GATE 2021 (ST), Q.15

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Question

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A fair die is rolled twice independently. Let X and Y denote the outcomes of the first and second roll, respectively. Then $E(X + Y \mid (X - Y)^2 = 1)$ equals

PDF of X+Y

PDF of sum of random variables X and Y given their individual PDFs can be calculated using

1. Convolution
2. Characteristic Function

PDF of X and Y

X and Y are two independent random variables that can take the values 1, 2, 3, 4, 5, 6.

$$\Pr(X = k) = \frac{1}{6}, 1 \leq k \leq 6 \tag{1}$$

$$\Pr(Y = k) = \frac{1}{6}, 1 \leq k \leq 6 \tag{2}$$

Convolution

The general formula for the distribution of the sum Z=X+Y of two discrete random variables is

$$\Pr(Z = z) = \sum_{k=-\infty}^{\infty} \Pr(X = k, Y = z - k) \tag{3}$$

If X and Y are independent

$$\Pr(Z = z) = \sum_{k=-\infty}^{\infty} \Pr(X = k) \times \Pr(Y = z - k) \tag{4}$$

PDF of X+Y using convolution

$$\Pr\left(X+Y=n\right)$$

$$= \sum_{k=n-6}^{n-1} \Pr\left(X=k\right) \times \Pr\left(Y=n-k\right), 1 \le k \le 6 \tag{5}$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{6} \times \frac{1}{6}, 1 \le k \le 6 \tag{6}$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{36}, 1 \le k \le 6 \tag{7}$$

$$= \begin{cases} \frac{n-1}{36} & , 2 \le n \le 7 \\ \frac{13-n}{36} & , 8 \le n \le 12 \end{cases} \tag{8}$$

PDF of X+Y using convolution

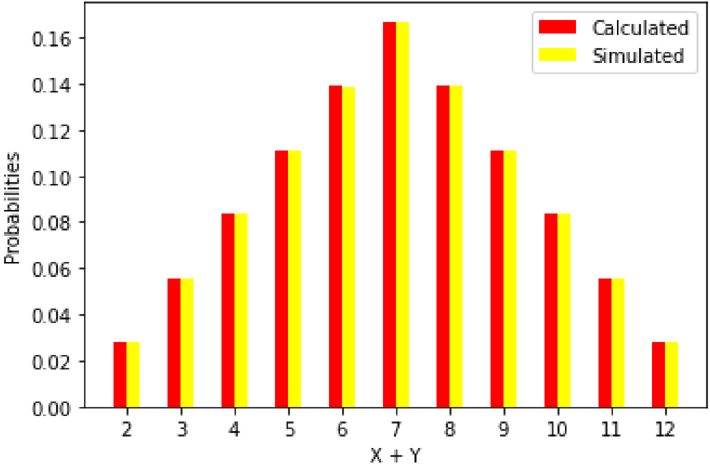

Figure: Plot of PMF for X+Y

PDF of X-Y using convolution

$$\Pr\left(X - Y = n\right)$$

$$= \sum_{k=n+1}^{n+6} \Pr\left(X = k\right) \times \Pr\left(Y = k - n\right), 1 \le k \le 6 \tag{9}$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{6} \times \frac{1}{6}, 1 \le k \le 6 \tag{10}$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{36}, 1 \le k \le 6 \tag{11}$$

$$= \begin{cases} \frac{n+6}{36} & , -5 \le n \le 0 \\ \frac{6-n}{36} & , 1 \le n \le 5 \end{cases} \tag{12}$$

PDF of X-Y using convolution

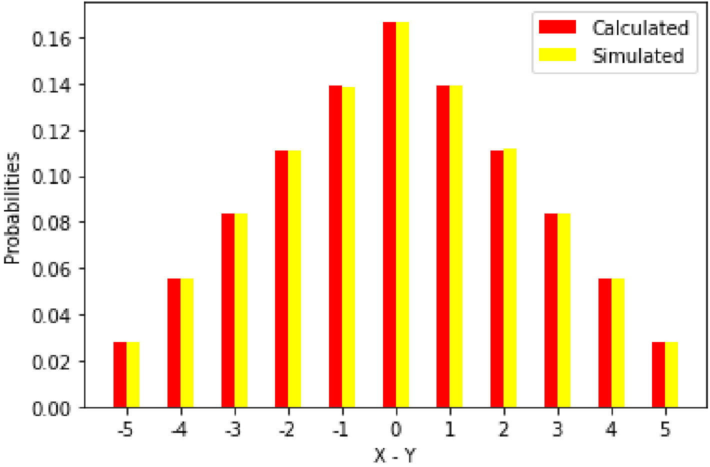

Figure: Plot of PMF for X-Y

Expectation value

Properties

$$E(X) = \sum x_i \times \Pr(X = x_i) \tag{13}$$

$$\Pr(A|B) = \frac{\Pr(A, B)}{\Pr(B)} \tag{14}$$

$E\left(X + Y \mid (X - Y)^2 = 1\right)$

$$= \sum n \times \Pr\left(X + Y = n \mid (X - Y)^2 = 1\right) \tag{15}$$

$$= \sum n \times \frac{\Pr\left(X + Y = n, (X - Y)^2 = 1\right)}{\Pr((X - Y)^2 = 1)} \tag{16}$$

Solution contd.

$$E\left(X+Y \mid (X-Y)^2=1\right)$$

$$= \sum n \times \frac{\Pr\left(X+Y=n,(X-Y)=1\right)}{\Pr\left((X-Y)=1\right)} \times \Pr\left((X-Y)=1 | (X-Y)^2=1\right) + \sum n \times \frac{\Pr\left(X+Y=n,(X-Y)=-1\right)}{\Pr\left((X-Y)=-1\right)} \times \Pr\left((X-Y)=1 \mid (X-Y)^2=1\right) \tag{17}$$

$$= \frac{\Pr\left((X-Y)=1 \mid (X-Y)^2=1\right)}{\Pr\left((X-Y)=1\right)} \times \sum n \times \Pr\left(X+Y=n,(X-Y)=1\right) + \frac{\Pr\left((X-Y)=-1 \mid (X-Y)^2=1\right)}{\Pr\left((X-Y)=-1\right)}$$

Solution contd.

Using equations (**??**) and (**??**) in (**??**)
We get,
$E\left(X+Y \mid (X-Y)^2=1\right)$

$$= \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) + \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) \tag{19}$$

$$= 7 \tag{20}$$