Comprehensive Analogy of Existing Methods Correlating Point and Line Objects in the Plane and Direction, Angle, Distance and Incidence Constraints between them

Vriddhi Pai

Abstract

Combinatorial characterization of constraint systems when more than one type of constraints is involved is to date a challenging problem in geometry. For common constraints like distances, there exists a combinatorial characterization such as the combinatorial characterization of 2-D distance constraint systems given by Laman and for directions constraints, a combinatorial characterization for Parallel Drawings. However, even for these existing solutions, the characterization conditions often cannot be extended for characterization in higher dimensions or d-dimensions. For instance, Laman's conditions for combinatorial characterization of rigidity in a framework has not been proven to work the same in d- dimensions. Recent developments in geometry have combined geometric constraints in combinations of two constraints such as length-direction, length-angle and even three constraint combinations of length-direction-angle systems. The usual approach, as seen in a lot of proposed work thus far, in systems involving more than two constraints has been to combine two of the constraints as a single constraint and making the third constraint the main constraint connecting the points in framework. One such example discussed that involves three constraints in its proposed design considers Euclidean distance between the points and the directions between them as single combined direction-length constraint connected via angle constraints. This paper surveys such proposed systems at length in order to compare and contrast the various approaches of such frameworks to highlight the key technical challenges and open questions thereby suggesting scope of future developments, both in terms of extending current work to other, novel frameworks or refining the existing constraint systems to facilitate extendibility to higher dimensions.

Keywords: *Combinatorial characterization, Constraint systems, Parallel drawings, Euclidean*

1. Introduction

Graphs with geometrical constraints are natural models for a multitude of applications, including Computer-Aided Design, Flexibility in molecules and even Sensor networks [5]. Given a graph and prescribed lengths for its edges, determining whether the graph has a straight-line visualization in Euclidean d-dimensional space with the given lengths is one of the basic problems in geometry. Such a visualization is referred to as a framework. When a framework is constructed, there are many questions surrounding it that need to be answered about its design including its uniqueness, both with respect to local translation and rotation and globally. One important question about the design is the rigidity. One such conjecture by Euler has been given according to which every 3-dimensional polyhedron, when visualized as a panel-and-hinge framework is rigid. Laman has famously provided the combinatorial characterization for rigidity in 2-dimensional realization of a generic framework. Jackson and Jordan further provided conditions for combinatorial characterization for global rigidity of graphs in 2 dimensions. Whiteley extended on the existing work and provided conditions for combinatorial characterization for purely direction constraint systems in d-dimensions and proposed a direction-length constraint system's characterization in

2-dimensions. Characterization of angle constraint system emerged in more recent work where angle constraints are considered in pure angle constraint systems to answer important questions about general angle rigidity in 2-dimensions and higher.

The proposed work in multiple papers reviewed and discussed in this paper can be categorized under developments under three major categories of ‘models’ that summarize the overall patterns of constraint combinations and essentially give a perspective on the context of the developments: Point-line Incidence and Line Direction, Point-line Distance and Line-Line Angle Model and Point-Line Incidence and Line-Angle Model.

Point-line Incidence and Line Direction revolves around two constraints: lengths and directions. Distance rigidity is defined via distance preservation of translational and rotational motions of a framework. In common knowledge, to determine whether a framework is distance rigid, two methods have been in use: First, testing the rank of the distance rigidity matrix which is derived from infinitesimal rigidity concept of distance motions. Second method is essentially a combinatorial verification given by Laman’s theorem and works only for generic frameworks.
The developments in direction constraint systems followed distance systems where conditions to check for direction rigidity are similar to those used for distance rigidity.

Point-line Distance and Line-Line Angle Model explores the distance and direction constraints as a unified constraint in an instance of a point-line framework while angles are the third constraints explored as part of the same framework. Angle rigidity is explored by extending the idea of a rigidity matrix used for direction and length frameworks to develop an angle rigidity matrix based on infinitesimally rigid angle motions in the framework.

Since the developments under angle constraint systems are still limited, Point-Line-Incidence and Line-Angle Model delves deeper into pure angle constraint systems to develop conditions on characterization of specific instances of general angle constraint systems and then works extending the characterization conditions to general angle constraint systems.

The paper is organized to give a logical overview of the topic intended to be explored across 2 major sections hereafter: Section 2, the Literature Review gives a detail insight into all papers that fall under the three categories of models as described above in the same order of relevance as listed. The literature review covers approaches, motivation/inspiration for those approaches, theoretical background or proofs supporting the applicability and authenticity of those approaches, methodologies utilized and a comparison of how the theme of a paper(s) compares to/is relevant to/is an extension of/is more effective than the development described in another paper(s) that worked with similar resources/ on similar problem. Section 3 summarizes the overall observations from the literature and discusses it to highlight the open challenges and obstacles evident in individual results and provide scope to extend the critical analysis to give way to future developments that could benefit the topic.

2. Literature Review

In [1], authors Servatius and Whiteley address the then tentatively explored and visualized idea of a plane configuration such that its realization, called a plane design is a homogeneous system capable of handling individualities of more than one constraint involved amongst the geometric objects that make up the design. This was needed to be addressed in order to answer important open questions of global uniqueness and local uniqueness of the design and other questions about appropriateness of a specific combination of constraints suited for a design to achieve the said uniqueness. With the help of original existing theories of infinitesimal rigidity of plane frameworks- configuration of points in the pane constrained only by lengths and parallel drawings-where constraints involved are directions alone, a new design is hypothesized combining directions and lengths as constraints within a single system giving a plane design. The novel system treats the length constraints as ordinary edges between the points and directions as the edges with two arrowheads in the interior. The main focus is to prove the new design 'robust'- where small changes in terms of translations or changes in the design parameters yield a design with identical "stiffness" to the original design. The authors used the theoretical concept of limiting designs as their main tool in the proof. Limiting designs express the direction-length system as a constraint matrix showing identical row dependencies as the original matrix of the design after a series of translations and parameter changes in the system. The robust design is able to achieve constraint independence and local uniqueness, however, the achieving global uniqueness is still an open problem. The paper further addresses the broader unsolved problem of a design with lengths between the pairs of points and angles as additional constraints between the lines. The problem here is the absence of a polynomial-time algorithm that gives a combinatorial characterization of the system of these constraints. The paper solves a special case of this problem where they propose that if the angles are linked together as a connected set among the attached edges in the realization, the proposed direction-length design could be extended to angle design. This is purely conjectured and the proof for the same not stated.

In [5], the authors Jackson and Keevash extend the design of direction-length frameworks from [1] to focus on the unsolved global uniqueness and rigidity of the system. The direction-length system in [1] expressed as a constraint graph is redefined in this paper as mixed graph which essentially labels the edges as 'length' or 'direction'. A mixed graph essentially partitions its edge set to subsets of edges- labelled L for holding edge lengths and D for holding direction edges of the system. The main focus of this paper was to obtain a combinatorial characterization of boundedness of the direction-length framework generalized to a d-dimensional system. The authors then use this so obtain conditions for characterizing global rigidity of generic direction-length frameworks. The paper uses Whiteley's [6] characterization of global rigidity for purely-direction constrained system guaranteeing global rigidness of the design in d-dimension in attempt to extend it to give an identical combinatorial characterization for direction-length design. The robustness property of the framework as per [1] is utilized to claim that there is an absolute bound for the dimeter of any framework that satisfies the identical direction-length constraints of the original, untranslated, unaltered framework. Boundedness is simply the rigidity condition of the augmented framework determinable via computing the rank of the corresponding rigidity matrix of the system. The rank of the rigidity matrix thereby is the main tool in giving the d-dimensional combinatorial characterization of a generic direction-length framework. The authors also use existing quadratic-time algorithms, another open problem in [1] and utilize them in detection of generic rigidity for direction-length frameworks. The authors address that unless the framework is generic, the rigidity and infinitesimal rigidity of the framework may not be equivalent and prove it by introducing an augmented graph out of the mixed graph by utilizing the relationship between infinitesimal motions and equivalent realizations of the augmented graph. This becomes an intermediate result of a mixed graph which guarantees rigidity if and only if infinitesimal rigidity is achieved by the realization of the augmented graph subtype of a mixed direction-length graph. To give a characterization of boundedness and hence rigidity of the direction-length frameworks, the paper introduces an instance or special case of such frameworks, called the ball-direction frameworks. The tool to give a boundness characterization is the known method of computing rank of its corresponding matrix. As result of the boundedness test, the paper concludes that a direction-length framework's boundedness can be calculated by calculating the rank of its rigidity matrix which can be extended to the generic boundedness of a framework on the same lines. Further, addressing the problem of proving global rigidity for frameworks, this paper extends Hendrickson's [7] redundant rigidity conditions on an instance of a purely directional graph with no length constraints. Considering two disjoint realizations of the graph instance in d-dimensions and proves them direction-equivalent and direction-congruent (through translation or dilation) and concludes that any such framework whose realizations show equivalence and congruence is globally direction rigid. Finally, the paper again extended Whiteley's [1] result to characterize graphs which are globally direction rigid in d-dimension. However, Hendrickson's redundant rigid realization conditions are not applicable to direction-length frameworks since the proof depends on a compactness argument which is invalid if the graph is unbounded and direction adds arbitrary dilations in the framework thus disallowing boundedness to hold.

Chapter 5 of the dissertation in [8] takes a step back from existing work on directions and lengths as common constraints in plane frameworks to explore pure angle constraints in a framework. It takes on a known open problem of non-existing combinatorial characterization for frameworks involving angular constraints and attempts to give a combinatorial characterization of a specific subset of cases of angle constraint systems. The proof provided in this paper applies to a specific set of instances of angle constraint systems and has not been shown to be extendible to general angle constraint systems and thus remains a locally verifiable solution. For the specific cases, the paper uses the idea of gradual construction which says that beginning with a pair of points, a point with at most two angles connecting it with the constructed system can be added to the realization at each step until all such points are incorporated. Such a system is then considered gradually constructible. The paper further theorizes that a gradually constructible system is usually generically independent but not always based on an instance where three angles, instead of two were involved in the angle constraint system which then continued to show generic independence even though it was not gradually constructible. It is then further concluded that additional dependencies like implicit angle cycle apart from gradual construction help decide the generic independence of the angle constraint system. Since implicit angle cycle is not the only dependency guaranteeing generic independence in an angle constraint system, the paper concludes with a proposed novel rigidity conjecture applicable to general angle constraint systems and justifies it with extensions of observations from characterizing rigidity for specific examples of angle constraint systems.

Mixing more than two constraints within a homogeneous geometric system and being able to characterize important properties such as rigidity and global uniqueness in visualizing such a system was still a conjectured concept in Whiteley's paper where the authors managed to come up with a unified framework involving lines and directions as constraints between points within a single system. Incorporating angles as one of the constraints was only proposed to be solved locally for a class of cases, with the solution not applicable to a general visualization of a system with lines, directions and angles. In [8], the dissertation chapter took to taking the characterization of systems with purely angles as constraints locally. [9] takes on the challenges posed by both works by working on the point-line framework in 2 dimensions. This framework is a collection of points and lines in the plane linked by pairwise constraints that fix angles between the pairs and also some point-line and point-point distances. The point-line graph has vertices reimagines as a combination of points and lines and the edge is composed of angles as main constraints connecting the vertices. The paper guarantees rigidity in the point-line framework which is obtainable from the point-line graph by assigning coordinates to the points and lines. The conditions for rigidity utilized in the proofs of this paper borrow context from the 2-dimensional rigidity characterization conditions given by Laman's Theorem for bar-joint frameworks. [10]. The point-line graph is closely related to the bar-joint framework in the rigidity context such that every generic realization of the graph as a 2-dimensional bar-joint framework always guarantees rigidity. The paper takes the instance of 2-dimensional rigidity matroid to characterize independence for concluding rigidity properties. The characterization utilizes two count matroids generated as instances of the point-line graph. The maximum independent set is obtained by the union of the two matroids using the existing Edmonds's algorithm [11]. The augmentation of the two count matroids, seen as independent sets to give the final maximal set union has been further implemented using existing approaches given by Berg and Jordan. [12]. Even though the base concepts and rigidity conditions given by Laman's Theorem were the concepts that led to necessary proof developments in the paper and the results essentially generalize the proof given by Laman (for the case where the framework has no line-vertices), the proof presented in the paper doesn't employ recursive construction formulated in Laman's proof [10] where the edge set maintains independence in the generic bar-joint rigidity matroid. The alternative proof presented for the point-line framework in the paper follows Whiteley's [1] technique of direct construction of the framework maintain the property that the edge set is independent in the generic rigidity matroid whenever it is independent in the edge set of the point-line graph.

The authors Chen, Cao and Li in [13] redefined the term angularity in context of a “multi-point” framework employing angle constraints. Angularity, just as in the dissertation produced in [8] discussed earlier, is meant to describe a purely angular system or a point-angular system (where points are nodes) where the set of nodes are embedded in a Euclidean space while the constraints connecting the nodes are angles. Angularity describes the rigidity property of the framework where under proper angle constraints, the angularity can only translate, rotate or scale as a whole when more than one of its nodes or vertices are perturbed. The angularity conditions developed in this paper, compared to the conjectures and proofs presented in [8], do not bear the extendibility to a global uniqueness in the system and only guarantee local uniqueness. The authors have attempted to segue from the way distance rigidity and bearing/direction rigidity, both have been described in similar works in the past. ‘Distance rigidity’ is generally expressed in terms of distance constraints that are quadratic with respect to the end nodes that connect them. The properties that define the distance rigidity involve distance preservation of rotational and translational motions of a given multi-point framework. Direction rigidity or ‘bearing’ rigidity, wherein the structure of the framework is governed by the inter-point directions has its direction constraint always linearly associated to the end points’ positions, while the description of the directions depends directly on the necessity of a general global coordinate system, this guaranteeing global uniqueness and global rigidity in the system. Angle rigidity theory accommodates angle constraints either linearly or quadratically with respect to end points’ positions without the knowledge of a global coordinate system, thus making it not necessarily global rigid. The authors express this via a theoretic proof in context of an instance of a framework involving a purely angular constraint system. This is again similar to the approach in the dissertation presented earlier in this paper where through theoretic proofs, the author established inextendibility of a purely angular constrained system guaranteeing global rigidity. Furthermore, angularity describes a combinatorial structure relating pairs of vertices as its main tool in defining rigidity which is different from a multi-point framework involving a distance where edges of the graph essentially determine the distance constraint between two points corresponding to the two vertices adjacent to the edges in the graph. Since global angle rigidity is merely questioned in the context of this paper’s scope, the authors have conjectured the global rigidity by establishing some necessary and sufficient conditions. The tool/method used to establish this is the existing type-I vertex addition performed in a sequence from a generically angle rigid 3-vertex angularity. However, these conditions are mere propositions without a theoretically bolstered proof. The paper also describes the infinitesimal angle rigidity with respect to the kernel of a well-defined angular rigidity matrix which is analogous to the infinitesimal property establishment in distance rigidity in a system. In analogizing their own concepts of angle rigidity and infinitesimal angular rigidity, the authors underscore how they are completely disjoint properties in context of an angular constraint system. Any such system can have both properties or one of them based on the model itself. The establishment of general angle rigidity and combinatorial conditions for minimal and infinitesimal angle rigidity are still unaddressed in the paper and remain areas for future developments.

3. Conclusion

3.1 Summary of Challenges and Open Problems

The various papers discussed at length in the previous section of this paper primarily deal with distances, directions and angles as constraints in geometric systems, either as standalone constraints in a system or in combinations of two and even all three within a proposed homogeneous system. The main goal of this paper was to explore the developments around the existing challenge of mixing a combination of disjoint geometric constraints in a single system which necessarily bound the objects in the system yet tend to cause inconsistencies when considered for characterization of the system on distinguishing properties like rigidity, robustness, uniqueness etc., both locally and globally. The first paper discussed in the previous section of this paper was by Servatius and Whiteley who discussed the properties of existing designs using purely lengths and purely directions in different systems and proposed a novel homogeneous system that unifies both lengths and directions as constraints. They were able to achieve desired independence in the design of the proposed system locally but could not justify the same conditions in the scope of extendibility to global independence and uniqueness. They also addressed how mixing angles as a third constraint can be challenging and attempted to propose a system combining the three; this solution was theoretically justified to work for a special case of general angle-length design. However, a polynomial time algorithm or a direct combinatorial algorithm for such a system is still left as an unresolved problem. This is one of the few papers that attempted to hypothesize a system involving more than two constraints in combination. Another paper that attempted mixing multiple constraints together was the work by Jackson and Owen on point-line frameworks. Their solution for a mixed constraint system was very similar to the one by Whiteley except instead of direction and lengths combined as one constraint with angles as the other, they made a combination with points and lines as one constraint while pairwise constraints between vertices of the underlying graph were angles; the other constraint. This solution had a polynomial time characterization which was left unresolved in Whiteley's approach. The characterization conditions proposed by the authors in this paper stood out since they did not directly extend from Laman's characterization conditions for rigidity unlike a lot of other approaches. Another paper that worked on bounded direction-length frameworks attempted to address the open problem about combinatorial characterization of d-dimensional frameworks in both the previous papers. They obtained the conditions for global rigidity characterization in the context of direction-length frameworks. Their theory on boundedness of the 2-dimensional generic, global rigid direction-length framework with was not however proven for correctness and thus remains a conjecture, not addressed in subsequent papers thus far. Two of the papers addressed the least explored geometric constraint among all discussed this far- angles, however, in context of pure angular systems. One of them being the dissertation under point-line incidence and line-line angle addressed the absence of combinatorial characterization of pure angular constraint systems but was able to characterize a subset of the generic angle constraint system in context of the specific instances discussed in the dissertation. The non-extendibility of the solution to general angle constraint system and no developments thus far on the conjecture provided in the paper for making this extendibility possible continues to keep this as an open problem. The other paper on angle rigidity to stabilize planar transformations worked on a purely angular constraint system but in context of a multipoint framework. The approach presented considered chose signed angles over usually explored scalar angles in most works which set it apart from other works on angular rigidity. Moreover, the authors separated angle rigidity in experimental setup and application from the works of Laman. However, the conditions presented fulfilled angle rigidity locally and did not extend to global angular rigidity. The two important open problems in this paper include proving global angle rigidity by extending conditions presented in the paper or an altered version and proposing combinatorial characterization for minimal and infinitesimal rigidity in context of angle rigidity.

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