AI1103 Assignment-4

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A fair die is rolled twice independently. Let X and Y denote the outcomes of the first and second roll, respectively. Then $E(X + Y \mid (X − Y)^2 = 1)$ equals

SOLUTION

X and Y are two independent random variables that can take the values 1, 2, 3, 4, 5, 6.

$$\Pr(X = k) = \frac{1}{6}, 1 \le k \le 6 \tag{0.0.1}$$

$$\Pr(Y = k) = \frac{1}{6}, 1 \le k \le 6 \tag{0.0.2}$$

On using convolution for discrete random variables,
$\Pr(X + Y = n)$

$$= \sum_{k=n-6}^{n-1} \Pr(X = k, Y = n - k), 1 \le k \le 6 \tag{0.0.3}$$

Since X and Y are independent random variables,

$$= \sum_{k=n-6}^{n-1} \Pr(X = k) \times \Pr(Y = n - k), 1 \le k \le 6 \tag{0.0.4}$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{36}, 1 \le k \le 6 \tag{0.0.5}$$

$$= \begin{cases} n - 1 & \text{if } 2 \le n \le 7 \\ 13 - n & \text{if } 8 \le n \le 12 \end{cases} \tag{0.0.6}$$

$\Pr(X − Y = n)$

$$= \sum_{k=n+1}^{n+6} \Pr(X = k, Y = k - n), 1 \le k \le 6 \tag{0.0.7}$$

Since X and Y are independent random variables,

$$= \sum_{k=n+1}^{n+6} \Pr(X = k) \times \Pr(Y = k - n), 1 \le k \le 6 \tag{0.0.8}$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{36}, 1 \le k \le 6 \tag{0.0.9}$$

$$= \begin{cases} n + 6 & \text{if } -5 \le n \le 0 \\ 6 - n & \text{if } 1 \le n \le 5 \end{cases} \tag{0.0.10}$$

$E\left(X + Y \mid (X − Y)^2 = 1\right)$

$$= \sum n \times \Pr\left(X + Y = n \mid (X − Y)^2 = 1\right) \tag{0.0.11}$$

$$= \sum n \times \frac{\Pr\left(X + Y = n, (X − Y)^2 = 1\right)}{\Pr\left((X − Y)^2 = 1\right)} \tag{0.0.12}$$

$$\begin{aligned} &= \sum n \times \frac{\Pr(X + Y = n, (X − Y) = 1)}{\Pr((X − Y) = 1)} \\ &\quad \times \Pr\left((X − Y) = 1 \mid (X − Y)^2 = 1\right) \\ &+ \sum n \times \frac{\Pr(X + Y = n, (X − Y) = -1)}{\Pr((X − Y) = -1)} \\ &\quad \times \Pr\left((X − Y) = 1 \mid (X − Y)^2 = 1\right) \end{aligned} \tag{0.0.13}$$

$$\begin{aligned} &= \frac{\Pr\left((X − Y) = 1 \mid (X − Y)^2 = 1\right)}{\Pr((X − Y) = 1)} \\ &\times \sum n \times \Pr(X + Y = n, (X − Y) = 1) \\ &+ \frac{\Pr\left((X − Y) = -1 \mid (X − Y)^2 = 1\right)}{\Pr((X − Y) = -1)} \\ &\times \sum n \times \Pr(X + Y = n, (X − Y) = -1) \end{aligned} \tag{0.0.14}$$

Using equations (0.0.5) and (0.0.9) in (0.0.14)
We get,
$$E\left(X + Y \mid (X - Y)^2 = 1\right)$$

$$= \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) + \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) \qquad (0.0.15)$$
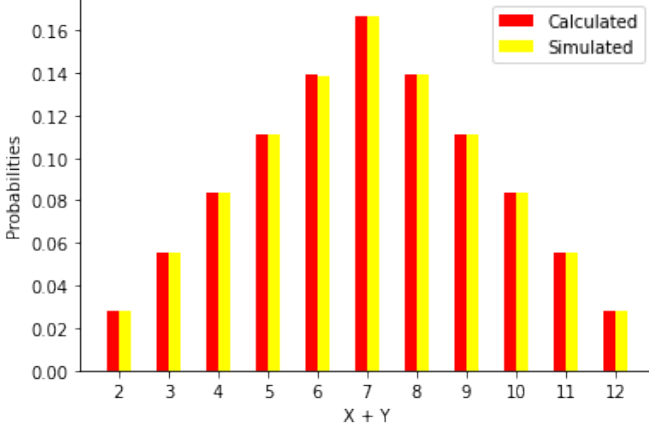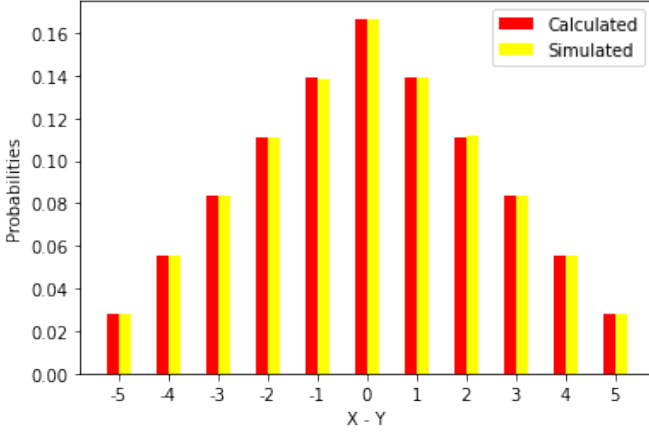
$$= 7 \qquad\qquad\qquad (0.0.16)$$

Fig. 0: Plot of PMF for X+Y

Fig. 0: Plot of PMF for X-Y