AI1103 Assignment-4

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Download all python codes from

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https://github.com/vrahul02/AI1103-Probability-and-Random-Variables/tree/main/Assignment-4/Codes
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and latex-tikz codes from

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https://github.com/vrahul02/AI1103-Probability-and-Random-Variables/tree/main/Assignment-4/Assignment-4.tex
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Problem GATE 2021 (ST), Q.15

A fair die is rolled twice independently. Let X and Y denote the outcomes of the first and second roll, respectively. Then $E(X+Y \mid (X-Y)^2 = 1)$ equals

Solution

X and Y are two independent random variables that can take the values 1, 2, 3, 4, 5, 6.

$$\Pr(X = k) = \frac{1}{6}, 1 \le k \le 6 \quad (0.0.1)$$

$$\Pr(Y = k) = \frac{1}{6}, 1 \le k \le 6 \quad (0.0.2)$$

On using convolution for discrete random variables, $\Pr(X+Y=n)$

$$= \sum_{k=n-6}^{n-1} \Pr(X = k, Y = n-k), 1 \le k \le 6 \quad (0.0.3)$$

Since X and Y are independent random variables,

$$= \sum_{k=n-6}^{n-1} \Pr(X = k) \times \Pr(Y = n-k), 1 \le k \le 6 \quad (0.0.4)$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{6} \times \frac{1}{6}, 1 \le k \le 6 \quad (0.0.5)$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{36}, 1 \le k \le 6 \quad (0.0.6)$$

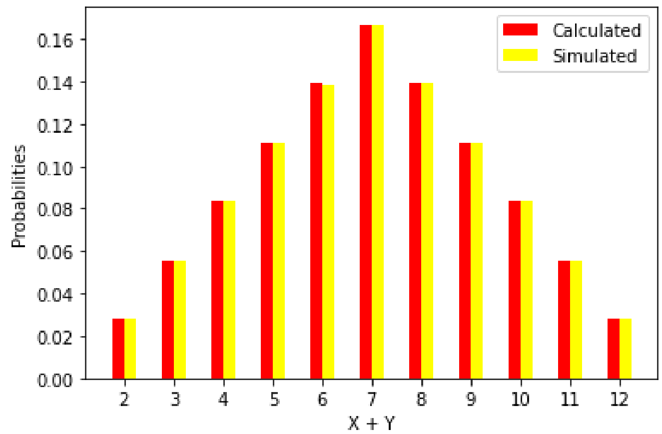

Fig. 0: Plot of PMF for X+Y

$\Pr(X-Y=n)$

$$= \sum_{k=n+1}^{n+6} \Pr(X = k, Y = k-n), 1 \le k \le 6 \quad (0.0.7)$$

Since X and Y are independent random variables,

$$= \sum_{k=n+1}^{n+6} \Pr(X = k) \times \Pr(Y = k-n), 1 \le k \le 6 \quad (0.0.8)$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{6} \times \frac{1}{6}, 1 \le k \le 6 \quad (0.0.9)$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{36}, 1 \le k \le 6 \quad (0.0.10)$$

$$(X-Y)^2 = 1 \quad (0.0.11)$$

$$X-Y = +1,\ X-Y = -1 \quad (0.0.12)$$

X+Y=n	3	5	7	9	11
Outcomes	(1,2), (2,1)	(2,3), (3,2)	(3,4), (4,3)	(4,5), (5,4)	(5,6), (6,5)

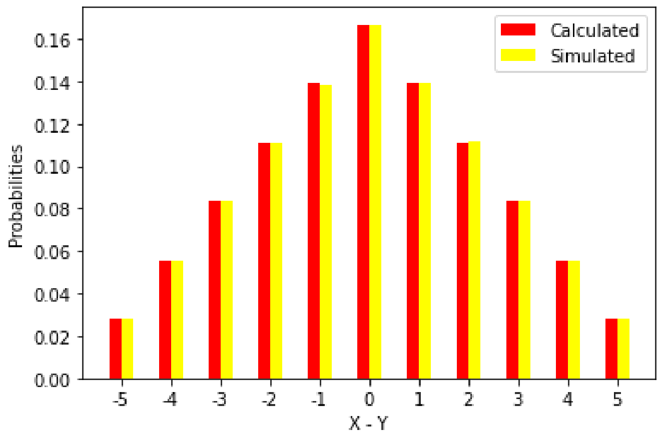

Fig. 0: Plot of PMF for X-Y

$$E\left(X+Y \mid (X-Y)^2=1\right)$$

$$=\sum n \times \Pr\left(X+Y=n \mid (X-Y)^2=1\right) \tag{0.0.13}$$

$$=\sum n \times \frac{\Pr\left(X+Y=n,(X-Y)^2=1\right)}{\Pr\left((X-Y)^2=1\right)} \tag{0.0.14}$$

$$\begin{aligned}
&=\sum n \times \frac{\Pr\left(X+Y=n,(X-Y)=1\right)}{\Pr\left((X-Y)=1\right)} \\
&\qquad \times \Pr\left((X-Y)=1|(X-Y)^2=1\right) \\
&+\sum n \times \frac{\Pr\left(X+Y=n,(X-Y)=-1\right)}{\Pr\left((X-Y)=-1\right)} \\
&\qquad \times \Pr\left((X-Y)=1 \mid (X-Y)^2=1\right)
\end{aligned} \tag{0.0.15}$$

$$\begin{aligned}
&=\frac{\Pr\left((X-Y)=1 \mid (X-Y)^2=1\right)}{\Pr\left((X-Y)=1\right)} \\
&\times \sum n \times \Pr\left(X+Y=n,(X-Y)=1\right) \\
&+\frac{\Pr\left((X-Y)=-1 \mid (X-Y)^2=1\right)}{\Pr\left((X-Y)=-1\right)} \\
&\times \sum n \times \Pr\left(X+Y=n,(X-Y)=-1\right)
\end{aligned} \tag{0.0.16}$$

Using equations (0.0.6) and (0.0.10)
We get,

$$E\left(X+Y \mid (X-Y)^2=1\right)$$

$$=\left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) + \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) \tag{0.0.17}$$

$$=7 \tag{0.0.18}$$