Simplified Reducibility Proofs of Church-Rosser for β- and $\beta\eta$-reduction

Fairouz Kamareddine and Vincent Rahli

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Abstract

Reducibility has been used to prove a number of properties in the λ-calculus and is well known to offer on one hand very general proofs which can be applied to a number of instantiations, and on the other hand, to be quite mysterious and inflexible. It has, amongst other things, been used along with the so called method of parallel reductions to prove the Church-Rosser property. In this paper, we concentrate on using the methods of reducibility and of parallel reductions for proving Church-Rosser for both β- and $\beta\eta$-reduction. Our contributions are two fold:

- We give a simple proof of CR for β-reduction which unlike the common proofs in the literature, avoids any type machinery and is solely carried out in a completely untyped setting.
- We give a new proof of CR for $\beta\eta$-reduction which is a generalisation of our simple proof for β-reduction.

1 Introduction

Reducibility is a method based on realisability semantics [7], developed by Tait [11] in order to prove normalisation of some functional theories. The idea is to interpret types by sets of λ-terms closed under some properties. Since its introduction, this method has gone through a number of improvements and generalisations. In particular, Krivine [10] uses reducibility to prove the strong normalisation (SN) of his intersection type system called system D. Koletsos [8] generalises and extends Krivine's method to prove that the set of simply typed λ-terms holds the Church-Rosser property (CR, also called confluence) w.r.t. β-reduction. Although it is well known that β-reduction satisfies CR, reducibility proofs of CR are in line with proofs of SN and hence, one can establish both SN and CR using the same method. Moreover, CR proofs can be quite involved (proofs solely via parallel reduction are very lengthy). So, reducibility proofs can help within the same machinery to prove the most important properties of a λ-calculus (such as SN, CR or standardisation).

In this paper we use reducibility for proving CR for both β- and $\beta\eta$-reduction. We give a proof of CR for β-reduction which is simpler that the one given by Ghilezan and Kunčak [4] and introduce a new proof of CR for $\beta\eta$-reduction which is a generalisation of our simple proof for β-reduction. The CR theorem is a strong form of a theorem stated by Church and Rosser [3] proving the consistency of the λ-calculus. A binary relation $\mathcal{R}$ (where $\mathcal{R}^*$ stands for its reflexive and transitive closure) on the λ-calculus satisfies CR iff for any λ-terms M, M_1, M_2 such that $M\mathcal{R}^*M_1 \wedge M\mathcal{R}^*M_2$ there exists M_3 such that $M_1\mathcal{R}^*M_3 \wedge M_2\mathcal{R}^*M_3$.

As in a number of other works [9, 6, 4], our method to prove CR for a given set of terms w.r.t. a reduction relation (we consider both β-reduction and $\beta\eta$-reduction) consists in two main steps:

- Our first step, based on a "simplified" reducibility method, differs from the "common" reducibility method because we do not relate (even if this relation exists) the given set of terms to a set of typable terms in some type system (such as the systems D or D^{Ω} [10] or the Simply Typed Lambda Calculus). This simplification enables us to get rid of all the machinery involved in a type system (the definitions of types, typing rules, environments, etc.). As it is crucial to a reducibility method to use a soundness result, our method also needs a soundness result. However, we replace type interpretations by simple sets of terms which bear no relation to types.

- The second step of our method consists in reducing the problem of the confluence of the λ-calculus w.r.t. the considered reduction relation to the problem of the confluence of the defined set of terms w.r.t. the defined reduction. This second step is done using a rather short method of parallel reductions by defining a new simple reduction (whose reflexive closure is equal to the considered reduction) and by proving it to be confluent.

To achieve their goals, all of [9, 6, 4] use the notion of developments. Both Koletsos and Stavrinos [9] as well as Kamareddine and Rahli [6] use a complicated handing of developments. On the other hand, Ghilezan and Kunčak [4] as well as this article are based on some weaker and sufficient notions of developments. Although this article was developed as a simplification of the work done by Koletsos and Stavrinos [9] and by Kamareddine and Rahli [6], it can be regarded as a simplification and generalisation of the work done by Ghilezan and Kunčak [4].

In section 2, we compare our solution to the related work in the literature, especially to the one of Ghilezan and Kunčak [4] and Koletsos and Stavrinos [9]. In section 3 we introduce the needed machinery about the λ-calculus and our weak form of developments. In section 4 we prove the Church-Rosser of the λ-calculus w.r.t. β-reduction. In section 5 we prove the Church-Rosser of the λ-calculus w.r.t. $\beta\eta$-reduction. Finally, we conclude in section 6. Omitted proofs can be found in appendix A.

2 Related Work and Comparison

In this section we compare our proposal in this paper to a number of the confluence proof methods in the literature [4, 6, 1, 9]. In this section and only in this section, we consider the confluence property w.r.t. the β-reduction. In the Figures 1, 2 and 3, an arrow labelled with o or β stands for $\to_o^*$ or $\to_\beta^*$ respectively. An arrow labelled with Ψ or $|-|_{cd}$ stands for the application of the function with the same name to the term at the start of the arrow.

In Figure 1 we recall the proof of Ghilezan and Kunčak [4] for the confluence of the untyped λ-calculus w.r.t. to β-reduction. This proof, based on a parallel reduction method, uses the confluence w.r.t. another reduction $\to_I$ whose transitive closure is equal to $\to_\beta^*$. The reduction $\to_I$ is defined as $\tau^{-1}\circ \to_\beta^* \circ\tau$ where:

- $\tau = \to_o^* \circ\Psi$
- $\to_o$ is the compatible closure of the rule $(o) : f(g(\lambda x.M))N \to_o (\lambda x.M)N$
- Ψ is defined on the λ-calculus by: $\Psi(x) = x$, $\Psi(\lambda x.M) = g(\lambda x.\Psi(M))$ and $\Psi(MN) = f\Psi(M)\Psi(N)$, where f and g are two constants (see remark 3.3).

The relation τ enables to "freeze" some β-redexes and the potential β-redexes (the other applications) of a term (in fact, τ does more, because Ψ does more by encapsulating the λ-abstractions using g which is needed by Ghilezan and Kunčak to

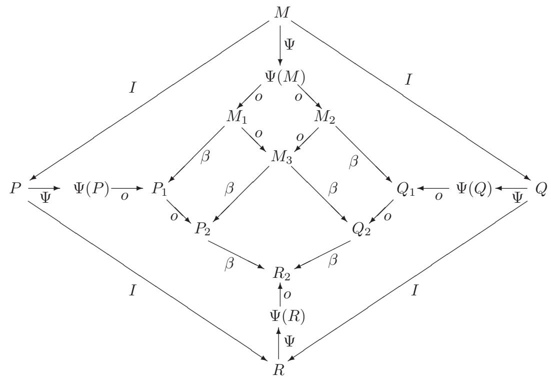

Figure 1: The method of Ghilezan and Kunčak for the confluence of $\rightarrow_I$

prove the typability of a defined set of terms in the Simply Typed Lambda Calculus). The reduction τ^{-1} is equivalent to our (and the one of Krivine [10] before) erasure function $|-|_{cd}$ (see below), which "unfreezes" the redexes in a term. By definition of $M \rightarrow_I P$, there exist M_1 and P_1 such that $\Phi(M) \rightarrow_o^* M_1 \rightarrow_\beta^* P_1$ and $\Phi(P) \rightarrow_o^* P_1$ (left part of the figure). By definition of $M \rightarrow_I Q$, there exist M_2 and Q_1 such that $\Phi(M) \rightarrow_o^* M_2 \rightarrow_\beta^* Q_1$ and $\Phi(Q) \rightarrow_o^* Q_1$ (right part of the figure). Because M_1 can be different from M_2, a confluence lemma for the $\rightarrow_o$ reduction and a commutation lemma for the reductions $\rightarrow_o^*$ and $\rightarrow_\beta^*$ are needed. The central part of the figure is due to the confluence of the terms typable in the Simply Typed Lambda Calculus. However, the confluence of the Simple Typed Lambda Calculus is not provided because the result has already been proved many times in the literature. For example, as cited by Ghilezan and Kunčak, Koletsos [8] proved this result using a reducibility method. Hence, when combined with Koletsos's proof of the confluence of the Simple Typed Lambda Calculus, Ghilezan and Kunčak's method can be regarded as the combination of a reducibility method and a method of parallel reductions.

The reduction $\rightarrow_I$ (designed by Ghilezan and Kunčak [4]) defines a development without specifying explicitly the set of redexes which are allowed to be reduced and their residuals (as done for example by Barendregt al. [1] , and which differ from the "common" one as defined for example by Barendregt [2] or Hindley [5]). Let us consider the reduction $M \rightarrow_I P$ (unfolded above). First, the function Ψ blocks all the redexes in M. Then $\rightarrow_o^*$ enables to set the set of redexes which are allowed to be reduced in M without explicitly naming them, by unblocking some redexes in $\Psi(M)$. The reduction $M_1 \rightarrow_\beta^* P_1$ reduces the allowed redexes. And finally in $\Psi(P) \rightarrow_o^* P_1$, the reduction $\rightarrow_o^*$ sets the set of residuals of the set of redexes in M_1 without naming them.

The gap in the work of Krivine [10] or Koletsos and Stavrinos [9] is about the treatment of the occurrences of β-redexes. In these works, occurrences are treated intuitively and not formally. So, the work turns out to be much more complicated than it seems when one wants to "formally" prove the results (see [6]), or even just define the developments. Ghilezan and Kunčak [4] do not face the same problem. The reduction $\rightarrow_o^*$ enables to unblock a certain set of β-redexes without explicitly specifying the set of unblocked redexes. In the work of Ghilezan and Kunčak, as in the work of Barendregt et al. [1] for example, a development of a term is defined without explicit control on the set of occurrences of reduced β-redexes, which is not needed.

Although Ghilezan and Kunčak [4] consider a simpler definition of developments than the "common" one, the scheme of their proof method is exactly the

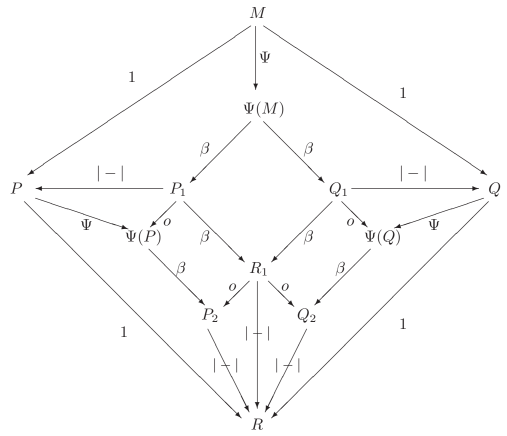

Figure 2: Our method for the confluence of $\to_1$

one followed by Koletsos and Stavrinos [9]. Koletsos and Stavrinos consider the following "common" definition of developments: there exists a development from M to N iff $\langle M, s_1\rangle \to_d^* \langle N, s_2\rangle$ where s_1 is a set of redexes in M and s_2 is the set of residuals of s_1 in N (where $\to_d^*$ is a new (complex) reduction relation based on $\to_\beta^*$). Their proof of the confluence of developments uses, among other things, the following claim: if $\langle M, s_1\rangle \to_d^* \langle N, s_2\rangle$ then there exists s_4 such that $\langle M, s_1 \cup s_3\rangle \to_d^* \langle N, s_2 \cup s_4\rangle$, where s_3 is a set of redexes of M. It is useful to prove that if $\langle M, s_1\rangle \to_d^* \langle M_1, s_1'\rangle$ and $\langle M, s_2\rangle \to_d^* \langle M_2, s_2'\rangle$ then there exist s_1'' and s_2'' such that $\langle M, s_1 \cup s_2\rangle \to_d^* \langle M_1, s_1' \cup s_2''\rangle$ and $\langle M, s_2 \cup s_1\rangle \to_d^* \langle M_2, s_2' \cup s_1''\rangle$. This corresponds to the proof of the confluence of $\to_o^*$ of Ghilezan and Kunčak, which is useful to get the reductions ($\Psi(M) \to_o^* M_1 \to_o^* M_3 \to_\beta^* P_2$ and $\Psi(P) \to_o^* P_1 \to_o^* P_2$) and ($\Psi(M) \to_o^* M_2 \to_o^* M_3 \to_\beta^* Q_2$ and $\Psi(Q) \to_o^* Q_1 \to_o^* Q_2$). Ghilezan and Kunčak emphasised this more strongly than Koletsos and Stavrinos.

We have to notice that the major difference between the methods of Ghilezan and Kunčak [4] and Barendregt et al. [1] is how developments are proved confluent. Barendregt et al. too give a definition of developments without explicitly naming an occurrence of a redex (no set of occurrences is defined), introducing among other things, a second abstraction $\underline{\lambda}$. The correspondence between the untyped λ-calculus and the calculus with this second abstraction is similar to the correspondence between the untyped λ-calculus and the marked calculus introduced by Krivine and reused in other works [10, 4, 9, 6]. The result obtained by Barendregt et al. is based on the finiteness (which is a termination result) and the confluence of developments.

In Figure 2 we draw the diagram of our method to prove the confluence of the λ-calculus. By definition of $M \to_1 P$, there exists P_1 such that $\Phi(M) \to_\beta^* P_1$ and $|P_1|_{cd} = P$ (left part of the figure). By definition of $M \to_1 Q$, there exists Q_1 such that $\Phi(M) \to_\beta^* Q_1$ and $|Q_1|_{cd} = Q$ (right part of the figure). Moreover $P_1 \to_o^* \Psi_{cd}(|P_1|_{cd})$ and $Q_1 \to_o^* \Psi_{cd}(|Q_1|_{cd})$. So, because P_1 and $\Psi_{cd}(|P_1|_{cd})$ might be different (as for Q_1 and $\Psi_{cd}(|Q_1|_{cd})$), as Ghilezan and Kunčak [4], we need a commutation result for the reductions $\to_\beta^*$ and $\to_o^*$. Then, the whole diagram commutes because $|P_2|_{cd} = |R_1|_{cd} = |Q_2|_{cd}$. As in the Figure 1, the central part is due to the confluence of a defined set of terms (in both cases typable in the Simply Typed Lambda Calculus, even if we do not use this fact because we do not use types). We do not need to prove the confluence of the reduction $\to_o^*$, because we use the following property: if $M \to_o^* N$ then $|M|_{cd} = |N|_{cd}$ (bottom of the figure).

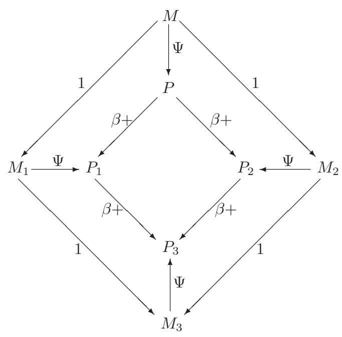

Figure 3: What we would like to get

So, as we can see in Figure 2, we can get rid of the erasure function and use instead $\Psi^{-1} \circ \to_o^*$ and the confluence of $\to_o^*$ which, we think, is more intuitive.

Our method is also based on some kind of weak developments, where all the β-redexes are let unblocked and where all the potential β-redexes (all the other applications) are blocked. In this paper we define two weak developments such as the reduction $\to_I$ defined by Ghilezan and Kunčak. They are called $\to_1$ for the β case and $\to_2$ for the $\beta\eta$ case. In that way, we do not need the reduction $\to_o^*$ to unblock some redexes in order to perform some reductions. But, it does not seem possible to get rid of the work done by this reduction. Indeed, our choice implies the introduction of some other material which turns out to be identical to the reduction $\to_o^*$ (which is why we called our reduction $\to_o^*$ too). Both the two different methods need the introduction of some equivalent material, but not at the same place. the reduction $\to_o^*$ is used by Ghilezan and Kunčak to unblock some redexes in order to enable some reductions whereas we use the reduction $\to_o^*$ to unblock some redexes which turned to be blocked after some reductions.

As we can see in these two figures, because the occurrences of redexes are not explicitly taken into consideration, the function Ψ (which enables to embed a term in a simply typed term, by blocking redexes or potential future redexes) needs to block or let unblocked all the redexes of a term. If all the redexes are blocked by Ψ, the reduction $\to_o$ is needed before being able to perform some reductions (see Figure 1). In this case some technical results are needed such as the confluence of $\to_o$. In the other case (Ψ let unblocked all the redexes), because a term with all its redexes unblocked does not necessarily reduce to a term with all its redexes unblocked, some technical results on $\to_o$ are also needed as we previously explained (see Figure 2).

Finally, We have to notice that the just described methods [4, 9, 6, 1] follow the proof scheme depicted in Figure 3. In this figure the reduction $\beta+$ stands for a reduction based on the β-reduction, such as developments. But, depending on how they are defined, developments may need the introduction of a huge machinery to deal with occurrences of redexes [9, 6]. So, the central part, even if still obtained by a simple reducibility method (whether or not using a type system such as the Simply Typed Lambda Calculus), may turn out to be very complicated [9, 6]. Hence, a better solution should be as depicted in Figure 3 with a simple proof of the confluence of a calculus w.r.t. the reduction $\beta+$. We still have to find out if it is possible to perform more simplifications on the proof given by Ghilezan and Kunčak [4] or on the present proof, because our attempt to do so in this article only partially succeed (we do need some "complicated" definitions and lemmas as depicted in the lower half of the Figure 2).

3 The Formal Machinery

In this section we provide some known formal machinery and introduce new definitions and lemmas that are necessary for the paper. Let n, m be metavariables which range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. We take as convention that if a metavariable v ranges over a set s then the metavariables v_i such that $i \geq 0$ and the metavariables $v', v'', etc.$ also range over s.

A binary relation is a set of pairs. Let *rel* range over binary relations. If $\langle x, y\rangle \in rel$ then we sometimes write it $x\ rel\ y$. Let $\mathrm{dom}(rel) = \{x \mid \langle x, y\rangle \in rel\}$ and $\mathrm{ran}(rel) = \{y \mid \langle x, y\rangle \in rel\}$. A function is a binary relation *fun* such that if $\{\langle x, y\rangle, \langle x, z\rangle\} \subseteq fun$ then $y = z$. Let *fun* range over functions. Let $s \to s' = \{fun \mid \mathrm{dom}(fun) \subseteq s \wedge \mathrm{ran}(fun) \subseteq s'\}$.

Given n sets $s_1, \ldots, s_n$, where $n \geq 2$, $s_1 \times \ldots \times s_n$ stands for the set of all the tuples built on the sets $s_1, \ldots, s_n$. If $x \in s_1 \times \ldots \times s_n$, then $x = \langle x_1, \ldots, x_n\rangle$ such that $x_i \in s_i$ for all $i \in \{1, \ldots, n\}$.

3.1 Background on the λ-calculus

This section consists of one long definition of some familiar (mostly standard) concepts of the λ-calculus and one basic lemma.

Definition 3.1.

1. Let x, y, z range over Var, a countable infinite set of variables. The set of terms of the λ-calculus is defined as follows:

$$M \in \Lambda ::= x \mid (\lambda x.M) \mid (M_1 M_2)$$

   We let M, N, P, Q, R range over Λ. We call a term of the form $\lambda x.M$, a λ-abstraction or just abstraction. We call a term of the form $M_1 M_2$ an application. We assume the usual definition of subterms and write $N \subseteq M$ if N is a subterm of M ($M \subseteq M$). We also assume the usual convention for parenthesis and omit these when no confusion arises. In particular, we write $MN_0 \cdots N_n$ instead of $(\cdots((MN_0)N_1) \cdots N_{n-1})N_n$.

   We take terms modulo α-conversion and use the Barendregt convention (BC) where the names of bound variables differ from the free ones. When two terms M and N are equal (modulo α), we write $M = N$. We write $\mathrm{fv}(M)$ for the set of the free variables of term M.

2. Let $n \geq 0$. We define $M^n(N)$, by induction on n, as follows: $M^0(N) = N$ and $M^{n+1}(N) = M(M^n(N))$.

3. We define as usual the substitution $M[x := N]$ of N for all free occurrences of x in M. We let $M[x_1 := N_1, \ldots, x_n := N_n]$ be the simultaneous substitution of N_i for all free occurrences of x_i in M for $1 \leq i \leq n$.

4. We assume the usual definition of compatibility (see the last line of Figure 4). For $r \in \{\beta, \beta\eta\}$, we define the reduction relation $\to_r$ on Λ as the least compatible relation closed under rule $(r) : L \to_r R$ below, and we call L an *r-redex* and R the *r-contractum* of L (or the L r-contractum).

   - (β): $(\lambda x.M)N \to_\beta M[x := N]$.
   - (η): $\lambda x.Mx \to_\eta M$ where $x \notin \mathrm{fv}(M)$.

   We define $\to_{\beta\eta} = \to_\beta \cup \to_\eta$.

let $\mathcal{R}$ be a binary relation on Λ.

$$\frac{}{M \mathrel{\mathcal{R}} M}\ (\mathit{refl}) \qquad \frac{M_1 \mathrel{\mathcal{R}} M_2 \quad M_2 \mathrel{\mathcal{R}} M_3}{M_1 \mathrel{\mathcal{R}} M_3}\ (\mathit{tr})$$

$$\frac{P \mathrel{\mathcal{R}} Q}{\lambda x.P \mathrel{\mathcal{R}} \lambda x.Q}\ (\mathit{abs}) \quad \frac{Q \mathrel{\mathcal{R}} Q'}{PQ \mathrel{\mathcal{R}} PQ'}\ (\mathit{app}_1) \quad \frac{P \mathrel{\mathcal{R}} P'}{PQ \mathrel{\mathcal{R}} P'Q}\ (\mathit{app}_2)$$

Figure 4: Closure rules

5. Let $r \in \{\to_\beta, \to_\eta, \to_{\beta\eta}\}$. We use $\to_r^*$ to denote the reflexive transitive closure (see the rules (*refl*) and (*tr*) of Figure 4) of $\to_r$. We let $\simeq_r$ denote the equivalence relation induced by $\to_r$. If the r-reduction from M to N is in k steps, we write $M \to_r^k N$.

6. Let $r \in \{\beta, \beta\eta\}$ and $n \geq 2$. A term $(\lambda x.M_1')M_2' \dots M_n'$ is a *direct r-reduct* of $(\lambda x.M_1)M_2 \dots M_n$ iff $\forall i \in \{1, \dots, n\}, M_i \to_r^* M_i'$.

7. Let $r \in \{\beta, \beta\eta\}$. We say that M has the Church-Rosser property for r (has r-CR) if whenever $M \to_r^* M_1$ and $M \to_r^* M_2$ then there exists M_3 such that $M_1 \to_r^* M_3$ and $M_2 \to_r^* M_3$. We define $\mathsf{CR}^r = \{M \mid M \text{ has } r\text{-CR}\}$. We define $\mathsf{CR}^r_\to = \{M \in \mathsf{CR}^r \mid \forall N \in \mathsf{CR}^r.\ MN \in \mathsf{CR}^r\}$. We use CR to denote CR^β and $\mathsf{CR}_\to$ to denote $\mathsf{CR}^\beta_\to$.

8. We define the set SAT of the saturated sets as follows:
$\mathsf{SAT} = \{s \subseteq \Lambda \mid n \geq 0 \wedge M[x := N]P_1 \dots P_n \in s \Rightarrow (\lambda x.M)NP_1 \dots P_n \in s\}$.

9. We define the set VAR of the set satisfying the variable property as follows:
$\mathsf{VAR} = \{s \subseteq \Lambda \mid n \geq 0 \wedge (\forall i \in \{1, \dots, n\}.\ M_i \in s) \Rightarrow xM_1 \cdots M_n \in s\}$ □

Lemma 3.2. *Let $r \in \{\beta, \beta\eta\}$.*

1. *If $M \to_r^* N$ and $P \to_r^* Q$ then $M[x := P] \to_r^* N[x := Q]$.*
2. $\mathrm{fv}(M[x := N]) \subseteq \mathrm{fv}((\lambda x.M)N)$.
3. *If $M \to_r^* N$ then $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$.*
4. *If $n \geq 0$, $Q = (\lambda x.M)NN_1 \dots N_n \to_r^k P$ and P is not a direct r-reduct of Q then (a) $k \geq 1$, (b) if $k = 1$ then $P = M[x := N]N_1 \dots N_n$ and (c) there exists a direct r-reduct $(\lambda x.M')N'N_1' \dots N_n'$ of Q such that $M'[x := N']N_1' \dots N_n' \to_r^* P$.*
5. *Let $n \geq 0$ and $(\lambda x.M)NN_1 \dots N_n \to_r^* P$. There exists P' such that $P \to_r^* P'$ and $M[x := N]N_1 \dots N_n \to_r^* P'$.*
6. $\mathsf{CR}^r \in \mathsf{SAT}$.
7. $\mathsf{CR}^r \in \mathsf{VAR}$.
8. *If $M \in \mathsf{CR}$ then $\lambda x.M \in \mathsf{CR}$.* □

3.2 Pseudo Development Definitions

Throughout, we take c and d to be two distinct metavariables ranging over Var.

Remark 3.3. Such c and d are usually given not as metavariables, but as new variables or constants [4, 9, 10]. We noted that this usual way leads to problems.

For example, Ghilezan and Kunčak [4], call c and d, f and g and are introduced as "predefined constants" not belonging to the λ-calculus. But the function Ψ defined by Ghilezan and Kunčak (similar to our function Ψ_{cd}) is proved to be a function from Λ to $\Lambda_0 \subset \Lambda$ where Λ_0 is a set of terms typable in the Simply Typed Lambda Calculus in a certain environment. So, it is obvious that their function Ψ does not associate a term in Λ_0 to each term in Λ since Ψ adds some f and g to the terms.

Moreover, typing environments (contexts) are defined as sets of type assignments of the form $x : \phi$ where x is a term variable. Later, some contexts are built with type assignments of the form $f : \phi$, but f is not defined as a term variable. More generally, the introduction of a new variable or a new constant implies that the considered type system has to be defined on the new calculus.

Koletsos and Stavrinos [9], define two sets CR and CR_0 which turn out to be equal to ours. Among other things, $\mathsf{CR}, \mathsf{CR}_0 \subseteq \Lambda$. Koletsos and Stavrinos prove that each term typable in the type system D has the Church-Rosser property. But it is not specified on which set of terms this result is stated. The proof of this statement fails, for example, for terms with free variables not belonging to the set of variables of the initial λ-calculus (c is defined as a variable not belonging to this set), since the proof uses the fact that the free variables of the term belong to the set CR_0. But, further, this statement is used for a term which may contain some c. □

As started by Krivine [10] and followed by many others [4, 9, 6], we use c and d to "freeze" some current or potential redexes (applications which are not currently redexes but which will become redexes after some substitutions). The following two parametrised calculi (with parameters c and d) are the "frozen" calculi based on the λ-calculus where some reductions are blocked by the use of c and d. For example $(\lambda x.xy)(\lambda z.z) \to_\beta (\lambda z.z)y \to_\beta y$, but $(\lambda x.cxy)(d(\lambda z.z)) \to_\beta c(d(\lambda z.z))y$ which does not reduce. In this example we remark that in fact c and d are not only needed to "freeze" potential redexes, but, as we will see below, they are also both needed to get our soundness results (lemma 4.4 and 5.6). Or, as proved by Ghilezan and Kunčak (for a calculus similar to the first of the two ones presented below), to get the typability of these calculi in the Simply Typed Lambda Calculus. It is easy to see that $\Lambda^{\beta}_{cd} \subset \Lambda^{\beta\eta}_{cd} \subset \Lambda$.

Definition 3.4 (Λ^{β}_{cd}, $\Lambda^{\beta\eta}_{cd}$)**.** Let $\bar{x}, \bar{y} \in \mathsf{Var}_{cd} = \mathsf{Var} \setminus \{c, d\}$.

$$\bar{M} \in \Lambda^{\beta}_{cd} ::= \bar{x} \mid d(\lambda\bar{x}.\bar{M}) \mid (\lambda\bar{x}.\bar{M}_1)\bar{M}_2 \mid c\bar{M}_1\bar{M}_2$$

$$\bar{M} \in \Lambda^{\beta\eta}_{cd} ::= \bar{x} \mid d(c\bar{M}) \mid d(\lambda\bar{x}.\bar{M}) \mid (\lambda\bar{x}.\bar{M}_1)\bar{M}_2 \mid c\bar{M}_1\bar{M}_2$$

We let $\bar{M}, \bar{N}, \bar{P}, \bar{Q}, \bar{R}$ range over $\Lambda^{\beta\eta}_{cd}$ □

We now define the function which "freezes" the potential redexes. The difference with similar definitions in the literature [4, 9, 10, 6], is that with our definition (third clause below), the current β-redexes of a term are all "unfrozen". Furthermore, our definition does not freeze the current η-redexes and does not freeze the potential η-redexes which are not current η-redexes. For example, $M = d(\lambda x.(\lambda y.czx)z)$ does not contain any η-redex but contains a potential η-redex, since $M \to_\beta d(\lambda x.czx) = N$ and N contains a η-redex. As we will see in this paper, it is not necessary to "freeze" the potential η-redexes.

Definition 3.5 ($\Psi_{cd}(-)$). $\Psi_{cd}(-)$ is defined as follows:

1. $\Psi_{cd}(x) = x$
2. $\Psi_{cd}(\lambda x.N) = d(\lambda x.\Psi_{cd}(N))$, where $x \notin \{c, d\}$
3. $\Psi_{cd}((\lambda x.N)Q) = (\lambda x.\Psi_{cd}(N))\Psi_{cd}(Q)$, where $x \notin \{c, d\}$
4. If P is not a λ-abstraction then $\Psi_{cd}(PQ) = c\Psi_{cd}(P)\Psi_{cd}(Q)$. $\square$

Similarly to those given in the literature [9, 10, 6], the following erasure function enables us to erase every c and d from a term of the "frozen" calculi Λ^{β}_{cd} and $\Lambda^{\beta\eta}_{cd}$.

Definition 3.6 ($|-|_{cd}$). $|-|_{cd} : \Lambda \to \Lambda$, is defined as follows:

- $|x|_{cd} = x$
- $|\lambda x.N|_{cd} = \lambda x.|N|_{cd}$, where $x \notin \{c, d\}$
- If $P \in \{c, d\}$ then $|PQ|_{cd} = |Q|_{cd}$
- If $P \notin \{c, d\}$ then $|PQ|_{cd} = |P|_{cd}|Q|_{cd}$. $\square$

The next definition introduces the reduction $\to_o$ which is a kind of partial erasure. This reduction turns out to be a simplification and a generalisation (in order to handle the $\beta\eta$-reduction) of the reduction, named also $\to_o$, defined by Ghilezan and Kunčak [4]. Note that a term in Λ^{β}_{cd} never reduces by the compatible closure of the rule (dc). But this rule is introduced in order to handle the $\beta\eta$ case.

Definition 3.7 ($\to_o$). Let the reduction relation $\to_o$ on Λ be the least compatible relation closed under the following rules:

- $(cd) : c(dM) \to_o M$.
- $(dc) : d(cM) \to_o M$.

As usual $\to^*_o$ is the reflexive and transitive closure of $\to_o$. $\square$

Notation 3.8. Let $(d \circ c)^0(M)$ stand for M and if $n \geq 0$, let $(d \circ c)^{n+1}(M)$ stand for $d(c((d \circ c)^n(M)))$. $\square$

Definition 3.9 (weak developments: $\to_1$, $\to_2$). Let M such that $c, d \notin \mathrm{fv}(M)$ and $\langle r, s\rangle \in \{\langle 1, \beta\rangle, \langle 2, \beta\eta\rangle\}$.

$$M \to_r N \iff \exists P.\ \Psi_{cd}(M) \to^*_s P \wedge |P|_{cd} = N$$

As usual, $\to^*_r$ is the reflexive and transitive closure of $\to_r$. $\square$

4 A simple Church-Rosser proof for β-reduction

Koletsos and Stavrinos [9] gave a proof of the Church-Rosser property for the set of terms typable in the intersection type system called system D [10] w.r.t. β-reduction and showed that this can be used to establish confluence of β-developments without using strong normalisation. Ghilezan and Kunčak [4] gave a proof of the Church-Rosser property for the set of terms typable in Simply Typed Lambda Calculus w.r.t. β-reduction and showed that this can be used to establish confluence of a weak form of β-developments without using strong normalisation.

The first aim of this section, was to simplify the proof of Koletsos and Stavrinos [9]. During this simplification, we obtained a proof that bears some resemblance to the proof of Ghilezan and Kunčak [4] but that is much simpler. The second aim of this section is to provide a framework for our main result: the extension to the case $\beta\eta$ where we give a simple proof of Church-Rosser for $\beta\eta$-reduction (section 5).

The next two lemmas are useful technicalities related to the reduction $\to_o$ and to the set of terms Λ^β_{cd}.

Lemma 4.1.

1. *If $\bar{M} \in \Lambda^\beta_{cd}$ and $\bar{M} \to_o N$ then $N \in \Lambda^\beta_{cd}$, $\bar{M} \notin \mathsf{Var}_{cd}$, $|\bar{M}|_{cd} = |N|_{cd}$, $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$ and:*
   - *if $\bar{M} = d(\lambda\bar{x}.\bar{P})$ then $N = d(\lambda\bar{x}.P')$ such that $\bar{P} \to_o P'$.*
   - *if $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$ then $N = (\lambda\bar{x}.P')\bar{Q}$ such that $\bar{P} \to_o P'$ or $N = (\lambda\bar{x}.\bar{P})Q'$ such that $\bar{Q} \to_o Q'$.*
   - *if $\bar{M} = c\bar{P}\bar{Q}$ then $N = cP'\bar{Q}$ such that $\bar{P} \to_o P'$ or $N = c\bar{P}Q'$ such that $\bar{Q} \to^*_o Q'$ or $\bar{P} = d(\lambda\bar{x}.\bar{R})$ and $N = (\lambda\bar{x}.\bar{R})\bar{Q}$.*
2. *If $\bar{M} \in \Lambda^\beta_{cd}$ and $\bar{M} \to^*_o d(\lambda x.Q)$ then $\bar{M} = d(\lambda x.P)$ and $P \to^*_o Q$.*
3. *If $\bar{M} \in \Lambda^\beta_{cd}$ and $\bar{M} \to^*_o N$ then $N \in \Lambda^\beta_{cd}$, $|\bar{M}|_{cd} = |N|_{cd}$, $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$ and*
   - *if $\bar{M} \in \mathsf{Var}_{cd}$ then $N = \bar{M}$.*
   - *if $\bar{M} = d(\lambda\bar{x}.\bar{P})$ then $N = d(\lambda\bar{x}.P')$ such that $\bar{P} \to^*_o P'$.*
   - *if $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$ then $N = (\lambda\bar{x}.P')Q'$ such that $\bar{P} \to^*_o P'$ and $\bar{Q} \to^*_o Q'$.*
   - *if $\bar{M} = c\bar{P}\bar{Q}$ then $N = cP'Q'$ such that $\bar{P} \to^*_o P'$ and $\bar{Q} \to^*_o Q'$ or $\bar{P} = d(\lambda\bar{x}.\bar{R})$ and $N = (\lambda\bar{x}.R')Q'$ such that $\bar{R} \to^*_o R'$ and $\bar{Q} \to^*_o Q'$.*

□

Lemma 4.2.

1. *$\mathrm{fv}(M) \setminus \{c,d\} = \mathrm{fv}(\Psi_{cd}(M)) \setminus \{c,d\}$.*
2. *If $\bar{M} \in \Lambda^\beta_{cd}$ then $\mathrm{fv}(|\bar{M}|_{cd}) = \mathrm{fv}(\bar{M}) \setminus \{c,d\}$.*
3. *If $\bar{M} \in \Lambda^\beta_{cd}$ and $|\bar{M}|_{cd} = \lambda\bar{x}.N$ then $\bar{M} = d(\lambda\bar{x}.\bar{P})$ and $|\bar{P}|_{cd} = N$.* □

The next lemma states that the function $\Psi_{cd}(-)$ associates to each term of the untyped λ-calculus (which does not contain c and d) a term in the language Λ^β_{cd}.

Lemma 4.3. *Let $M \in \Lambda$ such that $c, d \notin \mathrm{fv}(M)$ then $\Psi_{cd}(M) \in \Lambda^\beta_{cd}$.* □

The next lemma, as part of our "simplified" reducibility method, states the soundness of our simple calculus based on the set of terms Λ^β_{cd} w.r.t. our simple interpretation based on the set CR (as we can see in the proof of this lemma available in the extended version on the authors web pages), we also use the set $\mathsf{CR}_\to$ which correspond to the interpretation of an arrow type in the work done for example by Koletsos [8]) using among other things the saturation of the set CR (note that this lemma does not involve any type system).

Lemma 4.4. *If $\bar{M} \in \Lambda^\beta_{cd}$, $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \{x_1, \ldots, x_n\}$ and for all $i \in \{1, \ldots, n\}$, $M_i \in \mathsf{CR}$ then $\bar{M}[x_1 := M_1, \ldots, x_n := M_n] \in \mathsf{CR}$.* □

We are now able to prove that each term in Λ^β_{cd} is Church-Rosser (w.r.t. β-reduction), using the previous lemma.

Corollary 4.5. $\Lambda^\beta_{cd} \subseteq \mathsf{CR}$. □

Proof. Let $\bar{M} \in \Lambda^{\beta}_{cd}$ and $\mathrm{fv}(\bar{M})\backslash\{c, d\} = \{x_1, \ldots, x_n\}$. By lemma 3.2.7, $x_1, \ldots, x_n \in \mathsf{CR}$. So by lemma 4.4, $\bar{M} \in \mathsf{CR}$. □

Here is another lemma containing needed technicalities:

Lemma 4.6.

1. *If $\bar{M}, \bar{N} \in \Lambda^{\beta}_{cd}$ and $\bar{x} \in \mathsf{Var}_{cd}$ then $\bar{M}[\bar{x} := \bar{N}] \in \Lambda^{\beta}_{cd}$.*
2. *$|\Psi_{cd}(M)|_{cd} = M$.*
3. *If $\bar{M}, \bar{N} \in \Lambda^{\beta}_{cd}$ and $\bar{x} \in \mathsf{Var}_{cd}$ then $|\bar{M}[\bar{x} := \bar{N}]|_{cd} = |\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}]$.*
4. *If $\bar{M} \in \Lambda^{\beta}_{cd}$ and $\bar{M} \to^*_\beta N$ then $N \in \Lambda^{\beta}_{cd}$ and $|\bar{M}|_{cd} \to^*_\beta |N|_{cd}$.*
5. *If $c, d \notin \mathrm{fv}(M)$ and $\Psi_{cd}(M) \to^*_\beta N$ then $M \to_\beta |N|_{cd}$.* □

The next lemma is a key lemma of the method of parallel reductions. It states that the reflexive and transitive closure of $\to_\beta$ is equal to the reflexive and transitive closure of $\to_1$.

Lemma 4.7. *Let $c, d \notin \mathrm{fv}(M)$, then $M \to^*_\beta N \iff M \to^*_1 N$.* □

The next lemma constitutes important properties of the reduction $\to^*_o$. The first property states that for $\bar{M} \in \Lambda^{\beta}_{cd}$, $\Psi_{cd}(|\bar{M}|_{cd})$ is an "unfrozen" version of $\bar{M}$ (not totally "unfrozen", but some "frozen" redexes of M are "unfrozen" in $\Psi_{cd}(|\bar{M}|_{cd})$). The fourth property states that we can simulate the reduction of a term in Λ^{β}_{cd} from a partially "unfrozen" version of it. The fifth property is a technical result needed to prove the confluence of the $\to^*_1$ reduction. The proof of this result is based on properties of the reduction $\to^*_o$. This result is needed since in the proof of the confluence of the $\to^*_1$ reduction, we need to build a reduction $\to^*_1$ of length two from a $\to^*_\beta$ reduction of a term in Λ^{β}_{cd}. We saw that the function Ψ_{cd} associates to a term of the λ-calculus a term in the calculus Λ^{β}_{cd}. The problem comes from the fact that, even if such a term always reduces to a term belonging to Λ^{β}_{cd}, the reduct is not always the image of a term under the function Ψ_{cd}. We fill the gap thanks to the reduction $\to^*_o$ and its properties. The seventh property is the confluence of the λ-calculus w.r.t. $\to^*_1$ reduction.

Lemma 4.8.

1. *If $\bar{M} \in \Lambda^{\beta}_{cd}$ then $\bar{M} \to^*_o \Psi_{cd}(|\bar{M}|_{cd})$.*
2. *Let $\bar{M}, \bar{N} \in \Lambda^{\beta}_{cd}$ and $\bar{x} \in \mathsf{Var}_{cd}$. If $\bar{M} \to^*_o M'$ and $N \to^*_o N'$ then $\bar{M}[\bar{x} := \bar{N}] \to^*_o M'[\bar{x} := N']$.*
3. *If $\bar{M}_1 \in \Lambda^{\beta}_{cd}$, $\bar{M}_1 \to_\beta N_1$ and $\bar{M}_1 \to^*_o M_2$ then there exists N_2 such that $M_2 \to_\beta N_2$ and $N_1 \to^*_o N_2$.*
4. *If $\bar{M}_1 \in \Lambda^{\beta}_{cd}$, $\bar{M}_1 \to^*_\beta N_1$ and $\bar{M}_1 \to^*_o M_2$ then there exists N_2 such that $M_2 \to^*_\beta N_2$ and $N_1 \to^*_o N_2$.*
5. *Let $\bar{M} \in \Lambda^{\beta}_{cd}$. If $\bar{M} \to^*_\beta N$ and $|\bar{M}|_{cd} = P$, then there exists $Q \in \Lambda^{\beta}_{cd}$ such that $\Psi_{cd}(P) \to^*_\beta Q$ and $|Q|_{cd} = |N|_{cd}$.*
6. *Let $M \in \Lambda$ such that $c, d \notin \mathrm{fv}(M)$. If $M \to_1 M_1$ and $M \to_1 M_2$ then there exists M_3 such that $M_1 \to_1 M_3$ and $M_2 \to_1 M_3$.*
7. *Let $M \in \Lambda$ such that $c, d \notin \mathrm{fv}(M)$. If $M \to^*_1 M_1$ and $M \to^*_1 M_2$ then there exists M_3 such that $M_1 \to^*_1 M_3$ and $M_2 \to^*_1 M_3$.* □

The confluence of the λ-calculus w.r.t. β-reduction is now proved using the confluence of the λ-calculus w.r.t. $\to_1^*$ reduction and the equality between $\to_\beta^*$ and $\to_1^*$.

Theorem 4.9. $\Lambda = \mathsf{CR}$. □

Proof. $\mathsf{CR} \subseteq \Lambda$ is trivial, we only prove $\Lambda \subseteq \mathsf{CR}$. Let $M, M_1, M_2 \in \Lambda$ such that $M \to_\beta^* M_1$ and $M \to_\beta^* M_2$ and $c, d \notin \mathrm{fv}(M)$. By lemma 2, $c, d \notin \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2)$. By lemma 4.7, $M \to_1^* M_1$ and $M \to_1^* M_2$. By lemma 4.8.7, there exists M_3 such that $M_1 \to_1^* M_3$ and $M_2 \to_1^* M_3$. By lemma 4.7, $M_1 \to_\beta^* M_3$ and $M_2 \to_\beta^* M_3$. □

5 A simple Church-Rosser proof for $\beta\eta$-reduction

Now that we stated the principal steps of the method of the Church-Rosser property of the untyped λ-calculus w.r.t. β-reduction, we will generalise it to $\beta\eta$-reduction following the same steps and using the $\Lambda_{cd}^{\beta\eta}$ language. this generalisation can be regarded as an extension of the method of Ghilezan and Kunčak [4] and a simplification of the method of Kamareddine and Rahli. [6].

Lemma 5.1.

1. *If $\bar{M} \in \Lambda_{cd}^{\beta\eta}$ and $\bar{M} \to_o N$ then $N \in \Lambda_{cd}^{\beta\eta}$, $|\bar{M}|_{cd} = |N|_{cd}$, $\mathrm{fv}(\bar{M}) \setminus \{c, d\} = \mathrm{fv}(N) \setminus \{c, d\}$, $\bar{M} \notin \mathsf{Var}_{cd}$ and:*
   - *if $\bar{M} = d(\lambda\bar{x}.\bar{P})$ then $N = d(\lambda\bar{x}.P')$ such that $\bar{P} \to_o P'$.*
   - *if $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$ then $N = (\lambda\bar{x}.P')\bar{Q}$ such that $\bar{P} \to_o P'$ or $N = (\lambda\bar{x}.\bar{P})Q'$ such that $\bar{Q} \to_o Q'$.*
   - *if $\bar{M} = c\bar{P}\bar{Q}$ then $N = cP'\bar{Q}$ such that $\bar{P} \to_o P'$ or $N = c\bar{P}Q'$ such that $\bar{Q} \to_o^* Q'$ or $\bar{P} = d(\lambda\bar{x}.\bar{R})$ and $N = (\lambda\bar{x}.\bar{R})\bar{Q}$.*
   - *if $\bar{M} = d(c\bar{P})$ then $N = \bar{P}$ or $N = d(cP')$ such that $\bar{P} \to_o P'$.*

2. *If $\bar{M} \in \Lambda_{cd}^{\beta\eta}$, $n \geq 0$ and $\bar{M} \to_o^* (d \circ c)^n(d(\lambda x.Q))$ then $\bar{M} = (d \circ c)^m(d(\lambda x.P))$ such that $m \geq n$ and $P \to_o^* Q$.*

3. *If $\bar{M} \in \Lambda_{cd}^{\beta\eta}$ and $\bar{M} \to_o^* N$ then $N \in \Lambda_{cd}^{\beta\eta}$, $|\bar{M}|_{cd} = |N|_{cd}$, $\mathrm{fv}(\bar{M}) \setminus \{c, d\} = \mathrm{fv}(N) \setminus \{c, d\}$ and:*
   - *If $\bar{M} \in \mathsf{Var}_{cd}$ then $\bar{M} = N$.*
   - *If $\bar{M} = d(\lambda\bar{x}.\bar{P})$ then $N = d(\lambda\bar{x}.Q)$ such that $\bar{P} \to_o^* Q$.*
   - *If $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$ then $N = (\lambda\bar{x}.P')Q'$ such that $\bar{P} \to_o^* P'$ and $\bar{Q} \to_o^* Q'$.*
   - *If $\bar{M} = c\bar{P}\bar{Q}$ then $N = cP'Q'$ such that $\bar{P} \to_o^* P'$ and $\bar{Q} \to_o^* Q'$ or $\bar{P} = (d \circ c)^n(d(\lambda\bar{x}.\bar{P}_1))$ and $N = (\lambda\bar{x}.P_1')Q'$ such that $n \geq 0$, $\bar{x} \in \mathsf{Var}_{cd}$, $\bar{P}_1 \in \Lambda_{cd}^{\beta\eta}$, $\bar{P}_1 \to_o^* P_1'$ and $\bar{Q} \to_o^* Q'$.*
   - *If $\bar{M} = (d \circ c)^n(\bar{P})$ such that $n \geq 0$ then $N = (d \circ c)^m(Q)$ such that $m \leq n$ and $\bar{P} \to_o^* Q$.*

□

Lemma 5.2.

1. *If $\bar{M} \in \Lambda_{cd}^{\beta\eta}$ then $\mathrm{fv}(\bar{M}) \setminus \{c, d\} = \mathrm{fv}(|\bar{M}|_{cd})$.*

2. *If $\lambda x.M \to_{\beta\eta}^* N$ then:*
   - *Either $N = \lambda x.M'$ such that $M \to_{\beta\eta}^* M'$.*

- *Or* $M \rightarrow^*_{\beta\eta} Nx$ *such that* $x \notin \mathrm{fv}(N)$.

3. *If* $x \notin \mathrm{fv}(M)$ *and* $Mx \rightarrow^*_{\beta\eta} N$ *then* $M \rightarrow^*_{\beta\eta} P$ *and:*
   - *Either* $N = Px$.
   - *Or* $P = \lambda x.N$.

$\square$

Lemma 5.3. *If* $M \in \mathsf{CR}^{\beta\eta}$ *then* $\lambda x.M \in \mathsf{CR}^{\beta\eta}$. $\square$

Lemma 5.4.

1. *If* $\bar{M} \in \Lambda^{\beta\eta}_{cd}$, $\bar{x} \in \mathsf{Var}_{cd}$ *and* $|\bar{M}|_{cd} = \lambda\bar{x}.N$ *then* $\bar{M} = (d \circ c)^n(d(\lambda\bar{x}.\bar{P}))$ *where* $n \geq 0$, $\bar{P} \in \Lambda^{\beta\eta}_{cd}$ *and* $|\bar{P}|_{cd} = N$.
2. *If* $\bar{M}, \bar{N} \in \Lambda^{\beta\eta}_{cd}$ *and* $\bar{x} \in \mathsf{Var}_{cd}$ *then* $\bar{M}[\bar{x} := \bar{N}] \in \Lambda^{\beta\eta}_{cd}$. $\square$

The next lemma states that the function Ψ_{cd} associates to each term of the λ-calculus (which does not contain the variables c and d) a term in the language $\Lambda^{\beta\eta}_{cd}$. This result is trivial, since $\Lambda^{\beta}_{cd} \subset \Lambda^{\beta\eta}_{cd}$.

Lemma 5.5. *If* $c, d \notin \mathrm{fv}(M)$ *then* $\Psi_{cd}(M) \in \Lambda^{\beta\eta}_{cd}$. $\square$

Proof. By lemma 4.3, $\Psi_{cd}(M) \in \Lambda^{\beta}_{cd}$. Since $\Lambda^{\beta}_{cd} \subset \Lambda^{\beta\eta}_{cd}$ then $\Psi_{cd}(M) \in \Lambda^{\beta\eta}_{cd}$. $\square$

The next lemma, as part of our "simplified" reducibility method, states the soundness of our simple calculus based on the set of term $\Lambda^{\beta\eta}_{cd}$ w.r.t. our simple interpretation based on the set $\mathsf{CR}^{\beta\eta}$ (as we can see in the proof of this lemma, we also use the set $\mathsf{CR}^{\beta\eta}_{\rightarrow}$ which corresponds to the interpretation of an arrow type in the work done for example by Koletsos [8]) using among other things the saturation of the set $\mathsf{CR}^{\beta\eta}$ (note that as for lemma 4.4, this lemma does not involve any type system).

Lemma 5.6. *If* $\bar{M} \in \Lambda^{\beta\eta}_{cd}$, $\mathrm{fv}(\bar{M}) \setminus \{c, d\} = \{x_1, \ldots, x_n\}$ *and for all* $i \in \{1, \ldots, n\}$, $M_i \in \mathsf{CR}^{\beta\eta}$ *then* $\bar{M}[x_1 := M_1, \ldots, x_n := M_n] \in \mathsf{CR}^{\beta\eta}$. $\square$

We are now able to prove that each term in the $\Lambda^{\beta\eta}_{cd}$ calculus is Church-Rosser (w.r.t. the $\beta\eta$-reduction), using the previous lemma.

Corollary 5.7. $\Lambda^{\beta\eta}_{cd} \subseteq \mathsf{CR}^{\beta\eta}$. $\square$

Proof. Let $\bar{M} \in \Lambda^{\beta\eta}_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c, d\} = \{x_1, \ldots, x_n\}$. By lemma 3.2.7, $x_1, \ldots, x_n \in \mathsf{CR}^{\beta\eta}$. So by lemma 5.6, $\bar{M} \in \mathsf{CR}^{\beta\eta}$. $\square$

Lemma 5.8. *Let* $x \notin \mathrm{fv}(P) \cup \mathrm{fv}(y)$. *If for all* $N \in \Lambda$ *such that* $x \notin \mathrm{fv}(N)$, $M \neq Nx$ *then for all* $N \in \Lambda$ *such that* $x \notin \mathrm{fv}(N), M[y := P] \neq Nx$. $\square$

Lemma 5.9. *If* $\bar{x} \in \mathsf{Var}_{cd}$ *and* $\bar{M}, \bar{N} \in \Lambda^{\beta\eta}_{cd}$ *then* $|\bar{M}[\bar{x} := \bar{N}]|_{cd} = |\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}]$. $\square$

Lemma 5.10. *If* $\bar{M} \in \Lambda^{\beta\eta}_{cd}$ *and* $\bar{M} \rightarrow^*_{\beta\eta} N$ *then* $N \in \Lambda^{\beta\eta}_{cd}$ *and* $|\bar{M}|_{cd} \rightarrow^*_{\beta\eta} |N|_{cd}$. $\square$

Corollary 5.11. *Let* $M \in \Lambda$ *such that* $c, d \notin \mathrm{fv}(M)$. *If* $\Psi_{cd}(M) \rightarrow^*_{\beta\eta} N$ *then* $M \rightarrow^*_{\beta\eta} |N|_{cd}$. $\square$

Proof. By lemma 5.5, $\Psi_{cd}(M) \in \Lambda^{\beta\eta}_{cd}$. By lemma 5.10, $|\Psi_{cd}(M)|_{cd} \rightarrow^*_{\beta\eta} |N|_{cd}$. By lemma 4.6.2, $M \rightarrow^*_{\beta\eta} |N|_{cd}$. $\square$

The next lemma is a key lemma of the parallel reduction method. It states that the reflexive and transitive closure of $\rightarrow_{\beta\eta}$ is equal to the reflexive and transitive closure of $\rightarrow_2$.

Lemma 5.12. *Let $c, d \notin \mathrm{fv}(M)$, then $M \rightarrow^*_{\beta\eta} N \iff M \rightarrow^*_2 N$.* $\square$

The next lemma states that for $M \in \Lambda^{\beta\eta}_{cd}$, $\Psi_{cd}(|M|_{cd})$ is an "unfrozen" version of M (not totally "unfrozen", but some "frozen" redexes of M are "unfroze" in $\Psi_{cd}(|M|_{cd})$).

Lemma 5.13. *If $\bar{M} \in \Lambda^{\beta\eta}_{cd}$ then $\bar{M} \rightarrow^*_o \Psi_{cd}(|\bar{M}|_{cd})$.* $\square$

Lemma 5.14. *Let $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{M}, \bar{N} \in \Lambda^{\beta\eta}_{cd}$. If $\bar{M} \rightarrow^*_o M'$ and $N \rightarrow^*_o N'$ then $\bar{M}[\bar{x} := \bar{N}] \rightarrow^*_o M'[\bar{x} := N']$.* $\square$

Lemma 5.15. *If $\bar{M_1} \in \Lambda^{\beta\eta}_{cd}$, $\bar{M_1} \rightarrow_{\beta\eta} N_1$ and $\bar{M_1} \rightarrow^*_o M_2$ then there exists N_2 such that $M_2 \rightarrow_{\beta\eta} N_2$ and $N_1 \rightarrow^*_o N_2$.* $\square$

The next lemma states that we can simulate the reduction of a term in $\Lambda^{\beta\eta}_{cd}$ from an "unfrozen" version of it.

Lemma 5.16. *If $\bar{M_1} \in \Lambda^{\beta\eta}_{cd}$ such that $\bar{M_1} \rightarrow^*_{\beta\eta} N_1$ and $\bar{M_1} \rightarrow^*_o M_2$ then there exists N_2 such that $M_2 \rightarrow^*_{\beta\eta} N_2$ and $N_1 \rightarrow^*_o N_2$.* $\square$

Proof. Easy by lemma 5.15. $\square$

The next result is a technical result needed to prove the confluence of the $\rightarrow^*_2$ reduction. The proof of this result is based on properties of the reduction $\rightarrow^*_o$. This lemma is needed since in the proof of the confluence of the $\rightarrow^*_2$ reduction, we need to build a reduction $\rightarrow^*_2$ of length two from a $\rightarrow^*_{\beta\eta}$ reduction of a term in $\Lambda^{\beta\eta}_{cd}$. We saw that the function Ψ_{cd} associates to a term of the λ-calculus a term in the calculus $\Lambda^{\beta\eta}_{cd}$. The problem comes from the fact that, even if a such term always reduces to a term belonging to $\Lambda^{\beta\eta}_{cd}$, the reduct is not always the image of a term under the function Ψ_{cd}. We fill the gap thanks to the reduction $\rightarrow^*_o$ and its properties

Corollary 5.17. *Let $M \in \Lambda^{\beta\eta}_{cd}$. If $M \rightarrow^*_{\beta\eta} N$ and $|M|_{cd} = P$, then there exists Q such that $\Psi_{cd}(P) \rightarrow^*_{\beta\eta} Q$ and $|Q|_{cd} = |N|_{cd}$.* $\square$

Proof. By lemma 5.13, $M \rightarrow^*_o \Psi_{cd}(|M|_{cd})$. By lemma 5.16, there exists Q such that $\Psi_{cd}(|M|_{cd}) \rightarrow^*_{\beta\eta} Q$ and $N \rightarrow^*_o Q$. By lemma 5.1.3, $|Q|_{cd} = |N|_{cd}$. $\square$

Lemma 5.18. *Let $M \in \Lambda$ such that $c, d \notin \mathrm{fv}(M)$. If $M \rightarrow_2 M_1$ and $M \rightarrow_2 M_2$ then there exists M_3 such that $M_1 \rightarrow_2 M_3$ and $M_2 \rightarrow_2 M_3$.* $\square$

Proof. By definition, there exist P_1, P_2 such that $\Psi_{cd}(M) \rightarrow^*_{\beta\eta} P_1$, $\Psi_{cd}(M) \rightarrow^*_{\beta\eta} P_2$, $|P_1|_{cd} = M_1$ and $|P_2|_{cd} = M_2$. By lemma 5.5, $\Psi_{cd}(M) \in \Lambda^{\beta\eta}_{cd}$. So by corollary 5.7, there exists P_3 such that $P_1 \rightarrow^*_{\beta\eta} P_3$ and $P_2 \rightarrow^*_{\beta\eta} P_3$. Let $M_3 = |P_3|_{cd}$. By lemma 5.10, $P_1, P_2 \in \Lambda^{\beta\eta}_{cd}$. By corollary 5.17, there exist Q_1, Q_2 such that $\Psi_{cd}(M_1) \rightarrow^*_{\beta\eta} Q_1$, $\Psi_{cd}(M_2) \rightarrow^*_{\beta\eta} Q_2$, and $|Q_1|_{cd} = M_3 = |Q_2|_{cd}$. By lemma 5.2.1, $c, d \notin\ fvM_1 \cup \mathrm{fv}(M_2)$. So $M_1 \rightarrow_2 M_3$ and $M_2 \rightarrow_2 M_3$. $\square$

It is then easy to deduce the confluence of the λ-calculus w.r.t. $\rightarrow^*_2$ reduction:

Lemma 5.19. *Let $M \in \Lambda$ such that $c, d \notin \mathrm{fv}(M)$. If $M \rightarrow^*_2 M_1$ and $M \rightarrow^*_2 M_2$ then there exists M_3 such that $M_1 \rightarrow^*_2 M_3$ and $M_2 \rightarrow^*_2 M_3$.* $\square$

Proof. Easy by lemma 5.18 $\square$

The confluence of the λ-calculus w.r.t. $\beta\eta$-reduction is then proved using the confluence of the λ-calculus w.r.t. $\rightarrow_2^*$ reduction and the equality between $\rightarrow_{\beta\eta}^*$ and $\rightarrow_2^*$.

Theorem 5.20. $\Lambda = \mathsf{CR}^{\beta\eta}$. $\square$

Proof. $\mathsf{CR}^{\beta\eta} \subseteq \Lambda$ is trivial, we only prove $\Lambda \subseteq \mathsf{CR}^{\beta\eta}$. Let $M, M_1, M_2 \in \Lambda$ and $c, d \notin \text{fv}(M)$ such that $M \rightarrow_{\beta\eta}^* M_1$ and $M \rightarrow_{\beta\eta}^* M_2$. By lemma 3.2.3, $c, d \notin \text{fv}(M_1) \cup \text{fv}(M_2)$. By lemma 5.12, $M \rightarrow_2^* M_1$ and $M \rightarrow_2^* M_2$. By lemma 5.19, there exists M_3 such that $M_1 \rightarrow_2^* M_3$ and $M_2 \rightarrow_2^* M_3$. By lemma 5.12, $M_1 \rightarrow_{\beta\eta}^* M_3$ and $M_2 \rightarrow_{\beta\eta}^* M_3$. $\square$

6 Conclusion

Although our work derives from the one done by Koletsos and Stavrinos [9] and Kamareddine and Rahli [6], it turned out that it is also a simplification and generalisation of the work done by Ghilezan and Kunčak [4]. Because the work we achieved is more similar to the one of Ghilezan and Kunčak, we adapted some of our notations to theirs and focused our comparisons with the related work to their work.

Thereby, the two improvements of the present article can be regarded as the simplification of the work done by Ghilezan and Kunčak [4] by getting rid of all the type machinery and the extension of the defined method to the $\beta\eta$-reduction.

As explained above, the main lines of our proof are: the definition of some weak developments, the proof of the confluence of a simple calculus w.r.t. the considered reduction (β or $\beta\eta$) using a simplified reducibility method, the proof of the confluence of the defined developments and the proof of the equality between the reflexive and transitive closure of the developments and the reflexive and transitive closure of the considered reduction using a method of parallel reductions.

We think that the definitions of developments presented by Ghilezan and Kunčak [4] or in this paper would be hard to simplify further. But, as we pointed out in section 2, finding a simpler definition of developments (or similar reduction) might help simplifying further this kind of proof.

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A Proofs

of Lemma 3.2. 1. If $r = \beta\eta$, the proof is by induction on the length of the reduction $M \to^*_{\beta\eta} N$.

- If $M = N$ then $M[x := P] = N[x := P]$. We prove that $N[x := P] \to^*_{\beta\eta} N[x := Q]$ by induction on the structure of N.
  - Let $N \in \mathsf{Var}$. If $N = x$ then $N[x := P] = P \to^*_{\beta\eta} Q = N[x := Q]$, else $N[x := P] = N = N[x := Q]$.
  - Let $N = \lambda y.N'$. By IH, $N[x := P] = \lambda y.N'[x := P] \to^*_{\beta\eta} \lambda y.N'[x := Q] = N[x := Q]$ such that $y \notin \mathrm{fv}(PQx)$.
  - Let $N = N_1N_2$. By IH, $N[x := P] = N_1[x := P]N_2[x := P] \to^*_{\beta\eta} N_1[x := Q]N_2[x := Q] = N[x := Q]$.
- Let $M \to^*_{\beta\eta} M' \to_{\beta\eta} N$. By IH, $M[x := P] \to^*_{\beta\eta} M'[x := Q]$. We prove that $M'[x := Q] \to_{\beta\eta} N[x := Q]$ by induction on the structure of M'.
  - Let $M' \in \mathsf{Var}$ then nothing to prove since M' does not reduce.
  - Let $M' = \lambda y.M'_1$.
    - Either $N = \lambda y.M'_2$ such that $M'_1 \to_{\beta\eta} M'_2$. By IH, $M'_1[x := Q] \to_{\beta\eta} M'_2[x := Q]$. So $M'[x := Q] = \lambda y.M'_1[x := Q] \to_{\beta\eta} \lambda y.M'_2[x := Q] = N[x := Q]$ such that $y \notin \mathrm{fv}(Qx)$.
    - Or $M'_1 = Ny$ such that $y \notin \mathrm{fv}(N)$. So $M'[x := Q] = \lambda y.N[x := Q]y \to_\eta N[x := Q]$ such that $y \notin \mathrm{fv}(Qx)$.
  - Let $M' = M_1M_2$.
    - Either $N = M'_1M_2$ such that $M_1 \to_{\beta\eta} M'_1$. By IH, $M_1[x := Q] \to_{\beta\eta} M'_1[x := Q]$. So $M'[x := Q] = M_1[x := Q]M_2[x := Q] \to_{\beta\eta} M'_1[x := Q]M_2[x := Q] = N[x := Q]$.
    - Or $N = M_1M'_2$ such that $M_2 \to_{\beta\eta} M'_2$. By IH, $M_2[x := Q] \to_{\beta\eta} M'_2[x := Q]$, so $M'[x := Q] = M_1[x := Q]M_2[x := Q] \to_{\beta\eta} M_1[x := Q]M'_2[x := Q] = N[x := Q]$.
    - Or $M_1 = \lambda y.M'_1$ and $N = M'_1[y := M_2]$. So, $M'[x := Q] = (\lambda y.M'_1[x := Q])M_2[x := Q] \to_\beta M'_1[x := Q][y := M_2[x := Q]] = N[x := Q]$ by the well known substitution lemma and such that $y \notin \mathrm{fv}(Qx)$.

If $r = \beta$, the proof is by induction on the length of the reduction $M \to^*_\beta N$.

- If $M = N$ then $M[x := P] = N[x := P]$. We prove that $N[x := P] \to^*_\beta N[x := Q]$ by induction on the structure of N.
  - Let $N \in \mathsf{Var}$. If $N = x$ then $N[x := P] = P \to^*_\beta Q = N[x := Q]$, else $N[x := P] = N = N[x := Q]$.
  - Let $N = \lambda y.N'$. By IH, $N[x := P] = \lambda y.N'[x := P] \to^*_\beta \lambda y.N'[x := Q] = N[x := Q]$ such that $y \notin \mathrm{fv}(PQx)$.
  - Let $N = N_1N_2$. By IH, $N[x := P] = N_1[x := P]N_2[x := P] \to^*_\beta N_1[x := Q]N_2[x := Q] = N[x := Q]$.
- Let $M \to^*_\beta M' \to_\beta N$. By IH, $M[x := P] \to^*_\beta M'[x := Q]$. We prove that $M'[x := Q] \to_\beta N[x := Q]$ by induction on the structure of M'.
  - Let $M' \in \mathsf{Var}$ then nothing to prove since M' does not reduce.
  - Let $M' = \lambda y.M'_1$. Then $N = \lambda y.M'_2$ such that $M'_1 \to_\beta M'_2$. By IH, $M'_1[x := Q] \to_\beta M'_2[x := Q]$, so $M'[x := Q] = \lambda y.M'_1[x := Q] \to_\beta \lambda y.M'_2[x := Q] = N[x := Q]$ such that $y \notin \mathrm{fv}(Qx)$.
  - Let $M' = M_1M_2$.

- Either $N = M_1'M_2$ such that $M_1 \to_\beta M_1'$. By IH, $M_1[x := Q] \to_\beta M_1'[x := Q]$, so $M'[x := Q] = M_1[x := Q]M_2[x := Q] \to_\beta M_1'[x := Q]M_2[x := Q] = N[x := Q]$.
- Or $N = M_1M_2'$ such that $M_2 \to_\beta M_2'$. By IH, $M_2[x := Q] \to_\beta M_2'[x := Q]$, so $M'[x := Q] = M_1[x := Q]M_2[x := Q] \to_\beta M_1[x := Q]M_2'[x := Q] = N[x := Q]$.
- Or $M_1 = \lambda y.M_1'$ and $N = M_1'[y := M_2]$. So, $M'[x := Q] = (\lambda y.M_1'[x := Q])M_2[x := Q] \to_\beta M_1'[x := Q][y := M_2[x := Q]] = N[x := Q]$ by the well known substitution lemma and such that $y \notin \text{fv}(Qx)$.

2. We prove this lemma by induction on the structure of M.

   - Let $M \in \mathsf{Var}$ then either $M = x$ and so $\text{fv}(M[x := N]) = \text{fv}(N) = \text{fv}((\lambda x.M)N)$. Or $M \neq x$ and so $\text{fv}(M[x := N]) = \text{fv}(M) \subseteq \text{fv}(M) \cup \text{fv}(N) = \text{fv}((\lambda x.M)N)$.
   - Let $M = \lambda y.P$ then $\text{fv}(M[x := N]) = \text{fv}(\lambda y.P[x := N]) = \text{fv}(P[x := N]) \setminus \{y\} \subseteq^{IH} \text{fv}((\lambda x.P)N) \setminus \{y\} = \text{fv}((\lambda x.M)N)$ such that $y \notin \text{fv}(Nx)$.
   - let $M = P_1P_2$ then $\text{fv}(M[x := N]) = \text{fv}(P_1[x := N]) \cup \text{fv}(P_2[x := N]) \subseteq^{IH} \text{fv}((\lambda x.P_1)N) \cup \text{fv}((\lambda x.P_2)N) = \text{fv}((\lambda x.M)N)$.

3. We prove this lemma by induction on the length of the reduction $M \to^*_{\beta\eta} N$.

   - Let $M = N$ then $\text{fv}(M) = \text{fv}(N)$.
   - Let $M \to^*_{\beta\eta} M' \to_{\beta\eta} N$. By IH, $\text{fv}(M') \subseteq \text{fv}(M)$. We prove that $\text{fv}(N) \subseteq \text{fv}(M')$ by induction on the structure of M'.
     - Let $M' \in \mathsf{Var}$ then nothing to prove since M' does not reduce.
     - Let $M' = \lambda x.P$.
       - Either $N = \lambda x.Q$ such that $P \to_{\beta\eta} Q$. By IH, $\text{fv}(Q) \subseteq \text{fv}(P)$. So $\text{fv}(N) \subseteq \text{fv}(M')$.
       - Or $P = Nx$ such that $x \notin \text{fv}(N)$. So $\text{fv}(N) = \text{fv}(M')$.
     - Let $M' = P_1P_2$.
       - Either $N = P_1'P_2$ such that $P_1 \to_{\beta\eta} P_1'$. By IH, $\text{fv}(P_1') \subseteq \text{fv}(P_1)$, so $\text{fv}(N) \subseteq \text{fv}(M')$.
       - Or $N = P_1P_2'$ such that $P_2 \to_{\beta\eta} P_2'$. By IH, $\text{fv}(P_2') \subseteq \text{fv}(P_2)$, so $\text{fv}(N) \subseteq \text{fv}(M')$.
       - Or $P_1 = \lambda x.P_0$ and $N = P_0[x := P_2]$. By 2.2, $\text{fv}(N) \subseteq \text{fv}(M')$.

   A corollary of this result is that if $M \to^*_\beta N$ then $\text{fv}(N) \subseteq \text{fv}(M)$.

4. (a) If $k = 0$ then $P = Q$ is a direct r-reduct of Q, absurd.

   (b) Assume $k = 1$, we prove $P = M[x := N]N_1 \dots N_n$ by induction on $n \geq 0$.

   - Let $n = 0$ and $r = \beta$. The proof is by case on $Q = (\lambda x.M)N \to_\beta P$.
     - If $(\lambda x.M)N \to_\beta M[x := N]$ then we are done.
     - If $(\lambda x.M)N \to_\beta (\lambda x.M')N = P$ such that $M \to_\beta M'$ then P is a direct β-reduct of $(\lambda x.M)N$, absurd.
     - If $(\lambda x.M)N \to_\beta (\lambda x.M)N' = P$ such that $N \to_\beta N'$ then P is a direct β-reduct of $(\lambda x.M)N$, absurd.
   - Let $n = 0$ and $r = \beta\eta$. The proof is by case on $Q = (\lambda x.M)N \to_{\beta\eta} P$.
     - If $(\lambda x.M)N \to_\beta M[x := N]$, then we are done.
     - If $\lambda x.M \to_{\beta\eta} R$ and $P = RN$ then:

- $\cdot$ Either $R = \lambda x.M'$ such that $M \to_{\beta\eta} M'$. So P is a direct $\beta\eta$-reduct of $(\lambda x.M)N$, absurd.
- $\cdot$ Or $M = Rx$ and $x \notin FV(R)$. Hence, $P = RN = M[x := N]$ and we are done.

* If $N \to_{\beta\eta} N'$ and $P = (\lambda x.M)N'$ then P is a direct $\beta\eta$-reduct of $(\lambda x.M)N$, absurd.

– Let $n = m + 1$ where $m \geq 0$. By case on $Q = (\lambda x.M)NN_1 \dots N_n \to_r P$.

* Either $(\lambda x.M)NN_1 \dots N_m \to_r R$ and $P = RN_n$.
  - $\cdot$ If R is a direct r-reduct of $(\lambda x.M)NN_1 \dots N_m$ then P is a direct r-reduct of $(\lambda x.M)NN_1 \dots N_n$, absurd.
  - $\cdot$ Else it is done by IH.
* Or $N_n \to_r N'_n$ and $P = (\lambda x.M)NN_1 \dots N_m N'_n$ is a direct r-reduct of $(\lambda x.M)NN_0 \dots N_n$, absurd.

(c) We prove the statement by induction on $k \geq 1$.

– If $k = 1$ then it is done since by (b) $P = M[x := N]N_1 \dots N_n$.

– Else, let $k \geq 1$ and $Q = (\lambda x.M)NN_1 \dots N_n \to_r^k R \to_r P$.

* If R is a direct r-reduct of Q, then $R = (\lambda x.M')N'N'_1 \dots N'_n$, such that $M \to_r^* M'$, $N \to_r^* N'$ and for all $i \in \{1, \dots, n\}$, $N_i \to_r^* N'_i$. Since P is not a direct r-reduct of Q, P is not a direct r-reduct of R. Hence by (b), $P = M'[x := N']N'_1 \dots N'_n$.
* Else, by IH, there exists a direct r-reduct $(\lambda x.M')N'N'_1 \dots N'_n$ of Q such that $M'[x := N']N'_1 \dots N'_n \to_r^* R \to_r P$.

5. If P is a direct r-reduct of $(\lambda x.M)NN_1 \dots N_n$ then $P = (\lambda x.M')N'N'_1 \dots N'_n$ such that $M \to_r^* M'$, $N \to_r^* N'$ and for all $i \in \{1, \dots, n\}$, $N_i \to_r^* N'_i$. So $P \to_r M'[x := N']N'_1 \dots N'_n$ and $M[x := N]N_1 \dots N_n \to_r^* M'[x := N']N'_1 \dots N'_n$, by lemma 1. If P is not a direct r-reduct of $(\lambda x.M)NN_1 \dots N_n$ then by lemma 4.4, there exists a direct r-reduct, $(\lambda x.M')N'N'_1 \dots N'_n$ of $(\lambda x.M)NN_1 \dots N_n$ such that $M \to_r^* M'$, $N \to_r^* N'$, for all $i \in \{1, \dots, n\}$, $N_i \to_r^* N'_i$ and $M'[x := N']N'_1 \dots N'_n \to_r^* P$. Finally, by lemma 1, $M[x := N]N_1 \dots N_n \to_r^* M'[x := N']N'_1 \dots N'_n \to_r^* P$, .

6. Let $n \geq 0$, $M[x := N]N_1 \dots N_n \in \mathsf{CR}^r$, $(\lambda x.M)NN_1 \dots N_n \to_r^* M_1$ and $(\lambda x.M)NN_1 \dots N_n \to_r^* M_2$. By lemma 4.5, there exist M'_1 and M'_2 such that $M_1 \to_r^* M'_1$, $M[x := N]N_1 \dots N_n \to_r^* M'_1$, $M_2 \to_r^* M'_2$ and $M[x := N]N_1 \dots N_n \to_r^* M'_2$. Then we conclude using $M[x := N]N_1 \dots N_n \in \mathsf{CR}^r$.

7 Let $n \geq 0$ and for all $i \in \{1, \dots, n\}$, $M_i \in \mathsf{CR}^r$. First we prove that if $xM_1 \cdots M_n \to_r^* N$ then $N = xM'_1 \cdots M'_n$ such that for all $i \in \{1, \dots, n\}$, $M_i \to_r^* M'_i$. We prove the result by induction on the length of the reduction $xM_1 \cdots M_n \to_r^* N$.

– Let $xM_1 \cdots M_n = N$ then it is done

– Let $xM_1 \cdots M_n \to_r^* N' \to_r N$. By IH, $N' = xM'_1 \cdots M'_n$ such that for all $i \in \{1, \dots, n\}$, $M_i \to_r^* M'_i$. We prove the result by induction on n.

* Let $n = 0$ then it is done since x does not reduce by $\to_r$.
* Let $n = m + 1$ such that $m \geq 0$. By compatibility:
  - $\cdot$ Either $N = PM'_n$ such that $xM'_1 \cdots M'_m \to_r P$ Then by IH $P = xM''_1 \cdots M''_m$ such that for all $i \in \{1, \dots, m\}$, $M'_i \to_r^* M''_i$. So it is done.
  - $\cdot$ Or $N = xM'_1 \cdots M'_m M''_n$ such that $M'_n \to_r M''_n$ then it is done.

8. Let $\lambda x.M \rightarrow_\beta^* P_1$ and $\lambda x.M \rightarrow_\beta^* P_2$ then $P_1 = \lambda x.M_1$ and $P_2 = \lambda x.M_2$ such that $M \rightarrow_\beta^* M_1$ and $M \rightarrow_\beta^* M_2$. By hypothesis, there exists M_3 such that $M_1 \rightarrow_\beta^* M_3$ and $M_2 \rightarrow_\beta^* M_3$. So $P_1 \rightarrow_\beta^* \lambda x.M_3$ and $P_2 \rightarrow_\beta^* \lambda x.M_3$. $\square$

of lemma 4.1.

1 By induction on the structure of M.

- Let $\bar{M} \in \mathsf{Var}_{cd}$ then $\bar{M}$ does not reduce by $\rightarrow_o$.
- Let $\bar{M} = d(\lambda\bar{x}.\bar{P})$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P} \in \Lambda_{cd}^\beta$, then by compatibility $\bar{M} = d(\lambda\bar{x}.\bar{P}) \rightarrow_o d(\lambda\bar{x}.P') = N$ such that $\bar{P} \rightarrow_o P'$. By IH, $P' \in \Lambda_{cd}^\beta$, $|\bar{P}|_{cd} = |P'|_{cd}$ and $\mathrm{fv}(\bar{P})\setminus\{c,d\} = \mathrm{fv}(P')\setminus\{c,d\}$. So $N = d(\lambda\bar{x}.P') \in \Lambda_{cd}^\beta$, $|\bar{M}|_{cd} = \lambda\bar{x}.|\bar{P}|_{cd} = \lambda\bar{x}.|P'|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(\bar{P}) \setminus \{c,d,\bar{x}\} = \mathrm{fv}(P') \setminus \{c,d,\bar{x}\} = \mathrm{fv}(N)$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P},\bar{Q} \in \Lambda_{cd}^\beta$, then by compatibility:
  - Either $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q} \rightarrow_o (\lambda\bar{x}.P')\bar{Q} = N$ such that $\bar{P} \rightarrow_o P'$. By IH, $P' \in \Lambda_{cd}^\beta$, $|\bar{P}|_{cd} = |P'|_{cd}$ and $\mathrm{fv}(\bar{P}) \setminus \{c,d\} = \mathrm{fv}(P') \setminus \{c,d\}$. So $N = (\lambda\bar{x}.P')Q \in \Lambda_{cd}^\beta$, $|\bar{M}|_{cd} = (\lambda\bar{x}.|\bar{P}|_{cd})|\bar{Q}|_{cd} = (\lambda\bar{x}.|P'|_{cd})|\bar{Q}|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = (\mathrm{fv}(\bar{P}) \setminus \{c,d,\bar{x}\}) \cup (\mathrm{fv}(\bar{Q}) \setminus \{c,d\}) = (\mathrm{fv}(P') \setminus \{c,d,\bar{x}\}) \cup (\mathrm{fv}(\bar{Q}) \setminus \{c,d\}) = \mathrm{fv}(N)$.
  - Or $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q} \rightarrow_o (\lambda\bar{x}.\bar{P})Q' = N$ where $\bar{Q} \rightarrow_o Q'$. By IH, $Q' \in \Lambda_{cd}^\beta$, $|\bar{Q}|_{cd} = |Q'|_{cd}$ and $\mathrm{fv}(\bar{Q}) \setminus \{c,d\} = \mathrm{fv}(Q') \setminus \{c,d\}$. So $N \in \Lambda_{cd}^\beta$, $|\bar{M}|_{cd} = (\lambda\bar{x}.|\bar{P}|_{cd})|\bar{Q}|_{cd} = (\lambda\bar{x}.|\bar{P}|_{cd})|Q'|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = (\mathrm{fv}(\lambda\bar{x}.\bar{P}) \cup \mathrm{fv}(\bar{Q})) \setminus \{c,d\} = (\mathrm{fv}(\lambda\bar{x}.\bar{P}) \cup \mathrm{fv}(Q')) \setminus \{c,d\} = \mathrm{fv}(N)$.
- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P},\bar{Q} \in \Lambda_{cd}^\beta$, then by compatibility:
  - Either $\bar{M} = c\bar{P}\bar{Q} \rightarrow_o cP'\bar{Q} = N$ such that $\bar{P} \rightarrow_o P'$. By IH, $P' \in \Lambda_{cd}^\beta$, $|\bar{P}|_{cd} = |P'|_{cd}$ and $\mathrm{fv}(\bar{P}) \setminus \{c,d\} = \mathrm{fv}(P') \setminus \{c,d\}$. So, $N \in \Lambda_{cd}^\beta$, $|\bar{M}|_{cd} = |\bar{P}|_{cd}|\bar{Q}|_{cd} = |P'|_{cd}|\bar{Q}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = (\mathrm{fv}(\bar{P}) \cup \mathrm{fv}(\bar{Q})) \setminus \{c,d\} = (\mathrm{fv}(P') \cup \mathrm{fv}(\bar{Q})) \setminus \{c,d\} = \mathrm{fv}(N)$.
  - Or $\bar{M} = c\bar{P}\bar{Q} \rightarrow_o P'\bar{Q} = N$ such that $\bar{P} = dP'$. Since $\bar{P} \in \Lambda_{cd}^\beta$, $P' = (\lambda\bar{x}.\bar{R}')$ where $\bar{R}' \in \Lambda_{cd}^\beta$ and $\bar{x} \in \mathsf{Var}_{cd}$. Hence $N \in \Lambda_{cd}^\beta$, $|\bar{M}|_{cd} = |P'|_{cd}|\bar{Q}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = (\mathrm{fv}(P') \cup \mathrm{fv}(\bar{Q})) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.
  - Or $\bar{M} = c\bar{P}\bar{Q} \rightarrow_o c\bar{P}Q' = N$ such that $\bar{Q} \rightarrow_o Q'$. By IH, $Q' \in \Lambda_{cd}^\beta$, $|\bar{Q}|_{cd} = |Q'|_{cd}$ and $\mathrm{fv}(\bar{Q}) \setminus \{c,d\} = \mathrm{fv}(Q') \setminus \{c,d\}$. So $N \in \Lambda_{cd}^\beta$, $|\bar{M}|_{cd} = |\bar{P}|_{cd}|\bar{Q}|_{cd} = |\bar{P}|_{cd}|Q'|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = (\mathrm{fv}(\bar{P}) \cup \mathrm{fv}(\bar{Q})) \setminus \{c,d\} = (\mathrm{fv}(\bar{P}) \cup \mathrm{fv}(Q')) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.

2 By induction on the length of the reduction $\bar{M} \rightarrow_o^* d(\lambda x.Q)$.

- If $\bar{M} = d(\lambda x.Q)$ then it is done.
- Let $\bar{M} \rightarrow_o M' \rightarrow_o^* d(\lambda x.Q)$. By lemma 4.1.1, $M' \in \Lambda_{cd}^\beta$. By IH, $M' = d(\lambda x.R)$ such that $R \rightarrow_o^* Q$. Since $\bar{M} \in \Lambda_{cd}^\beta$, by case on $\bar{M}$ and by lemma 4.1.1, $M = d(\lambda x.P)$ and $P \rightarrow_o R$. So $P \rightarrow_o^* Q$.

3 We prove this lemma by induction on the length of the reduction $\bar{M} \rightarrow_o^* N$.

- Let $\bar{M} = N$ then it is done.

- Let $\bar{M} \to_o M' \to_o^* N$. By lemma 4.1.1, $M' \in \Lambda_{cd}^{\beta}$, $|\bar{M}|_{cd} = |M'|_{cd}$, $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(M') \setminus \{c,d\}$. By IH, $N \in \Lambda_{cd}^{\beta}$, $|M'|_{cd} = |N|_{cd}$ and $\mathrm{fv}(M') \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$. So $|\bar{M}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$. By case on the structure of $\bar{M}$ and lemma 4.1.1:
  - If $\bar{M} \in \mathsf{Var}_{cd}$ then nothing to prove since $\bar{M}$ does not reduce by $\to_o$.
  - If $\bar{M} = d(\lambda\bar{x}.\bar{P})$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P} \in \Lambda_{cd}^{\beta}$ then $M' = d(\lambda\bar{x}.P')$ such that $\bar{P} \to_o P'$. By IH, $N = d(\lambda x.P'')$ such that $P' \to_o^* P''$, so $\bar{P} \to_o^* P''$.
  - If $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$, $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda_{cd}^{\beta}$ then:
    * Either $M' = (\lambda\bar{x}.P')\bar{Q}$ such that $\bar{P} \to_o P'$. By IH, $N = (\lambda x.P'')Q'$ such that $P' \to_o^* P''$ and $\bar{Q} \to_o^* Q'$, so $\bar{P} \to_o^* P''$.
    * Or $M' = (\lambda\bar{x}.\bar{P})Q'$ such that $\bar{Q} \to_o Q'$. By IH, $N = (\lambda\bar{x}.P')Q''$ such that $\bar{P} \to_o^* P'$ and $Q' \to_o^* Q''$, so $\bar{Q} \to_o^* Q''$.
  - If $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda_{cd}^{\beta}$ then:
    * Either $M' = cP'\bar{Q}$ such that $\bar{P} \to_o P'$. By IH:
      · Either $N = cP''Q' \in \Lambda_{cd}^{\beta}$ such that $P' \to_o^* P''$ and $\bar{Q} \to_o^* Q'$, so $\bar{P} \to_o^* P''$.
      · Or $P' = d(\lambda\bar{x}.\bar{R}')$ and $N = (\lambda\bar{x}.R'')Q'$ such that $\bar{R}' \to_o^* R''$ and $\bar{Q} \to_o^* Q'$. By lemma 4.1.2, $\bar{P} = d(\lambda\bar{x}.\bar{R})$ such that $\bar{R} \to_o^* \bar{R}'$. So $\bar{R} \to_o^* R''$.
    * Or $M' = c\bar{P}Q'$ such that $\bar{Q} \to_o Q'$. By IH:
      · Either $N = cP'Q''$ such that $\bar{P} \to_o^* P'$ and $Q' \to_o^* Q''$, so $\bar{Q} \to_o^* Q''$.
      · Or $\bar{P} = d(\lambda\bar{x}.\bar{R})$ and $N = (\lambda\bar{x}.R')Q''$ such that $\bar{R} \to_o^* R'$ and $Q' \to_o^* Q''$. So $\bar{Q} \to_o^* Q''$.
    * Or $\bar{P} = d(\lambda\bar{x}.\bar{R})$ and $M' = (\lambda\bar{x}.\bar{R})\bar{Q}$. By IH, $N = (\lambda\bar{x}.R')Q'$ such that $\bar{R} \to_o^* R'$ and $\bar{Q} \to_o^* Q'$.

$\square$

of lemma 4.2.

1 By induction on the structure of M.

- Let $M \in \mathsf{Var}$, so it is done since $\Psi_{cd}(M) = M$.
- Let $M = \lambda x.N$, then $\mathrm{fv}(M) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d,x\} =^{IH} \mathrm{fv}(\Psi_{cd}(N)) \setminus \{c,d,x\} = \mathrm{fv}(\lambda x.\Psi_{cd}(N)) \setminus \{c,d\} = \mathrm{fv}(d(\lambda x.\Psi_{cd}(N))) \setminus \{c,d\} = \mathrm{fv}(\Psi_{cd}(M)) \setminus \{c,d\}$ such that $x \notin \{c,d\}$.
- Let $M = PQ$.
  - If $P = \lambda x.N$ then $\mathrm{fv}(M) \setminus \{c,d\} = (\mathrm{fv}(N) \setminus \{c,d,x\}) \cup (\mathrm{fv}(Q) \setminus \{c,d\}) =^{IH} (\mathrm{fv}(\Psi_{cd}(N)) \setminus \{c,d,x\}) \cup (\mathrm{fv}(\Psi_{cd}(Q)) \setminus \{c,d\}) = (\mathrm{fv}(\Psi_{cd}(\lambda x.N)) \setminus \{c,d\}) \cup (\mathrm{fv}(\Psi_{cd}(Q)) \setminus \{c,d\}) = \mathrm{fv}((\lambda x.\Psi_{cd}(N))\Psi_{cd}(Q)) \setminus \{c,d\} = \mathrm{fv}(\Psi_{cd}(M)) \setminus \{c,d\}$ such that $x \notin \{c,d\}$.
  - Else, $\mathrm{fv}(M) \setminus \{c,d\} = (\mathrm{fv}(P) \setminus \{c,d\}) \cup (\mathrm{fv}(Q) \setminus \{c,d\}) =^{IH} (\mathrm{fv}(\Psi_{cd}(P)) \setminus \{c,d\}) \cup (\mathrm{fv}(\Psi_{cd}(Q)) \setminus \{c,d\}) = \mathrm{fv}(c\Psi_{cd}(N)\Psi_{cd}(Q)) \setminus \{c,d\} = \mathrm{fv}(\Psi_{cd}(M)) \setminus \{c,d\}$.

2 By induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$ then it is done since $|\bar{M}|_{cd} = \bar{M}$.
- Let $\bar{M} = d(\lambda\bar{x}.\bar{N})$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{N} \in \Lambda_{cd}^{\beta}$. Then, $|\bar{M}|_{cd} = \lambda\bar{x}.|\bar{N}|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(\bar{N}) \setminus \{c,d,\bar{x}\} =^{IH} \mathrm{fv}(|\bar{N}|_{cd}) \setminus \{\bar{x}\} = \mathrm{fv}(|\bar{M}|_{cd})$.

- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda^{\beta}_{cd}$. Then $|\bar{M}|_{cd} = (\lambda\bar{x}.|\bar{P}|_{cd})|\bar{Q}|_{cd}$ and $\mathrm{fv}(\bar{M})\setminus\{c,d\} = (\mathrm{fv}(\bar{P})\setminus\{c,d,\bar{x}\})\cup(\mathrm{fv}(\bar{Q})\setminus\{c,d\}) =^{IH} (\mathrm{fv}(|\bar{P}|_{cd}) \setminus \{\bar{x}\}) \cup \mathrm{fv}(|\bar{Q}|_{cd}) = \mathrm{fv}(|\bar{M}|_{cd})$.
- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda^{\beta}_{cd}$. Then $|\bar{M}|_{cd} = |\bar{P}|_{cd}|\bar{Q}|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = (\mathrm{fv}(\bar{P}) \setminus \{c,d\}) \cup (\mathrm{fv}(\bar{Q}) \setminus \{c,d\}) =^{IH} \mathrm{fv}(|\bar{P}|_{cd}) \cup \mathrm{fv}(|\bar{Q}|_{cd}) = \mathrm{fv}(|\bar{M}|_{cd})$.

3 By induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$ then nothing to prove since $|\bar{M}|_{cd} = \bar{M} \neq \lambda\bar{x}.N$.
- Let $\bar{M} = d(\lambda\bar{x}.\bar{P})$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P} \in \Lambda^{\beta}_{cd}$, so it is done since $|\bar{M}|_{cd} = \lambda\bar{x}.|\bar{P}|_{cd}$.
- Let $\bar{M} = (\lambda\bar{y}.\bar{P})\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda^{\beta}_{cd}$ and $\bar{y} \in \mathsf{Var}_{cd}$, then nothing to prove since $|\bar{M}|_{cd} = (\lambda\bar{y}.|\bar{P}|_{cd})|\bar{Q}|_{cd} \neq \lambda\bar{x}.N$.
- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda^{\beta}_{cd}$, then nothing to prove since $|\bar{M}|_{cd} = |\bar{P}|_{cd}|\bar{Q}|_{cd} \neq \lambda\bar{x}.N$.

□

of lemma 4.3. By induction on the structure of M.

- Let $M \in \mathsf{Var}$, so $\Psi_{cd}(M) = M \in \mathsf{Var}_{cd}$, since $M \notin \{c,d\}$.
- Let $M = \lambda x.N$. By IH, $\Psi_{cd}(N) \in \Lambda^{\beta}_{cd}$, so $\Psi_{cd}(M) = d(\lambda x.\Psi_{cd}(N)) \in \Lambda^{\beta}_{cd}$ such that $x \notin \{c,d\}$.
- Let $M = PQ$.
  - If $P = \lambda x.N$ then $\Psi_{cd}(M) = (\lambda x.\Psi_{cd}(N))\Psi_{cd}(Q)$ such that $x \notin \{c,d\}$. By IH, $\Psi_{cd}(N), \Psi_{cd}(Q) \in \Lambda^{\beta}_{cd}$, so $\Psi_{cd}(M) \in \Lambda^{\beta}_{cd}$.
  - Else $\Psi_{cd}(M) = c\Psi_{cd}(P)\Psi_{cd}(Q)$. By IH, $\Psi_{cd}(P), \Psi_{cd}(Q) \in \Lambda^{\beta}_{cd}$, so $\Psi_{cd}(M) \in \Lambda^{\beta}_{cd}$.

□

of lemma 4.4. We prove the result by induction on the structure of $\bar{M}$:

- Let $\bar{M} = x \in \mathsf{Var}_{cd}$ and $M \in \mathsf{CR}$ then $x[x := M] = M \in \mathsf{CR}$.
- Let $\bar{M} = d(\lambda\bar{x}.\bar{N})$. Let $\mathrm{fv}(\bar{N}) \setminus \{c,d,\bar{x}\} = \{x_1, \dots, x_n\}$ and $M_1, \dots, M_n, M \in \mathsf{CR}$.
  - If $\bar{x} \in \mathrm{fv}(\bar{N})$ then by IH, $\bar{N}[x_1 := M_1, \dots, x_n := M_n, \bar{x} := M] \in \mathsf{CR}$. By lemma 3.2.6, $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n]M \in \mathsf{CR}$, such that $\bar{x} \notin \mathrm{fv}(M_1)\cup\dots\cup\mathrm{fv}(M_n)$. By IH and because by lemma 3.2.7 $\bar{x} \in \mathsf{CR}$, we obtain $\bar{N}[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}$. By lemma 3.2.8, $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}$. So $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}_{\rightarrow}$.
  - Else, by IH, $\bar{N}[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}$. By lemma 3.2.6, $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n]M \in \mathsf{CR}$, such that $\bar{x} \notin \mathrm{fv}(M_1) \cup \dots \cup \mathrm{fv}(M_n)$. By lemma 3.2.8, $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}$. So $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}_{\rightarrow}$

  So, by lemma 3.2.7 $(d(\lambda\bar{x}.\bar{N}))[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}$.

- Let $\bar{M} = c\bar{P}\bar{Q}$. Let $\mathrm{fv}(\bar{P}) \setminus \{c,d\} = \{x_1,\ldots,x_n,x'_1,\ldots,x'_{n_1}\}$, $\mathrm{fv}(\bar{Q}) \setminus \{c,d\} = \{x_1,\ldots,x_n,x''_1,\ldots,x''_{n_2}\}$ and $M_1,\ldots,M_n,M'_1,\ldots,M'_{n_1},M''_1,\ldots,M''_{n_2} \in \mathsf{CR}$ and $\{x'_1,\ldots,x'_{n_1}\} \cap \{x''_1,\ldots,x''_{n_2}\} = \varnothing$. By IH, $\bar{P}[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}]$, $\bar{Q}[x_1 := M_1,\ldots,x_n := M_n, x''_1 := M''_1,\ldots,x''_{n_2} := M''_{n_2}] \in \mathsf{CR}$. So, by lemma 3.2.7, $(c\bar{P}\bar{Q})[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}, x''_1 := M''_1,\ldots,x''_{n_2} := M''_{n_2}] \in \mathsf{CR}$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$. Let $\mathrm{fv}(\bar{P}) \setminus \{c,d,\bar{x}\} = \{x_1,\ldots,x_n,x'_1,\ldots,x'_{n_1}\}$, $\mathrm{fv}(\bar{Q}) \setminus \{c,d\} = \{x_1,\ldots,x_n,x''_1,\ldots,x''_{n_2}\}$ and $M_1,\ldots,M_n,M'_1,\ldots,M'_{n_1},M''_1,\ldots,M''_{n_2} \in \mathsf{CR}$ and $\{x'_1,\ldots,x'_{n_1}\} \cap \{x''_1,\ldots,x''_{n_2}\} = \varnothing$. By IH, $\bar{Q}[x_1 := M_1,\ldots,x_n := M_n, x''_1 := M''_1,\ldots,x''_{n_2} := M''_{n_2}] \in \mathsf{CR}$.
  - If $\bar{x} \in \mathrm{fv}(\bar{P})$ then by IH, $\bar{P}[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}, \bar{x} := M] \in \mathsf{CR}$. By lemma 3.2.6, $(\lambda\bar{x}.\bar{P})[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}]M \in \mathsf{CR}$, such that $\bar{x} \notin \mathrm{fv}(M_1) \cup \cdots \cup \mathrm{fv}(M_n) \cup \mathrm{fv}(M'_1) \cup \cdots \cup \mathrm{fv}(M'_{n_1})$. By IH and because by lemma 3.2.7 $\bar{x} \in \mathsf{CR}$, we obtain $\bar{P}[x_1 := M_1,\ldots,x_n := M_n] \in \mathsf{CR}$. By lemma 3.2.8, $(\lambda\bar{x}.\bar{P})[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}] \in \mathsf{CR}$. So $(\lambda\bar{x}.\bar{P})[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}] \in \mathsf{CR}_{\to}$.
  - Else, by IH, $\bar{P}[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}] \in \mathsf{CR}$. By lemma 3.2.6, $(\lambda\bar{x}.\bar{P})[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}]M \in \mathsf{CR}$, such that $\bar{x} \notin \mathrm{fv}(M_1) \cup \cdots \cup \mathrm{fv}(M_n) \cup \mathrm{fv}(M'_1) \cup \cdots \cup \mathrm{fv}(M'_{n_1})$. By lemma 3.2.8, $(\lambda\bar{x}.\bar{P})[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}] \in \mathsf{CR}$. So $(\lambda\bar{x}.\bar{P})[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}] \in \mathsf{CR}_{\to}$

  So, by definition, $((\lambda\bar{x}.\bar{P})\bar{Q})[x_1 := M_1,\ldots,x_n := M_n, x'_1 := M'_1,\ldots,x'_{n_1} := M'_{n_1}, x''_1 := M''_1,\ldots,x''_{n_2} := M''_{n_2}] \in \mathsf{CR}$. □

of lemma 4.6.

1 By induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$. Either $\bar{M} = \bar{x}$, then $\bar{M}[\bar{x} := \bar{N}] = \bar{N} \in \Lambda^{\beta}_{cd}$. Or, $\bar{M} \neq \bar{x}$ and so $\bar{M}[\bar{x} := \bar{N}] = \bar{M} \in \Lambda^{\beta}_{cd}$.
- Let $\bar{M} = d(\lambda\bar{y}.\bar{P})$ such that $\bar{y} \in \mathsf{Var}_{cd}$ and $\bar{P} \in \Lambda^{\beta}_{cd}$. By IH, $\bar{P}[\bar{x} := \bar{N}] \in \Lambda^{\beta}_{cd}$. Then, $\bar{M}[\bar{x} := \bar{N}] = d(\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}]) \in \Lambda^{\beta}_{cd}$ such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.
- Let $\bar{M} = (\lambda\bar{y}.\bar{P})\bar{Q}$ such that $\bar{y} \in \mathsf{Var}_{cd}$ and $\bar{P},\bar{Q} \in \Lambda^{\beta}_{cd}$. By IH, $\bar{P}[\bar{x} := \bar{N}], \bar{Q}[\bar{x} := \bar{N}] \in \Lambda^{\beta}_{cd}$. Then, $\bar{M}[\bar{x} := \bar{N}] = (\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}])\bar{Q}[\bar{x} := \bar{N}] \in \Lambda^{\beta}_{cd}$ such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.
- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P},\bar{Q} \in \Lambda^{\beta}_{cd}$. By IH, $\bar{P}[\bar{x} := \bar{N}], \bar{Q}[\bar{x} := \bar{N}] \in \Lambda^{\beta}_{cd}$. Then, $\bar{M}[\bar{x} := \bar{N}] = c\bar{P}[\bar{x} := \bar{N}]\bar{Q}[\bar{x} := \bar{N}] \in \Lambda^{\beta}_{cd}$.

2 By induction on the structure of M.

- Let $M \in \mathsf{Var}$, then $\Psi_{cd}(M) = M$ and $|M|_{cd} = M$.
- Let $M = \lambda x.N$, then $|\Psi_{cd}(M)|_{cd} = |d(\lambda x.\Psi_{cd}(N))|_{cd} = |\lambda x.\Psi_{cd}(N)|_{cd} = \lambda x.|\Psi_{cd}(N)|_{cd} =^{IH} \lambda x.N$ such that $x \notin \{c,d\}$.
- Let $M = PQ$.
  - If, $P = \lambda x.N$, then $|\Psi_{cd}(M)|_{cd} = |(\lambda x.\Psi_{cd}(N))\Psi_{cd}(Q)|_{cd} = |\lambda x.\Psi_{cd}(N)|_{cd}|\Psi_{cd}(Q)|_{cd} = (\lambda x.|\Psi_{cd}(N)|_{cd})|\Psi_{cd}(Q)|_{cd} =^{IH} M$ such that $x \notin \{c,d\}$.

– Else, $|\Psi_{cd}(M)|_{cd} = |c\Psi_{cd}(P)\Psi_{cd}(Q)|_{cd} = |\Psi_{cd}(P)|_{cd}|\Psi_{cd}(Q)|_{cd} =^{IH} M$

3 We prove the lemma by induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$ then $|\bar{M}|_{cd} = \bar{M}$. If $\bar{M} = x$ then $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] = |\bar{N}|_{cd} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$. Else, $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] = \bar{M} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$.
- Let $\bar{M} = d(\lambda\bar{y}.\bar{P})$, such that $\bar{P} \in \Lambda^{\beta}_{cd}$ and $\bar{y} \in \mathsf{Var}_{cd}$, then $|\bar{M}|_{cd} = \lambda\bar{y}.|\bar{P}|_{cd}$. So, $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{4.2.2} \lambda\bar{y}.|\bar{P}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{IH} \lambda\bar{y}.|\bar{P}[\bar{x} := \bar{N}]|_{cd} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$ such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.
- Let $M = (\lambda\bar{y}.\bar{P})\bar{Q}$, such that $\bar{y} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda^{\beta}_{cd}$, then $|\bar{M}|_{cd} = (\lambda\bar{y}.|\bar{P}|_{cd})|\bar{Q}|_{cd}$. So, $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{4.2.2} (\lambda\bar{y}.|\bar{P}|_{cd}[\bar{x} := |\bar{N}|_{cd}])|\bar{Q}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{IH} (\lambda\bar{y}.|\bar{P}[\bar{x} := \bar{N}]|_{cd})|\bar{Q}[\bar{x} := \bar{N}]|_{cd} = |(\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}])\bar{Q}[\bar{x} := \bar{N}]|_{cd} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$ such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.
- Let $\bar{M} = c\bar{P}\bar{Q}$, such that $\bar{P}, \bar{Q} \in \Lambda^{\beta}_{cd}$, then $|\bar{M}|_{cd} = |\bar{P}|_{cd}|\bar{Q}|_{cd}$. So, $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] = |\bar{P}|_{cd}[\bar{x} := |\bar{N}|_{cd}]|\bar{Q}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{IH} |\bar{P}[\bar{x} := \bar{N}]|_{cd}|\bar{Q}[\bar{x} := \bar{N}]|_{cd} = |c\bar{P}[\bar{x} := \bar{N}]\bar{Q}[\bar{x} := \bar{N}]|_{cd} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$.

4 We prove the lemma by induction on the length of the derivation $\bar{M} \to^*_\beta N$.

- let $\bar{M} = N$ then it is done.
- Let $\bar{M} \to^*_\beta M' \to_\beta N$. By IH, $M' \in \Lambda^{\beta}_{cd}$ and $|\bar{M}|_{cd} \to^*_\beta |M'|_{cd}$. We prove that $N \in \Lambda^{\beta}_{cd}$ and $|M'|_{cd} \to_\beta |N|_{cd}$ by induction on the structure of M'.
  - Let $M' \in \mathsf{Var}_{cd}$ then nothing to prove since M' does not reduce.
  - Let $M' = d(\lambda x.P)$ such that $x \in \mathsf{Var}_{cd}$ and $P \in \Lambda^{\beta}_{cd}$, so by compatibility $N = d(\lambda x.P')$ such that $P \to_\beta P'$. By IH, $P' \in \Lambda^{\beta}_{cd}$ and $|P|_{cd} \to_\beta |P'|_{cd}$. So, $N \in \Lambda^{\beta}_{cd}$ and $|M'|_{cd} = \lambda x.|P|_{cd} \to_\beta \lambda x.|P'|_{cd} = |N|_{cd}$.
  - Let $M' = (\lambda x.P)Q$ such that $x \in \mathsf{Var}_{cd}$ and $P, Q \in \Lambda^{\beta}_{cd}$. By compatibility:
    * Either $N = (\lambda x.P')Q$ such that $P \to_\beta P'$. By IH, $P' \in \Lambda^{\beta}_{cd}$ and $|P|_{cd} \to_\beta |P'|_{cd}$. So, $N \in \Lambda^{\beta}_{cd}$ and $|M'|_{cd} = (\lambda x.|P|_{cd})|Q|_{cd} \to_\beta (\lambda x.|P'|_{cd})|Q|_{cd} = |N|_{cd}$.
    * Or $N = (\lambda x.P)Q'$ such that $Q \to_\beta Q'$. By IH, $Q' \in \Lambda^{\beta}_{cd}$ and $|Q|_{cd} \to_\beta |Q'|_{cd}$. So, $N \in \Lambda^{\beta}_{cd}$ and $|M'|_{cd} = (\lambda x.|P|_{cd})|Q|_{cd} \to_\beta (\lambda x.|P|_{cd})|Q'|_{cd} = |N|_{cd}$.
    * Or $N = P[x := Q]$. So, by lemma 4.6.1, $N \in \Lambda^{\beta}_{cd}$ and $|M'|_{cd} = (\lambda x.|P|_{cd})|Q|_{cd} \to_\beta |P|_{cd}[x := |Q|_{cd}] =^{4.6.3} |P[x := Q]|_{cd}$.
  - Let $M' = cPQ$ such that $P, Q \in \Lambda^{\beta}_{cd}$. By compatibility:
    * Either $N = cP'Q$ such that $P \to_\beta P'$. By IH, $P' \in \Lambda^{\beta}_{cd}$ and $|P|_{cd} \to_\beta |P'|_{cd}$. So, $N \in \Lambda^{\beta}_{cd}$ and $|M'|_{cd} = |P|_{cd}|Q|_{cd} \to_\beta |P'|_{cd}|Q|_{cd} = |N|_{cd}$.
    * Or $N = cPQ'$ such that $Q \to_\beta Q'$. By IH, $Q' \in \Lambda^{\beta}_{cd}$ and $|Q|_{cd} \to_\beta |Q'|_{cd}$. So, $N \in \Lambda^{\beta}_{cd}$ and $|M'|_{cd} = |P|_{cd}|Q|_{cd} \to_\beta |P|_{cd}|Q'|_{cd} = |N|_{cd}$.

5 By lemma 4.3, $\Psi_{cd}(M) \in \Lambda^{\beta}_{cd}$. By lemma 4.6.4, $|\Psi_{cd}(M)|_{cd} \to^*_\beta |N|_{cd}$. By lemma 4.6.2, $M \to^*_\beta |N|_{cd}$. □

of lemma 4.7.

⇒) Let $M \to_\beta^* N$. We prove that $M \to_1^* N$ by induction on the size of the reduction $M \to_\beta^* N$.

- If $M = N$, then it is done since $M \to_1^* N$.
- If $M \to_\beta^* M' \to_\beta N$. By IH, $M \to_1^* M'$. We prove that $M' \to_1 N$ by induction on the structure of M'. By lemma 2, $c, d \notin \text{fv}(M')$.
  - Let $M' \in \mathsf{Var}$. Noting to prove since M' does not reduce.
  - Let $M' = \lambda x.P$, then by compatibility $N = \lambda x.P'$ and $P \to_\beta P'$. By IH, $P \to_1 P'$. So $\Psi_{cd}(P) \to_\beta^* Q$ and $|Q|_{cd} = P'$. So $\Psi_{cd}(\lambda x.P) = d(\lambda x.\Psi_{cd}(P)) \to_\beta^* d(\lambda x.Q)$ and $|d(\lambda x.Q)|_{cd} = \lambda x.|Q|_{cd} = \lambda x.P'$, such that $x \notin \{c, d\}$. Hence, $\lambda x.P \to_1 \lambda x.P'$.
  - Let $M' = PQ$.
    * If $P = \lambda x.P_1$ such that $x \notin \{c, d\}$ then by compatibility:
      · Either $N = (\lambda x.P_2)Q$ such that $P_1 \to_\beta P_2$. By IH, $P_1 \to_1 P_2$. So, $\Psi_{cd}(P_1) \to_\beta^* P_1'$ and $|P_1'|_{cd} = P_2$. So, $\Psi_{cd}(M') = (\lambda x.\Psi_{cd}(P_1))\Psi_{cd}(Q) \to_\beta^* (\lambda x.P_1')\Psi_{cd}(Q)$ and $|(\lambda x.P_1')\Psi_{cd}(Q)|_{cd} =^{4.6.2} (\lambda x.|P_1'|_{cd})Q = (\lambda x.P_2)Q = N$. Hence, $M' \to_1 N$.
      · Or $N = (\lambda x.P_1)Q_1$ such that $Q \to_\beta Q_1$. By IH, $Q \to_1 Q_1$. So, $\Psi_{cd}(Q) \to_\beta^* Q_2$ and $|Q_2|_{cd} = Q_1$. So, $\Psi_{cd}(M') = (\lambda x.\Psi_{cd}(P_1))\Psi_{cd}(Q) \to_\beta^* (\lambda x.\Psi_{cd}(P_1))Q_2$ and $|(\lambda x.\Psi_{cd}(P_1))Q_2|_{cd} =^{4.6.2} (\lambda x.P_1)|Q_2|_{cd} = (\lambda x.P_1)Q_1 = N$. Hence, $M' \to_1 N$.
      · Or $N = P_1[x := Q]$. So, $\Psi_{cd}(M') = (\lambda x.\Psi_{cd}(P_1))\Psi_{cd}(Q) \to_\beta \Psi_{cd}(P_1)[x := \Psi_{cd}(Q)]$ and $|\Psi_{cd}(P_1)[x := \Psi_{cd}(Q)]|_{cd} =^{4.6.3,4.3} |\Psi_{cd}(P_1)|_{cd}[x := |\Psi_{cd}(Q)|_{cd}] =^{4.6.2} P_1[x := Q]$. Hence, $M' \to_1 N$.
    * Else, by compatibility:
      · Either $N = P'Q$ such that $P \to_\beta P'$. By IH, $P \to_1 P'$. So, $\Psi_{cd}(P) \to_\beta^* P_1$ and $|P_1|_{cd} = P'$. So, $\Psi_{cd}(M') = c\Psi_{cd}(P)\Psi_{cd}(Q) \to_\beta^* cP_1\Psi_{cd}(Q)$ and $|cP_1\Psi_{cd}(Q)|_{cd} =^{4.6.2} |P_1|_{cd}Q = P'Q = N$. So $M' \to_1 N$.
      · Or $N = PQ'$ such that $Q \to_\beta Q'$. By IH, $Q \to_1 Q'$. So, $\Psi_{cd}(Q) \to_\beta^* Q_1$ and $|Q_1|_{cd} = Q'$. So, $\Psi_{cd}(M') = c\Psi_{cd}(P)\Psi_{cd}(Q) \to_\beta^* c\Psi_{cd}(P)Q_1$ and $|c\Psi_{cd}(P)Q_1|_{cd} =^{4.6.2} P|Q_1|_{cd} = PQ' = N$. So $M' \to_1 N$.

⇐) Let $M \to_1^* N$. We prove that $M \to_\beta^* N$ by induction on the size of the derivation $M \to_1^* N$.

- Let $M = N$, then it is done since $M \to_\beta^* N$.
- Let $M \to_1^* M' \to_1 N$. By IH, $M \to_\beta^* M'$. By lemma 2, $c, d \notin \text{fv}(M')$. Since $M' \to_1 N$, $\exists P \in \Lambda$ such that $\Psi_{cd}(M') \to_\beta^* P$ and $|P|_{cd} = N$. By lemma 4.6.5 $M' \to_\beta^* N$.

□

of lemma 4.8.

1 By induction on the structure of M.

- Let $\bar{M} \in \mathsf{Var}_{cd}$ then it is done since $\Psi_{cd}(|\bar{M}|_{cd}) = \Psi_{cd}(\bar{M}) = \bar{M}$.

- Let $\bar{M} = d(\lambda\bar{x}.\bar{P})$, such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P} \in \Lambda^{\beta}_{cd}$, then $|\bar{M}|_{cd} = \lambda\bar{x}.|\bar{P}|_{cd}$ and $\Psi_{cd}(|\bar{M}|_{cd}) = d(\lambda\bar{x}.\Psi_{cd}(|\bar{P}|_{cd}))$. By IH, $\bar{P} \to^*_o \Psi_{cd}(|\bar{P}|_{cd})$, so by compatibility $\bar{M} \to^*_o \Psi_{cd}(|\bar{M}|_{cd})$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$, such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda^{\beta}_{cd}$, then $|\bar{M}|_{cd} = (\lambda\bar{x}.|\bar{P}|_{cd})|\bar{Q}|_{cd}$ and $\Psi_{cd}(|\bar{M}|_{cd}) = (\lambda\bar{x}.\Psi_{cd}(|\bar{P}|_{cd}))\Psi_{cd}(|\bar{Q}|_{cd})$. By IH, $\bar{P} \to^*_o \Psi_{cd}(|\bar{P}|_{cd})$ and $\bar{Q} \to^*_o \Psi_{cd}(|\bar{Q}|_{cd})$, so by compatibility $\bar{M} \to^*_o \Psi_{cd}(|\bar{M}|_{cd})$.
- Let $\bar{M} = c\bar{P}\bar{Q}$, such that $\bar{P}, \bar{Q} \in \Lambda^{\beta}_{cd}$, then $|\bar{M}|_{cd} = |\bar{P}|_{cd}|\bar{Q}|_{cd}$.
  - If $|\bar{P}|_{cd} = \lambda\bar{x}.N$, then by lemma 4.2.3, $\bar{P} = d(\lambda\bar{x}.\bar{P}')$, such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}' \in \Lambda^{\beta}_{cd}$, and $|\bar{P}'|_{cd} = N$. So, $\Psi_{cd}(|\bar{M}|_{cd}) = (\lambda\bar{x}.\Psi_{cd}(|\bar{P}'|_{cd}))\Psi_{cd}(|\bar{Q}|_{cd})$. By IH, $\bar{P}' \to^*_o \Psi_{cd}(|\bar{P}'|_{cd})$ and $\bar{Q} \to^*_o \Psi_{cd}(|\bar{Q}|_{cd})$, so by compatibility $\bar{M} \to_o (\lambda\bar{x}.\bar{P}')\bar{Q} \to^*_o \Psi_{cd}(|\bar{M}|_{cd})$.
  - Else, $\Psi_{cd}(|\bar{M}|_{cd}) = c\Psi_{cd}(|\bar{P}|_{cd})\Psi_{cd}(|\bar{Q}|_{cd})$. By IH, $\bar{P} \to^*_o \Psi_{cd}(|\bar{P}|_{cd})$ and $\bar{Q} \to^*_o \Psi_{cd}(|\bar{Q}|_{cd})$, so by compatibility $\bar{M} \to^*_o \Psi_{cd}(|\bar{M}|_{cd})$.

2 By induction on the structure of M.

- let $\bar{M} \in \mathsf{Var}_{cd}$ then $\bar{M}$ does not reduce. If $\bar{M} = \bar{x}$ then $\bar{M}[\bar{x} := \bar{N}] = \bar{N} \to^*_o N' = \bar{M}[\bar{x} := N']$. Else, $\bar{M}[\bar{x} := \bar{N}] = \bar{M} = \bar{M}[\bar{x} := N']$.
- Let $\bar{M} = d(\lambda\bar{y}.\bar{P})$ such that $\bar{P} \in \Lambda^{\beta}_{cd}$ and $\bar{y} \in \mathsf{Var}_{cd}$. By lemma 4.1.3, $M' = d(\lambda\bar{y}.P')$ such that $\bar{P} \to^*_o P'$. By IH, $\bar{M}[\bar{x} := \bar{N}] = d(\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}]) \to^*_o d(\lambda\bar{y}.P'[\bar{x} := N']) = M'[\bar{x} := N']$ such that $\bar{y} \notin \mathrm{fv}(N) \cup \{\bar{x}\}$ (by lemma 4.1.3, $y \notin \mathrm{fv}(N')$).
- Let $M = (\lambda\bar{y}.\bar{P})\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda^{\beta}_{cd}$ and $\bar{y} \in \mathsf{Var}_{cd}$. By lemma 4.1.3 $M' = (\lambda\bar{y}.P')Q'$ such that $\bar{P} \to^*_o P'$ and $\bar{Q} \to^*_o Q'$. By IH, $\bar{M}[\bar{x} := \bar{N}] = (\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}])\bar{Q}[\bar{x} := \bar{N}] \to^*_o (\lambda\bar{y}.P'[\bar{x} := N'])Q'[\bar{x} := N'] = M'[\bar{x} := N']$ such that $\bar{y} \notin \mathrm{fv}(N) \cup \{\bar{x}\}$ (by lemma 4.1.3, $y \notin \mathrm{fv}(N')$).
- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda^{\beta}_{cd}$. by lemma 4.1.3:
  - Either $M' = cP'Q'$ such that $\bar{P} \to^*_o P'$ and $\bar{Q} \to^*_o Q'$. So by IH, $\bar{M}[\bar{x} := \bar{N}] = c\bar{P}[\bar{x} := \bar{N}]\bar{Q}[\bar{x} := \bar{N}] \to^*_o cP'[\bar{x} := N']Q'[\bar{x} := N'] = M'[\bar{x} := N']$.
  - Or $\bar{P} = d(\lambda\bar{y}.\bar{R})$ and $M' = (\lambda\bar{y}.R')Q'$ such that $\bar{R} \to^*_o R'$ and $\bar{Q} \to^*_o Q'$. Since $\bar{P} \in \Lambda^{\beta}_{cd}$, $\bar{y} \in \mathsf{Var}_{cd}$ and $\bar{R} \in \Lambda^{\beta}_{cd}$. By IH, $\bar{M}[\bar{x} := \bar{N}] = c(d(\lambda\bar{y}.\bar{R}[\bar{x} := \bar{N}]))\bar{Q}[\bar{x} := \bar{N}] \to_o (\lambda\bar{y}.\bar{R}[\bar{x} := \bar{N}])\bar{Q}[\bar{x} := \bar{N}] \to^*_o (\lambda\bar{y}.R'[\bar{x} := N'])Q'[\bar{x} := N'] = M'[x: = N']$, such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$ (by lemma 4.1.3, $y \notin \mathrm{fv}(N')$).

3 By induction on the structure of $\bar{M}_1$.

- Let $\bar{M}_1 \in \mathsf{Var}_{cd}$ then nothing to prove since $\bar{M}_1$ does not reduce.
- Let $\bar{M}_1 = d(\lambda\bar{x}.\bar{P}_1)$ such that $\bar{P}_1 \in \Lambda^{\beta}_{cd}$ and $\bar{x} \in \mathsf{Var}_{cd}$, then by lemma 4.1.3, $M_2 = d(\lambda\bar{x}.P_2)$ such that $\bar{P}_1 \to^*_o P_2$ and by compatibility $N_1 = d(\lambda\bar{x}.Q_1)$ such that $\bar{P}_1 \to_\beta Q_1$. By IH, there exists Q_2 such that $P_2 \to_\beta Q_2$ and $Q_1 \to^*_o Q_2$. So it is done with $N_2 = d(\lambda\bar{x}.Q_2)$.
- let $\bar{M}_1 = (\lambda\bar{x}.\bar{P}_1)\bar{Q}_1$ such that $\bar{P}_1, \bar{Q}_1 \in \Lambda^{\beta}_{cd}$ and $\bar{x} \in \mathsf{Var}_{cd}$ then by lemma 4.1.3, $M_2 = (\lambda\bar{x}.P_2)Q_2$ such that $\bar{P}_1 \to^*_o P_2$ and $\bar{Q}_1 \to^*_o Q_2$. By compatibility:
  - Either $N_1 = (\lambda\bar{x}.P_1')\bar{Q}_1$ such that $\bar{P}_1 \to_\beta P_1'$. By IH, there exist P_2' such that $P_2 \to_\beta P_2'$ and $P_1' \to^*_o P_2'$. So it is done with $N_2 = (\lambda\bar{x}.P_2')Q_2$.

- Or $N_1 = (\lambda\bar{x}.\bar{P}_1)Q'_1$ such that $\bar{Q}_1 \to_\beta Q'_1$. By IH, there exists Q'_2 such that $Q_2 \to_\beta Q'_2$ and $Q'_1 \to_o^* P'_2$. So it is done with $N_2 = (\lambda\bar{x}.P_2)Q'_2$.
- Or $N_1 = \bar{P}_1[\bar{x} := \bar{Q}_1]$. By lemma 4.8.2, it is done with $N_2 = P_2[\bar{x} := Q_2]$.

- Let $\bar{M}_1 = c\bar{P}_1\bar{Q}_1$ such that $\bar{P}_1, \bar{Q}_1 \in \Lambda^\beta_{cd}$. By lemma 4.1.3:
  - Either $M_2 = cP_2Q_2$ such that $\bar{P}_1 \to_o^* P_2$ and $\bar{Q}_1 \to_o^* Q_2$. By compatibility:
    * Either $N_1 = cP'_1\bar{Q}_1$ such that $\bar{P}_1 \to_\beta P'_1$. By IH, there exists P'_2 such that $P_2 \to_\beta P'_2$ and $P'_1 \to_o^* P'_2$. So it is done with $N_2 = cP'_2Q_2$.
    * Or $N_1 = c\bar{P}_1Q'_1$ such that $\bar{Q}_1 \to_\beta Q'_1$. By IH, there exists Q'_2 such that $Q_2 \to_\beta Q'_2$ and $Q'_1 \to_o^* Q'_2$. So it is done with $N_2 = cP_2Q'_2$.
  - Or $\bar{P}_1 = d(\lambda\bar{x}.\bar{R}_1)$ and $M_2 = (\lambda x.R_2)Q_2$ such that $\bar{R}_1 \to_o^* R_2$ and $Q_1 \to_o^* Q_2$. By compatibility:
    * Either $N_1 = c(d(\lambda\bar{x}.R'_1))\bar{Q}_1$ such that $\bar{R}_1 \to_\beta R'_1$. By IH, there exists R'_2 such that $R_2 \to_\beta R'_2$ and $R'_1 \to_o^* R'_2$. So it is done with $N_2 = (\lambda\bar{x}.R'_2)Q_2$.
    * Or $N_1 = c(d(\lambda\bar{x}.\bar{R}_1))Q'_1$ such that $\bar{Q}_1 \to_\beta Q'_1$. By IH, there exists Q'_2 such that $Q_2 \to_\beta Q'_2$ and $Q'_1 \to_o^* Q'_2$. So it is done with $N_2 = (\lambda\bar{x}.R_2)Q'_2$.

4 By induction on the length of the reduction $M_1 \to_\beta^* N_1$ using lemma 4.8.3.

5 By lemma 4.8.1, $M \to_o^* \Psi_{cd}(|M|_{cd})$. By lemma 4.8.4 and lemma 1.3, there exists $Q \in \Lambda^\beta_{cd}$ such that $\Psi_{cd}(|M|_{cd}) \to_\beta^* Q$ and $N \to_o^* Q$. By lemma 4.6.4, $N \in \Lambda^\beta_{cd}$ and by lemma 4.1.3, $|Q|_{cd} = |N|_{cd}$.

6 By definition, there exist P_1, P_2 such that $\Psi_{cd}(M) \to_\beta^* P_1$, $\Psi_{cd}(M) \to_\beta^* P_2$, $|P_1|_{cd} = M_1$ and $|P_2|_{cd} = M_2$. By lemma 4.3, $\Psi_{cd}(M) \in \Lambda^\beta_{cd}$. So by corollary 4.5, there exists P_3 such that $P_1 \to_\beta^* P_3$ and $P_2 \to_\beta^* P_3$. Let $M_3 = |P_3|_{cd}$. By lemma 4.6.4, $P_1, P_2 \in \Lambda^\beta_{cd}$. By lemma 4.8.5, there exist $Q_1, Q_2 \in \Lambda^\beta_{cd}$ such that $\Psi_{cd}(M_1) \to_\beta^* Q_1$, $\Psi_{cd}(M_2) \to_\beta^* Q_2$ and $|Q_1|_{cd} = M_3 = |Q_2|_{cd}$. By lemma 4.2.2, $c, d \notin \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2)$. So $M_1 \to_1 M_3$ and $M_2 \to_1 M_3$.

7 By lemma 4.8.6 □

of lemma 5.1.

1 By induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$. Done since $\bar{M}$ does not reduce.
- Let $\bar{M} = d(\lambda\bar{x}.\bar{P})$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P} \in \Lambda^{\beta\eta}_{cd}$, then by compatibility $\bar{M} = d(\lambda\bar{x}.\bar{P}) \to_o d\lambda\bar{x}.P' = N$ such that $\bar{P} \to_o P'$. By IH, $P' \in \Lambda^{\beta\eta}_{cd}$, $|\bar{P}|_{cd} = |P'|_{cd}$ and $\mathrm{fv}(\bar{P}) \setminus \{c,d\} = \mathrm{fv}(P') \setminus \{c,d\}$. So $N \in \Lambda^{\beta\eta}_{cd}$, $|\bar{M}|_{cd} = \lambda\bar{x}.|\bar{P}|_{cd} = \lambda\bar{x}.|P'|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(\bar{P}) \setminus \{c,d,\bar{x}\} = \mathrm{fv}(P') \setminus \{c,d,\bar{x}\} = \mathrm{fv}(N)$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda^{\beta\eta}_{cd}$, then by compatibility:
  - Either $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q} \to_o (\lambda\bar{x}.P')\bar{Q} = N$ such that $\bar{P} \to_o P'$. By IH, $P' \in \Lambda^{\beta\eta}_{cd}$, $|\bar{P}|_{cd} = |P'|_{cd}$ and $\mathrm{fv}(\bar{P}) \setminus \{c,d\} = \mathrm{fv}(P') \setminus \{c,d\}$. So, $N \in \Lambda^{\beta\eta}_{cd}$, $|\bar{M}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.

- Or $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q} \to_o (\lambda\bar{x}.\bar{P})Q'$ such that $\bar{Q} \to_o Q'$. By IH, $Q' \in \Lambda^{\beta\eta}_{cd}$, $|\bar{Q}|_{cd} = |Q'|_{cd}$ and $\mathrm{fv}(\bar{Q}) \setminus \{c,d\} = \mathrm{fv}(Q') \setminus \{c,d\}$. So, $N \in \Lambda^{\beta\eta}_{cd}$, $|\bar{M}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.

- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $P, \bar{Q} \in \Lambda^{\beta\eta}_{cd}$, then by compatibility:
  - Either $\bar{M} = c\bar{P}\bar{Q} \to_o cP'\bar{Q} = N$ such that $\bar{P} \to_o P'$. By IH, $P' \in \Lambda^{\beta\eta}_{cd}$, $|\bar{P}|_{cd} = |P'|_{cd}$ and $\mathrm{fv}(\bar{P}) \setminus \{c,d\} = \mathrm{fv}(P') \setminus \{c,d\}$. So, $N \in \Lambda^{\beta\eta}_{cd}$, $|\bar{M}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.
  - Or $\bar{M} = c\bar{P}\bar{Q} \to R\bar{Q} = N$ such that $\bar{P} = dR$. Since $\bar{P} \in \Lambda^{\beta\eta}_{cd}$:
    * Either $R = (\lambda\bar{x}.\bar{R}')$. So $R\bar{Q} = (\lambda\bar{x}.\bar{R}')\bar{Q} \in \Lambda^{\beta\eta}_{cd}$, $|\bar{M}|_{cd} = |R|_{cd}|\bar{Q}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = (\mathrm{fv}(R) \cup \mathrm{fv}(\bar{Q})) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.
    * Or $R = c\bar{R}'$. So $R\bar{Q} = c\bar{R}'\bar{Q} \in \Lambda^{\beta\eta}_{cd}$, $\bar{P} \to_o R'$, $|\bar{M}|_{cd} = |R|_{cd}|\bar{Q}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = (\mathrm{fv}(R) \cup \mathrm{fv}(\bar{Q})) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.
  - Or $\bar{M} = c\bar{P}\bar{Q} \to_o c\bar{P}Q'$ such that $\bar{Q} \to_o Q'$. By IH, $Q' \in \Lambda^{\beta\eta}_{cd}$, $|\bar{Q}|_{cd} = |Q'|_{cd}$ and $\mathrm{fv}(\bar{Q}) \setminus \{c,d\} = \mathrm{fv}(Q') \setminus \{c,d\}$. So, $N \in \Lambda^{\beta\eta}_{cd}$, $|\bar{M}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.

- Let $\bar{M} = d(c\bar{P})$ such that $\bar{P} \in \Lambda^{\beta\eta}_{cd}$, then by compatibility:
  - Either $\bar{M} = d(c\bar{P}) \to_o \bar{P} = N$. Then $N \in \Lambda^{\beta\eta}_{cd}$, $|\bar{M}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.
  - Or $\bar{M} = d(c\bar{P}) \to_o dR = N$ such that $\bar{P} = dR = N$. Then $N \in \Lambda^{\beta\eta}_{cd}$, $|\bar{M}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.
  - Or $\bar{M} = d(c\bar{P}) \to_o d(cP') = N$ such that $\bar{P} \to_o P'$. By IH, $P' \in \Lambda^{\beta\eta}_{cd}$, $|\bar{P}|_{cd} = |P'|_{cd}$ and $\mathrm{fv}(\bar{P}) \setminus \{c,d\} = \mathrm{fv}(P') \setminus \{c,d\}$. So, $N \in \Lambda^{\beta\eta}_{cd}$, $|\bar{M}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$.

2 By induction on the length of the reduction $\bar{M} \to_o^* (d \circ c)^n(d(\lambda x.Q))$.

- If $\bar{M} = (d \circ c)^n(d(\lambda x.Q))$ then it is done.
- Let $\bar{M} \to_o M' \to_o^* (d \circ c)^n(d(\lambda x.Q))$. By IH, $M' = (d \circ c)^k(d(\lambda x.R))$ such that $k \geq n$ and $R \to_o^* Q$. We prove the statement by induction on k.
  - If $k = 0$ then $M' = d(\lambda x.R)$. So by case on M and lemma 5.1.1, either $M = d(\lambda x.P)$ such that $P \to_o R$ or $M = d(c(d(\lambda x.R)))$. In both cases $P \to_o^* Q$.
  - If $k = j + 1$ such that $j \geq 0$ then $M' = d(c((d \circ c)^j(d(\lambda x.R))))$. So by case on M and lemma 5.1.1, either $M = (d \circ c)^{k+1}(d(\lambda x.R))$ or $M = d(c(M_0))$ and $M_0 \to_o (d \circ c)^j(d(\lambda x.R))$. By IH, $M_0 = (d \circ c)^m(d(\lambda x.P))$, such that $m \geq j$ and $P \to_o^* R$. So $M = (d \circ c)^{m+1}(d(\lambda x.P))$ and $m + 1 \geq k$.

3 By induction on the length of the reduction $\bar{M} \to_o^* N$.

- Let $\bar{M} = N$ then it is done.
- Let $\bar{M} \to_o M' \to_o^* N$. By lemma 4.1.1, $M' \in \Lambda^{\beta\eta}_{cd}$, $|\bar{M}|_{cd} = |M'|_{cd}$, $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(M') \setminus \{c,d\}$. By IH, $N \in \Lambda^{\beta\eta}_{cd}$, $|M'|_{cd} = |N|_{cd}$ and $\mathrm{fv}(M') \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$. So $|\bar{M}|_{cd} = |N|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c,d\} = \mathrm{fv}(N) \setminus \{c,d\}$. By case on the structure of $\bar{M}$ and lemma 4.1.1:
  - If $\bar{M} \in \mathsf{Var}_{cd}$ then it is done because $\bar{M}$ does not reduce.
  - If $\bar{M} = d(\lambda\bar{x}.\bar{P})$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P} \in \Lambda^{\beta\eta}_{cd}$ then $M' = d(\lambda\bar{x}.P')$ such that $\bar{P} \to_o P'$. By IH, $N = d(\lambda\bar{x}.P'')$ such that $P' \to_o^* P''$, so $\bar{P} \to_o^* P''$.

- If $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda^{\beta\eta}_{cd}$ then:
  - Either $M' = (\lambda\bar{x}.P')\bar{Q}$ such that $\bar{P} \to_o P'$. By IH, $N = (\lambda\bar{x}.P'')Q'$ such that $P' \to^*_o P''$ and $\bar{Q} \to^*_o Q'$, so $\bar{P} \to^*_o P''$.
  - Or $M' = (\lambda\bar{x}.\bar{P})Q'$ such that $\bar{Q} \to_o Q'$. By IH, $N = (\lambda\bar{x}.P')Q''$ such that $\bar{P} \to^*_o P'$ and $Q' \to^*_o Q''$, so $\bar{Q} \to^*_o Q''$.
- If $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda^{\beta\eta}_{cd}$ then:
  - Either $M' = cP'\bar{Q}$ such that $\bar{P} \to_o P'$. By IH:
    - either $N = cP''Q'$ such that $P' \to^*_o P''$ and $\bar{Q} \to^*_o Q'$, so $\bar{P} \to^*_o P''$.
    - or $P' = (d \circ c)^n(d(\lambda\bar{x}.\bar{R}'))$ and $N = (\lambda\bar{x}.R'')Q'$ such that $n \geq 0$, $\bar{x} \in \mathsf{Var}_{cd}$, $\bar{R}' \in \Lambda^{\beta\eta}_{cd}$, $\bar{R}' \to^*_o R''$ and $\bar{Q} \to^*_o Q'$. By lemma 5.1.2, $\bar{P} = (d \circ c)^m(d(\lambda\bar{x}.R))$ such that $m \geq n$ and $R \to^*_o \bar{R}'$, so $R \to^*_o R''$.
  - Or $M' = c\bar{P}Q'$ such that $\bar{Q} \to_o Q'$. By IH:
    - either $N = cP'Q''$ such that $\bar{P} \to^*_o P'$ and $Q' \to^*_o Q''$, so $\bar{Q} \to^*_o Q''$.
    - or $\bar{P} = (d \circ c)^n(d(\lambda\bar{x}.\bar{R}))$ and $N = (\lambda\bar{x}.R')Q''$ such that $\bar{R} \in \Lambda^{\beta\eta}_{cd}$, $\bar{x} \in \mathsf{Var}_{cd}$, $n \geq 0$, $\bar{R} \to^*_o R'$ and $Q' \to^*_o Q''$. So $\bar{Q} \to^*_o Q''$.
  - Or $\bar{P} = d(\lambda\bar{x}.\bar{R})$ and $M' = (\lambda\bar{x}.\bar{R})\bar{Q}$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{R} \in \Lambda^{\beta\eta}_{cd}$. By IH, $N = (\lambda\bar{x}.R')Q'$ such that $\bar{R} \to^*_o R'$ and $\bar{Q} \to^*_o Q'$.
- If $\bar{M} = d(c\bar{P})$ such that $\bar{P} \in \Lambda^{\beta\eta}_{cd}$ then:
  - Either $\bar{P} = M'$, so we are done.
  - Or $M' = d(cP')$ such that $\bar{P} \to_o P'$ and it is done by IH.

□

of lemma 5.2.

1 By induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$ then it is done since $|\bar{M}|_{cd} = \bar{M}$.
- Let $\bar{M} = d(\lambda\bar{x}.\bar{N})$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{N} \in \Lambda^{\beta\eta}_{cd}$. Then, $|\bar{M}|_{cd} = \lambda\bar{x}.|\bar{N}|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c, d\} = \mathrm{fv}(\bar{N}) \setminus \{c, d, \bar{x}\} =^{IH} \mathrm{fv}(|\bar{N}|_{cd}) \setminus \{\bar{x}\} = \mathrm{fv}(|\bar{M}|_{cd})$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda^{\beta\eta}_{cd}$. Then $|\bar{M}|_{cd} = (\lambda\bar{x}.|\bar{P}|_{cd})|\bar{Q}|_{cd}$ and $\mathrm{fv}(\bar{M})\setminus\{c, d\} = (\mathrm{fv}(\bar{P})\setminus\{c, d, \bar{x}\}) \cup (\mathrm{fv}(\bar{Q})\setminus\{c, d\}) =^{IH} (\mathrm{fv}(|\bar{P}|_{cd}) \setminus \{\bar{x}\}) \cup \mathrm{fv}(|\bar{Q}|_{cd}) = \mathrm{fv}(|\bar{M}|_{cd})$.
- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda^{\beta\eta}_{cd}$. Then $|\bar{M}|_{cd} = |\bar{P}|_{cd}|\bar{Q}|_{cd}$ and $\mathrm{fv}(\bar{M}) \setminus \{c, d\} = (\mathrm{fv}(\bar{P}) \setminus \{c, d\}) \cup (\mathrm{fv}(\bar{Q}) \setminus \{c, d\}) =^{IH} \mathrm{fv}(|\bar{P}|_{cd}) \cup \mathrm{fv}(|\bar{Q}|_{cd}) = \mathrm{fv}(|\bar{M}|_{cd})$.
- Let $\bar{M} = d(c\bar{N})$ such that $\bar{N} \in \Lambda^{\beta\eta}_{cd}$. Then, $|\bar{M}|_{cd} = |\bar{N}|_{cd}$ and $\mathrm{fv}(M) \setminus \{c, d\} = \mathrm{fv}(\bar{N}) \setminus \{c, d\} =^{IH} \mathrm{fv}(|\bar{N}|_{cd}) = \mathrm{fv}(|\bar{M}|_{cd})$.

2 By induction on the length of the reduction $\lambda x.M \to^*_{\beta\eta} N$.

- Let $\lambda x.M = N$ then it is done.
- Let $\lambda x.M \to^*_{\beta\eta} P \to_{\beta\eta} N$. By IH:
  - Either $P = \lambda x.Q$ such that $M \to^*_{\beta\eta} Q$. Then, by compatibility:
    - Either $Q = Nx$ such that $x \notin \mathrm{fv}(N)$. So it is done since $M \to^*_{\beta\eta} Nx$.

* Or $N = \lambda x.M'$ such that $Q \to_{\beta\eta} M'$. So it is done since $M \to^*_{\beta\eta} M'$.
  - Or $M \to^*_{\beta\eta} Px$ such that $x \notin \mathrm{fv}(P)$. So $M \to^*_{\beta\eta} Nx$ and it is done since by lemma 3.2.3, $x \notin \mathrm{fv}(N)$.

3 By induction on the length of the reduction $Mx \to^*_{\beta\eta} N$.

- Let $N = Mx$ then it is done.
- Let $Mx \to^*_{\beta\eta} P \to_{\beta\eta} N$. Then by IH, $M \to^*_{\beta\eta} Q$ (by lemma 2, $x \notin \mathrm{fv}(Q)$) and:
  - Either $P = Qx$. Then, by compatibility:
    * Either $N = Q'x$ such that $Q \to_{\beta\eta} Q'$. So it is done since $M \to^*_{\beta\eta} Q'$.
    * Or $Q = \lambda y.Q'$ and $N = Q'[y := x]$. So $M \to^*_{\beta\eta} \lambda y.Q' = \lambda x.N$.
  - Or $Q = \lambda x.P$. So it is done since $M \to^*_{\beta\eta} Q = \lambda x.P \to_{\beta\eta} \lambda x.N$.

□

of lemma 5.3. Let $\lambda x.M \to^*_{\beta\eta} P_1$ and $\lambda x.M \to^*_{\beta\eta} P_2$. By lemma 5.2.2:

- Either $P_1 = \lambda x.Q_1$ such that $M \to^*_{\beta\eta} Q_1$ and $P_2 = \lambda x.Q_2$ such that $M \to^*_{\beta\eta} Q_2$. So by hypothesis there exists Q_3 such that $Q_1 \to^*_{\beta\eta} Q_3$ and $Q_2 \to^*_{\beta\eta} Q_3$, hence, $P_1 \to^*_{\beta\eta} \lambda x.Q_3$ and $P_2 \to^*_{\beta\eta} \lambda x.Q_3$.

- Or $P_1 = \lambda x.Q_1$ such that $M \to^*_{\beta\eta} Q_1$ and $M \to^*_{\beta\eta} P_2x$ such that $x \notin \mathrm{fv}(P_2)$. By hypothesis there exists Q_3 such that $Q_1 \to^*_{\beta\eta} Q_3$ and $P_2x \to^*_{\beta\eta} Q_3$. So, by lemma 5.2.3 $P_2 \to^*_{\beta\eta} Q_2$ (by lemma 3.2.3, $x \notin \mathrm{fv}(Q_2)$) and:

  - Either $Q_3 = Q_2x$. So $P_1 = \lambda x.Q_1 \to^*_{\beta\eta} \lambda x.Q_3 = \lambda x.Q_2x \to_\eta Q_2$.
  - Or $Q_2 = \lambda x.Q_3$. So it is done since $P_1 = \lambda x.Q_1 \to^*_{\beta\eta} \lambda x.Q_3$.

- Or $M \to^*_{\beta\eta} P_1x$ such that $x \notin \mathrm{fv}(P_1)$ and $P_2 = \lambda x.Q_2$ such that $M \to^*_{\beta\eta} Q_2$. This case is similar to the previous one.

- Or $M \to^*_{\beta\eta} P_1x$ such that $x \notin \mathrm{fv}(P_1)$ and $M \to^*_{\beta\eta} P_2x$ such that $x \notin \mathrm{fv}(P_2)$. So by hypothesis, there exists Q_3 such that $P_1x \to^*_{\beta\eta} Q_3$ and $P_2x \to^*_{\beta\eta} Q_3$. By lemma 5.2.3, $P_1 \to^*_{\beta\eta} Q_1$, $P_2 \to^*_{\beta\eta} Q_2$ (by lemma 2, $x \notin \mathrm{fv}(Q_1) \cup \mathrm{fv}(Q_2)$) and:

  - Either $Q_3 = Q_1x$ and $Q_3 = Q_2x$ so $Q_1 = Q_2$.
  - Or $Q_3 = Q_1x$ and $Q_2 = \lambda x.Q_3$ so $Q_2 \to_\eta Q_1$.
  - Or $Q_1 = \lambda x.Q_3$ and $Q_3 = Q_2x$ so $Q_1 \to_\eta Q_2$.
  - Or $Q_1 = \lambda x.Q_3$ and $Q_2 = \lambda x.Q_3$ so $Q_1 = Q_2$.

□

of lemma 5.4.

1 By induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$ then it is done because $|\bar{M}|_{cd} = \bar{M} \neq \lambda\bar{x}.N$.
- Let $\bar{M} = d(c\bar{M}')$ such that $\bar{M}' \in \Lambda^{\beta\eta}_{cd}$ and $|\bar{M}|_{cd} = |\bar{M}'|_{cd} = \lambda\bar{x}.N$. Then by IH, $\bar{M}' = (d \circ c)^n(d(\lambda\bar{x}.\bar{P}))$ where $n \geq 0$, $\bar{P} \in \Lambda^{\beta\eta}_{cd}$ and $|\bar{P}|_{cd} = N$. So, $\bar{M} = (d \circ c)^{n+1}(d(\lambda\bar{x}.\bar{P}))$.

- Let $\bar{M} = d(\lambda\bar{x}.\bar{M}')$ such that $\bar{M}' \in \Lambda_{cd}^{\beta\eta}$ and $|\bar{M}|_{cd} = \lambda\bar{x}.|\bar{M}'|_{cd} = \lambda\bar{x}.N$, then it is done since $|\bar{M}'|_{cd} = N$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda_{cd}^{\beta\eta}$, then it is done because $|\bar{M}|_{cd} = (\lambda\bar{x}.|\bar{P}|_{cd})|\bar{Q}|_{cd} \neq \lambda\bar{x}.N$.
- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda_{cd}^{\beta\eta}$, then it is done because $|\bar{M}|_{cd} = |\bar{P}|_{cd}|\bar{Q}|_{cd} \neq \lambda\bar{x}.N$.

2 By induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$. Either $\bar{M} = \bar{x}$, then $\bar{M}[\bar{x} := \bar{N}] = \bar{N} \in \Lambda_{cd}^{\beta\eta}$. Or, $\bar{M} \neq \bar{x}$ and so $\bar{M}[\bar{x} := \bar{N}] = \bar{M} \in \Lambda_{cd}^{\beta\eta}$.
- Let $\bar{M} = d(c\bar{P})$ such that $\bar{P} \in \Lambda_{cd}^{\beta\eta}$. By IH, $\bar{P}[\bar{x} := \bar{N}] \in \Lambda_{cd}^{\beta\eta}$. Then, $\bar{M}[\bar{x} := \bar{N}] = d(c\bar{P}[\bar{x} := \bar{N}]) \in \Lambda_{cd}^{\beta\eta}$.
- Let $\bar{M} = d(\lambda\bar{y}.\bar{P})$ such that $\bar{y} \in \mathsf{Var}_{cd}$ and $\bar{P} \in \Lambda_{cd}^{\beta\eta}$. By IH, $\bar{P}[\bar{x} := \bar{N}] \in \Lambda_{cd}^{\beta\eta}$. Then, $\bar{M}[\bar{x} := \bar{N}] = d(\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}]) \in \Lambda_{cd}^{\beta\eta}$, such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.
- Let $\bar{M} = (\lambda\bar{y}.\bar{P})\bar{Q}$ such that $\bar{y} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda_{cd}^{\beta\eta}$. By IH, $\bar{P}[\bar{x} := \bar{N}], \bar{Q}[\bar{x} := \bar{N}] \in \Lambda_{cd}^{\beta\eta}$. Then, $\bar{M}[\bar{x} := \bar{N}] = (\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}])\bar{Q}[\bar{x} := \bar{N}] \in \Lambda_{cd}^{\beta\eta}$, such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.
- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda_{cd}^{\beta\eta}$. By IH, $\bar{P}[\bar{x} := \bar{N}], \bar{Q}[\bar{x} := \bar{N}] \in \Lambda_{cd}^{\beta\eta}$. Then, $\bar{M}[\bar{x} := \bar{N}] = c\bar{P}[\bar{x} := \bar{N}]\bar{Q}[\bar{x} := \bar{N}] \in \Lambda_{cd}^{\beta\eta}$.

□

of lemma 5.6. We prove the result by induction on the structure of $\bar{M}$:

- Let $\bar{M} = \bar{x} \in \mathsf{Var}_{cd}$ and $M \in \mathsf{CR}^{\beta\eta}$ then $\bar{x}[\bar{x} := M] = M \in \mathsf{CR}^{\beta\eta}$.
- Let $\bar{M} = d(\lambda\bar{x}.\bar{N})$. Let $\mathrm{fv}(\bar{N}) \setminus \{c, d, \bar{x}\} = \{x_1, \dots, x_n\}$ and $M_1, \dots, M_n, M \in \mathsf{CR}^{\beta\eta}$.
  - If $\bar{x} \in \mathrm{fv}(\bar{N})$ then by IH, $\bar{N}[x_1 := M_1, \dots, x_n := M_n, \bar{x} := M] \in \mathsf{CR}^{\beta\eta}$. By lemma 3.2.6, $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n]M \in \mathsf{CR}^{\beta\eta}$, such that $\bar{x} \notin \mathrm{fv}(M_1) \cup \dots \cup \mathrm{fv}(M_n)$. By IH and because by lemma 3.2.7 $\bar{x} \in \mathsf{CR}^{\beta\eta}$, we obtain $\bar{N}[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}$. By lemma 5.3, $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}^{\beta\eta}$. So $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}_{\rightarrow}^{\beta\eta}$.
  - Else, by IH, $\bar{N}[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}^{\beta\eta}$. By lemma 3.2.6, $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n]M \in \mathsf{CR}^{\beta\eta}$, such that $\bar{x} \notin \mathrm{fv}(M_1) \cup \dots \cup \mathrm{fv}(M_n)$. By lemma 5.3, $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}^{\beta\eta}$. So $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}_{\rightarrow}^{\beta\eta}$

  So, by lemma 3.2.7 $(d(\lambda\bar{x}.\bar{N}))[x_1 := M_1, \dots, x_n := M_n] \in \mathsf{CR}^{\beta\eta}$.
- Let $\bar{M} = c\bar{P}\bar{Q}$. Let $\mathrm{fv}(\bar{P}) \setminus \{c, d\} = \{x_1, \dots, x_n, x'_1, \dots, x'_{n_1}\}$, $\mathrm{fv}(\bar{Q}) \setminus \{c, d\} = \{x_1, \dots, x_n, x''_1, \dots, x''_{n_2}\}$ and $M_1, \dots, M_n, M'_1, \dots, M'_{n_1}, M''_1, \dots, M''_{n_2} \in \mathsf{CR}^{\beta\eta}$ and $\{x'_1, \dots, x'_{n_1}\} \cap \{x''_1, \dots, x''_{n_2}\} = \varnothing$. By IH, $\bar{P}[x_1 := M_1, \dots, x_n := M_n, x'_1 := M'_1, \dots, x'_{n_1} := M'_{n_1}]$, $\bar{Q}[x_1 := M_1, \dots, x_n := M_n, x''_1 := M''_1, \dots, x''_{n_2} := M''_{n_2}] \in \mathsf{CR}^{\beta\eta}$. So, by lemma 3.2.7, $(c\bar{P}\bar{Q})[x_1 := M_1, \dots, x_n := M_n, x'_1 := M'_1, \dots, x'_{n_1} := M'_{n_1}, x''_1 := M''_1, \dots, x''_{n_2} := M''_{n_2}] \in \mathsf{CR}^{\beta\eta}$.

- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$. Let $\mathrm{fv}(\bar{P}) \setminus \{c, d, \bar{x}\} = \{x_1, \ldots, x_n, x'_1, \ldots, x'_{n_1}\}$, $\mathrm{fv}(\bar{Q}) \setminus \{c, d\} = \{x_1, \ldots, x_n, x''_1, \ldots, x''_{n_2}\}$ and $M_1, \ldots, M_n, M'_1, \ldots, M'_{n_1}, M''_1, \ldots, M''_{n_2} \in \mathsf{CR}^{\beta\eta}$ and $\{x'_1, \ldots, x'_{n_1}\} \cap \{x''_1, \ldots, x''_{n_2}\} = \varnothing$. By IH, $\bar{Q}[x_1 := M_1, \ldots, x_n := M_n, x''_1 := M''_1, \ldots, x''_{n_2} := M''_{n_2}] \in \mathsf{CR}^{\beta\eta}$.
  - If $\bar{x} \in \mathrm{fv}(\bar{P})$ then by IH, $\bar{P}[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}, \bar{x} := M] \in \mathsf{CR}^{\beta\eta}$. By lemma 3.2.6, $(\lambda\bar{x}.\bar{P})[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}]M \in \mathsf{CR}^{\beta\eta}$, such that $\bar{x} \notin \mathrm{fv}(M_1) \cup \cdots \cup \mathrm{fv}(M_n) \cup \mathrm{fv}(M'_1) \cup \cdots \cup \mathrm{fv}(M'_{n_1})$. By IH and because by lemma 3.2.7 $\bar{x} \in \mathsf{CR}^{\beta\eta}$, we obtain $\bar{P}[x_1 := M_1, \ldots, x_n := M_n] \in \mathsf{CR}^{\beta\eta}$. By lemma 5.3, $(\lambda\bar{x}.\bar{P})[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}] \in \mathsf{CR}^{\beta\eta}$. So $(\lambda\bar{x}.\bar{P})[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}] \in \mathsf{CR}^{\beta\eta}_{\rightarrow}$.
  - Else, by IH, $\bar{P}[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}] \in \mathsf{CR}^{\beta\eta}$. By lemma 3.2.6, $(\lambda\bar{x}.\bar{P})[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}]M \in \mathsf{CR}^{\beta\eta}$, such that $\bar{x} \notin \mathrm{fv}(M_1) \cup \cdots \cup \mathrm{fv}(M_n) \cup \mathrm{fv}(M'_1) \cup \cdots \cup \mathrm{fv}(M'_{n_1})$. By lemma 5.3, $(\lambda\bar{x}.\bar{P})[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}] \in \mathsf{CR}^{\beta\eta}$. So $(\lambda\bar{x}.\bar{P})[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}] \in \mathsf{CR}^{\beta\eta}_{\rightarrow}$

  So, by definition, $((\lambda\bar{x}.\bar{P})\bar{Q})[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}, x''_1 := M''_1, \ldots, x''_{n_2} := M''_{n_2}] \in \mathsf{CR}^{\beta\eta}$.

- Let $\bar{M} = d(c\bar{P})$. Let $\mathrm{fv}(\bar{P}) \setminus \{c, d\} = \{x_1, \ldots, x_n\}$ and $M_1, \ldots, M_n \in \mathsf{CR}^{\beta\eta}$. By IH, $\bar{P}[x_1 := M_1, \ldots, x_n := M_n] \in \mathsf{CR}^{\beta\eta}$. So, by lemma 3.2.7 twice, $(d(c\bar{P}))[x_1 := M_1, \ldots, x_n := M_n] \in \mathsf{CR}^{\beta\eta}$. □

of lemma 5.8. We prove this lemma by induction on the structure of M.

- Let $M \in \mathsf{Var}$.
  - If $M = y$ then $M[y := P] = P$ and because $x \notin \mathrm{fv}(P)$, then for all $N \in \Lambda$, $P \neq Nx$.
  - If $M \neq y$ then $M[y := P] = M$ and for all $N \in \Lambda$, $M \neq Nx$.
- Let $M = \lambda z.Q$ then $M[y := P] = \lambda z.Q[y := P]$ such that $z \notin \mathrm{fv}(P) \cup \mathrm{fv}(y)$ and for all $N \in \Lambda$, $M[y := P] \neq Nx$.
- Let $M = M_1M_2$ so $M[y := P] = M_1[y := P]M_2[y := P]$. Because for all $N \in \Lambda$ such that $x \notin \mathrm{fv}(N)$, $M \neq Nx$, we have $x \in \mathrm{fv}(M_1)$ or $M_2 \neq x$.
  - Let $x \in \mathrm{fv}(M_1)$ then, since $x \neq y$, $x \in M_1[y := P]$. So for all $N \in \Lambda$ such that $x \notin \mathrm{fv}(N)$, $M[y := P] \neq Nx$.
  - Let $M_2 \neq x$. We prove that $M_2[y := P] \neq x$ by induction on the structure of M_2, and so for all $N \in \Lambda$, $M[y := P] \neq Nx$.
    * Let $M_2 \in \mathsf{Var}$.
      · Let $M_2 = y$ then $M_2[y := P] = P$. Because $x \notin \mathrm{fv}(P)$, $P \neq x$.
      · Let $M_2 \in \mathsf{Var} \setminus \{x, y\}$ then $M_2[y := P] = M_2 \neq x$.
    * Let $M_2 = \lambda z.M_3$ then $M_2[y := P] = \lambda z.M_3[y := P] \neq x$ such that $z \notin \mathrm{fv}(P) \cup \mathrm{fv}(y)$
    * Let $M_2 = M_3M_4$ then $M_2[y := P] = M_3[y := P]M_4[y := P] \neq x$

□

of lemma 5.9. We prove the lemma by induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$ then $|\bar{M}|_{cd} = \bar{M}$. If $\bar{M} = \bar{x}$ then $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] = |\bar{N}|_{cd} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$. Else, $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] = \bar{M} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$.

- Let $\bar{M} = d(c\bar{P})$ such that $\bar{P} \in \Lambda^{\beta\eta}_{cd}$, then $|\bar{M}|_{cd} = |\bar{P}|_{cd}$. So, $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] = |\bar{P}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{IH} |\bar{P}[\bar{x} := \bar{N}]|_{cd} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$.

- Let $\bar{M} = d(\lambda\bar{y}.\bar{P})$, such that $\bar{P} \in \Lambda^{\beta\eta}_{cd}$ and $\bar{y} \in \mathsf{Var}_{cd}$, then $|\bar{M}|_{cd} = \lambda\bar{y}.|\bar{P}|_{cd}$. So, $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{5.2.1} \lambda\bar{y}.|\bar{P}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{IH} \lambda\bar{y}.|\bar{P}[\bar{x} := \bar{N}]|_{cd} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$, such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.

- Let $\bar{M} = (\lambda\bar{y}.\bar{P})\bar{Q}$, such that $\bar{y} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda^{\beta\eta}_{cd}$, then $|\bar{M}|_{cd} = (\lambda\bar{y}.|\bar{P}|_{cd})|\bar{Q}|_{cd}$. So, $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{5.2.1} (\lambda\bar{y}.|\bar{P}|_{cd}[\bar{x} := |\bar{N}|_{cd}])|\bar{Q}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{IH} (\lambda\bar{y}.|\bar{P}[\bar{x} := \bar{N}]|_{cd})|\bar{Q}[\bar{x} := \bar{N}]|_{cd} = |(\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}])\bar{Q}[\bar{x} := \bar{N}]|_{cd} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$, such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.

- Let $\bar{M} = c\bar{P}\bar{Q}$, such that $\bar{P}, \bar{Q} \in \Lambda^{\beta\eta}_{cd}$, then $|\bar{M}|_{cd} = |\bar{P}|_{cd}|\bar{Q}|_{cd}$. So, $|\bar{M}|_{cd}[\bar{x} := |\bar{N}|_{cd}] = |\bar{P}|_{cd}[\bar{x} := |\bar{N}|_{cd}]|\bar{Q}|_{cd}[\bar{x} := |\bar{N}|_{cd}] =^{IH} |\bar{P}[\bar{x} := \bar{N}]|_{cd}|\bar{Q}[\bar{x} := \bar{N}]|_{cd} = |c\bar{P}[\bar{x} := \bar{N}]\bar{Q}[\bar{x} := \bar{N}]|_{cd} = |\bar{M}[\bar{x} := \bar{N}]|_{cd}$. □

of lemma 5.10. We prove the lemma by induction on the length of the derivation $\bar{M} \to^*_{\beta\eta} N$.

- let $\bar{M} = N$ then it is done.

- Let $\bar{M} \to^*_{\beta\eta} M' \to_{\beta\eta} N$. By IH, $M' \in \Lambda^{\beta\eta}_{cd}$ and $|\bar{M}|_{cd} \to^*_{\beta\eta} |M'|_{cd}$. We prove that $N \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} \to_{\beta\eta} |N|_{cd}$ by induction on the structure of M'.
  - Let $M' \in \mathsf{Var}_{cd}$ then it is done because M' does not reduce.
  - Let $M' = d(cP)$ such that $P \in \Lambda^{\beta\eta}_{cd}$, so by compatibility $N = d(cP')$ such that $P \to_{\beta\eta} P'$. By IH, $P' \in \Lambda^{\beta\eta}_{cd}$ and $|P|_{cd} \to_{\beta\eta} |P'|_{cd}$. So, $N \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = |P|_{cd} \to_{\beta\eta} |P'|_{cd} = |N|_{cd}$.
  - Let $M' = d(\lambda x.P)$ such that $x \in \mathsf{Var}_{cd}$ and $P \in \Lambda^{\beta\eta}_{cd}$. By compatibility:
    * Either $N = d(\lambda x.P')$ such that $P \to_{\beta\eta} P'$. By IH, $P' \in \Lambda^{\beta\eta}_{cd}$ and $|P|_{cd} \to_{\beta\eta} |P'|_{cd}$. So, $N \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = \lambda x.|P|_{cd} \to_{\beta\eta} \lambda x.|P'|_{cd} = |N|_{cd}$.
    * Or $P = Qx$ such that $x \notin \mathrm{fv}(Q)$ and $N = dQ$. Because $P \in \Lambda^{\beta\eta}_{cd}$, by case on P, either $Q = cQ'$ such that $Q' \in \Lambda^{\beta\eta}_{cd}$, so $N = d(cQ') \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = \lambda x.|Q'|_{cd}x \to_\eta |Q'|_{cd} = |N|_{cd}$ because by lemma 5.2.1, $x \notin fv|Q'|_{cd}$. Or $Q = \lambda y.Q'$ such that $y \in \mathsf{Var}_{cd}$ and $Q' \in \Lambda^{\beta\eta}_{cd}$, so $N = d(\lambda y.Q') \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = \lambda x.|Q|_{cd}x \to_\eta |Q|_{cd} = |N|_{cd}$ because by lemma 5.2.1, $x \notin \mathrm{fv}(|Q'|_{cd})$, such that $y \neq x$ and so $x \notin \mathrm{fv}(|Q|_{cd})$.
  - Let $M' = (\lambda x.P)Q$ such that $x \in \mathsf{Var}_{cd}$ and $P, Q \in \Lambda^{\beta\eta}_{cd}$. By compatibility:
    * Either $N = P_1Q$ such that $\lambda x.P \to_{\beta\eta} P_1$.
      · Either $P_1 = \lambda x.P'$ such that $P \to_{\beta\eta} P'$. By IH, $P' \in \Lambda^{\beta\eta}_{cd}$ and $|P|_{cd} \to_{\beta\eta} |P'|_{cd}$ and so $N = (\lambda x.P')Q \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = (\lambda x.|P|_{cd})|Q|_{cd} \to_{\beta\eta} (\lambda x.|P'|_{cd})|Q|_{cd} = |N|_{cd}$.
      · Or $P = P_1x$ such that $x \notin \mathrm{fv}(P_1)$. Because $P \in \Lambda^{\beta\eta}_{cd}$, either $P_1 = cP_2$ such that $P_2 \in \Lambda^{\beta\eta}_{cd}$ so $N = cP_2Q \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = (\lambda x.|P_2|_{cd}x)|Q|_{cd} \to_\eta |P_2|_{cd}|Q|_{cd} = |N|_{cd}$ because by lemma 5.2.1, $x \notin fv|P_2|_{cd}$. Or $P_1 = \lambda y.P_2$ such

that $P_2 \in \Lambda^{\beta\eta}_{cd}$ and $y \in \mathsf{Var}_{cd}$, so $N = (\lambda y.P_2)Q \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = (\lambda x.|P_1|_{cd}x)|Q|_{cd} \to_{\beta\eta} |P_1|_{cd}|Q|_{cd} = |N|_{cd}$ because by lemma 5.2.1, $x \notin \mathrm{fv}(|P_2|_{cd})$, such that $y \neq x$ and so $x \notin \mathrm{fv}(|P_1|_{cd})$.

    * Or $N = (\lambda x.P)Q'$ such that $Q \to_{\beta\eta} Q'$. By IH, $Q' \in \Lambda^{\beta\eta}_{cd}$ and $|Q|_{cd} \to_{\beta\eta} |Q'|_{cd}$. So, $N \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = (\lambda x.|P|_{cd})|Q|_{cd} \to_{\beta\eta} (\lambda x.|P|_{cd})|Q'|_{cd} = |N|_{cd}$.
    * Or $N = P[x := Q]$. So, by lemma 5.4.2, $N \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = (\lambda x.|P|_{cd})|Q|_{cd} \to_{\beta} |P|_{cd}[x := |Q|_{cd}] =^{5.9} |P[x := Q]|_{cd}$.

  – Let $M' = cPQ$ such that $P, Q \in \Lambda^{\beta\eta}_{cd}$. By compatibility:

    * Either $N = cP'Q$ such that $P \to_{\beta\eta} P'$. By IH, $P' \in \Lambda^{\beta\eta}_{cd}$ and $|P|_{cd} \to_{\beta\eta} |P'|_{cd}$. So, $N \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = |P|_{cd}|Q|_{cd} \to_{\beta\eta} |P'|_{cd}|Q|_{cd} = |N|_{cd}$.
    * Or $N = cPQ'$ such that $Q \to_{\beta\eta} Q'$. By IH, $Q' \in \Lambda^{\beta\eta}_{cd}$ and $|Q|_{cd} \to_{\beta\eta} |Q'|_{cd}$. So, $N \in \Lambda^{\beta\eta}_{cd}$ and $|M'|_{cd} = |P|_{cd}|Q|_{cd} \to_{\beta\eta} |P|_{cd}|Q'|_{cd} = |N|_{cd}$.

$\square$

of lemma 5.12.

⇒) Let $M \to^*_{\beta\eta} N$. We prove that $M \to^*_2 N$ by induction on the size of the reduction $M \to^*_{\beta\eta} N$.

▼ If $M = N$, then it is done since $M \to^*_2 N$.

▼ If $M \to^*_{\beta\eta} M' \to_{\beta\eta} N$. By IH, $M \to^*_2 M'$. We prove that $M' \to_2 N$ by induction on the structure of M'. By lemma 3.2.3, $c, d \notin \mathrm{fv}(M')$.

- Let $M' \in \mathsf{Var}$. It is done because M' does not reduce.
- Let $M' = \lambda x.P$ such that $x \notin \{c, d\}$. By compatibility:
  – Either $N = \lambda x.P'$ such that $P \to_{\beta\eta} P'$. By IH, $P \to_2 P'$. So there exists Q such that $\Psi_{cd}(P) \to^*_{\beta\eta} Q$ and $|Q|_{cd} = P'$. Then $\Psi_{cd}(M') = d(\lambda x.\Psi_{cd}(P)) \to^*_{\beta\eta} d(\lambda x.Q)$ and $|d(\lambda x.Q)|_{cd} = \lambda x.|Q|_{cd} = \lambda x.P'$. Hence, $M' \to_2 N$.
  – Or $P = Nx$ such that $x \notin \mathrm{fv}(N)$. By lemma 4.2.1, $x \notin \mathrm{fv}(\Psi_{cd}(N))$.
    * If $N = \lambda y.N_1$ where $y \notin \{c, d\}$, then $\Psi_{cd}(M') = d(\lambda x.\Psi_{cd}(P)) = d(\lambda x.(\lambda y.\Psi_{cd}(N_1))x) \to_\eta d(\lambda y.\Psi_{cd}(N_1))$ (because $x \notin \mathrm{fv}(\Psi_{cd}(N))$ implies that $x \notin \mathrm{fv}(\lambda y.\Psi_{cd}(N_1)))$ and $|d(\lambda y.\Psi_{cd}(N_1))|_{cd} = \lambda y.|\Psi_{cd}(N_1)|_{cd} =^{4.6.2} \lambda y.N_1 = N$. Hence, $M' \to_2 N$.
    * Else, $\Psi_{cd}(M') = d(\lambda x.\Psi_{cd}(P)) = d(\lambda x.c\Psi_{cd}(N)x) \to_\eta d(c\Psi_{cd}(N))$ and $|d(c\Psi_{cd}(N))|_{cd} = |\Psi_{cd}(N)|_{cd} =^{4.6.2} N$. Hence, $M' \to_2 N$.
- Let $M' = PQ$.
  – If $P = \lambda x.P_1$, such that $x \notin \{c, d\}$ then by compatibility:
    * Either $N = P_0Q$ such that $P \to_{\beta\eta} P_0$. By lemma 3.2.3, $c, d \notin \mathrm{fv}(P_0)$.
      · Either $P_0 = \lambda x.P_2$ and $P_1 \to_{\beta\eta} P_2$. By IH, $P_1 \to_2 P_2$. So, there exists P_1' such that $\Psi_{cd}(P_1) \to^*_{\beta\eta} P_1'$ and $|P_1'|_{cd} = P_2$. So, $\Psi_{cd}(M') = (\lambda x.\Psi_{cd}(P_1))\Psi_{cd}(Q) \to^*_{\beta\eta} (\lambda x.P_1')\Psi_{cd}(Q)$ and $|(\lambda x.P_1')\Psi_{cd}(Q)|_{cd} =^{4.6.2} (\lambda x.|P_1'|_{cd})Q = (\lambda x.P_2)Q = N$. Hence, $M' \to_2 N$.

- $\cdot$ Or, $P_1 = P_0x$ such that $x \notin \text{fv}(P_0)$. By lemma 4.2.1, $x \notin \text{fv}(\Psi_{cd}(P_0))$. If $P_0 = \lambda y.R$ then $\Psi_{cd}(M') = (\lambda x.(\lambda y.\Psi_{cd}(R))x)\Psi_{cd}(Q) \to_\eta (\lambda y.\Psi_{cd}(R))\Psi_{cd}(Q) = \Psi_{cd}(N)$, such that $y \notin \{c, d\}$ and because $x \notin \text{fv}(\Psi_{cd}(P_0))$ implies that $x \notin \text{fv}(\lambda y.\Psi_{cd}(R))$. Else, $\Psi_{cd}(M') = (\lambda x.c\Psi_{cd}(P_0)x)\Psi_{cd}(Q) \to_\eta c\Psi_{cd}(P_0)\Psi_{cd}(Q) = \Psi_{cd}(N)$. In both cases $|\Psi_{cd}(N)|_{cd} =^{4.6.2} N$, and so, $M' \to_2 N$.
  - $*$ Or $N = (\lambda x.P_1)Q_1$ such that $Q \to_{\beta\eta} Q_1$. By IH, $Q \to_2 Q_1$. So, there exists Q_2 sucht that $\Psi_{cd}(Q) \to^*_{\beta\eta} Q_2$ and $|Q_2|_{cd} = Q_1$. So, $\Psi_{cd}(M') = (\lambda x.\Psi_{cd}(P_1))\Psi_{cd}(Q) \to^*_{\beta\eta} (\lambda x.\Psi_{cd}(P_1))Q_2$ and $|(\lambda x.\Psi_{cd}(P_1))Q_2|_{cd} =^{4.6.2} (\lambda x.P_1)|Q_2|_{cd} = (\lambda x.P_1)Q_1 = N$. Hence, $M' \to_2 N$.
  - $*$ Or $N = P_1[x := Q]$. So, $\Psi_{cd}(M') = (\lambda x.\Psi_{cd}(P_1))\Psi_{cd}(Q) \to_\beta \Psi_{cd}(P_1)[x := \Psi_{cd}(Q)]$. and $|\Psi_{cd}(P_1)[x := \Psi_{cd}(Q)]|_{cd} =^{5.9} |\Psi_{cd}(P_1)|_{cd}[x := |\Psi_{cd}(Q)|_{cd}] =^{4.6.2} P_1[x := Q]$. Hence, $M' \to_1 N$.
- – Else,
  - $*$ Either $N = P'Q$ such that $P \to_{\beta\eta} P'$. By IH, $P \to_2 P'$. So, there exists P_1 such that $\Psi_{cd}(P) \to^*_{\beta\eta} P_1$ and $|P_1|_{cd} = P'$. So, $\Psi_{cd}(M') = c\Psi_{cd}(P)\Psi_{cd}(Q) \to^*_{\beta\eta} cP_1\Psi_{cd}(Q)$ and $|cP_1\Psi_{cd}(Q)|_{cd} =^{4.6.2} |P_1|_{cd}Q = P'Q = N$. So $M' \to_2 N$.
  - $*$ Or $N = PQ'$ such that $Q \to_{\beta\eta} Q'$. By IH, $Q \to_2 Q'$. So, there exists Q_1 suh that $\Psi_{cd}(Q) \to^*_{\beta\eta} Q_1$ and $|Q_1|_{cd} = Q'$. So, $\Psi_{cd}(M') = c\Psi_{cd}(P)\Psi_{cd}(Q) \to^*_\beta c\Psi_{cd}(P)Q_1$ and $|c\Psi_{cd}(P)Q_1|_{cd} =^{4.6.2} P|Q_1|_{cd} = PQ' = N$. So $M' \to_2 N$.

$\Leftarrow$) Let $M \to^*_2 N$. We prove that $M \to^*_{\beta\eta} N$ by induction on the size of the derivation $M \to^*_2 N$.

- Let $M = N$, then it is done because $M \to^*_{\beta\eta} N$.
- Let $M \to^*_2 M' \to_2 N$. By IH, $M \to^*_{\beta\eta} M'$. By lemma 3.2.3, $c, d \notin \text{fv}(M')$. Since $M' \to_2 N$, there exists P such that $\Psi_{cd}(M') \to^*_{\beta\eta} P$ and $|P|_{cd} = N$. By corollary 5.11, $M' \to^*_{\beta\eta} N$.

□

of lemma 5.13. We prove this lemma by induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$ then it is done since $\Psi_{cd}(|\bar{M}|_{cd}) = \Psi_{cd}(\bar{M}) = \bar{M}$.
- Let $\bar{M} = d(c\bar{P})$ such that $\bar{P} \in \Lambda^{\beta\eta}_{cd}$ then $|\bar{M}|_{cd} = |\bar{P}|_{cd}$. By IH, $\bar{M} \to_o \bar{P} \to^*_o \Psi_{cd}(|\bar{P}|_{cd}) = \Psi_{cd}(|\bar{M}|_{cd})$.
- Let $\bar{M} = d(\lambda\bar{x}.\bar{P})$, such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P} \in \Lambda^{\beta\eta}_{cd}$, then $|\bar{M}|_{cd} = \lambda\bar{x}.|\bar{P}|_{cd}$ and $\Psi_{cd}(|\bar{M}|_{cd}) = d(\lambda\bar{x}.\Psi_{cd}(|\bar{P}|_{cd}))$. By IH, $\bar{M} = d(\lambda\bar{x}.\bar{P}) \to^*_o d(\lambda\bar{x}.\Psi_{cd}(|\bar{P}|_{cd})) = \Psi_{cd}(|\bar{M}|_{cd})$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$, such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda^{\beta\eta}_{cd}$, then $|\bar{M}|_{cd} = (\lambda\bar{x}.|\bar{P}|_{cd})|\bar{Q}|_{cd}$ and $\Psi_{cd}(|\bar{M}|_{cd}) = (\lambda\bar{x}.\Psi_{cd}(|\bar{P}|_{cd}))\Psi_{cd}(|\bar{Q}|_{cd})$. By IH, $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q} \to^*_o (\lambda\bar{x}.\Psi_{cd}(|\bar{P}|_{cd}))\Psi_{cd}(|\bar{Q}|_{cd}) = \Psi_{cd}(|\bar{M}|_{cd})$.
- Let $\bar{M} = c\bar{P}\bar{Q}$, such that $\bar{P}, \bar{Q} \in \Lambda^{\beta\eta}_{cd}$, then $|\bar{M}|_{cd} = |\bar{P}|_{cd}|\bar{Q}|_{cd}$.
  - If $|\bar{P}|_{cd} = \lambda x.N$, then by lemma 5.4.1, $\bar{P} = (d \circ c)^n(d(\lambda x.P'))$ such that $n \geq 0$, $P' \in \Lambda^{\beta\eta}_{cd}$, $|P'|_{cd} = N$ and $x \in \mathsf{Var}_{cd}$. So, $\Psi_{cd}(|\bar{M}|_{cd}) = (\lambda x.\Psi_{cd}(|P'|_{cd}))\Psi_{cd}(|\bar{Q}|_{cd})$. By IH, $\bar{M} = c((d \circ c)^n(d(\lambda x.P')))\bar{Q} \to^*_o (\lambda x.P')Q \to^*_o (\lambda x.\Psi_{cd}(|P'|_{cd}))\Psi_{cd}(|\bar{Q}|_{cd}) = \Psi_{cd}(|\bar{M}|_{cd})$.

– Else, $\Psi_{cd}(|\bar{M}|_{cd}) = c\Psi_{cd}(|\bar{P}|_{cd})\Psi_{cd}(|\bar{Q}|_{cd})$. By IH, $\bar{M} = c\bar{P}\bar{Q} \rightarrow_o^* c\Psi_{cd}(|\bar{P}|_{cd})\Psi_{cd}(|\bar{Q}|_{cd}) = \Psi_{cd}(|\bar{M}|_{cd})$.

□

of lemma 5.14. We prove this lemma by induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_{cd}$ then by lemma 5.1.3, $M' = \bar{M}$. If $\bar{M} = x$ then $\bar{M}[\bar{x} := \bar{N}] = \bar{N} \rightarrow_o^* N' = M'[\bar{x} := N']$. Else $\bar{M}[\bar{x} := \bar{N}] = \bar{M} = M' = M'[\bar{x} := N']$.

- Let $\bar{M} = d(c\bar{P})$ such that $\bar{P} \in \Lambda_{cd}^{\beta\eta}$ then by lemma 5.1.3, $M' = (d \circ c)^n(P')$ such that $n \leq 1$ and $\bar{P} \rightarrow_o^* P'$. So, by IH, $\bar{M}[\bar{x} := \bar{N}] = d(c\bar{P}[\bar{x} := \bar{N}]) \rightarrow_o^* d(cP'[\bar{x} := N']) \rightarrow_o^* M'[\bar{x} := N']$ (the reduction $d(cP'[\bar{x} := N']) \rightarrow_o^* M'[\bar{x} := N']$ is of length 0 or 1).

- Let $\bar{M} = d(\lambda\bar{y}.\bar{P})$ such that $\bar{y} \in \mathsf{Var}_{cd}$ and $\bar{P} \in \Lambda_{cd}^{\beta\eta}$. So, by lemma 5.1.3, $M' = d(\lambda\bar{y}.P')$ such that $\bar{P} \rightarrow_o^* P'$. So, by IH, $\bar{M}[\bar{x} := \bar{N}] = d(\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}]) \rightarrow_o^* d(\lambda\bar{y}.P'[\bar{x} := N']) = M'[\bar{x} := N']$ such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$ so by lemma 5.1.3, $\bar{y} \notin \mathrm{fv}(N')$.

- Let $\bar{M} = (\lambda\bar{y}.\bar{P})\bar{Q}$ such that $\bar{y} \in \mathsf{Var}_{cd}$ and $\bar{P}, \bar{Q} \in \Lambda_{cd}^{\beta\eta}$. Then, by lemma 5.1.3, $M' = (\lambda\bar{y}.P')Q'$ such that $\bar{P} \rightarrow_o^* P'$ and $\bar{Q} \rightarrow_o^* Q'$. So by IH, $\bar{M}[\bar{x} := \bar{N}] = (\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}])\bar{Q}[\bar{x} := \bar{N}] \rightarrow_o^* (\lambda\bar{y}.P'[\bar{x} := N'])Q'[\bar{x} := N'] = M'[\bar{x} := N']$ such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$ so by lemma 5.1.3, $\bar{y} \notin \mathrm{fv}(N')$.

- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda_{cd}^{\beta\eta}$. By lemma 5.1.3:
  - Either $\bar{P} = (d \circ c)^n(d(\lambda\bar{y}.\bar{R}))$ and $M' = (\lambda\bar{y}.R')Q'$ such that $\bar{R} \in \Lambda_{cd}^{\beta\eta}$, $\bar{y} \in \mathsf{Var}_{cd}$, $n \geq 0$, $\bar{R} \rightarrow_o^* R'$ and $\bar{Q} \rightarrow_o^* Q'$. So by IH, $\bar{M}[\bar{x} := \bar{N}] = c((d \circ c)^n(d(\lambda\bar{y}.\bar{R}[\bar{x} := \bar{N}])))\bar{Q}[\bar{x} := \bar{N}] \rightarrow_o^* (\lambda\bar{y}.\bar{R}[\bar{x} := \bar{N}])\bar{Q}[\bar{x} := \bar{N}] \rightarrow_o^* (\lambda\bar{y}.R'[\bar{x} := N'])Q'[\bar{x} := N'] = M'[\bar{x} := N']$ such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$ so by lemma 5.1.3, $\bar{y} \notin \mathrm{fv}(N')$.
  - Or $M' = cP'Q'$ such that $\bar{P} \rightarrow_o^* P'$ and $\bar{Q} \rightarrow_o^* Q'$. So by IH, $\bar{M}[\bar{x} := \bar{N}] = c\bar{P}[\bar{x} := \bar{N}]\bar{Q}[\bar{x} := \bar{N}] \rightarrow_o^* cP'[\bar{x} := N']Q'[\bar{x} := N'] = M'[\bar{x} := N']$.

□

of lemma 5.15. We prove this lemma by induction on the structure of M_1.

▼ Let $\bar{M}_1 \in \mathsf{Var}_{cd}$, then it is done because $\bar{M}_1$ does not reduce.

▼ Let $\bar{M}_1 = d(c\bar{P}_1)$ such that $\bar{P}_1 \in \Lambda_{cd}^{\beta\eta}$. Then by compatibility $N_1 = d(cP_1')$ such that $\bar{P}_1 \rightarrow_{\beta\eta} P_1'$. By lemma 5.1.3, $M_2 = (d \circ c)^n(P_2)$ such that $n \leq 1$ and $\bar{P}_1 \rightarrow_o^* P_2$. By IH, there exists P_2' such that $P_2 \rightarrow_{\beta\eta} P_2'$ and $P_1' \rightarrow_o^* P_2'$. So $M_2 = (d \circ c)^n(P_2) \rightarrow_{\beta\eta} (d \circ c)^n(P_2') = N_2$ and $N_1 \rightarrow_o^* N_2$.

▼ Let $\bar{M}_1 = d(\lambda\bar{x}.\bar{P}_1)$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}_1 \in \Lambda_{cd}^{\beta\eta}$. Then, by lemma 5.1.3, $M_2 = d(\lambda\bar{x}.P_2)$ such that $\bar{P}_1 \rightarrow_o^* P_2$. By compatibility:
  - Either $N_1 = d(\lambda\bar{x}.P_1')$ such that $\bar{P}_1 \rightarrow_{\beta\eta} P_1'$. By IH, there exits P_2' such that $P_2 \rightarrow_{\beta\eta} P_2'$ and $P_1' \rightarrow_o^* P_2'$. So $M_2 = d(\lambda\bar{x}.P_2) \rightarrow_{\beta\eta} d(\lambda x.P_2') = N_2$ and $N_1 \rightarrow_o^* N_2$.
  - Or $\bar{P}_1 = Q\bar{x}$ such that $\bar{x} \notin \mathrm{fv}(Q)$ and $N_1 = dQ$. Because $\bar{P}_1 \in \Lambda_{cd}^{\beta\eta}$:
    - Either $Q = cQ_1$ such that $Q_1 \in \Lambda_{cd}^{\beta\eta}$. Since $\bar{P}_1 \rightarrow_o^* P_2$ and $\bar{P}_1 = cQ_1\bar{x}$, then by lemma 5.1.3:

* Either $P_2 = cQ_2\bar{x}$ such that $Q_1 \to_o^* Q_2$. By lemma 5.1.3, $\bar{x} \notin \mathrm{fv}(Q_2)$, so $M_2 = d(\lambda\bar{x}.cQ_2\bar{x}) \to_\eta d(cQ_2) = N_2$ and $N_1 = d(cQ_1) \to_o^* N_2$.
* Or, $Q_1 = (d \circ c)^n(d(\lambda y.R_1))$ and $P_2 = (\lambda y.R_2)\bar{x}$ such that $y \in \mathsf{Var}_{cd}$, $R_1 \in \Lambda_{cd}^{\beta\eta}$, $n \geq 0$ and $R_1 \to_o^* R_2$. By lemma 5.1.3, $\bar{x} \notin \mathrm{fv}(\lambda y.R_2)$, so $M_2 \to_\eta d(\lambda y.R_2) = N_2$ and $N_1 \to_o^* N_2$.

– Or $Q = \lambda y.Q_1$ such that $y \in \mathsf{Var}_{cd}$ and $Q_1 \in \Lambda_{cd}^{\beta\eta}$. Since $\bar{P}_1 \to_o^* P_2$ and $\bar{P}_1 = (\lambda y.Q_1)\bar{x}$ then by lemma 5.1.3, $P_2 = (\lambda y.Q_2)\bar{x}$ such that $Q_1 \to_o^* Q_2$. By lemma 5.1.3, $\bar{x} \notin \mathrm{fv}(\lambda y.Q_2)$, so $M_2 \to_\eta d(\lambda y.Q_2) = N_2$ and $N_1 \to_o^* N_2$.

▼ Let $\bar{M}_1 = (\lambda\bar{x}.\bar{P}_1)\bar{Q}_1$ such that $\bar{x} \in \mathsf{Var}_{cd}$ and $\bar{P}_1, \bar{Q}_1 \in \Lambda_{cd}^{\beta\eta}$. By lemma 5.1.3, $M_2 = (\lambda x.P_2)Q_2$ such that $\bar{P}_1 \to_o^* P_2$ and $\bar{Q}_1 \to_o^* Q_2$. By compatibility:

- Either, $N_1 = \bar{P}_1[\bar{x} := \bar{Q}_1]$. We have, $M_2 \to_\beta P_2[\bar{x} := Q_2]$ and by lemma 5.14, $N_1 \to_o^* P_2[x := Q_2]$.
- Or, $N_1 = (\lambda\bar{x}.P_1')\bar{Q}_1$ such that $\bar{P}_1 \to_{\beta\eta} P_1'$. By IH, there exists P_2' such that $P_2 \to_{\beta\eta} P_2'$ and $P_1' \to_o^* P_2'$. So, $M_2 = (\lambda\bar{x}.P_2)Q_2 \to_{\beta\eta} (\lambda\bar{x}.P_2')Q_2 = N_2$ and $N_1 \to_o^* N_2$.
- Or $\bar{P}_1 = R_1\bar{x}$ such that $\bar{x} \notin \mathrm{fv}(R_1)$ and $N_1 = R_1\bar{Q}_1$. Since $\bar{P}_1 \in \Lambda_{cd}^{\beta\eta}$:
  – Either $R_1 = cR_1'$ such that $R_1' \in \Lambda_{cd}^{\beta\eta}$. Since $\bar{P}_1 \to_o^* P_2$ and $\bar{P}_1 = cR_1'\bar{x}$, then by lemma 5.1.3:
    * Either $P_2 = cR_2\bar{x}$ such that $R_1' \to_o^* R_2$. By lemma 5.1.3, $\bar{x} \notin \mathrm{fv}(R_2)$, so $M_2 \to_\eta cR_2Q_2 = N_2$ and $N_1 \to_o^* N_2$.
    * Or, $R_1' = (d \circ c)^n(d(\lambda y.R_1''))$ and $P_2 = (\lambda y.R_2)\bar{x}$ such that $y \in \mathsf{Var}_{cd}$, $R_1'' \in \Lambda_{cd}^{\beta\eta}$, $n \geq 0$ and $R_1'' \to_o^* R_2$. By lemma 5.1.3, $\bar{x} \notin \mathrm{fv}(\lambda y.R_2)$, so $M_2 \to_\eta (\lambda y.R_2)Q_2 = N_2$ and $N_1 \to_o^* N_2$.
  – Or $R_1 = \lambda y.R_1'$ such that $y \in \mathsf{Var}_{cd}$ and $R_1' \in \Lambda_{cd}^{\beta\eta}$. Because $\bar{P}_1 \to_o^* P_2$ and $\bar{P}_1 = (\lambda y.R_1')\bar{x}$ then by lemma 5.1.3, $P_2 = (\lambda y.R_2')\bar{x}$ such that $R_1' \to_o^* R_2'$. By lemma 5.1.3, $\bar{x} \notin \mathrm{fv}(\lambda y.R_2')$, so $M_2 \to_\eta (\lambda y.R_2')Q_2 = N_2$ and $N_1 \to_o^* N_2$.
- Or, $N_1 = (\lambda\bar{x}.\bar{P}_1)Q_1'$ such that $\bar{Q}_1 \to_{\beta\eta} Q_1'$. By IH, there exist Q_2' such that $Q_2 \to_{\beta\eta} Q_2'$ and $Q_1' \to_o^* Q_2'$. So, $M_2 = (\lambda\bar{x}.P_2)Q_2 \to_{\beta\eta} (\lambda\bar{x}.P_2)Q_2' = N_2$ and $N_1 \to_o^* N_2$.

▼ Let $\bar{M}_1 = c\bar{P}_1\bar{Q}_1$ such that $\bar{P}_1, \bar{Q}_1 \in \Lambda_{cd}^{\beta\eta}$. By lemma 5.1.3:

- Either $\bar{P}_1 = (d \circ c)^n(d(\lambda x.P_0))$ and $M_2 = (\lambda x.P_2)Q_2$ such that $x \in \mathsf{Var}_{cd}$, $P_0 \in \Lambda_{cd}^{\beta\eta}$, $P_0 \to_o^* P_2$, $\bar{Q}_1 \to_o^* Q_2$ and $n \geq 0$. By compatibility:
  – Either, $N_1 = c((d \circ c)^n(d(\lambda y.P_0')))\bar{Q}_1$ such that $P_0 \to_{\beta\eta} P_0'$. By IH, there exists P_2' such that $P_2 \to_{\beta\eta} P_2'$ and $P_0' \to_o^* P_2'$. So, $M_2 = (\lambda x.P_2)Q_2 \to_{\beta\eta} (\lambda x.P_2')Q_2 = N_2$ and $N_1 \to_o^* N_2$.
  – Or, $N_1 = c((d \circ c)^n(d(\lambda y.P_0)))Q_1'$ such that $\bar{Q}_1 \to_{\beta\eta} Q_1'$. By IH, there exists Q_2' such that $Q_2 \to_{\beta\eta} Q_2'$ and $Q_1' \to_o^* Q_2'$. So, $M_2 = (\lambda x.P_2)Q_2 \to_{\beta\eta} (\lambda x.P_2)Q_2' = N_2$ and $N_1 \to_o^* N_2$.
  – Or, $P_0 = R_0x$ such that $x \notin \mathrm{fv}(R_0)$ and $N_1 = c((d \circ c)^n(dR_0))Q_1$. Since $P_0 \in \Lambda_{cd}^{\beta\eta}$:
    * Either $R_0 = cR_1$ such that $R_1 \in \Lambda_{cd}^{\beta\eta}$. Since $P_0 \to_o^* P_2$ and $P_0 = cR_1x$, then by lemma 5.1.3:
      · Either $P_2 = cR_2x$ such that $R_1 \to_o^* R_2$. By lemma 5.1.3, $x \notin \mathrm{fv}(R_2)$, so $M_2 \to_\eta cR_2Q_2 = N_2$ and $N_1 \to_o^* N_2$.

        · Or, $R_1 = (d \circ c)^m(d(\lambda y.R'_1))$ and $P_2 = (\lambda y.R_2)\bar{x}$ such that $y \in \mathsf{Var}_{cd}$, $m \geq 0$, $R'_1 \in \Lambda^{\beta\eta}_{cd}$ and $R'_1 \to^*_o R_2$. By lemma 5.1.3, $\bar{x} \notin \mathrm{fv}(\lambda y.R_2)$, so $M_2 \to_\eta (\lambda y.R_2)Q_2 = N_2$ and $N_1 \to^*_o N_2$.

      * Or $R_0 = \lambda y.R_1$ such that $y \in \mathsf{Var}_{cd}$ and $R_1 \in \Lambda^{\beta\eta}_{cd}$. Since $P_0 \to^*_o P_2$ and $P_0 = (\lambda y.R_1)\bar{x}$ then by lemma 5.1.3, $P_2 = (\lambda y.R_2)\bar{x}$ such that $R_1 \to^*_o R_2$. By lemma 5.1.3, $\bar{x} \notin \mathrm{fv}(\lambda y.R_2)$, so $M_2 \to_\eta (\lambda y.R_2)Q_2 = N_2$ and $N_1 \to^*_o N_2$.

- Or, $M_2 = cP_2Q_2$ such that $\bar{P}_1 \to^*_o P_2$ and $\bar{Q}_1 \to^*_o Q_2$. By compatibility:
  - Either, $N_1 = cP'_1\bar{Q}_1$ such that $\bar{P}_1 \to_{\beta\eta} P'_1$. By IH, there exists P'_2 such that $P_2 \to_{\beta\eta} P'_2$ and $P'_1 \to^*_o P'_2$. So, $M_2 = cP_2Q_2 \to_{\beta\eta} cP'_2Q_2 = N_2$ and $N_1 \to^*_o N_2$.
  - Or, $N_1 = c\bar{P}_1Q'_1$ such that $\bar{Q}_1 \to_{\beta\eta} Q'_1$. By IH, there exists Q'_2 such that $Q_2 \to_{\beta\eta} Q'_2$ and $Q'_1 \to^*_o Q'_2$. So, $M_2 = cP_2Q_2 \to_{\beta\eta} cP_2Q'_2 = N_2$ and $N_1 \to^*_o N_2$.

$\square$