

AI1103 Assignment-4

V Rahul - AI20BTECH11030

Download all python codes from

<https://github.com/vrahul02/AI1103-Probability-and-Random-Variables/tree/main/Assignment-4/Codes>

and latex-tikz codes from

<https://github.com/vrahul02/AI1103-Probability-and-Random-Variables/tree/main/Assignment-4/Assignment-4.tex>

PROBLEM GATE 2021 (ST), Q.15

A fair die is rolled twice independently. Let X and Y denote the outcomes of the first and second roll, respectively. Then $E(X + Y | (X - Y)^2 = 1)$ equals

SOLUTION

X and Y are two independent random variables that can take the values 1, 2, 3, 4, 5, 6.

$$\Pr(X = k) = \frac{1}{6}, 1 \leq k \leq 6 \quad (0.0.1)$$

$$\Pr(Y = k) = \frac{1}{6}, 1 \leq k \leq 6 \quad (0.0.2)$$

$$(X - Y)^2 = 1 \quad (0.0.3)$$

$$X - Y = +1, X - Y = -1 \quad (0.0.4)$$

X+Y=n	3	5	7	9	11
Outcomes	(1,2), (2,1)	(2,3), (3,2)	(3,4), (4,3)	(4,5), (5,4)	(5,6), (6,5)

$$E(X + Y | (X - Y)^2 = 1) = \sum n \times \Pr(X + Y = n | (X - Y)^2 = 1) \quad (0.0.5)$$

$$= \sum n \times \frac{\Pr(X + Y = n, (X - Y)^2 = 1)}{\Pr((X - Y)^2 = 1)} \quad (0.0.6)$$

$$= \sum n \times \frac{\Pr(X + Y = n, (X - Y) = 1)}{\Pr((X - Y) = 1)} \times \Pr((X - Y) = 1 | (X - Y)^2 = 1) + \sum n \times \frac{\Pr(X + Y = n, (X - Y) = -1)}{\Pr((X - Y) = -1)} \times \Pr((X - Y) = -1 | (X - Y)^2 = 1) \quad (0.0.7)$$

$$= \frac{\Pr((X - Y) = 1 | (X - Y)^2 = 1)}{\Pr((X - Y) = 1)} \times \sum n \times \Pr(X + Y = n, (X - Y) = 1) + \frac{\Pr((X - Y) = -1 | (X - Y)^2 = 1)}{\Pr((X - Y) = -1)} \times \sum n \times \Pr(X + Y = n, (X - Y) = -1) \quad (0.0.8)$$

On using convolution for discrete random variables, $\Pr(X + Y = n)$

$$= \sum_{k=n-6}^{n-1} \Pr(X = k, Y = n - k), 1 \leq k \leq 6 \quad (0.0.9)$$

Since X and Y are independent random variables,

$$= \sum_{k=n-6}^{n-1} \Pr(X = k) \times \Pr(Y = n - k), 1 \leq k \leq 6 \quad (0.0.10)$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{6} \times \frac{1}{6}, 1 \leq k \leq 6 \quad (0.0.11)$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{36}, 1 \leq k \leq 6 \quad (0.0.12)$$

$\Pr(X - Y = n)$

$$= \sum_{k=n+1}^{n+6} \Pr(X = k, Y = k - n), 1 \leq k \leq 6 \quad (0.0.13)$$

Since X and Y are independent random variables,

$$= \sum_{k=n+1}^{n+6} \Pr(X = k) \times \Pr(Y = k - n), 1 \leq k \leq 6 \quad (0.0.14)$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{6} \times \frac{1}{6}, 1 \leq k \leq 6 \quad (0.0.15)$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{36}, 1 \leq k \leq 6 \quad (0.0.16)$$

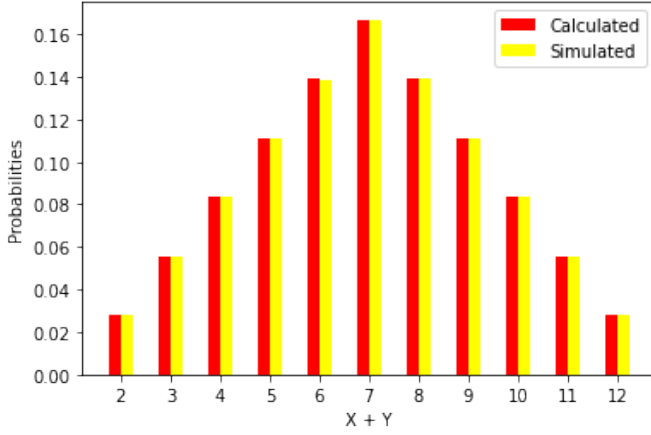


Fig. 0: Plot of PMF for $X+Y$

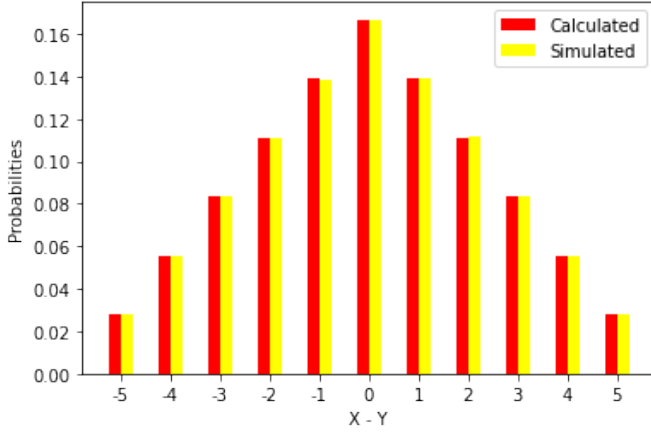


Fig. 0: Plot of PMF for $X-Y$

Using equations (0.0.12) and (0.0.16) in (0.0.8)

We get,

$$E(X + Y | (X - Y)^2 = 1)$$

$$= \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) + \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) \quad (0.0.17)$$

$$= 7 \quad (0.0.18)$$