λ-calculi and Church-Rosser property

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A bit of History
formalisation of Mathematics (and functions)

- First formalisation of the concept of function by Frege (1879, premises of the formalisation of Mathematics)
- Discovery of some **paradoxes** in Mathematics (around 1900).
- Functions can be applied to any function: reflexiveness.
- Formalization of the concept of **type** by Russel to restrict application of functions (1908).
- Formalisation of Mathematics: design of logical systems

- Design of the **λ-calculus** by Church as part of a formal system for logic and functions (1932).
- The full system was inconsistent:
  - Church uses the type free λ-calculus to investigate functions (successful model for computation);
  - Church adds simple types $(\mathrm{int}, \mathrm{int} \rightarrow \mathrm{int})$ to λ-calculus in a system with logical axioms to deal with logic and function.
- Functions are studied as **computational rules** rather than as sets of pairs.

The λ-Calculus

"The traditional theory of λ-calculus relies on β-reduction, that is the capture by a function of its argument followed by the process of substituting this argument to the places where it is used." *(Hugo Herbelin)*

Let Var be a countably infinite set of variables and $x, y, z, f \in \text{Var}$.

$$M, N \in \Lambda ::= x \mid (\lambda x.M) \mid (MN)$$

Let Var be a countably infinite set of variables and $x, y, z, f \in$ Var.

$$M, N \in \Lambda ::= x \mid (\lambda x.M) \mid (MN)$$

Natural Numbers: Church Numerals

$$\underline{0} := \lambda fx.x$$

$$\underline{1} := \lambda fx.fx$$

$$\underline{2} := \lambda fx.f(fx)$$

$$\underline{3} := \lambda fx.f(f(fx))$$

$\underline{0} := \lambda fx.x$ means $\underline{0}$ is a notation for $\lambda fx.x$

the **β-reduction** is the compatible closure† of the following rule:

$$(\lambda x.M_1)M_2 \rightarrow_\beta M_1[x := M_2]$$

†if $M_1 \rightarrow_\beta M_2$ then $\lambda x.M_1 \rightarrow_\beta \lambda x.M_2$ and $M_1 N \rightarrow_\beta M_2 N$ and $N M_1 \rightarrow_\beta N M_2$

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$$(\lambda x.M_1)M_2 \rightarrow_\beta M_1[x := M_2]$$

Natural Numbers: Church Numerals

$$2 + 1 = 3 \qquad \text{or} \qquad 2 + 1 \rightarrow 3$$

$$\mathrm{Succ} := \lambda yfx.f(yfx)$$

$$\mathrm{Plus} := \lambda y.y\ \mathrm{Succ}$$

$$\mathrm{Plus}\ \underline{2}\ \underline{1} \rightarrow_\beta^* \underline{3}$$

† if $M_1 \rightarrow_\beta M_2$ then $\lambda x.M_1 \rightarrow_\beta \lambda x.M_2$ and $M_1 N \rightarrow_\beta M_2 N$ and $NM_1 \rightarrow_\beta NM_2$

$\mathrm{Succ} := \lambda yfx.f(yfx)$ (term in the λ-calculus)

and not

$\{\langle 0,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \dots\}$ (set of pairs)

The theory $\boldsymbol{\lambda}$: the set of terms Λ and the equivalence relation $=_\beta$

The equivalence relation:

$$M =_\beta M'$$

is defined as the symmetric, transitive and reflexive closure of:

$$M =_\beta M' \qquad \text{if} \qquad M \rightarrow_\beta M'$$

Is the theory $\boldsymbol{\lambda}$ consistent?
($\boldsymbol{\lambda}$ is consistent iff there exists M, N such that $M =_\beta N$ is not provable in $\boldsymbol{\lambda}$)

The Church-Rosser Property

in arithmetic

Intuition:

$$
\begin{tikzcd}
 & 4+2+3 \arrow[dl] \arrow[dr] & \\
6+3 \arrow[dr] & & 4+5 \arrow[dl] \\
 & 9 &
\end{tikzcd}
$$

The Church-Rosser property is also called the confluence property.

$\mathit{Plus}\ \underline{2}\ \underline{1} \rightarrow_\beta^* \underline{3}$

Is that always the case?

Does $\mathrm{Plus}\ \underline{2}\ \underline{1} \rightarrow_\beta^* \underline{n}$ for $n \in \mathbb{N} \setminus \{3\}$?

Theorem

The λ-calculus satisfies the Church-Rosser property

First, proved by Church and Rosser in 1936 for a restricted version of the λ-calculus [CR36].

Since then, proved by many people (the most common one by Tait and Martin-Löf using the parallel reductions method).

The Church-Rosser Theorem
consistency of the λ-calculus

Assume

$$\text{for all } M_1, N_2 \in \Lambda, M_1 =_\beta M_2$$

By the Church-Rosser theorem

$$\text{there exists } M_3 \text{ such that } M_1 \rightarrow^*_\beta M_3 \text{ and } M_2 \rightarrow^*_\beta M_3$$

But, for example:

$$\underline{0} \not\rightarrow_\beta \text{ and } \underline{1} \not\rightarrow_\beta$$

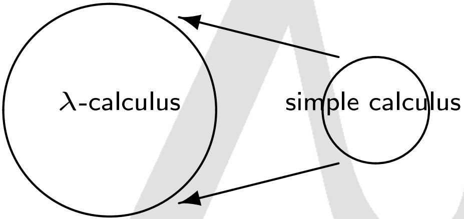
So

$$\underline{0} \neq_\beta \underline{1}$$

(recall: $\underline{0} := \lambda fx.x$ and $\underline{1} := \lambda fx.fx$)

The Church-Rosser Theorem
how do we prove that?

- Embedding into a simpler calculus (**parallel reduction**):
  - By modifying the reduction relation.
  - By modifying the set of terms
    (for example: based on a subset of the set of terms of the λ-calculus).



The extensional theory $\boldsymbol{\lambda} + \mathbf{ext}$ is defined as follows:

The theory $\boldsymbol{\lambda}$ is extended with the following rule:

$$M =_{\beta\eta} N \text{ if } Mx =_{\beta\eta} Nx$$

Is the theory $\boldsymbol{\lambda} + \mathbf{ext}$ consistent?

The **η-reduction** is the compatible closure of the rule:

$$\lambda x.Mx \rightarrow_\eta M, \qquad \text{if } x \notin \mathrm{fv}(M)$$

(example: $x \in \mathrm{fv}(\lambda y.xy)$ and $y, z \notin \mathrm{fv}(\lambda y.xy)$)

Intuition for this rule:

$$(\lambda x.Mx)N \rightarrow_\beta MN \text{ and } (\lambda x.Mx)N \rightarrow_\eta MN, \qquad \text{if } x \notin \mathrm{fv}(M)$$

The theory $\boldsymbol{\lambda} + \mathbf{ext}$ is equal to $\boldsymbol{\lambda}$ extended with the axiom scheme:

$$\lambda x.Mx =_{\beta\eta} M, \qquad \text{if } x \notin \mathrm{fv}(M)$$

Extensionality
consistency

Theorem

The λ-calculus satisfies the Church-Rosser property w.r.t. the $\beta\eta$-reduction.

Proved by:

- Hindley in 1974 [Hin74]
- Klop in 1980 [Klo80]
- Kamareddine and Rahli in 2008 [KR08]

We say that M has a normal form if there exists N such that $M \rightarrow^{*}_{\beta\eta} N$ and for all N', $N \not\rightarrow_{\beta\eta} N'$.

Theorem [Bar84]

Suppose M, N have a normal form. Then either $M =_{\beta\eta} N$ is provable in $\boldsymbol{\lambda} + \mathbf{ext}$ or $\boldsymbol{\lambda} + \mathbf{ext} + M =_{\beta\eta} N$ is inconsistent.

The theory $\boldsymbol{\lambda} + \mathbf{ext} + M =_{\beta\eta} N$ is the theory $\boldsymbol{\lambda} + \mathbf{ext}$ augmented with the axiom $M =_{\beta\eta} N$.

For example:

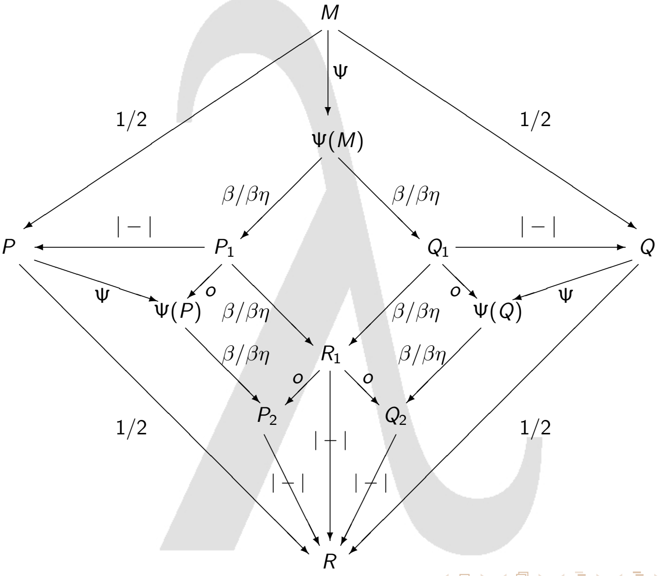
- Let $c, d \in \mathsf{Var}$. Let $\bar{x} \in \mathsf{Var}_{cd} = \mathsf{Var} \setminus \{c, d\}$.
  Let the parametrised set of terms $\Lambda_{cd}^{\beta\eta}$ be defined as follows:

$$\bar{M} \in \Lambda_{cd}^{\beta\eta} ::= \bar{x} \mid d(c\bar{M}) \mid d(\lambda\bar{x}.\bar{M}) \mid (\lambda\bar{x}.\bar{M}_1)\bar{M}_2 \mid c\bar{M}_1\bar{M}_2$$

- Why this calculus? To impose a control on the reductions.
- This calculus satisfies the Church-Rosser property w.r.t. the $\beta\eta$-reduction.
  (By reducibility: method based on a semantic argument.)
- With an embedding of the λ-calculus in this calculus we prove that the λ-calculus satisfies the Church-Rosser property w.r.t. the $\beta\eta$-reduction.
  (Using the method of parallel reduction.)

Extensionality

one of our proof scheme for both β- and $\beta\eta$-reduction



Extensionality

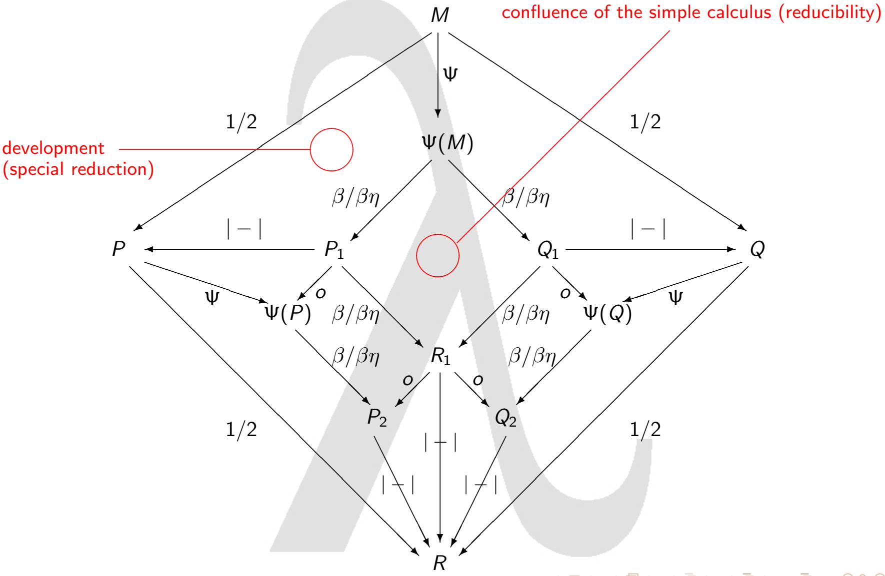
one of our proof scheme for both β- and $\beta\eta$-reduction

$$
\begin{tikzcd}
 & & & & M \arrow[ddllll, "1/2"'] \arrow[d, "\Psi"] \arrow[ddrrrr, "1/2"] & & & & \\
 & & & & \Psi(M) \arrow[dll, "\beta/\beta\eta"'] \arrow[drr, "\beta/\beta\eta"] & & & & \\
P \arrow[dr, "\Psi"'] \arrow[ddddrrrr, "1/2"'] & & P_1 \arrow[ll, "|-|"'] \arrow[dl, "o"] \arrow[ddrr, "\beta/\beta\eta"] & & & & Q_1 \arrow[rr, "|-|"] \arrow[dr, "o"] \arrow[ddll, "\beta/\beta\eta"'] & & Q \arrow[dl, "\Psi"] \arrow[ddddllll, "1/2"] \\
 & \Psi(P) \arrow[ddrr, "\beta/\beta\eta"] & & & & & & \Psi(Q) \arrow[ddll, "\beta/\beta\eta"'] & \\
 & & & & R_1 \arrow[dl, "o"'] \arrow[dr, "o"] \arrow[dd, "|-|"] & & & & \\
 & & & P_2 \arrow[dr, "|-|"] & & Q_2 \arrow[dl, "|-|"'] & & & \\
 & & & & R & & & &
\end{tikzcd}
$$

confluence of the simple calculus (reducibility)

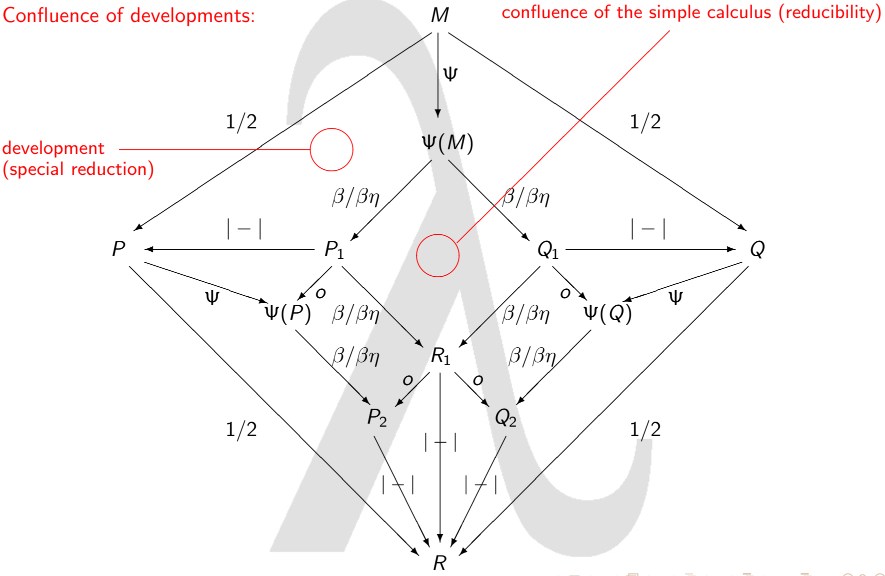
Extensionality

one of our proof scheme for both β- and $\beta\eta$-reduction



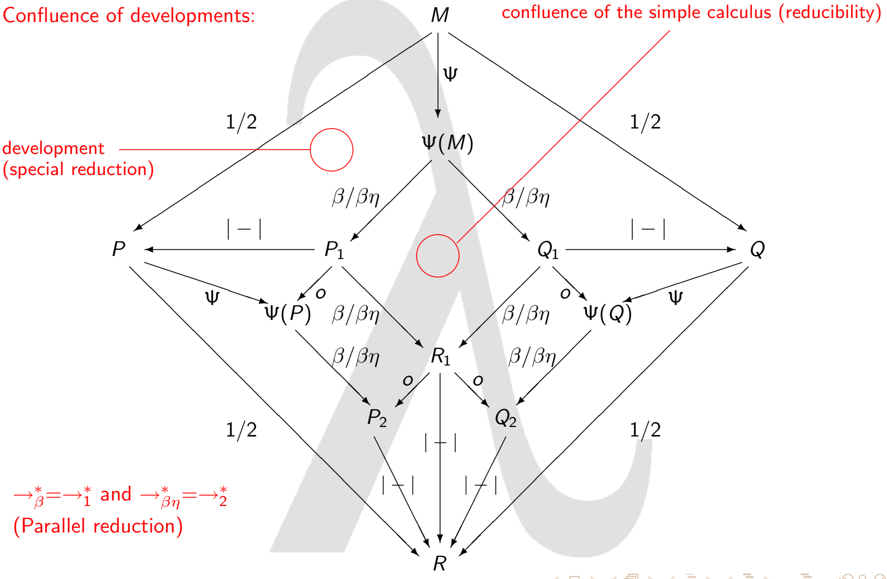
Extensionality

one of our proof scheme for both β- and $\beta\eta$-reduction



Extensionality

one of our proof scheme for both β- and $\beta\eta$-reduction



Our proof is a simplification and extension of the Ghilezan and Kunčak's proof of the Church-Rosser property w.r.t. the β-reduction:

$$
\begin{tikzcd}
 & & & & & M \arrow[d, "\Psi"] \arrow[ddddlllll, "I"'] \arrow[ddddrrrrr, "I"] & & & & & \\
 & & & & & \Psi(M) \arrow[dl, "o"] \arrow[dr, "o"'] & & & & & \\
 & & & & M_1 \arrow[dr, "o"] \arrow[ddll, "\beta"] & & M_2 \arrow[dl, "o"'] \arrow[ddrr, "\beta"'] & & & & \\
 & & & & & M_3 \arrow[ddll, "\beta"] \arrow[ddrr, "\beta"'] & & & & & \\
P \arrow[r, "\Psi"'] \arrow[ddddrrrrr, "I"'] & \Psi(P) \arrow[r, "o"'] & P_1 \arrow[dr, "o"'] & & & & & & Q_1 \arrow[dl, "o"] & \Psi(Q) \arrow[l, "o"] & Q \arrow[l, "\Psi"] \arrow[ddddlllll, "I"] \\
 & & & P_2 \arrow[drr, "\beta"'] & & & & Q_2 \arrow[dll, "\beta"] & & & \\
 & & & & & R_2 & & & & & \\
 & & & & & \Psi(R) \arrow[u, "o"'] & & & & & \\
 & & & & & R \arrow[u, "\Psi"'] & & & & &
\end{tikzcd}
$$

A general proof scheme

Both of the previous proof methods follow this proof scheme:

$$\begin{tikzcd}
 & & M \arrow[d, "\Psi"] \arrow[ddll, "1"'] \arrow[ddrr, "1"] & & \\
 & & P \arrow[dl, "\beta+"'] \arrow[dr, "\beta+"] & & \\
M_1 \arrow[r, "\Psi"] \arrow[ddrr, "1"'] & P_1 \arrow[dr, "\beta+"'] & & P_2 \arrow[dl, "\beta+"] & M_2 \arrow[l, "\Psi"'] \arrow[ddll, "1"] \\
 & & P_3 & & \\
 & & M_3 \arrow[u, "\Psi"'] & &
\end{tikzcd}$$

Remark

For now, this proof scheme is suitable for the Church-Rosser property only. Earlier attempts to find a common proof scheme for different properties of the lambda-calculus failed.

What else? What next?

Some of the other projects in which the ULTRA group is involved

- **Semantics** [KNRW08b, KNRW08a]: Soundness and completeness of semantics based on realisability, w.r.t. some type systems (intersection type systems with expansion variables). Project: find a complete semantics for an intersection type system with expansion (the whole mechanism).
- **Reducibility** [KRW08, KR08]: Study and simplification of proofs of properties of the lambda-calculus. Project: find a general proof scheme suitable for different properties of the lambda-calculus (such as Church-Rosser/normalisation/standardisation).
- **Type Error Slicing** [HW04]: The aim is to accurately identify the location of a type error of a piece of code (for a SML-based programming language), by providing a minimal and necessary set of points in the piece of code (a slice). Project: Extension to a bigger language.

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