Independent Study-1

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Chapter 1

Curves

1.1 What are Curves?

The most basic mathematical representation of curves is given by the definition that it is a collection of points given by a Cartesian equation For example fig 3.6. Curves represented as such are called Level curves.

$$\mathcal{C} = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, f(x, y) = c\right\}$$

We should note that curves can be defined in any $\mathbb{R}^n$ but we will consider the two dimensional curves unless specified otherwise. However in differential geometry there's another way of representation of a curve that we usually prefer and that is the Parametrized form. In this definition we look at the curve as a path traced out by a moving point.

Definition 1.1.1 *A parametrized curve in $\mathbb{R}^n$ is a map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, for some α, β with $-\infty \leq \alpha < \beta \leq \infty$.*

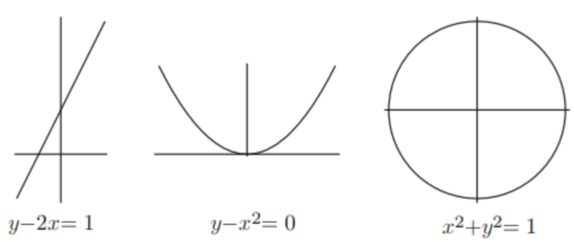

Figure 1.1: Curves

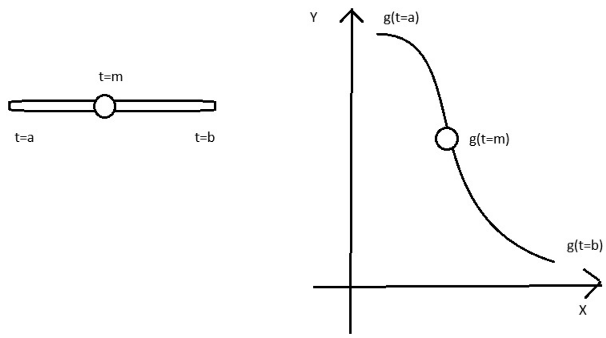

Figure 1.2: Paramterization

The way to typically think about this is having a knob in R and moving that knob to find the corresponding point moving on the plane. There are certain quantities that we can calculate for curves such as tangents and curvatures. The tangent of a curve can be perceived as the speed of the point moving along the curve if we think of curve as the road and imagine the point travelling on the road. Mathematically if γ is a parametrized curve, its first derivative $\dot{\gamma}(t)$ is the velocity of the point γ at the point $\gamma(t)$.

$$\dot{\gamma}(t) = \lim_{\delta t \to 0} \frac{\gamma(t+\delta t) - \gamma(t)}{\delta t}$$

For example consider the parabola given by $\gamma(t) = (t, t^2)$ to find the tangent vector at any point on the curve we will take it's derivative with respect to t. So we have $\dot{\gamma}(t) = (1, 2t)$ We need to mention reparameterization before we move ahead to define curvature of the curve. Reparameterization of a curve can be thought of as just changing the speed with which the point moves on the road(curve). Formally it is defined as follows.

$\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ is a reparametrization $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ if $\exists$ smooth bijective map

$$\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$$

(the reparametrization map) such that the inverse map $\phi^{-1} : (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is also smooth and

$$\tilde{\boldsymbol{\gamma}}(\tilde{t}) = \gamma(\phi(\tilde{t})) \quad \text{for all} \quad \tilde{t} \in (\tilde{\alpha}, \tilde{\beta}).$$

We have two parameterizations for our parabola

$$\gamma(t) = (t, t^2), \tilde{\gamma}(\tilde{t}) = (2\tilde{t}, 4\tilde{t^2})$$

$$\phi(\tilde{t}) = 2t$$

$$\phi^{-1}(t) = \tilde{t}/2$$

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}) = \gamma(2t) = (2t, 4t^2)$$

If t $\in$ (-1,1) then $\tilde{t} \in$ (-1/2,1/2) We usually consider unit speed parameterization of a curve that is curves such that the speed of the point is one at each instant on the curve. Given a unit speed curve we always have that the tangent and curvature are perpendicular to each other. For now just consider that curvature is interpreted as the double derivative of $\gamma(t)$.

$$||\dot{\gamma(t)}|| = 1 \forall t \in (\alpha, \beta)$$

$$\dot{\gamma(t)}\dot{\gamma(t)} = 1$$

taking derivative with respect to t we have

$$2\dot{\gamma(t)}\ddot{\gamma(t)} = 0$$

hence we prove that they are indeed perpendicular. We can always find a unit parameterization for a regular curve, a regular curve is a curve with all points s.t. $||\dot{\gamma(t)}|| \neq 0$. Consider $\tilde{\gamma}(\tilde{t}) = \gamma(t)$ then we have

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}\tilde{t}} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}$$

$$||\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}\tilde{t}}|| = ||\frac{\mathrm{d}\gamma}{\mathrm{d}t}|||\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}|$$

Here we can clearly see that for $||\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}\tilde{t}}|| = 1$ we must have $||\frac{\mathrm{d}\gamma}{\mathrm{d}t}|| \neq 0$ Hence, there exists a unit speed parameterization of a curve only if the curve is regular. Note that it can be simply proved that parameterization of a regular curve is always regular in a very similar fashion. The unit speed parameterization is given by arc length.

$$\gamma(t) \to \gamma(t + \Delta t)$$

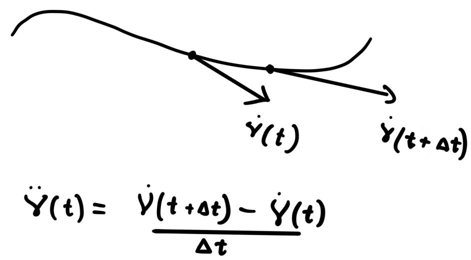

Figure 1.3: Curvature

For a unit speed curve the velocity will change even though speed will not, so change in velocity is

$$\dot{\gamma}_1 \to \dot{\gamma}_2$$

Change in the direction of velocity is only by the acceleration in direction perpendicular to the tangent Fig 1.3. Acceleration for a unit speed curve

$$\ddot{\gamma}(t) = \lim_{\Delta t \to 0} \|\dot{\gamma}(t + \Delta t) - \dot{\gamma}(t)\| / \Delta t$$

Note that sometimes it becomes difficult to find the unit speed parameterization of the curve and hence we also have a general formula for the curvature of any regular curve given by

$$\kappa = \frac{||\ddot{\gamma} \times \dot{\gamma}||}{||\dot{\gamma}||^3}$$

Chapter 2

Surfaces

2.1 What are surfaces?

The easiest way to imagine a surface which will be insightful for our discussion further is considering a two dimensional plane and bending it to form a certain object, the outer surface of this object that you're looking at is the surface of the object. A rather more objective way of defining the subject in matter is, a surface is a subset of R^3 that looks like a piece of R^2 in the vicinity of any given point. The surface of earth although spherical looks like a flat two dimensional plane to us because we are point-sized compared to the hugeness of earth, our vision, our vicinity is up to the horizon. Ancient people used to even think that the earth is flat and that they will fall if they go too far.

2.1.1 Formal Definition

Definition 2.1.1 *A subset $\mathcal{S}$ of $\mathbb{R}^3$ is a surface if, for every point $\mathbf{p} \in \mathcal{S}$, there is an open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing $\mathbf{p}$ such that $\mathcal{S} \cap W$ is* **homeomorphic** *to U.*

A homeomorphism $\sigma : U \to \mathcal{S} \cap W$ (mapping between the green patches in Fig 2.1)as in definition is called a surface patch or parameterization of the open subset $\mathcal{S} \cap W$.

Just like when we talked about curves, paramterization is just a way of representing the surface mathematically in terms of a function or one can

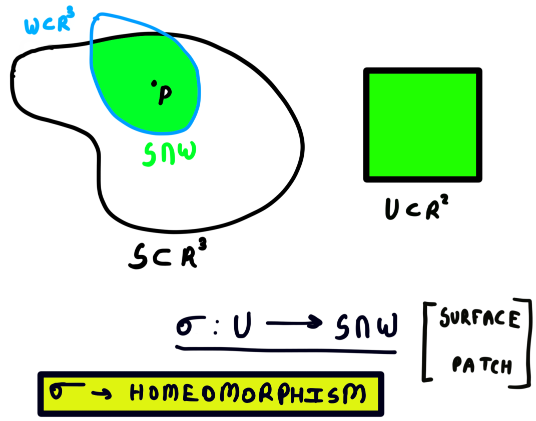

Figure 2.1: Definition of surface

think of it as a way of wrapping a plane sheet on a part of the surface.

Homeomorphism is a continuous bijective mapping with a continuous inverse. Think about creating a map of a part of the earth. What should we make sure that is preserved in the mapping ? Ofcourse each point on the surface of earth should map to a single point on the map. If I have a path on the surface then there should be a corresponding path on the map and vice versa, hence the above conditions.
Consider a circular cylinder represented paramtrically as $\sigma(u,v) = (\cos u, \sin u, v)$, to get a full cylinder we should have $0 \leq u \leq 2\pi$ but we know that U should be open so equality should not come and hence we are missing a line on the vertical body of the cylinder. We cannot increase the length of the interval as well because then it will be not bijective. Think about it like wrapping a long sheet around a cylinder, we don't want the sheet to rewrap at any instant. For these reasons while defining the whole cylinder is not possible by a single surface patch and we can define another surface patch that covers the previous remaining line. So the representation of the cylinder can be defined by two surfaces patches $\sigma|_U, \sigma|_{\tilde{U}}$ where $U = \{(u,v) \in \mathbb{R}^2 | 0 \leq u \leq 2\pi\}$ and $\tilde{U} = \{(u,v) \in \mathbb{R}^2 | -\pi \leq u \leq \pi\}$. Both functions are defined as above with following domain. Remember that we can define various overlapping patches for a surface but the surface patch itself should be a homeomorphism. Smooth surfaces In most of our analysis when we talk about surfaces

we consider smooth surfaces. Let's formally define what smooth surfaces are. We know the definition that a function f is smooth if each component of f has smooth partial derivatives of all orders. Now, we say that a surface patch is regular if σ representing it is smooth and has $\sigma_u \times \sigma_v$ non zero at each point or we can say that it should have a normal vector at each point(it also implies that it has tangent plane at that point). Places on the surface patch where $\sigma_u \times \sigma_v$ can be zero are typically points where we have cusps or sharp turns, where we cannot have a tangent plane. We will formally define tangent plane and normal to the tangent plane later.

Definition 2.1.2 *If $\mathcal{S}$ is a surface, an allowable surface patch for $\mathcal{S}$ is a regular surface patch $\sigma : U \to \mathbb{R}^3$ such that σ is a homeomorphism from U to an open subset of S. A smooth surface is a surface such that, for each point on the surface there is an allowable surface patch.*

Remember that smoothness is the property of representation and not of the surface itself. We can have non smooth representations of smooth surfaces and vice versa.

2.1.2 Transition Maps

Analogous to the idea of reparametrization of curves we can have different representations of the same part of the surface by different surface patches. Maps converting one surface representation to another is known as transition map. If we have two surface patches $\sigma : U \to S \cap W$ and $\tilde{\sigma} : \tilde{U} \to S \cap \tilde{W}$ such that there is some common surface patch between two then we can say that there exists $a \in S \cap W \cap \tilde{W}$. Now corresponding to this common area we will have $V \subseteq U$ and $\tilde{V} \subseteq \tilde{U}$. The mapping between V and $\tilde{V}$ given by

$$\phi = \sigma^{-1} \circ \sigma : V \to \tilde{V}$$

is a composite homeomorphism and is called transition map.

2.1.3 Smooth Maps

Surfaces are just subsets of a Euclidean space so we can think of functions which have it's domain and range as surfaces. So let's say I have a funtion $f : \mathcal{S}_1 \to \mathcal{S}_2$ how to look at the differentiability of f? The classic way to imagine this is that we go on $\mathcal{S}_1$ and perturbate the point and see how the corresponding point on $\mathcal{S}_2$ perturbates. But the problem here is that the directions in which the point can be perturbated is restricted, to avoid this we define the differentiability in terms of the domains of the surfaces which is two dimensional and we have all possible directions to move. This idea is very important and one of the main reasons why we define surfaces as such. So if I want to see the differentiability at a point p on $\mathcal{S}_1$ and it is contained in the surface patch σ_1 and the corresponding point f(p) is contained in σ_2 then we have that $\sigma_2^{-1} \circ f \circ \sigma_1 : U_1 \to U_2$ is smooth implies that f is smooth. This is true because f has to be smooth for the before quantity to be smooth and we have brought our differentiability condition in terms of 2 dimensional subspace where it is easy to comprehend, however we also have to show that this remains unaffected with reparameterization. Now it can be shown that this will happen when the transition maps are smooth.

Definition 2.1.3 *If a smooth map $f : S_1 \to S_2$ is smooth, bijective with a smooth inverse map than it is known as a diffeomorphism.*

Diffeomorphism is a general definition and is not limited to just mappings between surfaces. Consider the function $f(x) = x^3$ where $x \in \mathbb{R}$, it is a diffeomorphism. We have a slightly modified definition for local diffeomorphism. A smooth map $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a local diffeomorphism if for any point $p \in \mathcal{S}_1$, there is an open subset O of $\mathcal{S}_1$ such that $f(O)$ is an open subset of $\mathcal{S}_2$ and $f|_O : \mathcal{S}_1 \to \mathcal{S}_2$ is a diffeomorphism.

2.2 Tangents and Normals

Surfaces are often considered as collection of curves. If we go by our previous definition of curves as path traced out by point then in surfaces we have infinite number number of possible paths at an instance for a point to move. Our goal is to find the set of all possible directions for the point to move on the surface. Imagine that we arbitrarily took a path on our surface patch $\sigma : U \to \mathcal{S} \cap W$ at point p and moved along it. We must have a continuous path in U, we can represent this curve as $\gamma(t) = (u(t), v(t))$, corresponding to this curve we can write the equation of our original curve on the surface as $\alpha(t) = \sigma(u(t), v(t))$. Differentiating w.r.t. t we get (denoting d/dt by a dot)

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}$$

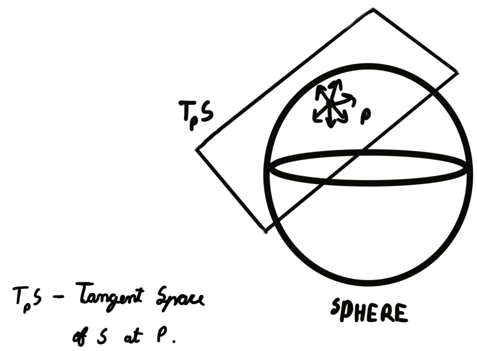

Figure 2.2: Tangent Space

We can see from the equation that the tangent space of the surface at any point p is spanned by $T_{\mathbf{p}}\mathcal{S}$ is spanned by σ_u and σ_v. It is beneficial to think about what these σ_u and σ_v are. They are the tangent vectors to the curve formed by only changing the $u(t)$, keeping the $v(t) =$ constant and vice versa.
This we discussed the tangent space of a surface or the derivative of the surface patch but what about the derivative of smooth maps between surfaces? We can also look at it following a similar approach. We will find that the derivative of f at a point p, $D_{\mathbf{p}}f : T_{\mathbf{p}}\mathcal{S} \rightarrow T_{\mathbf{p}}\tilde{\mathcal{S}}$ is the map such that $D_{\mathbf{p}}f(w) = \tilde{w}$ for any tangent vector $w \in T_{\mathbf{p}}\mathcal{S} Fig 2.2$.

At each point p on the surface we have a tangent plane $T_{\mathrm{p}}\mathcal{S}$, the unit normal vector to this plane is the unit normal vector at point p (Fig 2.3).

$$\mathbf{N}_{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|}$$

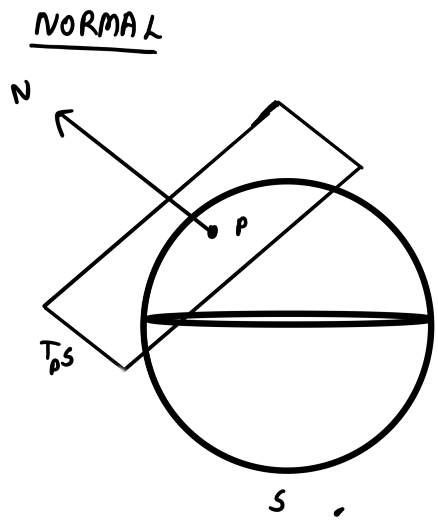

Figure 2.3: Normal to the surface

Chapter 3

Fundamental Forms

3.1 First Fundamental Form

Observe Fig 3.1 we want to measure the length of the curve from the first point t_0 to t_1 that is divided into very small straight lines.

$$S(t) = \sum_{i=0}^{n} ||\gamma(t_i + \Delta t) - \gamma(t_i)||$$

We know that

$$\lim_{\Delta t \to 0} ||\dot{\gamma}|| = \frac{||\gamma(t_i + \Delta t) - \gamma(t_i)||}{|\Delta t|}$$

Figure 3.1: Length of the Curve

Therefore we can also write,

$$S(t) = \int_{t_0}^{t_1} \langle \dot{\gamma}, \dot{\gamma} \rangle^2 dt$$

Now we know that,

$$\gamma(\dot{t_0}) \in T_{\mathbf{p}}\mathcal{S} \rightarrow \gamma(\dot{t_0}) = \lambda \boldsymbol{\sigma}_u + \mu \boldsymbol{\sigma}_v$$

$$||\dot{\gamma}|| = (\dot{\gamma}.\dot{\gamma})^{1/2}$$

$$\langle \dot{\gamma}, \dot{\gamma} \rangle = \lambda^2 \langle \sigma_u, \sigma_u \rangle + 2\lambda\mu \langle \sigma_u, \sigma_v \rangle + \mu^2 \langle \sigma_v, \sigma_v \rangle$$

So we can write

$$E = ||\sigma_u||^2, F = \langle \sigma_u, \sigma_v \rangle, G = ||\sigma_v||^2$$

We define maps $du : T_{\mathrm{p}}\mathcal{S} \rightarrow \mathbb{R}$ and $dv : T_{\mathbf{p}}\mathcal{S} \rightarrow \mathbb{R}$ such that $du(v) = \lambda_1$, $dv(v) = \mu_1$ and E=$\langle \sigma_u, \sigma_u \rangle$, F=$\langle \sigma_u, \sigma_v \rangle$, G=$\langle \sigma_v, \sigma_v \rangle$ then for v in $T_{\mathbf{p}}\mathcal{S}$ we can rewrite the equation as

$$\langle v, v \rangle = Edu(v)^2 + Fdu(v)dv(v) + Gdv(v)^2$$

So equation of FFF is

$$Edu^2 + Fdudv + Gdv^2$$

The general form of FFF is

$$\langle v, w \rangle^1 = \begin{bmatrix} du(v) & dv(v) \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} du(w) \\ dv(w) \end{bmatrix}$$

The hard task is to find the length of the curve on the surface which can be made easier by transforming the curve on the surface to a curve in two dimensional surface in terms of its local coordinates (u, v). The First Fundamental Form also has other implications which are discussed next. It is worthwhile to mention here that the dot product to be defined here in terms of the matrix is only possible if the matrix is positive definite.
Notice that $EG - F^2 \geq 0$ because $||\sigma_u||^2||\sigma_v||^2 \geq ||\sigma_u.\sigma_v||^2$

$$\begin{bmatrix} E - \lambda & F \\ F & G - \lambda \end{bmatrix}$$

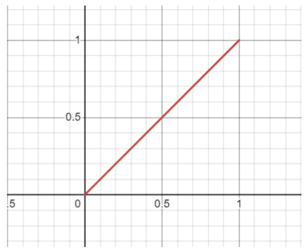

Figure 3.2: Line

and if we go on to find the roots of the equation for eigen values we have

$$\lambda_1, \lambda_2 = E + G \pm \sqrt{(E+G)^2 - 4(EG - F^2)}$$

roots are always positive because value in the square root is always less than $E + G$. Hence we proved that the matrix is semidefinite. Say you have a curve $\gamma(t) = (u(t), v(t)), u(t) = t, v(t) = t$ where $0 \geq t \geq 1$ in 3-d space and you project it on the surface $\sigma = (u, v, uv)$, the equation of the curve is $\alpha(t) = (t, t, t^2)$, now our task is to find the length of this curve. We will be again transforming this curve in terms of the local coordinates in 2-dimensional space, ofcourse it will not be the same straight line as before because now the curve's length has changed and it should be preserved and the curve might not even be a straight line(Fig 3.2, 3.3). So we go on to write the FFF of the curve which is

$$(1 + v^2)du^2 + 2vududv + (1 + u^2)dv^2$$

substituting $v = t, u = t, du = 1, dv = 1$ we have

$$S(t) = \int_0^1 \sqrt{(2 + 4t^2)} \approx 2.7$$

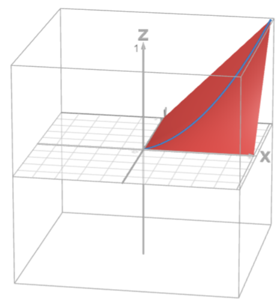

Figure 3.3: Line projected on the surface

3.1.1 Maps

Maps are one of the most important implications that come out of differential geometry. We will discuss the two most important maps here, namely Isometries and Conformal maps. The common link between them is that they are defined from the first fundamental form.

Isometries

Draw a curve of your choice on a sheet of paper. Imagine that sheet of paper rolled into a cylindrical shape without crumpling. Take a long strip of paper and curve it to make a wavy structure. Each time you increase a wave in this strip of paper we have a different surface however the length of the curve drawn remains the same because essentially you are not crumpling or folding the paper. Actually there's a a special way to represent these surfaces know as ruled surfaces, there you define a surface just based on the a curve, the curve will be the one that runs along the length of the strip. The important thing to notice here was that there are surfaces which preserve the length of the curves on the surfaces or so to speak the first fundamental form of the surfaces. These surfaces are called isometries of each other. The mappings between two surfaces which preserve the length of the curves are know as local isometry. The reason that it is local is because it is restricted to locally diffeomorphic opensets. It does not preserve the length of the curves globally, however if f is a diffeomorphism then we can say that the local isometry becomes a global isometry. In general f is a local isometry if an only if

$$\langle D_{\mathbf{p}} f(\mathbf{v}), D_{\mathbf{p}} f(\mathbf{w})\rangle_{f(\mathbf{p})} = \langle \mathbf{v}, \mathbf{w}\rangle_{\mathbf{p}}$$

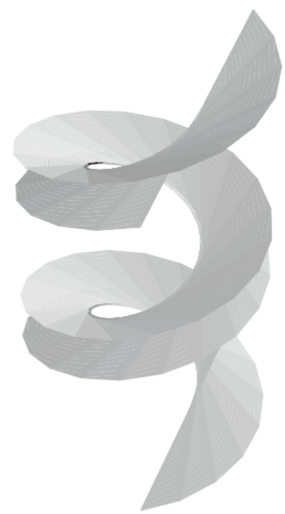

Figure 3.4: This is a Tangent Developable, also Isometric to plane

for all $p \in \mathcal{S}_1$ and $\mathbf{v}, \mathbf{w} \in T_{\mathrm{p}}\mathcal{S}_1$

Conformal maps

There can be no isometry between sphere and a plane, imagine wrapping a plane around a sphere without crumpling it but we still need planar maps of complex surfaces like sphere. We must've seen the classic map of the globe of Earth where Greenland is approximately the size of the African continent which is wrong because in reality Greenland is about seventeen times smaller than Africa. That map is a Mercator projection, so as we cannot have a map that perfectly represents the globe we find maps which sacrifice a few details but also preserve something to be useful enough. For example the Mercator projection that is in discussion preserves the direction(angles). This means that if you're standing in the ocean and represent that point on the globe and you measure the angle at which a country is with respect to you then that angle is the same if you look at your corresponding point on the plane map and the angle at which the country is. In the context of differential geometry these maps are known as conformal maps. Putting this condition in mathematical equations we get

$$\frac{\langle \dot{\gamma}, \dot{\tilde{\gamma}} \rangle}{\langle \dot{\gamma}, \dot{\gamma} \rangle^{1/2} \langle \dot{\tilde{\gamma}}, \dot{\tilde{\gamma}} \rangle^{1/2}} = \frac{f^* \langle \dot{\gamma}, \dot{\tilde{\gamma}} \rangle}{f^* \langle \dot{\gamma}, \dot{\gamma} \rangle^{1/2} f^* \langle \dot{\tilde{\gamma}}, \dot{\tilde{\gamma}} \rangle^{1/2}}$$

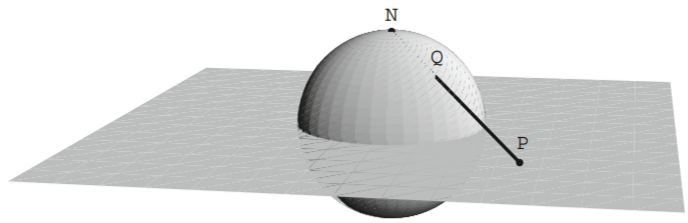

Figure 3.5: Stereographic Projections are classic examples of Conformal Maps

where $\left\langle D_{\mathbf{p}} f(\dot{\gamma}), D_{\mathbf{p}} f(\dot{\tilde{\gamma}})\right\rangle_{f(\mathbf{p})} = f^{*}\langle\dot{\gamma}, \dot{\tilde{\gamma}}\rangle_{\mathbf{p}}$

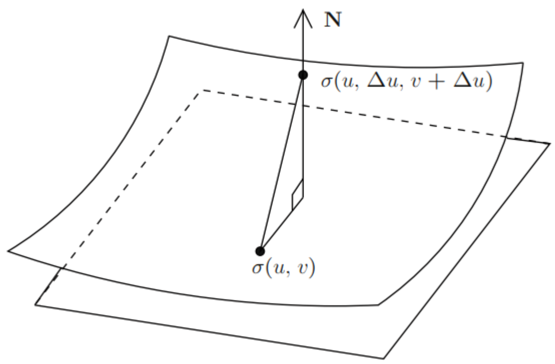

Figure 3.6: Curvature in Surface

3.2 Second Fundamental Form

Analogous to curvature of the curves we look at the curvature as to how much the surface moves away from the tangent plane. So if we have two points ($\boldsymbol{\sigma}(u+\Delta u, v+\Delta v)$ and $\boldsymbol{\sigma}(u,v)$) then the curvature in the direction ($\boldsymbol{\sigma}(u+\Delta u, v+\Delta v) - \boldsymbol{\sigma}(u,v)$) is given by

$$(\boldsymbol{\sigma}(u+\Delta u, v+\Delta v) - \boldsymbol{\sigma}(u,v)) \cdot \mathbf{N}$$

By the two variable form of Taylor's theorem,

$$\boldsymbol{\sigma}(u+\Delta u, v+\Delta v) - \boldsymbol{\sigma}(u,v)$$

is equal to

$$\boldsymbol{\sigma}_u \Delta u + \boldsymbol{\sigma}_v \Delta v + \frac{1}{2}\left(\boldsymbol{\sigma}_{uu}(\Delta u)^2 + 2\boldsymbol{\sigma}_{uv}\Delta u \Delta v + \boldsymbol{\sigma}_{vv}(\Delta v)^2\right) + \text{ remainder},$$

Now $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are tangent to the surface, hence perpendicular to $\mathbf{N}$, therefore we have

$$\frac{1}{2}\left(L(\Delta u)^2 + 2M\Delta u \Delta v + N(\Delta v)^2\right) + \text{ remainder},$$

where

$$L = \boldsymbol{\sigma}_{uu} \cdot \mathbf{N}, \quad M = \boldsymbol{\sigma}_{uv} \cdot \mathbf{N}, \quad N = \boldsymbol{\sigma}_{vv} \cdot \mathbf{N}.$$

The important observation to make here is that when we are considering two points and finding the curvature, we are essentially considering a direction

or a curve on the surface and finding the curvature in that direction. Let's take an example to make the idea more sound.
Let's find the SFF of

$$\sigma(u,v) = (u, v, u^2 + v^2)$$

$$\boldsymbol{\sigma}_u = (1,0,2u), \boldsymbol{\sigma}_v = (0,1,2v), \mathbf{N} = \lambda(-2u,-2v,1), \lambda = \frac{1}{\sqrt{1+4u^2+4v^2}};$$

$$\boldsymbol{\sigma}_{uu} = (0,0,2), \boldsymbol{\sigma}_{uv} = \mathbf{0}, \boldsymbol{\sigma}_{vv} = (0,0,2)$$

$L = 2\lambda, M = 0, N = 2\lambda$, and the second fundamental form is

$$2\lambda\left(du^2 + dv^2\right)$$

Now using the second fundamental form we will find the curvature of the curve on the surface. With the above surface we are considering a curve $\gamma(t) = (2t, t)$ on the surface and trying to find it's curvature at t=0 using SFF.
Substituting u=2t,v=t, du=2,dv=1 we have the formula as

$$10/\sqrt{1+16t^2+4t^2}$$

So the curvature at origin becomes 10.

Weingarten Maps

Another way to look at curvature is to look at the way the normal vector varies with time. It is intuitive to notice that the normal vector will change more when we move in the direction where the surface is more curved. If our surface is just a plane then the normal vector does not change the pointing direction hence the curvature is zero. So can derive a general form of SFF using this idea. We will consider the values of unit normal vector at each point we will use gauss map to serve the purpose. The $GaussMap$ is a map from $\mathcal{S}$ to the unit sphere $\mathcal{S}^2$ that assigns to any point p on the surface a point $N_p \in \mathcal{S}^2$, where N_p is the unit normal of surface at p. The rate at which N varies on the surface in different directions can then be given by the derivative of the Gauss map, it will then give the metric of how much the unit normal vector changes as the point changes on the surface. We have the derivative of gauss map as

$$D_{\mathbf{p}}G : T_{\mathbf{p}}\mathcal{S} \to T_{G(\mathbf{p})}\mathcal{S}^2$$

We can observe that $T_{G(\mathbf{p})}\mathcal{S}^2$ is the same as $T_{\mathbf{p}}\mathcal{S}$ so we can write

$$D_{\mathbf{p}}G : T_{\mathbf{p}}\mathcal{S} \to T_{\mathbf{p}}\mathcal{S}$$

The weingarten map is actually the negative of this derivative of gaussian map just by convention.

Definition 3.2.1 *Let* $\mathbf{p}$ *be a point on the the surface* $\mathcal{S}$. *The Wiengarten Map* $\mathcal{W}_{\mathbf{p}}, \mathcal{S}$ *of* $\mathcal{S}$ *at* $\mathbf{p}$ *is defined by*

$$\mathcal{W}_{\mathbf{p}}, \mathcal{S} = -D_{\mathbf{p}}G$$

The second fundamental form of the surface $\mathcal{S}$ *at* $\mathbf{p} \in \mathcal{S}$ *is the bilinear form on* $T_{\mathbf{p}}\mathcal{S}$ *given by*

$$\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{p},\mathcal{S}} = \langle \mathcal{W}_{\mathbf{p},\mathcal{S}}(\mathbf{v}), \mathbf{w} \rangle_{\mathbf{p},\mathcal{S}}, \quad \mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$$

Now we are equipped to derive the general form of SFF. The general form of SFF, for a point p and two vectors v,w $\in$ the tangent plane at that point we have the general formulae for second fundamental form as.

$$\langle \mathbf{v}, \mathbf{w} \rangle^2 = Ldu(\mathbf{v})du(\mathbf{w}) + M(du(\mathbf{v})dv(\mathbf{w}) + du(\mathbf{w})dv(\mathbf{v})) + Ndv(\mathbf{v})dv(\mathbf{w})$$

We can write this in matrix form as

$$\langle v, w \rangle^2 = \begin{bmatrix} du(v) & dv(v) \end{bmatrix} \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} du(w) \\ dv(w) \end{bmatrix}$$

Now there are some algebraic steps that one has to follow to get the general form of SFF from the Weigarten map definition which we are skipping in this report.

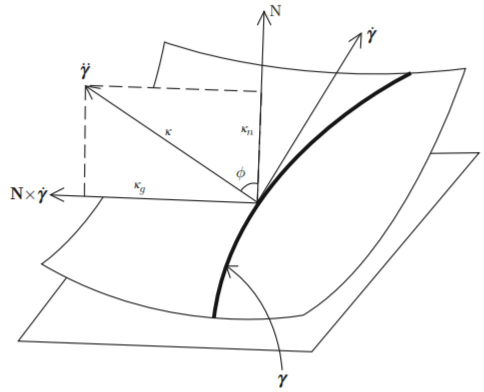

Figure 3.7: Normal and Geodesic Curvature

3.2.1 Different Curvatures Metrics

The point has two directional accelerations on a sloped and curved road. One is in the normal direction when it moves away from the tangent plane. Second is when it changes direction in the tangent plane. γ is a unit speed curve then $\dot{\gamma}$ and $\ddot{\gamma}$ are perpendicular, also $\dot{\gamma} \in$ tangent plane so perpendicular with N. Hence we have N, $\dot{\gamma}$ and $N \times \dot{\gamma}$ are mutually perpendicular. We have

$$\ddot{\gamma} = \kappa_n \mathbf{N} + \kappa_g \mathbf{N} \times \dot{\gamma}$$

κ_n and κ_g are normal and geodesic curvatures respectively. Look at the Fig **??** taken from Presely's book If the surface were not there and the curve was just a space curve in three dimension we would have another metric which measured how much the curve moves out of the plane as the curve would also move in the third dimension. We should keep in mind that the curvature we find by the SFF is essentially the normal curvature. If γ is the unit speed curve along which we are finding the curvature then the value of $\kappa_n = \langle \dot{\gamma}, \dot{\gamma} \rangle$. Now that we have the Weingarten Map(W) as the linear map that gives us measure of curvature at any point we can look talk about different metrics like gaussian curvature and mean curvature which are defined as $K = det(W)$ and $H = 1/2(trace(W))$ respectively. Deriving W is also not a big task and can be derived by following simple algebraic steps and it turns out that it is given by the matrix $\mathcal{F}_I \mathcal{F}_{II}$ where

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Now it will be worthwhile to give an example to show how the values are actually calculated in a unit sphere. For a unit sphere, we can parametrize the surface by spherical coordinates (θ, ϕ), where θ is the azimuthal angle and ϕ is the polar angle.

The parametric equations for a unit sphere are:

$$\sigma(\phi, \theta) = (\sin(\phi)\cos(\theta) \ \sin(\phi)\sin(\theta) \ \cos(\phi))$$

To find the Weingarten map, we first need to compute the first fundamental form (metric tensor) and then the second fundamental form. The Weingarten map is then given by the product of the inverse of the first fundamental form and the second fundamental form.

1. First Fundamental Form (Metric Tensor):

$$E = \langle \sigma_\theta, \sigma_\theta \rangle = 1$$

$$F = \langle \sigma_\theta, \sigma_\phi \rangle = 0$$

$$G = \langle \sigma_\phi, \sigma_\phi \rangle = \sin^2(\phi)$$

2. Second Fundamental Form:

$$L = \langle \sigma_{\theta\theta}, \mathbf{n} \rangle = \cos(\phi)$$

$$M = \langle \sigma_{\theta\phi}, \mathbf{n} \rangle = 0$$

$$N = \langle \sigma_{\phi\phi}, \mathbf{n} \rangle = -\sin(\phi)$$

Where $\mathbf{n}$ is the normal vector to the sphere, which is just the position vector normalized.

3. Weingarten Map:

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} \times \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \cos(\phi) & 0 \\ 0 & -\sin(\phi) \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2(\phi)} \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi) & 0 \\ 0 & -\sin(\phi) \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & \csc^2(\phi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi) & 0 \\ 0 & -\sin(\phi) \end{pmatrix} \times \begin{pmatrix} \sin^2(\phi) & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi)\sin^2(\phi) & 0 \\ 0 & -\sin(\phi) \end{pmatrix}$$

Now, we can compute the Gaussian curvature (K) and the mean curvature (H) from the coefficients of the Weingarten map:

$$K = \det(\text{Weingarten Map}) = \cos(\phi)\sin^2(\phi)\times(-\sin(\phi)) = -\sin(\phi)\cos(\phi)\sin^2(\phi)$$

$$H = \frac{1}{2}\mathrm{tr}(\text{Weingarten Map}) = \frac{1}{2}(\cos(\phi)\sin^2(\phi) - \sin(\phi))$$

These formulas give the Gaussian and mean curvature of the unit sphere as functions of the polar angle ϕ. One important thing to notice here is that the Weingarten map does not depend on the latitudes of the sphere. It depends only on the longitudinal position of the point on the sphere. Another way to look at it is that the longitudes are the great circles which are geodesics and hence the geodesic curvature is absent and the curvature is just the normal curvature that we get by the Weingarten Map.

Chapter 4

References

Images 1.1,3.1,3.2,3.3,3.4,3.5,3.6,3.7 are referenced from Elementary Differential Geometry, Pressley, Springer, 2010 which was the main book that we read for the study. We also studied parts of Differential geometry of curves and surfaces, M. do Carmo. Prentice Hall, 1976.