AI1103 Assignment-4

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Download all python codes from

```
https://github.com/vrahul02/AI1103−Probability−and−Random−Variables/tree/main/Assignment−4/Codes
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and latex-tikz codes from

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https://github.com/vrahul02/AI1103−Probability−and−Random−Variables/tree/main/Assignment−4/Assignment−4.tex
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Problem GATE 2021 (ST), Q.15

A fair die is rolled twice independently. Let X and Y denote the outcomes of the first and second roll, respectively. Then $E(X + Y \mid (X - Y)^2 = 1)$ equals

Solution

X and Y are two independent random variables that can take the values 1, 2, 3, 4, 5, 6.

$$\Pr(X = k) = \frac{1}{6}, 1 \le k \le 6 \tag{0.0.1}$$

$$\Pr(Y = k) = \frac{1}{6}, 1 \le k \le 6 \tag{0.0.2}$$

Lemma 0.1. *The PMF of X+Y is given by*

$$\Pr(X + Y = n) = \begin{cases} \frac{n-1}{36} & , 2 \le n \le 7 \\ \frac{13-n}{36} & , 8 \le n \le 12 \end{cases} \tag{0.0.3}$$

Proof. Using convolution for discrete random variables,
$\Pr(X + Y = n)$

$$= \sum_{k=n-6}^{n-1} \Pr(X = k, Y = n - k), 1 \le k \le 6 \tag{0.0.4}$$

Since X and Y are independent,

$$= \sum_{k=n-6}^{n-1} \Pr(X = k) \times \Pr(Y = n - k), 1 \le k \le 6 \tag{0.0.5}$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{36}, 1 \le k \le 6 \tag{0.0.6}$$

$$= \begin{cases} \frac{n-1}{36} & , 2 \le n \le 7 \\ \frac{13-n}{36} & , 8 \le n \le 12 \end{cases} \tag{0.0.7}$$

□

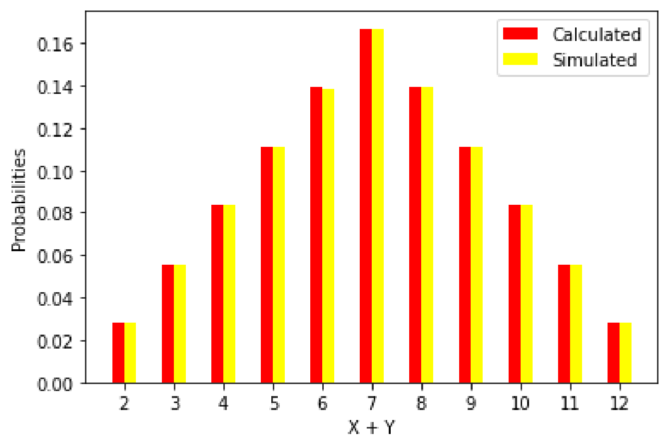

Fig. 0: Plot of PMF for X+Y

Lemma 0.2. *The PMF of X-Y is given by*

$$\Pr(X - Y = n) = \begin{cases} \frac{n+6}{36} & , -5 \le n \le 0 \\ \frac{6-n}{36} & , 1 \le n \le 5 \end{cases} \tag{0.0.8}$$

Proof. Using convolution for discrete random variables,
$\Pr(X - Y = n)$

$$= \sum_{k=n+1}^{n+6} \Pr(X = k, Y = k - n), 1 \le k \le 6 \tag{0.0.9}$$

Since X and Y are independent,

$$= \sum_{k=n+1}^{n+6} \Pr(X = k) \times \Pr(Y = k - n), 1 \le k \le 6 \tag{0.0.10}$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{36}, 1 \leq k \leq 6 \tag{0.0.11}$$

$$= \begin{cases} \frac{n+6}{36} & , -5 \leq n \leq 0 \\ \frac{6-n}{36} & , 1 \leq n \leq 5 \end{cases} \tag{0.0.12}$$

$\square$

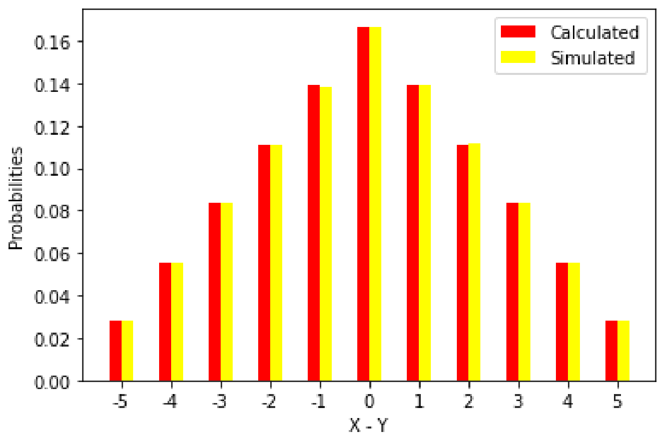

Fig. 0: Plot of PMF for X-Y

$$E\left(X + Y \mid (X - Y)^2 = 1\right)$$

$$= \sum n \times \Pr\left(X + Y = n \mid (X - Y)^2 = 1\right) \tag{0.0.13}$$

$$= \sum n \times \frac{\Pr\left(X + Y = n, (X - Y)^2 = 1\right)}{\Pr\left((X - Y)^2 = 1\right)} \tag{0.0.14}$$

$$\begin{aligned} &= \sum n \times \frac{\Pr\left(X + Y = n, (X - Y) = 1\right)}{\Pr\left((X - Y) = 1\right)} \\ &\quad \times \Pr\left((X - Y) = 1 | (X - Y)^2 = 1\right) \\ &+ \sum n \times \frac{\Pr\left(X + Y = n, (X - Y) = -1\right)}{\Pr\left((X - Y) = -1\right)} \\ &\quad \times \Pr\left((X - Y) = 1 \mid (X - Y)^2 = 1\right) \end{aligned} \tag{0.0.15}$$

$$\begin{aligned} &= \frac{\Pr\left((X - Y) = 1 \mid (X - Y)^2 = 1\right)}{\Pr\left((X - Y) = 1\right)} \\ &\times \sum n \times \Pr\left(X + Y = n, (X - Y) = 1\right) \\ &+ \frac{\Pr\left((X - Y) = -1 \mid (X - Y)^2 = 1\right)}{\Pr\left((X - Y) = -1\right)} \\ &\times \sum n \times \Pr\left(X + Y = n, (X - Y) = -1\right) \end{aligned} \tag{0.0.16}$$

Using equations (0.0.7) and (0.0.12) in (0.0.16)

We get,

$$E\left(X + Y \mid (X - Y)^2 = 1\right)$$

$$= \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) + \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) \tag{0.0.17}$$

$$= 7 \tag{0.0.18}$$