Curves and Surfaces

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What are Curves?

- A Level Curve($\mathcal{C}$) is a set points defined as:

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$$

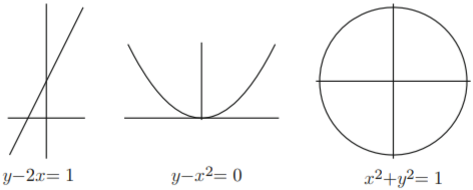


Figure: Examples of Curves

Parameterization of a Curve

- A curve can also be viewed as the path traced out by a moving point in time.
- A parametrized curve in $\mathbb{R}^n$ is a map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, for some α, β with $-\infty \leq \alpha < \beta \leq \infty$.
- All the curves in this presentation are considered to be smooth. ($d^n\gamma/dt^n$ exists for $n > 0$)

Parameterization of a Curve

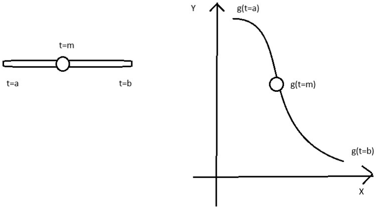


Figure: Parameterization

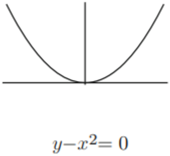


Figure: $\gamma : (-\infty, \infty) \to \mathbb{R}^2, \quad \gamma(t) = \left(t, t^2\right)$.

Velocity of the point

If γ is a parametrized curve, its first derivative $\dot{\gamma}(t)$ is the velocity of the point γ at the point $\gamma(t)$.

$$\dot{\gamma}(t) = \frac{\gamma(t+\delta t) - \gamma(t)}{\delta t}$$

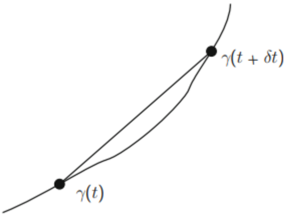
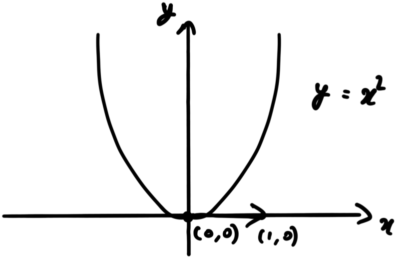


Figure: Tangent

Example

Computing Tangent



$$\gamma(t) = (t, t^2), \quad t \in (-\infty, \infty)$$

$$\dot{\gamma}(t) = (1, 2t) \rightarrow \dot{\gamma}(0) = (1, 0)$$

Figure: Computing Tangent

Reparameterization(Tuning the speed of the point)

$\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ is a reparametrization $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ if $\exists$ smooth bijective map

$$\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$$

(the reparametrization map) such that the inverse map $\phi^{-1} : (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is also smooth and

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) \quad \text{for all} \quad \tilde{t} \in (\tilde{\alpha}, \tilde{\beta}).$$

Example

We have two parameterizations for our parabola

$$\gamma(t) = (t, t^2), \tilde{\gamma}(\tilde{t}) = (2\tilde{t}, 4\tilde{t^2})$$

$$\phi(\tilde{t}) = 2t$$

$$\phi^{-1}(t) = \tilde{t}/2$$

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}) = \gamma(2t) = (2t, 4t^2)$$

If t $\in$ (-1,1) then $\tilde{t} \in$ (-1/2,1/2)

Unit speed Curve

- Point can move with a unit speed everywhere on the curve

Every regular curve($||\dot{\gamma}(t)|| \neq 0, \forall t \in (a,b)$) has a unit speed reparameterization. So there exists a $\phi(\tilde{t})$ such that

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) \quad \forall \quad \tilde{t} \in (\tilde{a}, \tilde{b}).$$

and

$$||\dot{\tilde{\gamma}}(\tilde{t})|| = 1, \forall \tilde{t} \in (\tilde{a}, \tilde{b})$$

.

Curvature in Unit Speed Curve(Acceleration)

$$\gamma(t) \to \gamma(t + \Delta t)$$

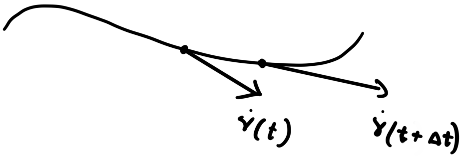
For a unit speed curve the velocity will change even though speed will not, so change in velocity is

$$\dot{\gamma}_1 \to \dot{\gamma}_2$$

Acceleration for a unit speed curve

$$\ddot{\gamma}(t) = \lim_{\Delta t \to 0} \|\dot{\gamma}(t + \Delta t) - \dot{\gamma}(t)\| / \Delta t$$

Curvature



$$\ddot{\gamma}(t) = \frac{\dot{\gamma}(t+\Delta t) - \dot{\gamma}(t)}{\Delta t}\Bigg|_{\Delta t \to 0}$$

Figure: Intuition of Curvature

Example

$$\gamma(t) = (R\cos t,\ R\sin t)$$

$$\dot\gamma(t) = (-R\sin t,\ R\cos t)$$

$\uparrow$

not unit speed

$$\phi(\tilde t) = \frac{t}{R}$$

$$\tilde\gamma(\tilde t) = \gamma(\phi(\tilde t))$$

$$\tilde\gamma(\tilde t) = \left(R\cos\frac{\tilde t}{R},\ R\sin\frac{\tilde t}{R}\right)$$

$$\dot{\tilde\gamma}(\tilde t) = \left(-\sin\frac{\tilde t}{R},\ \cos\frac{\tilde t}{R}\right)$$

$$\ddot{\tilde\gamma}(t) = \left(-\frac{1}{R}\cos\frac{\tilde t}{R},\ -\frac{1}{R}\sin\frac{\tilde t}{R}\right)$$

$$\boxed{\|\ddot\gamma\| = \frac{1}{R}}$$

Figure: Curvature of a circle with radius R

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Surfaces(Collection of Curves)

Definition

A subset $\mathcal{S}$ of $\mathbb{R}^3$ is a surface if, for every point $\mathbf{p} \in \mathcal{S}$, there is an open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing $\mathbf{p}$ such that $\mathcal{S} \cap W$ is homeomorphic to U.

Example

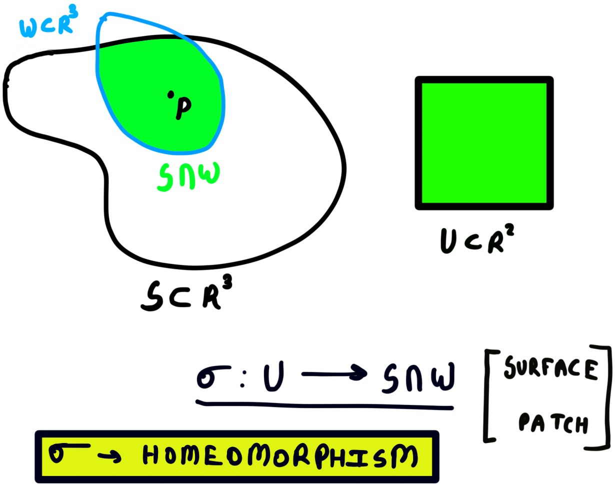


Figure: Surface Patch

Why Homeomorphism?

Map(Atlas) of the small portion of earth, Requirements?

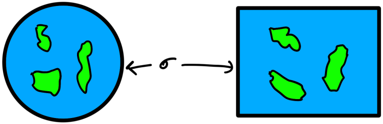


Figure: Globe Atlas

- Bijective :- Each point on the Globe should map to each point on the Atlas.

Why Homeomorphism?

- Continuous :- If the point is moving continuously on the globe then the path on the atlas should also be continuous.

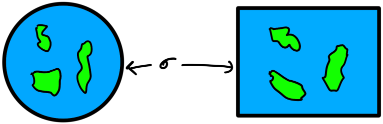


Figure: Globe Atlas

Why Homeomorphism?

- Inverse Continuous :- If a continuous path on the Atlas is given then we should be able to map it on the Globe.

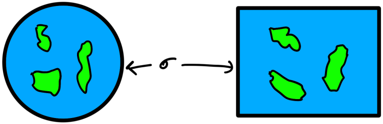
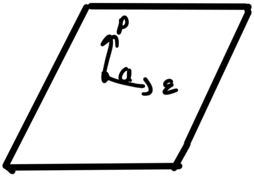


Figure: Globe Atlas

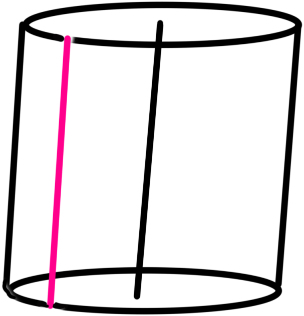
Example of surfaces



$$\boxed{\sigma(u,v) = a + u\vec{p} + v\vec{q}}$$

$$\|p\|^2 = \|q\|^2 = 1$$

$$p \cdot q = 0$$



$$\boxed{\sigma(u,v) = (\cos u, \sin u, v)}$$

$$-1 < v < 1 \quad , \quad 0 < u < 2\pi$$

$$\sigma(u=0, v) = (1, 0, v)$$

Tangent Space(Moving on a surface)

- Consider a point on the surface and consider all possible curves(roads for the point) on the surface passing through that point.

- Tangents to these curves give us the tangent space of the surface at point P. (All the directions possible to move)

Tangent Space

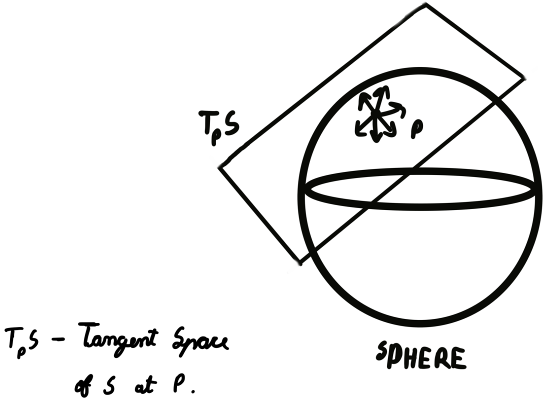


Figure: Tangent Space

Finding the Tangent Space(all possible Direction of roads)

The tangent space to $\mathcal{S}$ at $\mathbf{p}$ is the vector subspace of $\mathbb{R}^3$ spanned by the vectors $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ (the derivatives are evaluated at the point $(u_0, v_0) \in U$ such that $\boldsymbol{\sigma}(u_0, v_0) = \mathbf{p}$).

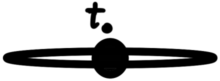
(Next slides: Proving this statement)

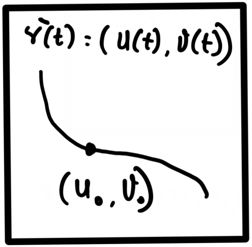
Finding the Tangent space(all possible Directions of roads)

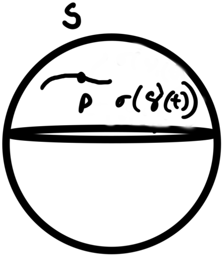
- We start off with an arbitrary smooth curve on the surface
- We see that this curve makes a corresponding curve in U as well(Why?)
- Parameterization of the curve in U is $\tilde{\gamma}(t) = (u(t), v(t))$ then the parameterization of the corresponding curve on the surface will be $\gamma(t) = \sigma(u(t), v(t))$.

Finding Tangent Space

$\gamma(t_0) = (u_0, v_0)$, $\sigma(u_0, v_0) = p$







$$\sigma(\gamma(t))$$
$$\parallel$$
$$\gamma(t) = \sigma(u(t), v(t))$$

curve on the surface.

Figure: Finding the Tangent Space

Finding the Tangent space(all possible Directions of roads)

- Denoting d/dt by a dot,

$$\dot{\gamma} = \boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}$$

- We can see from the equation that $T_{\mathbf{p}}\mathcal{S}$ is spanned by σ_u and σ_v.

(Next Slides: Observing what this means)

Analysing the equation of tangent

- What are σ_u and σ_v?

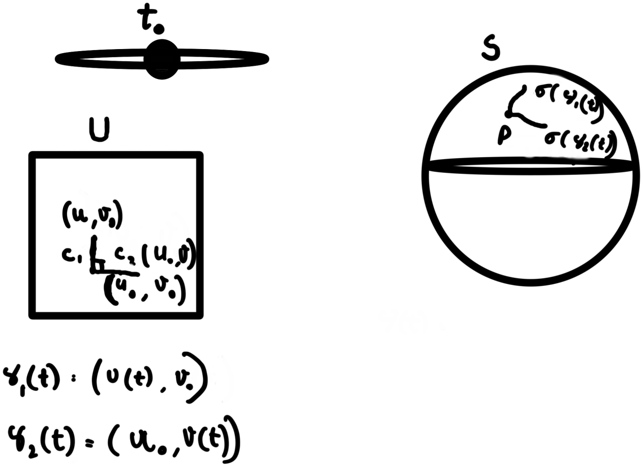


Figure: Spanning Tangent Vectors

Analysing the equation of tangent

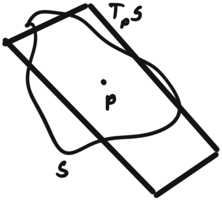
- What are $\dot{u}$ and $\dot{v}$?
- They depend on our choice of curve and are different for different tangent vectors.

Example

$$\sigma(u,v) : (u, v, u^2 - v^2) \quad , \quad (1,1,0) \to b$$

$$\sigma_u = (1,0,2u) = (1,0,2)$$

$$\sigma_v : (0,1,-2v) : (0,1,-2)$$



$$N : \sigma_u \times \sigma_v = (-2,2,1)$$

Equaⁿ of plane

$$-2(x-1) + 2(y-1) + 1(z-0) = 0$$

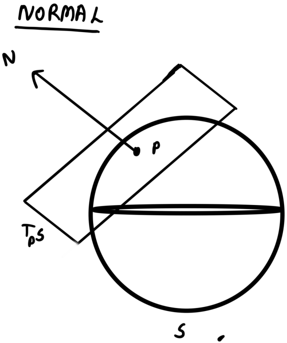
$$\boxed{-2x + 2y + z = 0}$$

Figure: Finding Tangent Space

Normals to the surface

At each point p on the surface we have a tangent plane $T_{\mathrm{p}}\mathcal{S}$, the unit normal vector to this plane is the unit normal vector at point p.

$$\mathbf{N}_{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|}$$



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Length of a curve on the surface

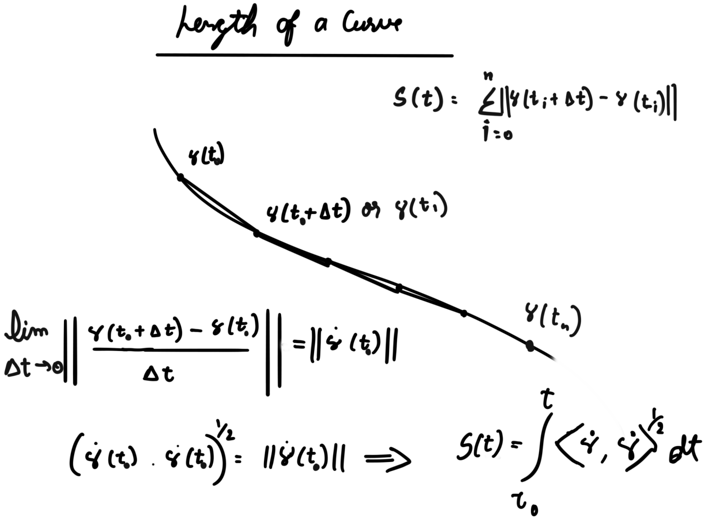


Figure: Curve on Surface

Length of a curve on the surface

$$\dot{\gamma(t_0)} \in T_{\mathbf{p}}\mathcal{S} \rightarrow \dot{\gamma(t_0)} = \lambda \boldsymbol{\sigma}_u + \mu \boldsymbol{\sigma}_v$$

$$||\dot{\gamma}|| = (\dot{\gamma}.\dot{\gamma})^{1/2}$$

$$\langle \dot{\gamma}, \dot{\gamma} \rangle = \lambda^2 \langle \sigma_u, \sigma_u \rangle + 2\lambda\mu \langle \sigma_u, \sigma_v \rangle + \mu^2 \langle \sigma_v, \sigma_v \rangle$$

Now we write

$$E = ||\sigma_u||^2, F = \langle \sigma_u, \sigma_v \rangle, G = ||\sigma_v||^2$$

First Fundamental Form

We define maps $du : T_{\mathrm{p}}\mathcal{S} \to \mathbb{R}$ and $dv : T_{\mathbf{p}}\mathcal{S} \to \mathbb{R}$ such that $du(v) = \lambda_1$, $dv(v) = \mu_1$ and E=$\langle \sigma_u, \sigma_u \rangle$, F=$\langle \sigma_u, \sigma_v \rangle$, G=$\langle \sigma_v, \sigma_v \rangle$ then for v in $T_{\mathbf{p}}\mathcal{S}$ we can rewrite the equation as

$$\langle v, v \rangle = Edu(v)^2 + Fdu(v)dv(v) + Gdv(v)^2$$

So equation of FFF is

$$Edu^2 + Fdudv + Gdv^2$$

General Form of FFF

$$\langle v, w \rangle^1 = \begin{bmatrix} du(v) & dv(v) \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} du(w) \\ dv(w) \end{bmatrix}$$

if we have

$$v = \lambda_1 \sigma_u + \mu_1 \sigma_v, w = \lambda_2 \sigma_u + \mu_2 \sigma_v$$

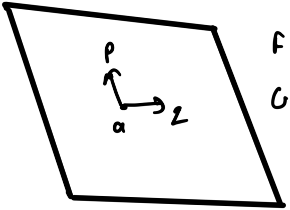
After doing the matrix multiplication we get:-

$$\langle v, w \rangle^1 = \lambda_1 \lambda_2 \langle \sigma_u, \sigma_u \rangle + (\lambda_1 \mu_2 + \lambda_2 \mu_1) \langle \sigma_u, \sigma_v \rangle + \mu_1 \mu_2 \langle \sigma_v, \sigma_v \rangle$$

Example

$$\sigma(u,v) = a + up + vq$$

$$\sigma_u = p \qquad \sigma_v = q$$



$$E = \|\sigma_u\|^2 = \|p\|^2 = 1$$

$$F = (\sigma_u \cdot \sigma_v) = (p \cdot q) = 0$$

$$G = \|\sigma_v\|^2 = \|q\|^2 = 1$$

$$\text{FFF} :- \; E\,du^2 + F\,du\,dv + G\,dv^2$$

$$p \perp q, \quad \|p\| = \|q\| = 1$$

$$\boxed{\text{FFF} = du^2 + dv^2}$$

Figure: First Fundamental Form of Plane

Using FFF to find the length of the curve on Surface

We have taken a curve $\gamma(t)$=(t,t) and a surface $\sigma(u,v) = (u,v,uv)$

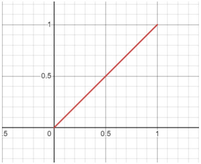


Figure: Curve

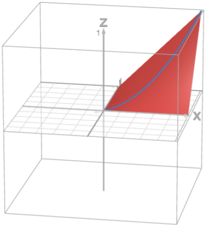


Figure: Curve on Surface

Using FFF to find the length of the curve on Surface

We had the curve $\gamma(t) = (t, t)$; $u(t) = t$, $v(t) = t$

$0 < t < 1$

we have the surface $\sigma(u, v) = (u, v, uv)$

$$\text{FFF} \rightarrow (1+v^2)\,du^2 + 2vu\,dv\,du + (1+u^2)\,dv^2$$

Substituting $v = t$, $u = t$, $du = 1$, $dv = 1$ we have

$$s(t) = \int_0^1 (1 + t^2 + 2t^2 + 1 + t^2)^{\frac{1}{2}}\,dt$$

$$= \int_0^1 (2 + 4t^2)^{\frac{1}{2}}\,dt$$

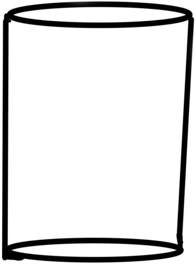
Figure: Finding Length of the Curve

Isometries

- We can wrap a plane sheet of paper on a cylinder without crumpling it, the lengths of the curves are equal on both the surfaces as it is one and the same.

- Two surfaces are locally isometric if they have the same fundamental form.

Isometries



$$\sigma(u,v) = (\cos u, \sin u, v)$$

$$\sigma_u \cdot \sigma_u = \begin{bmatrix} -\sin u & \cos u & 0 \end{bmatrix} \begin{bmatrix} -\sin u \\ \cos u \\ 0 \end{bmatrix} = 1$$

$$\sigma_v \cdot \sigma_u = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin u \\ \cos u \\ 0 \end{bmatrix} = 0$$

$$\sigma_v \cdot \sigma_v = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1$$

$$\boxed{FFF := du^2 + dv^2}$$

Figure: FFF of Cylinder

Curvature of a Surface(How much sloped are the roads?)

A surface can be thought of as a collection of curves locally, also if we look at a point then there are various curves with different curvature. How do we comment about the curvature of the surface?

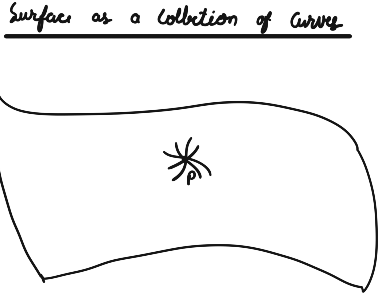


Figure: Surface is locally a collection of curves

Second Fundamental Form

- Analogous to curvature of the curves we look at the curvature as to how much the surface moves away from the tangent plane.
- So if we have two points $(\boldsymbol{\sigma}(u + \Delta u, v + \Delta v)$ and $\boldsymbol{\sigma}(u, v))$ then the curvature in the direction $(\boldsymbol{\sigma}(u + \Delta u, v + \Delta v) - \boldsymbol{\sigma}(u, v))$ is given by

$$(\boldsymbol{\sigma}(u + \Delta u, v + \Delta v) - \boldsymbol{\sigma}(u, v)) \cdot \mathbf{N}$$

Second Fundamental Form

The point performed two motions.

Curvature in direction $[\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v)] = [\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v)] \cdot \vec{N}$

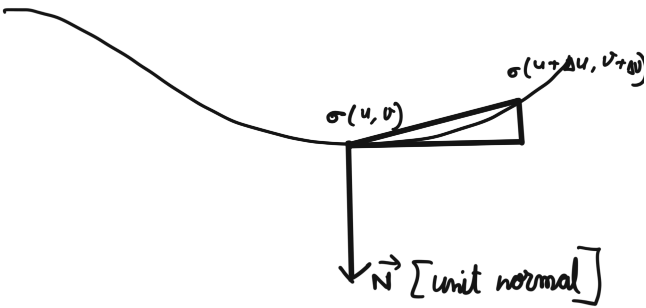


Figure: SFF

Second Fundamental Form

By the two variable form of Taylor's theorem,

$$\boldsymbol{\sigma}(u + \Delta u, v + \Delta v) - \boldsymbol{\sigma}(u, v)$$

is equal to

$$\boldsymbol{\sigma}_u \Delta u + \boldsymbol{\sigma}_v \Delta v + \frac{1}{2}\left(\boldsymbol{\sigma}_{uu}(\Delta u)^2 + 2\boldsymbol{\sigma}_{uv}\Delta u \Delta v + \boldsymbol{\sigma}_{vv}(\Delta v)^2\right) + \text{ remainder},$$

Second Fundamental Form

Now $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are tangent to the surface, hence perpendicular to $\mathbf{N}$, therefore we have

$$\frac{1}{2}\left(L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2\right) + \text{ remainder},$$

where

$$L = \boldsymbol{\sigma}_{uu} \cdot \mathbf{N}, \quad M = \boldsymbol{\sigma}_{uv} \cdot \mathbf{N}, \quad N = \boldsymbol{\sigma}_{vv} \cdot \mathbf{N}.$$

Example

$$\sigma(u, v) = (u, v, u^2 + v^2)$$

$$\boldsymbol{\sigma}_u = (1, 0, 2u), \boldsymbol{\sigma}_v = (0, 1, 2v),$$

N=$\lambda(-2u, -2v, 1), \lambda = \frac{1}{\sqrt{1+4u^2+4v^2}}$;$\boldsymbol{\sigma}_{uu} = (0, 0, 2), \boldsymbol{\sigma}_{uv} = \mathbf{0}, \boldsymbol{\sigma}_{vv} = (0, 0, 2)$
$L = 2\lambda, M = 0, N = 2\lambda$, and the second fundamental form is

$$2\lambda \left(du^2 + dv^2\right)$$

.

Finding Curvature at a Point using SFF

With the above surface we are considering a curve $\gamma(t) = (2t, t)$ on the surface and trying to find it's curvature at t=0 using SFF.

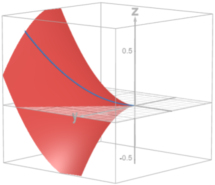


Figure: Curve on the surface

Substituting u=2t,v=t, du=2,dv=1 we have the formula as

$$10/\sqrt{1 + 16t^2 + 4t^2}$$

So the curvature at origin becomes 10.

General Form of SFF

For a point p and two vectors v,w $\in$ the tangent plane at that point we have the general formulae for second fundamental form as.

$$\langle \mathbf{v}, \mathbf{w} \rangle = L du(\mathbf{v}) du(\mathbf{w}) + M(du(\mathbf{v}) dv(\mathbf{w}) + du(\mathbf{w}) dv(\mathbf{v})) + N dv(\mathbf{v}) dv(\mathbf{w})$$

This gives an idea about relative curvature getting an getting a geometric interpretation requires Weingarten maps.

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- Elementary Differential Geometry, Pressley, Springer, 2010
- Differential geometry of curves and surfaces, M. do Carmo. Prentice Hall, 1976.

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Normal Curvature and Geodesic Curvature

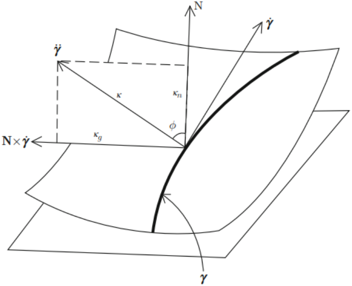

Figure: Normal and Geodesic Curvature

The point has two directional accelerations on a sloped and curved road.

- One is in the normal direction when it moves away from the tangent plane.
- Second is when it changes direction in the tangent plane.

Normal Curvature and Geodesic Curvature

γ is a unit speed curve then $\dot{\gamma}$ and $\ddot{\gamma}$ are perpendicular, also $\dot{\gamma} \in$ tangent plane so perpendicular with N. Hence we have N, $\dot{\gamma}$ and $N \times \dot{\gamma}$ are mutually perpendicular. We have

$$\ddot{\gamma} = \kappa_n \mathbf{N} + \kappa_g \mathbf{N} \times \dot{\gamma}$$

κ_n and κ_g are normal and geodesic curvatures respectively

Curvature for a general curve

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

Any parameterization of a regular curve is regular

Any parameterization of a regular curve is regular ; $t = \phi(\tilde{t})$

$\phi(\psi(t)) = t$, $\psi = \phi^{-1}$

$$\frac{d\phi}{d\tilde{t}} \cdot \frac{d\psi}{dt} = 1$$

can't be zero

$$\frac{d\tilde{\gamma}}{dt} = \frac{d\gamma}{dt} \cdot \frac{d\phi}{d\tilde{t}}$$

Unit speed Curve

Unit speed curve

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\|\, du \implies \frac{ds}{dt} = \|\dot{\gamma}(u)\|$$

$$\tilde{\gamma}(u(t)) = \gamma(t)$$ [Reparameterization mapping has a smooth inverse]

$u = \phi^{-1}$

$$\left\|\frac{d\tilde{\gamma}}{du}\right\| \left|\frac{du}{dt}\right| = \left\|\frac{d\gamma}{dt}\right\|$$

$$\left|\frac{du}{dt}\right| = \frac{ds}{dt} \implies u = \pm s + c$$

Figure: Parameterizing a unit speed curve