Notes on Topics in Analyis

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1. Isoperimetric Inequalities

Wiestrass Theorem :

(*Real Analysis*) If $f : [0,1] \to \mathbb{R}$ is a continuous function and $\epsilon > 0$ then $\exists$ a polynomial P such that $|f(x) - P(x)| < \epsilon$, whenever $x \in [0,1]$.

(*Functional Analysis*) Let $C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is cts}\}$ and $d(f,g) = \sup\limits_{x\in[0,1]} |f(x) - g(x)|$ be the metric. If P = set of all polynomials, then P is dense in $C[0,1]$.

Proof:

Let $f : [0,1] \to \mathbb{R}$ is continuous and $P_n(x) = \sum\limits_{k=0}^{n} \binom{n}{k}\, x^k(1-x)^{n-k}\, f\left(\frac{k}{n}\right)$ for $n \geq 1$.

Consider $P_{n,x}(k) = \binom{n}{k}\, x^k(1-x)^{n-k}$, this satisfies

$$P_{n,x}(k) \geq 0 \qquad \sum_{k=0}^{n} P_{n,x}(k) = 1 \qquad \sum_{k=0}^{n} \frac{k}{n} P_{n,x}(k) = x$$

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 P_{n,x}(k) = \frac{x(1-x)}{n} \leq \frac{1}{4n}$$

For any $x \in [0,1]$ and consider,

$$\begin{aligned}
|f(x) - P_n(x) &= \left|\sum_{k=0}^{n} P_{n,x}(k)\ \left[f(x) - f\left(\frac{k}{n}\right)\right]\right| \\
&\leq \sum_{k=0}^{n} P_{n,x}(k)\ \left|f(x) - f\left(\frac{k}{n}\right)\right| \\
&\leq \sum_{k:|\frac{k}{n}-x|\leq\delta} P_{n,x}(k)\left|f(x) - f\left(\frac{k}{n}\right)\right| + \sum_{k:|\frac{k}{n}-x|>\delta} P_{n,x}(k)\left|f(x) - f\left(\frac{k}{n}\right)\right| \\
&\leq \omega_f(\delta) \sum_{k:|\frac{k}{n}-x|\leq\delta} P_{n,x}(k) + 2\|f\|_{\sup} \sum_{k:|\frac{k}{n}-x|>\delta} P_{n,x}(k)
\end{aligned}$$

But we can also observe that,

$$1^{st} \text{ term} \leq \omega_f(\delta) \qquad \left[\because \sum_{k=0} nP_{n,x}(k) = 1\right]$$

$$2^{nd} \text{ term} \leq \frac{2}{\delta^2}\|f\|_{\sup} \sum_{k:|\frac{k}{n}-x|>\delta} \left(\frac{k}{n} - x\right) P_{n,x}(k) \leq \frac{1}{2\delta^2 n}\|f|_{\sup} \qquad [\because \text{Chebyshev's Inequality}]$$

For a given $\epsilon > 0$, choose $\delta > 0$ such that $\omega_f(\delta) < \epsilon/2$ and pick n large such that $\frac{1}{2\delta^2 n}\|f\|_{sup} < \epsilon/2$. $\square$

Feyer's Theorem:

The set of all trignometric polynomials is dense in $C(S^1)$.