \overline{e}_{n_1} \ldots \overline{e}_{n_m} V'$. Then $U^{-L} = U'$, $V^{-L} = V'$ and $U' = V'$. Thus $U = V$.
2., 3. and 4. By induction on L and using 1.
5. Because $(\overline{e}_i U)^{-i} = U$.
6. By definition. □

Our partition of the set $\mathcal{V}_2$ as above will enable us to give in the next definition useful infinite sets which will contain type environments that will play a crucial role in one particular type interpretation.

Definition 8.2. 1. Let $L \in \mathcal{L}_{\mathbb{N}}$. We denote $\mathbb{G}^L = \{(y^L : U) \mid U \in \mathbb{U}_3^L \text{ and } y^L \in \mathbb{V}_U\}$ and $\mathbb{H}^L = \bigcup_{K \succeq L} \mathbb{G}^K$. Note that $\mathbb{G}^L$ and $\mathbb{H}^L$ are not type environments because they are infinite sets.

2. Let $L \in \mathcal{L}_{\mathbb{N}}$, $M \in \mathcal{M}_3$ and $U \in \mathbb{U}_3$, we write:
   - $M : \langle \mathbb{H}^L \vdash_3 U \rangle$ if there is a type environment $\Gamma \subset \mathbb{H}^L$ where $M : \langle \Gamma \vdash_3 U \rangle$

- $M : \langle \mathbb{H}^L \vdash_3^* U\rangle$ if $M \rhd^*_{\beta\eta} N$ and $N : \langle \mathbb{H}^L \vdash_3 U\rangle$

Lemma 8.2. 1. If $\Gamma \subset \mathbb{H}^L$ then $\mathrm{OK}(\Gamma)$.

2. If $\Gamma \subset \mathbb{H}^L$ then $\overline{e}_i\Gamma \subset \mathbb{H}^{i::L}$.
3. If $\Gamma \subset \mathbb{H}^{i::L}$ then $\Gamma^{-i} \subset \mathbb{H}^L$.
4. If $\Gamma_1 \subset \mathbb{H}^L$, $\Gamma_2 \subset \mathbb{H}^K$ and $L \preceq K$ then $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$.

Proof:
1. Let $x^K : U \in \Gamma$ then $U \in \mathbb{U}^K$ and so $\mathrm{d}(U) = K$. 2. and 3. are by lemma 8.1. 4. First note that by 1., $\Gamma_1 \sqcap \Gamma_2$ is well defined. $\mathbb{H}^K \subseteq \mathbb{H}^L$. Let $(x^R : U_1 \sqcap U_2) \in \Gamma_1 \sqcap \Gamma_2$ where $(x^R : U_1) \in \Gamma_1 \subset \mathbb{H}^L$ and $(x^R : U_2) \in \Gamma_2 \subset \mathbb{H}^K \subseteq \mathbb{H}^L$, then $\mathrm{d}(U_1) = \mathrm{d}(U_2) = R$ and $x^R \in \mathbb{V}_{U_1} \cap \mathbb{V}_{U_2}$. Hence, by lemma 8.1, $U_1 = U_2$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \cup \Gamma_2 \subset \mathbb{H}^L$. □

For every $L \in \mathcal{L}_{\mathbb{N}}$, we define the set of terms of degree L which contain some free variable x^K where $x \in \mathcal{V}_1$ and $K \succeq L$.

Definition 8.3. For every $L \in \mathcal{L}_{\mathbb{N}}$, let $\mathcal{O}^L = \{M \in \mathcal{M}_3^L \mid x^K \in \mathrm{FV}(M),\, x \in \mathcal{V}_1 \text{ and } K \succeq L\}$. It is easy to see that, for every $L \in \mathcal{L}_{\mathbb{N}}$ and $x \in \mathcal{V}_1$, $\mathcal{N}_x^L \subseteq \mathcal{O}^L$.

Lemma 8.3. 1. $(\mathcal{O}^L)^{+i} = \mathcal{O}^{i::L}$.

2. If $y \in \mathcal{V}_2$ and $(My^K) \in \mathcal{O}^L$, then $M \in \mathcal{O}^L$
3. If $M \in \mathcal{O}^L$, $M \diamond N$ and $L \preceq K = \mathrm{d}(N)$, then $MN \in \mathcal{O}^L$.
4. If $\mathrm{d}(M) = L$, $L \preceq K$, $M \diamond N$ and $N \in \mathcal{O}^K$, then $MN \in \mathcal{O}^L$.

The crucial interpretation $\mathbb{I}$ for the proof of completeness is given as follows:

Definition 8.4. 1. Let $\mathbb{I}_{\beta\eta}$ be the $\beta\eta$-interpretation defined by: for all type variables a, $\mathbb{I}_{\beta\eta}(a) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash_3^* a\rangle\}$.

2. Let $\mathbb{I}_\beta$ be the β-interpretation defined by: for all type variables a, $\mathbb{I}_\beta(a) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash_3 a\rangle\}$.
3. Let $\mathbb{I}_{eh}$ be the h-interpretation defined by: for all type variables a, $\mathbb{I}_h(a) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash_3 a\rangle\}$.

The next crucial lemma shows that $\mathbb{I}$ is an interpretation and that the interpretation of a type of order L contains terms of order L which are typable in these special environments which are parts of the infinite sets of Definition 8.2.

Lemma 8.4. Let $r \in \{\beta\eta, \beta, h\}$ and $r' \in \{\beta, h\}$

1. If $\mathbb{I}_r \in r$-int and $a \in \mathcal{A}$ then $\mathbb{I}_r(a)$ is r-saturated and for all $x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathbb{I}_r(a)$.

2. If $U \in \mathbb{U}_3$ and $\mathrm{d}(U) = L$, then $\mathbb{I}_{\beta\eta}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash_3^* U\rangle\}$.

3. If $U \in \mathbb{U}_3$ and $\mathrm{d}(U) = L$, then $\mathbb{I}_{r'}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash_3 U\rangle\}$.

Now, we use this crucial $\mathbb{I}$ to establish completeness of our semantics.

Theorem 8.1. (Completeness of $\vdash_3$)
Let $U \in \mathbb{U}_3$ such that $\mathrm{d}(U) = L$.

1. $[U]_{\beta\eta_3} = \{M \in \mathcal{M}_3^L \mid M \text{ closed}, M \rhd_{\beta\eta}^* N \text{ and } N : \langle () \vdash_3 U\rangle\}$.

2. $[U]_{\beta_3} = [U]_{h_3} = \{M \in \mathcal{M}_3^L \mid M : \langle () \vdash_3 U\rangle\}$.

3. $[U]_{\beta\eta_3}$ is stable by reduction. I.e., If $M \in [U]_{\beta\eta_3}$ and $M \rhd_{\beta\eta}^* N$ then $N \in [U]_{\beta\eta_3}$.

Proof:
Let $r \in \{\beta, h, \beta\eta\}$.

1. Let $M \in [U]_{\beta\eta_3}$. Then M is a closed term and $M \in \mathbb{I}_{\beta\eta}(U)$. Hence, by Lemma 8.4, $M \in \mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash_3^* U\rangle\}$. Since M is closed, $M \notin \mathcal{O}^L$. Hence, $M \in \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash_3^* U\rangle$ and so, $M \rhd_{\beta\eta}^* N$ and $N : \langle \Gamma \vdash_3 U\rangle$ where $\Gamma \subset \mathbb{H}^L$. By Theorem 2.1, N is closed and, by Lemma 4.1.3a, $N : \langle () \vdash_3 U\rangle$.

   Conversely, take M closed such that $M \rhd_{\beta}^* N$ and $N : \langle () \vdash_3 U\rangle$. Let $\mathcal{I} \in \beta$-int. By Lemma 6.3, $N \in \mathcal{I}(U)$. By Lemma 6.2.1, $\mathcal{I}(U)$ is $\beta\eta$-saturated. Hence, $M \in \mathcal{I}(U)$. Thus $M \in [U]_{\beta\eta_3}$.

2. Let $M \in [U]_{\beta_3}$. Then M is a closed term and $M \in \mathbb{I}_{\beta}(U)$. Hence, by Lemma 8.4, $M \in \mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash_3 U\rangle\}$. Since M is closed, $M \notin \mathcal{O}^L$. Hence, $M \in \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash_3 U\rangle\}$ and so, $M : \langle \Gamma \vdash_3 U\rangle$ where $\Gamma \subset \mathbb{H}^L$. By Lemma 4.1.3a, $N : \langle () \vdash_3 U\rangle$.

   Conversely, take M such that $M : \langle () \vdash_3 U\rangle$. By Lemma 4.1.3a, M is closed. Let $\mathcal{I} \in \beta$-int. By Lemma 6.3, $M \in \mathcal{I}(U)$. Thus $M \in [U]_{\beta_3}$.

   It is easy to see that $[U]_{\beta_3} = [U]_{h_3}$.

3. Let $M \in [U]_{\beta\eta_3}$ such that $M \rhd_{\beta\eta}^* N$. By 1, M is closed, $M \rhd_{\beta\eta}^* P$ and $P : \langle () \vdash_3 U\rangle$. By confluence Theorem 2.2, there is Q such that $P \rhd_{\beta\eta}^* Q$ and $N \rhd_{\beta\eta}^* Q$. By subject reduction Theorem 5.1, $Q : \langle () \vdash_3 U\rangle$. By Theorem 2.1, N is closed and, by 1, $N \in [U]_{\beta\eta_3}$.

□

9. Conclusion

Expansion may be viewed to work like a multi-layered simultaneous substitution. Moreover, expansion is a crucial part of a procedure for calculating principal typings and helps support compositional type inference. Because the early definitions of expansion were complicated, expansion variables (E-variables) were introduced to simplify and mechanize expansion. The aim of this paper is to give a complete semantics for intersection type systems with expansion variables.

We studied first the $\lambda I^{\mathbb{N}}$-calculus, an indexed version of the λI-calculus. This indexed version was typed using first a basic intersection type system with expansion variables but without an intersection elimination rule, and then using an intersection type system with expansion variables and an elimination rule.

We gave a realisability semantics for both type systems showing that the first type system is not complete in the sense that there are types whose semantic meaning is not the set of $\lambda I^{\mathbb{N}}$-terms having this type. In particular, we showed that $\lambda y^0.y^0$ is in the semantic meaning of $(a \sqcap b) \to a$ but it is not possible to give $\lambda y^0.y^0$ the type $(a \sqcap b) \to a$. The main reason for the failure of completeness in the first system is associated with the failure of the subject reduction property for this first system. Hence, we moved to the second system which we showed to have the desirable properties of subject reduction and expansion and strong normalisation. However, for this second system, we showed again that completeness fails if we use more than one expansion variable but that completeness succeeds if we restrict the system to one single expansion variable.

In order to overcome the problems of completeness, we changed our realisability semantics from one which uses indices as natural number to one that uses the indices as lists of natural numbers. The new semantics is more complex and we lose the elegance of the first (especially in being able to define the so-called good terms and good types). However, we show that this second semantics has all the desirable properties of a type systems and it handles all of the lambda calculus (not simply the λI-calculus). We also show that this second semantics is complete when any number (including infinite) of expansion variables is used. As far as we know, our work constitutes the first study of a denotational semantics of intersection type systems with E-variables (using realizability or any other approach) and of the difficulties involved.

In this article we are not interested in a denotational semantics or at least we are not interested in an extensional lambda model interpreting the terms of the untyped lambda-calculus. Instead, we are interested in building a realisability semantics by defining sets of realisers (functions/programs satisfying the requirements of some specification) of types. Such a model would help to highlight the relation between typable terms of the untyped lambda-calculus and types w.r.t. a type system. Moreover, interpreting types in a model helps to understand the meaning of a type (w.r.t. the model) which is defined as a purely syntactic form and is clearly used as a meaningful expression (as the integer type, whatever its notation is, which is always used as the type of each integer). An arrow type expresses functionality. In that way, models based on lambda-models have been built for intersection type systems [8]. In these works, intersection types (introduced to be able to type more terms than in the Simply Typed Lambda Calculus) are interpreted by set-theoretical intersection of meanings. Even if expansion variables have been introduced to give a simple formalisation of the expansion mechanism, i.e. as a syntactic object, we are interested in the meaning of such a syntactic object. We are particularly interested in answering a number of questions which include:

1. What does an expansion variable applied to a type stand for?

2. What are the realisers of such a type?

3. How can the relation between terms and types be described w.r.t. a type system?

4. How can we extend models such as the one given in [12] to a type system with expansion?

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A. Proofs

A.1. Syntax of the calculi

Proof:
[of Lemma 2.2] The only if direction is by definition. The if direction, for each of 1. and 2. is by cases on the derivation $\lambda x^n.M$ is good respectively $M_1 M_2$ is good. $\square$

Lemma A.1. Let $i \in \{1, 2, 3\}$.

1. On $\mathcal{M}_i$, $\diamond$ is reflexive and symmetric but not transitive.
2. Let $M, N, M', N' \in \mathcal{M}_i$ such that M' is a subterm of M and N' is a subterm of N. If $M \diamond N$, then $M' \diamond N'$.
3. (a) Let $M, (N_1 N_2) \in \mathcal{M}_i$. We have $M \diamond \{N_1, N_2\}$ iff $M \diamond (N_1 N_2)$.
   (b) Let $M, \lambda x^I.N \in \mathcal{M}_i$. We have $M \diamond N$ iff $M \diamond (\lambda x^I.N)$.
   (c) Let $M, N[(x_i^{I_i} := N_i)_p] \in \mathcal{M}_i$ and $\mathcal{X} = \{N\} \cup \{N_i \mid 1 \leq i \leq p\} \subset \mathcal{M}_i$. If $M \diamond \mathcal{X}$, then $M \diamond N[(x_i^{I_i} := N_i)_p]$.
4. Let $M_1[(x_i^{I_i} := N_i)_p], M_2[(x_i^{I_i} := N_i)_p] \in \mathcal{M}_i$ and $\mathcal{X} = \{M_1, M_2\} \cup \{N_i \mid 1 \leq i \leq p\}$. If $\diamond\mathcal{X}$ then $M_1[(x_i^{I_i} := N_i)_p] \diamond M_2[(x_i^{I_i} := N_i)_p]$.
5. Let $i \in \{1, 2\}$ and $M \in \mathcal{M}_i$. We have: $\mathrm{d}(M) = \min(I \mid x^I$ occurs in $M)$.
6. Let $\mathcal{X} = \{M\} \cup \{N_i \mid 1 \leq i \leq p\} \subset \mathcal{M}_i$. We have:
   (a) $\diamond\mathcal{X}$ and $\forall\, 1 \leq i \leq p$, $\mathrm{d}(N_i) = I_i$ iff $M[(x_i^{I_i} := N_i)_p] \in \mathcal{M}_i$.
   (b) If $\diamond\mathcal{X}$ and $\forall\, 1 \leq i \leq p$, $\mathrm{d}(N_i) = I_i$, then $\mathrm{d}(M[(x_i^{I_i} := N_i)_p]) = \mathrm{d}(M)$.
7. Let $M, N, P \in \mathcal{M}_i$. If $\diamond\{M, N, P\}$, $\mathrm{d}(N) = I$, $\mathrm{d}(P) = J$ and $x^I \notin \mathrm{FV}(P) \cup \{y^J\}$, then $M[x^I := N][y^J := P] = M[y^J := P][x^I := N[y^J := P]]$.
8. Let $M, N, P \in \mathcal{M}_i$. If $M \diamond P$ and $\mathrm{FV}(M) = \mathrm{FV}(N)$ then $N \diamond P$.
9. Let $i \in \{1, 2\}$ and $M, N \in \mathcal{M}_i$ where $\mathrm{d}(N) = n$ and $x^I \in \mathrm{FV}(M)$. We have:
   $M[x^n := N]$ is good iff M and N are good and $M \diamond N$.

Proof:

1. For reflexivity, we show by induction on $M \in \mathcal{M}_i$ that if $x^I, x^J \in \mathrm{FV}(M)$, then $I = J$. Symmetry is by definition of $\diamond$. For failure of transitivity take z^1, y^2 and z^2 for the case $i \in \{1, 2\}$ and $z^{\oslash}$, $y^{(1)}$ and $z^{(1)}$ for the case $i = 3$.
2. Let $x^I \in \mathrm{FV}(M')$ and $x^J \in \mathrm{FV}(N')$. If $x^I \in \mathrm{FV}(M)$ and $x^J \in \mathrm{FV}(N)$ use $M \diamond N$. The cases a) $x^I \in \mathrm{FV}(M)$ and λx^J occurs in N and b) λx^I occurs in M and $x^J \in \mathrm{FV}(N)$ are not possible by BC. Finally, if λx^I occurs in M and λx^J occurs in N, then by BC, $I = J$.

3. Simple check of the $\diamond$ condition using 2.

4. By 3c, $M_1 \diamond M_2[(x_i^{I_i} := N_i)_p]$ and $N_j \diamond M_2[(x_i^{I_i} := N_i)_p] \; \forall\, 1 \leq j \leq p$, and, by 3c again and by 1, $M_1[(x_i^{I_i} := N_i)_p] \diamond M_2[(x_i^{I_i} := N_i)_p]$.

5. By induction on M.

6. 6a is by definition of substitution. 6b is by induction on M.

7. By induction on M using 3c and 6a.

8. If $x^I \in \mathrm{FV}(N) = \mathrm{FV}(M)$ and $x^J \in \mathrm{FV}(P)$ then since $M \diamond P$, $I = J$.

9. By induction on M.

   – By definition of substitution, $x^n[x^n := N]$ is good iff x^n and N are good and $x^n \diamond N$.
   – $(\lambda y^m.M')[x^n := N]$ is good $\Leftrightarrow \lambda y^m.M'[x^n := N]$ is good and $y^m \in FV(M') \setminus FV(N)$ (since $\lambda y^m.M' \in \mathcal{M}_1$ using BC) $\Leftrightarrow^{lemma\ 2.2} M'[x^n := N]$ is good, $y^m \in FV(M'[x^n := N])$ and $y^m \in FV(M') \setminus FV(N) \Leftrightarrow^{IH} M'$ and N are good, $M' \diamond N$, $y^m \in FV(M'[x^n := N])$ and $y^m \in FV(M') \setminus FV(N) \Leftrightarrow^{3b\&lemma\ 2.2} \lambda y^m.M'$ and N are good and $\lambda y^m.M' \diamond N$.
   – $(M_1M_2)[x^n := N]$ is good $\Leftrightarrow M_1[x^n := N]M_2[x^n := N]$ is good and $\diamond\{M_1, M_2, N\}$ (since $(M_1M_2)[x^n := N] \in \mathcal{M}_i$) $\Leftrightarrow^{6b\&lemma\ 2.2} M_1[x^n := N]$ and $M_2[x^n := N]$ are good, $M_1[x^n := N] \diamond M_2[x^n := N]$, $\diamond\{M_1, M_2, N\}$ and $\mathrm{d}(M_1) = \mathrm{d}(M_1[x^n := N]) \leq \mathrm{d}(M_2[x^n := N]) = \mathrm{d}(M_2) \Leftrightarrow^{IH} M_1$, M_2 and N are good, $\diamond\{M_1, M_2, N\}$ and $\mathrm{d}(M_1) \leq \mathrm{d}(M_2) \Leftrightarrow^{3a\&lemma\ 2.2} M_1$, M_2 and N are good and $(M_1M_2) \diamond N$.

$\square$

Lemma A.2. Let $i \in \{1, 2, 3\}$, $\blacktriangleright \in \{\rhd, \rhd^*\}$, $r \in \{\beta, \beta\eta, h\}$, $p \geq 0$ and $M, N, P, N_1, \ldots, N_p \in \mathcal{M}_i$.

1. If $M \blacktriangleright_r N$, $P \blacktriangleright_r Q$ and $M \diamond P$, then $N \diamond Q$.

2. If $M \blacktriangleright_r N$, $M \diamond P$ and $\mathrm{d}(P) = I$, then $M[x^I := P] \blacktriangleright_r N[x^I := P]$.

3. If $N \blacktriangleright_r P$, $M \diamond N$ and $\mathrm{d}(N) = I$, then $M[x^I := N] \blacktriangleright_r^* M[x^I := P]$.

4. If $M \rhd_r^* N$, $P \rhd_r^* P'$, $M \diamond P$ and $\mathrm{d}(P) = I$, then $M[x^I := P] \rhd_r^* N[x^I := P']$.

Proof:

1. Note that, by lemma 2.1, $\mathrm{FV}(N) \subseteq \mathrm{FV}(M)$ and $\mathrm{FV}(Q) \subseteq \mathrm{FV}(P)$.

2. Note that, by lemma 1, $N \diamond P$. Case $\rhd_r$ is by induction on M using lemmas A.1.6b and A.1.7. Case $\rhd_r^*$ is by induction on the length of $M \rhd_r^* N$ using the result for case $\rhd_r$.

3. Note that, by lemma 1, $M \diamond P$ and by lemma 2.1, $\mathrm{d}(P) = \mathrm{d}(N) = I$. Case $\rhd_r$ is by induction on M. Case $\rhd_r^*$ is by induction on the length of $M \rhd_r^* N$ using the result for case $\rhd_r$.

4. Use 2 and 3.

□

The next lemma shows that the lifting of a term to higher or lower degrees, is a well behaved operation with respect to all that matters (free variables, reduction, joinability, substitution, etc.).

Lemma A.3. Let $p \geq 0$, $i \in \{1, 2\}$ and $M, N, N_1, N_2, \ldots, N_p \in \mathcal{M}_i$.

1. (a) $\mathrm{d}(M^+) = \mathrm{d}(M) + 1$, $(M^+)^- = M$ and $x^n \in \mathrm{FV}(M^+)$ iff $x^{n-1} \in \mathrm{FV}(M)$.
   (b) If $\mathrm{d}(M) > 0$, then $M^- \in \mathcal{M}_i$, $\mathrm{d}(M^-) = \mathrm{d}(M) - 1$, $(M^-)^+ = M$ and $x^n \in \mathrm{FV}(M^-)$ iff $x^{n+1} \in \mathrm{FV}(M)$.
   (c) Let $\mathcal{X} \subset \mathcal{M}_i$. Then,
      i. $\diamond\mathcal{X}$ iff $\diamond\mathcal{X}^+$.
      ii. If $\mathrm{d}(\mathcal{X}) > 0$ then $\diamond\mathcal{X}$ iff $\diamond\mathcal{X}^-$.
      iii. $M \in \mathcal{X}^+$ iff ($M^- \in \mathcal{X}$ and $\mathrm{d}(M) > 0$).
   (d) M is good iff M^+ is good.
   (e) If $\mathrm{d}(M) > 0$ then M is good iff M^- is good.
2. Let $\mathcal{X} = \{M\} \cup \{N_i \mid 1 \leq i \leq p\} \subset \mathcal{M}_i$.
   If $\diamond\mathcal{X}$, then $(M[(x_i^{n_i} := N_i)_p])^+ = M^+[(x_i^{n_i+1} := N_i^+)_p]$.
3. If $\mathrm{d}(M), \mathrm{d}(N) > 0$, and $M \diamond N$, then $(M[x^{n+1} := N])^- = M^-[x^n := N^-]$.

Proof:

1. 1a and 1b are by induction on M. For 1(c)i use 1a. For 1(c)ii use 1b. As to 1(c)iii, if $M \in \mathcal{X}^+$, then $M = P^+$ where $P \in \mathcal{X}$ and by 1a, $\mathrm{d}(M) = \mathrm{d}(P) + 1 > 0$ and $M^- = (P^+)^- = P$. Hence, $M^- \in \mathcal{X}$ and $\mathrm{d}(M) > 0$. On the other hand, if $M^- \in \mathcal{X}$ and $\mathrm{d}(M) > 0$ then by 1b, $M = P^+$ and $(M^-)^+ = M \in \mathcal{X}^+$. Moreover, 1d is by induction on M using 1a, 1(c)i and lemma 2.2. Finally, for 1e, by 1b and 1d, $M = (M^-)^+ \in \mathbb{M} \Leftrightarrow M^- \in \mathbb{M}$.
2. By induction on M (by 1(c)i and lemma A.1.6, we have $M[(x_i^{n_i} := N_i)_p] \in \mathcal{M}_i$ and $M^+[(x_i^{n_i+1} := N_i^+)_p] \in \mathcal{M}_i$).
3. By induction on M (by 1(c)ii and lemma A.1.6, we have $M[x^{n+1} := N] \in \mathcal{M}_i$ and $M^-[x^n := N^-] \in \mathcal{M}_i$).

□

Lemma A.4. Let $r \in \{\eta, \beta\eta\}$, $\blacktriangleright \in \{\rhd, \rhd^*\}$, $p \geq 0$, $i \in \{1, 2\}$ and $M, N \in \mathcal{M}_i$.

1. If $M \blacktriangleright_r N$, then $M^+ \blacktriangleright_r N^+$.
2. If $\mathrm{d}(M) > 0$ and $M \blacktriangleright_r N$, then $M^- \blacktriangleright_r N^-$.
3. If $M \blacktriangleright_r N^+$, then $M^- \blacktriangleright_r N$.

4. If $M^+ \blacktriangleright_r N$, then $M \blacktriangleright_r N^-$.

Proof:

1. The case $r \in \{\eta\}$ and $\blacktriangleright = \rhd$ is by induction on $M \rhd_r N$ using lemma A.5, for case $\rhd_{\beta\eta}$ use the results for $\rhd_\beta$ (lemma A.5) and $\rhd_\eta$, case $\rhd_r^*$ is by induction on the length of $M \rhd_r^* N$ using the result for case $\rhd_r$.
2. Similar to 1.
3. By lemma 2.1.2, lemma A.5 and 2 above, $M^- \blacktriangleright N$.
4. Similar to 3.

□

Lemma A.5. Let $\blacktriangleright' \in \{\rhd_\beta^*, \rhd_\eta^*, \rhd_{\beta\eta}^*\}$, $\blacktriangleright \in \{\rhd_\beta, \rhd_\eta, \rhd_{\beta\eta}, \rhd_h, \rhd_\beta^*, \rhd_\eta^*, \rhd_{\beta\eta}^*, \rhd_h^*\}$, $i, p \geq 0$ and $M, N, N_1, \ldots, N_p \in \mathcal{M}_3$. We have:

1. $M^{+i} \in \mathcal{M}_3$ and $\mathrm{d}(M^{+i}) = i :: \mathrm{d}(M)$ and x^K occurs in M^{+i} iff $K = i :: L$ and x^L occurs in M.
2. $M \diamond N$ iff $M^{+i} \diamond N^{+i}$.
3. Let $\mathcal{X} \subseteq \mathcal{M}_3$ then $\diamond\mathcal{X}$ iff $\diamond\mathcal{X}^{+i}$.
4. $(M^{+i})^{-i} = M$.
5. If $\diamond\{M\} \cup \{N_j \mid j \in \{1, \ldots, p\}\}$ and for all $j \in \{1, \ldots, p\}$, $\mathrm{d}(N_j) = L_j$ then $(M[(x_j^{L_j} := N_j)_p])^{+i} = M^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.
6. If $M \blacktriangleright N$, then $M^{+i} \blacktriangleright N^{+i}$.
7. If $\mathrm{d}(M) = i :: L$, then:
   (a) $M = P^{+i}$ for some $P \in \mathcal{M}_3$, $\mathrm{d}(M^{-i}) = L$ and $(M^{-i})^{+i} = M$.
   (b) If $\forall 1 \leq j \leq p, \mathrm{d}(N_j) = i :: K_j$ and $\diamond\{M\} \cup \{N_j \mid j \in \{1, \ldots, p\}\}$ then $(M[(x_j^{i::K_j} := N_j)_p])^{-i} = M^{-i}[(x_j^{K_j} := N_j^{-i})_p]$.
   (c) If $M \blacktriangleright N$ then $M^{-i} \blacktriangleright N^{-i}$.
8. If $M \blacktriangleright N^{+i}$, then there is $P \in \mathcal{M}_3$ such that $M = P^{+i}$ and $P \blacktriangleright N$.
9. If $M^{+i} \blacktriangleright N$, then there is $P \in \mathcal{M}_3$ such that $N = P^{+i}$ and $M \blacktriangleright P$.

Proof:

1 We only prove the lemma by induction on M:

- If $M = x^L$ then $M^{+i} = x^{i::L} \in \mathcal{M}_3$ and $\mathrm{d}(x^{i::L}) = i :: L = i :: \mathrm{d}(x^L)$.

- If $M = \lambda x^L.M_1$ then $M_1 \in \mathcal{M}_3$, $L \succeq \mathrm{d}(M_1)$ and $M^{+i} = \lambda x^{i::L}.M_1^{+i}$. By IH, $M_1^{+i} \in \mathcal{M}_3$ and $\mathrm{d}(M_1^{+i}) = i :: \mathrm{d}(M_1)$ and x^K occurs in M_1^{+i} iff $K = i :: K'$ and $y^{K'}$ occurs in M_1. So $i :: L \succeq i :: \mathrm{d}(M_1) = \mathrm{d}(M_1^{+i})$. Hence, $\lambda x^{i::L}.M_1^{+i} \in \mathcal{M}_3$. Moreover, $\mathrm{d}(M^{+i}) = \mathrm{d}(M_1^{+i}) = i :: \mathrm{d}(M_1) = i :: \mathrm{d}(M)$. If y^K occurs in M^{+i} then either $y^K = x^{i::L}$, so it is done because x^L occurs in M. Or y^K occurs in M_1^{+i}. By IH, $K = i :: K'$ and $y^{K'}$ occurs in M_1. So $y^{K'}$ occurs in M. If y^K occurs in M then either $y^K = x^L$ and then $y^{i::K}$ occurs in M^{+i}. Or y^K occurs in M_1. Then by IH, $y^{i::K}$ occurs in M_1^{+i}. So, $y^{i::K}$ occurs in M^{+i}.

- If $M = M_1 M_2$ then $M_1, M_2 \in \mathcal{M}_3$, $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$, $M_1 \diamond M_2$ and $M^{+i} = M_1^{+i} M_2^{+i}$. By IH, $M_1^{+i}, M_2^{+i} \in \mathcal{M}_3$, $\mathrm{d}(M_1^{+i}) = i :: \mathrm{d}(M_1)$, $\mathrm{d}(M_2^{+i}) = i :: \mathrm{d}(M_2)$, y^K occurs in M_1^{+i} iff $K = i :: K'$ and $y^{K'}$ occurs in M_1, and y^K occurs in M_2^{+i} iff $K = i :: K'$ and $y^{K'}$ occurs in M_2. Let $x^L \in \mathrm{FV}(M_1^{+i})$ and $x^K \in \mathrm{FV}(M_2^{+i})$ then, using IH, $L = i :: L'$, $K = i :: K'$, $x^{L'}$ occurs in M_1 and $x^{K'}$ occurs in M_2. Using $M_1 \diamond M_2$, we obtain $L' = K'$, so $L = K$. Hence, $M_1^{+i} \diamond M_2^{+i}$. Because $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$, then $\mathrm{d}(M_1^{+i}) = i :: \mathrm{d}(M_1) \preceq i :: \mathrm{d}(M_2) = \mathrm{d}(M_2^{+i})$. So, $M^{+i} \in \mathcal{M}_3$. Moreover, $\mathrm{d}(M^{+1}) = \mathrm{d}(M_1^{+i}) = i :: \mathrm{d}(M_1) = i :: \mathrm{d}(M)$. If x^L occurs in M^{+i} then either x^L occurs in M_1^{+i} and using IH, $L = i :: L'$ and $x^{L'}$ occurs in M_1, so $x^{L'}$ occurs in M. Or x^L occurs in M_2^{+i} and using IH, $L = i :: L'$ and $x^{L'}$ occurs in M_2, so $x^{L'}$ occurs in M. If x^L occurs in M then either x^L occurs in M_1 so by IH $x^{i::L}$ occurs in M_1^{+i}, hence $x^{i::L}$ occurs in M^{+i}. Or x^L occurs in M_2 so by IH $x^{i::L}$ occurs in M_2^{+i}, hence $x^{i::L}$ occurs in M^{+i}.

2 Assume $M \diamond N$. Let $x^L \in \mathrm{FV}(M^{+i})$ and $x^K \in \mathrm{FV}(N^{+i})$ then by lemma A.5.1, $L = i :: L'$, $K = i :: K'$, $x^{L'} \in \mathrm{FV}(M)$ and $x^{K'} \in \mathrm{FV}(N)$. Using $M \diamond N$ we obtain $K' = L'$ and so $K = L$.

Assume $M^{+i} \diamond N^{+i}$. Let $x^L \in \mathrm{FV}(M)$ and $x^K \in \mathrm{FV}(N)$, then by lemma A.5.1, $x^{i::L} \in \mathrm{FV}(M^{+i})$ and $x^{i::K} \in \mathrm{FV}(N^{+i})$. Using $M^{+i} \diamond N^{+i}$ we obtain $i :: K = i :: L$ and so $K = L$.

3 Let $\mathcal{X} \subseteq \mathcal{M}_3$.

Assume $\diamond\mathcal{X}$. Let $M, N \in \mathcal{X}^{+i}$. Then by definition, $M = P^{+i}$ and $N = Q^{+i}$ such that $P, Q \in \mathcal{X}$. Because by hypothesis $P \diamond Q$ then by lemma A.5.2, $M \diamond N$.

Assume $\diamond\mathcal{X}^{+i}$. Let $M, N \in \mathcal{X}$ then $M^{+i}, N^{+i} \in \mathcal{X}^{+i}$. Because by hypothesis $M^{+i} \diamond N^{+i}$ then by lemma A.5.2, $M \diamond N$.

4 By lemma A.5.1, $M^{+i} \in \mathcal{M}_3$ and $\mathrm{d}(M^{+i}) = i :: \mathrm{d}(M)$. We prove the lemma by induction on M.

- Let $M = x^L$ then $M^{+i} = x^{i::L}$ and $(M^{+i})^{-i} = x^L$.
- Let $M = \lambda x^L.M_1$ such that $M_1 \in \mathcal{M}_3$ and $L \succeq \mathrm{d}(M_1)$. Then, $(M^{+i})^{-i} = (\lambda x^{i::L}.M_1^{+i})^{-i} = \lambda x^L.(M_1^{+i})^{-i} =^{IH} \lambda x^L.M_1$.
- Let $M = M_1 M_2$ such that $M_1, M_2 \in \mathcal{M}_3$, $M_1 \diamond M_2$ and $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$. Then, $(M^{+i})^{-i} = (M_1^{+i} M_2^{+i})^{-i} = (M_1^{+i})^{-i}(M_2^{+i})^{-i} =^{IH} M_1 M_2$.

5 By 3, $\diamond\{M^{+i}\} \cup \{N_j^{+i} \mid j \in \{1, \ldots, p\}\}$. By 1. and lemma A.1.6a, $M[(x_j^{L_j} := N_j)_p]$ and $M^{+i}[(x_j^{i::L_j} := N_j^{+i})_p] \in \mathcal{M}_3$. By induction on M:

- Let $M = y^K$. If $\forall 1 \leq j \leq p, y^K \neq x_j^{L_j}$ then $y^K[(x_j^{L_j} := N_j)_p] = y^K$. Hence $(y^K[(x_j^{L_j} := N_j)_p])^{+i} = y^{i::K} = y^{i::K}[(x_j^{i::L_j} := N_j^{+i})_p]$. If $\exists 1 \leq j \leq p, y^K = x_j^{L_j}$ then $y^K[(x_j^{L_j} := N_j)_p] = N_j$. Hence $(y^K[(x_j^{L_j} := N_j)_p])^{+i} = N_j^{+i} = y^{i::K}[(x_j^{i::L_j} := N_j^{+i})_p]$.
- Let $M = \lambda y^K.M_1$. Then $M[(x_j^{L_j} := N_j)_p] = \lambda y^K.M_1[(x_j^{L_j} := N_j)_p]$ where $\forall 1 \leq j \leq p, y^K \notin \mathrm{FV}(N_j) \cup \{x_j^{L_j}\}$. By lemma A.1.2, $\diamond\{M_1\} \cup \{N_j \mid j \in \{1, \ldots, p\}\}$. By IH, $(M_1[(x_j^{L_j} := N_j)_p])^{+i} = M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.
  Hence, $(M[(x_j^{L_j} := N_j)_p])^{+i} = \lambda y^{i::K}.(M_1[(x_j^{L_j} := N_j)_p])^{+i} =$
  $\lambda y^{i::K}.M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p] = (\lambda y^K.M_1)^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.
- Let $M = M_1 M_2$. $M[(x_j^{L_j} := N_j)_p] = M_1[(x_j^{L_j} := N_j)_p]M_2[(x_j^{L_j} := N_j)_p]$. By lemma A.1.2, $\diamond\{M_1\} \cup \{N_j \mid j \in \{1, \ldots, p\}\}$ and $\diamond\{M_2\} \cup \{N_j \mid j \in \{1, \ldots, p\}\}$. By IH, $(M_1[(x_j^{L_j} := N_j)_p])^{+i} = M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$ and $(M_2[(x_j^{L_j} := N_j)_p])^{+i} = M_2^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.
  Hence $(M[(x_j^{L_j} := N_j)_p])^{+i} = (M_1[(x_j^{L_j} := N_j)_p])^{+i}(M_2[(x_j^{L_j} := N_j)_p])^{+i} =$
  $M_1^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]M_2^{+i}[(x_j^{i::L_j} := N_j^{+i})_p] = M^{+i}[(x_j^{i::L_j} := N_j^{+i})_p]$.

6 By lemma A.5.1, if $M, N \in \mathcal{M}_3$ then $M^{+i}, N^{+i} \in \mathcal{M}_3$.

- Let $\blacktriangleright$ be $\rhd_\beta$. By induction on $M \rhd_\beta N$.
  - Let $M = (\lambda x^L.M_1)M_2 \rhd_\beta M_1[x^L := M_2] = N$ where $\mathrm{d}(M_2) = L$, then by lemma A.5.1, $\mathrm{d}(M_2^{+i}) = i :: L$ and $M^{+i} = (\lambda x^{i::L}.M_1^{+i})M_2^{+i} \rhd_\beta M_1^{+i}[x^{i::L} := M_2^{+i}] = (M_1[x^L := M_2])^{+i}$.
  - Let $M = \lambda x^L.M_1 \rhd_\beta \lambda x^L.N_1 = N$ such that $M_1 \rhd_\beta N_1$. By IH, $M_1^{+i} \rhd_\beta N_1^{+i}$, hence $M^{+i} = \lambda x^{i::L}.M_1^{+i} \rhd_\beta \lambda x^{i::L} N_1^{+i} = N^{+i}$.
  - Let $M = M_1 M_2 \rhd_\beta N_1 M_2 = N$ such that $M_1 \rhd_\beta N_1$. By IH, $M_1^{+i} \rhd_\beta N_1^{+i}$, hence $M^{+i} = M_1^{+i} M_2^{+i} \rhd_\beta N_1^{+i} M_2^{+i} = N^{+i}$.
  - Let $M = M_1 M_2 \rhd_\beta M_1 N_2 = N$ such that $M_2 \rhd_\beta N_2$. By IH, $M_2^{+i} \rhd_\beta N_2^{+i}$, hence $M^{+i} = M_1^{+i} M_2^{+i} \rhd_\beta N_1^{+i} M_2^{+i} = N^{+i}$.
- Let $\blacktriangleright$ be $\rhd_\beta^*$. By induction on $\rhd_\beta^*$ using $\rhd_\beta$.
- Let $\blacktriangleright$ be $\rhd_\eta$. We only do the base case. The inductive cases are as for $\rhd_\beta$. Let $M = \lambda x^L.N x^L \rhd_\eta N$ where $x^L \notin \mathrm{FV}(N)$. By lemma A.5.1, $x^{i::L} \notin \mathrm{FV}(N^{+i})$ Then $M^{+i} = \lambda x^{i::L}.N^{+i} x^{i::L} \rhd_\eta N^{+i}$.
- Let $\blacktriangleright$ be $\rhd_\eta^*$. By induction on $\rhd_\eta^*$ using $\rhd_\eta$.
- Let $\blacktriangleright$ be $\rhd_{\beta\eta}$, $\rhd_{\beta\eta}^*$, $\rhd_h$ or $\rhd_h^*$. By the previous items.

7 (a) By induction on M:

- Let $M = y^{i::L}$ then $y^L \in \mathcal{M}_3$ and $\mathrm{d}((y^{i::L})^{-i}) = \mathrm{d}(y^L) = L$ and $((y^{i::L})^{-i})^{+i} = y^{i::L}$.
- Let $M = \lambda y^K.M_1$ such that $M_1 \in \mathcal{M}_3$ and $K \succeq \mathrm{d}(M_1)$. Because $\mathrm{d}(M_1) = \mathrm{d}(M) = i :: L$, by IH, $M_1 = P^{+i}$ for some $P \in \mathcal{M}_3$, $\mathrm{d}(M_1^{-i}) = L$ and $(M_1^{-i})^{+i} = M_1$. Because, $K \succeq i :: L$ then $K = i :: L :: K'$ for some K'. Let $Q = \lambda y^{L::K'}.P$. Because

$P =^{A.5.4} (P^{+i})^{-i} = M_1^{-i}$, then $\mathrm{d}(P) = L$. Because $L \preceq L :: K'$, then $Q \in \mathcal{M}_3$ and $Q^{+i} = M$. Moreover, $\mathrm{d}(M^{-i}) =^{A.5.4} \mathrm{d}(Q) = \mathrm{d}(P) = L$ and $(M^{-i})^{+i} = P^{+i} = M$.

- Let $M = M_1 M_2$ such that $M_1, M_2 \in \mathcal{M}_3$, $M_1 \diamond M_2$ and $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$. Then $\mathrm{d}(M) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$, so $\mathrm{d}(M_2) = i :: L :: L'$ for some L'. By IH $M_1 = P_1^{+i}$ for some $P_1 \in \mathcal{M}_3$, $\mathrm{d}(M_1^{-i}) = L$ and $(M_1^{-i})^{+i} = M_1$. Again by IH, $M_2 = P_2^{+i}$ for some $P_2 \in \mathcal{M}_3$, $\mathrm{d}(M_2^{-i}) = L :: L'$ and $(M_2^{-i})^{+i} = M_2$. If $y^{K_1} \in \mathrm{FV}(P_1)$ and $y^{K_2} \in \mathrm{FV}(P_2)$, then by lemma A.5.1, $K_1' = i :: K_1$, $K_2' = i :: K_2$, $x^{K_1'} \in \mathrm{FV}(M_1)$ and $x^{K_2'} \in \mathrm{FV}(M_2)$. Thus $K_1' = K_2'$, so $K_1 = K_2$ and $P_1 \diamond P_2$. Because $\mathrm{d}(P_1) = \mathrm{d}(M_1^{-i}) = L \preceq L :: L' = \mathrm{d}(M_2^{-i}) = \mathrm{d}(P_2)$ then $Q = P_1 P_2 \in \mathcal{M}_3$ and $Q^{+i} = (P_1 P_2)^{+i} = P_1^{+i} P_2^{+i} = M$. Moreover, $\mathrm{d}(M^{-i}) =^{A.5.4} \mathrm{d}(Q) = \mathrm{d}(P_1) = L$ and $(M^{-i})^{+i} = Q^{+i} = M$.

(b) By the previous item, there exist $M', N_1', \ldots, N_n' \in \mathcal{M}_3$ such that $M = M'^{+i}$ and for all $j \in \{1, \ldots, p\}$, $N_j = N_j'^{+i}$. So by lemma A.5.3, $\diamond\{M'\} \cup \{N_j' \mid j \in \{1, \ldots, p\}\}$. By lemma A.5.4, $M^{-i} = M'$ and for all $j \in \{1, \ldots, p\}$, $N_j^{-i} = N_j'$. So, $\diamond\{M^{-i}\} \cup \{N_j^{-i} \mid j \in \{1, \ldots, p\}\}$. By lemma A.1.6a, $M[(x_j^{i::K_j} := N_j)_p], M^{-i}[(x_j^{K_j} := N_j^{-i})_p] \in \mathcal{M}_3$ and $\mathrm{d}(M[(x_j^{i::K_j} := N_j)_p]) = \mathrm{d}(M) = i :: L$. We prove the result by induction on M:

- Let $M = y^{i::L}$. If $\forall 1 \leq j \leq p, y^{i::L} \neq x_j^{i::K_j}$ then $y^{i::L}[(x_j^{i::K_j} := N_j)_p] = y^{i::L}$. Hence $(y^{i::L}[(x_j^{i::K_j} := N_j)_p])^{-i} = y^L = y^L[(x_j^{K_j} := N_j^{-i})_p]$. If $\exists 1 \leq j \leq p, y^{i::L} = x_j^{i::K_j}$ then $y^{i::L}[(x_j^{i::K_j} := N_j)_p] = N_j$. Hence $(y^{i::L}[(x_j^{i::K_j} := N_j)_p])^{-i} = N_j^{-i} = y^L[(x_j^{K_j} := N_j^{-i})_p]$.
- Let $M = \lambda y^K.M_1$ such that $M_1 \in \mathcal{M}_3$ and $K \succeq \mathrm{d}(M_1)$. Then, $M[(x_j^{i::K_j} := N_j)_p] = \lambda y^K.M_1[(x_j^{i::K_j} := N_j)_p]$ where $\forall 1 \leq j \leq p, y^K \notin \mathrm{FV}(N_j) \cup \{x_j^{i::K_j}\}$. By lemma A.1.2, $\diamond\{M_1\} \cup \{N_j \mid j \in \{1, \ldots, p\}\}$. By definition $\mathrm{d}(M) = \mathrm{d}(M_1)$. By IH, $(M_1[(x_j^{i::K_j} := N_j)_p])^{-i} = M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p]$. Because $\mathrm{d}(M_1) = i :: L \preceq K$, $K = i :: L :: K'$ for some K'.
Hence, $(M[(x_j^{i::K_j} := N_j)_p])^{-i} = \lambda y^{L::K'}.(M_1[(x_j^{i::K_j} := N_j)_p])^{-i} =$
$\lambda y^{L::K'}.M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p] = (\lambda y^K.M_1)^{-i}[(x_j^{K_j} := N_j^{-i})_p]$.
- Let $M = M_1 M_2$ such that $M_1, M_2 \in \mathcal{M}_3$, $M_1 \diamond M_2$ and $\mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$. Then, $M[(x_j^{i::K_j} := N_j)_p] = M_1[(x_j^{i::K_j} := N_j)_p] M_2[(x_j^{i::K_j} := N_j)_p]$. By lemma A.1.2, $\diamond\{M_1\} \cup \{N_j \mid j \in \{1, \ldots, p\}\}$ and $\diamond\{M_2\} \cup \{N_j \mid j \in \{1, \ldots, p\}\}$. By definition $\mathrm{d}(M) = \mathrm{d}(M_1) \preceq \mathrm{d}(M_2)$. So $\mathrm{d}(M_2) = i :: L :: L'$ for some L'. By IH, $(M_1[(x_j^{i::K_j} := N_j)_p])^{-i} = M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p]$ and $(M_2[(x_j^{i::K_j} := N_j)_p])^{-i} = M_2^{-i}[(x_j^{K_j} := N_j^{-i})_p]$. Hence
$(M[(x_j^{i::K_j} := N_j)_p])^{-i} = (M_1[(x_j^{i::K_j} := N_j)_p])^{-i}(M_2[(x_j^{i::K_j} := N_j)_p])^{-i}$
$= M_1^{-i}[(x_j^{K_j} := N_j^{-i})_p] M_2^{-i}[(x_j^{K_j} := N_j^{-i})_p] = M^{-i}[(x_j^{K_j} := N_j^{-i})_p]$.

(c) Using lemma A.5.4, lemma 2.1 and the first item, we prove that $M^{-i}, N^{-i} \in \mathcal{M}_3$.

- Let $\blacktriangleright$ be $\rhd_\beta$. By induction on $M \rhd_\beta N$.
  - Let $M = (\lambda x^K.M_1)M_2 \rhd_\beta M_1[x^K := M_2] = N$ where $\mathrm{d}(M_2) = K$. Because

$M \in \mathcal{M}_3$ then $M_1 \in \mathcal{M}_3$. Because $i :: L = \mathrm{d}(M) = \mathrm{d}(M_1) \preceq K$, then $K = i :: L :: K'$. By lemma A.5.7, $\mathrm{d}(M_2^{-i}) = L :: K'$. So $M^{-i} = (\lambda x^{L::K'}.M_1^{-i})M_2^{-i} \rhd_\beta M_1^{-i}[x^{L::K'} := M_2^{-i}] = (M_1[x^K := M_2])^{-i}$.

  - Let $M = \lambda x^K.M_1 \rhd_\beta \lambda x^K.N_1 = N$ such that $M_1 \rhd_\beta N_1$. Because $M \in \mathcal{M}_3$, $M_1 \in \mathcal{M}_3$ and $K \succeq \mathrm{d}(M_1)$. By definition $\mathrm{d}(M) = \mathrm{d}(M_1)$. Because $i :: L = \mathrm{d}(M_1) \preceq K$, $K = i :: L :: K'$ for some K'. By IH, $M_1^{-i} \rhd_\beta N_1^{-i}$, hence $M^{-i} = \lambda x^{L::K'}.M_1^{-i} \rhd_\beta \lambda x^{L::K'} N_1^{-i} = N^{-i}$.
  - Let $M = M_1M_2 \rhd_\beta N_1M_2 = N$ such that $M_1 \rhd_\beta N_1$. Because $M \in \mathcal{M}_3$ then $M_1 \in \mathcal{M}_3$. By definition $\mathrm{d}(M) = \mathrm{d}(M_1) = i :: L$. By IH, $M_1^{-i} \rhd_\beta N_1^{-i}$, hence $M^{-i} = M_1^{-i}M_2^{-i} \rhd_\beta N_1^{-i}M_2^{-i} = N^{-i}$.
  - Let $M = M_1M_2 \rhd_\beta M_1N_2 = N$ such that $M_2 \rhd_\beta N_2$. Because $M \in \mathcal{M}_3$ then $M_2 \in \mathcal{M}_3$. By definition $\mathrm{d}(M_2) \succeq \mathrm{d}(M_1) = \mathrm{d}(M) = i :: L$. So $\mathrm{d}(M_2) = i :: L :: L'$ for some L'. By IH, $M_2^{-i} \rhd_\beta N_2^{-i}$, hence $M^{-i} = M_1^{-i}M_2^{-i} \rhd_\beta N_1^{-i}M_2^{-i} = N^{-i}$.

- Let $\blacktriangleright$ be $\rhd_\beta^*$. By induction on $\rhd_\beta^*$. using $\rhd_\beta$.
- Let $\blacktriangleright$ be $\rhd_\eta$. We only do the base case. The inductive cases are as for $\rhd_\beta$. Let $M = \lambda x^K.Nx^K \rhd_\eta N$ where $x^K \notin \mathrm{FV}(N)$. Because $i :: L = \mathrm{d}(M) = \mathrm{d}(N) \preceq K$, then $K = i :: L :: K'$ for some K'. By lemma A.5.7, $N = N'^{+i}$ for some $N' \in \mathcal{M}_3$. By lemma A.5.7, $N' = N^{-i}$. By lemma A.5.1, $x^{L::K'} \notin \mathrm{FV}(N^{-i})$. Then $M^{-i} = \lambda x^{L::K'}.N^{-i}x^{L::K'} \rhd_\eta N^{-i}$.
- Let $\blacktriangleright$ be $\rhd_\eta^*$. By induction on $\rhd_\eta^*$ using $\rhd_\eta$.
- Let $\blacktriangleright$ be $\rhd_{\beta\eta}$, $\rhd_{\beta\eta}^*$, $\rhd_h$ or $\rhd_h^*$. By the previous items.

8 By lemma A.5.1, $\mathrm{d}(N^{+i}) = i :: \mathrm{d}(N)$. By lemma 2.1, $\mathrm{d}(M) = \mathrm{d}(N^{+i})$. By lemma A.5.7, $M = M'^{+i}$ such that $M' \in \mathcal{M}_3$. By lemma A.5.7, $M' =^{A.5.4} (M'^{+i})^{-i} = M^{-i} \blacktriangleright (N^{+i})^{-i} =^{A.5.4} N$.

9 By lemma A.5.1, $\mathrm{d}(M^{+i}) = i :: \mathrm{d}(M)$. By lemma 2.1, $\mathrm{d}(M^{+i}) = \mathrm{d}(N)$. By lemma A.5.7, $N = N'^{+i}$ such that $N' \in \mathcal{M}_3$. By lemma A.5.7, $M =^{A.5.4} (M^{+i})^{-i} \blacktriangleright N^{-i} = (N'^{+i})^{-i} =^{A.5.4} N'$.

□

A.2. Confluence of $\rhd_\beta^*$ and $\rhd_{\beta\eta}^*$

In this section we establish the confluence of $\rhd_\beta^*$ and $\rhd_{\beta\eta}^*$ using the standard parallel reduction method.

Definition A.1. Let $r \in \{\beta, \beta\eta\}$. We define the binary relation $\stackrel{\rho_r}{\to}$ on $\mathcal{M}_i$, where $i \in \{1, 2, 3\}$, by:

- $M \stackrel{\rho_r}{\to} M$
- If $M \stackrel{\rho_r}{\to} M'$ and $\lambda x^I M, \lambda x^I.M' \in \mathcal{M}_i$, then $\lambda x^I.M \stackrel{\rho_\beta}{\to} \lambda x^I.M'$.
- If $M \stackrel{\rho_r}{\to} M'$, $N \stackrel{\rho_\beta}{\to} N'$ and $MN, M'N' \in \mathcal{M}_i$ then $MN \stackrel{\rho_r}{\to} M'N'$
- If $M \stackrel{\rho_r}{\to} M'$, $N \stackrel{\rho_r}{\to} N'$ and $(\lambda x^I.M)N, M'[x^I := N'] \in \mathcal{M}_i$, then $(\lambda x^I.M)N \stackrel{\rho_r}{\to} M'[x^I := N']$
- If $M \stackrel{\rho_{\beta\eta}}{\to} M'$, $x^I \notin FV(M)$ and $\lambda x^I.Mx^I \in \mathcal{M}_i$, then $\lambda x^I.Mx^I \stackrel{\rho_{\beta\eta}}{\to} M'$

We denote the transitive closure of $\stackrel{\rho_r}{\to}$ by $\stackrel{\rho_r}{\twoheadrightarrow}$. When $M \stackrel{\rho_r}{\to} N$ (resp. $M \stackrel{\rho_r}{\twoheadrightarrow} N$), we can also write $N \stackrel{\rho_r}{\leftarrow} M$ (resp. $N \stackrel{\rho_r}{\twoheadleftarrow} M$). If $R, R' \in \{\stackrel{\rho_r}{\to}, \stackrel{\rho_r}{\twoheadrightarrow}, \stackrel{\rho_r}{\leftarrow}, \stackrel{\rho_r}{\twoheadleftarrow}\}$, we write $M_1 R M_2 R' M_3$ instead of $M_1 R M_2$ and $M_2 R' M_3$.

Lemma A.6. Let $r \in \{\beta, \beta\eta\}$, $i \in \{1, 2, 3\}$ and $M \in \mathcal{M}_i$.

1. If $M \rhd_r M'$, then $M \stackrel{\rho_r}{\to} M'$.
2. If $M \stackrel{\rho_r}{\to} M'$, then $M' \in \mathcal{M}_i$, $M \rhd_r^* M'$, $\mathrm{FV}(M') \subseteq \mathrm{FV}(M)$, $\mathrm{d}(M) = \mathrm{d}(M')$ and if $i \in \{1, 2\}$, $\mathrm{FV}(M') = \mathrm{FV}(M)$.
3. If $M \stackrel{\rho_r}{\to} M'$, $N \stackrel{\rho_r}{\to} N'$ and $M \diamond N$, then $M' \diamond N'$.

Proof:
1. By induction on the derivation of $M \rhd_r M'$.
2. By induction on the derivation of $M \stackrel{\rho_r}{\to} M'$ using lemmas 2.1 and A.2.4.
3. $M' \diamond N'$ since by 2, $\mathrm{FV}(M') \subseteq \mathrm{FV}(M)$ and $\mathrm{FV}(N') \subseteq \mathrm{FV}(N')$ and $M \diamond N$. □

Lemma A.7. Let $r \in \{\beta, \beta\eta\}$, $i \in \{1, 2, 3\}$ $M, N \in \mathcal{M}_i$, $N \stackrel{\rho_r}{\to} N'$, $\mathrm{d}(N) = I$ and $M \diamond N$. We have:

1. $M[x^I := N] \stackrel{\rho_r}{\to} M[x^I := N']$.
2. If $M \stackrel{\rho_r}{\to} M'$ then $M[x^I := N] \stackrel{\rho_r}{\to} M'[x^I := N']$.

Proof:
By lemma A.6.2, $\mathrm{d}(N') = \mathrm{d}(N) = I$. By lemma A.6.3, $M \diamond N'$.
1. By induction on M using lemmas A.1.2, A.1.4 and A.1.6a.
2. By lemma A.6.3, $M' \diamond N'$. By induction on $M \stackrel{\rho_r}{\to} M'$ using 1, lemmas A.1.2, A.1.4, A.1.6a and A.6.3. We only do one interesting case where $(\lambda y^J.M_1)M_2 \stackrel{\rho_\beta}{\to} M_1'[y^J := M_2']$, $M_1 \stackrel{\rho_\beta}{\to} M_1'$, $M_2 \stackrel{\rho_\beta}{\to} M_2'$, $(\lambda y^J.M_1)M_2, M_1'[y^J := M_2']$. By definition, $M_1, M_2 \in \mathcal{M}_i$. By lemma A.6.2, $M_1', M_2' \in \mathcal{M}_i$. By lemma A.1.6a, $M_1' \diamond M_2'$ and $\mathrm{d}(M_2') = J$. By lemma A.1.2, $M_1 \diamond N$ and $M_2 \diamond N$. By lemma A.6.3, $M_1' \diamond N$ and $M_2' \diamond N$. Then:

a. $M_1[x^I := N] \stackrel{\rho_\beta}{\to} M_1'[x^I := N']$, by IH and lemma A.1.2.

b. $M_2[x^I := N] \stackrel{\rho_\beta}{\to} M_2'[x^I := N']$, by IH and lemma A.1.2.

c. $M_1[x^I := N] \diamond M_2[x^I := N]$, by lemmas A.1.2 and A.1.4.

d. $M_1'[x^I := N'] \diamond M_2'[x^I := N']$, by a., b., c., and lemma A.6.3.

e. $y^J \notin FV(N') \cup \{x^I\}$, by BC and by lemma A.1.7,
$M_1'[x^I := N'][y^J := M_2'[x^I := N']] = M_1'[y^J := M_2'][x^I := N']$.

Hence, $(\lambda y^J.M_1[x^I := N])M_2[x^I := N] \stackrel{\rho_\beta}{\to} M_1'[x^I := N'][y^J := M_2'[x^I := N']]$
and so, $((\lambda y^J.M_1)M_2)[x^I := N] \stackrel{\rho_\beta}{\to} M_1'[y^J := M_2'][x^I := N']$. □

Lemma A.8. Let $r \in \{\beta, \beta\eta\}$, $i \in \{1,2,3\}$ and $M \in \mathcal{M}_i$.

1. If $M = x^I \overset{\rho_r}{\to} N$, then $N = x^I$.
2. If $M = \lambda x^I.P \overset{\rho_\beta}{\to} N$, then $N = \lambda x^I.P'$ where $P \overset{\rho_\beta}{\to} P'$.
3. If $M = \lambda x^I.P \overset{\rho_{\beta\eta}}{\to} N$ then one of the following holds:
   - $N = \lambda x^I.P'$ where $P \overset{\rho_{\beta\eta}}{\to} P'$.
   - $P = P'x^I$ where $x^I \notin \mathrm{FV}(P') \cup \mathrm{FV}(N)$, $\mathrm{d}(P') \leq n$ and $P' \overset{\rho_{\beta\eta}}{\to} N$.
4. If $M = PQ \overset{\rho_r}{\to} N$, then one of the following holds:
   - $N = P'Q'$, $P \overset{\rho_r}{\to} P'$, $Q \overset{\rho_r}{\to} Q'$, $P \diamond Q$, and $P' \diamond Q'$.
   - $P = \lambda x^I.P'$, $N = P''[x^I := Q']$, $\mathrm{d}(Q) = \mathrm{d}(Q') = I$, $P' \overset{\rho_r}{\to} P''$, $Q \overset{\rho_r}{\to} Q'$, $P' \diamond Q$ and $P'' \diamond Q'$.

Proof:
1. By induction on the derivation of $x^I \overset{\rho_r}{\to} N$.
2. By induction on the derivation of $\lambda x^I.P \overset{\rho_\beta}{\to} N$ using lemma A.6.2.
3. By induction on the derivation of $\lambda x^I.P \overset{\rho_{\beta\eta}}{\to} N$ using lemma A.6.2.
4. By induction on the derivation of $PQ \overset{\rho_r}{\to} N$ using lemma A.6.2 and A.6.3. □

Lemma A.9. Let $r \in \{\beta, \beta\eta\}$, $i \in \{1,2,3\}$ and $M, M_1, M_2 \in \mathcal{M}_i$.

1. If $M_2 \overset{\rho_r}{\leftarrow} M \overset{\rho_r}{\to} M_1$, then there is $M' \in \mathcal{M}_i$ such that $M_2 \overset{\rho_r}{\to} M' \overset{\rho_r}{\leftarrow} M_1$.
2. If $M_2 \overset{\rho_r}{\twoheadleftarrow} M \overset{\rho_r}{\twoheadrightarrow} M_1$, then there is $M' \in \mathcal{M}_i$ such that $M_2 \overset{\rho_r}{\twoheadrightarrow} M' \overset{\rho_r}{\twoheadleftarrow} M_1$.

Proof:
1. Both cases by induction on M. We only do the $\beta\eta$ case making discriminate use of lemma A.8.

- If $M = x^I$, by lemma A.8, $M_1 = M_2 = x^I$. Take $M' = x^I$.
- If $N_2P_2 \overset{\rho_{\beta\eta}}{\leftarrow} NP \overset{\rho_{\beta\eta}}{\to} N_1P_1$ where $N_2 \overset{\rho_{\beta\eta}}{\leftarrow} N \overset{\rho_{\beta\eta}}{\to} N_1$ and $P_2 \overset{\rho_{\beta\eta}}{\leftarrow} P \overset{\rho_{\beta\eta}}{\to} P_1$. Then, by IH, $\exists N', P' \in \mathcal{M}_i$ such that $N_2 \overset{\rho_{\beta\eta}}{\to} N' \overset{\rho_{\beta\eta}}{\leftarrow} N_1$ and $P_2 \overset{\rho_{\beta\eta}}{\to} P' \overset{\rho_{\beta\eta}}{\leftarrow} P_1$. By definition, $N_1 \diamond P_1$. By lemma A.6.2, $\mathrm{d}(N_1) = \mathrm{d}(N')$ and $\mathrm{d}(P_1) = \mathrm{d}(P')$. By lemma A.6.3, $N' \diamond P'$. If $i \in \{1,2\}$ then $N'P' \in \mathcal{M}_i$. If $i = 3$ then $\mathrm{d}(N_1) \preceq \mathrm{d}(P_1)$, so $\mathrm{d}(N') \preceq \mathrm{d}(P')$ and $N'P' \in \mathcal{M}_i$. Hence, $N_2P_2 \overset{\rho_{\beta\eta}}{\to} N'P' \overset{\rho_{\beta\eta}}{\leftarrow} N_1P_1$.
- If $P_1[x^I := Q_1] \overset{\rho_{\beta\eta}}{\leftarrow} (\lambda x^I.P)Q \overset{\rho_{\beta\eta}}{\to} P_2[x^I := Q_2]$ where $P_1 \overset{\rho_{\beta\eta}}{\leftarrow} P \overset{\rho_{\beta\eta}}{\to} P_2$ and $Q_1 \overset{\rho_{\beta\eta}}{\leftarrow} Q \overset{\rho_{\beta\eta}}{\to} Q_2$. Then, by IH, $\exists P', Q' \in \mathcal{M}_i$ where $P_1 \overset{\rho_{\beta\eta}}{\to} P' \overset{\rho_{\beta\eta}}{\leftarrow} P_2$ and $Q_1 \overset{\rho_{\beta\eta}}{\to} Q' \overset{\rho_{\beta\eta}}{\leftarrow} Q_2$. By lemma A.1.6a, $\mathrm{d}(Q_1) = \mathrm{d}(Q_2) = I$, $P_1 \diamond Q_1$ and $P_2 \diamond Q_2$.
  Hence, by lemma A.7.2, $P_1[x^I := Q_1] \overset{\rho_{\beta\eta}}{\to} P'[x^I := Q'] \overset{\rho_{\beta\eta}}{\leftarrow} P_2[x^I := Q_2]$.

- If $(\lambda x^I.P_1)Q_1 \overset{\rho\beta\eta}{\leftarrow} (\lambda x^I.P)Q \overset{\rho\beta\eta}{\rightarrow} P_2[x^I := Q_2]$ where $P \overset{\rho\beta\eta}{\rightarrow} P_1$, $P \overset{\rho\beta\eta}{\rightarrow} P_2$, $Q_1 \overset{\rho\beta\eta}{\leftarrow} Q \overset{\rho\beta\eta}{\rightarrow} Q_2$. By IH, $\exists P', Q' \in \mathcal{M}_i$ such that $P_1 \overset{\rho\beta\eta}{\rightarrow} P' \overset{\rho\beta\eta}{\leftarrow} P_2$ and $Q_1 \overset{\rho\beta\eta}{\rightarrow} Q' \overset{\rho\beta\eta}{\leftarrow} Q_2$. By lemma A.1.1 and lemma A.1.2, $P \diamond Q$. By lemma A.6.3, $P' \diamond Q'$. By lemma A.1.6a, $\mathrm{d}(Q_2) = I$ and $P_2 \diamond Q_2$. By lemma A.6.2, $\mathrm{d}(Q') = I$. By lemma A.1.6a, $P'[x^I := Q'] \in \mathcal{M}_i$. Hence, $(\lambda x^n.P_1)Q_1 \overset{\rho\beta\eta}{\rightarrow} P'[x^n := Q']$ and by lemma A.7.2, $P_2[x^n := Q_2] \overset{\rho\beta\eta}{\rightarrow} P'[x^n := Q']$.

- If $P_1Q_1 \overset{\rho\beta\eta}{\leftarrow} (\lambda x^I.Px^I)Q \overset{\rho\beta\eta}{\rightarrow} P_2[x^I := Q_2]$ where $P \overset{\rho\beta\eta}{\rightarrow} P_1$, $Px^I \overset{\rho\beta\eta}{\rightarrow} P_2$, $Q_1 \overset{\rho\beta\eta}{\leftarrow} Q \overset{\rho\beta\eta}{\rightarrow} Q_2$ and $x^I \notin \mathrm{FV}(P)$. By lemma A.1.6a, $\mathrm{d}(Q_2) = I$. By lemma A.6.2, $\mathrm{d}(Q_1) = I$. By lemma A.1.1 and lemma A.1.2, $\diamond\{P, x^I, Q\}$. By lemma A.6.3, $\diamond\{P_1, x^I, Q_1\}$. By lemma A.6.2, $\mathrm{d}(P) = \mathrm{d}(P_1)$ and $x^I \notin \mathrm{FV}(P_1)$. If $i \in \{1,2\}$ then $P_1x^I \in \mathcal{M}_i$. If $i = 3$ then $\mathrm{d}(P) \preceq I$, so $\mathrm{d}(P_1) \preceq I$ and $Px^I \in \mathcal{M}_i$. Hence $Px^I \diamond Q$ and by lemma A.1.6a, $P_1Q_1 = (P_1x^I)[x^I := Q_1] \in \mathcal{M}_i$. Moreover, $Px^I \overset{\rho\beta\eta}{\rightarrow} P_1x^I$ and we conclude as in the third item.

- If $\lambda x^I.N_2 \overset{\rho\beta\eta}{\leftarrow} \lambda x^I.N \overset{\rho\beta\eta}{\rightarrow} \lambda x^I.N_1$ where $N_2 \overset{\rho\beta\eta}{\leftarrow} N \overset{\rho\beta\eta}{\rightarrow} N_1$. By IH, there is $N' \in \mathcal{M}_i$ such that $N_2 \overset{\rho\beta\eta}{\rightarrow} N' \overset{\rho\beta\eta}{\leftarrow} N_1$. If $i \in \{1,2\}$ then $x^I \in \mathrm{FV}(N_1)$, so by lemma A.6.2, $x^I \in \mathrm{FV}(N)$, hence $\lambda x^I.N' \in \mathcal{M}_i$. If $i = 3$ then by lemma A.6.2, $I \succeq \mathrm{d}(N_1) = \mathrm{d}(N')$, so $\lambda x^I.N' \in \mathcal{M}_i$. Hence, $\lambda x^n.N_2 \overset{\rho\beta\eta}{\rightarrow} \lambda x^n.N' \overset{\rho\beta\eta}{\leftarrow} \lambda x^n.N_1$.

- If $M_1 \overset{\rho\beta\eta}{\leftarrow} \lambda x^I.Px^I \overset{\rho\beta\eta}{\rightarrow} M_2$ where $M_1 \overset{\rho\beta\eta}{\leftarrow} P \overset{\rho\beta\eta}{\rightarrow} M_2$. By IH, there is $M' \in \mathcal{M}_i$ such that $M_2 \overset{\rho\beta\eta}{\rightarrow} M' \overset{\rho\beta\eta}{\leftarrow} M_1$.

- If $M_1 \overset{\rho\beta\eta}{\leftarrow} \lambda x^I.Px^I \overset{\rho\beta\eta}{\rightarrow} \lambda x^I.P'$, where $P \overset{\rho\beta\eta}{\rightarrow} M_1$, $Px^I \overset{\rho\beta\eta}{\rightarrow} P'$ and $x^I \notin \mathrm{FV}(P)$. By the $\diamond$ property, for all J, $x^J \notin \mathrm{FV}(P)$. By lemma A.8:

  - Either $P' = P''x^I$ and $P \overset{\rho\beta\eta}{\rightarrow} P''$. By IH, there is $M' \in \mathcal{M}_i$ such that $P'' \overset{\rho\beta\eta}{\rightarrow} M' \overset{\rho\beta\eta}{\leftarrow} M_1$. By lemma A.6.2, $x^I \notin FV(P'')$ and $\mathrm{d}(P'') \leq n$. Hence, $M_2 = \lambda x^I.P''x^I \overset{\rho\beta\eta}{\rightarrow} M' \overset{\rho\beta\eta}{\leftarrow} M_1$.
  - Or $P = \lambda y^I.P''$ and $P' = P'''[y^I := x^I]$ such that $P'' \overset{\rho\beta\eta}{\rightarrow} P'''$. If $i \in \{1,2\}$ then $y^I \in \mathrm{FV}(P'')$, so by lemma A.6.2, $y^I \in \mathrm{FV}(P''')$ and $\lambda y^I.M''' \in \mathcal{M}_i$. If $i = 3$ then by lemma A.6.2 and BC, $\mathrm{d}(P''') = \mathrm{d}(P'') \preceq I$ and for all J, $x^J \notin \mathrm{FV}(P''')$. So $\lambda y^I.M''' \in \mathcal{M}_i$. Hence, $P = \lambda y^I.P'' \overset{\rho\beta\eta}{\rightarrow} \lambda y^I.P'''$. Moreover, $\lambda x^I.P' = \lambda x^I.P'''[y^I := x^I] = \lambda y^I.P'''$. We conclude using as in the sixth item.

2. First show by induction on $M \overset{\rho_r}{\twoheadrightarrow} M_1$ (and using 1) that if $M_2 \overset{\rho_r}{\leftarrow} M \overset{\rho_r}{\twoheadrightarrow} M_1$, then there is $M' \in \mathcal{M}_i$ such that $M_2 \overset{\rho_r}{\twoheadrightarrow} M' \overset{\rho_r}{\leftarrow} M_1$. Then use this to show 2 by induction on $M \overset{\rho_r}{\twoheadrightarrow} M_2$. □

Proof:
[of Theorem 2.2]

1. By lemma A.9.2, $\overset{\rho_r}{\twoheadrightarrow}$ is confluent. By lemma A.6.1 and A.6.2, $M \overset{\rho_r}{\twoheadrightarrow} N$ iff $M \rhd_r^* N$. Then $\rhd_r^*$ is confluent.

2. If) is by definition of $\simeq_\beta$. Only if) is by induction on $M_1 \simeq_\beta M_2$ using 1.

□

A.3. Types

Proof:
[of Lemma 3.1]

1. The if direction is by definition. We only do the if direction.
   1a. By induction on the derivation of $U \to T$ good. 1b. By induction on the derivation of $U \sqcap V$ good. 1c. By induction on the derivation of eU good.

2. 2a. By induction on T. 2b. By induction on U. 2c. By induction on U. 2d. If) By 1. Only if) By 2, $\mathrm{d}(U) \geq 0 = \mathrm{d}(T)$. Hence, by 1, $U \to T$ is good.

□

A.4. Type systems

Lemma A.10. In the relevant context ($\mathbb{U}_2$, $\mathbb{T}_2$, $Env_{\mathbb{U}_2}$ or 2-typings), we have:

1. If $U \sqsubseteq V \sqcap a$, then $U = U' \sqcap a$.

2. Let $U_1 \sqsubseteq U_2$.

   (a) If U_2 is good and $\mathrm{d}(U_2) = n$, then $U_1 = \sqcap_{i=1}^{k} \vec{e}_{i(1:n)} T_i$, $U_2 = \sqcap_{j=1}^{p} \vec{e'}_{j(1:n)} T'_j$, where $p, k \geq 1$, $\forall 1 \leq i \leq k\ T_i \in \mathbb{T}_2$, $\forall 1 \leq j \leq p\ T'_j \in \mathbb{T}_2$ and $\forall 1 \leq j \leq p$, $\exists 1 \leq i \leq k$ such that $\vec{e}_{i(1:n)} = \vec{e'}_{j(1:n)}$ and $T_i \sqsubseteq T'_j$.

   (b) Let $U_1 = \sqcap_{i=1}^{k} \vec{e}_{i(1:n_i)} (V_i \to T_i)$ and $U_2 = \sqcap_{j=1}^{p} \vec{e'}_{j(1:m_j)} (V'_j \to T'_j)$. If U_1 is good and $\mathrm{d}(U_1) = n$ then $\forall i, j, n_i = m_j = n$ and $\forall 1 \leq j \leq p$, $\exists 1 \leq i \leq k$ such that $\vec{e}_{i(1:n)} = \vec{e'}_{j(1:n)}$, $V'_j \sqsubseteq V_i$ and $T_i \sqsubseteq T'_j$.

3. If $eU \sqsubseteq V$ then $V = eU'$ where $U \sqsubseteq U'$.

4. If $U \to T \sqsubseteq V$ and $U \to T$ is good, then $V = \sqcap_{i=1}^{p} (U_i \to T_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $U_i \sqsubseteq U$ and $T \sqsubseteq T_i$.

5. If $\sqcap_{i=1}^{k} \vec{e}_{i(1:n_i)} (V_i \to T_i) \sqsubseteq V$ where V is good, $\mathrm{d}(V) = n$ and $k \geq 1$ then $\forall i, n_i = n$ and $V = \sqcap_{i=1}^{p} \vec{e'}_{i(1:n)} (V'_i \to T'_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $\exists 1 \leq j \leq k$ such that $\vec{e}_{j(1:n)} = \vec{e'}_{i(1:n)}$, $V'_i \sqsubseteq V_j$ and $T_j \sqsubseteq T'_i$.

6. Let $\Phi_1 \sqsubseteq \Phi_2$.

   - $\mathrm{d}(\Phi_1) > 0$ iff $\mathrm{d}(\Phi_2) > 0$
   - Φ_1 is good iff Φ_2 is good.

7. If $U \sqsubseteq U'_1 \sqcap U'_2$ then $U = U_1 \sqcap U_2$ where $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.

8. If $\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2$ then $\Gamma = \Gamma_1 \sqcap \Gamma_2$ where $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.

Proof:
[of Lemma A.10]

1. By induction on $U \sqsubseteq V \sqcap a$.

2. By induction on the derivation of $U_1 \sqsubseteq U_2$ using lemmas 3.1. We do case tr of 2b.
If $\frac{\sqcap_{i=1}^{k}\vec{e}_{i(1:n_i)}(V_i \to T_i) \sqsubseteq V \quad V \sqsubseteq \sqcap_{j=1}^{p}\vec{e'}_{j(1:m_j)}(V'_j \to T'_j)}{\sqcap_{i=1}^{k}\vec{e}_{i(1:n_i)}(V_i \to T_i) \sqsubseteq \sqcap_{j=1}^{p}\vec{e'}_{j(1:m_j)}(V'_j \to T'_j)}$, then, by 2a, $\forall i, n_i = n$ and $V = \sqcap_{l=1}^{q}\vec{e''}_{l(1:n)}T''_l$ where $q \geq 1$ and $\forall 1 \leq l \leq q$, $\exists 1 \leq i \leq k$, such that $\vec{e''}_{l(1:n)} = \vec{e}_{i(1:n)}$ and $V_i \to T_i \sqsubseteq T''_l$. If $T''_l = a$, then, by 1, $V_i \to T_i = V' \sqcap a$. Absurd. Hence, $\forall 1 \leq l \leq q$, $T''_l = W_l \to T'''_l$ and $V = \sqcap_{l=1}^{q}\vec{e''}_{l(1:n)}(W_l \to T'''_l)$. By IH, $\forall 1 \leq l \leq q$, $\exists 1 \leq i \leq k$ such that $\vec{e}_{i(1:n)} = \vec{e''}_{l(1:n)}$, $W_l \sqsubseteq V_i$ and $T_i \sqsubseteq T'''_l$. Also, by IH, $\forall j, m_j = m$ and $\forall 1 \leq j \leq p$, $\exists 1 \leq l \leq q$, $\vec{e''}_{l(1:n)} = \vec{e'}_{j(1:n)}$, $V'_j \sqsubseteq W_l$ and $T'''_l \sqsubseteq T'_j$. Hence, $\forall 1 \leq j \leq p$, $\exists 1 \leq i \leq k$, such that $\vec{e'}_{j(1:n)} = \vec{e}_{i(1:n)}$, $V'_j \sqsubseteq V_i$ and $T_i \sqsubseteq T'_j$.

3. By induction on $eU \sqsubseteq V$.

4. By 2a, $V = \sqcap_{i=1}^{p}T'_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $U \to T \sqsubseteq T'_i$. If $T'_i = a$, then, by 1, $U \to T = U' \sqcap a$. Absurd. Hence, $T'_i = U_i \to T_i$. Hence, by 2b, $\forall 1 \leq i \leq p$, $U_i \sqsubseteq U$ and $T \sqsubseteq T_i$.

5. By 2a, $\forall i, n_i = n$ and $V = \sqcap_{i=1}^{p}\vec{e'}_{i(1:n)}T''_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $\exists 1 \leq j_i \leq k$ such that $\vec{e}_{j_i(1:n)} = \vec{e'}_{i(1:n)}$ and $V_{j_i} \to T_{j_i} \sqsubseteq T''_i$. Let $1 \leq i \leq p$. If $T''_i = a$, then, by 1, $V_{j_i} \to T_{j_i} = U' \sqcap a$. Absurd. Hence, $T''_i = V'_i \to T'_i$. By 4, $V'_i \sqsubseteq V_{j_i}$ and $T_{j_i} \sqsubseteq T'_i$. We are done.

6. Using previous items.

7. By induction on $U \sqsubseteq U'_1 \sqcap U'_2$.

   - Let $\frac{}{U'_1 \sqcap U'_2 \sqsubseteq U'_1 \sqcap U'_2}$. By ref, $U'_1 \sqsubseteq U'_1$ and $U'_2 \sqsubseteq U'_2$.
   - Let $\frac{U \sqsubseteq U'' \qquad U'' \sqsubseteq U'_1 \sqcap U'_2}{U \sqsubseteq U'_1 \sqcap U'_2}$. By IH, $U'' = U''_1 \sqcap U''_2$ such that $U''_1 \sqsubseteq U'_1$ and $U''_2 \sqsubseteq U'_2$. Again by IH, $U = U_1 \sqcap U_2$ such that $U_1 \sqsubseteq U''_1$ and $U_2 \sqsubseteq U''_2$. So by tr, $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.
   - Let $\frac{U\ good\ \ \&\ \ \mathrm{d}(U'_1 \sqcap U'_2) = \mathrm{d}(U)}{(U'_1 \sqcap U'_2) \sqcap U \sqsubseteq U'_1 \sqcap U'_2}$. By ref, $U'_1 \sqsubseteq U'_1$ and $U'_2 \sqsubseteq U'_2$. Moreover:
     * If $\mathrm{d}(U) = \mathrm{d}(U'_1 \sqcap U'_2) = \mathrm{d}(U'_1)$ then by $\sqcap_E$, $U'_1 \sqcap U \sqsubseteq U'_1$. We are done.
     * If $\mathrm{d}(U) = \mathrm{d}(U'_1 \sqcap U'_2) = \mathrm{d}(U'_2)$ then by $\sqcap_E$, $U'_2 \sqcap U \sqsubseteq U'_2$. We are done.
   - If $\frac{U_1 \sqsubseteq U'_1 \quad \& \quad U_2 \sqsubseteq U'_2}{U_1 \sqcap U_2 \sqsubseteq U'_1 \sqcap U'_2}$ there is nothing to prove.
   - If $\frac{U \sqsubseteq U'_1 \sqcap U'_2}{eU \sqsubseteq eU'_1 \sqcap eU'_2}$ then by IH $U = U_1 \sqcap U_2$ such that $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$. So, $eU = eU_1 \sqcap eU_2$ and by $\sqsubseteq_{exp}$, $eU_1 \sqsubseteq eU'_1$ and $eU_2 \sqsubseteq eU'_2$.

8. By induction on $\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2$.

   - Let $\frac{}{\Gamma'_1 \sqcap \Gamma'_2 \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2}$. By ref, $\Gamma'_1 \sqsubseteq \Gamma'_1$ and $\Gamma'_2 \sqsubseteq \Gamma'_2$.

– Let $\dfrac{\Gamma \sqsubseteq \Gamma'' \qquad \Gamma'' \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2}{\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2}$. By IH, $\Gamma'' = \Gamma''_1 \sqcap \Gamma''_2$ such that $\Gamma''_1 \sqsubseteq \Gamma'_1$ and $\Gamma''_2 \sqsubseteq \Gamma'_2$. Again by IH, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ such that $\Gamma_1 \sqsubseteq \Gamma''_1$ and $\Gamma_2 \sqsubseteq \Gamma''_2$. So by tr, $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.

– Let $\dfrac{U_1 \sqsubseteq U_2}{\Gamma, (y^n : U_1) \sqsubseteq \Gamma, (y^n : U_2)}$ where $\Gamma, (y^n : U_2) = \Gamma'_1 \sqcap \Gamma'_2$.

  * If $\Gamma'_1 = \Gamma''_1, (y^n : U'_2)$ and $\Gamma'_2 = \Gamma''_2, (y^n : U''_2)$ such that $U_2 = U'_2 \sqcap U''_2$, then by 7, $U_1 = U'_1 \sqcap U''_1$ such that $U'_1 \sqsubseteq U'_2$ and $U''_1 \sqsubseteq U''_2$. Hence $\Gamma = \Gamma''_1 \sqcap \Gamma''_2$ and $\Gamma, (y^n : U_1) = \Gamma_1 \sqcap \Gamma_2$ where $\Gamma_1 = \Gamma''_1, (y^n : U'_1)$ and $\Gamma_2 = \Gamma''_2, (y^n : U''_1)$ such that $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$ by $\sqsubseteq_c$.
  * If $y^n \notin dom(\Gamma'_1)$ then $\Gamma = \Gamma'_1 \sqcap \Gamma''_2$ where $\Gamma''_2, (y^n : U_2) = \Gamma'_2$. Hence, $\Gamma, (y^n : U_1) = \Gamma'_1 \sqcap \Gamma_2$ where $\Gamma_2 = \Gamma''_2, (y^n : U_1)$. By ref and $\sqsubseteq_c$, $\Gamma'_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.
  * If $y^n \notin dom(\Gamma'_2)$ then similar to the above case.

□

Lemma A.11. In the relevant context ($\mathbb{U}_3$, $\mathbb{T}_3$, $Env_{\mathbb{U}_3}$ or 3-typings), we have:

1. If $T \in \mathbb{T}_3$, then $\mathrm{d}(T) = \oslash$.
2. Let $U \in \mathbb{U}_3$. If $\mathrm{d}(U) = L = (n_i)_m$, then $U = \omega^L$ or $U = \vec{e}_L \sqcap_{i=1}^p T_i$ where $p \geq 1$ and for all $i \in \{1, \ldots, p\}$, $T_i \in \mathbb{T}_3$.
3. Let $U_1, U_2 \in \mathbb{U}_3$ and $U_1 \sqsubseteq U_2$.
   (a) If $U_1 = \omega^K$ then $U_2 = \omega^K$.
   (b) If $U_1 = \vec{e}_K U$ then $U_2 = \vec{e}_K U'$ and $U \sqsubseteq U'$.
   (c) If $U_2 = \vec{e}_K U$ then $U_1 = \vec{e}_K U'$ and $U \sqsubseteq U'$.
   (d) If $U_1 = \sqcap_{i=1}^p \vec{e}_K(U_i \to T_i)$ where $p \geq 1$ then $U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^q \vec{e}_K(U'_j \to T'_j)$ where $q \geq 1$ and for all $j \in \{1, \ldots, q\}$, there exists $i \in \{1, \ldots, p\}$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$.
4. If $U \in \mathbb{U}_3$ and $U \sqsubseteq U'_1 \sqcap U'_2$ then $U = U_1 \sqcap U_2$ where $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.
5. If $\Gamma \in Env_{\mathbb{U}_3}$ and $\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2$ then $\Gamma = \Gamma_1 \sqcap \Gamma_2$ where $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.

Proof:
[Of lemma A.11]

1. By definition.
2. By induction on U.
   - If $U = a$ ($\mathrm{d}(U) = \oslash$), nothing to prove.
   - If $U = V \to T$ ($\mathrm{d}(U) = \oslash$), nothing to prove.
   - If $U = \omega^L$, nothing to prove.
   - If $U = U_1 \sqcap U_2$ ($\mathrm{d}(U) = \mathrm{d}(U_1) = \mathrm{d}(U_2) = L$), by IH we have four cases:
     – If $U_1 = U_2 = \omega^L$ then $U = \omega^L$.

- If $U_1 = \omega^L$ and $U_2 = \vec{e}_L \sqcap_{i=1}^k T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k, T_i \in \mathbb{T}$ then $U = U_2$ (since ω^L is a neutral).
- If $U_2 = \omega^L$ and $U_1 = \vec{e}_L \sqcap_{i=1}^k T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k, T_i \in \mathbb{T}$ then $U = U_1$ (since ω^L is a neutral).
- If $U_1 = \vec{e}_L \sqcap_{i=1}^p T_i$ and $U_2 = \vec{e}_L \sqcap_{i=p+1}^{p+q} T_i$ where $p, q \geq 1$, $\forall 1 \leq i \leq p+q, T_i \in \mathbb{T}$ then $U = \vec{e}_L \sqcap_{i=1}^{p+q} T_i$.

- If $U = \overline{e}_{n_1} V$ ($L = \mathrm{d}(U) = n_1 :: \mathrm{d}(V) = n_1 :: K$), by IH we have two cases:
  - If $V = \omega^K$, $U = \overline{e}_{n_1}\omega^K = \omega^L$.
  - If $V = \vec{e}_K \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p, T_i \in \mathbb{T}$ then $U = \vec{e}_L \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p, T_i \in \mathbb{T}$.

3. (a) By induction on $U_1 \sqsubseteq U_2$.

   (b) By induction on K. We do the induction step. Let $U_1 = \overline{e}_i U$. By induction on $\overline{e}_i U \sqsubseteq U_2$ we obtain $U_2 = \overline{e}_i U'$ and $U \sqsubseteq U'$.

   (c) same proof as in the previous item.

   (d) By induction on $U_1 \sqsubseteq U_2$:

   - By ref, $U_1 = U_2$.
   - If $\dfrac{\sqcap_{i=1}^p \vec{e}_K(U_i \to T_i) \sqsubseteq U \qquad U \sqsubseteq U_2}{\sqcap_{i=1}^p \vec{e}_K(U_i \to T_i) \sqsubseteq U_2}$. If $U = \omega^K$ then by (b), $U_2 = \omega^K$. If $U = \sqcap_{j=1}^q \vec{e}_K(U'_j \to T'_j)$ where $q \geq 1$ and $\forall 1 \leq j \leq q$, $\exists 1 \leq i \leq p$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$ then by IH, $U_2 = \omega^K$ or $U_2 = \sqcap_{k=1}^r \vec{e}_K(U''_k \to T''_k)$ where $r \geq 1$ and $\forall 1 \leq k \leq r$, $\exists 1 \leq j \leq q$ such that $U''_k \sqsubseteq U'_j$ and $T'_j \sqsubseteq T''_k$. Hence, by tr, $\forall 1 \leq k \leq r$, $\exists 1 \leq i \leq p$ such that $U''_k \sqsubseteq U_i$ and $T_i \sqsubseteq T''_k$.
   - By $\sqcap_E$, $U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^q \vec{e}_K(U'_j \to T'_j)$ where $1 \leq q \leq p$ and $\forall 1 \leq j \leq q$, $\exists 1 \leq i \leq p$ such that $U_i = U'_j$ and $T_i = T'_j$.
   - Case $\sqcap$ is by IH.
   - Case $\to$ is trivial.
   - If $\dfrac{\sqcap_{i=1}^p \vec{e}_L(U_i \to T_i) \sqsubseteq U_2}{\sqcap_{i=1}^p \vec{e}_K(U_i \to T_i) \sqsubseteq \overline{e}_i U_2}$ where $K = i :: L$ then by IH, $U_2 = \omega^L$ and so $\overline{e}_i U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^q \vec{e}_L(U'_j \to T'_j)$ so $\overline{e}_i U_2 = \sqcap_{j=1}^q \vec{e}_K(U'_j \to T'_j)$ where $q \geq 1$ and $\forall 1 \leq j \leq q$, $\exists 1 \leq i \leq p$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$.

4. By induction on $U \sqsubseteq U'_1 \sqcap U'_2$.

   - Let $\dfrac{}{U'_1 \sqcap U'_2 \sqsubseteq U'_1 \sqcap U'_2}$. By ref, $U'_1 \sqsubseteq U'_1$ and $U'_2 \sqsubseteq U'_2$.
   - Let $\dfrac{U \sqsubseteq U'' \qquad U'' \sqsubseteq U'_1 \sqcap U'_2}{U \sqsubseteq U'_1 \sqcap U'_2}$. By IH, $U'' = U''_1 \sqcap U''_2$ such that $U''_1 \sqsubseteq U'_1$ and $U''_2 \sqsubseteq U'_2$. Again by IH, $U = U_1 \sqcap U_2$ such that $U_1 \sqsubseteq U''_1$ and $U_2 \sqsubseteq U''_2$. So by tr, $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.
   - Let $\dfrac{}{(U'_1 \sqcap U'_2) \sqcap U \sqsubseteq U'_1 \sqcap U'_2}$. By ref, $U'_1 \sqsubseteq U'_1$ and $U'_2 \sqsubseteq U'_2$. Moreover $\mathrm{d}(U) = \mathrm{d}(U'_1 \sqcap U'_2) = \mathrm{d}(U'_1)$ then by $\sqcap_E$, $U'_1 \sqcap U \sqsubseteq U'_1$.

- If $\dfrac{U_1 \sqsubseteq U_1' \quad U_2 \sqsubseteq U_2'}{U_1 \sqcap U_2 \sqsubseteq U_1' \sqcap U_2'}$ there is nothing to prove.
- $\dfrac{V_2 \sqsubseteq V_1 \quad T_1 \sqsubseteq T_2}{V_1 \to T_1 \sqsubseteq V_2 \to T_2}$ then $U_1' = U_2' = V_2 \to T_2$ and $U = U_1 \sqcap U_2$ such that $U_1 = U_2 = V_1 \to T_1$ and we are done.
- If $\dfrac{U \sqsubseteq U_1' \sqcap U_2'}{eU \sqsubseteq eU_1' \sqcap eU_2'}$ then by IH $U = U_1 \sqcap U_2$ such that $U_1 \sqsubseteq U_1'$ and $U_2 \sqsubseteq U_2'$. So, $eU = eU_1 \sqcap eU_2$ and by $\sqsubseteq_e$, $eU_1 \sqsubseteq eU_1'$ and $eU_2 \sqsubseteq eU_2'$.

5. By induction on $\Gamma \sqsubseteq \Gamma_1' \sqcap \Gamma_2'$.

   - Let $\dfrac{}{\Gamma_1' \sqcap \Gamma_2' \sqsubseteq \Gamma_1' \sqcap \Gamma_2'}$. By ref, $\Gamma_1' \sqsubseteq \Gamma_1'$ and $\Gamma_2' \sqsubseteq \Gamma_2'$.
   - Let $\dfrac{\Gamma \sqsubseteq \Gamma'' \quad \Gamma'' \sqsubseteq \Gamma_1' \sqcap \Gamma_2'}{\Gamma \sqsubseteq \Gamma_1' \sqcap \Gamma_2'}$. By IH, $\Gamma'' = \Gamma_1'' \sqcap \Gamma_2''$ such that $\Gamma_1'' \sqsubseteq \Gamma_1'$ and $\Gamma_2'' \sqsubseteq \Gamma_2'$. Again by IH, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ such that $\Gamma_1 \sqsubseteq \Gamma_1''$ and $\Gamma_2 \sqsubseteq \Gamma_2''$. So by tr, $\Gamma_1 \sqsubseteq \Gamma_1'$ and $\Gamma_2 \sqsubseteq \Gamma_2'$.
   - Let $\dfrac{U_1 \sqsubseteq U_2}{\Gamma, (y^L : U_1) \sqsubseteq \Gamma, (y^L : U_2)}$ where $\Gamma, (y^L : U_2) = \Gamma_1' \sqcap \Gamma_2'$.
     - If $\Gamma_1' = \Gamma_1'', (y^L : U_2')$ and $\Gamma_2' = \Gamma_2'', (y^L : U_2'')$ such that $U_2 = U_2' \sqcap U_2''$, then by 4, $U_1 = U_1' \sqcap U_1''$ such that $U_1' \sqsubseteq U_2'$ and $U_1'' \sqsubseteq U_2''$. Hence $\Gamma = \Gamma_1'' \sqcap \Gamma_2''$ and $\Gamma, (y^L : U_1) = \Gamma_1 \sqcap \Gamma_2$ where $\Gamma_1 = \Gamma_1'', (y^L : U_1')$ and $\Gamma_2 = \Gamma_2'', (y^L : U_1'')$ such that $\Gamma_1 \sqsubseteq \Gamma_1'$ and $\Gamma_2 \sqsubseteq \Gamma_2'$ by $\sqsubseteq_c$.
     - If $y^L \notin \mathrm{dom}(\Gamma_1')$ then $\Gamma = \Gamma_1' \sqcap \Gamma_2''$ where $\Gamma_2'', (y^L : U_2) = \Gamma_2'$. Hence, $\Gamma, (y^L : U_1) = \Gamma_1' \sqcap \Gamma_2$ where $\Gamma_2 = \Gamma_2'', (y^L : U_1)$. By ref and $\sqsubseteq_c$, $\Gamma_1' \sqsubseteq \Gamma_1'$ and $\Gamma_2 \sqsubseteq \Gamma_2'$.
     - If $y^L \notin \mathrm{dom}(\Gamma_2')$ then similar to the above case.

□

Proof:
[of Remark 4.1]

1. Let $M : \langle \Gamma_1 \vdash_j U_1 \rangle$ and $M : \langle \Gamma_2 \vdash_j U_2 \rangle$. By lemma 4.1.3a, $dom(\Gamma_1) = dom(\Gamma_2)$. Let $\Gamma_1 = (x_i^{I_i} : V_i)_n$ and $\Gamma_2 = (x_i^{I_i} : V_i')_n$. By lemma 4.1.2, $\forall 1 \leq i \leq n$, if $j = 2$, V_i and V_i' are good and $\mathrm{d}(V_i) = \mathrm{d}(V_i') = I_i$. By $\sqcap_E$, $V_i \sqcap V_i' \sqsubseteq V_i$ and $V_i \sqcap V_i' \sqsubseteq V_i'$. Hence, by lemma 4.1.2, $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq \Gamma_1$ and $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq \Gamma_2$ and by $\sqsubseteq$ and $\sqsubseteq_{\langle\rangle}$, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_j U_1 \rangle$ and $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_j U_2 \rangle$. Finally, by $\sqcap_I$, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_j U_1 \sqcap U_2 \rangle$.

2. By lemma 3.1.2, $U = \sqcap_{i=1}^{k} \vec{e}_{i(1:n)} T_i$ where $k \geq 1$, and $\forall 1 \leq i \leq k$, $T_i \in \mathbb{T}_2$ and T_i is good. Let $1 \leq i \leq k$. By lemma 3.1.2, $\mathrm{d}(T_i) = 0$ and by ax, $x^0 : \langle (x^0 : T_i) \vdash_2 T_i \rangle$. Hence, $x^n : \langle (x^n : \vec{e}_{i(1:n)} T_i) \vdash_2 \vec{e}_{i(1:n)} T_i \rangle$ by n applications of exp. Now, by $k-1$ applications of $\sqcap_i'$, $x^n : \langle (x^n : U) \vdash_2 U \rangle$.

3. By lemma A.11, either $U = \omega^L$ so by ω, $x^L : \langle (x^L : \omega^L) \vdash_3 \omega^L \rangle$. Or $U = \sqcap_{i=1}^{p} \vec{e}_L T_i$ where $p \geq 1$, and $\forall 1 \leq i \leq p$, $T_i \in \mathbb{T}_3$. Let $1 \leq i \leq p$. By ax, $x^{\oslash} : \langle (x^{\oslash} : T_i) \vdash_3 T_i \rangle$, hence by e, $x^L : \langle (x^L : \vec{e}_L T_i) \vdash_3 \vec{e}_L T_i \rangle$. Now, by $\sqcap_I'$, $x^L : \langle (x^L : U) \vdash_3 U \rangle$.

4. By $\sqcap_E$ and since $\omega^{\mathrm{d}(U)}$ is a neutral.

□

Lemma A.12. 1. $\mathrm{OK}(env^{\omega}_{M})$.

2. If $\mathrm{dom}(\Gamma) = \mathrm{FV}(M)$ and $\mathrm{OK}(\Gamma)$ then $\Gamma \sqsubseteq env^{\omega}_{M}$
3. If $\Gamma \diamond \Delta$ and $\mathrm{d}(\Gamma), \mathrm{d}(\Delta) \succeq K$, then $\Gamma^{-K} \diamond \Delta^{-K}$.
4. If $U \sqsubseteq U'$ and $\mathrm{d}(U) \succeq K$ then $U^{-K} \sqsubseteq U'^{-K}$.
5. If $\Gamma \sqsubseteq \Gamma'$ and $\mathrm{d}(\Gamma) \succeq K$ then $\Gamma^{-K} \sqsubseteq \Gamma'^{-K}$.
6. If $\mathrm{OK}(\Gamma_1)$, $\mathrm{OK}(\Gamma_2)$ then $\mathrm{OK}(\Gamma_1 \sqcap \Gamma_2)$.
7. If $\mathrm{OK}(\Gamma)$ then $\mathrm{OK}(e\Gamma)$.
8. If $\Gamma_1 \sqsubseteq \Gamma_2$ then ($\mathrm{d}(\Gamma_1) \succeq L$ iff $\mathrm{d}(\Gamma_2) \succeq L$) and ($\mathrm{OK}(\Gamma_1)$ iff $\mathrm{OK}(\Gamma_2)$).

Proof:

1. By definition, if $\mathrm{FV}(M) = \{x_1^{L_1}, \dots, x_n^{L_n}\}$ then $env^{\omega}_{M} = (x_i^{L_i} : \omega^{L_i})_n$ and by definition, for all $i \in \{1, \dots, n\}$, $\mathrm{d}(\omega^{L_i}) = L_i$. Moreover, if $x^L : U, x^L : V \in env^{\omega}_{M}$, then $U = \omega^L = V$.
2. Let $\mathrm{FV}(M) = \{x_1^{L_1}, \dots, x_n^{L_n}\}$ and $\Gamma = (x_i^{L_i} : U_i)_n$. By definition, $env^{\omega}_{M} = (x_i^{L_i} : \omega^{L_i})_n$. Because $\mathrm{OK}(\Gamma)$, then for all $i \in \{1, \dots, n\}$, $\mathrm{d}(U_i) = L_i$. Hence, by lemma 4.1.4 and 2, $\Gamma \sqsubseteq env^{\omega}_{M}$.
3. Let $x^{L_1} \in \mathrm{dom}(\Gamma^{-K})$ and $x^{L_2} \in \mathrm{dom}(\Delta^{-K})$, then $x^{K::L_1} \in \mathrm{dom}(\Gamma)$ and $x^{K::L_2} \in \mathrm{dom}(\Delta)$, hence $K :: L_1 = K :: L_2$ and so $L_1 = L_2$.
4. Let $\mathrm{d}(U) = L = K :: K'$. By lemma A.11:
   - If $U = \omega^L$ then by lemma A.11.3a, $U' = \omega^L$ and by ref, $U^{-K} = \omega^{K'} \sqsubseteq \omega^{K'} = U'^{-K}$.
   - If $U = \vec{e}_L \sqcap_{i=1}^{p} T_i$ where $p \geq 1$ and $\forall\ 1 \leq i \leq p$, $T_i \in \mathbb{T}$ then by lemma A.11.3b, $U' = \vec{e}_L V$ and $\sqcap_{i=1}^{p} T_i \sqsubseteq V$. Hence, by $\sqsubseteq_e$, $U^{-K} = \vec{e}_{K'} \sqcap_{i=1}^{p} T_i \sqsubseteq \vec{e}_{K'} V = U'^{-K}$.
5. Let $\Gamma = (x_i^{L_i} : U_i)_n$, so by lemma 4.1.2, $\Gamma' = (x_i^{L_i} : U'_i)_n$ and $\forall 1 \leq i \leq n$, $U_i \sqsubseteq U'_i$. Because $\mathrm{d}(\Gamma) \succeq K$, then by definition $\forall 1 \leq i \leq n$, $\mathrm{d}(U_i) \succeq K$. By lemma 4.1.4, $\forall i \in \{1, \dots, n\}$, $U_i^{-K} \sqsubseteq {U'}_i^{-K}$ and by lemma 4.1.2, $\Gamma^{-K} \sqsubseteq \Gamma'^{-K}$.
6. Let $\Gamma_1 = (x_i^{L_i} : U_i)_n, \Gamma'_1$ and $\Gamma_2 = (x_i^{L_i} : U'_i)_n, \Gamma'_2$ such that $\mathrm{dom}(\Gamma'_1) \cap \mathrm{dom}(\Gamma'_2)$. Then, by hypotheses, for all $i \in \{1, \dots, n\}$, $\mathrm{d}(U_i) = L_i = \mathrm{d}(U'_i)$. Then $\Gamma_1 \sqcap \Gamma_2 = (x_i^{L_i} : U_i \sqcap U'_i)_n, \Gamma'_1, \Gamma'_2$ is well defined. Moreover, for all $x^L : U \in \Gamma'_1$, $\mathrm{d}(U) = L$ and for all $x^L : U \in \Gamma'_2$, $\mathrm{d}(U) = L$ and for all $i \in \{1, \dots, n\}$, $\mathrm{d}(U_i \sqcap U'_i) = \mathrm{d}(U_i) = L_i = \mathrm{d}(U'_i)$.
7. Let $\Gamma = (x_j^{L_j} : U_j)_n$ then by hypothesis, for all $j \in \{1, \dots, n\}$, $\mathrm{d}(U_j) = L_j$ and $\overline{e}_i\Gamma = (x_j^{i::L_j} : \overline{e}_i U_j)$. So, for all $j \in \{1, \dots, n\}$, $\mathrm{d}(\overline{e}_i U_j) = i :: \mathrm{d}(U_j) = i :: L_j$.

8. By lemma 4.1.2, $\Gamma_1 = (x_i^{L_i} : U_i)_n$ and $\Gamma_2 = (x_i^{L_i} : U'_i)_n$ and for all $i \in \{1, \ldots, n\}$, $U_i \sqsubseteq U'_i$. By lemma 4.1.5, for all $i \in \{1, \ldots, n\}$, $\mathrm{d}(U_i) = \mathrm{d}(U'_i)$. Assume $\mathrm{d}(\Gamma_1) \succeq K$ then for all $i \in \{1, \ldots, n\}$, $\mathrm{d}(U_i) = \mathrm{d}(U'_i) \succeq K$ and $L_i \succeq K$, so $\mathrm{d}(\Gamma_2) \succeq K$. Assume $\mathrm{d}(\Gamma_2) \succeq K$ then for all $i \in \{1, \ldots, n\}$, $\mathrm{d}(U_i) = \mathrm{d}(U'_i) \succeq K$ and $L_i \succeq K$, so $\mathrm{d}(\Gamma_1) \succeq K$. Assume $\mathrm{OK}(\Gamma_1)$ then for all $i \in \{1, \ldots, n\}$, $L_i = \mathrm{d}(U_i) = \mathrm{d}(U'_i)$, so $\mathrm{OK}(\Gamma_2)$. Assume $\mathrm{OK}(\Gamma_2)$ then for all $i \in \{1, \ldots, n\}$, $L_i = \mathrm{d}(U'_i) = \mathrm{d}(U_i)$, so $\mathrm{OK}(\Gamma_1)$.

□

A.5. Subject reduction and subject expansion properties

Proof:
[of Lemma 5.1]

1. By induction on the derivation of $x^n : \langle \Gamma \vdash_1 T \rangle$.
2. First, we prove by induction on the derivation of $\lambda x^n.M : \langle \Gamma \vdash_1 T_1 \to T_2 \rangle$ that $\exists k \geq 1, \Gamma_1, \Gamma_2, \ldots, \Gamma_k$, such that $\Gamma = \Gamma_1 \sqcap \Gamma_2 \cdots \sqcap \Gamma_k$ and $\forall\, 1 \leq i \leq k$, $M : \langle \Gamma_i, x^n : T_1 \vdash_1 T_2 \rangle$. We have two cases:
   - Case $\to_i$: take $k = 1$.
   - Case $\sqcap_i$: Let $\dfrac{\lambda x^n.M : \langle \Delta \vdash_1 T_1 \to T_2 \rangle \quad \lambda x^n.M : \langle \Delta' \vdash_1 T_1 \to T_2 \rangle}{\lambda x^n.M : \langle \Delta \sqcap \Delta' \vdash_1 T_1 \to T_2 \rangle}$. By IH, $\Delta = \Delta_1 \sqcap \cdots \sqcap \Delta_{k_1}$ and $\forall\, 1 \leq i \leq k_1$, $M : \langle \Delta_i, x^n : T_1 \vdash_1 T_2 \rangle$ and $\Delta' = \Delta'_1 \sqcap \cdots \sqcap \Delta'_{k_2}$ and $\forall\, 1 \leq j \leq k_2$, $M : \langle \Delta'_j, x^n : T_1 \vdash_1 T_2 \rangle$ and we are done.

   Now we prove 2. Since $\Gamma = \Gamma_1 \sqcap \Gamma_2 \cdots \sqcap \Gamma_k$ where $\forall\, 1 \leq i \leq k$, $M : \langle \Gamma_i, x^n : T_1 \vdash T_2 \rangle$, by $k-1$ applications of $\sqcap_i$ we get $M : \langle \Gamma, x^n : T_1 \vdash_1 T_2 \rangle$.
3. By induction on the derivation of $MN : \langle \Gamma \vdash_1 T \rangle$.

□

Proof:
[of Lemma 5.2] 1. By induction on the derivation of $x^n : \langle \Gamma \vdash_2 U \rangle$.
2. By induction on the derivation of $\lambda x^n.M : \langle \Gamma \vdash_2 U \rangle$. We have four cases:

- If $\dfrac{M : \langle \Gamma, x^n : U \vdash_2 T \rangle}{\lambda x^n.M : \langle \Gamma \vdash_2 U \to T \rangle}$, nothing to prove.
- Let $\dfrac{\lambda x^n.M : \langle \Gamma \vdash_2 U_1 \rangle \quad \lambda x^n.M : \langle \Gamma \vdash_2 U_2 \rangle}{\lambda x^n.M : \langle \Gamma \vdash_2 U_1 \sqcap U_2 \rangle}$. By lemma 4.1, $U_1 \sqcap U_2$ is good and $\mathrm{d}(U_1) = \mathrm{d}(U_2) = m$. By IH we have: $U_1 = \sqcap_{i=1}^{k} \vec{e}_{i(1:m)}(V_i \to T_i)$, $U_2 = \sqcap_{i=k+1}^{k+l} \vec{e}_{i(1:m)}(V_i \to T_i)$ (hence $U_1 \sqcap U_2 = \sqcap_{i=1}^{k+l} \vec{e}_{i(1:m)}(V_i \to T_i)$) where $k, l \geq 1$ and $\forall 1 \leq i \leq k+l$, $M : \langle \Gamma, x^n : \vec{e}_{i(1:m)} V_i \vdash_2 \vec{e}_{i(1:m)} T_i \rangle$. We are done.
- Let $\dfrac{\lambda x^n.M : \langle \Gamma \vdash_2 U \rangle}{\lambda x^{n+1}.M^+ : \langle e\Gamma \vdash_2 eU \rangle}$. By IH, $U = \sqcap_{i=1}^{k} \vec{e}_{i(1:m-1)}(V_i \to T_i)$ (since $\mathrm{d}(U) = m-1$) where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma, x^n : \vec{e}_{i(1:m-1)} V_i \vdash_2 \vec{e}_{i(1:m-1)} T_i \rangle$. By e, $\forall 1 \leq i \leq k$, $M^+ : \langle \Gamma, x^n + 1 : e\vec{e}_{i(1:m-1)} V_i \vdash_2 e\vec{e}_{i(1:m-1)} T_i \rangle$.

- Let $\dfrac{\lambda x^n.M : \langle \Gamma \vdash_2 U\rangle \quad \langle \Gamma \vdash_2 U\rangle \sqsubseteq \langle \Gamma' \vdash_2 U'\rangle}{\lambda x^n.M : \langle \Gamma' \vdash_2 U'\rangle}$. By lemma 4.1.3, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$. By Theorem 4.1, U, U' are good and $\mathrm{d}(U) = \mathrm{d}(U') = m$. By IH, $U = \sqcap_{i=1}^{k} \vec{e}_{i(1:m)}(V_i \to T_i)$, where $k \geq 1$ and $M : \langle \Gamma, x^n : \vec{e}_{i(1:m)}V_i \vdash_2 \vec{e}_{i(1:m)}T_i\rangle\ \forall\ 1 \leq i \leq k$. By lemma A.10, $U' = \sqcap_{i=1}^{p} \vec{e'}_{i(1:m)}(V'_i \to T'_i)$, where $p \geq 1$, and $\forall 1 \leq i \leq p$, $\exists 1 \leq j_i \leq k$ such that $\vec{e}_{j_i(1:m)} = \vec{e'}_{i(1:m)}$, $V'_i \sqsubseteq V_{j_i}$ and $T_{j_i} \sqsubseteq T'_i$. Let $1 \leq i \leq p$. Since $\langle \Gamma, x^n : \vec{e}_{j_i(1:m)}V_{j_i} \vdash_2 \vec{e}_{j_i(1:m)}T_{j_i}\rangle \sqsubseteq \langle \Gamma', x^n : \vec{e'}_{i(1:m)}V'_i \vdash_2 \vec{e'}_{i(1:m)}T'_i\rangle$, by lemma A.10, then $M : \langle \Gamma', x^n : \vec{e'}_{i(1:m)}V'_i \vdash_2 \vec{e'}_{i(1:m)}T'_i\rangle$.

$\square$

Lemma A.13. (Extra Generation for $\vdash_2$)

1. If $Mx^n : \langle \Gamma, x^n : U \vdash_2 V\rangle$, $\mathrm{d}(V) = 0$ and $x^n \notin FV(M)$, then $V = \sqcap_{i=1}^{k} T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma \vdash_2 U \to T_i\rangle$.
2. If $\lambda x^n.Mx^n : \langle \Gamma \vdash_2 U\rangle$ and $x^n \notin FV(M)$, then $M : \langle \Gamma \vdash_2 U\rangle$.

Proof:

1. By induction on the derivation of $Mx^n : \langle \Gamma, x^n : U \vdash_2 V\rangle$. We have three cases:
   - If $\dfrac{M : \langle \Gamma \vdash_2 U \to T\rangle \quad x^n : \langle x^n : V \vdash_2 U\rangle \quad \Gamma \diamond (x^n : V)\ \ V \sqsubseteq U}{Mx^n : \langle \Gamma, x^n : V \vdash_2 T\rangle}$ (using lemma 5.2.1 and lemma A.14), then since $U \to T \sqsubseteq V \to T$, we have $M : \langle \Gamma \vdash_2 V \to T\rangle$.
   - If $\dfrac{Mx^n : \langle \Gamma, x^n : U \vdash_2 U_1\rangle \quad Mx^n : \langle \Gamma, x^n : U \vdash_2 U_2\rangle}{Mx^n : \langle \Gamma, x^n : U \vdash_2 U_1 \sqcap U_2\rangle}$, by lemma 4.1 $U_1 \sqcap U_2$ is good and, by lemma 3.1.1b, $\mathrm{d}(U_1) = \mathrm{d}(U_2) = 0$. By IH, $U_1 = \sqcap_{i=1}^{k} T_i$, $U_2 = \sqcap_{i=k+1}^{k+l} T_i$, where $k, l \geq 1$ and $\forall\ 1 \leq i \leq k+l$, $M : \langle \Gamma \vdash_2 U \to T_i\rangle$.
   - If $\dfrac{Mx^n : \langle \Gamma, x^n : U \vdash_2 V\rangle \quad \langle \Gamma, x^n : U \vdash_2 V\rangle \sqsubseteq \langle \Gamma', x^n : U' \vdash_2 V'\rangle}{Mx^n : \langle \Gamma', x^n : U' \vdash_2 V'\rangle}$, by lemma 4.1, $\Gamma' \sqsubseteq \Gamma$, $U' \sqsubseteq U$ and $V \sqsubseteq V'$. By lemma A.10, $\mathrm{d}(V) = \mathrm{d}(V') = 0$. By IH, $V = \sqcap_{i=1}^{k} T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma \vdash_2 U \to T_i\rangle$. By lemma A.10.2 (since V is good by lemma 4.1), $V' = \sqcap_{i=1}^{p} T'_i$ where $1 \leq p \leq k$ and $\forall 1 \leq i \leq p$, $T_i \sqsubseteq T'_i$. Since for any $1 \leq i \leq p$, $\langle \Gamma \vdash_2 U \to T_i\rangle \sqsubseteq \langle \Gamma' \vdash_2 U' \to T'_i\rangle$, then $\forall 1 \leq i \leq p$, $M : \langle \Gamma' \vdash_2 U' \to T'_i\rangle$.
2. By lemma 4.1, $m = \mathrm{d}(U) = \mathrm{d}(\lambda x^n.Mx^n) = \mathrm{d}(Mx^n) \leq n$. Hence $n - m \geq 0$ and $\mathrm{d}(Mx^n) = \mathrm{d}(M) = m$. By lemma 5.2.2, $U = \sqcap_{i=1}^{k} \vec{e}_{i(1:m)}(V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $Mx^n : \langle \Gamma, x^n : \vec{e}_{i(1:m)}V_i \vdash_2 \vec{e}_{i(1:m)}T_i\rangle$.
   - If $m = 0$, then $\forall 1 \leq i \leq k$, $Mx^n : \langle \Gamma, x^n : V_i \vdash_2 T_i\rangle$ and $M : \langle \Gamma \vdash_2 V_i \to T_i\rangle$ by 1. Hence, by $k-1$ applications $\sqcap_i$, $M : \langle \Gamma \vdash_2 U\rangle$.
   - If $m > 0$, then, by lemma 4.1 and m-applications of lemma 4.1.3, $\forall 1 \leq i \leq k$, $M^{-m}x^{n-m} : \langle \Gamma^{-m}, x^{n-m} : V_i \vdash_2 T_i\rangle$ and, $M^{-m} : \langle \Gamma^{-m} \vdash_2 V_i \to T_i\rangle$ by 1. Now, by m-applications of exp, $M : \langle \Gamma \vdash_2 \vec{e}_{i(1:m)}(V_i \to T_i)\rangle$. Finally, by k-applications of $\sqcap_i$, $M : \langle \Gamma \vdash_2 U\rangle$.

$\square$

Proof:
[Of lemma 5.3]

1. By induction on the derivation $x^L : \langle \Gamma \vdash_3 U \rangle$. We have five cases:

   - If $\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash_3 T \rangle}$ then it is done using (ref).
   - If $\dfrac{}{x^L : \langle (x^L : \omega^L) \vdash_3 \omega^L \rangle}$ then it is done using (ref).
   - If $\dfrac{x^L : \langle \Gamma \vdash_3 U_1 \rangle \qquad x^L : \langle \Gamma \vdash_3 U_2 \rangle}{x^L : \langle \Gamma \vdash_3 U_1 \sqcap U_2 \rangle}$. By IH, $\Gamma = (x^L : V)$, $V \sqsubseteq U_1$ and $V \sqsubseteq U_2$, then by rule $\sqcap$, $V \sqsubseteq U_1 \sqcap U_2$.
   - If $\dfrac{x^L : \langle \Gamma \vdash_3 U \rangle}{x^{i::L} : \langle \overline{e}_i \Gamma \vdash_3 \overline{e}_i U \rangle}$. Then by IH, $\Gamma = (x^L : V)$ and $V \sqsubseteq U$, so $\overline{e}_i \Gamma = (x^{i::L} : \overline{e}_i V)$ and by $\sqsubseteq_e$, $\overline{e}_i V \sqsubseteq \overline{e}_i U$,
   - If $\dfrac{x^L : \langle \Gamma' \vdash_3 U' \rangle \quad \langle \Gamma' \vdash_3 U' \rangle \sqsubseteq \langle \Gamma \vdash_3 U \rangle}{x^L : \langle \Gamma \vdash_3 U \rangle}$. By lemma 4.1.3, $\Gamma \sqsubseteq \Gamma'$ and $U' \sqsubseteq U$ and, by IH, $\Gamma' = (x^L : V')$ and $V' \sqsubseteq U'$. Then, by lemma 4.1.2, $\Gamma = (x^L : V)$, $V \sqsubseteq V'$ and, by rule tr, $V \sqsubseteq U$.

2. By induction on the derivation $\lambda x^L.M : \langle \Gamma \vdash_3 U \rangle$. We have five cases:

   - If $\dfrac{}{\lambda x^L.M : \langle env^{\omega}_{\lambda x^L.M} \vdash_3 \omega^{\mathrm{d}(\lambda x^L.M)} \rangle}$ then it is done.
   - If $\dfrac{M : \langle \Gamma, x^L : U \vdash_3 T \rangle}{\lambda x^L.M : \langle \Gamma \vdash_3 U \to T \rangle}$ ($\mathrm{d}(U \to T) = \oslash$) then it is done.
   - If $\dfrac{\lambda x^L.M : \langle \Gamma \vdash_3 U_1 \rangle \quad \lambda x^L.M : \langle \Gamma \vdash_3 U_2 \rangle}{\lambda x^L.M : \langle \Gamma \vdash_3 U_1 \sqcap U_2 \rangle}$ then $\mathrm{d}(U_1 \sqcap U_2) = \mathrm{d}(U_1) = \mathrm{d}(U_2) = K$. By IH, we have four cases:
     - If $U_1 = U_2 = \omega^K$, then $U_1 \sqcap U_2 = \omega^K$.
     - If $U_1 = \omega^K$, $U_2 = \sqcap_{i=1}^p \vec{e}_K(V_i \to T_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $M : \langle \Gamma, x^L : \vec{e}_K V_i \vdash_3 \vec{e}_K T_i \rangle$, then $U_1 \sqcap U_2 = U_2$ (ω^K is a neutral element).
     - If $U_2 = \omega^K$, $U_1 = \sqcap_{i=1}^p \vec{e}_K(V_i \to T_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $M : \langle \Gamma, x^L : \vec{e}_K V_i \vdash_3 \vec{e}_K T_i \rangle$, then $U_1 \sqcap U_2 = U_1$ (ω^K is a neutral element).
     - If $U_1 = \sqcap_{i=1}^p \vec{e}_K(V_i \to T_i)$, $U_2 = \sqcap_{i=p+1}^{p+q} \vec{e}_K(V_i \to T_i)$ (hence $U_1 \sqcap U_2 = \sqcap_{i=1}^{p+q} \vec{e}_K(V_i \to T_i)$) where $p, q \geq 1$, $\forall 1 \leq i \leq p+q$, $M : \langle \Gamma, x^L : \vec{e}_K V_i \vdash_3 \vec{e}_K T_i \rangle$, we are done.
   - If $\dfrac{\lambda x^L.M : \langle \Gamma \vdash_3 U \rangle}{\lambda x^{i::L}.M^{+i} : \langle \overline{e}_i \Gamma \vdash_3 \overline{e}_i U \rangle}$. $\mathrm{d}(\overline{e}_i U) = i :: \mathrm{d}(U) = i :: K' = K$. By IH, we have two cases:
     - If $U = \omega^{K'}$ then $\overline{e}_i U = \omega^K$.
     - If $U = \sqcap_{j=1}^p \vec{e}_{K'}(V_j \to T_j)$, where $p \geq 1$ and for all $1 \leq j \leq p$, $M : \langle \Gamma, x^L : \vec{e}_{K'} V_j \vdash_3 \vec{e}_{K'} T_j \rangle$. So $\overline{e}_i U = \sqcap_{j=1}^p \vec{e}_K(V_j \to T_j)$ and by e, for all $1 \leq j \leq p$, $M^{+i} : \langle \overline{e}_i \Gamma, x^{i::L} : \vec{e}_K V_j \vdash_3 \vec{e}_K T_j \rangle$.

- Let $\dfrac{\lambda x^L.M : \langle \Gamma \vdash_3 U\rangle \quad \langle \Gamma \vdash_3 U\rangle \sqsubseteq \langle \Gamma' \vdash_3 U'\rangle}{\lambda x^L.M : \langle \Gamma' \vdash_3 U'\rangle}$. By lemma 4.1.3, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$ and by lemma 4.1.5 $\mathrm{d}(U) = \mathrm{d}(U') = K$. By IH, we have two cases:
  - If $U = \omega^K$, then, by lemma A.11.3a, $U' = \omega^K$.
  - If $U = \sqcap_{i=1}^{p} \vec{e}_K(V_i \to T_i)$, where $p \geq 1$ and for all $1 \leq i \leq p$ $M : \langle \Gamma, x^L : \vec{e}_K V_i \vdash_3 \vec{e}_K T_i\rangle$. By lemma A.11.3d:
    - Either $U' = \omega^K$.
    - Or $U' = \sqcap_{i=1}^{q} \vec{e}_K(V'_i \to T'_i)$, where $q \geq 1$ and $\forall 1 \leq i \leq q$, $\exists 1 \leq j_i \leq p$ such that $V'_i \sqsubseteq V_{j_i}$ and $T_{j_i} \sqsubseteq T'_i$. Let $1 \leq i \leq q$. Since, by lemma 4.1.3, $\langle \Gamma, x^L : \vec{e}_K V_{j_i} \vdash_3 \vec{e}_K T_{j_i}\rangle \sqsubseteq \langle \Gamma', x^L : \vec{e}_K V'_i \vdash_3 \vec{e}_K T'_i\rangle$, then $M : \langle \Gamma', x^L : \vec{e}_K V'_i \vdash_3 \vec{e}_K T'_i\rangle$.

3. Similar as the proof of 2.

4. By induction on the derivation $M\, x^L : \langle \Gamma, x^L : U \vdash_3 T\rangle$. We have two cases:

- Let $\dfrac{M : \langle \Gamma \vdash_3 V \to T\rangle \quad x^L : \langle (x^L : U) \vdash_3 V\rangle \quad \Gamma \diamond (x^L : U)}{M\, x^L : \langle \Gamma, (x^L : U) \vdash_3 T\rangle}$ (where, by 1. $U \sqsubseteq V$). Since $V \to T \sqsubseteq U \to T$, we have $M : \langle \Gamma \vdash_3 U \to T\rangle$.
- Let $\dfrac{M\ x^L : \langle \Gamma', (x^L : U') \vdash_3 V'\rangle \quad \langle \Gamma', (x^L : U') \vdash_3 V'\rangle \sqsubseteq \langle \Gamma, (x^L : U) \vdash_3 V\rangle}{M\ x^L : \langle \Gamma, (x^L : U) \vdash_3 V\rangle}$ (by lemma 4.1). By lemma 4.1, $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $V' \sqsubseteq V$. By IH, $M : \langle \Gamma' \vdash_3 U' \to V'\rangle$ and by $\sqsubseteq$, $M : \langle \Gamma \vdash_3 U \to V\rangle$.

$\square$

Lemma A.14. Let $i \in \{1, 2\}$ and $M : \langle \Gamma \vdash_i U\rangle$. We have:

1. If $M : \langle \Delta \vdash_i V\rangle$, then $dom(\Gamma) = dom(\Delta)$.
2. If $x^n : U_1 \in \Gamma$ and $y^m : U_2 \in \Gamma$, then:
   (a) If $x^n : U_1 \neq y^m : U_2$, then $x^n \neq y^m$.
   (b) If $x = y$, then $n = m$ and $U_1 = U_2$.
3. If $x^n : U_1 \in \Gamma$ and $y^m : U_2 \in \Gamma$ and $x^n : U_1 \neq y^m : U_2$, then $x \neq y$ and $x^n \neq y^m$.
4. Assume $N : \langle \Delta \vdash_i V\rangle$. We have $\Gamma \diamond \Delta$ iff $M \diamond N$.
5. If N is a subterm of M, then there are Δ, V such that $N : \langle \Delta \vdash_i V\rangle$.
6. If $\Gamma = \Gamma_1 \sqcap \Gamma_2 \sqcap \Gamma_3$, then $\Gamma_1 \diamond \Gamma_2$.

Proof:
1. Corollary of theorem 4.1.3a.
2. 2a. and 2a. are by induction on the derivation of $M : \langle \Gamma \vdash_i U\rangle$ using 1. in the $\sqcap_i$ case.
3. Corollary of 2.
4. Use theorem 4.1.3a.

5. By induction on the derivation of $M : \langle \Gamma \vdash_i U \rangle$.
6. Let $\Gamma = (x_i^{n_i} : U_i)_n$ and $x \in \mathcal{V}$. If $x^p \in dom(\Gamma_1) \subseteq dom(\Gamma)$ and $x^q \in dom(\Gamma_2) \subseteq dom(\Gamma)$, then by 2, p=q. $\square$

Proof:
[of Lemma 5.4]

- Case $\vdash_1$. By induction on the derivation of $M : \langle \Gamma \vdash_1 U \rangle$. The only interesting case is $\to_E$ where $M = (\lambda x^n.M_1)M_2$ is the subterm in question.
  Here, $\dfrac{\lambda x^n.M_1 : \langle \Gamma_1 \vdash_1 T_1 \to T_2 \rangle \quad M_2 : \langle \Gamma_2 \vdash T_1 \rangle \quad \Gamma_1 \diamond \Gamma_2}{(\lambda x^n.M_1)M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_1 T_2 \rangle}$.
  By Lemma 5.1.2, $M_1 : \langle \Gamma_1, x^n : T_1 \vdash_1 T_2 \rangle$. By lemma 4.1, $n = \mathrm{d}(T_1)$ and $\mathrm{d}(M_2) = \mathrm{d}(T_1)$. Hence, $n = \mathrm{d}(M_2)$ and $(\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2]$.

- Case $\vdash_2$. By lemma A.14, $(\lambda x^n.M_1)M_2$ is typable. By induction on the typing of $(\lambda x^n.M_1)M_2$. We consider only the rule $\to_E$: $\dfrac{\lambda x^n.M_1 : \langle \Gamma_1 \vdash_2 V \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash_2 V \rangle \quad \Gamma_1 \diamond \Gamma_2}{(\lambda x^n.M_1)M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle}$.
  By lemma 3.1.2, $\mathrm{d}(V \to T) = 0$. By Lemma 5.2.2, $V \to T = \sqcap_{i=1}^k (V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M_1 : \langle \Gamma_1, x^n : V_i \vdash_2 T_i \rangle$. Hence $k = 1$, $V_i = V$, $T_i = T$ and $M_1 : \langle \Gamma_1, x^n : V \vdash_2 T \rangle$. By lemma 4.1, V is good, $\mathrm{d}(M_2) = \mathrm{d}(V)$ and $\mathrm{d}(V) = n$. So, $\mathrm{d}(M_2) = n$ and $(\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2]$.

$\square$

Proof:
[of Lemma 5.5] By lemma 4.2.3, $\Gamma \diamond \Delta$.

1. Case $i = 2$. By induction on the derivation of $M : \langle \Gamma, x^n : U \vdash_2 V \rangle$.

   - If $\dfrac{T \ good}{x^0 : \langle (x^0 : T) \vdash_2 T \rangle}$ and $N : \langle \Delta \vdash_2 T \rangle$, then $N = x^0[x^0 := N] : \langle \Delta \vdash_2 T \rangle$.
   - Let $\dfrac{M : \langle \Gamma, x^n : U, y^m : U' \vdash_2 T \rangle}{\lambda y^m.M : \langle \Gamma, x^n : U \vdash_2 U' \to T \rangle}$. Since $\Gamma \diamond \Delta$, by BC, $\Gamma, y^m : U' \diamond \Delta$ and $y^m \notin dom(\Delta)$. By IH, $M[x^n := N] : \langle \Gamma \sqcap \Delta, y^m : U' \vdash_2 T \rangle$. By $\to_i$, $(\lambda y^m.M)[x^n := N] = \lambda y^m.M[x^n := N] : \langle \Gamma \sqcap \Delta \vdash_2 U' \to T \rangle$.
   - Let $\dfrac{M_1 : \langle \Gamma_1, x^n : U_1 \vdash_2 V \to T \rangle \quad M_2 : \langle \Gamma_2, x^n : U_2 \vdash_2 V \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2, x^n : U_1 \sqcap U_2 \vdash_2 T \rangle}$ where $x^n \in FV(M_1) \cap FV(M_2)$, $N : \langle \Delta \vdash_2 U_1 \sqcap U_2 \rangle$ and $(\Gamma_1 \sqcap \Gamma_2) \diamond \Delta$. By $\sqcap_E$ and $\sqsubseteq$, $N : \langle \Delta \vdash_2 U_1 \rangle$ and $N : \langle \Delta \vdash_2 U_2 \rangle$. Now use IH and $\to_E$.
     The cases $x^n \in FV(M_1) \setminus FV(M_2)$ or $x^n \in FV(M_2) \setminus FV(M_1)$ are easy.
   - If $\dfrac{M : \langle \Gamma, x^n : U \vdash_2 U_1 \rangle \quad M : \langle \Gamma, x^n : U \vdash_2 U_2 \rangle}{M : \langle \Gamma, x^n : U \vdash_2 U_1 \sqcap U_2 \rangle}$ use IH and $\sqcap_i$.
   - Let $\dfrac{M : \langle \Gamma, x^n : U \vdash_2 V \rangle}{M^+ : \langle e\Gamma, x^{n+1} : eU \vdash_2 eV \rangle}$ where $N : \langle \Delta \vdash_2 eU \rangle$ and $e\Gamma \diamond \Delta$. By lemma 4.1, $\mathrm{d}(N) = \mathrm{d}(eU) = \mathrm{d}(U) + 1 > 0$. Hence, by lemmas A.3.1 and 4.1.3, $N = P+$ and $P : \langle \Delta^- \vdash_2 U \rangle$. As $e\Gamma \diamond \Delta$, then $\Gamma \diamond \Delta^-$. By IH, $M[x^n := P] : \langle \Gamma \sqcap \Delta^- \vdash_2 V \rangle$. By e and lemma A.3.2, $M^+[x^{n+1} := N] : \langle e\Gamma \sqcap \Delta \vdash_2 eV \rangle$.

- Let $\dfrac{M : \langle \Gamma', x^n : U' \vdash_2 V' \rangle \quad \langle \Gamma', x^n : U' \vdash_2 V' \rangle \sqsubseteq \langle \Gamma, x^n : U \vdash_2 V \rangle}{M : \langle \Gamma, x^n : U \vdash_2 V \rangle}$ (note the use of lemma 4.1). By lemma 4.1, $dom(\Gamma) = dom(\Gamma')$, $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $V' \sqsubseteq V$. Hence $\Gamma' \diamond \Delta$, $N : \langle \Delta \vdash_2 U' \rangle$ and, by IH, $M[x^n := N] : \langle \Gamma' \sqcap \Delta \vdash_2 V' \rangle$. It is easy to show that $\Gamma \sqcap \Delta \sqsubseteq \Gamma' \sqcap \Delta$. Hence, $\langle \Gamma' \sqcap \Delta \vdash_2 V' \rangle \sqsubseteq \langle \Gamma \sqcap \Delta \vdash_2 V \rangle$ and $M[x^n := N] : \langle \Gamma \sqcap \Delta \vdash_2 V \rangle$.

2. Case $i = 3$. By lemma 4.1, $M, N \in \mathcal{M}_3$, $\mathrm{d}(N) = \mathrm{d}(U) = L$, $\mathrm{OK}(\Delta)$ and $\mathrm{OK}(\Gamma, x^L : U)$. By lemma 4.1.6, $\mathrm{OK}(\Gamma \sqcap \Delta)$. By lemma A.1.6, $M[x^L := N] \in \mathcal{M}_3$. By lemma 4.2.3a, $x^L \in \mathrm{FV}(M)$.

   We prove the lemma by induction on the derivation $M : \langle \Gamma, x^L : U \vdash_3 V \rangle$.

   - If $\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash_3 T \rangle}$ and $N : \langle \Delta \vdash_3 T \rangle$, then $x^{\oslash}[x^{\oslash} := N] = N : \langle \Delta \vdash_3 T \rangle$.
   - If $\dfrac{}{M : \langle env^{\omega}_{\mathrm{FV}(M)\setminus\{x^L\}}, (x^L : \omega^L) \vdash_3 \omega^{\mathrm{d}(M)} \rangle}$ and $N : \langle \Delta \vdash_3 \omega^L \rangle$ then by ω, $M[x^L := N] : \langle env^{\omega}_{M[x^L:=N]} \vdash_3 \omega^{\mathrm{d}(M[x^L:=N])} \rangle$. By lemma A.1.6 $\mathrm{d}(M[x^L := N]) = \mathrm{d}(M)$. Since $x^L \in \mathrm{FV}(M)$ (and so $\mathrm{FV}(M[x^L := N]) = (\mathrm{FV}(M) \setminus \{x^L\}) \cup \mathrm{FV}(N)$), by $\sqsubseteq$, $M[x^L := N] : \langle env^{\omega}_{\mathrm{FV}(M)\setminus\{x^L\}} \sqcap \Delta \vdash_3 \omega^{\mathrm{d}(M)} \rangle$.
   - Let $\dfrac{M : \langle \Gamma, x^L : U, y^K : U' \vdash_3 T \rangle}{\lambda y^K.M : \langle \Gamma, x^L : U \vdash_3 U' \to T \rangle}$ where $y^K \notin \mathrm{FV}(N) \cup \{x^L\}$.
     So $(\lambda y^K.M)[x^L := N] = \lambda y^K.M[x^L := N]$. By lemma A.1.2, $M \diamond N$. By IH, $M[x^L := N] : \langle \Gamma \sqcap \Delta, y^K : U' \vdash_3 T \rangle$. By $\to_I$, $(\lambda y^K.M)[x^L := N] : \langle \Gamma \sqcap \Delta \vdash_3 U' \to T \rangle$.
   - Let $\dfrac{M : \langle \Gamma, x^L : U \vdash_3 T \rangle \quad y^K \notin \mathrm{dom}(\Gamma, x^L : U)}{\lambda y^K.M : \langle \Gamma, x^L : U \vdash_3 \omega^K \to T \rangle}$ where $y^K \notin \mathrm{FV}(N) \cup \{x^L\}$.
     So $(\lambda y^K.M)[x^L := N] = \lambda y^K.M[x^L := N]$. By lemma A.1.2, $M \diamond N$. By lemma 4.2.3a, $\mathrm{FV}(N) = \mathrm{dom}(\Delta)$, so $y^K \notin \mathrm{dom}(\Delta)$. By IH, $M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash_3 T \rangle$. By $\to'_I$, $(\lambda y^K.M)[x^L := N] : \langle \Gamma \sqcap \Delta \vdash_3 \omega^K \to T \rangle$.
   - Let $\dfrac{M_1 : \langle \Gamma_1, x^L : U_1 \vdash_3 V \to T \rangle \quad M_2 : \langle \Gamma_2, x^L : U_2 \vdash_3 V \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2, x^L : U_1 \sqcap U_2 \vdash_3 T \rangle}$ (by lemma 4.2.3a) where $x^L \in \mathrm{FV}(M_1) \cap \mathrm{FV}(M_2)$, $N : \langle \Delta \vdash_3 U_1 \sqcap U_2 \rangle$. By lemma A.1.2, $M_1 \diamond N$ and $M_2 \diamond N$. By $\sqcap_E$ and $\sqsubseteq$, $N : \langle \Delta \vdash_3 U_1 \rangle$ and $N : \langle \Delta \vdash_3 U_2 \rangle$. Now use IH and $\to_E$ (using the fact that $\Gamma_1 \sqcap \Delta \diamond \Gamma_2 \sqcap \Delta$, by lemma 4.2.3a and lemma A.1.4).
     The cases $x^L \in \mathrm{FV}(M_1) \setminus \mathrm{FV}(M_2)$ or $x^L \in \mathrm{FV}(M_2) \setminus \mathrm{FV}(M_1)$ are similar.
   - If $\dfrac{M : \langle \Gamma, x^L : U \vdash_3 U_1 \rangle \quad M : \langle \Gamma, x^L : U \vdash_3 U_2 \rangle}{M : \langle \Gamma, x^L : U \vdash_3 U_1 \sqcap U_2 \rangle}$ use IH and $\sqcap_I$.
   - Let $\dfrac{M : \langle \Gamma, x^L : U \vdash_3 V \rangle}{M^{+i} : \langle \overline{e}_i\Gamma, x^{i::L} : \overline{e}_i U \vdash_3 \overline{e}_i V \rangle}$ and $N : \langle \Delta \vdash_3 \overline{e}_i U \rangle$. By lemma 4.1.2, $\mathrm{d}(M) = \mathrm{d}(\overline{e}_i U) = i :: \mathrm{d}(U)$. By lemma 4.1.3, $N^{-i} : \langle \Delta^{-i} \vdash_3 U \rangle$. By lemma A.5.7 and lemma A.5.2, $(N^{-i})^{+i} = N$ and $M \diamond N^{-i}$. By IH, $M[x^L := N^{-i}] : \langle \Gamma \sqcap \Delta^{-i} \vdash_3 V \rangle$. By e and lemma A.5.5, $M^{+i}[x^{i::L} := N] : \langle \overline{e}_i\Gamma \sqcap \Delta \vdash_3 \overline{e}_i V \rangle$.
   - Let $\dfrac{M : \langle \Gamma', x^L : U' \vdash_3 V' \rangle \quad \langle \Gamma', x^L : U' \vdash_3 V' \rangle \sqsubseteq \langle \Gamma, x^L : U \vdash_3 V \rangle}{M : \langle \Gamma, x^L : U \vdash_3 V \rangle}$ (lemma 4.1).
     By lemma 4.1, $\mathrm{dom}(\Gamma) = \mathrm{dom}(\Gamma')$, $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $V' \sqsubseteq V$. Hence $N : \langle \Delta \vdash_3 U' \rangle$

and, by IH, $M[x^L := N] : \langle \Gamma' \sqcap \Delta \vdash_3 V' \rangle$. It is easy to show that $\Gamma \sqcap \Delta \sqsubseteq \Gamma' \sqcap \Delta$. Hence, $\langle \Gamma' \sqcap \Delta \vdash_3 V' \rangle \sqsubseteq \langle \Gamma \sqcap \Delta \vdash_3 V \rangle$ and $M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash_3 V \rangle$.

$\square$

Lemma A.15. If $M : \langle \Gamma \vdash_2 U \rangle$ and $M \rhd_\beta N$, then $N : \langle \Gamma \vdash_2 U \rangle$.

Proof:
By induction on the derivation of $M : \langle \Gamma \vdash_2 U \rangle$. $\to_i$, $\sqcap_i$ and $\sqsubseteq$ are by IH. We give the remaining two cases.

- Let $\dfrac{M_1 : \langle \Gamma_1 \vdash_2 U \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash_2 U \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle}$. For the cases $N = M_1 N_2$ where $M_2 \rhd_\beta N_2$ or $N = N_1 M_2$ where $M_1 \rhd_\beta N_1$ use IH. Assume $M_1 = \lambda x^n.P$ and $M_1 M_2 = (\lambda x^n.P) M_2 \rhd_\beta P[x^n := M_2] = N$ where $\mathrm{d}(M_2) = n$. Since $\lambda x^n.P : \langle \Gamma_1 \vdash_2 U \to T \rangle$ and, by lemma 3.1.2.2a $\mathrm{d}(U \to T) = 0$, then, by lemma 5.2.2, $P : \langle \Gamma_1, x^n : U \vdash_2 T \rangle$. By lemma 5.5, $P[x^n := M_2] : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle$.

- Let $\dfrac{M : \langle \Gamma \vdash_2 U \rangle}{M^+ : \langle e\Gamma \vdash_2 eU \rangle}$. If $M^+ \rhd_\beta N$, then by lemma 2.1.2, $\mathrm{d}(M^+) = \mathrm{d}(N)$. By lemmas A.3.1a and A.5.7, $\mathrm{d}(N) > 0$, $N = P^+$ and $M \rhd_\beta P$. By IH, $P : \langle \Gamma \vdash_2 U \rangle$ and, by exp, $N : \langle e\Gamma \vdash_2 eU \rangle$.

$\square$

The next lemma will be used in the proof of subject expansion for β.

Lemma A.16. Let $(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U \rangle$ then $\Gamma = \Gamma_1 \sqcap \Gamma_2$ and $\exists V \in \mathbb{U}$ such that $M_1 : \langle \Gamma_1, (x^n : V) \vdash_2 U \rangle$ and $M_2 : \langle \Gamma_2 \vdash_2 V \rangle$.

Proof:
By induction on the derivation of $(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U \rangle$.

- Let $\dfrac{\lambda x^n.M_1 : \langle \Gamma_1 \vdash_2 V \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash_2 V \rangle \quad \Gamma_1 \diamond \Gamma_2}{(\lambda x^n.M_1)M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle}$.
Since $\mathrm{d}(V \to T) = 0$, by lemma 5.2.2 $M_1 : \langle \Gamma_1, (x^n : V) \vdash_2 T \rangle$.

- Let $\dfrac{(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U_1 \rangle \quad (\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U_2 \rangle}{(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U_1 \sqcap U_2 \rangle}$. By IH, $\Gamma = \Gamma_1 \sqcap \Gamma_2 = \Gamma'_1 \sqcap \Gamma'_2$, $\exists V, V' \in \mathbb{U}$, such that $M_1 : \langle \Gamma_1, (x^n : V) \vdash_2 U_1 \rangle$, $M_2 : \langle \Gamma_2 \vdash_2 V \rangle$, $M_1 : \langle \Gamma'_1, (x^n : V') \vdash_2 U_2 \rangle$ and $M_2 : \langle \Gamma'_2 \vdash_2 V' \rangle$. By lemma 4.1.2, $\Gamma_1, \Gamma_2, \Gamma'_1, \Gamma'_2, V$ and V' are all good. By lemma 4.1.3a, $dom(\Gamma_1, (x^n : V)) = \mathrm{FV}(M_1) = dom(\Gamma'_1, (x^n : V'))$ so $dom(\Gamma_1) = dom(\Gamma'_1)$ and $dom(\Gamma_2) = FV(M_2) = dom(\Gamma'_2)$. Hence,by $\sqcap_E$, and lemma 4.1, $\Gamma_1 \sqcap \Gamma'_1, (x^n : V \sqcap V') \sqsubseteq \Gamma_1, (x^n : V)$, $\Gamma_1 \sqcap \Gamma'_1, (x^n : V \sqcap V') \sqsubseteq \Gamma'_1, (x^n : V')$, $\Gamma_2 \sqcap \Gamma'_2 \sqsubseteq \Gamma_2$ and $\Gamma_2 \sqcap \Gamma'_2 \sqsubseteq \Gamma'_2$. By lemma 4.1.3 and $\sqsubseteq$, $M_1 : \langle \Gamma_1 \sqcap \Gamma'_1, (x^n : V \sqcap V') \vdash_2 U_1 \rangle$, $M_1 : \langle \Gamma_1 \sqcap \Gamma'_1, (x^n : V \sqcap V') \vdash_2 U_2 \rangle$, $M_2 : \langle \Gamma_2 \sqcap \Gamma'_2 \vdash_2 V \rangle$ and $M_2 : \langle \Gamma_2 \sqcap \Gamma'_2 \vdash_2 V' \rangle$. So by $\sqcap_i$, $M_1 : \langle \Gamma_1 \sqcap \Gamma'_1, (x^n : V \sqcap V') \vdash_2 U_1 \sqcap U_2 \rangle$ and $M_2 : \langle \Gamma_2 \sqcap \Gamma'_2 \vdash_2 V \sqcap V' \rangle$.

- Let $\dfrac{(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U \rangle}{(\lambda x^{n+1}.M_1^+)M_2^+ : \langle e\Gamma \vdash_2 eU \rangle}$. By IH, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ and $\exists V \in \mathbb{U}$, such that $M_1 : \langle \Gamma_1, (x^n : V) \vdash_2 U \rangle$ and $M_2 : \langle \Gamma_2 \vdash_2 V \rangle$. So by exp, $M_1^+ : \langle e\Gamma_1, (x^{n+1} : eV) \vdash_2 eU \rangle$ and $M_2^+ : \langle e\Gamma_2 \vdash_2 eV \rangle$.

- Let $\dfrac{(\lambda x^n.M_1)M_2 : \langle \Gamma' \vdash_2 U' \rangle \quad \langle \Gamma' \vdash_2 U' \rangle \sqsubseteq \langle \Gamma \vdash_2 U \rangle}{(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U \rangle}$. By lemma 4.1.3, $\Gamma \sqsubseteq \Gamma'$ and $U' \sqsubseteq U$. By IH, $\Gamma' = \Gamma'_1 \sqcap \Gamma'_2$ and $\exists V \in \mathbb{U}$, such that $M_1 : \langle \Gamma'_1, (x^n : V) \vdash_2 U' \rangle$ and $M_2 : \langle \Gamma'_2 \vdash_2 V \rangle$. By lemma A.10.8, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ such that $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$. So by $\sqsubseteq$, $M_1 : \langle \Gamma_1, (x^n : V) \vdash_2 U \rangle$ and $M_2 : \langle \Gamma_2 \vdash_2 V \rangle$.

$\square$

Now, we give the basic block in the proof of subject expansion for β.

Lemma A.17. If $N : \langle \Gamma \vdash_2 U \rangle$ and $M \rhd_\beta N$ then $M : \langle \Gamma \vdash_2 U \rangle$

Proof:
By induction on the derivation of $N : \langle \Gamma \vdash_2 U \rangle$.

- Let $\dfrac{T \ good}{x^0 : \langle (x^0 : T) \vdash_2 T \rangle}$ where $M \rhd_\beta x^0$. By cases on M, we can show that $M = (\lambda y^0.y^0)x^0$. Since T is good, by ax, $y^0 : \langle (y^0 : T) \vdash_2 T \rangle$, then by $\to_i$, $\lambda y^0.y^0 : \langle () \vdash_2 T \to T \rangle$, and so by $\to_E$, $(\lambda y^0.y^0)x^0 : \langle (x^0 : T) \vdash_2 T \rangle$.

- Let $\dfrac{N : \langle \Gamma, (x^n : U) \vdash_2 T \rangle}{\lambda x^n.N : \langle \Gamma \vdash_2 U \to T \rangle}$ where $M \rhd_\beta \lambda x^n.N$. By cases on M.

  - If M is a variable this is not possible.
  - If $M = \lambda x^n.M'$ such that $M' \rhd_\beta N$ and $x^n \in FV(M') \cap FV(N)$ then by IH, $M : \langle \Gamma, (x^n : U) \vdash_2 T \rangle$ and by $\to_i$, $M : \langle \Gamma \vdash_2 U \to T \rangle$.
  - If M is an application term then the reduction must be at the root. Hence, $M = (\lambda y^m.M_1)M_2 \rhd_\beta M_1[y^m := M_2] = \lambda x^n.N$ where $y^m \in FV(M_1)$. There are two cases (M_1 cannot be an application term):

    * If $M_1 = y^m$ then $M_2 = \lambda x^n.N$ and $\mathrm{d}(N) = m$. By lemma 4.1.2, $m = \mathrm{d}(N) = \mathrm{d}(T) = 0$. So $M = (\lambda y^0.y^0)(\lambda x^n.N)$. Since by lemma 4.1.2, $U \to T$ is good , by ax, $y^0 : \langle (y^0 : U \to T) \vdash_2 U \to T \rangle$, then by $\to_i$, $\lambda y^0.y^0 : \langle () \vdash_2 (U \to T) \to (U \to T) \rangle$, and so by $\to_E$, $(\lambda y^0.y^0)(\lambda x^n.N) : \langle \Gamma \vdash_2 U \to T \rangle$.
    * If $M_1 = \lambda x^n.M'_1$ then $M_1[y^m := M_2] = \lambda x^n.M'_1[y^m := M_2] = \lambda x^n.N$ and $\mathrm{d}(M_2) = m$. Since $(\lambda y^m.M'_1)M_2 \rhd_\beta M'_1[y^m := M_2] = N$, by IH, $(\lambda y^m.M'_1)M_2 : \langle \Gamma, (x^n : U) \vdash_2 T \rangle$. By lemma A.16, $\Gamma, (x^n : U) = \Gamma_1 \sqcap \Gamma_2$ and $\exists V \in \mathbb{U}$ such that $M'_1 : \langle \Gamma_1, (y^m : V) \vdash_2 T \rangle$ and $M_2 : \langle \Gamma_2 \vdash_2 V \rangle$. Since $M \in \mathcal{M}_2$, $y^m \in FV(M'_1)$ and so (since $x^n \notin FV(M_2)$), by lemma A.14 $\Gamma = \Gamma'_1 \sqcap \Gamma_2$ and $\Gamma_1 = \Gamma'_1, (x^n : U)$. Hence by $\to_i$, $\lambda x^n.M'_1 : \langle \Gamma'_1, (y^m : V) \vdash_2 U \to T \rangle$, again by $\to_i$, $\lambda y^m.\lambda x^n.M'_1 : \langle \Gamma'_1 \vdash_2 V \to U \to T \rangle$, and since by lemma A.14.6, $\Gamma'_1 \diamond \Gamma_2$, by $\to_E$, $M = (\lambda y^m.\lambda x^n.M'_1)M_2 : \langle \Gamma \vdash_2 U \to T \rangle$.

- Let $\dfrac{N_1 : \langle \Gamma_1 \vdash_2 U \to T \rangle \quad N_2 : \langle \Gamma_2 \vdash_2 U \rangle \qquad \Gamma_1 \diamond \Gamma_2}{N_1 N_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle}$ and $M \rhd_\beta N_1 N_2$.

  - If $M = M_1 N_2 \rhd_\beta N_1 N_2$ where $M_1 \diamond N_2$, $N_1 \diamond N_2$ and $M_1 \rhd_\beta N_1$ then by IH, $M_1 : \langle \Gamma_1 \vdash_2 U \to T \rangle$, and by $\to_E$, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle$.

- If $M = N_1 M_2 \rhd_\beta N_1 N_2$ where $N_1 \diamond M_2$, $N_1 \diamond N_2$ and $M_2 \rhd_\beta N_2$ then by IH, $M_2 : \langle \Gamma_2 \vdash_2 U \rangle$, and by $\to_E$, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle$.
- If $M = (\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2] = N_1 N_2$ where $\mathrm{d}(M_2) = n$ and $x^n \in FV(M_1)$. By cases on M_1 (M_1 cannot be an abstraction):
  - ∗ If $M_1 = x^n$ then $M_2 = N_1 N_2$, $\mathrm{d}(N_1 N_2) = n$ and $M = (\lambda x^0.x^0)(N_1 N_2)$. By lemma 4.1, $n = 0$ and T is good. By ax, $x^0 : \langle (x^0 : T) \vdash_2 T \rangle$, hence by $\to_i$, $\lambda x^0.x^0 : \langle () \vdash_2 T \to T \rangle$, and by $\to_E$, $(\lambda x^0.x^0)(N_1 N_2) : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle$.
  - ∗ If $M_1 = M_1' M_1''$ then $M_1[x^n := M_2] = M_1'[x^n := M_2] M_1''[x^n := M_2] = N_1 N_2$. So, $M_1'[x^n := M_2] = N_1$ and $M_1''[x^n := M_2] = N_2$.
    - · If $x^n \in FV(M_1')$ and $x^n \in FV(M_1'')$ then $(\lambda x^n.M_1')M_2 \rhd_\beta N_1$ and $(\lambda x^n.M_1'')M_2 \rhd_\beta N_2$. By IH, $(\lambda x^n.M_1')M_2 : \langle \Gamma_1 \vdash_2 U \to T \rangle$ and $(\lambda x^n.M_1'')M_2 : \langle \Gamma_2 \vdash_2 U \rangle$. By lemma A.16 twice, $\Gamma_1 = \Gamma_1' \sqcap \Gamma_1''$, $\Gamma_2 = \Gamma_2' \sqcap \Gamma_2''$, and $\exists V, V' \in \mathbb{U}$ such that $M_1' : \langle \Gamma_1', (x^n : V) \vdash_2 U \to T \rangle$, $M_2 : \langle \Gamma_1'' \vdash_2 V \rangle$, $M_1'' : \langle \Gamma_2', (x^n : V') \vdash_2 U \rangle$ and $M_2 : \langle \Gamma_2'' \vdash_2 V' \rangle$. By lemma 4.1.2, $\Gamma_1', \Gamma_2', \Gamma_1'', \Gamma_2'', V$ and V' are all good. By lemma A.14.1, $dom(\Gamma_1'') = FV(M_2) = dom(\Gamma_2'')$. Hence, by $\sqcap_E$, lemma 4.1, $\sqsubseteq$ and $\sqcap_i$, $M_2 : \langle \Gamma_1'' \sqcap \Gamma_2'' \vdash_2 V \sqcap V' \rangle$. Since by lemma A.14.6, $\Gamma_1' \diamond \Gamma_2'$, by $\to_E$, $M_1' M_1'' : \langle \Gamma_1' \sqcap \Gamma_2', (x^n : V \sqcap V') \vdash_2 T \rangle$. So by $\to_i$, $\lambda x^n.M_1' M_1'' : \langle \Gamma_1' \sqcap \Gamma_2' \vdash_2 (V \sqcap V') \to T \rangle$. Finally, by $\to_E$ and since by lemma A.14.6, $\Gamma_1' \sqcap \Gamma_2' \diamond \Gamma_1'' \sqcap \Gamma_2''$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1' \sqcap \Gamma_2' \sqcap \Gamma_1'' \sqcap \Gamma_2''$, $(\lambda x^n.M_1' M_1'')M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle$.
    - · If $x^n \in FV(M_1')$ and $x^n \notin FV(M_1'')$ then $M_1'[x^n := M_2] = N_1$ and $M_1'' = N_2$. We have $(\lambda x^n.M_1')M_2 \rhd_\beta N_1$, so by IH, $(\lambda x^n.M_1')M_2 : \langle \Gamma_1 \vdash_2 U \to T \rangle$. By lemma A.16, $\Gamma_1 = \Gamma_1' \sqcap \Gamma_1''$ and $\exists V \in \mathbb{U}$ such that $M_1' : \langle \Gamma_1', (x^n : V) \vdash_2 U \to T \rangle$ and $M_2 : \langle \Gamma_1'' \vdash_2 V \rangle$. Since by lemma A.14.6, $\Gamma_1' \diamond \Gamma_2$, by $\to_E$, $M_1' M_1'' : \langle \Gamma_1' \sqcap \Gamma_2, (x^n : V) \vdash_2 T \rangle$, and by $\to_i$, $\lambda x^n.M_1' M_1'' : \langle \Gamma_1' \sqcap \Gamma_2 \vdash_2 V \to T \rangle$. Finally, by $\to_E$ and since by lemma A.14.6, $\Gamma_1' \sqcap \Gamma_2 \diamond \Gamma_1''$, $(\lambda x^n.M_1' M_1'')M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle$.
    - · If $x^n \notin FV(M_1')$ and $x^n \in FV(M_1'')$ then the proof is similar to the previous case.

- Let $\dfrac{N : \langle \Gamma \vdash_2 U_1 \rangle \quad N : \langle \Gamma \vdash_2 U_2 \rangle}{N : \langle \Gamma \vdash_2 U_1 \sqcap U_2 \rangle}$ and $M \rhd_\beta N$. By IH, $M : \langle \Gamma \vdash_2 U_1 \rangle$ and $M : \langle \Gamma \vdash_2 U_2 \rangle$, hence by $\sqcap_i$, $M : \langle \Gamma \vdash_2 U_1 \sqcap U_2 \rangle$.
- Let $\dfrac{N : \langle \Gamma \vdash_2 U \rangle}{N^+ : \langle e\Gamma \vdash_2 eU \rangle}$ and $M \rhd_\beta N^+$. By lemma A.5.8, $M^- \rhd_\beta N$, and by IH, $M^- : \langle \Gamma \vdash_2 U \rangle$. By lemma A.3.1b, $(M^-)^+ = M$ and by exp, $M : \langle e\Gamma \vdash_2 eU \rangle$.
- Let $\dfrac{N : \langle \Gamma \vdash_2 U \rangle \quad \langle \Gamma \vdash_2 U \rangle \sqsubseteq \langle \Gamma' \vdash_2 U' \rangle}{N : \langle \Gamma' \vdash_2 U' \rangle}$ and $M \rhd_\beta N$. By IH, $M : \langle \Gamma \vdash_2 U \rangle$ and by $\sqsubseteq$ $M : \langle \Gamma' \vdash_2 U' \rangle$.

□

Proof:
[of lemma 5.7]

1. By induction on the length of the derivation of $M \rhd_\beta^* N$ using lemma A.15.

2. By induction on the length of the derivation of $M \rhd_\beta^* N$ using lemma A.17.

$\square$

The next lemma is needed in the proofs.

Lemma A.18. 1. If $\mathrm{FV}(N) \subseteq \mathrm{FV}(M)$, then $env_\omega^M \upharpoonright_N = env_\omega^N$.

2. If $\mathrm{OK}(\Gamma_1)$, $\mathrm{OK}(\Gamma_2)$, $\mathrm{FV}(M) \subseteq \mathrm{dom}(\Gamma_1)$ and $\mathrm{FV}(N) \subseteq \mathrm{dom}(\Gamma_2)$, then $(\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{MN} \sqsubseteq (\Gamma_1 \upharpoonright_M) \sqcap (\Gamma_2 \upharpoonright_N)$.
3. $\overline{e}_i(\Gamma \upharpoonright_M) = (\overline{e}_i\Gamma) \upharpoonright_{M^{+i}}$

Proof:

1. Easy.
2. First, note that $\mathrm{OK}(\Gamma_1 \sqcap \Gamma_2)$ by lemma 4.1.6, $\mathrm{OK}(\Gamma_1 \upharpoonright_M)$, $\mathrm{OK}(\Gamma_2 \upharpoonright_N)$ and $\mathrm{dom}((\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{MN}) = \mathrm{FV}(MN) = \mathrm{FV}(M) \cup \mathrm{FV}(N) = \mathrm{dom}(\Gamma_1 \upharpoonright_M) \cup \mathrm{dom}(\Gamma_2 \upharpoonright_N) = \mathrm{dom}((\Gamma_1 \upharpoonright_M) \sqcap (\Gamma_2 \upharpoonright_N))$. Now, we show by cases that if $(x^L : U_1) \in (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{MN}$ and $(x^L : U_2) \in (\Gamma_1 \upharpoonright_M) \sqcap (\Gamma_2 \upharpoonright_N)$ then $U_1 \sqsubseteq U_2$:
   - If $x^L \in \mathrm{FV}(M) \cap \mathrm{FV}(N)$ then $(x^L : U_1') \in \Gamma_1$, $(x^L : U_1'') \in \Gamma_2$ and $U_1 = U_1' \sqcap U_1'' = U_2$.
   - If $x^L \in \mathrm{FV}(M) \setminus \mathrm{FV}(N)$ then
     - If $x^L \in \mathrm{dom}(\Gamma_2)$ then $(x^L : U_2) \in \Gamma_1$, $(x^L : U_1') \in \Gamma_2$ and $U_1 = U_1' \sqcap U_2 \sqsubseteq U_2$.
     - If $x^L \notin \mathrm{dom}(\Gamma_2)$ then $(x^L : U_2) \in \Gamma_1$ and $U_1 = U_2$.
   - If $x^L \in \mathrm{FV}(N) \setminus \mathrm{FV}(M)$ then
     - If $x^L \in \mathrm{dom}(\Gamma_1)$ then $(x^L : U_1') \in \Gamma_1$, $(x^L : U_2) \in \Gamma_2$ and $U_1 = U_1' \sqcap U_2 \sqsubseteq U_2$.
     - If $x^L \notin \mathrm{dom}(\Gamma_1)$ then $x^L : U_2 \in \Gamma_2$ and $U_1 = U_2$.
3. Let $\Gamma = (x_j^{L_j} : U_j)_n$ and let $\mathrm{FV}(M) = \{y_1^{K_1}, \ldots, y_m^{K_m}\}$ where $m \leq n$ and $\forall 1 \leq k \leq m$ $\exists 1 \leq j \leq n$ such that $y_k^{K_k} = x_j^{L_j}$. So $\Gamma \upharpoonright_M = (y_k^{K_k} : U_k)_m$ and $\overline{e}_i(\Gamma \upharpoonright_M) = (y_k^{i::K_k} : \overline{e}_i U_k)_m$. Since $\overline{e}_i\Gamma = (x_j^{i::L_j} : \overline{e}_i U_j)_n$, $\mathrm{FV}(M^{+i}) = \{y_1^{i::K_1}, \ldots, y_m^{i::K_m}\}$ and $\forall 1 \leq k \leq m$ $\exists 1 \leq j \leq n$ such that $y_k^{i::K_k} = x_j^{i::L_j}$ then $(\overline{e}_i\Gamma) \upharpoonright_{M^{+i}} = (y_k^{i::K_k} : U_k)_m$.

$\square$

The next two theorems are needed in the proof of subject reduction.

Theorem A.1. If $M : \langle \Gamma \vdash_3 U \rangle$ and $M \rhd_\beta N$, then $N : \langle \Gamma \upharpoonright_N \vdash_3 U \rangle$.

Proof:
By induction on the derivation $M : \langle \Gamma \vdash_3 U \rangle$.

- Rule ω follows by theorem 2.1.2 and lemma A.18.1.

- If $\dfrac{M : \langle \Gamma, (x^L : U) \vdash_3 T\rangle}{\lambda x^L.M : \langle \Gamma \vdash_3 U \to T\rangle}$ then $N = \lambda x^L.N'$ and $M \rhd_\beta N'$. By IH, $N' : \langle (\Gamma, (x^L : U)) \upharpoonright_{N'} \vdash_3 T\rangle$. If $x^L \in \mathrm{FV}(N')$ then $N' : \langle \Gamma \upharpoonright_{\mathrm{FV}(N')\setminus\{x^L\}}, (x^L : U) \vdash_3 T\rangle$ and by $\to_I$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash_3 U \to T\rangle$. Else $N' : \langle \Gamma \upharpoonright_{\mathrm{FV}(N')\setminus\{x^L\}} \vdash_3 T\rangle$ so by $\to'_I$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash_3 \omega^L \to T\rangle$ and since by lemma 4.1.2 and lemma 4.1.4, $U \sqsubseteq \omega^L$, by $\sqsubseteq$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash_3 U \to T\rangle$.

- If $\dfrac{M : \langle \Gamma \vdash_3 T\rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash_3 \omega^L \to T\rangle}$ then $N = \lambda x^L N'$ and $M \rhd_\beta N'$. Since $x^L \notin \mathrm{FV}(M)$, by theorem 2.1.2, $x^L \notin \mathrm{FV}(N')$. By IH, $N' : \langle \Gamma \upharpoonright_{\mathrm{FV}(N')\setminus\{x^L\}} \vdash_3 T\rangle$ so by $\to'_I$, $\lambda x^L.N' : \langle \Gamma \upharpoonright_{\lambda x^L.N'} \vdash_3 \omega^L \to T\rangle$.

- If $\dfrac{M_1 : \langle \Gamma_1 \vdash_3 U \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash_3 U\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_3 T\rangle}$. Using lemma A.18.2, case $M_1 \rhd_\beta N_1$ and $N = N_1 M_2$ and case $M_2 \rhd_\beta N_2$ and $N = M_1 N_2$ are easy. Let $M_1 = \lambda x^L.M'_1$ and $N = M'_1[x^L := M_2]$. By lemma 4.2.3 and lemma A.1.2, $M'_1 \diamond M_2$. If $x^L \in FV(M'_1)$ then by lemma 5.2.2, $M'_1 : \langle \Gamma_1, x^L : U \vdash_3 T\rangle$. By lemma 5.5, $M'_1[x^L := M_2] : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_3 T\rangle$. If $x^L \notin FV(M'_1)$ then by lemma 5.3.3, $M'_1[x^L := M_2] = M'_1 : \langle \Gamma_1 \vdash_3 T\rangle$ and by $\sqsubseteq$, $N : \langle (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_N \vdash_3 T\rangle$.

- Case $\sqcap_I$ is by IH.

- If $\dfrac{M : \langle \Gamma \vdash_3 U\rangle}{M^{+i} : \langle \overline{e}_i \Gamma \vdash_3 \overline{e}_i U\rangle}$ and $M^{+i} \rhd_\beta N$, then by lemma A.5.9, there is $P \in \mathcal{M}_3$ such that $P^{+i} = N$ and $M \rhd_\beta P$. By IH, $P : \langle \Gamma \upharpoonright_P \vdash_3 U\rangle$ and by e and lemma A.18.3, $N : \langle (\overline{e}_i \Gamma) \upharpoonright_N \vdash_3 \overline{e}_i U\rangle$.

- If $\dfrac{M : \langle \Gamma \vdash_3 U\rangle \qquad \langle \Gamma \vdash_3 U\rangle \sqsubseteq \langle \Gamma' \vdash_3 U'\rangle}{M : \langle \Gamma' \vdash_3 U'\rangle}$ then by IH, lemma 4.1.3 and $\sqsubseteq$, $N : \langle \Gamma' \upharpoonright_N \vdash_3 U'\rangle$.

$\square$

Theorem A.2. If $M : \langle \Gamma \vdash_3 U\rangle$ and $M \rhd_\eta N$, then $N : \langle \Gamma \vdash_3 U\rangle$.

Proof:
By induction on the derivation $M : \langle \Gamma \vdash_3 U\rangle$.

- If $\dfrac{}{M : \langle env^\omega_M \vdash_3 \omega^{\mathrm{d}(M)}\rangle}$ then by lemma 2.1.1, $\mathrm{d}(M) = \mathrm{d}(N)$ and $\mathrm{FV}(M) = \mathrm{FV}(N)$ and by ω, $N : \langle env^\omega_M \vdash_3 \omega^{\mathrm{d}(M)}\rangle$.

- If $\dfrac{M : \langle \Gamma, (x^L : U) \vdash_3 T\rangle}{\lambda x^L.M : \langle \Gamma \vdash_3 U \to T\rangle}$ then we have two cases:
  - $M = N x^L$ and so by lemma 5.3.4, $N : \langle \Gamma \vdash_3 U \to T\rangle$.
  - $N = \lambda x^L.N'$ and $M \rhd_\eta N'$. By IH, $N' : \langle \Gamma, (x^L : U) \vdash_3 T\rangle$ and by $\to_I$, $N : \langle \Gamma \vdash_3 U \to T\rangle$.

- if $\dfrac{M : \langle \Gamma \vdash_3 T\rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash_3 \omega^L \to T\rangle}$ then $N = \lambda x^L.N'$ and $M \rhd_\eta N'$. By IH, $N' : \langle \Gamma \vdash_3 T\rangle$ and by $\to'_I$, $N : \langle \Gamma \vdash_3 \omega^L \to T\rangle$.

- If $\dfrac{M_1 : \langle \Gamma_1 \vdash_3 U \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash_3 U\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_3 T\rangle}$, then we have two cases:
  - $M_1 \rhd_\eta N_1$ and $N = N_1M_2$. By IH $N_1 : \langle \Gamma_1 \vdash_3 U \to T\rangle$ and by $\to_E$, $N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_3 T\rangle$.
  - $M_2 \rhd_\eta N_2$ and $N = M_1N_2$. By IH $N_2 : \langle \Gamma_2 \vdash_3 U\rangle$ and by $\to_E$, $N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_3 T\rangle$.
- Case $\sqcap_I$ is by IH and $\sqcap_I$.
- If $\dfrac{M : \langle \Gamma \vdash_3 U\rangle}{M^{+i} : \langle \overline{e}_i\Gamma \vdash_3 \overline{e}_iU\rangle}$ then by lemma A.5.9, there is $P \in \mathcal{M}_3$ such that $P^{+i} = N$ and $M \rhd_\eta P$. By IH, $P : \langle \Gamma \vdash_3 U\rangle$ and by e, $N : \langle \overline{e}_i\Gamma \vdash_3 \overline{e}_iU\rangle$.
- If $\dfrac{M : \langle \Gamma \vdash_3 U\rangle \qquad \langle \Gamma \vdash_3 U\rangle \sqsubseteq \langle \Gamma' \vdash_3 U'\rangle}{M : \langle \Gamma' \vdash_3 U'\rangle}$ then by IH, lemma 4.1.3 and $\sqsubseteq$, $N : \langle \Gamma' \vdash_3 U'\rangle$.

□

The next auxiliary lemma is needed in proofs.

Lemma A.19. Let $i \in \{1, 2\}$ and $M : \langle \Gamma \vdash_3 U\rangle$. We have:

1. If $(x^L : U_1) \in \Gamma$ and $(y^K : U_2) \in \Gamma$, then:
   (a) If $(x^L : U_1) \neq (y^K : U_2)$, then $x^L \neq y^K$.
   (b) If $x = y$, then $L = K$ and $U_1 = U_2$.
2. If $(x^L : U_1) \in \Gamma$ and $(y^K : U_2) \in \Gamma$ and $(x^L : U_1) \neq (y^K : U_2)$, then $x \neq y$ and $x^L \neq y^K$.

Proof:
1. If $x^L = Y^K$ then by definition $U_1 = U_2$, so $(x^L : U_1) = (y^K : U_2)$. By lemma 4.2.3a, $x^L, y^K \in \mathrm{FV}(M)$. By lemma A.1.1, $M \diamond M$. So, if $x = y$ then $L = K$ and by definition $U_1 = U_2$.
2. Corollary of 1. □ □

Proof:
[Of theorem 5.1] Proofs are by induction on derivations using theorem A.1 and theorem A.2. □

Proof:
[Of lemma 5.8] By lemma 4.1.2, $M[x^L := N] \in \mathcal{M}_3$, so by definition, $M, N \in \mathcal{M}_3$ and $M \diamond N$ and $\mathrm{d}(N) = L$. By induction on the derivation $M[x^L := N] : \langle \Gamma \vdash_3 U\rangle$.

- If $\dfrac{}{y^\oslash : \langle (y^\oslash : T) \vdash_3 T\rangle}$ then $M = x^\oslash$ and $N = y^\oslash$. By ax, $x^\oslash : \langle (x^\oslash : T) \vdash_3 T\rangle$.
- If $\dfrac{}{M[x^L := N] : \langle env^\omega_{M[x^L:=N]} \vdash_3 \omega^{\mathrm{d}(M[x^L:=N])}\rangle}$ then by lemma A.1.6b, $\mathrm{d}(M) = \mathrm{d}(M[x^L := N])$. By ω, $M : \langle env^\omega_{\mathrm{FV}(M)\setminus\{x^L\}}, (x^L : \omega^L) \vdash_3 \omega^{\mathrm{d}(M)}\rangle$ and $N : \langle env^\omega_N \vdash_3 \omega^L\rangle$ and because $\mathrm{FV}(M[x^L := N]) = (\mathrm{FV}(M) \setminus \{x^L\}) \cup \mathrm{FV}(N)$, $env^\omega_{\mathrm{FV}(M)\setminus\{x^L\}} \sqcap env^\omega_N = env^\omega_{M[x^L:=N]}$.

- If $\dfrac{M[x^L := N] : \langle \Gamma, (y^K : W) \vdash_3 T\rangle}{\lambda y^K.M[x^L := N] : \langle \Gamma \vdash_3 W \to T\rangle}$ where $y^K \not\in \mathrm{FV}(N) \cup \{x^L\}$. By IH, $\exists\, V$ and $\exists\, \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash_3 T\rangle$, $N : \langle \Gamma_2 \vdash_3 V\rangle$ and $\Gamma, y^K : W = \Gamma_1 \sqcap \Gamma_2$. By lemma 4.2.3a, $\mathrm{FV}(N) = \mathrm{dom}(\Gamma_2)$ and $\mathrm{FV}(M) = \mathrm{dom}(\Gamma_1) \cup \{y^K\}$. Since $y^K \in \mathrm{FV}(M)$ and $y^K \not\in \mathrm{FV}(N)$, $\Gamma_1 = \Delta_1, y^K : W$. Hence $M : \langle \Delta_1, y^K : W, x^L : V \vdash_3 T\rangle$. By rule $\to_I$, $\lambda y^K.M : \langle \Delta_1, x^L : V \vdash_3 W \to T\rangle$. Finally $\Gamma = \Delta_1 \sqcap \Gamma_2$.

- If $\dfrac{M[x^L := N] : \langle \Gamma \vdash_3 T\rangle \quad y^K \not\in \mathrm{dom}(\Gamma)}{\lambda y^K.M[x^L := N] : \langle \Gamma \vdash_3 \omega^K \to T\rangle}$ where $y^K \not\in \mathrm{FV}(N) \cup \{x^L\}$. By IH, $\exists\, V$ type and $\exists\, \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash_3 T\rangle$, $N : \langle \Gamma_2 \vdash_3 V\rangle$ and $\Gamma = \Gamma_1 \sqcap \Gamma_2$. Since $y^K \neq x^L$, $\lambda y^K.M : \langle \Gamma_1, x^L : V \vdash_3 \omega^K \to T\rangle$.

- If $\dfrac{M_1[x^L := N] : \langle \Gamma_1 \vdash_3 W \to T\rangle \quad M_2[x^L := N] : \langle \Gamma_2 \vdash_3 W\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1[x^L := N]\ M_2[x^L := N] : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_3 T\rangle}$ where $M = M_1 M_2$, then we have three cases:

  – If $x^L \in \mathrm{FV}(M_1) \cap \mathrm{FV}(M_2)$ then by IH, $\exists\, V_1, V_2$ types and $\exists\, \Delta_1$, Δ_2, Δ'_1, Δ'_2 type environments such that $M_1 : \langle \Delta_1, (x^L : V_1) \vdash_3 W \to T\rangle$, $M_2 : \langle \Delta'_1, (x^L : V_2) \vdash_3 W\rangle$, $N : \langle \Delta_2 \vdash_3 V_1\rangle$, $N : \langle \Delta'_2 \vdash_3 V_2\rangle$, $\Gamma_1 = \Delta_1 \sqcap \Delta_2$ and $\Gamma_2 = \Delta'_1 \sqcap \Delta'_2$. Because $\Gamma_1 \diamond \Gamma_2$, then $\Delta_1 \diamond \Delta'_1$ and $\Delta_2 \diamond \Delta'_2$ and because $\Delta_1, (x^L : V_1)$ and $\Delta'_1, (x^L : V_2)$ are type environments, by lemma A.19, $(\Delta_1, (x^L : V_1)) \diamond (\Delta'_1, (x^L : V_2))$. Then, by rules $\sqcap_I$ and $\to_E$, $M_1 M_2 : \langle \Delta_1 \sqcap \Delta'_1, (x^L : V_1 \sqcap V_2) \vdash_3 T\rangle$ and by $\sqcap'_I$, $N : \langle \Delta_2 \sqcap \Delta'_2 \vdash_3 V_1 \sqcap V_2\rangle$. Finally, $\Gamma_1 \sqcap \Gamma_2 = (\Delta_1 \sqcap \Delta_2) \sqcap (\Delta'_1 \sqcap \Delta'_2)$.
  – If $x^L \in \mathrm{FV}(M_1) \setminus \mathrm{FV}(M_2)$ then by IH, $\exists\, V$ types and $\exists\, \Delta_1$, Δ_1 type environments such that $M_1 : \langle \Delta_1, (x^L : V) \vdash_3 W \to T\rangle$, $N : \langle \Delta_2 \vdash_3 V\rangle$ and $\Gamma_1 = \Delta_1 \sqcap \Delta_2$. Since $\Gamma_1 \diamond \Gamma_2$, $\Delta_1 \diamond \Gamma_2$ and since $\Gamma_1 \sqcap \Gamma_2$ is a type environment, by lemma A.19, $(\Delta_1, (x^L : V)) \diamond \Gamma_2$. By $\to_E$, $M_1 M_2 : \langle \Delta_1 \sqcap \Gamma_2, (x^L : V) \vdash_3 T\rangle$ and $\Gamma_1 \sqcap \Gamma_2 = (\Delta_1 \sqcap \Delta_2) \sqcap \Gamma_2$.
  – If $x^L \in \mathrm{FV}(M_2) \setminus \mathrm{FV}(M_1)$ then by IH, $\exists\, V$ types and $\exists\, \Delta_1$, Δ_2 type environments such that $M_2 : \langle \Delta_1, (x^L : V) \vdash_3 W\rangle$, $N : \langle \Delta_2 \vdash_3 V\rangle$ and $\Gamma_2 = \Delta_1 \sqcap \Delta_2$. Since $\Gamma_1 \diamond \Gamma_2$, $\Gamma_1 \diamond \Delta_1$ and since $\Gamma_1 \sqcap \Gamma_2$ is a type environment, by lemma A.19, $(\Delta_1, (x^L : V)) \diamond \Gamma_1$. By $\to_E$, $M_1 M_2 : \langle \Gamma_1 \sqcap \Delta_1, (x^L : V) \vdash_3 T\rangle$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \sqcap (\Delta_1 \sqcap \Delta_2)$.

- Let $\dfrac{M[x^L := N] : \langle \Gamma \vdash_3 U_1\rangle \quad M[x^L := N] : \langle \Gamma \vdash_3 U_2\rangle}{M[x^L := N] : \langle \Gamma \vdash_3 U_1 \sqcap U_2\rangle}$. By IH, $\exists\, V_1, V_2$ types and $\exists\, \Delta_1$, Δ_2, Δ'_1, Δ'_2 type environments such that $M : \langle \Delta_1, x^L : V_1 \vdash_3 U_1\rangle$, $M : \langle \Delta'_1, x^L : V_2 \vdash_3 U_2\rangle$, $N : \langle \Delta_2 \vdash_3 V_1\rangle$, $N : \langle \Delta'_2 \vdash_3 V_2\rangle$, $\Gamma = \Delta_1 \sqcap \Delta_2$ and $\Gamma = \Delta'_1 \sqcap \Delta'_2$. Then, by rule $\sqcap'_I$, $M : \langle \Delta_1 \sqcap \Delta'_1, x^L : V_1 \sqcap V_2 \vdash_3 U_1 \sqcap U_2\rangle$ and $N : \langle \Delta_2 \sqcap \Delta'_2 \vdash_3 V_1 \sqcap V_2\rangle$. Finally, $\Gamma = (\Delta_1 \sqcap \Delta_2) \sqcap (\Delta'_1 \sqcap \Delta'_2)$.

- If $\dfrac{M[x^L := N] : \langle \Gamma \vdash_3 U\rangle}{M^{+j}[x^{j::L} := N^{+j}] : \langle \overline{e}_j\Gamma \vdash_3 \overline{e}_j U\rangle}$ then by IH, $\exists\, V$ type and $\exists\, \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash_3 U\rangle$, $N : \langle \Gamma_2 \vdash_3 V\rangle$ and $\Gamma = \Gamma_1 \sqcap \Gamma_2$.
So by e, $M^{+j} : \langle \overline{e}_j\Gamma_1, x^{j::L} : \overline{e}_j V \vdash_3 \overline{e}_j U\rangle$, $N : \langle \overline{e}_j\Gamma_2 \vdash_3 \overline{e}_j V\rangle$ and $\overline{e}_j\Gamma = \overline{e}_j\Gamma_1 \sqcap \overline{e}_j\Gamma_2$.

- If $\dfrac{M[x^L := N] : \langle \Gamma' \vdash_3 U'\rangle \quad \langle \Gamma' \vdash_3 U'\rangle \sqsubseteq \langle \Gamma \vdash_3 U\rangle}{M[x^L := N] : \langle \Gamma \vdash_3 U\rangle}$ then by lemma 4.1.2, $\Gamma \sqsubseteq \Gamma'$ and $U' \sqsubseteq U$.

By IH, $\exists\, V$ type and $\exists\, \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma'_1, x^L : V \vdash_3 U' \rangle$, $N : \langle \Gamma'_2 \vdash_3 V \rangle$ and $\Gamma' = \Gamma'_1 \sqcap \Gamma'_2$. Then by lemma A.11.5, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ and $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$. So by $\sqsubseteq$, $M : \langle \Gamma_1, x^L : V \vdash_3 U \rangle$ and $N : \langle \Gamma_2 \vdash_3 V \rangle$.

□

The next lemma is basic for the proof of subject expansion for β.

Lemma A.20. If $M[x^L := N] : \langle \Gamma \vdash_3 U \rangle$, $\mathrm{d}(U) = K$ and $L \succeq \mathrm{d}(M)$, $\mathcal{U} = \mathrm{FV}((\lambda x^L.M)N)$, then $(\lambda x^L.M)N : \langle \Gamma{\uparrow}^{\mathcal{U}} \vdash_3 U \rangle$.

Proof:
By lemma 4.1.2, $M[x^L := N] \in \mathcal{M}_3$, so $M, N \in \mathcal{M}_3$ and $M \diamond N$ and $\mathrm{d}(N) = L$. By definition $(\lambda x^L.M)N \in \mathcal{M}_3$. By lemma A.1.6b and theorem 4.1.2, $\mathrm{d}(\Gamma) \succeq \mathrm{d}(U) = K = \mathrm{d}(M[x^L := N]) = \mathrm{d}(M) = \mathrm{d}((\lambda x^L.M)N)$. So $L \succeq K$ and there exists K' such that $L = K :: K'$. We have two cases:

- If $x^L \in \mathrm{FV}(M)$, then, by lemma 5.8, $\exists\, V$ type and $\exists\, \Gamma_1, \Gamma_2$ type environments such that $M : \langle \Gamma_1, x^L : V \vdash_3 U \rangle$, $N : \langle \Gamma_2 \vdash_3 V \rangle$ and $\Gamma = \Gamma_1 \sqcap \Gamma_2$. By lemma 4.1.2, $\mathrm{OK}(\Gamma_1)$ and $\mathrm{OK}(\Gamma_2)$. By lemma 4.1.6, $\mathrm{OK}(\Gamma_1 \sqcap \Gamma_2)$. So, it is easy to prove, using lemma 4.1.1, that $\mathrm{OK}(\Gamma{\uparrow}^{\mathcal{U}})$. By lemma 4.2.3, $\Gamma_1, x^L : V \diamond \Gamma_2$, so $\Gamma_1 \diamond \Gamma_2$. By lemma 4.1.2, $\mathrm{d}(\Gamma_1) \succeq \mathrm{d}(M) = \mathrm{d}(U) = K$ and $L = \mathrm{d}(N) = \mathrm{d}(V) \preceq \mathrm{d}(\Gamma_2)$. By lemma A.11, we have two cases :
  - If $U = \omega^K$, then by lemma 4.2.2, $(\lambda x^L.M)N : \langle \Gamma{\uparrow}^{\mathcal{U}} \vdash_3 U \rangle$.
  - If $U = \vec{e}_K \sqcap_{i=1}^{p} T_i$ where $p \geq 1$ and $\forall\, 1 \leq i \leq p$, $T_i \in \mathbb{T}$, then by theorem 4.1.3, $M^{-K} : \langle \Gamma_1^{-K}, x^{K'} : V^{-K} \vdash_3 \sqcap_{i=1}^{p} T_i \rangle$. By $\sqsubseteq$, $\forall\, 1 \leq i \leq p$, $M^{-K} : \langle \Gamma_1^{-K}, x^{K'} : V^{-K} \vdash_3 T_i \rangle$, so by $\to_I$, $\lambda x^{K'}.M^{-K} : \langle \Gamma_1^{-K} \vdash_3 V^{-K} \to T_i \rangle$. Again by theorem 4.1.3, $N^{-K} : \langle \Gamma_2^{-K} \vdash_3 V^{-K} \rangle$ and since $\Gamma_1 \diamond \Gamma_2$, by lemma 4.1.3, $\Gamma_1^{-K} \diamond \Gamma_2^{-K}$, so by $\to_E$, $\forall\, 1 \leq i \leq p$, $(\lambda x^{K'}.M^{-K})N^{-K} : \langle \Gamma_1^{-K} \sqcap \Gamma_2^{-K} \vdash_3 T_i \rangle$. Finally by $\sqcap_I$ and e, $(\lambda x^L.M)N : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_3 U \rangle$, so $(\lambda x^L.M)N : \langle \Gamma{\uparrow}^{\mathcal{U}} \vdash_3 U \rangle$.
- If $x^L \notin \mathrm{FV}(M)$, then $M : \langle \Gamma \vdash_3 U \rangle$. By lemma 4.1.2, $\mathrm{OK}(\Gamma)$. So, it is easy to prove, using lemma 4.1.1, that $\mathrm{OK}(\Gamma{\uparrow}^{\mathcal{U}})$. By lemma A.11, we have two cases :
  - If $U = \omega^K$, then by lemma 4.2.2, $(\lambda x^L.M)N : \langle \Gamma{\uparrow}^{\mathcal{U}} \vdash_3 U \rangle$.
  - If $U = \vec{e}_K \sqcap_{i=1}^{p} T_i$ where $p \geq 1$ and $\forall\, 1 \leq i \leq p$, $T_i \in \mathbb{T}$, then by theorem 4.1.3, $M^{-K} : \langle \Gamma^{-K} \vdash_3 \sqcap_{i=1}^{p} T_i \rangle$. By $\sqsubseteq$, $\forall\, 1 \leq i \leq p$, $M^{-K} : \langle \Gamma^{-K} \vdash_3 T_i \rangle$. Using lemma A.5 and by induction on K, we can prove that $x^{K'} \notin \mathrm{FV}(M^{-K})$. So by lemma 4.2.3a, $x^{K'} \notin \mathrm{dom}(\Gamma^{-K})$. So by $\to'_I$, $\lambda x^{K'}.M^{-K} : \langle \Gamma^{-K} \vdash_3 \omega^{K'} \to T_i \rangle$. By (ω), $N^{-K} : \langle env^{\omega}_{N^{-K}} \vdash_3 \omega^{K'} \rangle$ and $N : \langle env^{\omega}_N \vdash_3 \omega^L \rangle$. By theorem 4.1.2, $\mathrm{d}(env^{\omega}_N) \succeq \mathrm{d}(N) = L$. By lemma 4.2.3, $\Gamma \diamond env^{\omega}_N$. By lemma 4.1.3, $\Gamma^{-K} \diamond env^{\omega}_{N^{-K}}$. By $\to_E$, $\forall\, 1 \leq i \leq p$, $(\lambda x^{K'}.M^{-K})N^{-K} : \langle \Gamma^{-K} \sqcap env^{\omega}_{N^{-K}} \vdash_3 T_i \rangle$. Finally by $\sqcap_I$ and e, $(\lambda x^L.M)N : \langle \Gamma \sqcap env^{\omega}_N \vdash_3 U \rangle$, so $(\lambda x^L.M)N : \langle \Gamma{\uparrow}^{\mathcal{U}} \vdash_3 U \rangle$.

□

Next, we give the main block for the proof of subject expansion for β.

Theorem A.3. If $N : \langle \Gamma \vdash_3 U \rangle$ and $M \rhd_\beta N$, then $M : \langle \Gamma{\uparrow}^{M} \vdash_3 U \rangle$.

Proof:
By induction on the derivation $N : \langle \Gamma \vdash_3 U \rangle$.

- If $\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash_3 T \rangle}$ and $M \rhd_\beta x^{\oslash}$, then $M = (\lambda y^K.M_1)M_2$ and $x^{\oslash} = M_1[y^K := M_2]$. Because $M \in \mathcal{M}_3$ then $K \succeq \mathrm{d}(M_1)$. By lemma A.20, $M : \langle (x^{\oslash} : T)\uparrow^M \vdash_3 T \rangle$.
- If $\dfrac{}{N : \langle env^{\omega}_N \vdash_3 \omega^{\mathrm{d}(N)} \rangle}$ and $M \rhd_\beta N$, then since by theorem 2.1.2, $\mathrm{FV}(N) \subseteq \mathrm{FV}(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$, $(env^{\omega}_N)\uparrow^M = env^{\omega}_M$. By ω, $M : \langle env^{\omega}_M \vdash_3 \omega^{\mathrm{d}(M)} \rangle$. Hence, $M : \langle (env^{N}_{\omega})\uparrow^M \vdash_3 \omega^{\mathrm{d}(N)} \rangle$.
- If $\dfrac{N : \langle \Gamma, x^L : U \vdash_3 T \rangle}{\lambda x^L.N : \langle \Gamma \vdash_3 U \to T \rangle}$ and $M \rhd_\beta \lambda x^L.N$, then we have two cases:
  - If $M = \lambda x.M'$ where $M' \rhd_\beta N$, then by IH, $M' : \langle (\Gamma, (x^L : U))\uparrow^{M'} \vdash_3 T \rangle$. Since by theorem 2.1.2 and lemma 4.2.3a, $x^L \in \mathrm{FV}(N) \subseteq \mathrm{FV}(M')$, then we have $(\Gamma, (x^L : U))\uparrow^{\mathrm{FV}(M')} = \Gamma\uparrow^{\mathrm{FV}(M')\setminus\{x^L\}}, (x^L : U)$ and $\Gamma\uparrow^{\mathrm{FV}(M')\setminus\{x^L\}} = \Gamma\uparrow^{\lambda x^L.M'}$. Hence, $M' : \langle \Gamma\uparrow^{\lambda x^L.M'}, (x^L : U) \vdash_3 T \rangle$ and finally, by $\to_I$, $\lambda x^L.M' : \langle \Gamma\uparrow^{\lambda x^L.M'} \vdash_3 U \to T \rangle$.
  - If $M = (\lambda y^K.M_1)M_2$ where $y^K \notin \mathrm{FV}(M_2)$ and $\lambda x^L.N = M_1[y^K := M_2]$, then, because $M \in \mathcal{M}_3$ then $K \succeq \mathrm{d}(M_1)$ and by lemma A.20, Because $M_1[y^K := M_2] : \langle \Gamma \vdash_3 U \to T \rangle$, we have $(\lambda y^K.M_1)M_2 : \langle \Gamma\uparrow^{(\lambda y^K.M_1)M_2} \vdash_3 U \to T \rangle$.
- If $\dfrac{N : \langle \Gamma \vdash_3 T \rangle \quad x^L \notin \mathrm{dom}(\Gamma)}{\lambda x^L.N : \langle \Gamma \vdash_3 \omega^L \to T \rangle}$ and $M \rhd_\beta N$ then similar to the above case.
- If $\dfrac{N_1 : \langle \Gamma_1 \vdash_3 U \to T \rangle \quad N_2 : \langle \Gamma_2 \vdash_3 U \rangle \quad \Gamma_1 \diamond \Gamma_2}{N_1\ N_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_3 T \rangle}$ and $M \rhd_\beta N_1 N_2$, we have three cases:
  - $M = M_1 N_2$ where $M_1 \rhd_\beta N_1$ and $M_1 \diamond N_2$. By IH, $M_1 : \langle \Gamma_1\uparrow^{M_1} \vdash_3 U \to T \rangle$. It is easy to show that $(\Gamma_1 \sqcap \Gamma_2)\uparrow^{M_1 N_2} = \Gamma_1\uparrow^{M_1} \sqcap \Gamma_2$. Since $M_1 \diamond N_2$, by lemma 4.2.3, $\Gamma_1\uparrow^{M_1} \diamond \Gamma_2$, hence use $\to_E$.
  - $M = N_1 M_2$ where $M_2 \rhd_\beta N_2$. Similar to the above case.
  - If $M = (\lambda x^L.M_1)M_2$ and $N_1 N_2 = M_1[x^L := M_2]$ then, because $M \in \mathcal{M}_3$ then $L \succeq \mathrm{d}(M_1)$ and by lemma A.20, $(\lambda x^L.M_1)M_2 : \langle (\Gamma_1 \sqcap \Gamma_2)\uparrow^{(\lambda x^L.M_1)M_2} \vdash_3 T \rangle$.
- If $\dfrac{N : \langle \Gamma \vdash_3 U_1 \rangle \qquad N : \langle \Gamma \vdash_3 U_2 \rangle}{N : \langle \Gamma \vdash_3 U_1 \sqcap U_2 \rangle}$ and $M \rhd_\beta N$ then use IH.
- If $\dfrac{N : \langle \Gamma \vdash_3 U \rangle}{N^{+j} : \langle \overline{e}_j \Gamma \vdash_3 \overline{e}_j U \rangle}$ then by lemma A.5.8 then there is $P \in \mathcal{M}_3$ such that $M = P^{+j}$ and $P \rhd_\beta N$. By IH, $P : \langle \Gamma\uparrow^P \vdash_3 U \rangle$ and by e, $M : \langle (\overline{e}_j \Gamma)\uparrow^M \vdash_3 \overline{e}_j U \rangle$.
- If $\dfrac{N : \langle \Gamma \vdash_3 U \rangle \qquad \langle \Gamma \vdash_3 U \rangle \sqsubseteq \langle \Gamma' \vdash_3 U' \rangle}{N : \langle \Gamma' \vdash_3 U' \rangle}$ and $M \rhd_\beta N$. By lemma 4.1.3, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$. It is easy to show that $\Gamma'\uparrow^M \sqsubseteq \Gamma\uparrow^M$ and hence by lemma 4.1.3, $\langle \Gamma\uparrow^M \vdash_3 U \rangle \sqsubseteq \langle \Gamma'\uparrow^M \vdash_3 U' \rangle$. By IH, $M\uparrow^M : \langle \Gamma \vdash_3 U \rangle$. Hence, by $\sqsubseteq_{\langle\rangle}$, we have $M : \langle \Gamma'\uparrow^M \vdash_3 U' \rangle$. $\square$

□

Proof:
[Of theorem 5.2] By induction on the length of the derivation $M \rhd_\beta^* N$ using theorem A.3 and the fact that if $\mathrm{FV}(P) \subseteq \mathrm{FV}(Q)$, then $(\Gamma\uparrow^P)\uparrow^Q = \Gamma\uparrow^Q$. □

A.6. Realisability

Proof:
[Of lemma 6.1]

1. easy.

2. If $M \rhd_r^* N^{+i}$ where $N \in \mathcal{X}$, then, by lemma 2.1.1, lemma A.5.7 and lemma A.5.8, $M = P^{+i}$ such that $P \in \mathcal{M}_3$ and $P \rhd_r N$. As $\mathcal{X}$ is r-saturated, $P \in \mathcal{X}$ and so $P^{+i} = M \in \mathcal{X}^{+i}$.

3. If $M \rhd_r^* N^+$ where $N \in \mathcal{X}$, then, by lemma 2.1.1, lemma A.3.1 and lemma A.4.3, $M = P^+$ and $P \rhd_\beta N$. As $\mathcal{X}$ is saturated, $P \in \mathcal{X}$ and so $P^+ = M \in \mathcal{X}^+$.

4. Let $i \in \{2, 3\}$, $M \in \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $N \rhd_r^* M$. If $P \in \mathcal{X}$ such that $P \diamond N$, then by lemma A.2.1, $P \diamond M$. So, by definition, $MP \in \mathcal{Y}$. Because $\mathcal{Y} \subseteq \mathcal{M}_i$, then $MP \in \mathcal{M}_i$. Hence, $\mathrm{d}(M) \preceq \mathrm{d}(P)$. By lemma 2.1, $\mathrm{d}(M) = \mathrm{d}(N)$. So $NP \in \mathcal{M}_i$ and $NP \rhd_r^* MP$. Because $MP \in \mathcal{Y}$ and $\mathcal{Y}$ is r-saturated, then $NP \in \mathcal{Y}$. Hence, $N \in \mathcal{X} \rightsquigarrow \mathcal{Y}$.

5. Let $M \in (\mathcal{X} \rightsquigarrow \mathcal{Y})^+$, then $M = N^+$ and $N \in \mathcal{X} \rightsquigarrow \mathcal{Y}$. If $P \in \mathcal{X}^+$ such that $M \diamond P$, then $P = Q^+$, $Q \in \mathcal{X}$ and $MP = N^+Q^+ = (NQ)^+$. By lemma A.3.1(c)i, $N \diamond Q$ and hence $NQ \in \mathcal{Y}$ and $MP \in \mathcal{Y}^+$. Thus $M \in \mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+$.

6. Let $M \in (\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i}$, then $M = N^{+i}$ and $N \in \mathcal{X} \rightsquigarrow \mathcal{Y}$. Let $P \in \mathcal{X}^{+i}$ such that $M \diamond P$. Then $P = Q^{+i}$ such that $Q \in \mathcal{X}$. Because $M \diamond P$ then by lemma A.5.2, $N \diamond Q$. So $NQ \in \mathcal{Y}$. Because $\mathcal{Y} \subseteq \mathcal{M}_3$ then $NQ \in \mathcal{M}_3$. Because $(NQ)^{+i} = N^{+i}Q^{+i} = MP$ then $MP \in \mathcal{Y}^{+i}$. Hence, $M \in \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$.

7. let $M \in \mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+$. There is $N \in \mathcal{X}^+$ such that $M \diamond N$. We have $MN \in \mathcal{Y}^+$, then $MN = P^+$ where $P \in \mathcal{Y}$. Hence, $M = M_1^+$. Let $N_1 \in \mathcal{X}$ such that $M_1 \diamond N_1$. By lemma A.3.1(c)i, $M \diamond N_1^+$ and we have $(M_1N_1)^+ = M_1^+N_1^+ \in \mathcal{Y}^+$. Hence $M_1N_1 \in \mathcal{Y}$. Thus $M_1 \in \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $M = M_1^+ \in (\mathcal{X} \rightsquigarrow \mathcal{Y})^+$.

8. Let $M \in \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$ such that $\mathcal{X}^{+i} \wr \mathcal{Y}^{+i}$. By hypothesis, there exists $P \in \mathcal{X}^{+i}$ such that $M \diamond P$. Then $MP \in \mathcal{Y}^{+i}$. Hence $MP = Q^{+i}$ such that $Q \in \mathcal{Y}$. Because $\mathcal{Y} \subseteq \mathcal{M}_3$ then $Q \in \mathcal{M}_3$ and by lemma A.5.1, $MP \in \mathcal{M}_3$. Hence by definition $M \in \mathcal{M}_3$ and by lemma A.5.1, $\mathrm{d}(M) = \mathrm{d}(Q^{+i}) = i :: \mathrm{d}(Q)$. So by lemma A.5.7, there exists $M_1 \in \mathcal{M}_3$ such that $M = M_1^{+i}$. Let $N_1 \in \mathcal{X}$ such that $M_1 \diamond N_1$. By definition $N_1^{+i} \in \mathcal{X}^{+i}$ and by lemma A.5.2, $M \diamond N_1^+$. So, $MN_1^{+i} \in \mathcal{Y}^{+i}$. So $MN_1^{+i} = M'^{+i}$ such that $M' \in \mathcal{Y}$. Because $\mathcal{Y} \subseteq \mathcal{M}_3$ then $M' \in \mathcal{M}_3$. By lemma A.5.1, $MN_1^{+i} \in \mathcal{M}_3$. So $M_1^{+i} \diamond N_1^{+i}$ and $\mathrm{d}(M_1^{+i}) \preceq \mathrm{d}(N_1^{+i})$. By lemma A.5.1 and lemma A.5.2, $M_1 \diamond N_1$ and $\mathrm{d}(M_1) \preceq \mathrm{d}(N_1)$. So $M_1N_1 \in \mathcal{M}_3$ and $(M_1N_1)^{+i} = M_1^{+i}N_1^{+i} \in \mathcal{Y}^{+i}$. Hence $M_1N_1 \in \mathcal{Y}$. Thus, $M_1 \in \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $M = M_1^{+i} \in (\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i}$.

9. If $M \rhd_\beta^* N$ and $N \in \mathbb{M} \cap \mathcal{M}_2^n$ then by lemma 2.1.2, $M \in \mathbb{M} \cap \mathcal{M}_2^n$.

$\square$

Proof:
[Of lemma 6.2]

1. 1a. By induction on U using lemma 6.1 and lemma 2.1.

   1b. We prove $\forall x \in \mathcal{V}_1, \mathcal{N}_x^L \subseteq \mathcal{I}(U) \subseteq \mathcal{M}_3^L$ by induction on U. Case $U = a$: by definition. Case $U = \omega^L$: We have $\forall x \in \mathcal{V}_1, \mathcal{N}_x^L \subseteq \mathcal{M}_3^L \subseteq \mathcal{M}_3^L$. Case $U = U_1 \sqcap U_2$ (resp. $U = \overline{e}_i V$): use IH since $\mathrm{d}(U_1) = \mathrm{d}(U_2)$ (resp. $\mathrm{d}(U) = i :: \mathrm{d}(V)$, $\forall x \in \mathcal{V}_1, (\mathcal{N}_x^K)^{+i} = \mathcal{N}_x^{i::K}$ and $(\mathcal{M}_3^K)^{+i} = \mathcal{M}_3^{i::K}$). Case $U = V \to T$: by definition, $K = \mathrm{d}(V) \succeq \mathrm{d}(T) = \oslash$.

   - Let $x \in \mathcal{V}_1$, $N_1, ..., N_k$ such that $\forall\ 1 \leq i \leq k$, $\mathrm{d}(N_i) \succeq \oslash$ and $\diamond\{x^\oslash, N_1, \ldots, N_k\}$ and let $N \in \mathcal{I}(V)$ such that $(x^\oslash N_1...N_k) \diamond N$. By IH, $\mathrm{d}(N) = K \succeq \oslash$. Again, by IH, $x^\oslash N_1...N_k N \in \mathcal{I}(T)$. Thus $x^\oslash N_1...N_k \in \mathcal{I}(V \to T)$.
   - Let $M \in \mathcal{I}(V \to T)$. Let $x \in \mathcal{V}_1$ such that $\forall L, x^L \notin \mathrm{FV}(M)$. By IH, $x^K \in \mathcal{I}(V)$, then $Mx^K \in \mathcal{I}(T)$ and, by IH, $\mathrm{d}(Mx^K) = \oslash$. Thus $\mathrm{d}(M) = \oslash$.

   1c Obviously, $x^n \in \mathcal{N}_x^n$. We prove $\mathcal{N}_x^n \subseteq \mathcal{I}(U) \subseteq \mathbb{M}^n$ by induction on U good. Case $U = a$: by definition. Case $U = U \sqcap V$ (resp. $U = eU'$): use IH since U, V are good and $\mathrm{d}(U) = \mathrm{d}(V)$ (resp. U' is good, $\mathrm{d}(U) = \mathrm{d}(U') + 1$, $(\mathcal{N}_x^n)^+ = \mathcal{N}_x^{n+1}$ and $(\mathcal{M}_2^n)^+ = \mathcal{M}_2^{n+1}$). Case $U = U \to T$: by definition, U, T are good and $m = \mathrm{d}(U) \geq \mathrm{d}(T) = n$.

   - Let $N_1, ..., N_k$ such that $x^n N_1...N_k \in \mathbb{M}$ (note that $\mathrm{d}(x^n N_1...N_k) = n$) and let $N \in \mathcal{I}(U)$ such that $(x^n N_1...N_k) \diamond N$ (hence $x^n N_1...N_k N \in \mathcal{M}_2$). By IH, $\mathrm{d}(N) = m \geq n$ and $N \in \mathbb{M}$. Hence, $x^n N_1...N_k N \in \mathbb{M}$ and $x^n N_1...N_k N \in \mathcal{N}_x^n$. By IH, $x^n N_1...N_k N \in \mathcal{I}(T)$. Thus $x^n N_1...N_k \in \mathcal{I}(U \to T)$.
   - Let $M \in \mathcal{I}(U \to T)$. Let $x \in \mathcal{V}_1$ such that $\forall\ p \in \mathbb{N}$, $x^p \notin FV(M)$. Hence, $M \diamond x^m$. By IH, $x^m \in \mathcal{I}(U)$. Then $M\ x^m \in \mathcal{I}(T)$, and so by IH $M\ x^m \in \mathbb{M}^n$. By lemma 2.2, M is good and $\mathrm{d}(M) \leq m$. Since $\mathrm{d}(M\ x^m) = \min(\mathrm{d}(M), m) = n$, $\mathrm{d}(M) = n$ and so $M \in \mathbb{M}^n$.

2. By induction of the derivation $U \sqsubseteq V$.

$\square$

Proof:
[of Lemma 6.3]

- Case $\vdash_1$ / $\vdash_2$: Let $i \in \{1, 2\}$. By induction on the derivation of $M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \rangle$. First note, by lemma 4.1, $\forall 1 \leq i \leq n$, U_i is good and $N_i \in \mathcal{M}_2$.

  – If $\dfrac{T\ good \quad \mathrm{d}(T) = n}{x^n : \langle (x^n : T) \vdash_i T \rangle}$ and $N \in \mathcal{I}(T)$, then $x^n[x^n := N] = N \in \mathcal{I}(T)$.

  – Let $\dfrac{M : \langle (x_i^{n_i} : U_i)_n, (x^m : U) \vdash_i T \rangle}{\lambda x^m.M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \to T \rangle}$ and $\forall 1 \leq i \leq n$, $N_i \in \mathcal{I}(U_i)$ where $(\lambda x^m.M)[(x_i^{n_i} := N_i)_n] \in \mathcal{M}_2$. Let $N \in \mathcal{I}(U)$ where $(\lambda x^m.M)[(x_i^{n_i} := N_i)_n] \diamond N$. Since $(\lambda x^m.M[(x_i^{n_i} := N_i)_n]) \diamond N$, by lemma A.1, $M[(x_i^{n_i} := N_i)_n] \diamond N$ and $M[(x_i^{n_i} := N_i)_n][x^m := N] =$

$M[(x_i^{n_i} := N_i)_n, x^m := N] \in \mathcal{M}_2$. Hence, by IH, $M[(x_i^{n_i} := N_i)_n, x^m := N] \in \mathcal{I}(T)$. By lemma 4.1, U, T are good and $\mathrm{d}(U) = m$. By lemma 1, $\mathrm{d}(N) = m$ and $(\lambda x^m.M[(x_1^{n_1} := N_1)_n])N \rhd_\beta M[(x_i^{n_i} := N_i)_n, x^m := N] \in \mathcal{I}(T)$. Since, by lemma 1 $\mathcal{I}(T)$ is saturated, then $(\lambda x^m.M[(x_1^{n_1} := N_1)_n])N \in \mathcal{I}(T)$ and hence $\lambda x^m.M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T) = \mathcal{I}(U \to T)$.

- Let $\dfrac{M_1 : \langle \Gamma_1 \vdash_i U \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash_i U \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_i T \rangle}$ where
  $\Gamma_1 = (x_i^{n_i} : U_i)_n, (y_j^{m_j} : V_j)_m, \Gamma_2 = (x_i^{n_i} : U'_i)_n, (z_k^{r_k} : W_k)_r$ and
  $\Gamma_1 \sqcap \Gamma_2 = (x_i^{n_i} : U_i \sqcap U'_i)_n, (y_j^{m_j} : V_j)_m, (z_k^{r_k} : W_k)_r$.
  Let $\forall\, 1 \leq i \leq n, P_i \in \mathcal{I}(U_i \sqcap U'_i), \forall\, 1 \leq j \leq m, Q_j \in \mathcal{I}(V_j)$ and $\forall\, 1 \leq k \leq r, R_k \in \mathcal{I}(W_k)$ where
  $(M_1M_2)[(x_i^{n_i} := P_i)_n, (y_j^{m_j} := Q_j)_m, (z_k^{r_k} := R_k)_r] \in \mathcal{M}_2$.
  Let $A = M_1[(x_i^{n_i} := P_i)_n, (y_j^{m_j} := Q_j)_m]$ and
  $B = M_2[(x_i^{n_i} := P_i)_n, (z_k^{r_k} := R_k)_r]$.
  By lemma A.14, $FV(M_1) = dom(\Gamma_1)$ and $FV(M_2) = dom(\Gamma_2)$. Hence,
  $(M_1M_2)[(x_i^{n_i} := P_i)_n, (y_j^{m_j} := Q_j)_m, (z_k^{r_k} := R_k)_r] = AB$.
  By lemma A.1, $A \in \mathcal{M}_2$, $B \in \mathcal{M}_2$, and $A \diamond B$.
  By IH, $A \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$ and $B \in \mathcal{I}(U)$.
  Hence, $AB = (M_1M_2)[(x_i^{n_i} := P_i)_n, (y_j^{m_j} := Q_j)_m, (z_k^{r_k} := R_k)_r] \in \mathcal{I}(T)$.
- Let $\dfrac{M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \rangle \quad M : \langle (x_i^{n_i} : V_i)_n \vdash_i V \rangle}{M : \langle (x_i^{n_i} : U_i \sqcap V_i)_n \vdash_i U \sqcap V \rangle}$ (note lemma A.14.4) and $\forall\, 1 \leq i \leq n$, $N_i \in \mathcal{I}(U_i \sqcap V_i) = \mathcal{I}(U_i) \cap \mathcal{I}(V_i)$ where $M[(x_i^{n_i} := N_i)_n] \in \mathcal{M}_2$. By IH, $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U)$ and $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(V)$. Hence, $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U \sqcap V)$.
- Let $\dfrac{M : \langle (x_i^{n_i} : T_i)_n \vdash_i U \rangle}{M^+ : \langle (x_i^{n_i+1} : eT_i)_n \vdash_i eU \rangle}$ and $\forall\ 1 \leq i \leq n$, $N_i \in \mathcal{I}(eT_i) = \mathcal{I}(T_i)^+$ where $M^+[(x_i^{n_i+1} := N_i)_n] \in \mathcal{M}_2$. Then $\forall\ 1 \leq i \leq n$, $N_i = P_i^+$ where $P_i \in \mathcal{I}(T_i)$. By lemmas A.1 and A.3.1(c)i, $\diamond\{M^+, N_1, \ldots, N_n\}$ and $\diamond\{M, P_1, \ldots, P_n\}$. Then, by lemma A.1, $M[(x_i^{n_i} := P_i)_n] \in \mathcal{M}_2$ and, by IH, $M[(x_i^{n_i} := P_i)_n] \in \mathcal{I}(U)$. Hence, by lemma A.5.5, $M^+[(x_i^{n_i+1} := P_i^+)_n] = (M[(x_i^{n_i} := P_i)_n])^+ \in \mathcal{I}(U)^+ = \mathcal{I}(eU)$.
- Let $\dfrac{M : \Phi \quad \Phi \sqsubseteq \Phi'}{M : \Phi'}$ where $\phi' = \langle (x_i^{n_i} : U_i)_n \vdash_2 U \rangle$. By lemma 4.1, we have $\Phi = \langle (x_i^{n_i} : U'_i)_n \vdash_2 U' \rangle$, where for every $1 \leq i \leq m$, $U_i \sqsubseteq U'_i$ and $U' \sqsubseteq U$. By lemma 2, $N_i \in \mathcal{I}(U'_i)$, then, by IH, $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U')$ and, by lemma 2, $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U)$.

- Case $\vdash_3$: By induction on the derivation $M : \langle (x_j^{L_j} : U_j)_n \vdash_3 U \rangle$.
  - If $\dfrac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash_3 T \rangle}$ and $N \in \mathcal{I}(T)$, then $x^{\oslash}[x^{\oslash} := N] = N \in \mathcal{I}(T)$.
  - If $\dfrac{}{M : \langle env_M^\omega \vdash_3 \omega^{\mathrm{d}(M)} \rangle}$. Let $env_M^\omega = (x_j^{L_j} : U_j)_n$ so $\mathrm{FV}(M) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$. Because, by lemma 4.1.2, for all $j \in \{1, \ldots, n\}$, $\mathrm{d}(U_j) = L_j$ by lemma 6.2.1, $\mathcal{I}(U_j) \subseteq \mathcal{M}_3^{L_j}$, hence, $\mathrm{d}(N_j) = L_j$. Because $M[(x_j^{L_j} := N_j)_n] \in \mathcal{M}_3$, then $\diamond\{M\} \cup \{N_i \mid i \in \{1, \ldots, n\}\}$.

Then, by lemma A.1.6, $\mathrm{d}(M[(x_j^{L_j} := N_j)_n]) = \mathrm{d}(M)$ and $M[(x_j^{L_j} := N_j)_n] \in \mathcal{M}_3^{\mathrm{d}(M)} = \mathcal{I}(\omega^{\mathrm{d}(M)})$.

– If $\dfrac{M : \langle (x_j^{L_j} : U_j)_n, (x^K : V) \vdash_3 T \rangle}{\lambda x^K.M : \langle (x_j^{L_j} : U_j)_n \vdash_3 V \to T \rangle}$, $\forall 1 \le j \le n$, $N_j \in \mathcal{I}(U_j)$ and $N \in \mathcal{I}(V)$ such that $(\lambda x^K.M)[(x_j^{L_j} := N_j)_n] \diamond N$. By lemma 4.1.2, $\mathrm{d}(V) = K$. We have, $(\lambda x^K.M)[(x_j^{L_j} := N_j)_n] = \lambda x^K.M[(x_j^{L_j} := N_j)_n]$, where $\forall 1 \le j \le n, y^K \notin \mathrm{FV}(N_j) \cup \{x_j^{L_j}\}$. Since $N \in \mathcal{I}(V)$ and by lemma 6.2.1, $\mathcal{I}(V) \subseteq \mathcal{M}_3^K$, $\mathrm{d}(N) = K$. By lemma A.1.3 and lemma A.1.6, $M[(x_j^{L_j} := N_j)_n] \diamond N$ and $M[(x_j^{L_j} := N_j)_n][x^K := N] = M[(x_j^{L_j} := N_j)_n, x^K := N] \in \mathcal{M}_3$. Hence, $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \in \mathcal{M}_3$ and $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \rhd_r M[(x_j^{L_j} := N_j)_n, (x^K := N)]$. By IH, $M[(x_j^{L_j} := N_j)_n, (x^K := N)] \in \mathcal{I}(T)$. Since, by lemma 6.2.1 $\mathcal{I}(T)$ is r-saturated, then $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \in \mathcal{I}(T)$ and so $\lambda x^K.M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V) \rightsquigarrow \mathcal{I}(T) = \mathcal{I}(V \to T)$.

– If $\dfrac{M : \langle (x_j^{L_j} : U_j)_n \vdash_3 T \rangle \quad x^K \notin \mathrm{dom}((x_j^{L_j} : U_j)_n)}{\lambda x^K.M : \langle (x_j^{L_j} : U_j)_n \vdash_3 \omega^K \to T \rangle}$, $\forall 1 \le j \le n$, $N_j \in \mathcal{I}(U_j)$ and $N \in \mathcal{I}(\omega^K)$ such that $(\lambda x^K.M)[(x_j^{L_j} := N_j)_n] \diamond N$. By lemma 4.2.3a, $x^K \notin \mathrm{FV}(M)$. We have, $(\lambda x^K.M)[(x_j^{L_j} := N_j)_n] = \lambda x^K.M[(x_j^{L_j} := N_j)_n]$, where $\forall 1 \le j \le n, x^K \notin \mathrm{FV}(N_j) \cup \{x_j^{L_j}\}$. Since $N \in \mathcal{I}(\omega^K)$ and by lemma 6.2.1, $\mathcal{I}(\omega^K) = \mathcal{M}_3^K$ then $\mathrm{d}(N) = K$. By lemma A.1.3 and lemma A.1.6, $M[(x_j^{L_j} := N_j)_n] \diamond N$ and $M[(x_j^{L_j} := N_j)_n][x^K := N] = M[(x_j^{L_j} := N_j)_n, x^K := N] = M[(x_j^{L_j} := N_j)_n] \in \mathcal{M}_3$. Hence, $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \in \mathcal{M}_3$ and $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \rhd_r M[(x_j^{L_j} := N_j)_n, (x^K := N)]$. By IH, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(T)$. Since, by lemma 6.2.1 $\mathcal{I}(T)$ is r-saturated, then $(\lambda x^K.M[(x_j^{L_j} := N_j)_n])N \in \mathcal{I}(T)$ and so $\lambda x^K.M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(\omega^K) \rightsquigarrow \mathcal{I}(T) = \mathcal{I}(\omega^K \to T)$.

– Let $\dfrac{M_1 : \langle \Gamma_1 \vdash_3 V \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash_3 V \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_3 T \rangle}$ where $\Gamma_1 = (x_j^{L_j} : U_j)_n, (y_j^{K_j} : V_j)_m$, $\Gamma_2 = (x_j^{L_j} : U'_j)_n, (z_j^{S_j} : W_j)_p$ such that $\{y_1^{K_1}, \dots, y_m^{K_m}\} \cap \{z_1^{S_1}, \dots, z_p^{S_p}\} = \emptyset$ and $\Gamma_1 \sqcap \Gamma_2 = (x_j^{L_j} : U_j \sqcap U'_j)_n, (y_j^{K_j} : V_j)_m, (z_j^{S_j} : W_j)_p$.
Let $\forall\ 1 \le j \le n, P_j \in \mathcal{I}(U_j \sqcap U'_j)$, $\forall\ 1 \le j \le m, Q_j \in \mathcal{I}(V_j)$ and $\forall\ 1 \le j \le p, R_j \in \mathcal{I}(W_j)$. So, for all $j \in \{1, \dots, n\}$, $P_j \in \mathcal{I}(U_j)$ and $P_j \in \mathcal{I}(U'_j)$. By hypothesis, $(M_1 M_2)[(x_j^{L_j} := P_j)_n, (y_j^{K_j} := Q_j)_m, (z_j^{S_j} := R_j)_p] = AB \in \mathcal{M}_3$ where using lemma 4.2.3a, $A = M_1[(x_j^{L_j} := P_j)_n, (y_j^{K_j} := Q_j)_m] \in \mathcal{M}_3$ and $B = M_2[(x_j^{L_j} := P_j)_n, (z_j^{S_j} := R_j)_p] \in \mathcal{M}_3$ and $A \diamond B$.
By IH, $A \in \mathcal{I}(V) \rightsquigarrow \mathcal{I}(T)$ and $B \in \mathcal{I}(V)$. Hence, $AB \in \mathcal{I}(T)$.

– Let $\dfrac{M : \langle (x_j^{L_j} : U_j)_n \vdash_3 V_1 \rangle \quad M : \langle (x_j^{L_j} : U_j)_n \vdash_3 V_2 \rangle}{M : \langle (x_j^{L_j} : U_j)_n \vdash_3 V_1 \sqcap V_2 \rangle}$. By IH, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V_1)$

and $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V_2)$. Hence, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(V_1 \sqcap V_2)$.

- Let $\dfrac{M : \langle (x_k^{L_k} : U_k)_n \vdash_3 U\rangle}{M^{+j} : \langle (x_k^{j::L_k} : \overline{e}_j U_k)_n \vdash_3 \overline{e}_j U\rangle}$ and $\forall\ 1 \leq k \leq n$, $N_k \in \mathcal{I}(\overline{e}_j U_k) = \mathcal{I}(U_k)^{+j}$. Then $\forall\ 1 \leq k \leq n$, $N_k = P_k^{+j}$ where $P_k \in \mathcal{I}(U_k)$. By lemma 6.2.1b, for all $k \in \{1, \ldots, n\}$, $P_k \in \mathcal{M}_3^{L_k}$. By the definition of the substitution, $\diamond\{M^{+j}\} \cup \{N_k \mid k \in \{1, \ldots, n\}\}$. By lemma A.5.3, $\diamond\{M\} \cup \{P_k \mid k \in \{1, \ldots, n\}\}$. By lemma A.1.6, $M[(x_k^{L_k} := P_k)_n] \in \mathcal{M}_3$. By IH, $M[(x_k^{L_k} := P_k)_n] \in \mathcal{I}(T)$. Hence, by lemma A.5, $M^{+j}[(x_k^{j::L_k} := N_k)_n] = (M[(x_k^{L_k} := P_k)_n])^{+j} \in \mathcal{I}(U)^{+j} = \mathcal{I}(\overline{e}_j U)$.
- Let $\dfrac{M : \Phi \quad \Phi \sqsubseteq \Phi'}{M : \Phi'}$ where $\Phi' = \langle (x_j^{L_j} : U_j)_n \vdash_3 U\rangle$. By lemma 4.1, we have $\Phi = \langle (x_j^{L_j} : U'_j)_n \vdash_3 U'\rangle$, where for every $1 \leq j \leq n$, $U_j \sqsubseteq U'_j$ and $U' \sqsubseteq U$. By lemma 6.2.2, $N_j \in \mathcal{I}(U'_j)$, then, by IH, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(U')$ and, by lemma 6.2.2, $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(U)$.

□

Next we give a lemma that will be used in the rest of the article.

Lemma A.21. 1. If $M[y^n := x^n] \rhd_\beta N$ then $M \rhd_\beta N'$ where $N = N'[y^n := x^n]$.

2. If $M[y^n := x^n]$ has a β-normal form then M has a β-normal form.
3. Let $k \geq 1$. If $M x_1^{n_1} ... x_k^{n_k}$ is normalizing, then M is normalizing.
4. Let $k \geq 1$, $1 \leq i \leq k$, $l \geq 0$, $x_i^{n_i} N_1 ... N_l$ be in normal form and M be closed. If $M x_1^{n_1} ... x_k^{n_k} \rhd_\beta^* x_i^{n_i} N_1 ... N_l$, then for some $m \geq i$ and $n \leq l$, $M \rhd_\beta^* \lambda x_1^{n_1} .... \lambda x_m^{n_m} . x_i^{n_i} M_1 ... M_n$ where $n + k = m + l$, $M_j \simeq_\beta N_j$ for every $1 \leq j \leq n$ and $N_{n+j} \simeq_\beta x_{m+j}^{n_{m+j}}$ for every $1 \leq j \leq k - m$.

Proof:

1. By induction on $M[y^n := x^n] \rhd_\beta N$.
2. $M[y^n := x^n] \rhd_\beta^* P$ where P is in β-normal form. The proof is by induction on $M[y^n := x^n] \rhd_\beta^* P$ using 1.
3. By induction on $k \geq 1$. We only prove the basic case. The proof is by cases.
   - If $M\, x_1^{n_1} \rhd_\beta^* M'\, x_1^{n_1}$ where $M'\, x_1^{n_1}$ is in β-normal form and $M \rhd_\beta^* M'$ then M' is in β-normal form and M is β-normalising.
   - If $M\, x_1^{n_1} \rhd_\beta^* (\lambda y^{n_1}.N)\, x_1^{n_1} \rhd_\beta N[y^{n_1} := x_1^{n_1}] \rhd_\beta^* P$ where P is in β-normal form and $M \rhd_\beta^* \lambda y^{n_1}.N$ then by 2, N has a β-normal form and so, $\lambda y^{n_1}.N$ has a β-normal form. Hence, M has a β-normal form.

4. By 3, M is normalizing, and, since M is closed, its normal form is $\lambda x_1^{n_1}....\lambda x_m^{n_m}.z^r M_1...M_n$ for $n, m \geq 0$.
Since by theorem 2.2, $x_i^{n_i} N_1...N_l \simeq_\beta (\lambda x_1^{n_1}....\lambda x_m^{n_m}.z^r M_1...M_n) x_1^{n_1}...x_k^{n_k}$ then $m \leq k$, $x_i^{n_i}\ N_1...N_l \simeq_\beta z^r M_1...M_n x_{m+1}^{n_{m+1}}...x_k^{n_k}$. Hence, $z^r = x_i^{n_i}$, $n \leq l$, $i \leq m$, $l = n+(k-(m+1))+1 = n+k-m$, $M_j \simeq_\beta N_j$ for every $1 \leq j \leq n$ and $N_{n+j} \simeq_\beta x_{m+j}^{n_{m+j}}$ for every $1 \leq j \leq k-m$.

□

Proof:

1. Let $y \in \mathcal{V}_2$ and take $\mathcal{X} = \{M \in \mathbb{M}^0 \mid M \rhd_\beta^* y^0$ or $M \rhd_\beta^* x^0 N_1...N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1\}$. $\mathcal{X}$ is saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^0 \subseteq \mathcal{X} \subseteq \mathbb{M}^0$. Let $\mathcal{I}$ be an β_1-interpretation such that $\mathcal{I}(a) = \mathcal{I}(b) = \mathcal{X}$. If $M \in [(a \sqcap b) \to a]_{\beta_1}$, then M is closed and $M \in \mathcal{X} \rightsquigarrow \mathcal{X}$. Since $M\ y^0 \in \mathcal{X}$ (as $y^0 \in \mathcal{X}$ and $M \diamond y^0$) and M is closed and $x^0 \neq y^0$, by lemma 2.1.2, $M\ y^0 \rhd_\beta^* y^0$. Hence, by lemma A.21.4, $M \rhd_\beta^* \lambda y^0.y^0$. By lemma 2.1, $\mathrm{d}(M) = \mathrm{d}(\lambda y^0.y^0) = 0$ and $M \in \mathbb{M}^0$.

   Conversely, let $M \in \mathbb{M}^0$ be closed and $M \rhd_\beta^* \lambda y^0.y^0$. Let $\mathcal{I}$ be an β_1-interpretation and $N \in \mathcal{I}(a \sqcap b)$ (hence $M \diamond N$). Since $\mathcal{I}(a)$ is saturated, $N \in \mathcal{I}(a)$ and $MN \rhd_\beta^* N$, then $MN \in \mathcal{I}(a)$ and hence $M \in \mathcal{I}(a \sqcap b) \rightsquigarrow \mathcal{I}(a)$. Hence, $M \in [(a \sqcap b) \to a]_{\beta_1}$.

2 If $\lambda y^0.y^0 : \langle () \vdash_1 (a \sqcap b) \to a \rangle$, then, by Lemma 5.1, $y^0 : \langle (y^0 : a \sqcap b) \vdash_1 a \rangle$ and again, by Lemma 5.1, $y^0 : a = y^0 : a \sqcap b$. Hence, $a = a \sqcap b$. Absurd.

3. Easy.

4. Let $y \in \mathcal{V}_2$ and $\mathcal{X} = \{M \in \mathcal{M}_3^\oslash \mid M \rhd_\beta^* x^\oslash N_1...N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1$ or $M \rhd_\beta^* y^\oslash\}$. $\mathcal{X}$ is β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^\oslash \subseteq \mathcal{X} \subseteq \mathcal{M}_3^\oslash$. Take a β_3-interpretation $\mathcal{I}$ such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Id_0]_{\beta_3}$, then M is closed and $M \in \mathcal{X} \rightsquigarrow \mathcal{X}$. Since $y^\oslash \in \mathcal{X}$ and $M \diamond y^\oslash$ then $My^\oslash \in \mathcal{X}$ and $My^\oslash \rhd_\beta^* x^\oslash N_1...N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1$ or $My^\oslash \rhd_\beta^* y^\oslash$. Since M is closed and $x^\oslash \neq y^\oslash$, by lemma 2.1.2, $My^\oslash \rhd_\beta^* y^\oslash$. Hence, by lemma A.21.4, $M \rhd_\beta^* \lambda y^\oslash.y^\oslash$ and, by lemma 2.1, $M \in \mathcal{M}_3^\oslash$.

   Conversely, let $M \in \mathcal{M}_3^\oslash$ such that M is closed and $M \rhd_\beta^* \lambda y^\oslash.y^\oslash$. Let $\mathcal{I}$ be an β_3-interpretation and $N \in \mathcal{I}(a)$ such that $M \diamond N$. By lemma 6.2.1b, $N \in \mathcal{M}_3^\oslash$, so $MN \in \mathcal{M}_3^\oslash$. Since $\mathcal{I}(a)$ is β-saturated and $MN \rhd_\beta^* N$, $MN \in \mathcal{I}(a)$ and hence $M \in \mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)$. Hence, $M \in [Id_0]_{\beta_3}$.

5. By lemma 6.4 and lemma 6.1, $[Id_1']_{\beta_3} = [\overline{e}_1 a \to \overline{e}_1 a]_{\beta_3} = [\overline{e}_1(a \to a)]_{\beta_3} = [Id_1] = [a \to a]_{\beta_3}^{+1} = [Id_0]_{\beta_3}^{+1}$. By 1., $[Id_0]_{\beta_3}^{+1} = \{M \in \mathcal{M}_3^{(1)} \mid M \rhd_\beta^* \lambda y^{(1)}.y^{(1)}\}$.

6. Let $y \in \mathcal{V}_2$, $\mathcal{X} = \{M \in \mathcal{M}_3^\oslash \mid M \rhd_\beta^* y^\oslash$ or $M \rhd_\beta^* x^\oslash N_1...N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1\}$ and $\mathcal{Y} = \{M \in \mathcal{M}_3^\oslash \mid M \rhd_\beta^* y^\oslash y^\oslash$ or $M \rhd_\beta^* x^\oslash N_1...N_k$ or $M \rhd_\beta^* y^\oslash(x^\oslash N_1...N_k)$ where $k \geq 0$ and $x \in \mathcal{V}_1\}$. $\mathcal{X}$, $\mathcal{Y}$ are β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^\oslash \subseteq \mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}_3^\oslash$. Let $\mathcal{I}$ be a β_3-interpretation such that $\mathcal{I}(a) = \mathcal{X}$ and $\mathcal{I}(b) = \mathcal{Y}$. If $M \in [D]_{\beta_3}$, then M is closed (hence $M \diamond y^\oslash$) and $M \in (\mathcal{X} \cap (\mathcal{X} \rightsquigarrow \mathcal{Y})) \rightsquigarrow \mathcal{Y}$. Since $y^\oslash \in \mathcal{X}$ and $y^\oslash \in \mathcal{X} \rightsquigarrow \mathcal{Y}$, $y^\oslash \in \mathcal{X} \cap (\mathcal{X} \rightsquigarrow \mathcal{Y})$ and $My^\oslash \in \mathcal{Y}$. Since $x^\oslash \neq y^\oslash$, by lemma 2.1.2, $My^\oslash \rhd_\beta^* y^\oslash y^\oslash$. Hence, by lemma A.21.4, $M \rhd_\beta^* \lambda y^\oslash.y^\oslash y^\oslash$ and, by lemma 2.1, $\mathrm{d}(M) = \oslash$ and $M \in \mathcal{M}_3^\oslash$.

Conversely, let $M \in \mathcal{M}_3^\oslash$ such that M is closed and $M \rhd_\beta^* \lambda y^\oslash.y^\oslash y^\oslash$. Let $\mathcal{I}$ be an β_3-interpretation and $N \in \mathcal{I}(a\sqcap(a \to b)) = \mathcal{I}(a)\cap(\mathcal{I}(a) \rightsquigarrow \mathcal{I}(b))$ such that $M\diamond N$. By lemma 6.2.1b and lemma A.1.1, $N \in \mathcal{M}_3^\oslash$ and $N \diamond N$. So $NN, MN \in \mathcal{M}_3^\oslash$. Since $\mathcal{I}(b)$ is β-saturated, $NN \in \mathcal{I}(b)$ and $MN \rhd_\beta^* NN$, we have $MN \in \mathcal{I}(b)$ and hence $M \in \mathcal{I}(a \sqcap (a \to b)) \rightsquigarrow \mathcal{I}(b)$. Therefore, $M \in [D]_{\beta_3}$.

7. Let $f, y \in \mathcal{V}_2$ and take $\mathcal{X} = \{M \in \mathcal{M}_3^\oslash \mid M \rhd_\beta^* (f^\oslash)^n(x^\oslash N_1...N_k)$ or $M \rhd_\beta^* (f^\oslash)^n y^\oslash$ where $k, n \geq 0$ and $x \in \mathcal{V}_1\}$. $\mathcal{X}$ is β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^\oslash \subseteq \mathcal{X} \subseteq \mathcal{M}_3^\oslash$. Let $\mathcal{I}$ be a β_3-interpretation such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Nat_0]_{\beta_3}$, then M is closed and $M \in (\mathcal{X} \rightsquigarrow \mathcal{X}) \rightsquigarrow (\mathcal{X} \rightsquigarrow \mathcal{X})$. We have $f^\oslash \in \mathcal{X} \rightsquigarrow \mathcal{X}$, $y^\oslash \in \mathcal{X}$ and $\diamond\{M, f^\oslash, y^\oslash\}$ then $Mf^\oslash y^\oslash \in \mathcal{X}$ and $Mf^\oslash y^\oslash \rhd_\beta^* (f^\oslash)^n(x^\oslash N_1...N_k)$ or $Mf^\oslash y^\oslash \rhd_\beta^* (f^\oslash)^n y^\oslash$ where $n \geq 0$ and $x \in \mathcal{V}_1$. Since M is closed and $\{x^\oslash\} \cap \{y^\oslash, f^\oslash\} = \emptyset$, by lemma 2.1.2, $Mf^\oslash y^\oslash \rhd_\beta^* (f^\oslash)^n y^\oslash$ where $n \geq 1$. Hence, by lemma A.21.4, $M \rhd_\beta^* \lambda f^\oslash.f^\oslash$ or $M \rhd_\beta^* \lambda f^\oslash.\lambda y^\oslash.(f^\oslash)^n y^\oslash$ where $n \geq 1$. Moreover, by lemma 2.1, $\mathrm{d}(M) = \oslash$ and $M \in \mathcal{M}_3^\oslash$.

   Conversely, let $M \in \mathcal{M}_3^\oslash$ such that M is closed and $M \rhd_\beta^* \lambda f^\oslash.f^\oslash$ or $M \rhd_\beta^* \lambda f^\oslash.\lambda y^\oslash.(f^\oslash)^n y^\oslash$ where $n \geq 1$. Let $\mathcal{I}$ be an β_3-interpretation, $N \in \mathcal{I}(a \to a) = \mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)$ and $N' \in \mathcal{I}(a)$ such that $\diamond\{M, N, N'\}$. By lemma 6.2.1b, $N, N' \in \mathcal{M}_3^\oslash$, so $MNN', (N)^m N' \in \mathcal{M}_3^\oslash$, where $m \geq 0$. We show, by induction on $m \geq 0$, that $(N)^m N' \in \mathcal{I}(a)$. Since $MNN' \rhd_\beta^* (N)^m N'$ where $m \geq 0$ and $(N)^m N' \in \mathcal{I}(a)$ which is β-saturated, then $MNN' \in \mathcal{I}(a)$. Hence, $M \in (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)) \to (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(a))$ and $M \in [Nat_0]_{\beta_3}$.

8. By lemma 6.4, $[Nat_1]_{\beta_3} = [\overline{e}_1 Nat_0]_{\beta_3} = [Nat_0]_{\beta_3}^{+1}$. By 7., $[Nat_1]_{\beta_3} = [Nat_0]_{\beta_3}^{+1} = \{M \in \mathcal{M}_3^{(1)} \mid M \rhd_\beta^* \lambda f^{(1)}.f^{(1)}$ or $M \rhd_\beta^* \lambda f^{(1)}.\lambda y^{(1)}.(f^{(1)})^n y^{(1)}$ where $n \geq 1\}$.

9. Let $f, y \in \mathcal{V}_2$ and take $\mathcal{X} = \{M \in \mathcal{M}_3^\oslash \mid M \rhd_\beta^* x^\oslash P_1...P_l$ or $M \rhd_\beta^* f^\oslash(x^\oslash Q_1...Q_n)$ or $M \rhd_\beta^* y^\oslash$ or $M \rhd_\beta^* f^\oslash y^{(1)}$ where $l, n \geq 0$ and $\mathrm{d}(Q_i) \succeq (1)\}$. $\mathcal{X}$ is β-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^\oslash \subseteq \mathcal{X} \subseteq \mathcal{M}_3^\oslash$. Let $\mathcal{I}$ be a β_3-interpretation such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Nat'_0]_{\beta_3}$, then M is closed and $M \in (\mathcal{X}^{+1} \rightsquigarrow \mathcal{X}) \rightsquigarrow (\mathcal{X}^{+1} \rightsquigarrow \mathcal{X})$. Let $N \in \mathcal{X}^{+1}$ such that $N \diamond f^\oslash$. We have $N \rhd_\beta^* x^{(1)} P_1^{+1}...P_k^{+1}$ or $N \rhd_\beta^* y^{(1)}$, then $f^\oslash N \rhd_\beta^* f^\oslash(x^{(1)} P_1^{+1}...P_k^{+1}) \in \mathcal{X}$ or $N \rhd_\beta^* f^\oslash y^{(1)} \in \mathcal{X}$, thus $f^\oslash \in \mathcal{X}^{+1} \rightsquigarrow \mathcal{X}$. We have $f^\oslash \in \mathcal{X}^{+1} \rightsquigarrow \mathcal{X}$, $y^{(1)} \in \mathcal{X}^{+1}$ and $\diamond\{M, f^\oslash, y^{(1)}\}$, then $Mf^\oslash y^{(1)} \in \mathcal{X}$. Since M is closed and $\{x^\oslash, x^{(1)}\} \cap \{y^{(1)}, f^\oslash\} = \emptyset$, by lemma 2.1.2, $Mf^\oslash y^{(1)} \rhd_\beta^* f^\oslash y^{(1)}$. Hence, by lemma A.21.4, $M \rhd_\beta^* \lambda f^\oslash.f^\oslash$ or $M \rhd_\beta^* \lambda f^\oslash.\lambda y^{(1)}.f^\oslash y^{(1)}$. Moreover, by lemma 2.1, $\mathrm{d}(M) = \oslash$ and $M \in \mathcal{M}_3^\oslash$.

   Conversely, let $M \in \mathcal{M}_3^\oslash$ such M is closed and $M \rhd_\beta^* \lambda f^\oslash.f^\oslash$ or $M \rhd_\beta^* \lambda f^\oslash.\lambda y^{(1)}.f^\oslash y^{(1)}$. Let $\mathcal{I}$ be an β-interpretation, $N \in \mathcal{I}(\overline{e}_1 a \to a) = \mathcal{I}(a)^{+1} \rightsquigarrow \mathcal{I}(a)$ and $N' \in \mathcal{I}(a)^{+1}$ where $\diamond\{M, N, N'\}$. By lemma 6.2.1b, $N \in \mathcal{M}_3^\oslash$ and $N' \in \mathcal{M}_3^{(1)}$, so $MNN', NN' \in \mathcal{M}_3^\oslash$. Since $MNN' \rhd_\beta^* NN'$, $NN' \in \mathcal{I}(a)$ and $\mathcal{I}(a)$ is β-saturated, then $MNN' \in \mathcal{I}(a)$. Hence, $M \in (\mathcal{I}(a)^{+1} \rightsquigarrow \mathcal{I}(a)) \to (\mathcal{I}(a)^{+1} \rightsquigarrow \mathcal{I}(a))$ and $M \in [Nat'_0]$.

$\square$

A.7. Completeness

Proof:
[of Remark 7.2] 1) For every interpretation $\mathcal{I}$, $\mathcal{I}(e_1 a \to a) = \mathcal{I}(e_2 a \to a) = \mathcal{I}(a)^+ \rightsquigarrow \mathcal{I}(a)$.

2) If $\lambda f^0.f^0 : \langle () \vdash_2 Nat''_0 \rangle$, by lemma 5.2.2 and 5.2.1, $f^0 : \langle f^0 : e_1 a \to a \vdash_2 e_2 a \to a \rangle$ and $e_1 a \to a \sqsubseteq e_2 a \to a$. Thus, by lemma A.10.4, $e_2 a \sqsubseteq e_1 a$. Again, by lemma A.10.1, $e_1 a = e_2 U$ where $a \sqsubseteq U$. This is impossible since $e_1 \neq e_2$. □

Proof:
[of Lemma 7.1] 1) By induction on U.

2) If $U^- = V^-$ then $\overline{e}_1 U^- = \overline{e}_1 V^-$ and by 1., $U = V$. □

Lemma A.22.

1. If $\mathrm{d}(U) = n$, then $\mathbb{V}_U$ is an infinite set of variables of degree n and if $y^n \in \mathbb{V}_U$, then $y \in \mathcal{V}_2$.
2. If $U \neq V$ and $\mathrm{d}(U) = \mathrm{d}(V) = n$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.
3. If $y^n \in \mathbb{V}_U$, then $y^{n+1} \in \mathbb{V}_{e_c U}$.
4. If $y^{n+1} \in \mathbb{V}_U$, then $y^n \in \mathbb{V}_{U^-}$.

Proof:
1. and 2. By induction on n and using lemma 7.1. 3. Because $(e_c U)^- = U$. 4. By definition. □

Lemma A.23.

1. If $\Gamma \subset \mathbb{H}^n$, then $e\Gamma \subset \mathbb{H}^{n+1}$.
2. If $\Gamma \subset \mathbb{H}^{n+1}$, then $\Gamma^- \subset \mathbb{H}^n$.
3. If $\Gamma_1 \subset \mathbb{H}^n$, $\Gamma_2 \subset \mathbb{H}^m$ and $m \geq n$, then $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^n$.

Proof:
1. resp. 2. By lemma A.22.3 resp. A.22.4. 3. First note that $\mathbb{H}^m \subseteq \mathbb{H}^n$. Let $(x^p : U_1 \sqcap U_2) \in \Gamma_1 \sqcap \Gamma_2$ where $(x^p : U_1) \in \Gamma_1 \subset \mathbb{H}^n$ and $(x^p : U_2) \in \Gamma_2 \subset \mathbb{H}^m \subseteq \mathbb{H}^n$, then $\mathrm{d}(U_1) = \mathrm{d}(U_2) = p$ and $x^p \in \mathbb{V}_{U_1} \cap \mathbb{V}_{U_2}$. Hence, by lemma A.22.2, $U_1 = U_2$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \cup \Gamma_2 \subset \mathbb{H}^n$. □

Lemma A.24.

1. $(\mathcal{V}^n)^+ = \mathcal{V}^{n+1}$.
2. If $y \in \mathcal{V}_2$ and $(M\ y^m) \in \mathcal{V}^n$, then $M \in \mathcal{V}^n$.
3. If $M \in \mathcal{V}^n$, $M \diamond N$, $N \in \mathbb{M}$ and $\mathrm{d}(N) = m \geq n$, then $MN \in \mathcal{V}^n$.
4. If $\mathrm{d}(M) = n$, $m \geq n$, $M \diamond N$, $M \in \mathbb{M}$ and $N \in \mathcal{V}^m$, then $MN \in \mathcal{V}^n$.

Proof:
Easy. □

Proof:
[of Lemma 7.2]

1. First we show that $\mathbb{I}(a)$ is saturated. Let $M \rhd_\beta^* N$ and $N \in \mathbb{I}(a)$.

   - If $N \in \mathcal{V}^0$ then $N \in \mathbb{M}^0$ and $\exists x^i$ such that $x \in \mathcal{V}_1$, $i \geq 0$ and $x^i \in FV(N)$. By lemma 6.1.9, $\mathbb{M}^0$ is saturated and so, $M \in \mathbb{M}^0$. By lemma 2.1.2, $FV(M) = FV(N)$ and so, $x^i \in FV(M)$. Hence, $M \in \mathcal{V}^0$
   - If $N \in \{M \in \mathcal{M}_2^0 \mid M : \langle \mathbb{H}^0 \vdash_2 a \rangle\}$ then $\exists \Gamma \subset \mathbb{H}^0$, such that $N : \langle \Gamma \vdash_2 a \rangle$. By subject expansion corollary 5.7, $M : \langle \Gamma \vdash_2 a \rangle$ and by lemma 2.1.2, $\mathrm{d}(M) = \mathrm{d}(N)$. Hence, $M \in \{M \in \mathcal{M}_2^0 \mid M : \langle \mathbb{H}^0 \vdash_2 a \rangle\}$.

   Now we show that $\forall x \in \mathcal{V}_1, \mathcal{N}_x^0 \subseteq \mathbb{I}(a) \subseteq \mathbb{M}^0$.

   - Let $x \in \mathcal{V}_1$ and $M \in \mathcal{N}_x^0$. Hence, $M = x^0 N_1 \ldots N_k \in \mathbb{M}^0$, and $x^0 \in FV(M)$. Thus, $M \in \mathcal{V}^0$.
   - Let $M \in \mathbb{I}(a)$. If $M \in \mathcal{V}^0$, then $M \in \mathbb{M}^0$. Else, $\exists \Gamma \subset \mathbb{H}^0$ where $M : \langle \Gamma \vdash_2 a \rangle$. Since by lemma 4.1, M is good and $\mathrm{d}(M) = \mathrm{d}(a) = 0$, $M \in \mathbb{M}^0$.

2. By induction on U good.

   - $U = a$: By definition of $\mathbb{I}$ and by 1.
   - $U = e_c V$: $\mathrm{d}(V) = n - 1$ and, by lemma 3.1, V is good. By IH and lemma A.24.1, $\mathbb{I}(e_c V) = (\mathbb{I}(V))^+ = (\mathcal{V}^{n-1} \cup \{M \in \mathbb{M}^{n-1} \mid M : \langle \mathbb{H}^{n-1} \vdash_2 V \rangle\})^+ = \mathcal{V}^n \cup (\{M \in \mathbb{M}^{n-1} \mid M : \langle \mathbb{H}^{n-1} \vdash_2 V \rangle\})^+$.
     - If $M \in \mathbb{M}^{n-1}$ and $M : \langle \mathbb{H}^{n-1} \vdash_2 V \rangle$, then $M : \langle \Gamma \vdash_2 V \rangle$ where $\Gamma \subset \mathbb{H}^{n-1}$. By exp and lemma A.23.1, $M^+ : \langle e_c \Gamma \vdash_2 e_c V \rangle$ and $e_c \Gamma \subset \mathbb{H}^n$. Thus by lemma 4.1.2, $M^+ \in \mathbb{M}^n$ and $M^+ : \langle \mathbb{H}^n \vdash_2 e_c V \rangle$.
     - If $M \in \mathbb{M}^n$ and $M : \langle \mathbb{H}^n \vdash_2 e_c V \rangle$, then $M : \langle \Gamma \vdash_2 e_c V \rangle$ where $\Gamma \subset \mathbb{H}^n$. By lemmas 4.1.3, and A.23.2, $M^- : \langle \Gamma^- \vdash_2 V \rangle$ and $\Gamma^- \subset \mathbb{H}^{n-1}$. Thus, by lemma A.3.(1b and 1d), $M = (M^-)^+$ and $M^- \in \mathbb{M}^{n-1}$. Hence, $M^- \in \{M \in \mathbb{M}^{n-1} \mid M : \langle \mathbb{H}^{n-1} \vdash_2 V \rangle\}$.

     Hence $(\{M \in \mathbb{M}^{n-1} \mid M : \langle \mathbb{H}^{n-1} \vdash_2 V \rangle\})^+ = \{M \in \mathbb{M}^n \mid M : \langle \mathbb{H}^n \vdash_2 U \rangle\}$ and $\mathbb{I}(U) = \mathcal{V}^n \cup \{M \in \mathbb{M}^n \mid M : \langle \mathbb{H}^n \vdash_2 U \rangle\}$.
   - $U = U_1 \sqcap U_2$: By lemma 3.1, U_1, U_2 are good and $\mathrm{d}(U_1) = \mathrm{d}(U_2) = n$. By IH, $\mathbb{I}(U_1 \sqcap U_2) = \mathbb{I}(U_1) \cap \mathbb{I}(U_2) =$
     $(\mathcal{V}^n \cup \{M \in \mathbb{M}^n \mid M : \langle \mathbb{H}^n \vdash_2 U_1 \rangle\}) \cap (\mathcal{V}^n \cup \{M \in \mathbb{M}^n \mid M : \langle \mathbb{H}^n \vdash_2 U_2 \rangle\}) =$
     $\mathcal{V}^n \cup (\{M \in \mathbb{M}^n \mid M : \langle \mathbb{H}^n \vdash_2 U_1 \rangle\} \cap \{M \in \mathbb{M}^n \mid M : \langle \mathbb{H}^n \vdash_2 U_2 \rangle\})$.
     - If $M \in \mathbb{M}^n$, $M : \langle \mathbb{H}^n \vdash_2 U_1 \rangle$ and $M : \langle \mathbb{H}^n \vdash_2 U_2 \rangle$, then $M : \langle \Gamma_1 \vdash_2 U_1 \rangle$ and $M : \langle \Gamma_2 \vdash_2 U_2 \rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{H}^n$. By remark 4.1, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 U_1 \sqcap U_2 \rangle$. Since by lemma A.23.3, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^n$, $M : \langle \mathbb{H}^n \vdash_2 U_1 \sqcap U_2 \rangle$.
     - If $M \in \mathbb{M}^n$ and $M : \langle \mathbb{H}^n \vdash_2 U_1 \sqcap U_2 \rangle$, then $M : \langle \Gamma \vdash_2 U_1 \sqcap U_2 \rangle$ where $\Gamma \subset \mathbb{H}^n$. By $\sqsubseteq$, $M : \langle \Gamma \vdash_2 U_1 \rangle$ and $M : \langle \Gamma \vdash_2 U_2 \rangle$. Hence, $M : \langle \mathbb{H}^n \vdash_2 U_1 \rangle$ and $M : \langle \mathbb{H}^n \vdash_2 U_2 \rangle$.

     We deduce that $\mathbb{I}(U_1 \sqcap U_2) = \mathcal{V}^n \cup \{M \in \mathbb{M}^n \mid M : \langle \mathbb{H}^n \vdash_2 U_1 \sqcap U_2 \rangle\}$.
   - $U = V \to T$: By lemmas 3.1.2, V, T are good and let $m = \mathrm{d}(V) \geq \mathrm{d}(T) = 0$. By IH, $\mathbb{I}(V) = \mathcal{V}^m \cup \{M \in \mathbb{M}^m \mid M : \langle \mathbb{H}^m \vdash_2 V \rangle\}$ and $\mathbb{I}(T) = \mathcal{V}^0 \cup \{M \in \mathbb{M}^0 \mid M : \langle \mathbb{H}^0 \vdash_2 T \rangle\}$. Note that $\mathbb{I}(V \to T) = \mathbb{I}(V) \rightsquigarrow \mathbb{I}(T)$.

- Let $M \in \mathbb{I}(V) \rightsquigarrow \mathbb{I}(T)$ and by lemma A.22.1, let $y^m \in \mathbb{V}_V$ such that $y \in \mathcal{V}_2$, and $\forall n, y^n \notin FV(M)$. Then $y^m \diamond M$. By remark 4.1, $y^m : \langle (y^m : V) \vdash_2 V\rangle$. Hence $y^m : \langle \mathbb{H}^m \vdash_2 V\rangle$ and so $y^m \in \mathbb{I}(V)$ and $My^m \in \mathbb{I}(T)$.
  - $*$ If $My^m \in \mathcal{V}^0$, then since $y \in \mathcal{V}_2$, by lemma A.24.2, $M \in \mathcal{V}^0$.
  - $*$ If $My^m \in \{M \in \mathbb{M}^0 \mid M : \langle \mathbb{H}^0 \vdash_2 T\rangle\}$ then $My^m \in \mathbb{M}^0$ and $My^m : \langle \mathbb{H}^0 \vdash_2 T\rangle$. So $My^m : \langle \Gamma \vdash_2 T\rangle$ where $\Gamma \subset \mathbb{H}^0$. Since $y^m \in FV(My^m)$ and since by lemma A.14, $dom(\Gamma) = FV(My^m)$, $\Gamma = \Gamma', (y^m : V')$ where by lemma 4.1.2, $\mathrm{d}(V') = m$. Since $(y^m : V') \in \mathbb{H}^0$, $\mathrm{d}(V') = m$ and $y^m \in \mathbb{V}_{V'}$, by lemma A.22.2, $V = V'$. So $My^m : \langle \Gamma', (y^m : V) \vdash_2 T\rangle$ and by lemma 5.2.1, $M : \langle \Gamma' \vdash_2 V \to T\rangle$ and by lemma 4.1.2, $M \in \mathbb{M}$ and $\mathrm{d}(M) = 0$. Since $\Gamma' \subset \mathbb{H}^0$, $M : \langle \mathbb{H}^0 \vdash_2 V \to T\rangle$. And so, $M \in \{M \in \mathbb{M}^0 \mid M : \langle \mathbb{H}^0 \vdash_2 V \to T\rangle\}$.
- Let $M \in \mathcal{V}^0 \cup \{M \in \mathbb{M}^0 \mid M : \langle \mathbb{H}^0 \vdash_2 V \to T\rangle\}$ and $N \in \mathbb{I}(V) = \mathcal{V}^m \cup \{M \in \mathbb{M}^m \mid M : \langle \mathbb{H}^m \vdash_2 V\rangle\}$ such that $M \diamond N$. Then, $\mathrm{d}(N) = m$.
  - $*$ Case $M \in \mathcal{V}^0$. Since $N \in \mathbb{M}$, by lemma A.24.3, $MN \in \mathcal{V}^0 \subseteq \mathbb{I}(T)$.
  - $*$ Case $M \in \{M \in \mathbb{M}^0 \mid M : \langle \mathbb{H}^0 \vdash_2 V \to T\rangle\}$, so $M \in \mathbb{M}^0$.
    - $\cdot$ If $N \in \mathcal{V}^m$, then, by lemma A.24.4, $MN \in \mathcal{V}^0 \subseteq \mathbb{I}(T)$.
    - $\cdot$ If $N \in \{M \in \mathbb{M}^m \mid M : \langle \mathbb{H}^m \vdash_2 V\rangle\}$, so $M : \langle \Gamma_1 \vdash_2 V \to T\rangle$ and $N : \langle \Gamma_2 \vdash_2 V\rangle$ where $\Gamma_1 \subset \mathbb{H}^0$ and $\Gamma_2 \subset \mathbb{H}^m$. Since $M \diamond N$ then by lemma A.14.4, $\Gamma_1 \diamond \Gamma_2$. So by $\to_E$, $MN : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle$. By lemma A.23.3, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^0$. Therefore $MN : \langle \mathbb{H}^0 \vdash_2 T\rangle$. By lemma 4.1, $MN \in \mathbb{M}^0$. Hence, $MN \in \{M \in \mathbb{M}^0 \mid M : \langle \mathbb{H}^0 \vdash_2 T\rangle\} \subseteq \mathbb{I}(T)$.

  Hence, $M \in \mathbb{I}(V \to T)$.

We deduce that $\mathbb{I}(V \to T) = \mathcal{V}^0 \cup \{M \in \mathbb{M}^0 \mid M : \langle \mathbb{H}^0 \vdash_2 V \to T\rangle\}$.

$\square$

Proof:
[of Theorem 7.1] Recall: $[U]_{\beta_2} = \{M \in \mathcal{M}_2 \mid M \text{ is closed and } M \in \bigcap_{\mathcal{I} \in \beta_2\text{-int}} \mathcal{I}(U)\}$.

1. Let $M \in [U]_{\beta_2}$. Then M is a closed term and $M \in \mathbb{I}(U)$. Hence, by lemma 7.2, $M \in \mathcal{V}^n \cup \{M \in \mathbb{M}^n \mid M : \langle \mathbb{H}^n \vdash_2 U\rangle\}$. Since M is closed, $M \notin \mathcal{V}^n$. Hence, $M \in \{M \in \mathbb{M}^n \mid M : \langle \mathbb{H}^n \vdash_2 U\rangle\}$ and so, $M : \langle \Gamma \vdash_2 U\rangle$ where $\Gamma \subset \mathbb{H}^n$. Since M is closed, by lemma A.14.1, $\Gamma = ()$ and $M : \langle () \vdash_2 U\rangle$.

   Conversely, let $M \in \mathbb{M}^n$ where $M : \langle () \vdash_2 U\rangle$. By lemma A.14.1, M is closed. Let $\mathcal{I}$ be a β_2-interpretation. By soundness lemma 6.3, $M \in \mathcal{I}(U)$. Thus $M \in [U]_{\beta_2}$.

2. Let $M \in [U]_{\beta_2}$ such that $M \rhd_\beta^* N$. By 1, $M \in \mathbb{M}^n$ and $M : \langle () \vdash_2 U\rangle$. By subject reduction corollary 5.7, $N : \langle () \vdash_2 U\rangle$. By lemma 2.1.2, $\mathrm{d}(N) = \mathrm{d}(M) = n$. By lemma 4.1.2, $N \in \mathbb{M}$. Hence, by 1, $N \in [U]_{\beta_2}$.

3. Let $N \in [U]_{\beta_2}$ such that $M \rhd_\beta^* N$. By 1, $N \in \mathbb{M}^n$ and $N : \langle () \vdash_2 U\rangle$. By subject expansion corollary 5.7, $M : \langle () \vdash_2 U\rangle$. By lemma 2.1.2, $\mathrm{d}(N) = \mathrm{d}(M) = n$. By lemma 4.1.2, $M \in \mathbb{M}$. Hence, by 1, $M \in [U]_{\beta_2}$.

$\square$

Proof:

1. We do two cases:
   Case $r = \beta\eta$. It is easy to see that $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{O}^{\oslash} \subseteq \mathbb{I}_{\beta\eta}(a)$. Now we show that $\mathbb{I}_{\beta\eta}(a)$ is $\beta\eta$-saturated. Let $M \rhd_{\beta\eta}^* N$ and $N \in \mathbb{I}_{\beta\eta}(a)$.

   - If $N \in \mathcal{O}^{\oslash}$ then $N \in \mathcal{M}_3^{\oslash}$ and $\exists L$ and $x \in \mathcal{V}_1$ such that $x^L \in \mathrm{FV}(N)$. By theorem 2.1.2, $\mathrm{FV}(N) \subseteq \mathrm{FV}(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$, hence, $M \in \mathcal{O}^{\oslash}$
   - If $N \in \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash^* a \rangle\}$ then $N \rhd_{\beta\eta}^* N'$ and $\exists \Gamma \subset \mathbb{H}^{\oslash}$, such that $N' : \langle \Gamma \vdash a \rangle$. Hence $M \rhd_{\beta\eta}^* N'$ and since by theorem 2.1.2, $\mathrm{d}(M) = \mathrm{d}(N')$, $M \in \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash^* a \rangle\}$.

   Case $r = \beta$. It is easy to see that $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{O}^{\oslash} \subseteq \mathbb{I}_{\beta}(a)$. Now we show that $\mathbb{I}_{\beta}(a)$ is β-saturated. Let $M \rhd_{\beta}^* N$ and $N \in \mathbb{I}_{\beta}(a)$.

   - If $N \in \mathcal{O}^{\oslash}$ then $N \in \mathcal{M}_3^{\oslash}$ and $\exists L$ and $x \in \mathcal{V}_1$ such that $x^L \in \mathrm{FV}(N)$. By theorem 2.1.2, $\mathrm{FV}(N) \subseteq \mathrm{FV}(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$, hence, $M \in \mathcal{O}^{\oslash}$
   - If $N \in \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash a \rangle\}$ then $\exists \Gamma \subset \mathbb{H}^{\oslash}$, such that $N : \langle \Gamma \vdash a \rangle$. By theorem 5.2, $M : \langle \Gamma\uparrow^M \vdash a \rangle$. Since by theorem 2.1.2, $\mathrm{FV}(N) \subseteq \mathrm{FV}(M)$, let $\mathrm{FV}(N) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$ and $\mathrm{FV}(M) = \mathrm{FV}(N) \cup \{x_{n+1}^{L_{n+1}}, \ldots, x_{n+m}^{L_{n+m}}\}$. So $\Gamma\uparrow^M = \Gamma, (x_{n+1}^{L_{n+1}} : \omega^{L_{n+1}}, \ldots, x_{n+m}^{L_{n+m}} : \omega^{L_{n+m}})$. $\forall n+1 \leq i \leq n+m$, let U_i such that $x_i^{L_i} \in \mathbb{V}_{U_i}$. Then $\Gamma, (x_{n+1}^{L_{n+1}} : U_{n+1}, \ldots, x_{n+m}^{L_{n+m}} : U_{n+m}) \subset \mathbb{H}^{\oslash}$ and by $\sqsubseteq$, $M : \langle \Gamma, (x_{n+1}^{L_{n+1}} : U_{n+1}, \ldots, x_{n+m}^{L_{n+m}} : U_{n+m}) \vdash a \rangle$. Thus $M : \langle \mathbb{H}^{\oslash} \vdash a \rangle$ and since by theorem 2.1.2, $\mathrm{d}(M) = \mathrm{d}(N)$, $M \in \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash a \rangle\}$.

2. By induction on U.

   - $U = a$: By definition of $\mathbb{I}_{\beta\eta}$.
   - $U = \omega^L$: By definition, $\mathbb{I}_{\beta\eta}(\omega^L) = \mathcal{M}_3^L$. Hence, $\mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash^* \omega^L \rangle\} \subseteq \mathbb{I}_{\beta\eta}(\omega^L)$.
     Let $M \in \mathbb{I}_{\beta\eta}(\omega^L)$ where $\mathrm{FV}(M) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$ then $M \in \mathcal{M}_3^L$. $\forall\, 1 \leq i \leq n$, let U_i the type such that $x_i^{L_i} \in \mathbb{V}_{U_i}$. Then $\Gamma = (x_i^{L_i} : U_i)_n \subset \mathbb{H}^L$. By lemma 4.2.2 and lemma 8.2, $M : \langle \Gamma \vdash \omega^L \rangle$. Hence $M : \langle \mathbb{H}^L \vdash \omega^L \rangle$. Therefore, $\mathbb{I}(\omega^L) \subseteq \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash^* \omega^L \rangle\}$.
     We deduce $\mathbb{I}_{\beta\eta}(\omega^L) = \mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash^* \omega^L \rangle\}$.
   - $U = \overline{e}_i V$: $L = i :: K$ and $\mathrm{d}(V) = K$. By IH and lemma 8.3, $\mathbb{I}_{\beta\eta}(\overline{e}_i V) = (\mathbb{I}_{\beta\eta}(V))^{+i} = (\mathcal{O}^K \cup \{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash^* V \rangle\})^{+i} =$
     $\mathcal{O}^L \cup (\{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash^* V \rangle\})^{+i}$.
     - If $M \in \mathcal{M}_3^K$ and $M : \langle \mathbb{H}^K \vdash^* V \rangle$, then $M \rhd_{\beta\eta}^* N$ and $N : \langle \Gamma \vdash V \rangle$ where $\Gamma \subset \mathbb{H}^K$. By e, lemmas A.5 and 8.2, $N^{+i} : \langle \overline{e}_i \Gamma \vdash \overline{e}_i V \rangle$, $M^{+i} \rhd_{\beta\eta}^* N^{+i}$ and $\overline{e}_i \Gamma \subset \mathbb{H}^L$. Thus $M^{+i} \in \mathcal{M}_3^L$ and $M^{+i} : \langle \mathbb{H}^L \vdash^* U \rangle$.

- If $M \in \mathcal{M}_3^L$ and $M : \langle \mathbb{H}^L \vdash^* U\rangle$, then $M \rhd^*_{\beta\eta} N$ and $N : \langle \Gamma \vdash U\rangle$ where $\Gamma \subset \mathbb{H}^L$. By lemmas A.5, 4.1, and 8.2, $M^{-i} \rhd^*_{\beta\eta} N^{-i}$, $N^{-i} : \langle \Gamma^{-i} \vdash V\rangle$ and $\Gamma^{-i} \subset \mathbb{H}^K$. Thus by lemma A.5, $M = (M^{-i})^{+i}$ and $M^{-i} \in \{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash^* V\rangle\}$.

Hence $(\{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash^* V\rangle\})^{+i} = \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash^* U\rangle\}$ and $\mathbb{I}_{\beta\eta}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash^* U\rangle\}$.

- $U = U_1 \sqcap U_2$: By IH, $\mathbb{I}_{\beta\eta}(U_1 \sqcap U_2) = \mathbb{I}_{\beta\eta}(U_1) \cap \mathbb{I}_{\beta\eta}(U_2) = (\mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash^* U_1\rangle\}) \cap (\mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash^* U_2\rangle\}) = \mathcal{O}^L \cup (\{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash^* U_1\rangle\} \cap \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash^* U_2\rangle\})$.
  - If $M \in \mathcal{M}_3^L$, $M : \langle \mathbb{H}^L \vdash^* U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash^* U_2\rangle$, then $M \rhd^*_{\beta\eta} N_1$, $M \rhd^*_{\beta\eta} N_2$, $N_1 : \langle \Gamma_1 \vdash U_1\rangle$ and $N_2 : \langle \Gamma_2 \vdash U_2\rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{H}^L$. By confluence theorem 2.2 and subject reduction theorem 5.1, $\exists M'$ such that $M \rhd^*_{\beta\eta} M'$, $M' : \langle \Gamma_1 \upharpoonright_{M'} \vdash U_1\rangle$ and $M' : \langle \Gamma_2 \upharpoonright_{M'} \vdash U_2\rangle$. Hence by Remark 4.1 and lemma 2.1 and lemma 4.2.3a and lemma A.18.2, $M' : \langle (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{M'} \vdash U_1 \sqcap U_2\rangle$ and, by lemma 8.2, $(\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{M'} \subseteq \Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$. Thus $M : \langle \mathbb{H}^L \vdash^* U_1 \sqcap U_2\rangle$.
  - If $M \in \mathcal{M}_3^L$ and $M : \langle \mathbb{H}^L \vdash^* U_1 \sqcap U_2\rangle$, then $M \rhd^*_{\beta\eta} N$, $N : \langle \Gamma \vdash U_1 \sqcap U_2\rangle$ and $\Gamma \subset \mathbb{H}^L$. By $\sqsubseteq$, $N : \langle \Gamma \vdash U_1\rangle$ and $N : \langle \Gamma \vdash U_2\rangle$.
    Hence, $M : \langle \mathbb{H}^L \vdash^* U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash^* U_2\rangle$.

  We deduce that $\mathbb{I}_{\beta\eta}(U_1 \sqcap T_2) = \mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash^* U_1 \sqcap U_2\rangle\}$.

- $U = V \to T$: Let $\mathrm{d}(T) = \oslash \preceq K = \mathrm{d}(V)$. By IH, $\mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash^* V\rangle\}$ and $\mathbb{I}_{\beta\eta}(T) = \mathcal{O}^\oslash \cup \{M \in \mathcal{M}_3^\oslash \mid M : \langle \mathbb{H}^\oslash \vdash^* T\rangle\}$. Note that $\mathbb{I}_{\beta\eta}(V \to T) = \mathbb{I}_{\beta\eta}(V) \rightsquigarrow \mathbb{I}_{\beta\eta}(T)$.
  - Let $M \in \mathbb{I}_{\beta\eta}(V) \rightsquigarrow \mathbb{I}_{\beta\eta}(T)$ and, by lemma 8.1, let $y^K \in \mathbb{V}_V$ such that $\forall K, y^K \notin \mathrm{FV}(M)$. Then $M \diamond y^K$. By remark 4.1, $y^K : \langle (y^K : V) \vdash^* V\rangle$. Hence $y^K : \langle \mathbb{H}^K \vdash^* V\rangle$. Thus, $y^K \in \mathbb{I}_{\beta\eta}(V)$ and $My^K \in \mathbb{I}_{\beta\eta}(T)$.
    - If $My^K \in \mathcal{O}^\oslash$, then since $y \in \mathcal{V}_2$, by lemma 8.3, $M \in \mathcal{O}^\oslash$.
    - If $My^K \in \{M \in \mathcal{M}_3^\oslash \mid M : \langle \mathbb{H}^\oslash \vdash^* T\rangle\}$ then $My^K \rhd^*_{\beta\eta} N$ and $N : \langle \Gamma \vdash T\rangle$ such that $\Gamma \subset \mathbb{H}^\oslash$, hence, $\lambda y^K.My^K \rhd^*_{\beta\eta} \lambda y^K.N$. We have two cases:
      - If $y^K \in \mathrm{dom}(\Gamma)$, then $\Gamma = \Delta, (y^K : V)$ and by $\to_I$, $\lambda y^K.N : \langle \Delta \vdash V \to T\rangle$.
      - If $y^K \notin \mathrm{dom}(\Gamma)$, let $\Delta = \Gamma$. By $\to'_I$, $\lambda y^K.N : \langle \Delta \vdash \omega^K \to T\rangle$. By $\sqsubseteq$, since $\langle \Delta \vdash \omega^K \to T\rangle \sqsubseteq \langle \Delta \vdash V \to T\rangle$, we have $\lambda y^K.N : \langle \Delta \vdash V \to T\rangle$.

      Note that $\Delta \subset \mathbb{H}^\oslash$. Since $\lambda y^K.My^K \rhd^*_{\beta\eta} M$ and $\lambda y^K.My^K \rhd^*_{\beta\eta} \lambda y^K.N$, by theorem 2.2 and theorem 5.1, there is M' such that $M \rhd^*_{\beta\eta} M'$, $\lambda y^K.N \rhd^*_{\beta\eta} M'$, $M' : \langle \Delta \upharpoonright_{M'} \vdash V \to T\rangle$. Since $\Delta \upharpoonright_{M'} \subseteq \Delta \subset \mathbb{H}^\oslash$, $M : \langle \mathbb{H}^\oslash \vdash^* V \to T\rangle$.
  - Let $M \in \mathcal{O}^\oslash \cup \{M \in \mathcal{M}_3^\oslash \mid M : \langle \mathbb{H}^\oslash \vdash^* V \to T\rangle\}$ and $N \in \mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash^* V\rangle\}$ such that $M \diamond N$. Then, $\mathrm{d}(N) = K \succeq \oslash = \mathrm{d}(M)$.
    - If $M \in \mathcal{O}^\oslash$, then, by lemma 8.3, $MN \in \mathcal{O}^\oslash$.
    - If $M \in \{M \in \mathcal{M}_3^\oslash \mid M : \langle \mathbb{H}^\oslash \vdash^* V \to T\rangle\}$, then
      - If $N \in \mathcal{O}^K$, then, by lemma 8.3, $MN \in \mathcal{O}^\oslash$.
      - If $N \in \{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash^* V\rangle\}$ then $M \rhd^*_{\beta\eta} M_1$, $N \rhd^*_{\beta\eta} N_1$, $M_1 : \langle \Gamma_1 \vdash V \to T\rangle$ and $N_1 : \langle \Gamma_2 \vdash V\rangle$ where $\Gamma_1 \subset \mathbb{H}^\oslash$ and $\Gamma_2 \subset \mathbb{H}^K$. By lemma A.5 and

theorem 2.1, $MN \rhd^*_{\beta\eta} M_1N_1$ and, by $\to_E$ and lemma 4.2.3, $M_1N_1 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T\rangle$. By lemma 8.2, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^{\oslash}$. Therefore $MN : \langle \mathbb{H}^{\oslash} \vdash^* T\rangle$.

We deduce that $\mathbb{I}_{\beta\eta}(V \to T) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash^* V \to T\rangle\}$.

3. We only do the case $r = \beta$. By induction on U.

   - $U = a$: By definition of $\mathbb{I}_\beta$.
   - $U = \omega^L$: By definition, $\mathbb{I}_\beta(\omega^L) = \mathcal{M}_3^L$. Hence, $\mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash \omega^L\rangle\} \subseteq \mathbb{I}_\beta(\omega^L)$.
     Let $M \in \mathbb{I}_\beta(\omega^L)$ where $\mathrm{FV}(M) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$ then $M \in \mathcal{M}_3^L$. $\forall\, 1 \le i \le n$, let U_i the type such that $x_i^{L_i} \in \mathbb{V}_{U_i}$. Then $\Gamma = (x_i^{L_i} : U_i)_n \subset \mathbb{H}^L$. By lemma 4.2.2 and lemma 8.2, $M : \langle \Gamma \vdash \omega^L\rangle$. Hence $M : \langle \mathbb{H}^L \vdash \omega^L\rangle$. Therefore, $\mathbb{I}(\omega^L) \subseteq \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash \omega^L\rangle\}$.
     We deduce $\mathbb{I}_\beta(\omega^L) = \mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash \omega^L\rangle\}$.
   - $U = \overline{e}_i V$: $L = i :: K$ and $\mathrm{d}(V) = K$. By IH and lemma 8.3, $\mathbb{I}_\beta(\overline{e}_i V) = (\mathbb{I}_\beta(V))^{+i} = (\mathcal{O}^K \cup \{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash V\rangle\})^{+i} =$
     $\mathcal{O}^L \cup (\{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash V\rangle\})^{+i}$.
     - If $M \in \mathcal{M}_3^K$ and $M : \langle \mathbb{H}^K \vdash V\rangle$, then $M : \langle \Gamma \vdash V\rangle$ where $\Gamma \subset \mathbb{H}^K$. By e and 8.2, $M^{+i} : \langle \overline{e}_i\Gamma \vdash \overline{e}_i V\rangle$ and $\overline{e}_i\Gamma \subset \mathbb{H}^L$. Thus $M^{+i} \in \mathcal{M}_3^L$ and $M^{+i} : \langle \mathbb{H}^L \vdash U\rangle$.
     - If $M \in \mathcal{M}_3^L$ and $M : \langle \mathbb{H}^L \vdash U\rangle$, then $M : \langle \Gamma \vdash U\rangle$ where $\Gamma \subset \mathbb{H}^L$. By lemmas 4.1, and 8.2, $M^{-i} : \langle \Gamma^{-i} \vdash V\rangle$ and $\Gamma^{-i} \subset \mathbb{H}^K$. Thus by lemma A.5, $M = (M^{-i})^{+i}$ and $M^{-i} \in \{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash V\rangle\}$.

     Hence $(\{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash V\rangle\})^{+i} = \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash U\rangle\}$ and $\mathbb{I}_\beta(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash U\rangle\}$.
   - $U = U_1 \sqcap U_2$: By IH, $\mathbb{I}_\beta(U_1 \sqcap U_2) = \mathbb{I}_\beta(U_1) \cap \mathbb{I}_\beta(U_2) = (\mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash U_1\rangle\}) \cap (\mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash U_2\rangle\}) = \mathcal{O}^L \cup (\{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash U_1\rangle\} \cap \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash U_2\rangle\})$.
     - If $M \in \mathcal{M}_3^L$, $M : \langle \mathbb{H}^L \vdash U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash U_2\rangle$, then $M : \langle \Gamma_1 \vdash U_1\rangle$ and $M : \langle \Gamma_2 \vdash U_2\rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{H}^L$. Hence by Remark 4.1, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash U_1 \sqcap U_2\rangle$ and, by lemma 8.2, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$. Thus $M : \langle \mathbb{H}^L \vdash U_1 \sqcap U_2\rangle$.
     - If $M \in \mathcal{M}_3^L$ and $M : \langle \mathbb{H}^L \vdash U_1 \sqcap U_2\rangle$, then $M : \langle \Gamma \vdash U_1 \sqcap U_2\rangle$ and $\Gamma \subset \mathbb{H}^L$. By $\sqsubseteq$, $M : \langle \Gamma \vdash U_1\rangle$ and $M : \langle \Gamma \vdash U_2\rangle$. Hence, $M : \langle \mathbb{H}^L \vdash U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash U_2\rangle$.

     We deduce that $\mathbb{I}_\beta(U_1 \sqcap T_2) = \mathcal{O}^L \cup \{M \in \mathcal{M}_3^L \mid M : \langle \mathbb{H}^L \vdash U_1 \sqcap U_2\rangle\}$.
   - $U = V \to T$: Let $\mathrm{d}(T) = \oslash \preceq K = \mathrm{d}(V)$. By IH, $\mathbb{I}_\beta(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash V\rangle\}$ and $\mathbb{I}_\beta(T) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash T\rangle\}$. Note that $\mathbb{I}_\beta(V \to T) = \mathbb{I}_\beta(V) \rightsquigarrow \mathbb{I}_\beta(T)$.
     - Let $M \in \mathbb{I}_\beta(V) \rightsquigarrow \mathbb{I}_\beta(T)$ and, by lemma 8.1, let $y^K \in \mathbb{V}_V$ such that $\forall K, y^K \notin \mathrm{FV}(M)$. Then $M \diamond y^K$. By remark 4.1, $y^K : \langle (y^K : V) \vdash^* V\rangle$. Hence $y^K : \langle \mathbb{H}^K \vdash V\rangle$. Thus, $y^K \in \mathbb{I}_\beta(V)$ and $My^K \in \mathbb{I}_\beta(T)$.
       - If $My^K \in \mathcal{O}^{\oslash}$, then since $y \in \mathcal{V}_2$, by lemma 8.3, $M \in \mathcal{O}^{\oslash}$.
       - If $My^K \in \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash T\rangle\}$ then $My^K : \langle \Gamma \vdash T\rangle$ such that $\Gamma \subset \mathbb{H}^{\oslash}$. Since by lemma 4.2.3a, $\mathrm{dom}(\Gamma) = \mathrm{FV}(My^K)$ and $y^K \in \mathrm{FV}(My^K)$,

$\Gamma = \Delta, (y^K : V')$. Since $(y^K : V') \in \mathbb{H}^{\oslash}$, by lemma 8.1, $V = V'$. So $My^K : \langle \Delta, (y^K : V) \vdash T \rangle$ and by lemma 5.2 $M : \langle \Delta \vdash V \to T \rangle$. Note that $\Delta \subset \mathbb{H}^{\oslash}$, hence $M : \langle \mathbb{H}^{\oslash} \vdash V \to T \rangle$.

- Let $M \in \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash V \to T \rangle\}$ and $N \in \mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash V \rangle\}$ such that $M \diamond N$. Then, $\mathrm{d}(N) = K \succeq \oslash = \mathrm{d}(M)$.
  - $*$ If $M \in \mathcal{O}^{\oslash}$, then, by lemma 8.3, $MN \in \mathcal{O}^{\oslash}$.
  - $*$ If $M \in \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash V \to T \rangle\}$, then
    - $\cdot$ If $N \in \mathcal{O}^K$, then, by lemma 8.3, $MN \in \mathcal{O}^{\oslash}$.
    - $\cdot$ If $N \in \{M \in \mathcal{M}_3^K \mid M : \langle \mathbb{H}^K \vdash V \rangle\}$ then $M : \langle \Gamma_1 \vdash V \to T \rangle$ and $N : \langle \Gamma_2 \vdash V \rangle$ where $\Gamma_1 \subset \mathbb{H}^{\oslash}$ and $\Gamma_2 \subset \mathbb{H}^K$. By $\to_E$ and lemma 4.2.3, $MN : \langle \Gamma_1 \sqcap \Gamma_2 \vdash T \rangle$. By lemma 8.2, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^{\oslash}$. Therefore $MN : \langle \mathbb{H}^{\oslash} \vdash T \rangle$.

We deduce that $\mathbb{I}_\beta(V \to T) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}_3^{\oslash} \mid M : \langle \mathbb{H}^{\oslash} \vdash V \to T \rangle\}$.

□