A complete realisability semantics for intersection types and infinite expansion variables

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Abstract. *Expansion* was invented at the end of the 1970s for calculating *principal typings* for λ-terms in type systems with intersection types. *Expansion variables* (E-variables) were invented at the end of the 1990s to simplify and help mechanize expansion. Recently, E-variables have been further simplified and generalized to also allow calculating other type operators than just intersection. There has been much work on denotational semantics for type systems with intersection types, but only one such work on type systems with E-variables. That work established that building a semantics for E-variables is very challenging. Because it is unclear how to devise a space of meanings for E-variables, that work developed instead a space of meanings for types that is hierarchical in the sense of having many levels. When assigning meanings to types, it made each use of E-variables simply change levels. However, although the leveled calculus proposed there helped identify the serious problems of giving a semantics for expansion variables, the sound realisability semantics given there, was only complete when one single E-variable is used. Furthermore, that paper did not allow the universal type ω since otherwise, it would need to assign ω to every level, which is impossible. In this paper, we are able to overcome the challenges of both completeness in the presence of an infinite number of expansion variables, and of allowing a universal type. We develop a realizability semantics where we allow an infinite number of exapansion variables and where ω is present. We show the soundness and completeness of our proposed semantics.

1 Introduction

Expansion is a crucial part of a procedure for calculating *principal typings* and thus helps support compositional type inference. As a simple example, the λ-term $M = (\lambda x.x(\lambda y.yz))$ can be assigned the typing $\Phi_1 = \langle (z : a) \vdash (((a \to b) \to b) \to c) \to c \rangle$, which happens to be its principal typing. The term M can also be assigned the typing $\Phi_2 = \langle (z : a_1 \sqcap a_2) \vdash (((a_1 \to b_1) \to b_1) \sqcap ((a_2 \to b_2) \to b_2) \to c) \to c \rangle$, and an expansion operation can obtain Φ_2 from Φ_1. Because the early definitions of expansion were complicated [4], E-variables were introduced in order to make the calculations easier to mechanize and reason about. For example, in System E [2], the typing Φ_1 from above is replaced by $\Phi_3 = \langle (z : ea) \vdash (e((a \to b) \to b) \to c) \rangle$, which differs from Φ_1 by the insertion of the E-variable e at two places, and Φ_2 can be obtained from Φ_3 by substituting for e the *expansion term* $E = (a := a_1, b := b_1) \sqcap (a := a_2, b := b_2)$.

Carlier and Wells [3] have surveyed the history of expansion and also E-variables. [9] showed that E-variables pose serious challenges for semantics. Most commonly, a type's semantics is given as a set of closed λ-terms with behavior related to the specification given by the type. In many kinds of semantics, the meaning of a type T is calculated by an expression $[T]_\nu$ that takes two parameters, the type T and also a valuation ν that assigns to type variables the same kind of meanings that are assigned to types. To extend this idea to types with E-variables, we would need to devise some space of possible meanings for E-variables. Given that a type $e\,T$ can be turned by expansion into a new type $S_1(T) \sqcap S_2(T)$, where S_1 and S_2 are arbitrary substitutions (in fact, they can be arbitrary further expansions), and that this can introduce an unbounded number of new variables (both E-variables and regular type variables), the situation is complicated.

This was the main motivation for [9] to develop a space of meanings for types that is hierarchical in the sense of having many levels. When assigning meanings to types, [9] ensured that each use of E-variables simply changes levels and that each E-variable acts as a kind of capsule that isolates parts of the λ-term being analyzed by the typing. The leveled calculus of [9] captured accurately the intuition behind E-variables: parts of the λ-term that are typed inside the uses of the E-variable-introduction typing rule for a particular E-variable e can interact with each other, and parts outside e can only pass the parts inside e around.

The semantic approach used in [9] is realisability semantics [5]. Realisability allows showing *soundness* in the sense that the meaning of a type T contains all closed λ-terms that can be assigned T as their result type. This has been shown useful in previous work for characterizing the behavior of typed λ-terms [10]. One also wants to show the converse of soundness which is called *completeness* [6–8], i.e., that every closed λ-term in the meaning of T can be assigned T as its result type. [9] showed that if more than one E-variable is used, the semantics is not complete. Furthermore, the levels used in [9] made it difficult to allow the universal type ω and this limited the study to the λI-calculus. In this paper, we are able to overcome the challenges of both completeness in the presence of an infinite number of expansion variables, and of allowing a universal type. We develop a realizability semantics where we allow the full λ-calculus, an infinite number of exapansion variables and where ω is present, and we show its soundness and completeness. We do so by introducing an indexed calculus as in [9]. However here, our indices are finite sequences of integers rather than single integers.

In Section 2 we give the full λ-calculus indexed with finite sequences of integers and show the confluence of β, $\beta\eta$ and weak head reduction on the indexed λ-calculus. In Section 3 we introduce two type systems for the indexed λ-calculus (both with the universal type ω). In the first, there are no restrictions on where the arrow occurs. In the second, arrows cannot occur to the left of intersections or expansions. In Section 4 we establish that subject reduction holds for $\vdash_2$ but fails for $\vdash_1$. In Section 5 we show that subject β-expansion holds for both $\vdash_1$ and $\vdash_2$ but that subject η-expansion fails. In Section 6 we introduce the realisability semantics and show its soundness for both $\vdash_1$ and $\vdash_2$. We also show that completeness fails for $\vdash_1$. In Section 7 we establish the completeness of $\vdash_2$ by introducing a special interpretation. We conclude in Section 8. The full article with proofs is available at: `http://www.macs.hw.ac.uk/~fairouz/papers/drafts/compsem-big.pdf`.

2 The pure $\lambda^{\mathcal{L}_{\mathbb{N}}}$-calculus

In this section we give the λ-calculus indexed with finite sequences of integers and show the confluence of β, $\beta\eta$ and weak head reduction on the indexed λ-calculus.

The next definition introduces the labels of the calculus.

Definition 1. *1. A label is a finite sequence of integers $L = (n_i)_{1 \leq i \leq l}$. We denote $\mathcal{L}_{\mathbb{N}}$ the set of labels and $\oslash$ the empty sequence of integers.*

2. If $L = (n_i)_{1 \leq i \leq l}$ and $m \in \mathbb{N}$, we use $m :: L$ to denote the sequence $(r_i)_{1 \leq i \leq l+1}$ where $r_1 = m$ and $\forall\ 2 \leq i \leq l+1$, $r_i = n_{i-1}$. In particular, $k :: \oslash = (k)$.

3. *If $L = (n_i)_{1 \leq i \leq n}$ and $K = (m_i)_{1 \leq i \leq m}$, we use $L :: K$ to denote the sequence $(r_i)_{1 \leq i \leq n+m}$ where $\forall\ 1 \leq i \leq n$, $r_i = n_i$ and $\forall\ n+1 \leq i \leq n+m$, $r_i = m_{i-n}$. In particular, $L :: \oslash = \oslash :: L = L$.*
4. *We define on $\mathcal{L}_{\mathbb{N}}$ a binary relation $\preceq$ by:*
   $\forall\ L_1, L_2 \in \mathcal{L}_{\mathbb{N}}$, $L_1 \preceq L_2$ (or $L_2 \succeq L_1$) if $\exists\ L_3 \in \mathcal{L}_{\mathbb{N}}$ such that $L_2 = L_1 :: L_3$.

Lemma 2. *$\preceq$ is an order relation on $\mathcal{L}_{\mathbb{N}}$.*

The next definition gives the syntax of the calculus and the notions of reduction.

Definition 3. *1. Let $\mathcal{V}$ be a denumerably infinite set of variables. The set of terms $\mathcal{M}$, the set of free variables $FV(M)$ of a term $M \in \mathcal{M}$, the degree function $d : \mathcal{M} \to \mathcal{L}_{\mathbb{N}}$ and the joinability $M \diamond N$ of terms M and N are defined by simultaneous induction as follows:*
   - *If $x \in \mathcal{V}$ and $L \in \mathcal{L}_{\mathbb{N}}$, then $x^L \in \mathcal{M}$, $FV(x^L) = \{x^L\}$ and $d(x^L) = L$.*
   - *If $M, N \in \mathcal{M}$, $d(M) \preceq d(N)$ and $M \diamond N$ (see below), then $M\ N \in \mathcal{M}$, $FV(MN) = FV(M) \cup FV(N)$ and $d(M\ N) = d(M)$.*
   - *If $x \in \mathcal{V}$, $M \in \mathcal{M}$ and $L \succeq d(M)$, then $\lambda x^L.M \in \mathcal{M}$, $FV(\lambda x^L.M) = FV(M) \setminus \{x^L\}$ and $d(\lambda x^L.M) = d(M)$.*

2.
   - *Let $M, N \in \mathcal{M}$. We say that M and N are joinable and write $M \diamond N$ iff $\forall x \in \mathcal{V}$, if $x^L \in FV(M)$ and $x^K \in FV(N)$, then $L = K$.*
   - *If $\mathcal{X} \subseteq \mathcal{M}$ such that $\forall M, N \in \mathcal{X}, M \diamond N$, we write, $\diamond\mathcal{X}$.*
   - *If $\mathcal{X} \subseteq \mathcal{M}$ and $M \in \mathcal{M}$ such that $\forall N \in \mathcal{X}, M \diamond N$, we write, $M \diamond \mathcal{X}$.*

   The $\diamond$ property ensures that in any term M, variables have unique degrees.
   We assume the usual definition ([1, 10]) of subterms and the usual convention for parentheses and their omittance. Note that every subterm of $M \in \mathcal{M}$ is also in $\mathcal{M}$. We let x, y, z, etc. range over $\mathcal{V}$ and $M, N, P, M_1, M_2, \ldots$ range over $\mathcal{M}$ and use $=$ for syntactic equality.
3. *The usual substitution $M[x^L := N]$ of $N \in \mathcal{M}$ for all free occurrences of x^L in $M \in \mathcal{M}$ only matters when $d(N) = L$. Similarly, $M[(x_i^{L_i} := N_i)_n]$, the simultaneous substitution of N_i for all free occurrences of $x_i^{L_i}$ in M only matters when $\forall\ 1 \leq i \leq n$, $d(N_i) = L_i$.*
4. *We take terms modulo α-conversion given by:*
   $\lambda x^L.M = \lambda y^L.(M[x^L := y^L])$ where $y^L \notin Fv(M)$.
   Moreover, we use the Barendregt convention (BC) where the names of bound variables differ from the free ones and where we rewrite terms so that not both λx^L and λx^K co-occur when $L \neq K$.
5. *A relation R on $\mathcal{M}$ is compatible iff for all $M, N, P \in \mathcal{M}$:*
   - *If MRN and $\lambda x^n.M, \lambda x^n.M \in \mathcal{M}$ then $(\lambda x^n.M)R(\lambda x^n.N)$.*
   - *If MRN and $MP, NP \in \mathcal{M}$ (resp. $PM, PN \in \mathcal{M}$), then $(MP)R(NP)$ (resp. $(PM)R(PN)$).*
6. *The reduction relation $\rhd_\beta$ on $\mathcal{M}$ is defined as the least compatible relation closed under the rule: $(\lambda x^L.M)N \rhd_\beta M[x^L := N]$ if $d(N) = L$*
7. *The reduction relation $\rhd_\eta$ on $\mathcal{M}$ is defined as the least compatible relation closed under the rule: $\lambda x^L.(M\ x^L) \rhd_\eta M$ if $x^L \notin FV(M)$*
8. *The weak head reduction $\rhd_h$ on $\mathcal{M}$ is defined by:*
   $(\lambda x^L.M)NN_1 \ldots N_n \rhd_h M[x^L := N]N_1 \ldots N_n$ where $n \geq 0$
9. *We let $\rhd_{\beta\eta} = \rhd_\beta \cup \rhd_\eta$. For $r \in \{\beta, \eta, h, \beta\eta\}$, we denote by $\rhd_r^*$ the reflexive and transitive closure of $\rhd_r$ and by $\simeq_r$ the equivalence relation induced by $\rhd_r^*$.*

Theorem 4. *Let $M \in \mathcal{M}$ and $r \in \{\beta, \beta\eta, h\}$.*

1. *If $M \rhd_\eta^* N$, then $N \in \mathcal{M}$, $FV(N) = FV(M)$ and $d(M) = d(N)$.*
2. *If $M \rhd_r^* N$, then $N \in \mathcal{M}$, $FV(N) \subseteq FV(M)$ and $d(M) = d(N)$.*

As expansions change the level of a term, labels in a term need to increase/decrease.

Definition 5. *Let $i \in \mathbb{N}$ and $M \in \mathcal{M}$.*

1. *We define M^{+i} by:*
   $\bullet (x^L)^{+i} = x^{i::L}$ $\quad \bullet (M_1\ M_2)^{+i} = M_1^{+i}\ M_2^{+i}$ $\quad \bullet (\lambda x^L.M)^{+i} = \lambda x^{i::L}.M^{+i}$
2. *If $d(M) = i :: L$, we define M^{-i} by:*
   $\bullet (x^{i::K})^{-i} = x^K$ $\quad \bullet (M_1\ M_2)^{-i} = M_1^{-i}\ M_2^{-i}$ $\quad \bullet (\lambda x^{i::K}.M)^{-i} = \lambda x^K.M^{-i}$

Normal forms are defined as usual.

Definition 6. 1. *$M \in \mathcal{M}$ is in β-normal form ($\beta\eta$-normal form, h-normal form resp.) if there is no $N \in \mathcal{M}$ such that $M \rhd_\beta N$ ($M \rhd_{\beta\eta} N$, $M \rhd_h N$ resp.).*
2. *$M \in \mathcal{M}$ is β-normalising ($\beta\eta$-normalising, h-normalising resp.) if there is an $N \in \mathcal{M}$ such that $M \rhd_\beta^* N$ ($M \rhd_{\beta\eta} N$, $M \rhd_h N$ resp.) and N is in β-normal form ($\beta\eta$-normal form, h-normal form resp.).*

Theorem 7 (Confluence). *Let $M, M_1, M_2 \in \mathcal{M}$ and $r \in \{\beta, \beta\eta, h\}$.*

1. *If $M \rhd_r^* M_1$ and $M \rhd_r^* M_2$, then there is M' such that $M_1 \rhd_r^* M'$ and $M_2 \rhd_r^* M'$.*
2. *$M_1 \simeq_r M_2$ iff there is a term M such that $M_1 \rhd_r^* M$ and $M_2 \rhd_r^* M$.*

3 Typing system

This paper studies two type systems for the indexed λ-calculus (both with the universal type ω). In the first, there are no restrictions on where the arrow occurs. In the second, arrows cannot occur to the left of intersections or expansions. We show that within a typing, degrees are well behaved.

The next two definitions introduce the two type systems.

Definition 8. 1. *Let $\mathcal{A}$ be a denumerably infinite set of atomic types and $\mathcal{E} = \{e_0, e_1, ...\}$ a denumerably infinite set of expansion variables. We define the sets of types $\mathbb{T}$, $\mathbb{U}$ and $\mathcal{T}$, such that $\mathbb{T} \subseteq \mathbb{U} \subseteq \mathcal{T}$, and the function $d : \mathcal{T} \to \mathcal{L}_\mathbb{N}$ by:*
   - *If $a \in \mathcal{A}$, then $a \in \mathbb{T}$ and $d(a) = \oslash$.*
   - *If $U \in \mathbb{U}$ and $T \in \mathbb{T}$, then $U \to T \in \mathbb{T}$ and $d(U \to T) = \oslash$.*
   - *If $U, T \in \mathcal{T}$ and $d(U) \succeq d(T)$, then $U \to T \in \mathcal{T}$ and $d(U \to T) = d(T)$.*
   - *If $L \in \mathcal{L}_\mathbb{N}$, then $\omega^L \in \mathbb{U}$ and $d(\omega^L) = L$.*
   - *If $U_1, U_2 \in \mathbb{U}$ and $d(U_1) = d(U_2)$, then $U_1 \sqcap U_2 \in \mathbb{U}$ and $d(U_1 \sqcap U_2) = d(U_1) = d(U_2)$.*
   - *$U \in \mathbb{U}$ and $e_i \in \mathcal{E}$, then $e_iU \in \mathbb{U}$ and $d(e_iU) = i :: d(U)$.*

   We let $T, T_1, T_2, T', \ldots$ range over $\mathbb{T}$, $U, V, W, U_1, V_1, U', \ldots$ range over $\mathbb{U}$ and $T, T_1, T_2, T', U, V, W, U_1, V_1, U', \ldots$ range over $\mathcal{T}$. We quotient types by taking $\sqcap$ to be commutative (i.e. $U_1 \sqcap U_2 = U_2 \sqcap U_1$), associative (i.e. $U_1 \sqcap (U_2 \sqcap U_3) = (U_1 \sqcap U_2) \sqcap U_3$), idempotent (i.e. $U \sqcap U = U$) distributivity of expansion variables to $\sqcap$ (i.e. $e_i(U_1 \sqcap U_2) = e_iU_1 \sqcap e_iU_2$) and to have ω^L neutral (i.e. $\omega^L \sqcap U = U$). We denote $U_n \sqcap U_{n+1} \ldots \sqcap U_m$ by $\sqcap_{i=n}^m U_i$ (when $n \leq m$). We also assume, $\forall i \geq 0, \forall K \in \mathcal{L}_\mathbb{N}, e_i\omega^K = \omega^{i::K}$.
2. *We denote $e_{i_1} \ldots e_{i_n}$ by $\boldsymbol{e}_K$, where $K = (i_1, \ldots, i_n)$ and $U_n \sqcap U_{n+1} \ldots \sqcap U_m$ by $\sqcap_{i=n}^m U_i$ (when $n \leq m$).*

Definition 9. 1. *A type environment is a set $\{x_i^{L_i} : U_i \ /\ 1 \leq i \leq n$ where $n \geq 0$, $d(U_i) = L_i$ and $\forall 1 \leq i, j \leq n$, if $i \neq j$ then $x_i^{L_i} \neq x_j^{L_j}\}$. We denote such environment (call it Γ) by $x_1^{L_1} : U_1, \ldots, x_n^{L_n} : U_n$ or simply by $(x_i^{L_i} : U_i)_n$ and define $dom(\Gamma) = \{x_i^{L_i} \ /\ 1 \leq i \leq l\}$. We use $\Gamma, \Delta, \Gamma_1, \ldots$ to range over environments and write $()$ for the empty environment.*
2. *If $M \in \mathcal{M}$ and $FV(M) = \{x_1^{L_1}, \ldots, x_n^{L_n}\}$, we denote env_M^ω the type environment $(x_i^{L_i} : \omega^{L_i})_n$.*

$$\frac{}{x^{\mathrm{d}(T)} : \langle (x^{\mathrm{d}(T)} : T) \vdash_1 T\rangle}\ (ax)$$

$$\frac{}{x^{\oslash} : \langle (x^{\oslash} : T) \vdash_2 T\rangle}\ (ax)$$

$$\frac{}{M : \langle env^{\omega}_M \vdash_i \omega^{\mathrm{d}(M)}\rangle}\ (\omega)$$

$$\frac{M : \langle \Gamma, (x^L : U) \vdash_i T\rangle}{\lambda x^L.M : \langle \Gamma \vdash_i U \to T\rangle}\ (\to_I)$$

$$\frac{M : \langle \Gamma \vdash_i T\rangle \quad x^L \notin dom(\Gamma)}{\lambda x^L.M : \langle \Gamma \vdash_i \omega^L \to T\rangle}\ (\to'_I)$$

$$\frac{M_1 : \langle \Gamma_1 \vdash_i U \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash_i U\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_i T\rangle}\ (\to_E)$$

$$\frac{M : \langle \Gamma_1 \vdash_1 U_1\rangle \qquad M : \langle \Gamma_2 \vdash_1 U_2\rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_1 U_1 \sqcap U_2\rangle}\ (\sqcap_I)$$

$$\frac{M : \langle \Gamma \vdash_2 U_1\rangle \qquad M : \langle \Gamma \vdash_2 U_2\rangle}{M : \langle \Gamma \vdash_2 U_1 \sqcap U_2\rangle}\ (\sqcap_I)$$

$$\frac{M : \langle \Gamma \vdash_i U\rangle}{M^{+j} : \langle e_j\Gamma \vdash_i e_j U\rangle}\ (e)$$

$$\frac{M : \langle \Gamma \vdash_2 U\rangle \qquad \langle \Gamma \vdash_2 U\rangle \sqsubseteq \langle \Gamma' \vdash_2 U'\rangle}{M : \langle \Gamma' \vdash_2 U'\rangle}\ (\sqsubseteq)$$

$$\frac{}{\Phi \sqsubseteq \Phi}\ (ref)$$

$$\frac{\Phi_1 \sqsubseteq \Phi_2 \quad \Phi_2 \sqsubseteq \Phi_3}{\Phi_1 \sqsubseteq \Phi_3}\ (tr)$$

$$\frac{}{U_1 \sqcap U_2 \sqsubseteq U_1}\ (\sqcap_E)$$

$$\frac{U_1 \sqsubseteq V_1 \quad U_2 \sqsubseteq V_2}{U_1 \sqcap U_2 \sqsubseteq V_1 \sqcap V_2}\ (\sqcap)$$

$$\frac{U_2 \sqsubseteq U_1 \quad T_1 \sqsubseteq T_2}{U_1 \to T_1 \sqsubseteq U_2 \to T_2}\ (\to)$$

$$\frac{U_1 \sqsubseteq U_2}{eU_1 \sqsubseteq eU_2}\ (\sqsubseteq_e)$$

$$\frac{U_1 \sqsubseteq U_2}{\Gamma, y^L : U_1 \sqsubseteq \Gamma, y^L : U_2}\ (\sqsubseteq_c)$$

$$\frac{U_1 \sqsubseteq U_2 \quad \Gamma_2 \sqsubseteq \Gamma_1}{\langle \Gamma_1 \vdash_2 U_1\rangle \sqsubseteq \langle \Gamma_2 \vdash_2 U_2\rangle}\ (\sqsubseteq_{\langle\rangle})$$

Fig. 1. Typing rules / Subtyping rules

3. *Let $\Gamma_1 = (x_i^{L_i} : U_i)_n, (y_i^{K_i} : V_i)_m$ and $\Gamma_2 = (x_i^{L_i} : U'_i)_n, (z_i^{R_i} : W_i)_k$. We denote $\Gamma_1 \sqcap \Gamma_2$ the type environment $(x_i^{L_i} : U_i \sqcap U'_i)_n, (y_i^{K_i} : V_i)_m, (z_i^{R_i} : W_i)_k$. Note that $dom(\Gamma_1 \sqcap \Gamma_2) = dom(\Gamma_1) \cup dom(\Gamma_2)$ and that, on environments, $\sqcap$ is commutative, associative and idempotent.*
4. *If $\Gamma = (x_i^{L_i} : U_i)_n$, $x^L \notin dom(\Gamma)$ and $U \in \mathbb{U}$ such that $d(U) = L$, then we denote $\Gamma, (x^L : U)$ the type environment $x_1^{L_1} : U_1, \dots, x_n^{L_n} : U_n, x^L : U$.*
5. *Let $\Gamma = (x_i^{L_i} : U_i)_{1 \le i \le n}$ and $e_j \in \mathcal{E}$. We denote $e_j\Gamma = (x_i^{j::L_i} : e_j U_i)_{1 \le i \le n}$. Note that $e_j(\Gamma_1 \sqcap \Gamma_2) = e_j\Gamma_1 \sqcap e_j\Gamma_2$.*
6. *We write $\Gamma_1 \diamond \Gamma_2$ if and only if $x^L \in dom(\Gamma_1)$ and $x^K \in dom(\Gamma_2)$ implies $K = L$.*
7. *The type system $\vdash_1$ (resp. $\vdash_2$) uses the set $\mathcal{T}$ (resp. $\mathbb{U}$). We follow [3] and write type judgements as $M : \langle \Gamma \vdash U\rangle$ instead of the traditional format of $\Gamma \vdash M : U$. For $i \in \{1,2\}$, the typing rules of $\vdash_i$ are (recall that when used for $\vdash_1$, U and T range over $\mathcal{T}$, and when used for $\vdash_2$, U ranges over $\mathbb{U}$ and T ranges over $\mathbb{T}$) given on the lefthand side of figure 7. In the last clause, the binary relation $\sqsubseteq$ is defined on U by the rules on the righthand side of figure 7. We let Φ denote types in $\mathbb{U}$, or environments Γ or typings $\langle \Gamma \vdash_2 U\rangle$. When $\Phi \sqsubseteq \Phi'$, then Φ and Φ' belong to the same set ($\mathbb{U}$/environments/typings).*
8. *If $L \in \mathcal{L}_{\mathbb{N}}$, $U \in \mathbb{U}$ and $\Gamma = (x_i^{L_i} : U_i)_n$ is a type environment, we say that:*
   - *$d(\Gamma) \succeq L$ if and only if $\forall\ 1 \le i \le n$, $d(U_i) = L_i \succeq L$.*
   - *$d(\langle \Gamma \vdash_i U\rangle) \succeq L$ if and only if $d(\Gamma) \succeq L$ and $d(U) \succeq L$.*

As we did for terms, we decrease the labels of types, environments and typings.

Definition 10. *1. If $d(U) \succeq L$, then if $L = \oslash$ then $U^{-L} = U$ else $L = i :: K$ and we inductively define the type U^{-L} as follows:*

$(U_1 \sqcap U_2)^{-i::K} = U_1^{-i::K} \sqcap U_2^{-i::K}$ $\qquad\qquad$ $(e_i U)^{-i::K} = U^{-K}$

If $d(U) \succeq L = (n_i)_k$ with $k \geq 1$, we write U^{-L} for $(\ldots(U^{-n_1})^{-n_2}\ldots)^{-n_k}$.
We write U^{-i} instead of $U^{-(i)}$.

2. *If $\Gamma = (x_i^{L_i} : U_i)_k$ and $d(\Gamma) \succeq L$, then $\forall 1 \leq i \leq k$, $L_i = L :: L'_i$ and we denote $\Gamma^{-L} = (x^{L'_i} : U_i^{-L})_k$.*
If $d(\Gamma) \succeq L = (n_i)_k$ with $k \geq 1$, we write Γ^{-L} for $(\ldots(\Gamma^{-n_1})^{-n_2}\ldots)^{-n_k}$.
We write Γ^{-i} instead of $\Gamma^{-(i)}$.
3. *If U is a type and Γ is a type environment such that $d(\Gamma) \succeq K$ and $d(U) \succeq K$, then we denote $(\langle \Gamma \vdash_i U \rangle)^{-K} = \langle \Gamma^{-K} \vdash_i U^{-K} \rangle$.*

The next lemma is informative about types and their degrees.

Lemma 11. *1. If $T \in \mathbb{T}$, then $d(T) = \oslash$.*

2. *Let $U \in \mathbb{U}$. If $d(U) = L = (n_i)_m$, then $U = \omega^L$ or $U = \mathbf{e}_L \sqcap_{i=1}^p T_i$ where $p \geq 1$ and $\forall\ 1 \leq i \leq p$, $T_i \in \mathbb{T}$.*
3. *Let $U_1 \sqsubseteq U_2$.*
   (a) *$d(U_1) = d(U_2)$.*
   (b) *If $U_1 = \omega^K$ then $U_2 = \omega^K$.*
   (c) *If $U_1 = \mathbf{e}_K U$ then $U_2 = \mathbf{e}_K U'$ and $U \sqsubseteq U'$.*
   (d) *If $U_2 = \mathbf{e}_K U$ then $U_1 = \mathbf{e}_K U'$ and $U \sqsubseteq U'$.*
   (e) *If $U_1 = \sqcap_{i=1}^p \mathbf{e}_K(U_i \to T_i)$ where $p \geq 1$ then $U_2 = \omega^K$ or $U_2 = \sqcap_{j=1}^q \mathbf{e}_K(U'_j \to T'_j)$ where $q \geq 1$ and $\forall 1 \leq j \leq q$, $\exists 1 \leq i \leq p$ such that $U'_j \sqsubseteq U_i$ and $T_i \sqsubseteq T'_j$.*
4. *$\forall U \in \mathbb{U}$ such that $d(U) = L$, $U \sqsubseteq \omega^L$.*
5. *If $U \sqsubseteq U'_1 \sqcap U'_2$ then $U = U_1 \sqcap U_2$ where $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.*
6. *If $\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2$ then $\Gamma = \Gamma_1 \sqcap \Gamma_2$ where $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$.*

The next lemma says how ordering and level decreasing propagate to environments.

Lemma 12. *1. If $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $x^L \notin dom(\Gamma)$ then $\Gamma, (x^L : U) \sqsubseteq \Gamma', (x^L : U')$.*

2. *$\Gamma \sqsubseteq \Gamma'$ iff $\Gamma = (x_i^{L_i} : U_i)_n$, $\Gamma' = (x_i^{L_i} : U'_i)_n$ and for every $1 \leq i \leq n$, $U_i \sqsubseteq U'_i$.*
3. *$\langle \Gamma \vdash_2 U \rangle \sqsubseteq \langle \Gamma' \vdash_2 U' \rangle$ iff $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$.*
4. *If $dom(\Gamma) = FV(M)$, then $\Gamma \sqsubseteq env_M^\omega$*
5. *If $\Gamma \diamond \Delta$ and $d(\Gamma), d(\Delta) \succeq K$, then $\Gamma^{-K} \diamond \Delta^{-K}$.*
6. *If $U \sqsubseteq U'$ and $d(U) \succeq K$ then $U^{-K} \sqsubseteq U'^{-K}$.*
7. *If $\Gamma \sqsubseteq \Gamma'$ and $d(\Gamma) \succeq K$ then $\Gamma^{-K} \sqsubseteq \Gamma'^{-K}$.*

The next lemma shows that we do not allow weakening in $\vdash_2$.

Lemma 13. *1. For every Γ and M such that $dom(\Gamma) = FV(M)$ and $d(M) = K$, we have $M : \langle \Gamma \vdash_2 \omega^K \rangle$.*

2. *Let $i \in \{1, 2\}$. If $M : \langle \Gamma \vdash_i U \rangle$, then $dom(\Gamma) = FV(M)$.*
3. *Let $i \in \{1, 2\}$, $M_1 : \langle \Gamma_1 \vdash_i U \rangle$ and $M_2 : \langle \Gamma_2 \vdash_i U \rangle$. $\Gamma_1 \diamond \Gamma_2$ if and only if $M_1 \diamond M_2$.*

Proof 1. By ω, $M : \langle env_M^\omega \vdash_2 \omega^K \rangle$. By lemma 12.4, $\Gamma \sqsubseteq env_M^\omega$. Hence, by $\sqsubseteq$ and $\sqsubseteq_{\langle\rangle}$, $M : \langle \Gamma \vdash_2 \omega^K \rangle$.
2. By induction on the derivation $M : \langle \Gamma \vdash_i U \rangle$.
3. If) Let $x^L \in dom(\Gamma_1)$ and $x^K \in dom(\Gamma_2)$ then by lemma 13.2, $x^L \in FV(M_1)$ and $x^K \in FV(M_2)$ so $\Gamma_1 \diamond \Gamma_2$. Only if) Let $x^L \in FV(M_1)$ and $x^K \in FV(M_2)$ then by lemma 13.2, $x^L \in dom(\Gamma_1)$ and $x^K \in dom(\Gamma_2)$ so $M_1 \diamond M_2$. $\square$

The next theorem states that within a typing, degrees are well behaved.

Theorem 14. *Let $i \in \{1, 2\}$ and $M : \langle \Gamma \vdash_i U \rangle$.*

1. *$d(\Gamma) \succeq d(U) = d(M)$.*
2. *If $i = 2$ and $d(U) \succeq K$ then $M^{-K} : \langle \Gamma^{-K} \vdash_2 U^{-K} \rangle$.*

Finally, here are two derivable typing rules.

Remark 15. 1. The rule $\dfrac{M : \langle \Gamma_1 \vdash_2 U_1 \rangle \qquad M : \langle \Gamma_2 \vdash_2 U_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 U_1 \sqcap U_2 \rangle}\ \sqcap'_I$ is derivable.

2. The rule $\dfrac{}{x^{d(U)} : \langle (x^{d(U)} : U) \vdash_2 U \rangle}\ ax'$ is derivable.

4 Subject reduction properties

In this section we show that subject reduction holds for $\vdash_2$ but fails for $\vdash_1$. The proof of subject reduction uses generation and substitution (hence the next lemmas).

Lemma 16 (Generation for $\vdash_1$).

1. *If $x^L : \langle \Gamma \vdash_1 T\rangle$, then $\Gamma = (x^L : T)$.*
2. *If $x^L \in FV(M)$ and $\lambda x^L.M : \langle \Gamma \vdash_1 T\rangle$, then $\exists K \succeq \oslash, T = \omega^K$ or $\exists n \geq 1, T = \sqcap_{i=1}^n (T_i \to T'_i)$ and $\forall 1 \leq i \leq n, M : \langle \Gamma, x^L : \sqcap_{i=1}^n T_i \vdash_1 \sqcap_{i=1}^n T'_i\rangle$.*
3. *If $x^L \notin FV(M)$ and $\lambda x^L.M : \langle \Gamma \vdash_1 T\rangle$, then $\exists K \succeq \oslash, T = \omega^K$ or $\exists n \geq 1, T = \sqcap_{i=1}^n (T_i \to T'_i)$ and $M : \langle \Gamma \vdash_1 \sqcap_{i=1}^n T'_i\rangle$.*
4. *If $MN : \langle \Gamma \vdash_1 T\rangle$ then $\Gamma = \Gamma_1 \sqcap \Gamma_2$, $T = \sqcap_{i=1}^n \boldsymbol{e}_{K_i} T_i$, $n \geq 1, \forall 1 \leq i \leq n, K_i \succeq \oslash$, $M : \langle \Gamma_1 \vdash_1 \sqcap_{i=1}^n \boldsymbol{e}_{K_i}(T'_i \to T_i)\rangle$ and $N : \langle \Gamma_2 \vdash_1 \sqcap_{i=1}^n \boldsymbol{e}_{K_i} T'_i\rangle$.*

Lemma 17 (Generation for $\vdash_2$).

1. *If $x^L : \langle \Gamma \vdash_2 U\rangle$, then $\Gamma = (x^L : V)$ and $V \sqsubseteq U$.*
2. *If $\lambda x^L.M : \langle \Gamma \vdash_2 U\rangle$, $x^L \in FV(M)$ and $d(U) = K$, then $U = \omega^K$ or $U = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $M : \langle \Gamma, x^L : \boldsymbol{e}_K V_i \vdash_2 \boldsymbol{e}_K T_i\rangle$.*
3. *If $\lambda x^L.M : \langle \Gamma \vdash_2 U\rangle$, $x^L \notin FV(M)$ and $d(U) = K$, then $U = \omega^K$ or $U = \sqcap_{i=1}^p \boldsymbol{e}_K(V_i \to T_i)$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $M : \langle \Gamma \vdash_2 \boldsymbol{e}_K T_i\rangle$.*
4. *If $M\ x^L : \langle \Gamma, (x^L : U) \vdash_2 T\rangle$ and $x^L \notin FV(M)$, then $M : \langle \Gamma \vdash_2 U \to T\rangle$.*

Lemma 18 (Substitution for $\vdash_2$). *If $M : \langle \Gamma, x^L : U \vdash_2 V\rangle$, $N : \langle \Delta \vdash_2 U\rangle$ and $\Gamma \diamond \Delta$ then $M[x^L := N] : \langle \Gamma \sqcap \Delta \vdash_2 V\rangle$.*

Since $\vdash_2$ does not allow weakening, we need the next definition since when a term is reduced, it may lose some of its free variables and hence will need to be typed in a smaller environment.

Definition 19. *If Γ is a type environment and $\mathcal{U} \subseteq dom(\Gamma)$, then we write $\Gamma \upharpoonright_{\mathcal{U}}$ for the restriction of Γ on the variables of $\mathcal{U}$. If $\mathcal{U} = FV(M)$ for a term M, we write $\Gamma \upharpoonright_M$ instead of $\Gamma \upharpoonright_{FV(M)}$.*

Theorem 20 (Subject reduction for $\vdash_2$). *If $M : \langle \Gamma \vdash_2 U\rangle$ and $M \rhd^*_{\beta\eta} N$, then $N : \langle \Gamma \upharpoonright_N \vdash_2 U\rangle$.*

Corollary 21. *1. If $M : \langle \Gamma \vdash_2 U\rangle$ and $M \rhd^*_{\beta} N$, then $N : \langle \Gamma \upharpoonright_N \vdash_2 U\rangle$.*
2. If $M : \langle \Gamma \vdash_2 U\rangle$ and $M \rhd^*_{h} N$, then $N : \langle \Gamma \upharpoonright_N \vdash_2 U\rangle$.

Lemma 22 (Subject h-reduction fails for $\vdash_1$). *Let a, b, c be different elements of $\mathcal{A}$. We have:*

1. $(\lambda x^\oslash.x^\oslash x^\oslash)(y^\oslash z^\oslash) \rhd_h (y^\oslash z^\oslash)(y^\oslash z^\oslash)$
2. $(\lambda x^\oslash.x^\oslash x^\oslash)(y^\oslash z^\oslash) : \langle y^\oslash : b \to ((a \to c) \sqcap a), z^\oslash : b \vdash_1 c\rangle$.
3. *It is not possible that* $(y^\oslash z^\oslash)(y^\oslash z^\oslash) : \langle y^\oslash : b \to ((a \to c) \sqcap a), z^\oslash : b \vdash_1 c\rangle$.
   Hence, the substitution and subject h-reduction lemmas fail for $\vdash_1$.

Proof 1. and 2. are easy. For 3., assume $(y^\oslash z^\oslash)(y^\oslash z^\oslash) : \langle y^\oslash : b \to ((a \to c) \sqcap a), z^\oslash : b \vdash_1 c\rangle$. By lemma 16.4 twice using lemmas 13, 14 and 16.1:

- $y^\oslash z^\oslash : \langle y^\oslash : b \to ((a \to c) \sqcap a), z^\oslash : b \vdash_1 \sqcap_{i=1}^n (T_i \to c)\rangle$.
- $y^\oslash : \langle y^\oslash : b \to ((a \to c) \sqcap a) \vdash_1 b \to (a \to c) \sqcap a\rangle$.
- $z^\oslash : \langle z^\oslash : b \vdash_1 b\rangle$.
- $\sqcap_{i=1}^n (T_i \to c) = (a \to c) \sqcap a$.

Hence $a = T_i \to c$ for some T_i. Absurd. $\square$

5 Subject expansion properties

In this section we show that subject β-expansion holds for both $\vdash_1$ and $\vdash_2$ but that subject η-expansion fails. The next lemma is needed for expansion.

Lemma 23. *If $M[x^L := N] : \langle \Gamma \vdash_i U \rangle$, $d(N) = L$ and $x^L \in FV(M)$ then $\exists$ V type such that $d(V) = L$ and $\exists$ Γ_1, Γ_2 type environments such that:*
$M : \langle \Gamma_1, x^L : V \vdash_i U \rangle \qquad N : \langle \Gamma_2 \vdash_i V \rangle \qquad \Gamma = \Gamma_1 \sqcap \Gamma_2$

Since more free variables might appear in the β-expansion of a term, the next definition gives a possible enlargement of an environment.

Definition 24. *Let $m \geq n$, $\Gamma = (x_i^{L_i} : U_i)_n$ and $\mathcal{U} = \{x_1^{L_1}, ..., x_m^{L_m}\}$. We write $\Gamma\uparrow^{\mathcal{U}}$ for $x_1^{L_1} : U_1, ..., x_n^{L_n} : U_n, x_{n+1}^{L_{n+1}} : \omega^{L_{n+1}}, ..., x_m^{L_m} : \omega^{L_m}$. If $dom(\Gamma) \subseteq FV(M)$, we write $\Gamma\uparrow^M$ instead of $\Gamma\uparrow^{FV(M)}$.*

Theorem 25 (Subject expansion for β). *If $N : \langle \Gamma \vdash_i U \rangle$ and $M \rhd_\beta^* N$, then $M : \langle \Gamma\uparrow^M \vdash_i U \rangle$.*

Corollary 26. *If $N : \langle \Gamma \vdash_i U \rangle$ and $M \rhd_h^* N$, then $M : \langle \Gamma\uparrow^M \vdash_i U \rangle$.*

Lemma 27 (Subject η-expansion fails for $\vdash_1$ and $\vdash_2$). *Let a be an element of $\mathcal{A}$. We have:*

1. $\lambda y^\oslash.\lambda x^\oslash.y^\oslash x^\oslash \rhd_\eta \lambda y^\oslash.y^\oslash$
2. $\lambda y^\oslash.y^\oslash : \langle () \vdash_i a \to a \rangle$.
3. *It is not possible that* $\lambda y^\oslash.\lambda x^\oslash.y^\oslash x^\oslash : \langle () \vdash_i a \to a \rangle$.
   Hence, the subject η-expansion lemmas fail for $\vdash_1$ and $\vdash_2$.

Proof 1. and 2. are easy. For 3., assume $\lambda y^\oslash.\lambda x^\oslash.y^\oslash x^\oslash : \langle () \vdash_i a \to a \rangle$.
- Let $i = 1$. By lemma 16.2, $\lambda x^\oslash.y^\oslash x^\oslash : \langle (y : a) \vdash_1 \to a \rangle$. Again, by lemma 16.2, $\exists K \succeq \oslash$ such that $a = \omega^K$ or $\exists n \geq 1$ such that $a = \sqcap_{i=1}^n (T_i \to T_i')$, absurd.
- Let $i = 2$. By lemma 17.2, $\lambda x^\oslash.y^\oslash x^\oslash : \langle (y : a) \vdash_1 \to a \rangle$. Again, by lemma 17.2, $a = \omega^\oslash$ or $\exists n \geq 1$ such that $a = \sqcap_{i=1}^n (U_i \to T_i)$, absurd.

$\square$

6 The realisability semantics

In this section we introduce the realisability semantics and show its soundness for both $\vdash_1$ and $\vdash_2$. We also show that completeness fails for $\vdash_1$.

Crucial to a realisability semantics is the notion of a saturated set:

Definition 28. *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}$.*

1. *We use $\mathcal{P}(\mathcal{X})$ to denote the powerset of $\mathcal{X}$, i.e. $\{\mathcal{Y} \ / \ \mathcal{Y} \subseteq \mathcal{X}\}$.*
2. *We define $\mathcal{X}^{+i} = \{M^{+i} \ / \ M \in \mathcal{X}\}$.*
3. *We define $\mathcal{X} \rightsquigarrow \mathcal{Y} = \{M \in \mathcal{M} \ / \ M\ N \in \mathcal{Y}$ for all $N \in \mathcal{X}$ such that $M \diamond N\}$.*
4. *We say that $\mathcal{X} \wr \mathcal{Y}$ iff $\forall M \in \mathcal{X} \rightsquigarrow \mathcal{Y}, \exists N \in \mathcal{X}$ such that $M \diamond N$.*
5. *For $r \in \{\beta, \beta\eta, h\}$, we say that $\mathcal{X}$ is r-saturated if whenever $M \rhd_r^* N$ and $N \in \mathcal{X}$, then $M \in \mathcal{X}$.*

Saturation is closed under intersection, lifting and arrows:

Lemma 29. *1. $(\mathcal{X} \cap \mathcal{Y})^{+i} = \mathcal{X}^{+i} \cap \mathcal{Y}^{+i}$.*

2. *If $\mathcal{X}, \mathcal{Y}$ are r-saturated sets, then $\mathcal{X} \cap \mathcal{Y}$ is r-saturated.*
3. *If $\mathcal{X}$ is r-saturated, then $\mathcal{X}^{+i}$ is r-saturated.*
4. *If $\mathcal{Y}$ is r-saturated, then, for every set $\mathcal{X}$, $\mathcal{X} \rightsquigarrow \mathcal{Y}$ is r-saturated.*

5. $(\mathcal{X} \rightsquigarrow \mathcal{Y})^{+i} \subseteq \mathcal{X}^{+i} \rightsquigarrow \mathcal{Y}^{+i}$.
6. *If* $\mathcal{X}^+ \wr \mathcal{Y}^+$, *then* $\mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+ \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})^+$.

We now give the basic step in our realisability semantics: the interpretations and meanings of types.

Definition 30. *Let* $\mathcal{V}_1$, $\mathcal{V}_2$ *be denumerably infinite,* $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ *and* $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$.

1. *Let* $L \in \mathcal{L}_{\mathbb{N}}$. *We define* $\mathcal{M}^L = \{M \in \mathcal{M} / d(M) = L\}$.
2. *Let* $x \in \mathcal{V}_1$. *We define* $\mathcal{N}_x^L = \{x^L\ N_1 ... N_k \in \mathcal{M} /\ k \geq 0\}$.
3. *Let* $r \in \{\beta, \beta\eta, h\}$. *An* r*-interpretation* $\mathcal{I} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M}^{\oslash})$ *is a function such that for all* $a \in \mathcal{A}$:
   - $\mathcal{I}(a)$ *is* r*-saturated* *and* • $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{I}(a)$.

   We extend an r*-interpretation* $\mathcal{I}$ *to* $\mathbb{U}$ *as follows:*
   - $\mathcal{I}(\omega^L) = \mathcal{M}^L$ • $\mathcal{I}(e_i U) = \mathcal{I}(U)^{+i}$
   - $\mathcal{I}(U_1 \sqcap U_2) = \mathcal{I}(U_1) \cap \mathcal{I}(U_2)$ • $\mathcal{I}(U \to T) = \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$
4. *Let* $U \in \mathcal{T}$ *and* $r \in \{\beta, \beta\eta, h\}$. *Define the* r*-interpretation of* U *by:*

$$[U]_r = \{M \in \mathcal{M}\ /\ M \textit{ is closed and } M \in \bigcap_{\mathcal{I}\ \ r-interpretation} \mathcal{I}(U)\}$$

Lemma 31. *Let* $r \in \{\beta, \beta\eta, h\}$.

1. (a) *For any* $U \in \mathbb{U}$ *and* r*-interpretation* $\mathcal{I}$, *we have* $\mathcal{I}(U)$ *is* r*-saturated.*
   (b) *If* $d(U) = L$ *and* $\mathcal{I}$ *is an* r*-interpretation, then* $\forall x \in \mathcal{V}_1, \mathcal{N}_x^L \subseteq \mathcal{I}(U) \subseteq \mathcal{M}^L$.
2. *Let* $r \in \{\beta, \beta\eta, h\}$. *If* $\mathcal{I}$ *be an* r*-interpretation and* $U \sqsubseteq V$, *then* $\mathcal{I}(U) \subseteq \mathcal{I}(V)$.

Here is the soundness lemma.

Lemma 32. *Let* $r \in \{\beta, \beta\eta, h\}$, $i \in \{1, 2\}$, $M : \langle (x_j^{L_j} : U_j)_n \vdash_i U \rangle$, $\mathcal{I}$ *be an* r*-interpretation and* $\forall\ 1 \leq j \leq n$, $N_j \in \mathcal{I}(U_j)$. *We have* $M[(x_j^{L_j} := N_j)_n] \in \mathcal{I}(U)$.

Corollary 33. *Let* $r \in \{\beta, \beta\eta, h\}$. *If* $M : \langle () \vdash U \rangle$, *then* $M \in [U]_r$.

Proof By lemma 32, $M \in \mathcal{I}(U)$ for any r-interpretation $\mathcal{I}$. By lemma 13, $FV(M) = dom(()) = \emptyset$ and hence M is closed. Therefore, $M \in [U]_r$. □

Lemma 34 (The meaning of types is closed under type operations). *Let* $r \in \{\beta, \beta\eta, h\}$. *On* $\mathcal{T}$ *(hence also on* $\mathbb{U}$*) the following hold:*

1. $[eU]_r = [U]_r^+$
2. $[U \sqcap V]_r = [U]_r \cap [V]_r$
3. *If* $U \to T \in \mathcal{T}$ *then for any interpretation* $\mathcal{I}$, $\mathcal{I}(U) \wr \mathcal{I}(T)$.
4. *On* $\mathcal{T}$ *only (since* $e_i U \to e_i T \notin \mathbb{U}$*), we have* $[e_i(U \to T)]_r = [e_i U \to e_i T]_r$.

Proof 1. and 2. are easy. 3. Let $d(U) = K$, $M \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$ and $x \in \mathcal{V}_1$ such that $\forall L, x^L \notin FV(M)$, hence $M \diamond x^K$ and $x^K \in \mathcal{I}(U)$. 4. By lemma 29 and 3., for any r-interpretation $\mathcal{I}$ we have $\mathcal{I}(e_i(U \to T)) = \mathcal{I}(e_i U \to e_i T)$. □

The next definition and lemma put the realisability semantics in use and will help us show that completeness fails for $\vdash_1$.

Definition 35 (Examples). *Let* $a, b \in \mathcal{A}$ *where* $a \neq b$. *We define:*

- $Id_0 = a \to a$, $Id_1 = e_1(a \to a)$ *and* $Id_1' = e_1 a \to e_1 a$.
- $D = (a \sqcap (a \to b)) \to b$.
- $Nat_0 = (a \to a) \to (a \to a)$, $Nat_1 = e_1((a \to a) \to (a \to a))$,
  $Nat_1' = e_1(a \to a) \to (e_1 a \to e_1 a)$ *and* $Nat_0' = (e_1 a \to a) \to (e_1 a \to a)$.

Moreover, if M, N *are terms and* $n \in \mathbb{N}$, *we define* $(M)^n\ N$ *by induction on* n: $(M)^0\ N = N$ *and* $(M)^{m+1}\ N = M\ ((M)^m\ N)$.

Lemma 36. *1.* $[Id_0]_\beta = \{M \in \mathcal{M}^\oslash \ / \ M \rhd^*_\beta \lambda y^\oslash y^\oslash\}$.
2. $[Id_1]_\beta = [Id'_1]_\beta = \{M \in \mathcal{M}^{(1)} \ / \ M \rhd^*_\beta \lambda y^{(1)}.y^{(1)}\}$. *(Note that* $Id'_1 \not\in \mathbb{U}$.*)*
3. $[D]_\beta = \{M \in \mathcal{M}^\oslash \ / \ M \rhd^*_\beta \lambda y^\oslash.y^\oslash y^\oslash\}$.
4. $[Nat_0]_\beta = \{M \in \mathcal{M}^\oslash \ / \ M \rhd^*_\beta \lambda f^\oslash.f^\oslash$ *or* $M \rhd^*_\beta \lambda f^\oslash.\lambda y^\oslash.(f^\oslash)^n y^\oslash$ *where* $n \geq 1\}$.
5. $[Nat_1]_\beta = [Nat'_1]_\beta = \{M \in \mathcal{M}^{(1)} \ / \ M \rhd^*_\beta \lambda f^{(1)}.f^{(1)}$ *or* $M \rhd^*_\beta \lambda f^{(1)}.\lambda x^{(1)}.(f^{(1)})^n y^{(1)}$ *where* $n \geq 1\}$. *(Note that* $Nat'_1 \not\in \mathbb{U}$.*)*
6. $[Nat'_0]_\beta = \{M \in \mathcal{M}^\oslash \ /M \rhd^*_\beta \lambda f^\oslash.f^\oslash$ *or* $M \rhd^*_\beta \lambda f^\oslash.\lambda y^{(1)}.f^\oslash y^{(1)}\}$.
7. $[(a \sqcap b) \to a]_\beta = \{M \in \mathcal{M}^\oslash \ /M \rhd^*_\beta \lambda y^\oslash.y^\oslash\}$.
8. It is not possible that $\lambda y^\oslash.y^\oslash : \langle () \vdash_1 (a \sqcap b) \to a\rangle$.
9. $\lambda y^\oslash.y^\oslash : \langle () \vdash_2 (a \sqcap b) \to a\rangle$.

Remark 37 (Failure of completeness for $\vdash_1$*).* Items 7. and 8. of lemma 36 show that we can not have a completeness result (a converse of corollary 33) for $\vdash_1$. To type the term $\lambda y^0.y^0$ by the type $(a \sqcap b) \to a$, we need an elimination rule for $\sqcap$ which we have in $\vdash_2$. However, we will see that we have completeness for $\vdash_2$ only if we are restricted to the use of one single expansion variable.

7 The completeness theorem

In this section we set out the machinery and prove that completeness holds for $\vdash_2$.

We need the following partition of the set of variables $\{y^L/y \in \mathcal{V}_2\}$.

Definition 38. *1. Let* $L \in \mathcal{L}_\mathbb{N}$. *We define* $\mathbb{U}^L = \{U \in \mathbb{U}/d(U) = L\}$ *and* $\mathcal{V}^L = \{x^L/x \in \mathcal{V}_2\}$.
2. Let $U \in \mathbb{U}$. *We inductively define a set of variables* $\mathbb{V}_U$ *as follows:*
- *If* $d(U) = \oslash$ *then:*
  - $\mathbb{V}_U$ *is an infinite set of variables of degree* $\oslash$.
  - *If* $U \neq V$ *and* $d(U) = d(V) = \oslash$, *then* $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.
  - $\bigcup_{U \in \mathbb{U}^\oslash} \mathbb{V}_U = \mathcal{V}^\oslash$.
- *If* $d(U) = L$, *then we put* $\mathbb{V}_U = \{y^L \ / \ y^\oslash \in \mathbb{V}_{U^{-L}}\}$.

Lemma 39. *1. If* $d(U), d(V) \succeq L$ *and* $U^{-L} = V^{-L}$, *then* $U = V$.
2. If $d(U) = L$, *then* $\mathbb{V}_U$ *is an infinite subset of* $\mathcal{V}^L$.
3. If $U \neq V$ *and* $d(U) = d(V) = L$, *then* $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.
4. $\bigcup_{U \in \mathbb{U}^L} \mathbb{V}_U = \mathcal{V}^L$.
5. If $y^L \in \mathbb{V}_U$, *then* $y^{i::L} \in \mathbb{V}_{e_i U}$.
6. If $y^{i::L} \in \mathbb{V}_U$, *then* $y^L \in \mathbb{V}_{U^{-i}}$.

Proof 1. If $L = (n_i)_m$, we have $U = e_{n_1}...e_{n_m}U'$ and $V = e_{n_1}...e_{n_m}V'$. Then $U^{-L} = U'$, $V^{-L} = V'$ and $U' = V'$. Thus $U = V$. 2. 3. and 4. By induction on L and using 1. 5. Because $(e_i U)^{-i} = U$. 6. By definition. □

Our partition of the set $\mathcal{V}_2$ as above will enable us to define useful infinite sets which will contain type environments that will play a crucial role in one particular type interpretation. These infinite sets and type environments are given in the next definition.

Definition 40. *1. Let* $L \in \mathcal{L}_\mathbb{N}$. *We denote* $\mathbb{G}^L = \{(y^L : U) \ / \ U \in \mathbb{U}^L$ *and* $y^L \in \mathbb{V}_U\}$ *and* $\mathbb{H}^L = \bigcup_{K \succeq L} \mathbb{G}^K$. *Note that* $\mathbb{G}^L$ *and* $\mathbb{H}^L$ *are not type environments because they are infinite sets.*
2. Let $L \in \mathcal{L}_\mathbb{N}$, $M \in \mathcal{M}$ *and* $U \in \mathbb{U}$, *we write:*
- $M : \langle \mathbb{H}^L \vdash_2 U\rangle$ *if there is a type environment* $\Gamma \subset \mathbb{H}^L$ *where* $M : \langle \Gamma \vdash_2 U\rangle$
- $M : \langle \mathbb{H}^L \vdash^*_2 U\rangle$ *if* $M \rhd^*_{\beta\eta} N$ *and* $N : \langle \mathbb{H}^L \vdash_2 U\rangle$

Lemma 41. *1. If* $\Gamma \subset \mathbb{H}^L$ *then* $e_i\Gamma \subset \mathbb{H}^{i::L}$.
2. If $\Gamma \subset \mathbb{H}^{i::L}$ *then* $\Gamma^{-i} \subset \mathbb{H}^L$.

3. If $\Gamma_1 \subset \mathbb{H}^L$, $\Gamma_2 \subset \mathbb{H}^K$ and $L \preceq K$ then $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$.

Proof 1. and 2. By lemma 39. 3. First note that $\mathbb{H}^K \subseteq \mathbb{H}^L$. Let $(x^R : U_1 \sqcap U_2) \in \Gamma_1 \sqcap \Gamma_2$ where $(x^R : U_1) \in \Gamma_1 \subset \mathbb{H}^L$ and $(x^R : U_2) \in \Gamma_2 \subset \mathbb{H}^K \subseteq \mathbb{H}^L$, then $\mathrm{d}(U_1) = \mathrm{d}(U_2) = R$ and $x^R \in \mathbb{V}_{U_1} \cap \mathbb{V}_{U_2}$. Hence, by lemma 39, $U_1 = U_2$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \cup \Gamma_2 \subset \mathbb{H}^L$. □

For every $L \in \mathcal{L}_{\mathbb{N}}$, we define the set of terms of degree L which contain some free variable x^K where $x \in \mathcal{V}_1$ and $K \succeq L$.

Definition 42. *For every $L \in \mathcal{L}_{\mathbb{N}}$, let $\mathcal{O}^L = \{M \in \mathcal{M}^L \ / \ x^K \in FV(M),\ x \in \mathcal{V}_1$ and $K \succeq L\}$. It is easy to see that, for every $L \in \mathcal{L}_{\mathbb{N}}$ and $x \in \mathcal{V}_1$, $\mathcal{N}_x^L \subseteq \mathcal{O}^L$.*

Lemma 43. *1. $(\mathcal{O}^L)^{+i} = \mathcal{O}^{i::L}$.*
2. If $y \in \mathcal{V}_2$ and $(My^K) \in \mathcal{O}^L$, then $M \in \mathcal{O}^L$
3. If $M \in \mathcal{O}^L$, $M \diamond N$ and $L \preceq K = d(N)$, then $MN \in \mathcal{O}^L$.
4. If $d(M) = L$, $L \preceq K$, $M \diamond N$ and $N \in \mathcal{O}^K$, then $MN \in \mathcal{O}^L$.

The crucial interpretation $\mathbb{I}$ for the proof of completeness is given as follows:

Definition 44. *1. Let $\mathbb{I}_{\beta\eta}$ be the $\beta\eta$-interpretation defined by: for all type variables a, $\mathbb{I}_{\beta\eta}(a) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2^* a\rangle\}$.*
2. Let $\mathbb{I}_{\beta}$ be the β-interpretation defined by: for all type variables a, $\mathbb{I}_{\beta}(a) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2 a\rangle\}$.
3. Let $\mathbb{I}_{eh}$ be the h-interpretation defined by: for all type variables a, $\mathbb{I}_{h}(a) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2 a\rangle\}$.

The next crucial lemma shows that $\mathbb{I}$ is indeed an interpretation and moreover, the interpretation of a type of order L contains terms of order L which are typable in these special environments which are parts of the infinite sets of definition 40.

Lemma 45. *Let $r \in \{\beta\eta, \beta, h\}$ and $r' \in \{\beta, h\}$*

1. $\mathbb{I}_r$ is an r-interpretation: $\forall a \in \mathcal{A}$, $\mathbb{I}_r(a)$ is r-saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathbb{I}_r(a)$.
2. If $U \in \mathbb{U}$ and $d(U) = L$, then $\mathbb{I}_{\beta\eta}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* U\rangle\}$.
3. If $U \in \mathbb{U}$ and $d(U) = L$, then $\mathbb{I}_{r'}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 U\rangle\}$.

Proof We give a proof sketch. The full proof is available at `http://www.macs.hw.ac.uk/~fairouz/papers/drafts/compsem-big.pdf`.
1. We do two cases:
Case $r = \beta\eta$. It is easy to see that $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{O}^{\oslash} \subseteq \mathbb{I}_{\beta\eta}(a)$. Now we show that $\mathbb{I}_{\beta\eta}(a)$ is $\beta\eta$-saturated. Let $M \rhd_{\beta\eta}^* N$ and $N \in \mathbb{I}_{\beta\eta}(a)$.

- If $N \in \mathcal{O}^{\oslash}$ then $N \in \mathcal{M}^{\oslash}$ and $\exists L$ and $x \in \mathcal{V}_1$ such that $x^L \in FV(N)$. By theorem 4.2, $FV(N) \subseteq FV(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$, hence, $M \in \mathcal{O}^{\oslash}$
- If $N \in \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2^* a\rangle\}$ then $N \rhd_{\beta\eta}^* N'$ and $\exists \Gamma \subset \mathbb{H}^{\oslash}$, such that $N' : \langle \Gamma \vdash_2 a\rangle$. Hence $M \rhd_{\beta\eta}^* N'$ and since by theorem 4.2, $\mathrm{d}(M) = \mathrm{d}(N')$, $M \in \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2^* a\rangle\}$.

Case $r = \beta$. It is easy to see that $\forall x \in \mathcal{V}_1, \mathcal{N}_x^{\oslash} \subseteq \mathcal{O}^{\oslash} \subseteq \mathbb{I}_{\beta}(a)$. Now we show that $\mathbb{I}_{\beta}(a)$ is β-saturated. Let $M \rhd_{\beta}^* N$ and $N \in \mathbb{I}_{\beta}(a)$.

- If $N \in \mathcal{O}^{\oslash}$ then $N \in \mathcal{M}^{\oslash}$ and $\exists L$ and $x \in \mathcal{V}_1$ such that $x^L \in FV(N)$. By theorem 4.2, $FV(N) \subseteq FV(M)$ and $\mathrm{d}(M) = \mathrm{d}(N)$, hence, $M \in \mathcal{O}^{\oslash}$
- If $N \in \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2 a\rangle\}$ then $\exists \Gamma \subset \mathbb{H}^{\oslash}$, such that $N : \langle \Gamma \vdash_2 a\rangle$. By theorem 25, $M : \langle \Gamma\uparrow^M \vdash_2 a\rangle$. Since by theorem 4.2, $FV(N) \subseteq FV(M)$, let $FV(N) = \{x_1^{L_1}, \dots, x_n^{L_n}\}$ and $FV(M) = FV(N) \cup \{x_{n+1}^{L_{n+1}}, \dots, x_{n+m}^{L_{n+m}}\}$. So $\Gamma\uparrow^M = \Gamma, (x_{n+1}^{L_{n+1}} : \omega^{L_{n+1}}, \dots, x_{n+m}^{L_{n+m}} : \omega^{L_{n+m}})$. $\forall n+1 \leq i \leq n+m$, let U_i such that $x_i \in \mathbb{V}_{U_i}$. Then $\Gamma, (x_{n+1}^{L_{n+1}} : U_{n+1}, \dots, x_{n+m}^{L_{n+m}} : U_{n+m}) \subset \mathbb{H}^{\oslash}$ and by $\sqsubseteq$, $M : \langle \Gamma, (x_{n+1}^{L_{n+1}} : U_{n+1}, \dots, x_{n+m}^{L_{n+m}} : U_{n+m}) \vdash_2 a\rangle$. Thus $M : \langle \mathbb{H}^{\oslash} \vdash_2 a\rangle$ and since by theorem 4.2, $\mathrm{d}(M) = \mathrm{d}(N)$, $M \in \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2 a\rangle\}$.

2. By induction on U.

- $U = a$: By definition of $\mathbb{I}_{\beta\eta}$.
- $U = \omega^L$: By definition, $\mathbb{I}_{\beta\eta}(\omega^L) = \mathcal{M}^L$. Hence, $\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* \omega^L \rangle\} \subseteq \mathbb{I}_{\beta\eta}(\omega^L)$.
  Let $M \in \mathbb{I}_{\beta\eta}(\omega^L)$ where $FV(M) = \{x_1^{L_1}, ..., x_n^{L_n}\}$. We have $M : \langle (x_i^{L_i} : \omega^{L_i})_n \vdash_2 \omega^L \rangle$ and $M \in \mathcal{M}^L$. $\forall\ 1 \leq i \leq n$, let U_i the type such that $x_i^{L_i} \in \mathbb{V}_{U_i}$. Then $\Gamma = (x_i^{L_i} : U_i)_n \subset \mathbb{H}^L$. By lemma 13, $M : \langle \Gamma \vdash_2 \omega^L \rangle$. Hence $M : \langle \mathbb{H}^L \vdash_2 \omega^L \rangle$. Therefore, $\mathbb{I}(\omega^L) \subseteq \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* \omega^L \rangle\}$.
  We deduce $\mathbb{I}_{\beta\eta}(\omega^L) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* \omega^L \rangle\}$.
- $U = e_i V$: $L = i :: K$ and $\mathrm{d}(V) = K$. By IH and lemma 43, $\mathbb{I}_{\beta\eta}(e_i V) = (\mathbb{I}_{\beta\eta}(V))^{+i} = (\mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2^* V \rangle\})^{+i} =$
  $\mathcal{O}^{i::L} \cup (\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2^* V \rangle\})^{+i}$. We can easily show that $(\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2^* V \rangle\})^{+i} = \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* U \rangle\}$ and $\mathbb{I}_{\beta\eta}(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* U \rangle\}$ (see `http://www.macs.hw.ac.uk/~fairouz/papers/drafts/compsem-big.pdf`).
- $U = U_1 \sqcap U_2$: By IH, $\mathbb{I}_{\beta\eta}(U_1 \sqcap U_2) = \mathbb{I}_{\beta\eta}(U_1) \cap \mathbb{I}_{\beta\eta}(U_2) = (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* U_1 \rangle\}) \cap (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* U_2 \rangle\}) = \mathcal{O}^L \cup (\{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* U_1 \rangle\} \cap \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* U_2 \rangle\})$.
  - If $M \in \mathcal{M}^L$, $M : \langle \mathbb{H}^L \vdash_2^* U_1 \rangle$ and $M : \langle \mathbb{H}^L \vdash_2^* U_2 \rangle$, then $M \rhd_{\beta\eta}^* N_1$, $M \rhd_{\beta\eta}^* N_2$, $N_1 : \langle \Gamma_1 \vdash_2 U_1 \rangle$ and $N_2 : \langle \Gamma_2 \vdash_2 U_2 \rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{H}^L$. By confluence theorem 7 and subject reduction theorem 20, $\exists M'$ such that $M \rhd_{\beta\eta}^* M'$, $M' : \langle \Gamma_1 \upharpoonright_{M'} \vdash_2 U_1 \rangle$ and $M' : \langle \Gamma_2 \upharpoonright_{M'} \vdash_2 U_2 \rangle$. Hence by Remark 15, $M' : \langle (\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{M'} \vdash_2 U_1 \sqcap U_2 \rangle$ and, by lemma 41, $(\Gamma_1 \sqcap \Gamma_2) \upharpoonright_{M'} \subseteq \Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$. Thus $M : \langle \mathbb{H}^L \vdash_2^* U_1 \sqcap U_2 \rangle$.
  - If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash_2^* U_1 \sqcap U_2 \rangle$, then $M \rhd_{\beta\eta}^* N$, $N : \langle \Gamma \vdash_2 U_1 \sqcap U_2 \rangle$ and $\Gamma \subset \mathbb{H}^L$. By $\sqsubseteq$, $N : \langle \Gamma \vdash_2 U_1 \rangle$ and $N : \langle \Gamma \vdash_2 U_2 \rangle$.
    Hence, $M : \langle \mathbb{H}^L \vdash_2^* U_1 \rangle$ and $M : \langle \mathbb{H}^L \vdash_2^* U_2 \rangle$.

  We deduce that $\mathbb{I}_{\beta\eta}(U_1 \sqcap T_2) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2^* U_1 \sqcap U_2 \rangle\}$.
- $U = V \to T$: Let $\mathrm{d}(T) = \oslash \preceq K = \mathrm{d}(V)$. By IH, $\mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2^* V \rangle\}$ and $\mathbb{I}_{\beta\eta}(T) = \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash_2^* T \rangle\}$. Note that $\mathbb{I}_{\beta\eta}(V \to T) = \mathbb{I}_{\beta\eta}(V) \rightsquigarrow \mathbb{I}_{\beta\eta}(T)$.
  - Let $M \in \mathbb{I}_{\beta\eta} V) \rightsquigarrow \mathbb{I}_{\beta\eta} T)$ and, by lemma 39, let $y^K \in \mathbb{V}_V$ such that $\forall K, y^K \notin FV(M)$. Then $M \diamond y^K$. By remark 15, $y^K : \langle (y^K : V) \vdash_2^* V \rangle$. Hence $y^K : \langle \mathbb{H}^K \vdash_2^* V \rangle$. Thus, $y^K \in \mathbb{I}_{\beta\eta}(V)$ and $My^K \in \mathbb{I}_{\beta\eta}(T)$.
    - If $My^K \in \mathcal{O}^\oslash$, then since $y \in \mathcal{V}_2$, by lemma 43, $M \in \mathcal{O}^\oslash$.
    - If $My^K \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash_2^* T \rangle\}$ then $My^K \rhd_{\beta\eta}^* N$ and $N : \langle \Gamma \vdash_2 T \rangle$, hence, $\lambda y^K.My^K \rhd_{\beta\eta}^* \lambda y^K.N$. We have two cases:
      - If $y^K \in dom(\Gamma)$, then $\Gamma = \Delta, (y^K : V)$ and by $\to_I$, $\lambda y^K.N : \langle \Delta \vdash_2 V \to T \rangle$.
      - If $y^K \notin dom(\Gamma)$, let $\Delta = \Gamma$. By $\to_I'$, $\lambda y^K.N : \langle \Delta \vdash_2 \omega^K \to T \rangle$. By $\sqsubseteq$, since $\langle \Delta \vdash_2 \omega^K \to T \rangle \sqsubseteq \langle \Delta \vdash_2 V \to T \rangle$, we have $\lambda y^K.N : \langle \Delta \vdash_2 V \to T \rangle$.

      Note that $\Delta \subset \mathbb{G}$. Since $\lambda y^K.My^K \rhd_{\beta\eta}^* M$ and $\lambda y^K.My^K \rhd_{\beta\eta}^* \lambda y^K.N$, by theorem 7 and theorem 20, there is M' such that $M \rhd_{\beta\eta}^* M'$, $\lambda y^K.N \rhd_{\beta\eta}^* M'$, $M' : \langle \Delta \upharpoonright_{M'} \vdash_2 V \to T \rangle$. Since $\Delta \upharpoonright_{M'} \subseteq \Delta \subset \mathbb{H}^\oslash$, $M : \langle \mathbb{H}^\oslash \vdash_2^* V \to T \rangle$.
  - Let $M \in \mathcal{O}^\oslash \cup \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash_2^* V \to T \rangle\}$ and $N \in \mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2^* V \rangle\}$ such that $M \diamond N$. Then, $\mathrm{d}(N) = K$.
    - If $M \in \mathcal{O}^\oslash$, then, by lemma 43, $MN \in \mathcal{O}^\oslash$.
    - If $M \in \{M \in \mathcal{M}^\oslash \ / \ M : \langle \mathbb{H}^\oslash \vdash_2^* V \to T \rangle\}$, then
      - If $N \in \mathcal{O}^K$, then, by lemma 43, $MN \in \mathcal{O}^\oslash$.

- · If $N \in \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2^* V\rangle\}$ then $M \rhd^*_{\beta\eta} M_1$, $N \rhd^*_{\beta\eta} N_1$, $M_1 : \langle \Gamma_1 \vdash_2 V \to T\rangle$ and $N_1 : \langle \Gamma_2 \vdash_2 V\rangle$ where $\Gamma_1 \subset \mathbb{H}^{\oslash}$ and $\Gamma_2 \subset \mathbb{H}^K$. Now, $MN \rhd^*_{\beta\eta} M_1N_1$ and, by $\to_E$, $M_1N_1 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle$ (again, see `http://www.macs.hw.ac.uk/~fairouz/papers/drafts/compsem-big.pdf`). By lemma 41, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^{\oslash}$. Therefore $MN : \langle \mathbb{H}^{\oslash} \vdash_2^* T\rangle$.

  We deduce that $\mathbb{I}_{\beta\eta}(V \to T) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2^* V \to T\rangle\}$.

3. We only do the case $r = \beta$. By induction on U.

- $U = a$: By definition of $\mathbb{I}_\beta$.
- $U = \omega^L$: By definition, $\mathbb{I}_\beta(\omega^L) = \mathcal{M}^L$. Hence, $\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 \omega^L\rangle\} \subseteq \mathbb{I}_\beta(\omega^L)$.
  Let $M \in \mathbb{I}_\beta(\omega^L)$ where $FV(M) = \{x_1^{L_1}, ..., x_n^{L_n}\}$. We have $M : \langle (x_i^{L_i} : \omega^{L_i})_n \vdash_2 \omega^L\rangle$ and $M \in \mathcal{M}^L$. $\forall\ 1 \leq i \leq n$, let U_i the type such that $x_i^{L_i} \in \mathbb{V}_{U_i}$. Then $\Gamma = (x_i^{L_i} : U_i)_n \subset \mathbb{H}^L$. By lemma 13, $M : \langle \Gamma \vdash_2 \omega^L\rangle$. Hence $M : \langle \mathbb{H}^L \vdash_2 \omega^L\rangle$. Therefore, $\mathbb{I}(\omega^L) \subseteq \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 \omega^L\rangle\}$.
  We deduce $\mathbb{I}_\beta(\omega^L) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 \omega^L\rangle\}$.
- $U = e_iV$: $L = i :: K$ and $\mathrm{d}(V) = K$. By IH and lemma 43, $\mathbb{I}_\beta(e_iV) = (\mathbb{I}_\beta(V))^{+i} = (\mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2 V\rangle\})^{+i} = \mathcal{O}^{i::L} \cup (\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2 V\rangle\})^{+i}$.
  - If $M \in \mathcal{M}^K$ and $M : \langle \mathbb{H}^K \vdash_2 V\rangle$, then $M : \langle \Gamma \vdash_2 V\rangle$ where $\Gamma \subset \mathbb{H}^K$. By e and 41, $M^{+i} : \langle e_i\Gamma \vdash_2 e_iV\rangle$ and $e_i\Gamma \subset \mathbb{H}^L$. Thus $M^{+i} \in \mathcal{M}^L$ and $M^{+i} : \langle \mathbb{H}^L \vdash_2 U\rangle$.
  - If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash_2 U\rangle$, then $M : \langle \Gamma \vdash_2 U\rangle$ where $\Gamma \subset \mathbb{H}^L$. By lemmas 12, and 41, $M^{-i} : \langle \Gamma^{-i} \vdash_2 V\rangle$ and $\Gamma^{-i} \subset \mathbb{H}^K$. Thus by $M = (M^{-i})^{+i}$ and $M^{-i} \in \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2 V\rangle\}$ (again, see `http://www.macs.hw.ac.uk/~fairouz/papers/drafts/compsem-big.pdf`).

  Hence $(\{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2 V\rangle\})^{+i} = \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 U\rangle\}$ and $\mathbb{I}_\beta(U) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 U\rangle\}$.
- $U = U_1 \sqcap U_2$: By IH, $\mathbb{I}_\beta(U_1 \sqcap U_2) = \mathbb{I}_\beta(U_1) \cap \mathbb{I}_\beta(U_2) = (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 U_1\rangle\}) \cap (\mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 U_2\rangle\}) = \mathcal{O}^L \cup (\{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 U_1\rangle\} \cap \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 U_2\rangle\})$.
  - If $M \in \mathcal{M}^L$, $M : \langle \mathbb{H}^L \vdash_2 U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash_2 U_2\rangle$, then $M : \langle \Gamma_1 \vdash_2 U_1\rangle$ and $M : \langle \Gamma_2 \vdash_2 U_2\rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{H}^L$. Hence by Remark 15, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 U_1 \sqcap U_2\rangle$ and, by lemma 41, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^L$. Thus $M : \langle \mathbb{H}^L \vdash_2 U_1 \sqcap U_2\rangle$.
  - If $M \in \mathcal{M}^L$ and $M : \langle \mathbb{H}^L \vdash_2 U_1 \sqcap U_2\rangle$, then $M : \langle \Gamma \vdash_2 U_1 \sqcap U_2\rangle$ and $\Gamma \subset \mathbb{H}^L$. By $\sqsubseteq$, $M : \langle \Gamma \vdash_2 U_1\rangle$ and $M : \langle \Gamma \vdash_2 U_2\rangle$. Hence, $M : \langle \mathbb{H}^L \vdash_2 U_1\rangle$ and $M : \langle \mathbb{H}^L \vdash_2 U_2\rangle$.

  We deduce that $\mathbb{I}_\beta(U_1 \sqcap T_2) = \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 U_1 \sqcap U_2\rangle\}$.
- $U = V \to T$: Let $\mathrm{d}(T) = \oslash \preceq K = \mathrm{d}(V)$. By IH, $\mathbb{I}_\beta(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2 V\rangle\}$ and $\mathbb{I}_\beta(T) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2 T\rangle\}$. Note that $\mathbb{I}_\beta(V \to T) = \mathbb{I}_\beta(V) \rightsquigarrow \mathbb{I}_\beta(T)$.
  - Let $M \in \mathbb{I}_\beta V) \rightsquigarrow \mathbb{I}_\beta T)$ and, by lemma 39, let $y^K \in \mathbb{V}_V$ such that $\forall K, y^K \notin FV(M)$. Then $M \diamond y^K$. By remark 15, $y^K : \langle (y^K : V) \vdash_2^* V\rangle$. Hence $y^K : \langle \mathbb{H}^K \vdash_2 V\rangle$. Thus, $y^K \in \mathbb{I}_\beta(V)$ and $My^K \in \mathbb{I}_\beta(T)$.
    - ∗ If $My^K \in \mathcal{O}^{\oslash}$, then since $y \in \mathcal{V}_2$, by lemma 43, $M \in \mathcal{O}^{\oslash}$.
    - ∗ If $My^K \in \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2 T\rangle\}$ then $My^K : \langle \Gamma \vdash_2 T\rangle$. Since by lemma 13, $dom(\Gamma) = FV(My^K)$ and $y^K \in FV(My^K)$, $\Gamma = \Delta, (y^K : V')$. Since $(y^K : V') \in \mathbb{H}^{\oslash}$, by lemma 39, $V = V'$. So $My^K : \langle \Delta, (y^K : V) \vdash_2 T\rangle$ and by lemma 17 $M : \langle \Delta \vdash_2 V \to T\rangle$. Note that $\Delta \subset \mathbb{H}^{\oslash}$, hence $M : \langle \mathbb{H}^{\oslash} \vdash_2 V \to T\rangle$.
  - Let $M \in \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2 V \to T\rangle\}$ and $N \in \mathbb{I}_{\beta\eta}(V) = \mathcal{O}^K \cup \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2 V\rangle\}$ such that $M \diamond N$. Then, $\mathrm{d}(N) = K$.

* If $M \in \mathcal{O}^{\oslash}$, then, by lemma 43, $MN \in \mathcal{O}^{\oslash}$.
* If $M \in \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2 V \to T\rangle\}$, then
  - If $N \in \mathcal{O}^K$, then, by lemma 43, $MN \in \mathcal{O}^{\oslash}$.
  - If $N \in \{M \in \mathcal{M}^K \ / \ M : \langle \mathbb{H}^K \vdash_2 V\rangle\}$ then $M : \langle \Gamma_1 \vdash_2 V \to T\rangle$ and $N : \langle \Gamma_2 \vdash_2 V\rangle$ where $\Gamma_1 \subset \mathbb{H}^{\oslash}$ and $\Gamma_2 \subset \mathbb{H}^K$. By $\to_E$, $MN : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle$. By lemma 41, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^{\oslash}$. Therefore $MN : \langle \mathbb{H}^{\oslash} \vdash_2 T\rangle$.

We deduce that $\mathbb{I}_\beta(V \to T) = \mathcal{O}^{\oslash} \cup \{M \in \mathcal{M}^{\oslash} \ / \ M : \langle \mathbb{H}^{\oslash} \vdash_2 V \to T\rangle\}$.

□

Now, we use this crucial $\mathbb{I}$ to establish completeness of our semantics.

Theorem 46 (Completeness of $\vdash_2$). *Let $U \in \mathbb{U}$ such that $d(U) = L$.*

1. $[U]_{\beta\eta} = \{M \in \mathcal{M}^L \ / \ M$ *closed,* $M \rhd^*_{\beta\eta} N$ *and* $N : \langle () \vdash_2 U\rangle\}$.
2. $[U]_\beta = [U]_h = \{M \in \mathcal{M}^L \ / \ M : \langle () \vdash_2 U\rangle\}$.
3. $[U]_{\beta\eta}$ *is stable by reduction. I.e., If* $M \in [U]_{\beta\eta}$ *and* $M \rhd^*_{\beta\eta} N$ *then* $N \in [U]_{\beta\eta}$.

Proof Let $r \in \{\beta, h, \beta\eta\}$. Recall that $[U]_r = \{M \in \mathcal{M} \ / \ M$ *is closed and* $M \in \bigcap_{\mathcal{I} \quad r-interpretation} \mathcal{I}(U)\}$.

1. Let $M \in [U]_{\beta\eta}$. Then M is a closed term and $M \in \mathbb{I}_{\beta\eta}(U)$. Hence, by lemma 45, $M \in \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^*_2 U\rangle\}$. Since M is closed, $M \notin \mathcal{O}^L$. Hence, $M \in \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash^*_2 U\rangle$ and so, $M \rhd^*_{\beta\eta} N$ and $N : \langle \Gamma \vdash_2 U\rangle$ where $\Gamma \subset \mathbb{H}^L$. By theorem 4, N is closed and, by lemma 13.2, $N : \langle () \vdash_2 U\rangle$.
   Conversely, take M closed such that $M \rhd^*_\beta N$ and $N : \langle () \vdash_2 U\rangle$. Let $\mathcal{I}$ be an $\beta\eta$-interpretation. By lemma 32, $N \in \mathcal{I}(U)$. By lemma 31.1, $\mathcal{I}(U)$ is $\beta\eta$-saturated. Hence, $M \in \mathcal{I}(U)$. Thus $M \in [U]$.
2. Let $M \in [U]_\beta$. Then M is a closed term and $M \in \mathbb{I}_\beta(U)$. Hence, by lemma 45, $M \in \mathcal{O}^L \cup \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 U\rangle\}$. Since M is closed, $M \notin \mathcal{O}^L$. Hence, $M \in \{M \in \mathcal{M}^L \ / \ M : \langle \mathbb{H}^L \vdash_2 U\rangle$ and so, $M : \langle \Gamma \vdash_2 U\rangle$ where $\Gamma \subset \mathbb{H}^L$. By lemma 13.2, $N : \langle () \vdash_2 U\rangle$.
   Conversely, take M such that $M : \langle () \vdash_2 U\rangle$. By lemma 13.2, M is closed. Let $\mathcal{I}$ be an β-interpretation. By lemma 32, $M \in \mathcal{I}(U)$. Thus $M \in [U]_\beta$.
   It is easy to see that $[U]_\beta = [U]_h$.
3. Let $M \in [U]$ such that $M \rhd^*_{\beta\eta} N$. By 1, M is closed, $M \rhd^*_{\beta\eta} P$ and $P : \langle () \vdash_2 U\rangle$. By confluence theorem 7, there is Q such that $P \rhd^*_{\beta\eta} Q$ and $N \rhd^*_{\beta\eta} Q$. By subject reduction theorem 20, $Q : \langle () \vdash_2 U\rangle$. By theorem 4, N is closed and, by 1, $N \in [U]$.

□

8 Conclusion

Expansion may be viewed to work like a multi-layered simultaneous substitution. Moreover, expansion is a crucial part of a procedure for calculating principal typings and helps support compositional type inference. Because the early definitions of expansion were complicated, expansion variables (E-variables) were invented to simplify and mechanize expansion. The aim of this paper is to give a denotational semantics for intersection type systems with exapansion variables.

Denotational semantics helps in reasoning about the properties of an entire type system and of specific typed terms. However, E-variables pose serious problems for semantics. Given that a type $e\,T$ can be turned by expansion into a new type $S_1(T) \sqcap S_2(T)$, where S_1 and S_2 are arbitrary substitutions (in fact, they can be arbitrary further expansions), and that this can introduce an unbounded number of new variables (both E-variables and regular type variables), it is difficult to give a semantics to expansion variables.

The only earlier attempt at giving a semantics for expansion variables could only handle the λI-calculus, did not allow a universal type, and was incomplete in the presence of more than one expansion variable. This paper overcomes these difficulties and gives a complete semantics for an intersection type system with an infinite number of expansion variables. It does so by introducing a labeled calculus where the labels are finite sequences of integers each of which represents a particular level at which the term can occur. These levels can be said to accurately capture the intuition behind E-variables: parts of the λ-term that are typed inside the uses of the E-variable-introduction typing rule for a particular E-variable e can interact with each other, and parts outside e can only pass the parts inside e around.

The proposed labeled calculus is typed using first a basic intersection type system with expansion variables but without an intersection elimination rule, and then using an intersection type system with expansion variables and an elimination rule. These two type systems (both with a universal type) illustrate that subject reduction and completeness fail in the abscence of an elimination rule.

We give a realisability semantics for both type systems showing that the first system is not complete in the sense that there are types whose semantic meaning is not the set of terms having this type. Then, we show that the second type system has the desirable properties of subject reduction, expansion and completeness (with infinite expansion variables).

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