AI1103 Assignment-4

V Rahul - AI20BTECH11030

Download all python codes from

```
https://github.com/vrahul02/AI1103-Probability-and-Random-Variables/tree/main/Assignment-4/Codes
```

and latex-tikz codes from

```
https://github.com/vrahul02/AI1103-Probability-and-Random-Variables/tree/main/Assignment-4/Assignment-4.tex
```

Problem GATE 2021 (ST), Q.15

A fair die is rolled twice independently. Let X and Y denote the outcomes of the first and second roll, respectively. Then $E(X + Y \mid (X - Y)^2 = 1)$ equals

Solution

X and Y are two independent random variables that can take the values 1, 2, 3, 4, 5, 6.

$$\Pr(X = k) = \frac{1}{6}, 1 \le k \le 6 \tag{0.0.1}$$

$$\Pr(Y = k) = \frac{1}{6}, 1 \le k \le 6 \tag{0.0.2}$$

On using convolution for discrete random variables, $\Pr(X + Y = n)$

$$= \sum_{k=n-6}^{n-1} \Pr(X = k, Y = n - k), 1 \le k \le 6 \tag{0.0.3}$$

Since X and Y are independent random variables,

$$= \sum_{k=n-6}^{n-1} \Pr(X = k) \times \Pr(Y = n - k), 1 \le k \le 6 \tag{0.0.4}$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{36}, 1 \le k \le 6 \tag{0.0.5}$$

$$= \begin{cases} n - 1 & \text{if } 2 \le n \le 7 \\ 13 - n & \text{if } 8 \le n \le 12 \end{cases} \tag{0.0.6}$$

$\Pr(X - Y = n)$

$$= \sum_{k=n+1}^{n+6} \Pr(X = k, Y = k - n), 1 \le k \le 6 \tag{0.0.7}$$

Since X and Y are independent random variables,

$$= \sum_{k=n+1}^{n+6} \Pr(X = k) \times \Pr(Y = k - n), 1 \le k \le 6 \tag{0.0.8}$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{36}, 1 \le k \le 6 \tag{0.0.9}$$

$$= \begin{cases} n + 6 & \text{if } -5 \le n \le 0 \\ 6 - n & \text{if } 1 \le n \le 5 \end{cases} \tag{0.0.10}$$

$E\left(X + Y \mid (X - Y)^2 = 1\right)$

$$= \sum n \times \Pr\left(X + Y = n \mid (X - Y)^2 = 1\right) \tag{0.0.11}$$

$$= \sum n \times \frac{\Pr\left(X + Y = n, (X - Y)^2 = 1\right)}{\Pr((X - Y)^2 = 1)} \tag{0.0.12}$$

$$\begin{aligned} &= \sum n \times \frac{\Pr(X + Y = n, (X - Y) = 1)}{\Pr((X - Y) = 1)} \\ &\quad \times \Pr\left((X - Y) = 1 | (X - Y)^2 = 1\right) \\ &+ \sum n \times \frac{\Pr(X + Y = n, (X - Y) = -1)}{\Pr((X - Y) = -1)} \\ &\quad \times \Pr\left((X - Y) = 1 \mid (X - Y)^2 = 1\right) \end{aligned} \tag{0.0.13}$$

$$\begin{aligned} &= \frac{\Pr\left((X - Y) = 1 \mid (X - Y)^2 = 1\right)}{\Pr((X - Y) = 1)} \\ &\quad \times \sum n \times \Pr(X + Y = n, (X - Y) = 1) \\ &\quad + \frac{\Pr\left((X - Y) = -1 \mid (X - Y)^2 = 1\right)}{\Pr((X - Y) = -1)} \\ &\quad \times \sum n \times \Pr(X + Y = n, (X - Y) = -1) \end{aligned} \tag{0.0.14}$$

Using equations (0.0.5) and (0.0.9) in (0.0.14)
We get,
$E\left(X+Y\mid (X-Y)^2=1\right)$

$$= \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) + \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) \tag{0.0.15}$$

$$= 7 \tag{0.0.16}$$

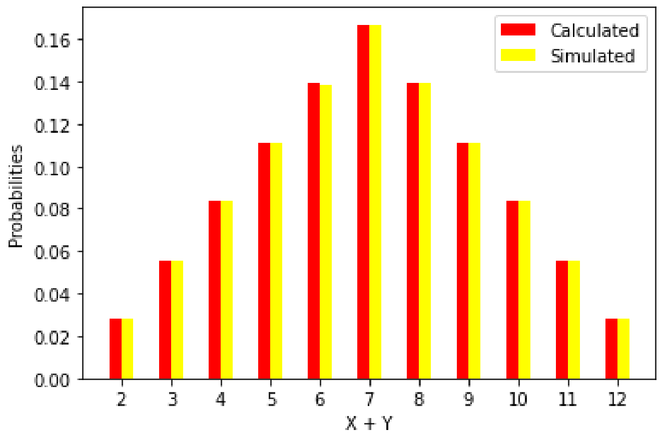

Fig. 0: Plot of PMF for X+Y

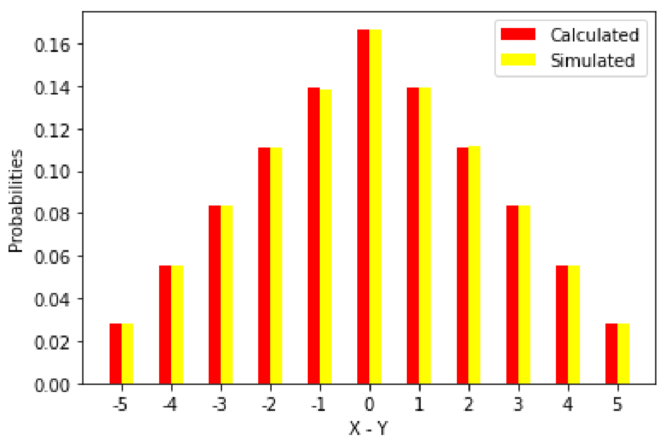

Fig. 0: Plot of PMF for X-Y