Market Design: Periodic Auction Market

Introduction

Context (1)

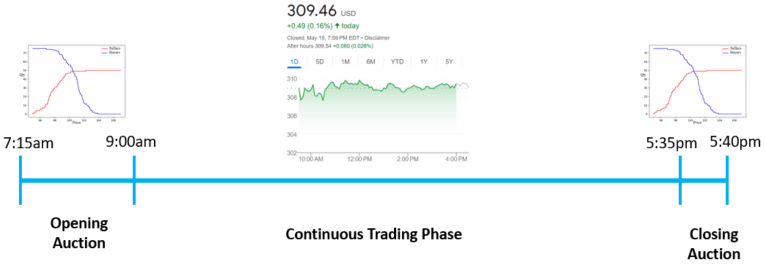


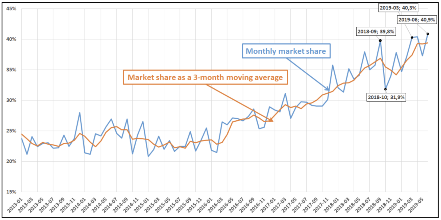
Figure – Trading day at Euronext

- **Auction phase** : Orders are aggregated [no transaction] - eligible orders executed at a unique clearing price at the end of the auction

Context (2)

- Opening & closing auctions → short events compared to the duration of the continuous trading time.
- An increasing concentration of share trading volumes at the end of the day → Markets Worldwide
- In France almost 40 % of the total volume traded during last 5 minutes of the trading day (closing auction) in 2019 vs 20% in 2015.
- CAC 40 shares (French stock market index)

Percentage of the auctions in the trading volumes on CAC 40 shares on Euronext Paris
(from 2013 to Q2 2019)

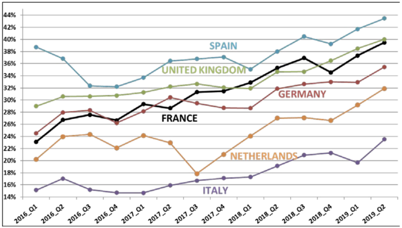


Source: AMF statistics

Context (3)

- The days on which quarterly derivatives expire are the days on which the share of the closing auction reaches its highest level
- The market share of auctions in Spain is 44 % in Q2 2019 and Italian market 22 % and increasing sharply.

Auction share on equities[4] in the main European markets
(on the historical stock exchanges only[5])



Source: Refinitiv

Context (4) : Why ?

- One possible reason : Willingness to avoid high-frequency trader arbitrageurs.
- Another possible reason : Auctions have the benefit to aggregate many orders and produce price that reflects market information

Context (5) : 2010 Flash Crash

Figure – The DIJA on May 6, 2010 (11 :00 AM - 4 :00 PM EDT)

- Euronxt introduces an auction mechanism to prevent flash crash : Reservation phase
- Market order causes +/- 15% variation of the midprice → Auction phase starts

Idea : Periodic Auctions Market

We replace a continuous traded phase by a sequence of periodic auctions.

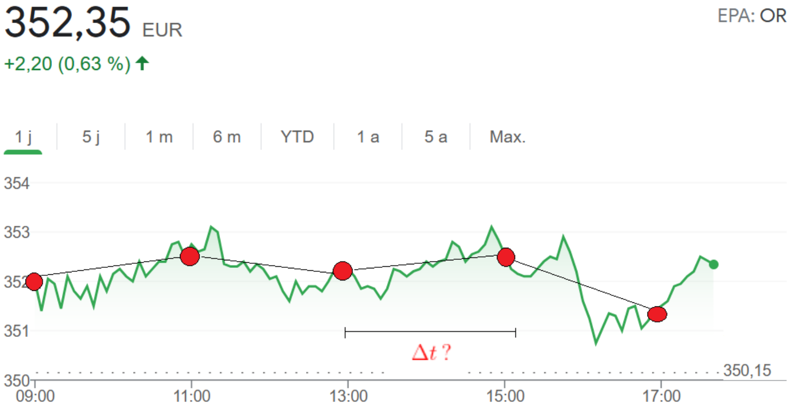


Figure – Discretize trading periods in financial markets

Schedule

1. **First model of periodic auctions** (Vincelot)
2. **Parameter Calibration** (Qingqing)
3. **Second model of periodic auction**s (Reine)
4. **Common framework for both models** (Vincelot)
5. Optimal auction duration does not always exist (Vincelot)
6. **New Calibration results** (Qingqing)
7. **Trading strategies in periodic auction markets** (Aajeya)
8. **Nash equilibrium with strategic market participants** (Qingqing)
9. **Conclusion** (Aajeya)

What is the optimal auction duration ? Existing models

- Fricke, D., & Gerig, A. (2018). *Too fast or too slow ? Determining the optimal speed of financial markets.*
- Jusselin, P., Mastrolia, T., & Rosenbaum, M. (2020). *Optimal auction duration : A price formation viewpoint.*

First Model of Periodic Auctions : Fricke & Gerig (2018)

Fricke & Gerig (2018) - Model Assumptions

$\rightarrow$ One security traded through a sequence of consecutive and independent auctions of duation $\tau > 0$

- **(A1)** A fixed number of traders at each auction $N = \lfloor \omega\tau \rfloor \geq 1$.
- **(A2)** The traders arrive at regular time intervals $\frac{1}{\omega}, \frac{2}{\omega}, ..., \frac{N}{\omega}$
  $\rightarrow \omega =$ traders arrival frequency

Fricke & Gerig (2018) - Model Assumptions

- **(A3)** Trader k has a demand schedule $p \mapsto D_k(p) = e(r_k - p)$
  - r_k = *reservation price* → Trader estimation of what the clearing price is
  - e = Volume/Price elasticity

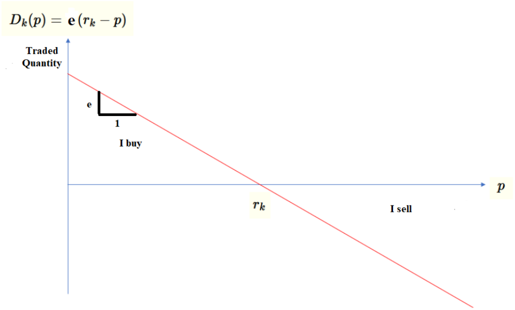


Figure – Demand Schedule

Fricke & Gerig (2018) - Model Assumptions

- The security has an unobservable efficient price $dm_t = \sqrt{\frac{3}{2}}\psi dB_t$
- The reservation price of each trader is a noisy sample of the efficient price :

$$\forall k \in \{1, ..., N\}, \qquad \underbrace{r_k}_{\text{reservation price}} = \underbrace{m_{\frac{k}{\omega}}}_{\text{efficient price}} + \underbrace{\varepsilon_k}_{\text{noise}},$$

where $m_{\frac{k}{\omega}} = \sqrt{\frac{3}{2}}\psi B_{\frac{k}{\omega}} \sim \mathcal{N}\left(0, \frac{3}{2}\psi^2 \frac{k}{\omega}\right)$ and $\varepsilon_k \sim \mathcal{N}(0, \sigma^2)$

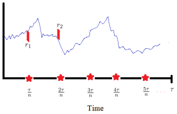


Figure – Demand Schedule

Fricke & Gerig (2018) - Model Assumptions

- **Model** : The volatility is independent of the trading volume

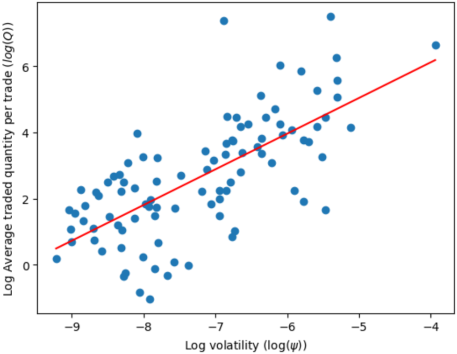


Figure – Log-average traded quantity vs. Log-volatility (97 French equities - June 2021)

- $D_k(p) = e\,(r_k - p) \longrightarrow e(\psi) \propto \psi^{\alpha}$

Fricke & Gerig (2018) - Auction Clearing Mechanism

Auction Clearing Price : Price that does not generate excess demand/supply :

$$\underbrace{\sum_{k \,|\, e(r_k - \overline{p}) < 0} -e\left(r_k - \overline{p}\right)}_{\text{buyers}} = \underbrace{\sum_{k \,|\, e(r_k - \overline{p}) \geq 0} e\left(r_k - \overline{p}\right)}_{\text{sellers}}$$

i.e.

$$\boxed{\overline{p} = \frac{1}{N} \sum_{k=1}^{N} r_k}$$

Fricke & Gerig (2018) - Market Quality

k-th trader liquidity risk :

$$\mathbf{L}^{(k)}(\tau) = \mathrm{Var}\left(\overline{p} - m_{\frac{k}{\omega}}\right)$$

Liquidity Risk :

$$\mathbf{L}(\tau) = \frac{1}{N}\sum_{k=1}^{N} \mathrm{Var}\left(\overline{p} - m_{\frac{k}{\omega}}\right)$$

Intuituvely :

- **If τ is too small** : not enough information aggregated in $\overline{p}$ to reflect what the efficient price is
- **If τ too large** : clearing price $\overline{p}$ might be far from the efficient price at the beginning of the auction

Optimal Auction Duration

$$\tau^* \in \underset{\tau \geq \frac{1}{\omega}}{\mathrm{Argmin}}\ \mathbf{L}(\tau)$$

Fricke & Gerig (2018) - Market Quality

Theorem 1 [Fricke & Gerig (2018)] Assuming that $\omega >> 1$ (large number of traders at each auction), we have :

$$\mathbf{L}(\tau) = \frac{\sigma^2}{\omega\tau} + \frac{\psi^2}{4}\tau$$

so :

$$\tau^* = \frac{2\sigma}{\psi\sqrt{\omega}}$$

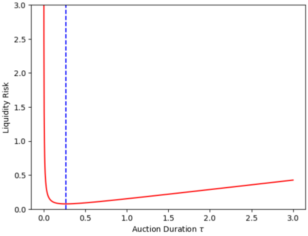


Figure – Theoretical liquidity risk for $(\sigma, \psi, \omega) = (1, 0.75, 100)$.

Fricke & Gerig (2018) - Agent-based simulation

- For $\tau \in [0, 2]$:
  - For n_{sim} in $\{1, ..., B\}$:
    - Simulate a brownian motion with variance $\frac{3}{2}\psi^2$ every $\frac{1}{\omega}$
    - Add a noise $N(0, \sigma^2)$ to the brownian motion samples to get the traders' reservation prices
    - Compute the clearing price and get a sample of the $\overline{p} - m_i$
  - Compute the liquidity risk $\frac{1}{N}\sum_{i=1}^{N} \text{Var}(\overline{p} - m_i)$ [empirical variance]

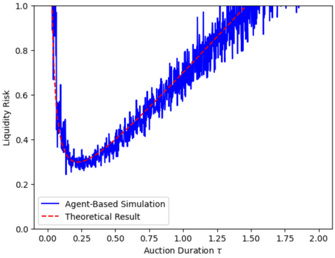


Figure – Simulation $((\omega, \psi, \sigma) = (30, 2, 1))$

Paper results do not hold for small values of ω !

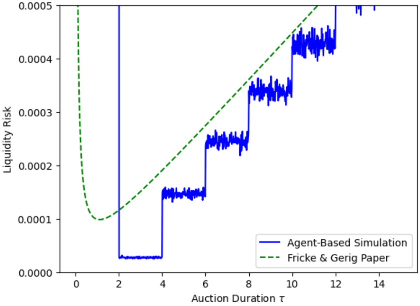


Figure – Agent-based simulation ($\omega = 0.5$, $\psi = 0.001$, $\sigma = 0.005$)

- Picewise-constant liquidity risk
- **Range** of optimal auction durations !

Contribution 1- Market Quality ▸▸ Proof

Contribution 1 : In the general case, we have :

$$\boxed{\mathbf{L}(\tau) := \left(\sigma^2 - \frac{\psi^2}{4\omega}\right)\frac{1}{\lfloor \omega\tau \rfloor} + \frac{\psi^2}{4\omega}\lfloor \omega\tau \rfloor}$$

and :

$$\boxed{\begin{cases} \tau^* \in \left[\frac{1}{\omega}N^*, \frac{1}{\omega}\left(N^*+1\right)\right) & \text{if } \omega \geq \frac{\psi^2}{4\sigma^2} \\ \tau^* = \frac{1}{\omega} & \text{if } \omega < \frac{\psi^2}{4\sigma^2} \end{cases}}$$

where $N^* \in \left\{\left\lfloor \sqrt{\frac{4\sigma^2\omega}{\psi^2} - 1} \right\rfloor, \left\lfloor \sqrt{\frac{4\sigma^2\omega}{\psi^2} - 1} \right\rfloor + 1\right\}$

$$\boxed{\tau^* \underset{\omega \to +\infty}{\sim} \frac{2\sigma}{\psi\sqrt{\omega}}}$$

Contribution 1 - Agent-based simulation (Rectified)

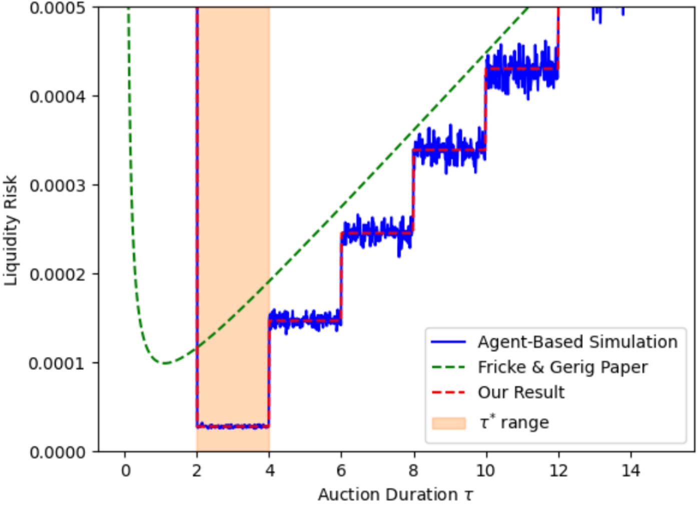


Figure – Agent-based simulation ($\omega = 0.5$, $\psi = 0.001$, $\sigma = 0.005$)

Parameters Calibration - Market Data

- No access to auction data

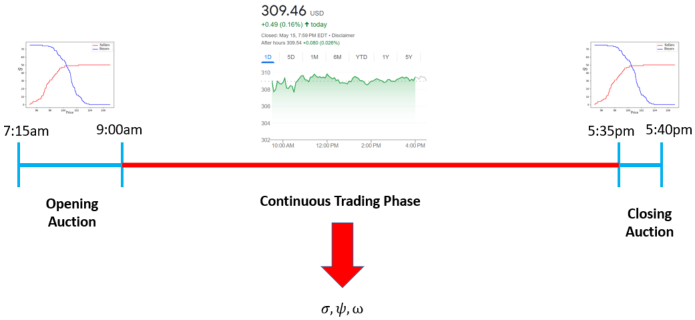


Figure – Parameter Calibration : Assumptions

- **Assumption :** Our calibrations of σ, ψ, ω are realistic values of the parameters if the continuous trading markets turned into periodic auction markets.

Parameters Calibration - Market Data

- 97 French equities
- 1-month dataset (06/01/2021 to 06/30/2021)
- State of the limit order book after each event [book depth = 1]

date_time	sequence_number	event_type	order_type	passive_or_aggressive_indicator	direction	price	quantity	transaction_price	...	l1_ask_size	l1_bid_price	l1_bid_size	Trade	tra
2021-06-01 09:00:31.675000+02:00	570637	CAME	Limit	NaN	B	73.30	200.0	0.0	...	316.0	73.55	100.0	False	
2021-06-01 09:00:31.756000+02:00	570638	CAME	Limit	NaN	B	73.45	140.0	0.0	...	316.0	73.55	100.0	False	
2021-06-01 09:00:31.757000+02:00	570639	REME	Limit	NaN	B	73.45	61.0	0.0	...	316.0	73.55	100.0	False	
2021-06-01 09:00:35.423000+02:00	570640	NEWO	Limit	NaN	B	73.25	200.0	0.0	...	316.0	73.55	100.0	False	
2021-06-01 09:00:35.423000+02:00	570641	NEWO	Limit	NaN	B	73.60	70.0	0.0	...	316.0	73.60	70.0	False	
...	...	...	...	...	...	...	...	...	...	...	...	...	...	
2021-06-30 17:29:59.903000+02:00	611750	CAME	Limit	NaN	S	73.00	1643.0	0.0	...	60.0	72.75	363.0	False	
2021-06-30 17:29:59.904000+02:00	611751	CAME	Limit	NaN	S	73.05	173.0	0.0	...	60.0	72.75	363.0	False	
2021-06-30 17:29:59.909000+02:00	611752	CAME	Limit	NaN	S	73.10	1656.0	0.0	...	60.0	72.75	363.0	False	
2021-06-30 17:29:59.910000+02:00	611753	CAME	Limit	NaN	S	72.95	512.0	0.0	...	60.0	72.75	363.0	False	
2021-06-30 17:29:59.911000+02:00	611754	CAME	Limit	NaN	S	73.00	357.0	0.0	...	NaN	NaN	NaN	False	

Figure – Limit order book data [ISIN : FR0000035081]

Parameters Calibration - Methodology

→ Using Fricke & Gerig paper calibration

- σ is estimated by the average relative spread during the trading day :

$$\hat{\sigma} = \frac{1}{T}\sum_{i=1}^{T}\frac{\text{AskPrice}_i - \text{BidPrice}_i}{\text{MidPrice}_i}$$

- We estimate ω by :

$$\hat{\omega} = \frac{\#\ \text{Number of new limit orders}}{\text{Trading phase duration}}$$

- ψ is estimated by the standard deviation of the midprice returns.

Parameters Calibration - Results

- **Median Lifetime** - 6 min 52 s
- **Mean Lifetime** - 17 min 17 s
- **Quantiles 10% - 90%** - 1min 22s | 52min 22s
- **Range** From 50.35s to 1hr 58min 40s

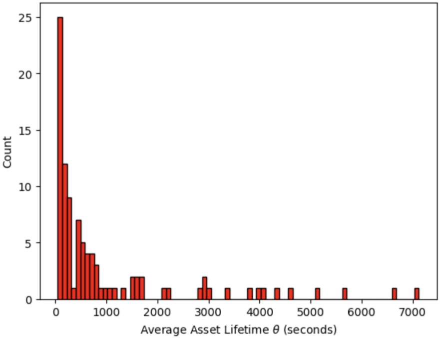


Figure – Average Lifetime of an Order

Parameters Calibration - Results

- **Median Auction Duration** - 14.48 s
- **Quantiles 10% - 90%** - 1.67 s | 1min 20s
- **Range** From 0.66 s to 36min

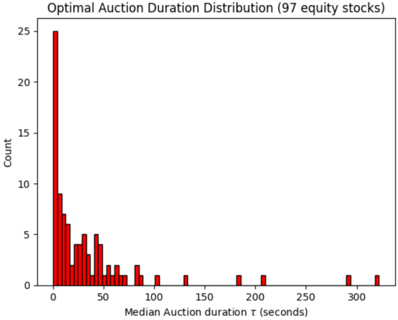


Figure – Optimal Auction Durations Distribution

Parameters Calibration - Results

Company Name	Optimal Auction Duration
TotalEnergies	0.66 s
Airbus	0.80 s
Engie	1.14 s

Company Name	Optimal Auction Duration
Hexaom	5 min 21 s
Cafom	4 min 53 s
Neurones	3 min 27

Figure – Highest vs Lowest median optimal auction durations

Second Model of Periodic Auctions : Rosenbaum et al.

Model 2 - Jusselin, Mastrolia & Rosenbaum (2020)

Key Idea - Add more randomness in the model & define a new market quality metric

- Participants who submit demand schedules ($=$ market makers) are modeled by a counting process $(\tilde{N}_t)_{t \geq 0}$.
- $t_1 \sim \mathcal{E}(\omega)$ $=$ arrival time of the first market maker $\rightarrow$ starts an auction for τ seconds
  - $(\tilde{N}_{t_1+t} - 1)_{t \in [0,\tau]}$ is a Poisson process w/ intensity $\omega > 0$
  - $\tilde{N}_{t_1+\tau} - 1 \sim \mathcal{P}(\omega(\tau + t_1))$ market makers during the auction
  - They arrive at uniform random times during the auction phase

Notations

- The other participants (= market takers) submit market orders with a fixed volume $v > 0$ during the auction phase.
  - Sell market orders $\rightarrow$ Poisson process $(\tilde{N}_t^a)_{t \in [t_1, t_1+\tau]}$
  - Buy market orders $\rightarrow$ Poisson process $(\tilde{N}_t^b)_{t \in [t_1, t_1+\tau]}$

Notations -

- N_τ number of market makers during an auction
- N_τ^a, N_τ^b number of market takers during an auction

Model 2 - Auction Clearing Price & Market Quality

Auction Clearing Price -

$$\underbrace{vN_\tau^a}_{\substack{\text{buy}\\ \text{market orders}}} + \underbrace{\sum_{k \mid e(r_k-\overline{p})<0} -e\left(r_k-\overline{p}\right)}_{\text{buyers}} = \underbrace{\sum_{k \mid e(r_k-\overline{p})\geq 0} e\left(r_k-\overline{p}\right)}_{\text{sellers}} + \underbrace{vN_\tau^b}_{\substack{\text{sell}\\ \text{market orders}}}$$

i.e.

$$\overline{p} = \frac{1}{N_\tau}\sum_{k=1}^{N} r_k + \underbrace{\frac{v\left(N_\tau^b - N_\tau^a\right)}{eN_\tau}}_{\substack{\text{linear}\\ \text{price impact}}}$$

Market Quality Metric - Clearing price formation quality :

$$\mathbf{C}(\tau) := \mathbf{E}\left[\left(\underbrace{\overline{p}}_{\substack{\text{clearing}\\ \text{price}}} - \underbrace{m_\tau}_{\substack{\text{efficient price at the}\\ \text{end of the auction}}}\right)^2\right]$$

Model 2 - Paper Results

- Derive analytic expressions for $\mathbf{C}(\tau)$:

$$\mathbf{C}(\tau) = \mathbf{C}^{\text{mid}}(\tau) + \frac{\mathbf{E}\left[\left(N_\tau^a - N_\tau^b\right)^2\right]}{e^2} \frac{e^{\nu\tau}\int_\tau^{+\infty} \nu e^{-\nu t} e^{-t} \int_0^{\omega t} \frac{1}{s}\int_0^s \frac{e^u - 1}{u}\, \mathrm{d}u\mathrm{d}}{1 - e^{-\omega\tau}\frac{\nu}{\nu+\omega}}$$

where

$$\mathbf{C}^{\text{mid}}(\tau) = \frac{e^{\nu\tau}\int_\tau^{+\infty} \nu e^{-\nu t}\left(\left(\psi^2\frac{t}{4}+\sigma^2\right)e^{-\omega t}\int_0^{\omega t}\frac{e^s-1}{s}\mathrm{d}s + \psi^2\frac{t}{2}\left(1-e^{-\omega t}\right)\right)\mathrm{d}t}{1-e^{-\omega\tau}\frac{\nu}{\nu+\omega}}$$

- The optimal auction duration is :

$$\tau^{opt} = \underset{\tau>0}{\text{Argmin}}\ \mathbf{C}(\tau)$$

- Their parameter calibration $\rightarrow \tau^* \in [2\text{min};\ 10\text{min}]$
- Hard to compare the two models (different metrics, parameters, calibrations)

Common Framework for Both Models

Contribution - Common framework for both models ▸▸ Proof

Under Fricke & Gerig paper assumptions :

$$\mathbf{C}(\tau) = \mathbf{E}\left[(\overline{p} - m_\tau)^2\right] = \mathrm{Var}\left(\overline{p} - m_\tau\right)$$

We get the following result :

Contribution 2.1 - When $N = \lfloor \omega\tau \rfloor \geq 1$, we get :

$$\mathbf{C}(\tau) = \left(\sigma^2 + \frac{\psi^2}{4\omega}\right)\frac{1}{\lfloor \omega\tau \rfloor} - \frac{\psi^2}{\omega}\lfloor \omega\tau \rfloor + \frac{3\psi^2}{2}\left(\tau - \frac{1}{2\omega}\right)$$

In particular :

$$\tau^* \underset{\omega\to+\infty}{\sim} \frac{1}{\sqrt{2}} \underbrace{\frac{2\sigma}{\psi\sqrt{\omega}}}_{\text{Fricke\& Gerig optimal duration}}$$

Common Framework - Agent-based simulation

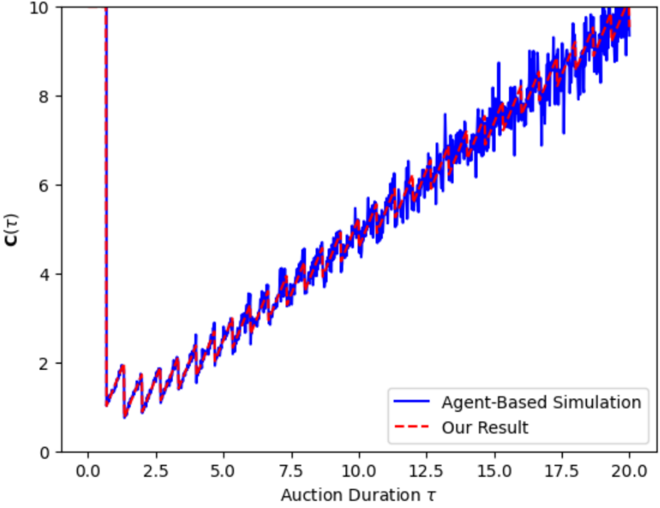


Figure – Agent-based simulation ($\omega = 1.5$, $\psi = 1$, $\sigma = 1$)

Contribution 2.2 - Common framework for both two models

Contribution - If the number of traders follows a Poisson distribution ($N_\tau - 1 \sim \mathcal{P}(\omega\tau)$), then :

$$\mathbf{L}(\tau) = \left(\sigma^2 - \frac{\psi^2}{4\omega}\right)\frac{1 - e^{-\omega\tau}}{\omega\tau} + \frac{\psi^2}{4\omega}(\omega\tau + 1)$$

$$\mathbf{C}(\tau) = \left(\sigma^2 + \frac{\psi^2}{4\omega}\right)\frac{1 - e^{-\omega\tau}}{\omega\tau} - \frac{\psi^2}{\omega}(\omega\tau + 1) + \frac{3\psi^2}{2}\left(\tau - \frac{1}{2\omega}\right)$$

Common Framework - Agent-based simulation

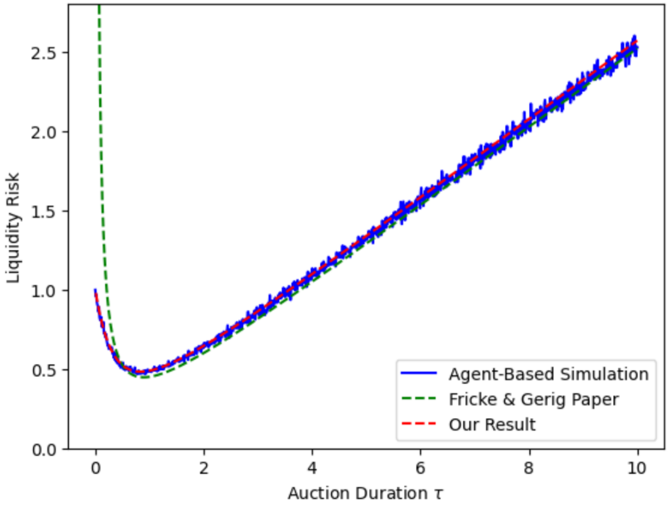
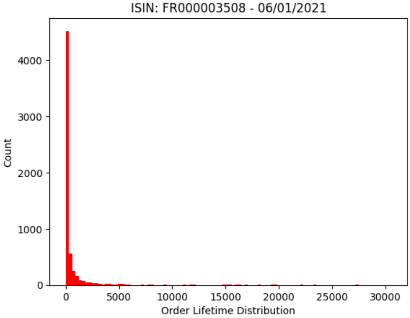


Figure – Agent-based simulation ($\omega = 1.5$, $\psi = 1$, $\sigma = 1$)

Common framework for both two models - Order Cancellation

- The lifetime of an order $\sim \mathcal{E}(\theta)$



- $N_\tau - 1 \sim \mathcal{P}\left(\omega\tau\right) \longrightarrow N_\tau^{(\theta)} - 1 \sim \mathcal{P}\left(\omega\left(\theta, \tau\right)\right)$ where

$$\omega\left(\theta, \tau\right) = \frac{\omega}{\theta}\left(1 - e^{-\theta\tau}\right)$$

- $\theta \to 0 : \omega\left(\theta, \tau\right) \to \omega$
- $\theta \to +\infty : \omega\left(\theta, \tau\right) \to 0$

Contribution - Common framework for both models

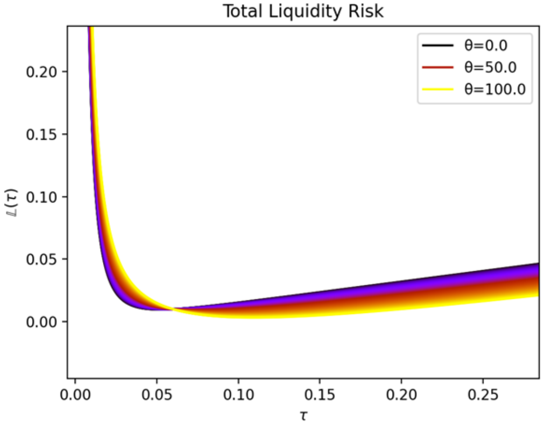


Figure – Effect of order cancellation on the liquidity risk $\mathbf{L}(\tau)$

- Order cancellation increases the optimal auction duration → we need to wait longer to get a meaningful price.

Common framework for both models - Random Arrival Times

- $t_1, \ldots, t_{N_\tau}$ arrival times of the traders' orders
- $t_1, \ldots, t_{N_\tau}$ are i.i.d with a density f_τ
- If the traders arrive at uniform times in $[0, \tau)$:

$$\forall t \in [0, \tau), \quad f_\tau(t) = \frac{1}{\tau} \mathbf{1}_{[0,\tau)}(t)$$

- If the traders mostly arrive at the beginning of the auction :

$$\forall t \in [0, \tau), \quad f_\tau(t) = \frac{\lambda e^{-\lambda t}}{e^{\lambda \tau} - 1}$$

with $\lambda > 0$

Question - What does **L** if the traders arrive at times $t_1, \ldots, t_{N_\tau} \sim f_\tau$?

Common framework for both models - Random Arrival Times ▸▸ Proof

Contribution - We get the following result :

$$\mathbf{L}(\tau) = \sigma^2 \frac{1 - e^{-\omega(\theta,\tau)}}{\omega(\theta,\tau)} + \frac{3}{2}\psi^2 \left(\int_0^{\tau} F_\tau \left(1 - F_\tau\right) \right) \left(1 - \frac{1 - e^{-\omega(\theta,\tau)}}{\omega(\theta,\tau)} \right)$$

where F_τ is the cdf of f_τ.

In particular, if $\theta \to 0$, then :

$$\mathbf{L}(\tau) = \sigma^2 \frac{1 - e^{-\omega\tau}}{\omega\tau} + \frac{3}{2}\psi^2 \left(\int_0^{\tau} F_\tau \left(1 - F_\tau\right) \right) \left(1 - \frac{1 - e^{-\omega\tau}}{\omega\tau} \right)$$

Contribution - Market Orders

Contribution - We get the following result :

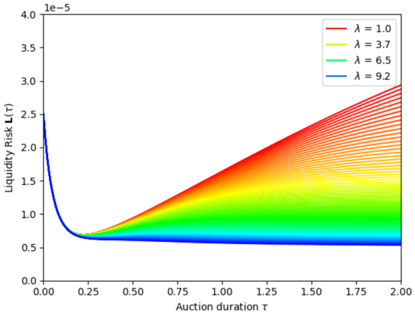
$$\mathbf{L}(\tau) = \sigma^2 \frac{1-e^{-\omega\tau}}{\omega\tau} + \frac{3}{2}\psi^2 \left(\int_0^{\tau} F_\tau \left(1 - F_\tau\right) \right) \left(1 - \frac{1-e^{-\omega\tau}}{\omega\tau} \right) + \frac{\mathbf{E}\left[\left(N_\tau^b - N_\tau^a \right)^2 \right]}{\omega\tau e^2} e^{-\omega\tau} \int_0^{\omega\tau} \frac{e^x - 1}{x} \mathrm{d}x.$$

Contribution - Does the optimal auction duration always exist ?

If the traders come at the beginning of the auction :

$$\forall t \in [0, \tau), \quad f_\tau(t) = \frac{\lambda e^{-\lambda t}}{1 - e^{-\lambda \tau}}$$

$$\mathbf{L}(\tau) = \sigma^2 \frac{1 - e^{-\omega \tau}}{\omega \tau} + \frac{3}{2}\psi^2 \frac{(\sinh(\lambda \tau) - \lambda \tau)\,\mathrm{csch}^2\left(\frac{\lambda \tau}{2}\right)}{4\lambda}\left(1 - \frac{1 - e^{-\omega \tau}}{\omega \tau}\right)$$

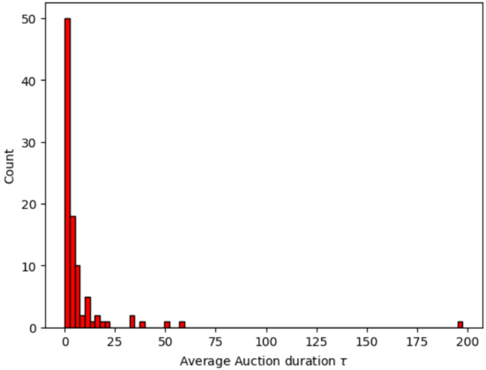


New Calibration - Assumptions & Calibration

- Uniform arrival times : $f_\tau = \frac{1}{\tau}\mathbf{1}_{[0,\tau)}$
- $$\theta = \frac{1}{\text{Average Lifetime of an order}}$$
- Keep the previous estimations of ψ, σ, ω.

New Calibration - Results

Median : 2.3s
Quantiles 10% | 90% : 0.3s | 16.2s



Trading Strategies in Periodic Auction Markets

Contribution 3 - Trading Strategies in Periodic Auction Markets

Assumptions -

- Auction duration τ^* fixed
- There is a fixed number of traders $K = \lfloor \omega\tau \rfloor \geq 1$ at each auction.
- The market contains :
  - $K = \lfloor \omega\tau \rfloor \geq 1$ (non-strategic) market makers who submit demand schedules
  - One strategic market participant who wants to maximize his profit.

Contribution 3 - Trading Strategies in Periodic Auction Markets

Contribution 3.1 - The clearing price $\overline{p}$ follows a normal distribution :

$$\overline{p} \sim \mathcal{N}\left(0\,,\underbrace{\left(\sigma^2+\frac{\psi^2}{4\omega}\right)\frac{1}{\lfloor\omega\tau\rfloor}+\frac{\psi^2}{4\omega}\left(2\lfloor\omega\tau\rfloor+3\right)}_{\overline{\sigma}^2(\psi,\sigma,\omega)}\right)$$

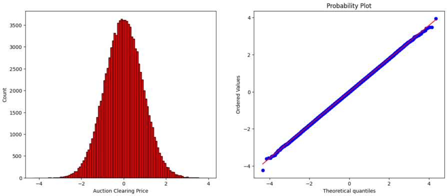


Figure – Clearing price distribution [100,000 simlulations]

Contribution 3 - Trading Strategies in Periodic Auction Markets

Trading strategy - Buy low $\rightarrow$ Sell high or Sell high $\rightarrow$ Buy low
Intuition - When the clearing price is in a low quantile q_α of its distribution, then the next price will be higher with a probability $\geq 1 - \alpha$.
Technical Issue -

- If we buy a quantity $Q > 0 \rightarrow$ linear price impact in $\frac{Q}{eK}$ (increase the price)
- If we sell a quantity $-Q < 0 \rightarrow$ linear price impact in $-\frac{Q}{eK}$ (decrease the price)

$\rightarrow$ There is a range of volumes Q for which the strategy is profitable
$\rightarrow$ Is there an optimal Q^* for which the profit is optimal ?

Contribution 3 - Trading Strategies in Periodic Auction Markets

Assume the strategic participant who has information on the market i.e $\bar{p} \sim \mathcal{N}(0, \bar{\sigma}^2)$ For $\alpha \in [0, \frac{1}{2}]$, let :

$$\begin{cases} q_\alpha := \text{quantile}_\alpha(\mathcal{N}(0; \bar{\sigma}^2)) = \bar{\sigma} q_\alpha^{0,1} \\ q_{1-\alpha} := \text{quantile}_{1-\alpha}(\mathcal{N}(0; \bar{\sigma}^2)) = \bar{\sigma} q_\alpha^{0,1} \end{cases}$$

Assume trader participates in M ≥ 1 consecutive auctions and $\bar{p}_i$ auction price without informed traders

Contribution 3 - Trading Strategies in Periodic Auction Markets

If $\bar{p}_i \leq q_\alpha$ [Buy low → Sell High] :

- Trader submits a market order to buy Q volume before the end of the auction : $p_i := \overline{p_i} + \frac{Q}{eK}$
- Next auction, trader submit selling order Q before the end of auction : $p_{i+1} := \overline{p_{i+1}} - \frac{Q}{eK}$ (where $\overline{p_i}, \overline{p_{i+1}} \sim \mathcal{N}(0, \bar{\sigma}^2)$)

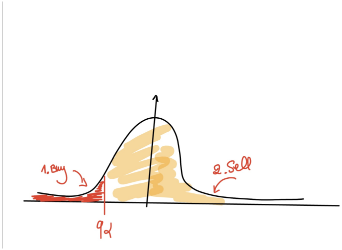


Figure – Clearing price distribution [Buy Low Sell High]

Contribution 3 - Trading Strategies in Periodic Auction Markets

If $\bar{p}_i \geq q_{1-\alpha}$ [Sell High $\rightarrow$ Buy Low] :

- Trader submits a market order to sell Q volume before the end of the auction : $p_i := \overline{p_i} - \frac{Q}{eK}$
- Next auction, trader submit buying order Q before the end of auction : $p_{i+1} := \overline{p_{i+1}} + \frac{Q}{eK}$
- If $\bar{p} \in (q_\alpha, q_{1-\alpha})$: the trader does not buy/sell $(p_i = \bar{p}_i \;\; \& \;\; p_{i+1} = \bar{p}_{i+1})$

Figure – Clearing price distribution [Sell High Buy Low]

Contribution 3 - Theoretical Profit

Consider two consecutive auctions with clearing prices $\overline{p}_1$ and $\overline{p}_2$. Let π_1 and π_2 be the respective profit after these auctions. We have :

$$\pi_1(Q) = \begin{cases} 0 & \text{if } \overline{p_1} \in (q_\alpha ; q_{1-\alpha}] \\ -Q\left(\overline{p_1} + \frac{Q}{eK}\right) & \text{if } \overline{p_1} \leq q_\alpha \\ Q\left(\overline{p_1} - \frac{Q}{eK}\right) & \text{if } \overline{p_1} > q_{1-\alpha} \end{cases} \tag{1}$$

and

$$\pi_2(Q) = \begin{cases} 0 & \text{if } \overline{p_1} \in (q_\alpha ; q_{1-\alpha}] \\ Q\left(\overline{p_2} - \frac{Q}{eK}\right) & \text{if } \overline{p_1} \leq q_\alpha \\ -Q\left(\overline{p_2} + \frac{Q}{eK}\right) & \text{if } \overline{p_1} > q_{1-\alpha} \end{cases} \tag{2}$$

By the strong law of large numbers, if $\pi_M(Q)$ is the average profit after M sequences of two consecutive auctions we get :

$$\frac{\pi_M(Q)}{M} \xrightarrow[M \to +\infty]{\text{a.s.}} \mathbf{E}\left[\pi_1(Q) + \pi_2(Q)\right]$$

Contribution 3 - Theoretical Profit

Contribution 3.2 - Let $\pi_M(Q)$ be the profit of the strategy after M auctions. We have :

$$\frac{\pi_M(Q)}{M} \underset{M\to+\infty}{\longrightarrow} \pi(Q) := 2\alpha Q\left[\frac{\overline{\sigma}}{\alpha}\phi\left(q_\alpha^{0,1}\right) - \frac{2Q}{eK}\right].$$

This profit is maximized for :

$$\boxed{Q^* = \frac{eK\overline{\sigma}}{4\sqrt{2\pi}\alpha^*}\exp\left(-\frac{\left(q_{\alpha^*}^{0,1}\right)^2}{2}\right)}$$

and leads to the optimal profit :

$$\boxed{\pi^*(Q^*) = \frac{eK\overline{\sigma}^2}{8\pi\alpha^*}\exp\left(-\frac{\left(q_{\alpha^*}^{0,1}\right)^2}{2}\right)}$$

where $\alpha^* \approx 0.27$.

Contribution 3 - Agent-based simulation

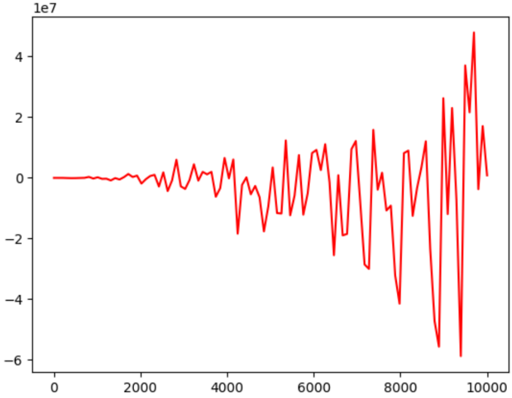


Figure – Trader profit - agent-based simulation, averaged over **1000 simulations** [x-axis : Q, y-axis : Empirical profit]

→ Huge variance when Q increases !

Contribution 3 - Theoretical Profit Variance

Contribution 3.3 - We have :

$$\boxed{\mathrm{Var}\left(\pi(Q)\right) \sim \frac{4Q^4}{e^2K^2}\left(1-4\alpha^2\right)}$$

So :

$$\boxed{\mathrm{Var}\left(\pi(Q)\right) = O\left(Q^4\right)}$$

Nash Equilibrium between Two Strategic Participants

Contribution 4 - Nash equilibrium among two strategic participants ?

Assumptions -

- $N \geq 1$ (non-strategic) market makers who submit their demand schedules
- 2 strategic market participants who use the trading strategy above, with respective quantiles $\alpha_1, \alpha_2 \in [0, \frac{1}{2}]$.

Cases -

- $\alpha_1 = \alpha_2 := \alpha$

$$\pi_i(Q_1, Q_2) = 2Q_i \left(\overline{\sigma}\phi\left(q^{0,1}_{\alpha_1}\right) - \frac{2}{eN}\left(Q_1 + Q_2\right)\right)$$

- $\alpha_1 \neq \alpha_2$; WLOG, assume that $0 \leq \alpha_1 < \alpha_2 \leq \frac{1}{2}$

$$\pi_1(Q_1, Q_2) = 2Q_1 \left(\overline{\sigma}\phi\left(q^{0,1}_{\alpha_1}\right) - \frac{2}{eN}\left(Q_1 + Q_2\right)\right)$$

$$\pi_2(Q_1, Q_2) = 2Q_2 \left(\overline{\sigma}\phi\left(q_{\alpha_2^{0,1}}\right) - \frac{2\left(\alpha_2 Q_2 + \alpha_1 Q_1\right)}{eN}\right)$$

Contribution 4 - Nash equilibrium among two strategic participants ?

Contribution 3.3 - There is a Nash equilibrium if and only if $\alpha_1 = \alpha_2 := \alpha$. It occurs when the two strategic traders use the trading strategy with volumes :

$$\boxed{Q_1^* = Q_2^* = \frac{eN\overline{\sigma}}{6\sqrt{2\pi\alpha}} \exp\left(-\frac{1}{2}\left(q_\alpha^{0,1}\right)^2\right)}$$

Comments -

- Generalized to $M \geq 2$ strategic market participants
- In practice, asymmetric information among strategic participants $\longrightarrow$ different estimates of α
- Market instability

Conclusion

Conclusion

- Our simulations suggest that current periodic auction durations used in practice are sub-optimal
- This is expected since the real market periodic auctions are very dependent on the parallel continuous trading market - an auction clearing price not landing within the bid-ask spread results in the auction being cancelled
- Several exchanges have expressed interest in switching completely to period auction markets (our study)
- It is essential to tune this auction duration to the correct value to ensure the efficiency of the market

APPENDIX

Contribution 1 ⏩ Th1

Let $i \in \{1, ..., K\}$. We have :

$$\begin{aligned}\operatorname{Var}\left(\overline{p} - m_{\frac{i}{\omega}}\right) &= \operatorname{Var}\left(\frac{1}{N}\sum_{k=1}^{N}\left(\varepsilon_k + m_{\frac{k}{\omega}}\right) - m_{\frac{i}{\omega}}\right) \\ &= \frac{\sigma^2}{N} + \operatorname{Var}\left(\frac{1}{N}\sum_{k=1}^{N}\left(m_{\frac{k}{\omega}} - m_{\frac{i}{\omega}}\right)\right)\end{aligned}$$

We notice that :

$$\begin{aligned}\operatorname{Var}\left(\frac{1}{N}\sum_{k=1}^{N}\left(m_{\frac{k}{\omega}} - m_{\frac{i}{\omega}}\right)\right) &= \operatorname{Var}\left(\sqrt{\frac{3}{2}}\frac{\psi}{N}\sum_{k=1}^{N}\left(B_{\frac{k}{\omega}} - B_{\frac{i}{\omega}}\right)\right) \\ &= \frac{3}{2}\frac{\psi^2}{N^2}\operatorname{Var}\left(\sum_{k=1}^{i-1} k\left(B_{\frac{k}{\omega}} - B_{\frac{k+1}{\omega}}\right) + \sum_{k=i+1}^{N}(N-k+1)\left(B_{\frac{k}{\omega}} - B_{\frac{k-1}{\omega}}\right)\right) \\ &= \frac{3}{2}\frac{N}{\omega}\frac{\psi^2}{N^3}\left(\sum_{k=1}^{i-1} k^2 + \sum_{k=1}^{N-i} k^2\right)\end{aligned}$$

Contribution 1 ⏩ Th1

After computing the sums, we get :

$$\mathrm{Var}\left(\overline{p} - m_{\frac{i}{\omega}}\right) = \frac{\sigma^2}{N} + \frac{N}{\omega}\frac{\psi^2}{2}\left[1 + \frac{3}{2N} + \frac{1}{2N^2} - \frac{3i}{N} - \frac{3i}{N^2} + \frac{3i^2}{N^2}\right]$$

So by definition :

$$\begin{aligned}
\mathbf{L}(\tau) &= \frac{1}{N}\sum_{i=1}^{N}\mathrm{Var}\left(\overline{p} - m_{\frac{i}{\omega}}\right) \\
&= \frac{\sigma^2}{N} + \frac{N}{\omega}\frac{\psi^2}{2}\left[1 + \frac{3}{2N} + \frac{1}{2N^2} - \frac{3}{2}\left(1 + \frac{1}{N}\right) - \frac{3(N+1)}{2N^2} + \frac{N(N+1)(2N+1)}{2N^3}\right] \\
&= \frac{\sigma^2}{N} + \frac{N}{\omega}\frac{\psi^2}{4}\left(1 - \frac{1}{N^2}\right) \\
&= \left(\sigma^2 - \frac{\psi^2}{4\omega}\right)\frac{1}{N} + \frac{\psi^2}{4\omega}N \\
&= \left(\sigma^2 - \frac{\psi^2}{4\omega}\right)\frac{1}{\lfloor\omega\tau\rfloor} + \frac{\psi^2}{4\omega}\lfloor\omega\tau\rfloor
\end{aligned}$$

Contribution 1 ▸ Th1

- If $\sigma^2 - \frac{\psi^2}{4\omega^2} < 0$, then $N \mapsto \mathbf{L}(N)$ is an increasing function of $N \in \mathbb{N}^*$ and it is minimized at $N^* = 1 = \lfloor \omega\tau^* \rfloor$ which implies that $\tau^* = \frac{1}{\omega}$.
- If $\sigma^2 - \frac{\psi^2}{4\omega^2} \geq 0$, then the function $x \in \mathbb{R}_+^* \mapsto \mathbb{R}$ is minimized at $x^* = \sqrt{\frac{4\sigma^2\omega}{\psi^2} - 1}$ so $N \in \mathbf{N}^* \mapsto \mathbf{L}(N)$ is minimized either at $\lfloor x^* \rfloor$ or $\lfloor x^* \rfloor + 1$. If the following inequality holds for $y = \frac{4\sigma^2\omega}{\psi^2} - 1$:

$$\frac{y}{\lfloor \sqrt{y} \rfloor} + \lfloor \sqrt{y} \rfloor < \frac{y}{\lfloor \sqrt{y} \rfloor + 1} + \lfloor \sqrt{y} \rfloor + 1,$$

we get that $\mathbf{L}$ is minimized at $N^* = \lfloor x^* \rfloor$. Since $N^* = \lfloor \omega\tau^* \rfloor$, by definition of the integer part : $\omega\tau^* - 1 < N^* \leq \omega\tau^*$, so :

$$\tau^* \in \left[\frac{1}{\omega} N^*, \frac{1}{\omega}(N^* + 1)\right)$$

Contribution 2 Th2

We have :

$$
\begin{aligned}
\mathrm{Var}\left(\overline{p} - m_\tau\right) &= \mathrm{Var}\left(\frac{1}{N}\sum_{k=1}^{N}\left(\varepsilon_k + m_{\frac{k}{\omega}}\right) - m_\tau\right) \\
&= \frac{\sigma^2}{N} + \mathrm{Var}\left(\frac{1}{N}\sum_{k=1}^{N}\left(m_{\frac{k}{\omega}} - m_\tau\right)\right) \\
&= \frac{\sigma^2}{N} + \frac{3\psi^2}{2N^2}\mathrm{Var}\left(\sum_{k=1}^{N-1} k\left(B_{\frac{k}{\omega}} - B_{\frac{k+1}{\omega}}\right) + N\left(B_{\frac{N}{\omega}} - m_\tau\right)\right) \\
&= \frac{\sigma^2}{N} + \frac{3\psi^2}{2N^2}\left(\frac{1}{\omega}\sum_{k=1}^{N-1} k^2 + N^2\left(\tau - \frac{N}{\omega}\right)\right) \\
&= \left(\sigma^2 + \frac{\psi^2}{4\omega}\right)\frac{1}{N} - \frac{\psi^2}{\omega}N + \frac{3\psi^2}{2}\left(\tau - \frac{1}{2\omega}\right) \\
&= \left(\sigma^2 + \frac{\psi^2}{4\omega}\right)\frac{1}{\lfloor \omega\tau \rfloor} - \frac{\psi^2}{\omega}\lfloor \omega\tau \rfloor + \frac{3\psi^2}{2}\left(\tau - \frac{1}{2\omega}\right)
\end{aligned}
$$

Contribution 2 Th2

We know that : $\lfloor y \rfloor = y + o(y)$ as $y \to +\infty$. Hence :

$$\mathbf{C}(\tau) \underset{\omega \to +\infty}{=} \left(\sigma^2 + \frac{\psi^2}{4\omega}\right)\frac{1}{\omega\tau} - \frac{\psi^2}{2}\tau - \frac{3\psi^2}{4\omega} + o_\omega(1)$$

So, **C** is asymptotically minimized at :

$$\tau^* \underset{\omega \to +\infty}{\sim} \frac{\sqrt{2}\sigma}{\psi\sqrt{\omega}}$$

Contribution 3.1

We have $\overline{p} = \frac{1}{N}\sum_{i=1}^{N} r_i$, where $r_i \sim \mathcal{N}\left(0, \sigma^2 + \frac{3}{2}\psi^2\frac{i}{\omega}\right)$, so $\overline{p}$ is normally distributed with mean 0 and variance :

$$\begin{aligned}
\mathrm{Var}\left(\overline{p}\right) &= \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N} g_i + \sqrt{\frac{3}{2}}\frac{\psi}{N}\sum_{i=1}^{N} B_{\frac{i}{\omega}}\right) \\
&= \frac{\sigma^2}{N} + \frac{3}{2}\psi^2 \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N} B_{\frac{i}{\omega}}\right) \\
&= \frac{\sigma^2}{N} + \frac{\psi^2}{4}\frac{N}{\omega}\left(2 + \frac{3}{N} + \frac{1}{N^2}\right) \\
&= \frac{\sigma^2}{N} + \frac{\psi^2}{4\omega}\left(2N + 3 + \frac{1}{N}\right) \\
&= \left(\sigma^2 + \frac{\psi^2}{4\omega}\right)\frac{1}{\lfloor \omega\tau \rfloor} + \frac{\psi^2}{4\omega}\left(2\lfloor \omega\tau \rfloor + 3\right)
\end{aligned}$$

Contribution 3.2 ⏩ Th3

Let $\mathbf{L}_n(\tau)$ be the liquidity risk knowing that $N_\tau = n \geq 1$. Then, by similar computations as above, we get :

$$\mathbf{L}_n(\tau) = \frac{\sigma^2}{n} + \frac{3}{2}\psi^2 \mathrm{Var}\left(\sum_{i=1}^{n}(B_{t_i} - B_{t_k})\right)$$

If $t_{(1)} \leq t_{(2)} \leq \ldots \leq t_{(n)}$, are the ordered statistics of $t_1, \ldots, t_n$, then :

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{n}\left(B_{t_i} - B_{t_k}\right)\right) &= \sum_{i=1}^{n}\sum_{k=1}^{i-1} k^2 \mathrm{Var}\left(B_{t_{(k)}} - B_{t_{(k+1)}}\right) + \sum_{i=1}^{n}\sum_{k=i}^{n}(n-k)^2 \mathrm{Var}\left(B_{t_{(k+1)}} - B_{t_{(k)}}\right) \\
&= \sum_{k=1}^{n-1}(n-k)k^2 \mathrm{Var}\left(B_{t_{(k)}} - B_{t_{(k+1)}}\right) + \sum_{k=1}^{n-1} k(n-k)^2 \mathrm{Var}\left(B_{t_{(k+1)}} - B_{t_{(k)}}\right) \\
&= n\sum_{k=1}^{n-1} k(n-k)\mathrm{Var}\left(B_{t_{(k+1)}} - B_{t_{(k)}}\right)
\end{aligned}$$

Contribution 3.2

We know that the distribution of the k-th ordered statistic is fiven by :

$$F_{t_{(k)}} = \sum_{i=k}^{n} \binom{n}{i} F_{\tau}(x)^{i} \left(1 - F_{\tau}(x)\right)^{n-i}$$

$$\begin{aligned}
\mathrm{Var}\left(B_{t_{(k+1)}} - B_{t_{(k)}}\right) &= \mathbf{E}\left[t_{(k+1)} - t_{(k)}\right] \\
&= \int_0^{\tau} x \frac{\mathrm{d}}{\mathrm{d}x}\left[F_{t_{(k+1)}} - F_{t_k}\right](x)\ \mathrm{d}x \\
&= \left[x\left(F_{t_{(k+1)}} - F_{t_{(k)}}\right)(x)\right]_0^{\tau} + \int_0^{\tau}\left[F_{t_{(k+1)}} - F_{t_{(k)}}\right](x)\mathrm{d}x \\
&= \int_0^{\tau} \binom{n}{k} F_{\tau}^{k}(1 - F_{\tau})^{n-k}.
\end{aligned}$$

Hence :

$$\begin{aligned}
\sum_{k=1}^{n} \mathrm{Var}\left(\sum_{i=1}^{n}(B_{t_i} - B_{t_k})\right) &= n\int_0^{\tau}\sum_{k=1}^{n-1} k(n-k)\binom{n}{k} F_{\tau}^{k}(1 - F_{\tau})^{n-k} \\
&= n^2(n-1)\int_0^{\tau} F_{\tau}(1 - F_{\tau}).
\end{aligned}$$

Contribution 3.2

Therefore, we get that :

$$\mathbf{L}_n(\tau) = \frac{\sigma^2}{n} + \psi^2 \left(\int_0^\tau F_\tau(1 - F_\tau) \right) \left(1 - \frac{1}{n} \right)$$

And the result follows by taking the expected value

$$\mathbf{L}(\tau) = \mathbf{E}\left[\mathbf{L}_{N_\tau}(\tau) \right]$$

where $N_\tau - 1 \sim \mathcal{P}(\omega\tau)$

Contribution 4.1

By the law of total expectation :

$$\begin{aligned}\mathbf{E}\left[\pi_1+\pi_2\right] &= \mathbf{E}\left[\pi_1+\pi_2 \,\middle|\, \overline{p_1} \le q_\alpha\right]\mathbf{P}\left(\overline{p_1} \le q_\alpha\right)\\ &+ \mathbf{E}\left[\pi_1+\pi_2 \,\middle|\, \overline{p_1} > q_{1-\alpha}\right]\mathbf{P}\left(\overline{p_1} > q_{1-\alpha}\right)\\ &+ \mathbf{E}\left[\pi_1+\pi_2 \,\middle|\, \overline{p_1} \in \left(q_\alpha; q_{1-\alpha}\right)\right]\mathbf{P}\left(\overline{p_1} \in \left(q_\alpha; q_{1-\alpha}\right]\right)\\ &= \alpha\left(\mathbf{E}\left[\pi_1+\pi_2 \,\middle|\, \overline{p_1} \le q_\alpha\right] + \mathbf{E}\left[\pi_1+\pi_2 \,\middle|\, \overline{p_1} > q_{1-\alpha}\right]\right)\end{aligned}$$

By the equations (1) and (2), we have :

$$\begin{aligned}\mathbf{E}\left[\pi_1+\pi_2 \,\middle|\, \overline{p_1} \le q_\alpha\right] &= \mathbf{E}\left[Q\left(\overline{p_2}-\overline{p_1}\right) \,\middle|\, \overline{p_1} \le q_\alpha\right] - \frac{2Q^2}{eN}\\ &= -Q\mathbf{E}\left[\overline{p_1} \,\middle|\, \overline{p_1} \le q_\alpha\right] - \frac{2Q^2}{eN}\end{aligned}$$

We know that $\overline{p_1} \sim \mathcal{N}\left(0, \overline{\sigma}^2\right)$ so :

$$\mathbf{E}\left[\overline{p_1} \,\middle|\, \overline{p_1} \le q_\alpha\right] = -\overline{\sigma}\frac{\phi\left(q_\alpha^{0,1}\right)}{\alpha}$$

where ϕ is the cdf of a standard gaussian distribution.

Contribution 4.2

So :

$$\mathbf{E}\left[\pi_1 + \pi_2 \,\middle|\, \overline{p_1} \leq q_\alpha\right] Q\left(\frac{\overline{\sigma}}{\alpha}\phi\left(q_\alpha^{0,1}\right) - \frac{2Q}{eN}\right)$$

and we get the same expression for $\mathbf{E}\left[\pi_1 + \pi_2 \,\middle|\, \overline{p_1} \in (q_\alpha; q_{1-\alpha})\right]$, i.e.

$$\frac{\pi_M(Q)}{M} \xrightarrow[M\to+\infty]{\text{a.s.}} \pi(Q) := 2\alpha Q\left[\frac{\overline{\sigma}}{\alpha}\phi\left(q_\alpha^{0,1}\right) - \frac{2Q}{eN}\right]$$

We note that $\pi(Q)$ is a quadratic polynomial in Q and it is maximized for :

$$Q^* = \frac{eN\overline{\sigma}}{4\sqrt{2\pi}\alpha}\exp\left(-\frac{1}{2}\left(q_\alpha^{0,1}\right)^2\right) \text{ and } \pi^*(Q^*) = \frac{eN\overline{\sigma}^2}{8\pi\alpha}\exp\left(-\frac{1}{2}\left(q_\alpha^{0,1}\right)^2\right)$$

The profit is maximized for α^* that maximizes $\alpha \in [0, \frac{1}{2}] \mapsto \frac{1}{\alpha}\exp\left(-\frac{1}{2}\left(q_\alpha^{0,1}\right)^2\right)$, i.e. for $\alpha \approx 0.27$.

Contribution 4.3

We do not detail the proof here, but the result is obtained by decomposing the profit as in the proof of Contribution 3.2 and using :

$$\mathrm{Var}\left(\pi(Q)\right) = \mathbf{E}\left[\pi^2(Q)\right] - \mathbf{E}\left[\pi(Q)\right]^2 .$$

Contribution 4.4

Assume that $\alpha_1 = \alpha_2 := \alpha$. Then the profit of the trader 1 after two consecutive auctions, respectively denoted by $\pi_1^{(1)}$ and $\pi_2^{(1)}$ is :

$$\pi_1^{(1)}(Q) = \begin{cases} 0 & \text{if } \overline{p_1} \in (q_\alpha; q_{1-\alpha}] \\ -Q_1\left(\overline{p_1} + \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} \leq q_\alpha \\ Q_1\left(\overline{p_1} - \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} > q_{1-\alpha} \end{cases}$$

and

$$\pi_2^{(1)}(Q) = \begin{cases} 0 & \text{if } \overline{p_1} \in (q_\alpha; q_{1-\alpha}] \\ Q_1\left(\overline{p_2} - \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} \leq q_\alpha \\ -Q_1\left(\overline{p_1} + \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} > q_{1-\alpha} \end{cases}$$

So the average profit of a sequence of 2 auctions for the trader I after a long time is :

$$\pi_1(Q_1, Q_2) = 2Q_1\left(\overline{\sigma}\phi\left(q_\alpha^{0,1}\right) - \frac{2}{eN}(Q_1 + Q_2)\right)$$

Contribution 4.5

Similar computations gives us the following profit for the second trader :

$$\pi_2(Q_1, Q_2) = 2Q_2 \left(\overline{\sigma}\phi\left(q_\alpha^{0,1}\right) - \frac{2}{eN}\left(Q_1 + Q_2\right) \right)$$

There is a Nash equilibrium among the two strategic participants iff there exists $Q_1^*, Q_2^* \geq 0$ such that :

$$\frac{\partial \pi_1}{\partial Q_1}\left(Q_1^*, Q_2^*\right) = \frac{\partial \pi_2}{\partial Q_2}\left(Q_1^*, Q_2^*\right) = 0.$$

Solving these equations gives us :

$$Q_1^* = Q_2^* = \frac{eN\overline{\sigma}}{6\sqrt{2\pi}\alpha} \exp\left(-\frac{1}{2}\left(q_\alpha^{0,1}\right)^2 \right).$$

Contribution 4.5

Now assume that $\alpha_1 \neq \alpha_2$. Without loss of generality, let's assume that $0 \leq \alpha_1 < \alpha_2 \leq \frac{1}{2}$. Then :

$$\pi_1^{(1)}(Q) = \begin{cases} 0 & \text{if } \overline{p_1} \in (q_{\alpha_1}; q_{1-\alpha_1}] \\ -Q_1\left(\overline{p_1} + \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} \leq q_{\alpha_1} \\ Q_1\left(\overline{p_1} - \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} > q_{1-\alpha_1} \end{cases}$$

and

$$\pi_2^{(1)}(Q) = \begin{cases} 0 & \text{if } \overline{p_1} \in (q_{\alpha_1}; q_{1-\alpha_1}] \\ Q_1\left(\overline{p_2} - \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} \leq q_{\alpha_1} \\ -Q_1\left(\overline{p_2} + \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} > q_{1-\alpha_1} \end{cases}$$

So :

$$\pi_1(Q_1, Q_2) = 2Q_1\left(\overline{\sigma}\phi\left(q_{\alpha_1}^{0,1}\right) - \frac{2}{eN}(Q_1 + Q_2)\right)$$

Contribution 4.5

$$\pi_1^{(1)}(Q) = \begin{cases} 0 & \text{if } \overline{p_1} \in (q_{\alpha_2}; q_{1-\alpha_2}] \\ -Q_2\left(\overline{p_1} + \frac{Q_2}{eN}\right) & \text{if } \overline{p_1} \in (q_{\alpha_1}, q_{\alpha_2}] \\ -Q_2\left(\overline{p_1} + \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} \le q_{\alpha_1} \\ Q_2\left(\overline{p_1} - \frac{Q_2}{eN}\right) & \text{if } \overline{p_1} \in (q_{1-\alpha_2}, q_{1-\alpha_1}) \\ Q_2\left(\overline{p_1} - \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} \ge q_{1-\alpha_1} \end{cases}$$

$$\pi_2^{(1)}(Q) = \begin{cases} 0 & \text{if } \overline{p_1} \in (q_{\alpha_2}; q_{1-\alpha_2}] \\ Q_2\left(\overline{p_2} - \frac{Q_2}{eN}\right) & \text{if } \overline{p_1} \in (q_{\alpha_1}, q_{\alpha_2}] \\ Q_2\left(\overline{p_2} - \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} \le q_{\alpha_1} \\ -Q_2\left(\overline{p_2} + \frac{Q_2}{eN}\right) & \text{if } \overline{p_1} \in (q_{1-\alpha_2}, q_{1-\alpha_1}) \\ -Q_2\left(\overline{p_2} + \frac{Q_1+Q_2}{eN}\right) & \text{if } \overline{p_1} \ge q_{1-\alpha_1} \end{cases}$$

So :

$$\pi_2(Q_1, Q_2) = 2Q_2\left(\overline{\sigma}\phi\left(q_{\alpha_2^{0,1}}\right) - \frac{2(\alpha_2 Q_2 + \alpha_1 Q_1)}{eN}\right)$$

Contribution 4.5

There is a Nash equilibrium among the two strategic participants iff there exists $Q_1^*, Q_2^* \geq 0$ such that :

$$\frac{\partial \pi_1}{\partial Q_1}\left(Q_1^*, Q_2^*\right) = \frac{\partial \pi_2}{\partial Q_2}\left(Q_1^*, Q_2^*\right) = 0.$$

Solving these equations gives us :

$$\begin{cases} Q_1^* = eN\overline{\sigma}\dfrac{\frac{1}{2}\phi\left(q_{\alpha_2}^{0,1}\right) - \alpha_2\phi\left(q_{\alpha_1}^{0,1}\right)}{\alpha_1 - 4\alpha_2} < 0 \\ Q_2^* = eN\overline{\sigma}\left(\frac{1}{2}\phi\left(q_{\alpha_2}^{0,1}\right) - \frac{\phi\left(q_{\alpha_2}^{0,1}\right) - 2\alpha_2\phi\left(q_{\alpha_1}^{0,1}\right)}{\alpha_1 - 4\alpha_2}\right) > 0 \end{cases}$$

So there is no Nash equilibrium among the two strategic participants if $\alpha_1 \neq \alpha_2$.

References

[1] Fricke, D., & Gerig, A. (2018). *Too fast or too slow ? Determining the optimal speed of financial markets.*

[2] Jusselin, P., Mastrolia, T., & Rosenbaum, M. (2020). *Optimal auction duration : A price formation viewpoint.*