Reducibility proofs in the λ-calculus

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Abstract. Reducibility has been used to prove a number of properties in the λ-calculus and is well known to offer on one hand very general proofs which can be applied to a number of instantiations, and on the other hand, to be quite mysterious and inflexible. In this paper, we look at two related but different results in the literature. We show that one such result (which aims at giving reducibility proofs of Church-Rosser, standardization and weak normalisation) faces serious problems which break the reducibility method and then we provide a proposal to partially repair the method. Then, we consider a second result whose purpose is to use reducibility to show Church-Rosser of β-developments (without needing to use strong normalisation). We extend the second result to encompass both βI- and $\beta\eta$-reduction rather than simply β-reduction.

1 Introduction

Reducibility is a method based on realizability semantics [6], developed by Tait [11] in order to prove normalization of some functional theories. The idea is to interpret types by sets of λ-terms closed under some properties. Since, this method has been generalized. Krivine uses it in [10] to prove the strong normalization of system D [2]. Koletsos proves in [8] that the set of simply typed λ-terms holds the Church-Rosser property. Some aspects of his method have been reused by Gallier in [3, 4] to prove some results such as the strong normalization of λ-terms that are typable in systems like D or D^{Ω}. In his work, Gallier states some conditions a property needs to satisfy in order to be enjoyed by some typable terms under some restrictions. Similarly, Ghilezan and Likavec [5] state some conditions a property on λ-terms has to satisfy to be held by some λ-terms typable in a system close to system D^{Ω}. [5] states a condition that a property needs to satisfy in order to step from "a λ-term typable, under some restrictions on types holds the property" to "a λ-term of the untyped lambda-calculus holds the property". If it works, [5] would provide an attractive method to establishing properties like Church-Rosser for all the untyped λ-terms, simply by showing easier conditions on typed terms. However, we show in this paper that both the method fails for the typed terms, and that the step of passing from typed to untyped terms fails. We provide a solution to repair the first result, however, the second result seems unrepairable.

Steps of establishing properties like Church-Rosser (or confluence) for typed λ-terms and concluding the properties for all the untyped λ-terms have been successfully exploited in the literature. Koletsos and Stravinos [9] use a reducibility method to state that λ-terms that are typable in system D hold the Church-Rosser property. Then, using this result together with a method based on β-developments [7, 10], they show that β-developments are Church-Rosser and this in turn will imply the confluence of the untyped λ-calculus. Although Klop proves the confluence of β-developments [1], his proof is based on strong normalisation whereas [9] only uses an embedding of β-developments in the reduction of typable λ-terms. In this paper, we apply the method of [9] to βI-reduction and then generalise the method to $\beta\eta$-reduction.

In section 2 we introduce the formal machinery. In section 3 we present the reducibility method of [5] and show that it fails at a number of important propositions which makes it inapplicable. In particular, we give counterexamples which show that all the conditions stated in [5] are satisfied, yet the the claimed property does not hold. In section 4 we provide subsets of types which we use to partially salvage the reducibility method of [5] and we show that this can now be correctly used to establish confluence, standardization and weak head normal forms but only for restricted sets of lambda terms and types. In section 5 we adapt the Church-Rosser proof of [9] to βI-reduction. In section 6 we generalise the method of [9] to handle $\beta\eta$-reduction. We conclude in section A. For space reasons we omit proofs. However, full proofs can be downloaded from the web page of the authors.

2 The Formal Machinery

In this section we provide some known formal machinery. Those familiar with the λ-calculus and type theory can skip this section. If a metavariable v ranges over a set S, we also let $v_1, v_2, \ldots, v', v'', \ldots$ range over S.

Definition 1 (Well-Known Backgrounds on λ-calculus).

1. *λ-terms are defined by $M \in \Lambda ::= x \mid (\lambda x.M) \mid (M_1 M_2)$. We let x, y, z, etc. range over $\mathcal{V}$, a denumerably infinite set of λ-term variables, and M, N, P, Q, etc. range over Λ. We assume the usual definition of subterms and write $N \subset M$ if N is a subterm of M. We assume the usual convention for parenthesis and omit these if no confusion arises. Hence, $M\ N_1 ... N_n$ stands for $(...((M\ N_1)\ N_2)...N_{n-1})\ N_n$.*
   We take terms modulo α-conversion and use the Barendregt convention (BC) where the names of bound variables differ from the free ones. When two terms M and N are equal (modulo α), we write $M = N$. We write $FV(M)$ for the set of the free variables of term M.

2. *We define* $M^n(N)$*, where* $n \geq 0$*, by induction on* n*, as following:* $M^0(N) = N$ *and* $M^{n+1}(N) = M(M^n(N))$.
3. *Term contexts are defined by* $C \in \mathcal{C} ::= \Box \mid \lambda x.C \mid CM \mid MC$*. We write* $C[M]$ *for the usual filling up of the context* C *with the term* M.
4. *The set* $\Lambda I \subset \Lambda$*, of terms of the* λI*-calculus is defined by the grammar:*
   (a) *If* $x \in \mathcal{V}$ *then* $x \in \Lambda I$.
   (b) *If* $x \in FV(M)$ *and* $M \in \Lambda I$ *then* $\lambda x.M \in \Lambda I$.
   (c) *If* $M, N \in \Lambda I$ *then* $MN \in \Lambda I$.
5. *We define as usual the substitution* $M[x := N]$ *of the term* N *for all free occurrences of* x *in the term* M*. Similarly, the substitution* $C[x := M]$ *of a term* N *for all free occurrences of* x *in the context* C *is defined as intended. We write* $M[(x_i := N_i)_1^n]$ *for the simultaneous substitution of* N_i *for all free occurrences of* x_i *in* M *for* $1 \leq i \leq n$.
6. *For* $r \in \{\beta, \beta I, \beta\eta\}$*, we define the reduction relation* $\to_r$ *on* Λ *as the least compatible relation closed under rule* $(r) : L \to_r R$ *below, and we call* L *as an* r*-redex and* R *the contractum of* L *(or the* L *contractum). We define* $\mathcal{R}^r$ *to be the set of* r*-redexes.*
   – (β)*:* $(\lambda x.M)N \to_\beta M[x := N]$.
   – (βI)*:* $(\lambda x.M)N \to_{\beta I} M[x := N]$ *when* $x \in FV(M)$.
   – (η)*:* $\lambda x.Mx \to_\eta M$ *when* $x \notin FV(M)$.
   We define $\mathcal{R}^{\beta\eta} = \mathcal{R}^\beta \cup \mathcal{R}^\eta$ *and* $\to_{\beta\eta} = \to_\beta \cup \to_\eta$.
7. *Let* $r \in \{\beta, \beta I, \beta\eta\}$*. Define* $\mathcal{R}^r_M = \{C \mid C \in \mathcal{C} \wedge \exists R \in \mathcal{R}^r, C[R] = M\}$*. If* $M \to_r N$ *by contracting the* r*-redex* R *in* $M = C[R]$ *then* $C \in \mathcal{R}^r_M$ *and we write* $M \xrightarrow{C}_r N$ *where* $N = C[R']$ *and* R' *is the* R *contractum.*
8. *Let* $M \in \Lambda$ *and* $\mathcal{F} \subseteq \Lambda$*.* $\mathcal{F} \restriction M = \{N \mid N \in \mathcal{F} \wedge N \subset M\}$.
9. *If* $M = \lambda x_1 \ldots . x_n.(\lambda x.M_0)M_1 \ldots M_m$ *where* $n \geq 0$ *and* $m \geq 1$*, we call* $(\lambda x.M_0)M_1$ *the* β*-head redex of* M*. We write* $M \to_h M'$ *(resp.* $M \to_i M'$*) if* M' *is obtained by reducing the* β*-head (resp. a non* β*-head) redex of* M.
10. *For any* r*, we use* $\to_r^*$ *to denote the reflexive transitive closure of* $\to_r$.
11. β*-normal forms and weakly* β*-normalizable terms are defined by:* $\mathsf{NF}_\beta = \{\lambda x_1 \ldots . \lambda x_n . x_0 N_1 \ldots N_m \mid n, m \geq 0 \wedge N_1, \ldots, N_m \in \mathsf{NF}_\beta\}$ *and* $\mathsf{WN}_\beta = \{M \in \Lambda \mid \exists N \in \mathsf{NF}_\beta, M \to_\beta^* N\}$.
12. *Let* $r \in \{\beta, \beta I, \beta\eta\}$*. We say that* M *has the Church-Rosser property for* r *(has* r*-*CR*) if whenever* $M \to_r^* M_1$ *and* $M \to_r^* M_2$*, there is an* M_3 *such that* $M_1 \to_r^* M_3$ *and* $M_2 \to_r^* M_3$*. Let* $\mathsf{CR}^r = \{M \mid M$ *has* r*-*$\mathsf{CR}\}$ *and* $\mathsf{CR}^r_0 = \{xM_1 \ldots M_n \mid n \geq 0 \wedge x \in \mathcal{V} \wedge (\forall i \in \{1, \ldots, n\}, M_i \in \mathsf{CR}^r)\}$*. We use* CR *to denote* CR^β *and* CR_0 *to denote* CR^β_0.
13. *Let* $r \in \{\beta, \beta I, \beta\eta\}$*. A term is a weak head normal form if it is an abstraction or if it starts with a variable. A term is weakly head normalizing if it reduces to a weak head normal form. let* $\mathsf{W}^r = \{M \in \Lambda \mid \exists n \geq$

$0, \exists x \in \mathcal{V}, \exists P, P_1, \ldots, P_n \in \Lambda, M \to_r^* \lambda x.P$ *or* $M \to_r^* xP_1 \ldots P_n\}$. *We use* W *to denote* W^β.

14. *We say that* M *has the standardization property if whenever* $M \to_\beta^* N$ *then there is an* M' *such that* $M \to_h^* M'$ *and* $M' \to_i^* N$. *Let* $\mathsf{S} = \{M \in \Lambda \mid M$ *has the standardization property*$\}$.

Let c be a metavariable ranging over $\mathcal{V}$. The next definition adapts Λ_c and the c-erasure of [10] to deal with βI- and $\beta\eta$-reduction.

Definition 2 (Backgrounds on developments).

1. *Let* $\mathcal{M}_c$ *range over* $\Lambda\eta_c, \Lambda I_c$ *defined as follows (note that* $\Lambda I_c \subset \Lambda I$*):*
   - *(R1) If* x *is a variable distinct form* c *then*
     1. $x \in \mathcal{M}_c$.
     2. *If* $M \in \Lambda I_c$ *and* $x \in FV(M)$ *then* $\lambda x.M \in \Lambda I_c$.
     3. *If* $M \in \Lambda\eta_c$ *then* $\lambda x.M[x := c(cx)] \in \Lambda\eta_c$.
     4. *If* $Nx \in \Lambda\eta_c$, $x \notin FV(N)$ *and* $N \neq c$ *then* $\lambda x.Nx \in \Lambda\eta_c$.
   - *(R2) If* $M, N \in \mathcal{M}_c$ *then* $cMN \in \mathcal{M}_c$.
   - *(R3) If* $M, N \in \mathcal{M}_c$ *and* M *is a* λ*-abstraction then* $MN \in \mathcal{M}_c$.
   - *(R4) If* $M \in \Lambda\eta_c$ *then* $cM \in \Lambda\eta_c$.
2. *Let* $M \in \mathcal{M}_c$. *We call* C *an* $\mathcal{M}_c$*-context if* $\exists R$ *such that* $C[R] = M$.
3. *Let* $M \in \Lambda$. *We define* $|M|^c$ *inductively as follows:*
   - $|x|^c = x$
   - $|\lambda x.N|^c = \lambda x.|N|^c$
   - $|cP|^c = |P|^c$
   - $|NP|^c = |N|^c|P|^c$ *if* $N \neq c$.
4. *Let* $C \in \mathcal{C}$. *We define* $|C|^c_{\mathcal{C}}$ *inductively as follows:*
   - $|\Box|^c_{\mathcal{C}} = \Box$
   - $|\lambda x.N|^c_{\mathcal{C}} = \lambda x.|C|^c_{\mathcal{C}}$
   - $|C'N|^c_{\mathcal{C}} = |C'|^c_{\mathcal{C}}|N|^c$
   - $|cC'|^c_{\mathcal{C}} = |C'|^c_{\mathcal{C}}$
   - $|NC'|^c_{\mathcal{C}} = |N|^c|C'|^c_{\mathcal{C}}$ *if* $N \neq c$

   Let $\mathcal{F} \subseteq \mathcal{C}$ *then we define* $|\mathcal{F}|^c_{\mathcal{C}} = \{|C|^c_{\mathcal{C}} \mid C \in \mathcal{F}\}$.

Definition 3 (Backgrounds on Type Systems). *Let* $i \in \{1, 2\}$.

1. *Let* $\mathcal{A}$ *be a denumerably infinite set of type variables and let* $\Omega \notin \mathcal{A}$ *be a constant type. The sets of types* $\mathsf{Type}^1 \subset \mathsf{Type}^2$ *are defined by:*
   $$\sigma^1, \tau^1 \in \mathsf{Type}^1 ::= \alpha \mid \sigma^1 \to \tau^1 \mid \sigma^1 \cap \tau^1$$
   $$\sigma^2, \tau^2 \in \mathsf{Type}^2 ::= \alpha \mid \sigma^2 \to \tau^2 \mid \sigma^2 \cap \tau^2 \mid \Omega$$
   We let α *range over* $\mathcal{A}$*;* σ^1, τ^1, ρ^1*, etc. range over* Type^1*;* σ^2, τ^2, ρ^2*, etc. range over* Type^2 *and* σ, τ, ρ*, etc. range over* Type^i.
2. *Let* $\mathcal{B}^i = \{\Gamma = \{x_1 : \sigma^i_1, \ldots, x_n : \sigma^i_n\} \mid n \geq 0$*, if* $k \neq j$ *then* $x_k \neq x_j\}$ *where* $i \in \{1, 2\}$ *and let* Γ, Δ *range over* $\mathcal{B}^i$. *We define* $DOM(\Gamma) = \{x \mid x : \sigma \in \Gamma\}$. *When* $x \notin DOM(\Gamma)$*, we write* $\Gamma, x : \sigma$ *for* $\Gamma \cup \{x : \sigma\}$. *We denote* $\Gamma = x_m : \sigma_m, \ldots, x_n : \sigma_n$ *where* $n \geq m \geq 0$*, by* $(x_i : \sigma_i)^m_n$. *If* $m = 1$*, we simply denote* Γ *by* $(x_i : \sigma_i)_n$. *If* $\Gamma_1 = (x_i : \sigma_i)_n, (y_i : \tau_i)_p$ *and* $\Gamma_2 = (x_i : \sigma'_i)_n, (z_i : \rho_i)_q$ *where* $x_1, \ldots, x_n$ *are the only shared variables, then* $\Gamma_1 \sqcap \Gamma_2 = (x_i : \sigma_i \cap \sigma'_i)_n, (y_i : \tau_i)_p, (z_i : \rho_i)_q$. *Let* $X \subseteq \mathcal{V}$. *We define* $\Gamma \upharpoonright X = \Gamma' \subseteq \Gamma$ *where* $DOM(\Gamma') = DOM(\Gamma) \cap X$.

3. *We now define* $\lambda\cap^1$, $\lambda\cap^2$, *and* D, D_I, *our four main type systems:*
   - *Referring to Figure 1, let* $\nabla_1 = \{(1),(2),(3),(4),(5),(6),(7),(8)\}$, $\nabla_2 = \nabla_1 \cup \{(9),(10)\}$, $i \in \{1,2\}$ *and* $\mathsf{Type}^{\nabla_i} = \mathsf{Type}^i$. $\leq^i$ *relates* $\sigma, \tau \in \mathsf{Type}^i$ *by axioms* ∇_i. *We write* $\sigma \sim^i \tau$ *if* $\sigma \leq^i \tau$ *and* $\tau \leq^i \sigma$.
   - • $\lambda\cap^i = \langle \Lambda, \mathsf{Type}^i, \vdash^i \rangle$ *such that* $\vdash^i$ *is the type derivability relation on* $\mathcal{B}^i$, Λ *and* Type^i *generated using the following typing rules of Figure 2:* (ax), $(\rightarrow_E)$, $(\rightarrow_I)$, $(\cap_I)$, $(\leq^1)$ *and, if* $i = 2$, (Ω) .
     - $D = \langle \Lambda, \mathsf{Type}^1, \vdash^{\beta\eta} \rangle$ *where* $\vdash^{\beta\eta}$ *is the type derivability relation on* $\mathcal{B}^1$, Λ *and* Type^1 *generated using the following typing rules of Figure 2: (ax),* $(\rightarrow_E)$, $(\rightarrow_I)$, $(\cap_I)$, $(\cap_{E1})$ *and* $(\cap_{E2})$.
     - $D_I = \langle \Lambda, \mathsf{Type}^1, \vdash^{\beta I} \rangle$ *where* $\vdash^{\beta I}$ *is the type derivability relation on* $\mathcal{B}^1$, Λ *and* Type^1 *generated using the following typing rule of Figure 2:* (ax^I), $(\rightarrow_{E^I})$, $(\rightarrow_I)$, $(\cap_I)$, $(\cap_{E1})$ *and* $(\cap_{E2})$. *Moreover, in this type system, we assume that* $\sigma \cap \sigma = \sigma$.

(1) $\sigma \leq \sigma$	(9) $\sigma \leq \Omega$
(2) $\sigma \leq \tau \wedge \tau \leq \rho \Rightarrow \sigma \leq \rho$	(10) $\sigma \rightarrow \Omega \leq \Omega \rightarrow \Omega$
(3) $\sigma \cap \tau \leq \sigma$	(11) $\sigma \leq \sigma \cap \sigma$
(4) $\sigma \cap \tau \leq \tau$	(12) $\Omega \leq \Omega \rightarrow \Omega$
(5) $(\sigma \rightarrow \tau) \cap (\sigma \rightarrow \rho) \leq \sigma \rightarrow (\tau \cap \rho)$	(13) $\sigma \rightarrow \tau \leq \Omega \rightarrow \Omega$
(6) $\sigma \leq \tau \wedge \sigma \leq \rho \Rightarrow \sigma \leq \tau \cap \rho$	
(7) $\sigma \leq \sigma' \wedge \tau \leq \tau' \Rightarrow \sigma \cap \tau \leq \sigma' \cap \tau'$	
(8) $\sigma \leq \sigma' \wedge \tau' \leq \tau \Rightarrow \sigma' \rightarrow \tau' \leq \sigma \rightarrow \tau$	

Fig. 1. Ordering axioms on types

$$\frac{}{\Gamma, x : \sigma \vdash x : \sigma}\;(ax) \qquad \frac{}{x : \sigma \vdash x : \sigma}\;(ax^I)$$

$$\frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}\;(\rightarrow_E) \qquad \frac{\Gamma_1 \vdash M : \sigma \rightarrow \tau \quad \Gamma_2 \vdash N : \sigma}{\Gamma_1 \sqcap \Gamma_2 \vdash MN : \tau}\;(\rightarrow_{E^I})$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \rightarrow \tau}\;(\rightarrow_I) \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau}\;(\cap_I)$$

$$\frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \sigma}\;(\cap_{E1}) \qquad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \tau}\;(\cap_{E2})$$

$$\frac{\Gamma \vdash M : \sigma \quad \sigma \leq^{\nabla} \tau}{\Gamma \vdash M : \tau}\;(\leq^{\nabla}) \qquad \frac{}{\Gamma \vdash M : \Omega}\;(\Omega)$$

Fig. 2. Typing rules

3 Problems of the reducibility method of [5]

We now introduce the method of [5] and explain its problems.

Definition 4 (Types/reducibility of [5]). *Let* $i \in \{1, 2\}$ *and* $\mathcal{P} \subseteq \Lambda$.

1. *The type interpretation* $[\![-]\!]^i : \mathsf{Type}^i \to 2^{\Lambda}$ *is defined by:*
   - $[\![\alpha]\!]^i = \mathcal{P}$, *where* $\alpha \in \mathcal{A}$ • $[\![\sigma \cap \tau]\!]^i = [\![\sigma]\!]^i \cap [\![\tau]\!]^i$ • $[\![\Omega]\!]^2 = \Lambda$
   - $[\![\sigma \to \tau]\!]^1 = \{M \in \Lambda | \forall N \in [\![\sigma]\!]^1, MN \in [\![\tau]\!]^1\}$
   - $[\![\sigma \to \tau]\!]^2 = \{M \in \mathcal{P} | \forall N \in [\![\sigma]\!]^2, MN \in [\![\tau]\!]^2\}$.
2. *A valuation is a function* $v : \mathcal{V} \to \Lambda$. *We let* $v(x := M)$ *be the function* v' *where* $v'(x) = M$ *and* $v'(y) = v(y)$ *if* $y \neq x$. *Let* $[\![-]\!]_v : \Lambda \to \Lambda$ *where* $[\![M]\!]_v = M[x_1 := v(x_1), \dots, x_n := v(x_n)]$ *for* $FV(M) = \{x_1, \dots, x_n\}$.
3. • $v \models^i M : \sigma$ *iff* $[\![M]\!]_v \in [\![\sigma]\!]^i$ • $v \models^i \Gamma$ *iff* $\forall (x : \sigma) \in \Gamma, v(x) \in [\![\sigma]\!]^i$
   • $\Gamma \models^i M : \sigma$ *iff* $\forall v \models^i \Gamma, v \models^i M : \sigma$
4. *Let* $\mathcal{X} \subseteq \Lambda$. *Define the following:*
   – $VAR^i(\mathcal{P}, \mathcal{X}) = \forall x, x \in \mathcal{X}$.
   – $SAT^1(\mathcal{P}, \mathcal{X}) = \forall M, \forall x, \forall N \in \mathcal{P}, M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X}$.
   – $SAT^2(\mathcal{P}, \mathcal{X}) = \forall M, \forall N, \forall x, M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X}$.
   – $CLO^1(\mathcal{P}, \mathcal{X}) = \forall M, \forall x, Mx \in \mathcal{X} \Rightarrow M \in \mathcal{P}$.
   – $CLO^2(\mathcal{P}, \mathcal{X}) = \forall M, \forall x, M \in \mathcal{X} \Rightarrow \lambda x.M \in \mathcal{P}$.
   – $VAR(\mathcal{P}, \mathcal{X}) = \forall x, \forall n \geq 0, \forall N_1, \dots, N_n \in \mathcal{P}, xN_1 \dots N_n \in \mathcal{X}$.
   – $SAT(\mathcal{P}, \mathcal{X}) = \forall M, \forall N, \forall x, \forall n \geq 0, \forall N_1, \dots, N_n \in \mathcal{P}$,
   $M[x := N]N_1 \dots N_n \in \mathcal{X} \Rightarrow (\lambda x.M)NN_1 \dots N_n \in \mathcal{X}$.
   – $CLO(\mathcal{P}, \mathcal{X}) = \forall M, \forall x, M \in \mathcal{X} \Rightarrow \lambda x.M \in \mathcal{P}$.
   – $INV(\mathcal{P}) = \forall M, \forall x, M \in \mathcal{P} \iff \lambda x.M \in \mathcal{P}$.
   For $\mathcal{R} \in \{VAR^i, SAT^i, CLO^i\}$, *let* $\mathcal{R}(\mathcal{P}) \iff \forall \sigma \in \mathsf{Type}^i, \mathcal{R}(\mathcal{P}, [\![\sigma]\!]^i)$.

Lemma 5 (Principal basic lemmas proved in [5]).
1. *If* $VAR^2(\mathcal{P})$, $SAT^2(\mathcal{P})$ *and* $CLO^2(\mathcal{P})$ *then* $\Gamma \vdash^2 M : \sigma \Rightarrow \Gamma \models^2 M : \sigma$
2. *If* $VAR^2(\mathcal{P})$, $SAT^2(\mathcal{P})$ *and* $CLO^2(\mathcal{P})$ *then* $\forall \sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \wedge \Gamma \vdash^2 M : \sigma \Rightarrow M \in \mathcal{P}$
3. *If* $CLO(\mathcal{P}, \mathcal{P}), \sigma \in \mathsf{Type}^2$ *and* $\sigma \not\sim^2 \Omega$ *then* $CLO^2(\mathcal{P}, [\![\sigma]\!]^2)$.

According to [5], VAR^i, SAT^i and CLO^i are sufficient to develop the reducibility method, and in order to prove these properties one needs stronger induction hypotheses which are easier to prove. [5] sets out to show that when $i = 2$, these stronger conditions are VAR, SAT and CLO. We show below that this attempt fail.

Lemma 6 (Lemma 3.16 of [5] is false).
Lemma 3.16 of [5] stated below is false.
Lemma 3.16: $VAR(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \to \tau \Rightarrow VAR(\mathcal{P}, [\![\sigma]\!]^2)$.

Proof. To show that this statement is false, we give the following counterexample. Let σ be $\alpha \to \Omega \to \alpha \not\sim^2 \Omega \to \tau$, where $\alpha \in \mathcal{A}$. $VAR(\mathcal{P}, [\![\sigma]\!]^2)$ is true if $\forall x \in \mathcal{V}, \forall n \geq 0, \forall N_1, \dots, N_n \in \mathcal{P}, xN_1 \dots N_n \in [\![\sigma]\!]^2$, in particular if $x \in [\![\sigma]\!]^2$, where $x \in \mathcal{V}$. Let $\mathcal{P}$ be the set of strong normalizing terms.

We have to notice that $VAR(\mathcal{P},\mathcal{P})$ is true. Since $x \in \mathcal{P}$, $xx \in [\![\Omega \to \alpha]\!]^2$. Since $\Delta\Delta \in \Lambda = [\![\Omega]\!]^2$, where $\Delta = \lambda x.xx$, $xx(\Delta\Delta) \in [\![\alpha]\!]^2 = \mathcal{P}$. But $\Delta\Delta \notin \mathcal{P}$, hence $xx(\Delta\Delta) \notin \mathcal{P}$, so $VAR(\mathcal{P}, [\![\sigma]\!]^2)$ is false. $\square$

Since the proof given in [5] for Lemma 3.18 does not go through and we have neither been able to prove nor disprove this lemma, we state:

Remark 7 (It is not clear that Lemma 3.18 of [5] holds).
It is not clear that the lemma of [5] stated below holds.
$SAT(\mathcal{P},\mathcal{P}) \Rightarrow \forall\sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \to \tau \Rightarrow SAT(\mathcal{P}, [\![\sigma]\!]^2)$.

Then, [5] gives the following proposition (Proposition 3.21) which is the reducibility method for typable terms:

$$\text{Let } VAR(\mathcal{P},\mathcal{P}),\ SAT(\mathcal{P},\mathcal{P}) \text{ and } CLO(\mathcal{P},\mathcal{P}), \text{ then}$$
$$\forall\sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \wedge \sigma \not\sim^2 \Omega \to \tau \wedge \Gamma \vdash^2 M : \sigma \Rightarrow M \in \mathcal{P}.$$

However, the proof of that proposition depends on two problematic lemmas (lemma 3.16 which we showed to fail in our lemma 6, and lemma 3.18 which according to remark 7 has not been proven). Below, we show that proposition 3.21 of [5] fails by giving a counterexample.

Lemma 8 (Proposition 3.21 of [5] fails). *Assume $VAR(\mathcal{P},\mathcal{P})$, $SAT(\mathcal{P},\mathcal{P})$ and $CLO(\mathcal{P},\mathcal{P})$. It is* **not** *the case that*
$\forall\sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \wedge \sigma \not\sim^2 \Omega \to \tau \wedge \Gamma \vdash^2 M : \sigma \Rightarrow M \in \mathcal{P}$.

Proof. Let $\mathcal{P}$ be WN_β of Definition 1 and $\Delta = \lambda x.xx$. Note that $\lambda y.\Delta\Delta \notin \mathsf{WN}_\beta$. Moreover, $\forall\rho \in \mathsf{Type}^2$, we can construct the typing judgment $\vdash^2 \lambda y.\Delta\Delta : \rho \to \Omega$. Let σ be $\rho \to \Omega$. Obviously, $\sigma \not\sim^2 \Omega$. Let $\tau \in \mathsf{Type}^2$.

If $\tau \not\sim^2 \Omega$ then obviously $\sigma = \rho \to \Omega \not\sim^2 \Omega \to \tau$.

If $\tau \sim^2 \Omega$ then let $\rho \not\sim^2 \Omega$. Obviously $\sigma = \rho \to \Omega \not\sim^2 \Omega \to \tau$.

Since $VAR(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$, $CLO(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ and $SAT(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ hold, we get a counterexample for Proposition 3.21 of [5]. $\square$

Finally, also the proof method for untyped terms given in [5] fails.

Lemma 9 (Proposition 3.23 of [5] fails). *Proposition 3.23 of [5] which states that "If $\mathcal{P} \subseteq \Lambda$ is invariant under abstraction, $VAR(\mathcal{P},\mathcal{P})$ and $SAT(\mathcal{P},\mathcal{P})$ then $\mathcal{P} = \Lambda$" fails.*

Proof. The proof given in [5] depends on Proposition 3.21 which we have shown to fail. As $VAR(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$, $SAT(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ and $INV(\mathsf{WN}_\beta)$, we have a counterexample for Proposition 3.23. $\square$

4 Salvaging the reducibility method of [5]

Remark 10. When proving lemma 5.1, properties $VAR^2(\mathcal{P})$, $SAT^2(\mathcal{P})$ and $CLO^2(\mathcal{P})$ were not needed for all types in Type^2. If $\Gamma \vdash^2 M : \sigma \to \tau$, we only need to have $VAR^2(\mathcal{P})$ for σ and $SAT^2(\mathcal{P})$ and $CLO^2(\mathcal{P})$ for τ.

Lemma 11. *If $\Gamma \vdash^2 M : \rho$ and (if $\rho = \sigma \to \tau$ then $VAR^2(\mathcal{P}, [\![\sigma]\!]^2)$, $SAT^2(\mathcal{P}, [\![\tau]\!]^2)$ and $CLO^2(\mathcal{P}, [\![\tau]\!]^2)$) then $\Gamma \models^2 M : \rho$.*

The next definition helps us salvage the reducibility method of [5].

Definition 12.
- $\sigma^{2+} \in \mathit{Type}^{2+} = \{\sigma \in \mathit{Type}^2 \mid \sigma \sim^2 \Omega\}$.
- $\sigma^{2-} \in \mathit{Type}^{2-} = \{\sigma \in \mathit{Type}^2 \mid \sigma \not\sim^2 \Omega\}$.
- $\sigma^{S_1} \in S_1 ::= \alpha \mid \sigma_1^{2+} \to \sigma_2^{2+} \mid \sigma^{2-} \to \sigma^{S_1} \mid \sigma^{S_1} \cap \sigma^{S_1}$.
- $\sigma^{S_2} \in S_2 ::= \Omega \to \Omega \mid \sigma^1$.

We let $\sigma, \tau, \rho, \sigma_1, \sigma_2, \ldots$ range over Type^1, Type^2, Type^{2+}, Type^{2-}, S_1 or S_2. It is easy to see that $S_2 \subseteq S_1$.

Using S_1, we can revise Lemmas 3.16 and 3.18 of [5] and deduce:

Corollary 13.
1. *$VAR(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in S_1, VAR^2(\mathcal{P}, [\![\sigma]\!]^2)$.*
2. *$SAT(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in S_1, SAT^2(\mathcal{P}, [\![\sigma]\!]^2)$.*

Remark 14. $\sigma \not\sim^2 \Omega$ is not a sufficient hypothesis in Proposition 3.21. We saw in remark 10 that if $\sigma = \tau \to \rho$, we need to have $CLO^2(\mathcal{P})$ only for ρ (not for all types in Type^2). Hence, since $CLO(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \Rightarrow CLO^2(\mathcal{P}, [\![\sigma]\!]^2)$, at least, we need to have $\rho \not\sim^2 \Omega$. The same remark holds for the hypothesis $\sigma \not\sim^2 \Omega \to \tau$. Similarly, the same remark holds if we replace $\sigma \not\sim^2 \Omega \wedge \sigma \not\sim^2 \Omega \to \tau$ by $\sigma \not\sim^2 \Omega \wedge \sigma \in S_1$.

Using S_1 in Proposition 3.21 of [5] does not help. If $VAR(\mathcal{P}, \mathcal{P})$, $SAT(\mathcal{P}, \mathcal{P})$ and $CLO(\mathcal{P}, \mathcal{P})$, then it is **not** the case that $\forall \sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \wedge \sigma \in S_1 \wedge \Gamma \vdash^2 M : \sigma \Rightarrow M \in \mathcal{P}$ (take the same counterexample given in the proof of Lemma 8 and choose $\rho = \Omega$). Since σ belongs to S_2 so to S_1. However, we can rescue the reducibility method for typable terms by:

Lemma 15. *Let $VAR(\mathcal{P}, \mathcal{P})$, $SAT(\mathcal{P}, \mathcal{P})$ and $CLO(\mathcal{P}, \mathcal{P})$, then $\forall \sigma \in \mathit{Type}^2, \sigma \not\sim^2 \Omega \wedge \Gamma \vdash^2 M : \sigma \wedge (\sigma = \tau \to \rho \Rightarrow \tau, \rho \in S_1 \wedge \rho \not\sim^2 \Omega) \Rightarrow M \in \mathcal{P}$.*

Proof. By lemmas 5.2, 5.3 and 11 and corollaries 13.1 and 13.2. □

Finally, [5] applied the method to confluence of β in Λ and standardisation in Λ by showing that the method of their Proposition 3.23 is applicable to the sets CR and S of Definition 2. It applied the method to the existence of weak head normal forms in $\lambda\cap^2$ (under some restrictions on types) by showing that the method of their Proposition 3.21 is applicable to the set W of Definition 2. However, since we showed in lemma 8 that proposition 3.21 fails, we need to review the applications and show where exactly they work. First, here is a lemma proven in [5].

Lemma 16. *$VAR(\mathcal{P},\mathcal{P})$, $SAT(\mathcal{P},\mathcal{P})$ and $CLO(\mathcal{P},\mathcal{P})$ for $\mathcal{P} \in \{\mathsf{CR},\mathsf{S},\mathsf{W}\}$.*

Finally, by lemmas 16 and 15 we get the following lemma:

Lemma 17. *Let $M \in \Lambda$. If $\exists \Gamma, \sigma$ such that $\Gamma \vdash^2 M : \sigma$ and ($\sigma = \tau \to \rho \Rightarrow \tau, \rho \in S_1 \wedge \rho \not\sqsubseteq^2 \Omega$) then $M \in \mathsf{CR}$, $M \in \mathsf{S}$, and $M \in \mathsf{W}$.*

5 Adapting the CR proof of [9] to βI-reduction

[9] gave a proof of Church-Rosser for β-reduction for system D given in Definition 3 and showed that this can be used to show confluence of β-developments without using strong normalisation. In this section, we adapt this proof to βI and set the formal ground for generalising the method of [9] for $\beta\eta$ in the next section. After giving the definition of βI-developments, we will introduce the type interpretation which will be used to establish Church-Rosser of both systems D and DI (for $\beta\eta$- resp. βI-reduction). Recall that c is a distinguished variable of $\mathcal{V}$.

The next definition, taken from [10] (and used in [9]) uses the variable c to destroy the βI-redexes of M which are not in the set $\mathcal{F}$ of βI-redex occurrences in M, and to neutralise applications so that they cannot be transformed into redexes after βI-reduction.

Definition 18 **($\Phi^{\beta I}(-,-)$).** *Let $M \in \Lambda I$, $c \notin FV(M)$, and $\mathcal{F} \subseteq \mathcal{R}^{\beta I}_M$.*

1. *If $M = x$ then $\mathcal{F} = \emptyset$ and $\Phi^{\beta I}(x,\mathcal{F}) = x$*
2. *If $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$ then $\Phi^{\beta I}(\lambda x.N,\mathcal{F}) = \lambda x.\Phi^{\beta I}(N,\mathcal{F}')$*
3. *If $M = NP$, $\mathcal{F}_1 = \{C \mid CP \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$ and $\mathcal{F}_2 = \{C \mid NC \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_P$ then $\Phi^{\beta I}(NP,\mathcal{F}) = \begin{cases} c\Phi^{\beta I}(N,\mathcal{F}_1)\Phi^{\beta I}(P,\mathcal{F}_2) \text{ if } \Box \notin \mathcal{F} \\ \Phi^{\beta I}(N,\mathcal{F}_1)\Phi^{\beta I}(P,\mathcal{F}_2) \quad \text{otherwise} \end{cases}$*

Lemma 19. *Let $M \in \Lambda I$, such that $c \notin FV(M)$, $\mathcal{F} \subseteq \mathcal{R}^{\beta I}_M$, $C \in \mathcal{F}$ and $M \xrightarrow{C}_{\beta I} M'$. Then, there exists an unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^{\beta I}(M,\mathcal{F}) \xrightarrow{C'}_{\beta I} \Phi^{\beta I}(M',\mathcal{F}')$ and $|C'|^c_{\mathcal{C}} = C$.*

We follow [10] and define the set of βI-residuals of a set of βI-redexes $\mathcal{F}$ relative to a sequence of βI-redexes.

Definition 20. *Let $M \in \Lambda I$, such that $c \notin FV(M)$ and let $\mathcal{F} \subseteq \mathcal{R}^{\beta I}_M$.*

1. *Let $C \in \mathcal{F}$ and $M \xrightarrow{C}_{\beta I} M'$. By lemma 19, $\exists \mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$ the unique set such that $\Phi^{\beta I}(M, \mathcal{F}) \xrightarrow{C'}_{\beta I} \Phi^{\beta I}(M', \mathcal{F}')$ and $|C'|^{\beta I} = C$. We call $\mathcal{F}'$ the set of βI-**residuals of $\mathcal{F}$ in** M' **relative to** C.*
2. *A one-step βI-development of $(M, \mathcal{F})$, denoted $(M, \mathcal{F}) \to_{\beta Id} (M', \mathcal{F}')$, is a βI-reduction $M \xrightarrow{C}_{\beta I} M'$ where $C \in \mathcal{F}$ and $\mathcal{F}'$ is the set of βI-residuals of $\mathcal{F}$ in M' relative to C. A βI-**development** is the transitive closure of a one-step βI-development. We write also $M \xrightarrow{\mathcal{F}}_{\beta Id} M_n$ for the βI-development $(M, \mathcal{F}) \to^*_{\beta Id} (M_n, \mathcal{F}_n)$.*

Definition 21. *1. Let $r \in \{\beta I, \beta\eta\}$. We define $[\![-]\!]^r : \mathsf{Type}^1 \to 2^\Lambda$ by:*

- *$[\![\alpha]\!]^r = CR^r$, where $\alpha \in \mathcal{A}$*
- *$[\![\sigma \cap \tau]\!]^r = [\![\sigma]\!]^r \cap [\![\tau]\!]^r$*
- *$[\![\sigma \to \tau]\!]^r = ([\![\sigma]\!]^r \Rightarrow [\![\tau]\!]^r) \cap CR^r = \{t \in CR | \forall u \in [\![\sigma]\!]^r, tu \in [\![\tau]\!]^r\}$.*

2. *$\mathcal{X} \subseteq \Lambda$ is saturated if $\forall n \geq 0, \forall M, N, M_1, \ldots, M_n \in \Lambda, \forall x \in \mathcal{V}$, $M[x := N]M_1 \ldots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$*
3. *$\mathcal{X} \subseteq \Lambda I$ is I-saturated if $\forall n \geq 0, \forall M, N, M_1, \ldots, M_n \in \Lambda, \forall x \in \mathcal{V}$, $x \in FV(M) \Rightarrow M[x := N]M_1 \ldots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$*

It turns out that if $\sigma \in \mathsf{Type}^1$ then $[\![\sigma]\!]^r$ is saturated and only contains Church-Rosser terms. [10] gave a proof for β-SN. [9] adapted Krivine's proof for β-Church-Rosser. We adapt [9] for $\beta\eta$-Church-Rosser.

First, we adapt the soundness lemma of [10] to both $\vdash^{\beta I}$ and $\vdash^{\beta\eta}$.

Lemma 22. *Let $r \in \{\beta I, \beta\eta\}$. If $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash^r M : \sigma$ and $\forall i \in \{1, \ldots, n\}$, $N_i \in [\![\sigma_i]\!]^r$ then $M[(x_i := N_i)^n_1] \in [\![\sigma]\!]^r$.*

Finally, we adapt a corollary from [9] to show that if $\Gamma \vdash^r M : \sigma$ then $M \in \mathsf{CR}^r$, for $r \in \{\beta I, \beta\eta\}$. In order to accommodate βI- and $\beta\eta$-reduction, the next lemma generalises a lemma given in [10] (and used in [9]). This lemma states that if $M \in \Lambda I_c$ (resp. $\Lambda\eta_c$) then M is typable in D (resp. D_I) and hence $M \in \mathsf{CR}^{\beta I}$ (resp. $M \in \mathsf{CR}^{\beta\eta}$).

Lemma 23. *Let $c \notin DOM(\Gamma) \supseteq FV(M) \setminus \{c\} = \{x_1, \ldots, x_n\}$.*

1. *If $M \in \Lambda I_c$ and $\Gamma' = \Gamma \upharpoonright FV(M)$, then $\exists \sigma, \tau \in \mathsf{Type}^1$ such that if $c \in FV(M)$ then $\Gamma', c : \sigma \vdash^{\beta I} M : \tau$, else $\Gamma' \vdash^{\beta I} M : \tau$.*
2. *If $M \in \Lambda\eta_c$ then $\exists \sigma, \tau \in \mathsf{Type}^1$ such that $\Gamma, c : \sigma \vdash^{\beta\eta} M : \tau$.*

The next lemma is an adaptation of the main theorem in [9] where as far as we know appears for the first time in [9].

Lemma 24 (confluence of the βI-developments). *Let $M \in \Lambda I$, such that $c \notin FV(M)$. If $M \xrightarrow{\mathcal{F}_1}_{\beta Id} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta Id} M_2$, then there exist $\mathcal{F}'_1 \subseteq \mathcal{R}^{\beta I}_{M_1}$, $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta I}_{M_2}$ and $M_3 \in \Lambda I$ such that $M_1 \xrightarrow{\mathcal{F}'_1}_{\beta Id} M_3$ and $M_2 \xrightarrow{\mathcal{F}'_2}_{\beta Id} M_3$.*

By the notation: $M \rightarrow_{1I} M' \iff \exists \mathcal{F}, \mathcal{F}', (M, \mathcal{F}) \rightarrow^*_{\beta Id} (M', \mathcal{F}')$ where $M, M' \in \Lambda I$, such that $c \notin FV(M)$, the transitive and reflexive closure of $\rightarrow_{\beta I}$ turns to be equal to the transitive and reflexive closure of $\rightarrow_{1I}$. We are now able to prove the (non-strict) inclusion of ΛI in $\mathsf{CR}^{\beta I}$ and so the equality between these two sets.

Lemma 25. *If $M \in \Lambda I$ such that $c \notin FV(M)$ then $M \in \mathsf{CR}^{\beta I}$.*

6 Generalisation of the method to $\beta\eta$-reduction

In this section we generalise the method of section 5 to $\beta\eta$-reduction. This generalisation is not trivial since when studying developments involving η-reduction we need closure under η-reduction of a defined set of frozen terms. For example, let $M = \lambda x.cNx \in \Lambda I_c$ where $x \notin FV(N)$ and $N \in \Lambda I_c$, then $M \rightarrow_\eta cN \notin \Lambda I_c$. For such reasons, we extended ΛI_c to $\Lambda\eta_c$. In this section, many of the notions used to prove Church-Rosser of βI-reduction will be extended to deal with $\beta\eta$-reduction. The next definition adapts definition 18 to deal with $\beta\eta$-reduction.

Definition 26 **($\Phi^{\beta\eta}(-,-), \Phi_0^{\beta\eta}(-,-)$).** *Let $c \notin FV(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$.*

(P1) If $M \in \mathcal{V} \setminus \{c\}$ then $\mathcal{F} = \emptyset$ and
$\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(M) \mid n > 0\} \qquad \Phi_0^{\beta\eta}(M, \mathcal{F}) = \{M\}$

(P2) If $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$ then:
$$\Phi^{\beta\eta}(M, \mathcal{F}) = \begin{cases} \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \wedge P \in \Phi^{\beta\eta}(N, \mathcal{F}')\} & \textit{if } \Box \notin \mathcal{F} \\ \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Phi_0^{\beta\eta}(N, \mathcal{F}')\} & \textit{otherwise} \end{cases}$$
$$\Phi_0^{\beta\eta}(M, \mathcal{F}) = \begin{cases} \{\lambda x.N'[x := c(cx)] \mid N' \in \Phi^{\beta\eta}(N, \mathcal{F}')\} & \textit{if } \Box \notin \mathcal{F} \\ \{\lambda x.N' \mid N' \in \Phi_0^{\beta\eta}(N, \mathcal{F}')\} & \textit{otherwise} \end{cases}$$

(P3) If $M = NP$, $\mathcal{F}_1 = \{C \mid CP \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$ and $\mathcal{F}_2 = \{C \mid NC \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta\eta}$ then:
$$\Phi^{\beta\eta}(M, \mathcal{F}) = \begin{cases} \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}(N, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}(P, \mathcal{F}_2)\} & \textit{if } \Box \notin \mathcal{F} \\ \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Phi_0^{\beta\eta}(N, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}(P, \mathcal{F}_2)\} & \textit{otherwise} \end{cases}$$
$$\Phi_0^{\beta\eta}(M, \mathcal{F}) = \begin{cases} \{cN'P' \mid N' \in \Phi^{\beta\eta}(N, \mathcal{F}_1) \wedge P' \in \Phi_0^{\beta\eta}(P, \mathcal{F}_2)\} & \textit{if } \Box \notin \mathcal{F} \\ \{N'P' \mid N' \in \Phi_0^{\beta\eta}(N, \mathcal{F}_1) \wedge P' \in \Phi_0^{\beta\eta}(P, \mathcal{F}_2) & \textit{otherwise} \end{cases}$$

Lemma 27. *Let $M \in \Lambda$, such that $c \notin FV(M)$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$, $C \in \mathcal{F}$ and $M \xrightarrow{C}_{\beta\eta} M'$. Then, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta\eta}$, such that $\forall N \in \Phi^{\beta\eta}(M, \mathcal{F}), \exists N' \in \Phi^{\beta\eta}(M', \mathcal{F}'), \exists C' \in \mathcal{R}_N^{\beta\eta}, N \xrightarrow{C'}_{\beta\eta} N'$ and $|C'|_{\mathcal{C}}^{c} = C$.*

Definition 28. *Let $M \in \Lambda$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$.*

1. *Let $C \in \mathcal{F}$ and $M \xrightarrow{C}_{\beta\eta} M'$. By lemma 27, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta\eta}$, such that $\forall N \in \Phi^{\beta\eta}(M, \mathcal{F})$, $\exists N' \in \Phi^{\beta\eta}(M', \mathcal{F}')$ and $\exists C' \in \mathcal{R}_N^{\beta\eta}$ such that $N \xrightarrow{C'}_{\beta\eta} N'$ and $|C'|^c = C$. We call $\mathcal{F}'$ the set of $\beta\eta$-**residuals of $\mathcal{F}$ in M' relative to C**.*

*2. Let $c \notin FV(M)$. A one-step $\beta\eta$-development of $(M, \mathcal{F})$, denoted $(M, \mathcal{F}) \to_{\beta\eta d} (M', \mathcal{F}')$, is a $\beta\eta$-reduction $M \xrightarrow{C}_{\beta\eta} M'$ where $C \in \mathcal{F}$ and $\mathcal{F}'$ is the set of $\beta\eta$-residuals of $\mathcal{F}$ in M' relative to C. A $\beta\eta$-**development** is the transitive closure of a one-step $\beta\eta$-development. We write also $M \xrightarrow{\mathcal{F}}_{\beta\eta d} M'$ for the $\beta\eta$-development $(M, \mathcal{F}) \to^*_{\beta\eta d} (M', \mathcal{F}')$.*

Lemma 29 (confluence of the $\beta\eta$-developments). *Let $M, M_1, M_2 \in \Lambda$. If $M \xrightarrow{\mathcal{F}_1}_{\beta\eta d} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta\eta d} M_2$, then there exists sets $\mathcal{F}'_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$ and a term $M_3 \in \Lambda$ such that $M_1 \xrightarrow{\mathcal{F}'_1}_{\beta\eta d} M_3$ and $M_2 \xrightarrow{\mathcal{F}'_2}_{\beta\eta d} M_3$.*

Considering the notation: $M \to_1 M' \iff \exists \mathcal{F}, \mathcal{F}', (M, \mathcal{F}) \to^*_{\beta\eta d} (M', \mathcal{F}')$, the transitive and reflexive closure of $\to_{\beta\eta}$ turns to be equal to the transitive and reflexive closure of $\to_1$. We are now able to prove the (non-strict) inclusion of Λ in $\mathsf{CR}^{\beta\eta}$ and the equality between these sets.

Lemma 30. *If $M \in \Lambda$ such that $c \notin FV(M)$ then $M \in \mathsf{CR}^{\beta\eta}$.*

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A Conclusion/comparison

Reducibility is a powerful method and has been applied to prove using a single method, a number of properties of the λ-calculus (CR, SN, etc.). This paper studied two reducibilty methods which exploit the passage from typed to untyped terms. We showed that the first method [5] fails in its aim and we have only been able to provide a partial solution. We adapted the second method [9] from β to βI-reduction and we generalised it to $\beta\eta$-reduction. There are differences in the typed systems chosen and the methods of reducibility used in [5, 9]. [9] uses system D [2], which has elimination rules for intersection types whereas [5] uses $\lambda\cap$ and $\lambda\cap^{\Omega}$ with subtyping. Moreover, [9] depends on the inclusion of typable λ-terms in the set of λ-terms possessing the CR property, whereas [5] proves the inclusion of typable terms in an arbitrary subset of the untyped λ-calculus closed by some properties. Moreover, [5] considers the $VAR(\mathcal{P})$, $SAT(\mathcal{P})$ and $CLO(\mathcal{P})$ whereas [9] uses standard reducibility methods through saturated sets. [9] proves the confluence of developments using the confluence of typable λ-terms in system D (the authors prove that even a simple type system is sufficient). The advantage of the proof of confluence of developments of [9] is that SN is not needed.

In [4], Gallier considers systems D and D^{Ω}. He states some properties which a set of λ-terms has to satisfy to include the terms typable in D or D^{Ω} (under some restrictions). He states that the terms typable in D^{Ω} by a "weakly nontrivial type" ($WNT ::= \mathcal{A} \mid \mathsf{Type}^2 \to WNT \mid WNT \cap WNT$) are weakly head normalizable. The "weakly nontrivial types" include types in our set S_1 since, for example, the type $\alpha \to \Omega \to \alpha$, where $\alpha \in \mathcal{A}$, does not belong to S_1 but is a "weakly nontrivial type". However, unlike Gallier we only restrict functional types. There are common properties with [5]: we can observe some trivial correspondences: (P4w) implies $CLO(\mathcal{P}, \mathcal{P})$, (P1) and (P3s) imply $VAR(\mathcal{P}, \mathcal{P})$, $SAT(\mathcal{P}, \mathcal{P})$ implies (P5n), and $VAR(\mathcal{P}, \mathcal{P})$ implies (P1). Gallier states some others properties held by the terms typable in D^{Ω} under some restriction (always on the use of the type Ω), and for different conditions on the properties, in order to be adapted to different cases. It is an attractive feature of [4] that all the conditions on properties have the same general shape. [3] considers quantifiers and other type constructors instead of intersection types.