Lab -3

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MC312, Modeling and Simulation

In this lab, we numerically and analytically analyze different models of population growth. At first, we studied a model, in which the rate constant is constant, and the solution comes out to be an exponential function. Then, we took a look at the cases where the rate constant was proportional to the instantaneous number who have adopted the product. At last, we also analyzeed if the rate constant has incorporated both the effects (mixed-influence).

I. INTRODUCTION

The number of births for a particular species in an environment is theoretically exponential, but some factors such as predators, diseases, and limited resources tend to limit the growth of the population to some extent. Therefore, it can be assumed that population growth will not be exponential in reality. **Carrying Capacity** is termed as the maximum number of organisms that the area can support for a particular organism.

II. LOGISTIC EQUATIONS

First, we'll discuss the unconstrained growth of the population, without considering any constraints. This can be solved easily, as it avoids major complications.

$$\frac{dP}{dt} = rP \Rightarrow P = P_0 \cdot e^{rt}$$

where

- P is the instantaneous population.
- P_0 is the initial population.
- r is the growth rate

Now, we need to make some changes in the first case (discussed above), incorporating all the factors we discussed above so that our model somewhat applies in reality. In the revised model, it should be noted that the population at any instant should be less than the carrying capacity. Let the initial population also be less than the carrying capacity. There must exist a time when the population is very much closer to the Carrying Capacity. At that instant, the population growth will almost be 0, which means the population remain almost constant, which is a direct indication that the number of births will be equal to the number of deaths.

For a population greater than the carrying capacity, the population should decrease, in the form of deaths, thus, death rate should be revised. In this situation, the P/M ratio will be a perfect indicator of the current population with respect to the carrying capacity. Therefore, the death rate should be revised to $r\frac{P}{M}$. The birth rate will remain the same. Therefore, we can write the instantaneous growth of the population as:

$$\frac{dP}{dt} = (BirthRate) - (DeathRate)$$

$$\frac{dP}{dt} = (rP) - \left(r \cdot \frac{P}{M}\right) P$$

$$\boxed{\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right)} \quad (1)$$

We know that, $\frac{\triangle D}{\triangle t} = (r\frac{P}{M})P$, where $\triangle D$ is the change in number of deaths. For population at time (t-1), number of deaths from (t-1) to (t) can be approximated as:

$$\frac{\triangle D}{\triangle t} = \left(r\frac{P(t-1)}{M}\right) \cdot P(t-1)$$

Let's assume $\triangle$ t to be equal to 1. We get the equation as:

$$\triangle D = \left(r\frac{P(t-\triangle t)}{M}\right) \cdot P(t-\triangle t)\triangle t$$

We can calculate the change in population from time $(t - \triangle t)$ to time (t), as the difference between number of births and the number of deaths over that period.

$$\triangle P = \text{births - deaths}$$

$$\triangle P = rP(t-\triangle t)\triangle t - \left(r\frac{P(t-\triangle t)}{M}\right) P(t-\triangle t)\triangle t$$

$$\boxed{\triangle P = P(t-\triangle t) \cdot r\triangle t\left(1 - \frac{P(t-\triangle t)}{M}\right)} \quad (2)$$

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(1) and (2) are called the Logistic equations. These equations match with real-time observations, as mentioned previously. Let's analyze logistic equations, by assuming M=100. If the initial population is less than M, the population will increase until it reaches M. On the contrary, if the initial population is greater than M, the population will decrease until it reaches M. If the initial population is exactly M, that means, $P/M = 1$, therefore, $\frac{dP}{dt} = 0$. Therefore, the system is at equilibrium. We can also state that, $P(t) = P(t - \triangle t)$ for all $t > 0$.

III. MODEL

We have the equation: $\boxed{\dot{N} = \alpha(t) \cdot (C - N(t))}$
where

- α is the coefficient of diffusion.
- C is the maximum number of potential users.
- $N(t)$ is the instantaneous number of people who have adopted the product till time t.

We will model and analyze the functions based on different functions of $\alpha(t)$:

A. External Influence Model

This is a model in which $\alpha(t)$ is a constant (say p), which captures people without being influenced by others. Substituting the value of $\alpha(t)$ into the above equation, we get:

$$\dot{N} = p(c - N(t))$$

To obtain it's maximum, $d^2N/dt^2 = 0$

$$p(0 - N'(t)) = 0 \Rightarrow p(0 - p(c - N(t)) = 0 \Rightarrow N(t) = c$$

Therefore, the maximum of $\dot{N}$ is obtained at $N = c$ or at $t = 0$.

If we go and solve the above equation, and try to analyze the function, we get:

$$\frac{dN}{dt} = p \cdot (C - N) => \frac{dN}{C - N} = p \cdot dt$$

Integrating the above equation, we get:

$$\ln(C - N|_{N_0}^{N} = p \cdot (t|_0^t$$

$$\boxed{(C - N) = (C - N_0) \cdot e^{pt}}$$

We can easily plot the exponential graph, as done in previous labs, using the analytical solution illustrated above.

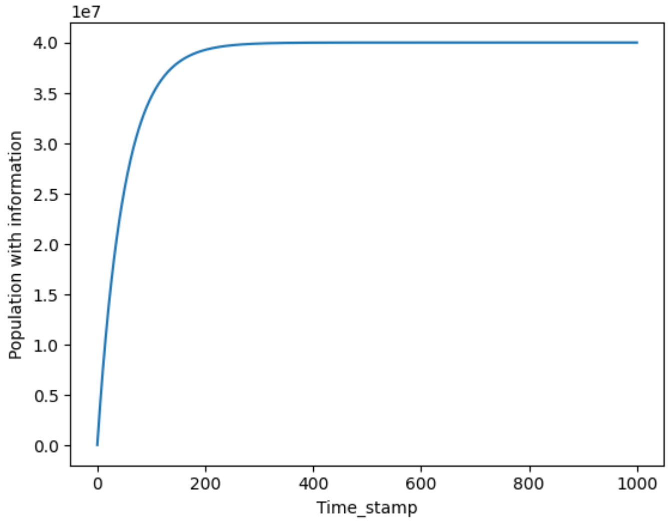

FIG. 1: Here we get an exponential increase like the newton's law of cooling. As the growth rate p which essentially refers to the rate at which information spreads is increased, we see that the population of people with the information reaches to it's maximum quickly which is also very intuitive. (Values assumed for C(Max$_p opulation = 40000000) andp = 0.02$)

B. Internal Influence Model

This is a model, in which α captures people influenced by others only. Mathematically, the model is given as:

$$\dot{N} = \frac{qN(t)}{C} \cdot (C - N(t))$$

Let the individuals who have been exposed to the information be called 'keepers'
Where:

- $\dot{N}$ is the rate of people who have adopted the product over time t.
- $N(t)$ is the number of people who have adopted the product at time t.
- q is a constant that represents the influence of existing keepers on the rate of new keepers.
- C is the maximum potential number of keepers (carrying capacity).

Now to solve the above differential equation for $N(t)$: Starting with the differential equation:

$$\dot{N} = \frac{qN(t)}{C} \cdot (C - N(t))$$

$$\frac{d\dot{N}}{dt} = \frac{q}{C} \cdot (C - 2N(t)) = 0$$

The maximum will occur at $N(t) = C/2$.

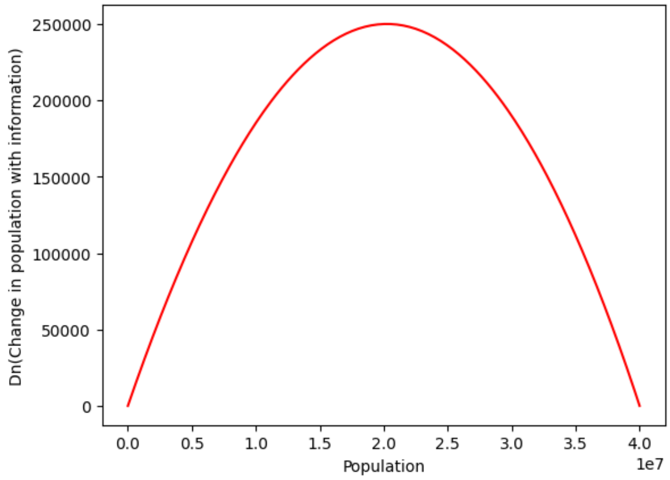

FIG. 2: The above plot signifies the number of people getting the information with their time stamps which a bell shaped curve. This can be easily understood from the differential equation that the curve will attain a maxima. But intuitively we see that the number of people who do not have the information will be very large so the information spreads rapidly as more and more people interact but at a point when information has already spread to many people and there are not enough people to spread the information as quickly as before then from that point the curve starts going downwards. From this point the information spread will not be as large because many people already have the information and less people don't have the information. When plotting the value of change in population with the information with respect to the total population with information we will find the maximum information will be passed when N(t)=C/2 which has been verified above analytically.

We can separate variables and integrate both sides:

$$\frac{dN}{N(C-N)} = \frac{q}{C}\,dt$$

Now integrate both sides:

$$\int \frac{dN}{N(C-N)} = \int \frac{q}{C}\,dt$$

The left-hand side integral can be solved using partial fraction decomposition:

$$\int \left(\frac{1}{N} + \frac{1}{C-N}\right) dN = \int \frac{q}{C}\,dt$$

$$\ln\left|\frac{N}{C-N}\right| = \frac{q}{C}t + K$$

$$\frac{N}{C-N} = e^{\frac{q}{C}t+K}$$

$$N = (C-N)\cdot e^{\frac{q}{C}t+K}$$

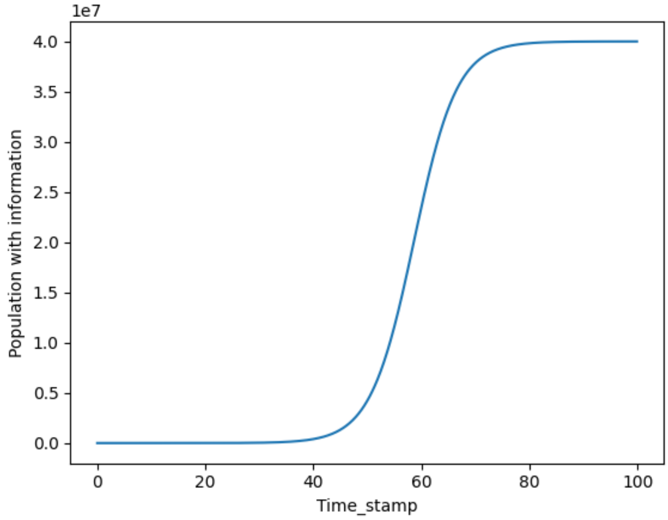

FIG. 3: The plot obtained here is a logistic curve, which is the interaction rate and signifies the rate at which people interact to spread the information. Intuitively, we would expect an increase in the rate that should result in faster spreading of information and faster attainment of the equilibrium or maximum population with the information. Also, we must mention that in this model every human is indifferent and has the same properties. Moreover, there are no external other factors involved.

$$N = C\cdot e^{\frac{q}{C}t+K} - N\cdot e^{\frac{q}{C}t+K}$$

$$N + N\cdot e^{\frac{q}{C}t+K} = C\cdot e^{\frac{q}{C}t+K}$$

$$N\cdot(1+e^{\frac{q}{C}t+K}) = C\cdot e^{\frac{q}{C}t+K}$$

$$N = \frac{C\cdot e^{\frac{q}{C}t+K}}{1+e^{\frac{q}{C}t+K}}$$

This is the analytical solution for the internal influence model.

The solution we derived is:

$$N(t) = \frac{C\cdot e^{\frac{q}{C}t+K}}{1+e^{\frac{q}{C}t+K}}$$

Assuming the values of C and q to be 40000000 and 0.25 respectively we can verify that the analytical solution matches the numerical results.

In summary, the $N(t)$ curve begins with a low starting point, increases rapidly, reaches a point of equilibrium, and then converges to the maximum potential number C of keepers. Hence we notice an S-shaped graph, as demonstrated above.

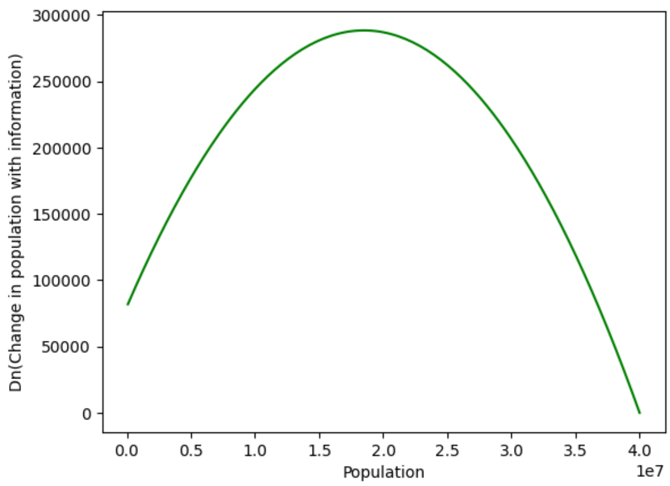

FIG. 4: As we can see, the maximum is shifted from $c/2$. This result can also been showed analytically as follows.

C. Mixed Influence Model

This model is applicable when we have interactions as well as some constantly increasing number of people(Mass media effect) getting the information which is an extension to both the problems discussed above. In this model, the peak shifts because there are two or we can say increased factors causing the change in the population of the people with information. So the peak which essentially means the time_stamp at which the change in the population of people with information is maximum and after which the change decreases because most of the people who are going to interact already have the information shifts leftwards because there is an increased rate caused due to the summation of both terms. Moreover, we also observe that the change in population with information is maximum at some point which is not N(t)= C/2 but occurs before it which is also evident from the analytical solution described below. The mathematical equation can be given as:

$$\frac{dN}{dt} = \left(p + \frac{qN(t)}{c}\right) \cdot (C - N(t))$$

$$\frac{dN}{dt} = pC + (q-p)N - qCN^2$$

Let's try to analyze the function to some extent. For maximum $\dot{N}$, $\frac{d^2N}{dt^2} = 0$ or $\frac{d\dot{N}}{dt} = 0$

$$0 + (q-p) - qC \cdot (2N) = 0 \Rightarrow N = \frac{c}{2}\left(1 - \frac{p}{q}\right)$$

It's maximum occur at $N(t) = \frac{c}{2}\left(1 - \frac{p}{q}\right)$

Contrary to the Internal Influence Model, we can easily see from the above equation, that the maximum is shifted from $\frac{c}{2}$.

Let's try to solve for N(t) from the original Mixed Influence Model equation.
To eliminate some constants from the equation:

$$\text{Let } x = \frac{N}{c}, \alpha = \frac{p}{q} \Rightarrow c \cdot \frac{dx}{dt} = qc(\alpha + x)(1 - x)$$

$$\frac{dx}{d(qt)} = (\alpha + x)(1 - x)$$

$$\text{Let } \tau = qt$$

$$\frac{dx}{d\tau} = (\alpha + x)(1 - x)$$

We converted such a complicated equation to a very simplified version, which can be solved easily. The solution of the above differential equation is:

$$\ln\left|\frac{\alpha + x}{1 - x}\right| = (\alpha + 1)\tau + k$$

By replacing with the original variables, we get the final equation as:

$$N(t) = c\left(\frac{e^{(p+q)t+k} - \alpha}{e^{(p+q)t+k} + 1}\right)$$

Substituting the values of C=40000000, p=0.02 and q=0.25 we verify the numerical results obtained match with the analytical solutions. Also the max value for change in population occurring at

$$N = \frac{c}{2}\left(1 - \frac{p}{q}\right)$$

will be at 18400000 which is just to the left of the previous point at 20000000.

IV. CONCLUSIONS

We analyzed different models of population growth, namely External, Internal and Mixed Influence Models. We analyzed the behaviour of the equations numerically and analytically of all the models briefly. Firstly, we considered the case when α is a constant. Then, we took linear dependence into account. Finally, we modelled the model close to the practical observation, that is, the Mixed Influence Model, which observes both effects.

[1] A. Shiflet and G. Shiflet, *Introduction to Computational Science: Modeling an Simulation for the Sciences*, Princeton University Press.3, 276 (2006).

[2] A. Einstein and N. Rosen, Phys. Rev.**48**, 73 (1935).