Realizing Continuity Using Stateful Computations

Liron Cohen and Vincent Rahli

February, 2023

Motivation

Continuity is a key component of intuitionistic logic

$$\begin{array}{l}\forall F : \mathscr{B} \to \mathbb{N}.\ \forall \alpha : \mathscr{B}.\ \exists n : \mathbb{N}.\ \forall \beta : \mathscr{B}. \\ \qquad (\alpha = \beta \in \mathscr{B}_n) \to (F(\alpha) = F(\beta) \in \mathbb{N})\end{array}$$

($\mathscr{B} = \mathbb{N}^{\mathbb{N}}$ **&** $\mathscr{B}_n = \mathbb{N}^{\mathbb{N}_n}$**)**

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Models exist for MLTT, System T, CTT, etc.

Used for example to prove that all real-valued functions on the unit interval are continuous.

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Typical methods to validate continuity:

- Forcing-based approaches (Coquand et al.)
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⚠ **Non-extensional** (Kreisel, Troesltra, Escardó and Xu)

For example: do $\lambda\alpha.0$ and $\lambda\alpha.\texttt{let}\ x = \alpha(10)\ \texttt{in}\ x - x$ have the same modulus of continuity?

This talk in 1 slide

type modality

- Beth
- Kripke
- open

$$\mathtt{CTT} \longrightarrow \mathtt{TT}_{\mathsf{C}}^{\square}$$

choice operator

- reference cells
- choice sequences

Realized continuity
using exceptions

Realized continuity
using stateful computations

- non-empty modality
- choices are numbers
- choices can be retrieved

(Syntactical account of the semantical methods)

$\mathrm{TT}^{\Box}_{\mathscr{C}}$: A Family of Extensional Type Theories

A family of extensional type theories parameterized by
a type modality $\Box$, and a choice type $\mathscr{C}$,
compatible with intuitionistic and classical principles

Formalized in Agda

$TT_{\mathscr{C}}^{\Box}$: A Family of Extensional Type Theories

Untyped call-by-name lambda-calculus

sequent calculus

realizability semantics

Extensional

Dependent types

$\mathrm{TT}^{\square}_{\mathscr{C}}$: Syntax

Core Syntax:

$T \in \mathtt{Type} ::= \mathbb{N} \mid \mathbb{U}_i \mid \mathbf{\Pi} x{:}t.t \mid \mathbf{\Sigma} x{:}t.t \mid t = t \in t \mid t{+}t \mid \ldots$

$v \in \mathtt{Value} ::= T \mid \star \mid \underline{n} \mid \lambda x.t \mid \langle t,t \rangle \mid \mathtt{inl}(t) \mid \mathtt{inr}(t) \mid \ldots$

$t \in \mathtt{Term} ::= x \mid v \mid t\ t \mid \mathtt{fix}(t) \mid \mathtt{let}\ x := t\ \mathtt{in}\ t$
$\quad \mid \mathtt{case}\ t\ \mathtt{of}\ \mathtt{inl}(x) \Rightarrow t\ |\ \mathtt{inr}(y) \Rightarrow t$
$\quad \mid \mathtt{let}\ x,y = t\ \mathtt{in}\ t \mid \mathtt{if}\ t{=}t\ \mathtt{then}\ t\ \mathtt{else}\ t \mid \ldots$

$\mathsf{TT}_{\mathscr{C}}^{\Box}$: World-Based Computations

Core Operational Semantics:

$$\begin{array}{lcll} w & \vdash & (\lambda x.t_1)\ t_2 & \longmapsto & t_1[x\backslash t_2] \\ w & \vdash & \texttt{let}\ x_1,x_2=\langle t_1,t_2\rangle\ \texttt{in}\ t & \longmapsto & t[x_1\backslash t_1;x_2\backslash t_2] \\ w & \vdash & \texttt{fix}(v) & \longmapsto & v\ \texttt{fix}(v) \end{array}$$

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where $w \in \mathscr{W}$ (a poset with ordering $\sqsubseteq$)

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So far we haven't used the world

$\mathsf{TT}_{\mathscr{C}}^{\square}$: Choice Operator

Additional Components

- $\mathscr{N}$: abstract type of choice names
- $\mathscr{C}$: abstract type of choices inhabited by $\boldsymbol{\kappa}_0 \neq \boldsymbol{\kappa}_1$
- a partial function: $\mathsf{choice} \in \mathscr{W} \to \mathscr{N} \to \mathscr{C}$

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Syntax

$$v \in \mathtt{Value} ::= \cdots \mid \delta \ \text{(choice name)}$$

$$t \in \mathtt{Term} ::= \cdots \mid {!t} \ \text{(reading)}$$

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Operational Semantics

$$w \vdash\, !\delta \mapsto \mathsf{choice}(w, \delta)$$

$\mathsf{TT}_{\mathscr{C}}^{\square}$: Inference Rules

Standard ETT rules:

$$\frac{\Gamma, x : A \vdash b : B[x] \qquad \Gamma \vdash \star : (A \in \mathbb{U}_i)}{\Gamma \vdash \lambda x.b : \mathbf{\Pi} a{:}A.B[a]} \qquad \cdots$$

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+ LEM for some $\square$ modalities (e.g., Beth)

+ $\neg$LEM for some $\square$ modalities (e.g., open)

$\mathrm{TT}_{\mathscr{C}}^{\square}$: Realizability semantics

An inductive relation that expresses type equality

$$w \vDash T_1 \equiv T_2$$

A recursive function that expresses equality in a type

$$w \vDash a \equiv b \in T$$

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For example (product types):

$w \vDash \mathbf{\Pi} x_1 : A_1 . B_1 \equiv \mathbf{\Pi} x_2 : A_2 . B_2$
$\iff$
$\forall w' \sqsupseteq w . w' \vDash A_1 \equiv A_2 \ \wedge$
$\forall w' \sqsupseteq w . \forall a_1, a_2 .\ w' \vDash a_1 \equiv a_2 \in A_1 \Rightarrow w' \vDash B_1[x_1 \backslash a_1] \equiv B_2[x_2 \backslash a_2]$

$\mathsf{TT}_{\mathscr{C}}^{\square}$: Modalities

An abstract modality on (the semantics of) types: $\square$

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Forcing interpretation: $\square_w(w'.w' \vDash T) \rightarrow w \vDash T$

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Properties (where $(w : \mathscr{W}), (P, Q : \mathscr{P}_w)$):

monotonicity of $\square$	$\forall w' \sqsupseteq w. \square_w P \to \square_{w'} P$
K, distribution axiom	$\square_w(P \to Q) \to \square_w P \to \square_w Q$
$C4$, i.e., $\square\square \to \square$	$\square_w(w'.\square_{w'} P) \to \square_w P$
$\forall \to \square$	$\forall^{\sqsubseteq}_w(P) \to \square_w P$
T, reflexivity axiom	$\forall (P : \mathbb{P}). \square_w(w'.P) \to P$

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$\forall \rightarrow \square$	$\forall_w^{\sqsubseteq}(P) \rightarrow \square_w P$
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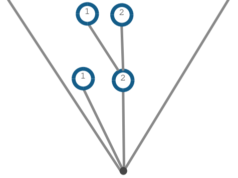
Enough to prove standard properties of the type system: consistency, symmetry, transitivity, etc.

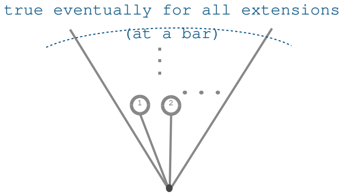
$TT^{\square}_{\mathscr{C}}$: Modalities

Kripke modality



$w \vDash T \iff$
$\forall w_1 \sqsupseteq w. w_1 \vDash T$
(modality: $\square_K(T)$)

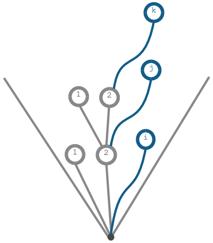
Beth modality



$w \vDash T \iff$
$\exists b \in \mathsf{bar}(w). \forall w_1 \in b.$
$\forall w_2 \sqsupseteq w_1. w_2 \vDash T$

(modality: $\square_B(T)$)

Open modality



$w \vDash T \iff$
$\forall w_1 \sqsupseteq w. \exists w_2 \sqsupseteq w_1.$
$\forall w_3 \sqsupseteq w_2. w_2 \vDash T$
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Modalities can be derived from **coverings**

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$C \in \mathsf{Covering}$ is a **covering** if:

- it is closed under binary intersections, union & subsets
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For example, Kripke, Beth, Open coverings

Continuity – Functions in $\mathbb{N}^{\mathscr{B}}$ only need initial segments

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Continuity axiom for numbers:

$$\forall F : \mathbb{N}^{\mathscr{B}}.\ \forall \alpha : \mathscr{B}.\ \exists n : \mathbb{N}.\ \forall \beta : \mathscr{B}.\ \alpha =_{\mathscr{B}_n} \beta \rightarrow F(\alpha) =_{\mathbb{N}} F(\beta)$$

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Uniform continuity theorem $(f \in [\alpha, \beta] \to \mathbb{R})$:

$$\forall \epsilon > 0. \exists \delta > 0. \forall x, y : [\alpha, \beta].\ |x - y| \leq \delta \to |f(x) - f(y)| \leq \epsilon$$

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False (Kreisel 62, Troelstra 77, Escardó & Xu 2015):

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Consistent with CTT if truncated (MSCS'17):

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Effectful computations following Longley's method:

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   let  exception e  in
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                 else  raise  e);
   true)  handle e ⇒ false
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Plus a loop until the modulus of continuity is reached

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Also consistent with $\mathsf{TT}^{\square}_{\mathscr{C}}$ (CSL'23):

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Again following Longley's method:

```
let r = ref 0 in
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More straightforward; No need for a loop

Continuity – Purity

Different moduli in extensions:

- $\lambda\alpha.\alpha(!\delta);0$
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? Are impure functions continuous?

? Can the modulus of continuity inhabit a variant of $\mathbb{N}$ where numbers are allowed to change in extensions?

Continuity – Purity

We require here functions to be pure ($\mathbf{\Pi_p}$):

Theorem (Continuity Principle)

The following continuity principle, is valid w.r.t. the above semantics:

$$\mathbf{\Pi_p} F{:}\mathscr{B} \to \mathbb{N}.\mathbf{\Pi_p}\alpha{:}\mathscr{B}.\downharpoonleft\mathbf{\Sigma} n{:}\mathbb{N}.\mathbf{\Pi_p}\beta{:}\mathscr{B}.$$
$$(\alpha = \beta \in \mathscr{B}_n) \to (F(\alpha) = F(\beta) \in \mathbb{N})$$

and is inhabited by the above computation, denoted $\texttt{mod}(F, \alpha)$

Continuity – Further Additional Components

Further Additional Components

- $\texttt{namefree}(t)$ states that t does not contain choices
- a function: $\mathsf{update} \in \mathscr{W} \rightarrow \mathscr{N} \rightarrow \mathscr{C} \rightarrow \mathscr{W}$

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Syntax

$$
\begin{aligned}
t \in \texttt{Term} &::= \cdots \boxed{\mid \boldsymbol{\nu} x.t} \boxed{\mid \texttt{choose}(t_1, t_2)} \\
&\mid \texttt{if } t_1 \texttt{ < } t_2 \texttt{ then } t_3 \texttt{ else } t_4 \mid t_1 + t_2 \\
T \in \texttt{Type} &::= \cdots \boxed{\mid \texttt{pure}} \mid t_1 \cap t_2 \mid \downharpoonleft t
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Operational Semantics

$$w, \mathsf{update}(w, \delta, t) \vdash \texttt{choose}(\delta, t) \mapsto \star$$

Continuity – Proof Steps

Step 1 (The Modulus is a Number)

If $\texttt{namefree}(F)$, $\texttt{namefree}(\alpha)$, $w \vDash F \in \mathbb{N}^{\mathscr{B}}$, *and* $w \vDash \alpha \in \mathscr{B}$, *for some world* w, *then* $w \vDash \texttt{mod}(F, \alpha) \in \mathbb{N}$

Continuity – Proof Steps

Step 1 (The Modulus is a Number)

If $\texttt{namefree}(F)$, $\texttt{namefree}(\alpha)$, $w \vDash F \in \mathbb{N}^{\mathscr{B}}$, *and* $w \vDash \alpha \in \mathscr{B}$, *for some world* w, *then* $w \vDash \texttt{mod}(F,\alpha) \in \mathbb{N}$

Step 2 (The Modulus is the Highest Number)

If $\texttt{mod}(F,\alpha) \Downarrow_{w'}^{w} \underline{n}$ *such that* $\texttt{mod}(F,\alpha)$ *generates a fresh name* δ, *then for any world* w_0 *occurring along this computation, it must be that* $\mathsf{choice}(w_0,\delta) \leq \mathsf{choice}(w',\delta)$.

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Step 3 (The Modulus is the Modulus)

If $w \vDash \alpha {\equiv} \beta \in \mathscr{B}_n$ *then* $w \vDash F(\alpha) {\equiv} F(\beta) \in \mathbb{N}$.

Summary

$\boxed{\mathrm{TT}_{\mathscr{C}}^{\Box}\text{: a type theory to program with effects}}$

$\Box \in \{\mathit{Kripke}, \mathit{Beth}, \mathit{Open}\}$
$\mathscr{C} \in \{\mathit{Ref}, \mathit{CS}\}$

$\boxed{\text{Simple reference-based computation of continuity}}$

What about impure functions?

Questions?