GATE 2021 (ST), Q.15

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Question

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A fair die is rolled twice independently. Let X and Y denote the outcomes of the first and second roll, respectively. Then $E(X + Y \mid (X - Y)^2 = 1)$ equals

PDF of X+Y

PDF of sum of random variables X and Y given their individual PDFs can be calculated using

1. Convolution
2. Characteristic Function

PDF of X and Y

X and Y are two independent random variables that can take the values 1, 2, 3, 4, 5, 6.

$$\Pr(X = k) = \frac{1}{6}, 1 \le k \le 6 \tag{1}$$

$$\Pr(Y = k) = \frac{1}{6}, 1 \le k \le 6 \tag{2}$$

Convolution

The general formula for the distribution of the sum Z=X+Y of two discrete random variables is

$$\Pr(Z = z) = \sum_{k=-\infty}^{\infty} \Pr(X = k, Y = z - k) \tag{3}$$

If X and Y are independent,

$$\Pr(Z = z) = \sum_{k=-\infty}^{\infty} \Pr(X = k) \times \Pr(Y = z - k) \tag{4}$$

The counterpart for independent continuous random variables X and Y with probability density functions f(x), g(x) is

$$\Pr(Z = z) = \int_{-\infty}^{+\infty} x \times f(t) \times g(z - t) dt \tag{5}$$

PDF of X+Y using convolution

$\Pr\left(X+Y=n\right)$

$$= \sum_{k=n-6}^{n-1} \Pr\left(X=k\right) \times \Pr\left(Y=n-k\right), 1 \le k \le 6 \tag{6}$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{6} \times \frac{1}{6}, 1 \le k \le 6 \tag{7}$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{36}, 1 \le k \le 6 \tag{8}$$

$$= \begin{cases} \frac{n-1}{36} & , 2 \le n \le 7 \\ \frac{13-n}{36} & , 8 \le n \le 12 \end{cases} \tag{9}$$

PDF of X+Y using convolution

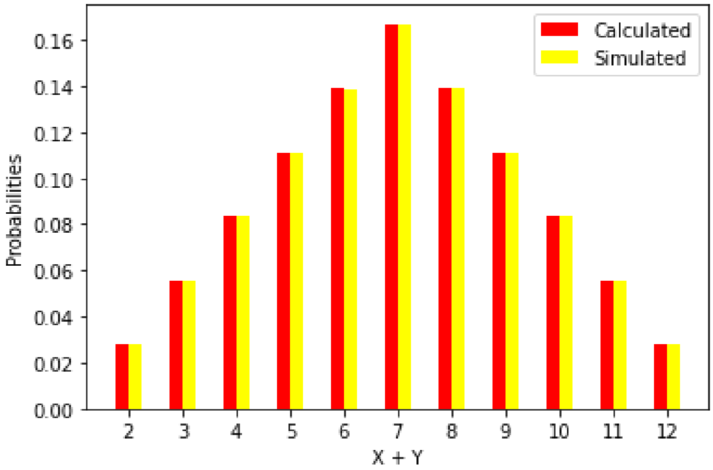

Figure: Plot of PMF for X+Y

PDF of X-Y using convolution

$$\Pr\left(X - Y = n\right)$$

$$= \sum_{k=n+1}^{n+6} \Pr\left(X = k\right) \times \Pr\left(Y = k - n\right), 1 \le k \le 6 \tag{10}$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{6} \times \frac{1}{6}, 1 \le k \le 6 \tag{11}$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{36}, 1 \le k \le 6 \tag{12}$$

$$= \begin{cases} \frac{n+6}{36} & , -5 \le n \le 0 \\ \frac{6-n}{36} & , 1 \le n \le 5 \end{cases} \tag{13}$$

PDF of X-Y using convolution

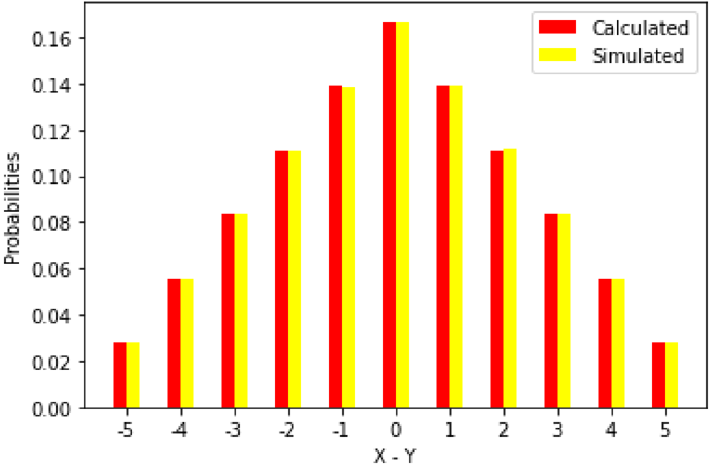

Figure: Plot of PMF for X-Y

Expectation value

Definition

1. The expectation value of X is often called as mean of X and is denoted by μ_x.
2. It is often considered as measure of central tendency.

Formulas

For discrete random variable X,

$$E(X) = \sum x_i \times \Pr(X = x_i) \tag{14}$$

For continuous random variable X having probability density function $f(x)$,

$$E(X) = \int_{-\infty}^{+\infty} x \times f(x)dx \tag{15}$$

Expectation value

Formulas

If X and Y are two discrete random variables,

$$E\left(g(x,y)\right) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} g(x_i, y_i) \times \Pr\left(X = x_i, Y = y_i\right) \tag{16}$$

If X and Y are two continuous random variables having joint density function $f(x,y)$,

$$E\left(g(x,y)\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \times f(x,y)dxdy \tag{17}$$

Expectation value

Properties

If c is any constant, then

$$E(cX) = c(X) \tag{18}$$

If X and Y are any random variables, then

$$E(X+Y) = E(X) + E(Y) \tag{19}$$

If X and Y are independent random variables, then

$$E(XY) = E(X) \times E(Y) \tag{20}$$

Solution contd.

Conditional Probability

$\Pr(X|Y) = \frac{\Pr(X,Y)}{\Pr(Y)}$

$E\left(X + Y \mid (X - Y)^2 = 1\right)$

$$= \sum n \times \Pr\left(X + Y = n \mid (X - Y)^2 = 1\right) \tag{21}$$

$$= \sum n \times \frac{\Pr\left(X + Y = n, (X - Y)^2 = 1\right)}{\Pr((X - Y)^2 = 1)} \tag{22}$$

Solution contd.

$$E\left(X+Y \mid (X-Y)^2=1\right)$$

$$\begin{aligned}
&= \sum n \times \frac{\Pr\left(X+Y=n, (X-Y)=1\right)}{\Pr\left((X-Y)=1\right)} \\
&\quad \times \Pr\left((X-Y)=1 | (X-Y)^2=1\right) \\
&+ \sum n \times \frac{\Pr\left(X+Y=n, (X-Y)=-1\right)}{\Pr\left((X-Y)=-1\right)} \\
&\quad \times \Pr\left((X-Y)=1 \mid (X-Y)^2=1\right)
\end{aligned} \tag{23}$$

$$\begin{aligned}
&= \frac{\Pr\left((X-Y)=1 \mid (X-Y)^2=1\right)}{\Pr\left((X-Y)=1\right)} \\
&\quad \times \sum n \times \Pr\left(X+Y=n, (X-Y)=1\right) \\
&+ \frac{\Pr\left((X-Y)=-1 \mid (X-Y)^2=1\right)}{\Pr\left((X-Y)=-1\right)} \\
&\quad \times \sum n \times \Pr\left(X+Y=n, (X-Y)=-1\right)
\end{aligned} \tag{24}$$

Solution contd.

Using equations (9) and (13) in (24)
We get,
$E\left(X + Y \mid (X - Y)^2 = 1\right)$

$$= \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) + \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right) \times \left(\frac{35}{36}\right) \tag{25}$$

$$= 7 \tag{26}$$