Single-View Geometry

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Image Analysis and Computer Vision

Book: HZ, chapter 2,3

Outline

- Trasformations in $\mathbb{P}^2$: Homographies
- $\boxed{\text{DLT}}$ for estimating Homographies
- Examples of Affine Rectification and Stitching

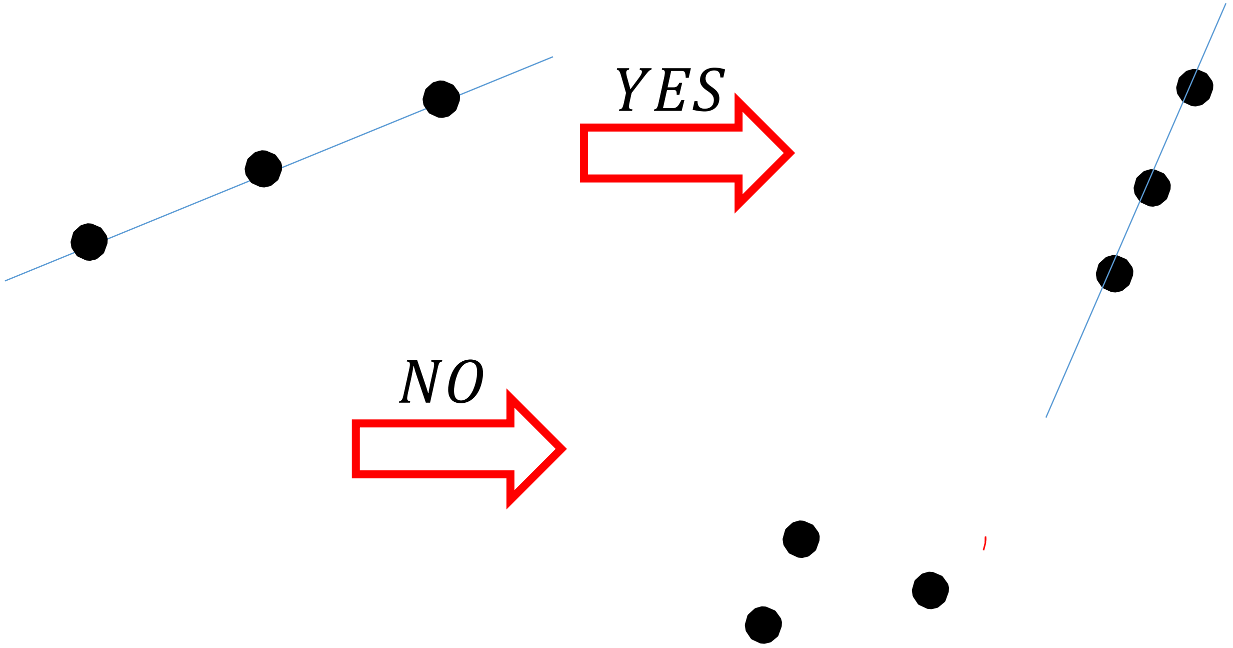
Transformations in $\mathbb{P}^2$

Homographies

The most general transformation we consider in $\mathbb{P}^2$

Definition A *projectivity* is an invertible mapping h from $\mathbb{P}^2$ to itself such that three points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ lie on the same line if and only if $h(\boldsymbol{x}_1)$, $h(\boldsymbol{x}_2)$ and $h(\boldsymbol{x}_3)$ do.

Rmk this property is called **collinearity**



Alternative names

- *Collineation*
- *Projective transformation*
- *Homography*

Homographies

Theorem 2.10. *A mapping* $h: \mathbb{P}^2 \to \mathbb{P}^2$ *is a projectivity if and only if there exists a non-singular* 3×3 *matrix* H *such that for any point in* $\mathbb{P}^2$ *represented by a vector* $\boldsymbol{x}$ *it is true that* $h(\boldsymbol{x}) = H\boldsymbol{x}$

$$H = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{bmatrix}$$

Each and every linear mapping in $\mathbb{P}^2$ is an homography, and only linear mapping are homographies

Homographies and Points in $\mathbb{P}^2$

From the theorem follows

(x', x) corresponding

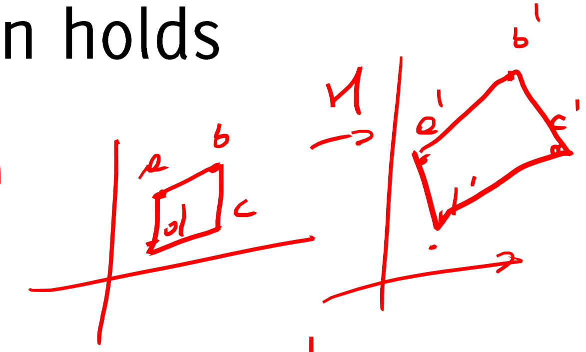
$$\boxed{\boldsymbol{x}' = H\boldsymbol{x}}$$

$x' = Hx$ $\quad h(x)$

Rmk: if we scale both $\boldsymbol{x}'$ and $\boldsymbol{x}$ by arbitrary factors the relation holds since we are in $\mathbb{P}^2$

$$\boxed{\boldsymbol{x}' = \lambda H\boldsymbol{x},} \qquad \underline{\forall \lambda \in \mathbb{R}\backslash\{0\}}$$

$x' = Hx' \; \forall \lambda$ $\quad (x', x)$



$a \to a'$ $\quad a' = Ha$

$a' = \lambda Ha$

$\lambda \neq 0$

Thus, H has 9 entries but only 8 degrees of freedom, since only the ratio between the elements counts. H is said to be an homogeneous matrix.

Homographies and Lines in $\mathbb{P}^2$

An homography transform **each line $\boldsymbol{l}$** in a line m such that:

$$m = (H^{-1})^{\intercal}\ l$$

$x \longrightarrow Hx$

$\ell \longrightarrow (H^{-1})^{T}\ell$

We say that points transform *contravariantly* and lines and conics transform *covariantly*.

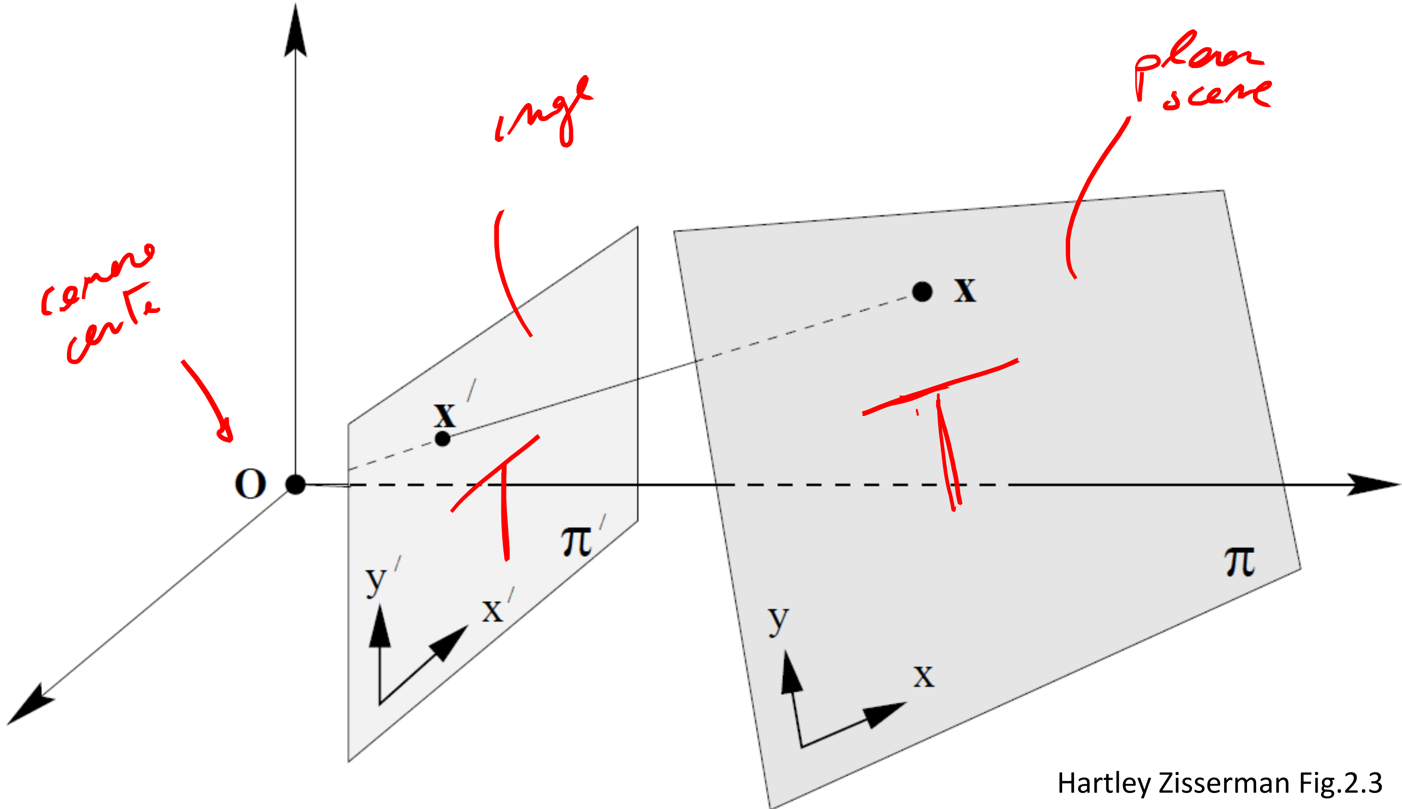
Mapping between planes: homography

Mapping between planes induced by a central projection is an homography, as this preserves collinearity

If a coordinate system is defined in each plane and points are represented in homogeneous coordinates, then the *central projection* mapping may be expressed by

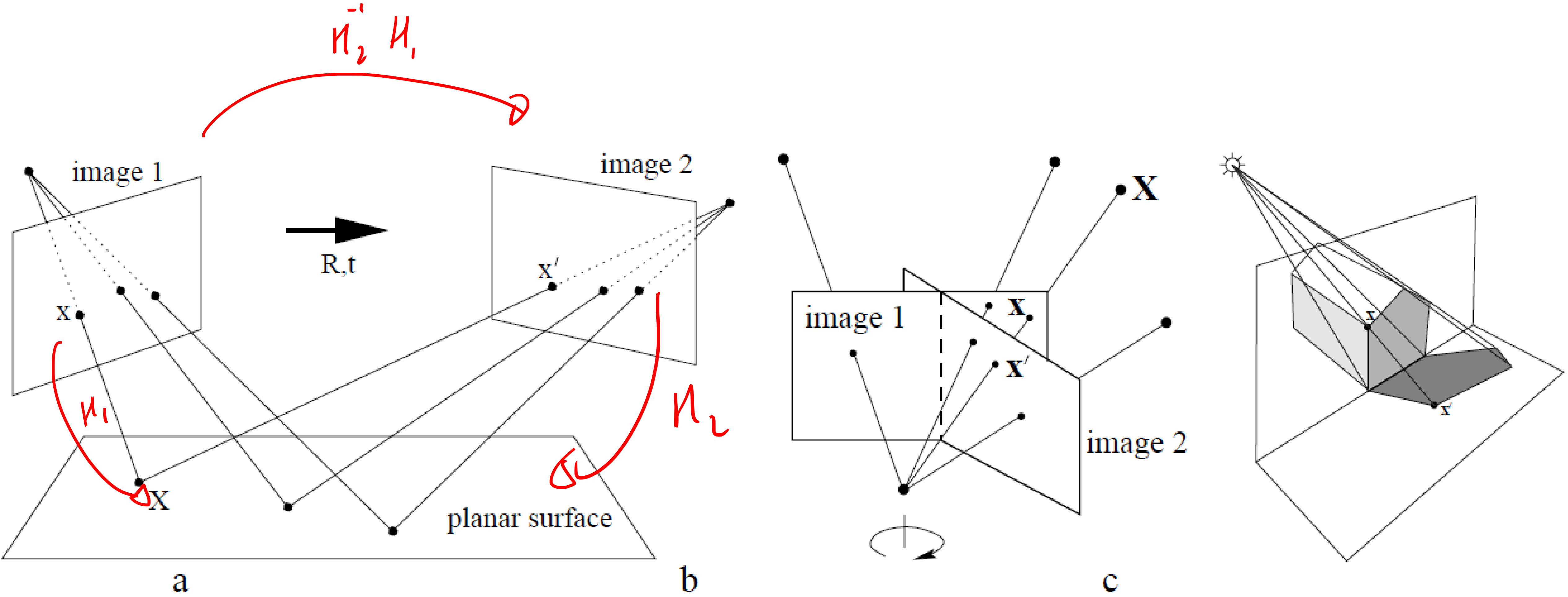
$$\boldsymbol{x}' = H\boldsymbol{x}$$

where H is a non-singular 3×3 matrix



Hartley Zisserman Fig.2.3

Other examples where homographies apply



Hartley Zisserman Fig.2.5

A Hierarchy of Transformations

Linear Transformation

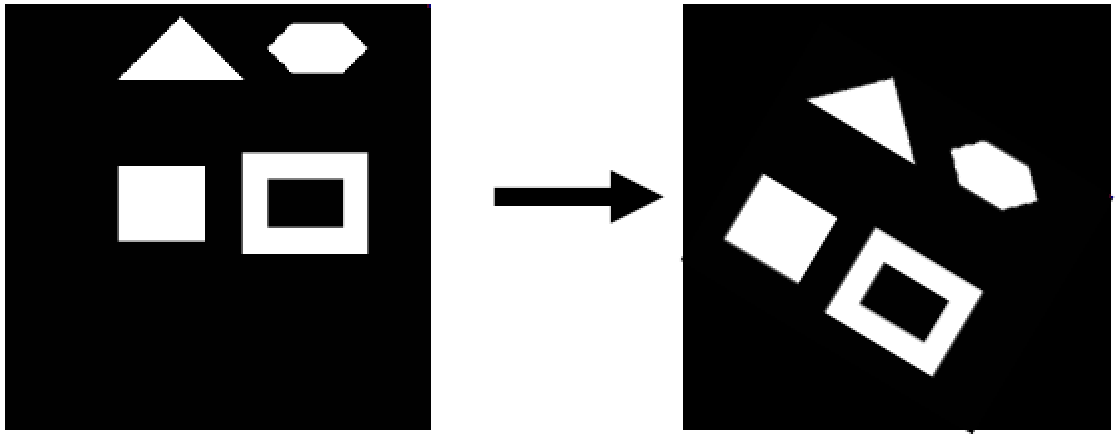
We consider linear transformations of $\mathbb{P}^2$,

- these can be expressed as a 3×3 matrix H,
- the homogeneous constraint applies

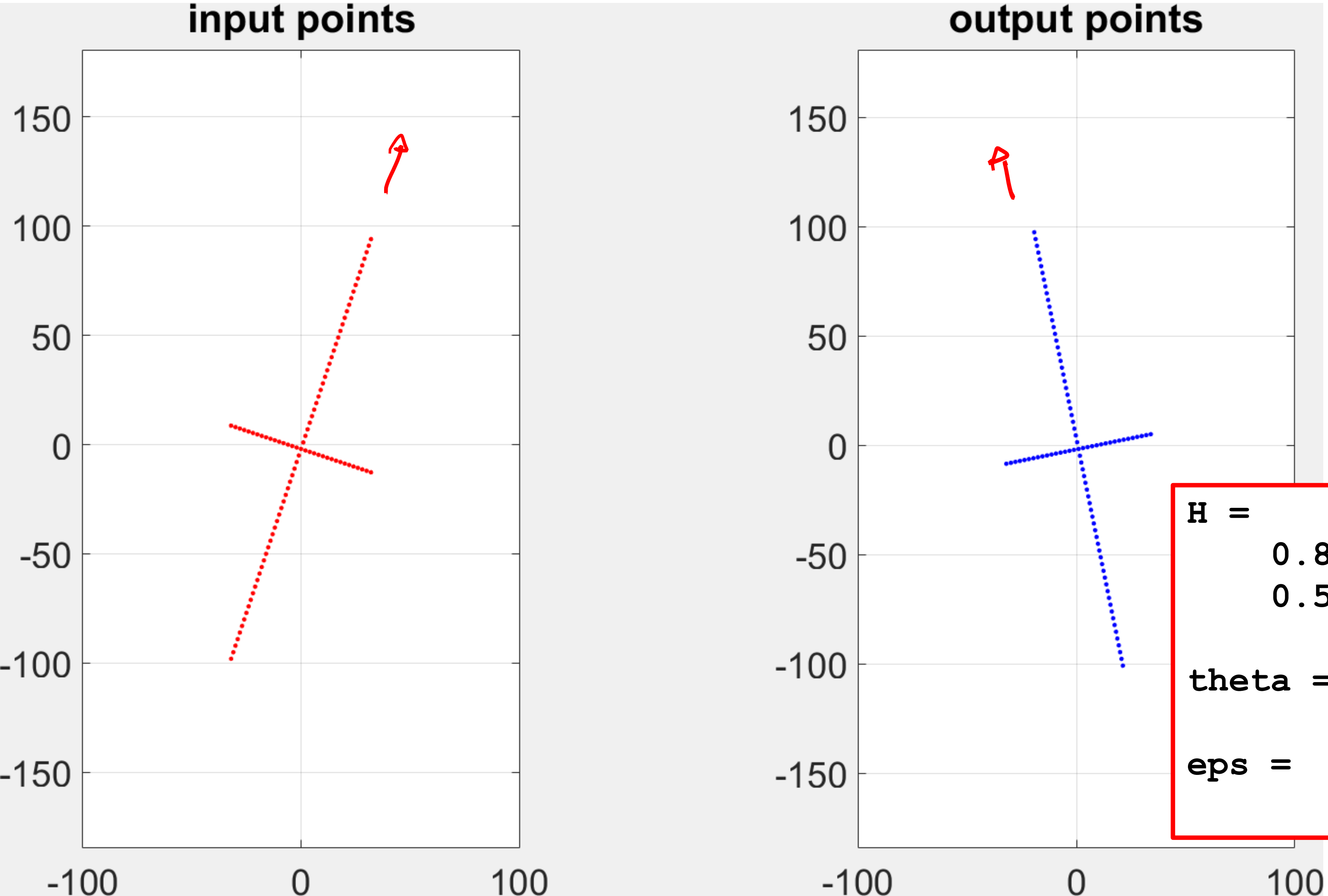
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Depending on the structure of H there are different class of transformations with a different number of degrees of freedom

Isometries



Isometries




```
H =
    0.8660   -0.5000         0
    0.5000    0.8660         0
         0         0    1.0000
theta =
    30
eps =
    1
```

Isometries

Isometries can be written as

$$\mathbf{x}' = H_I \mathbf{x}$$

where

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and $\epsilon = \pm 1$.

H 9 entries, 8 d.o.f.

R

t 3

Isometries

Isometries can be written as

$$\boldsymbol{x}' = H_I \boldsymbol{x}$$

A more compact representation is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where $\mathrm{R} \in O(2)$ is a rotation matrix, i.e. an orthogonal matrix

$$\mathrm{R}^\top\mathrm{R} = \mathrm{R}\mathrm{R}^\top = I_2$$

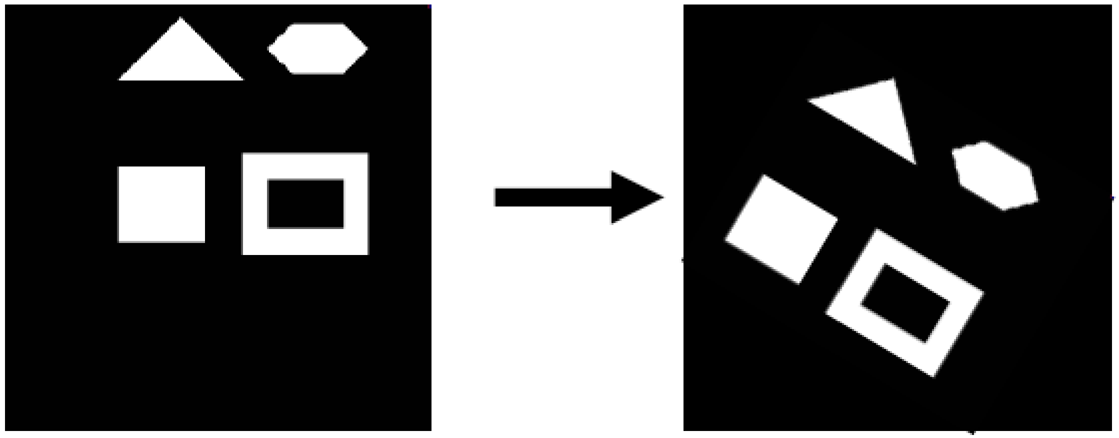
And $\mathbf{t} \in \mathbb{R}^{\mathbf{2}}$ is a translation vector .

Pure rotation $\mathbf{t} = [0; 0]$, pure translation $\mathrm{R} = I_2$

Isometries: Invariants

Isometries describe a 2D motion of a rigid object

Isometries preserves the angles, the distances, the areas.



Isometries: Remarks

Isometries have three degrees of freedom: the rotation angle θ and the translation vector $[t_x, t_y]$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rmk Isometries can be estimated from two point correspondences

Rmk When $\epsilon = 1$ the isometry preserves the orientation, when $\epsilon = -1$ the isometry reverses the orientation.

For instance $\mathrm{diag}(-1,1,1)$ is a reflection

Isometries: Remarks

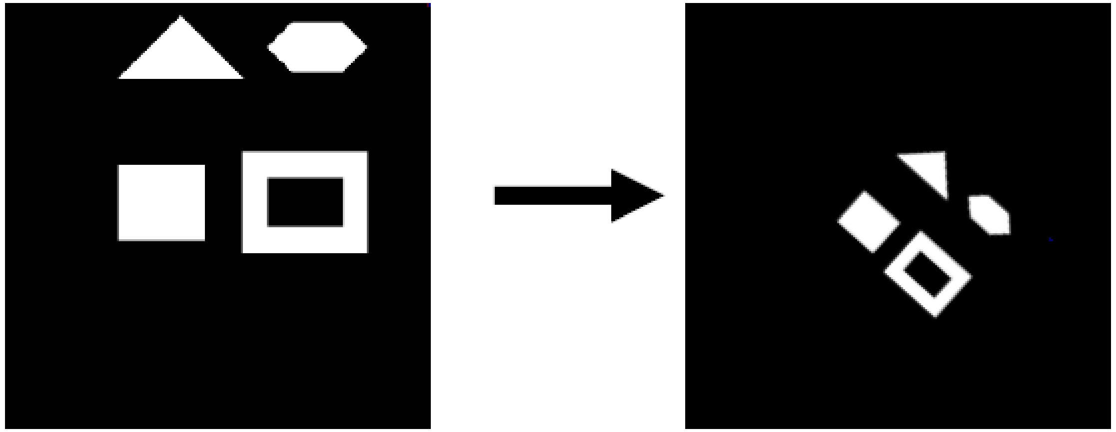
Isometries have three degrees of freedom: the rotation angle θ and the translation vector $[t_x, t_y]$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \epsilon \cos\theta & -\sin\theta & t_x \\ \epsilon \sin\theta & \cos\theta & t_y \\ \boxed{0 \quad\quad 0} & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

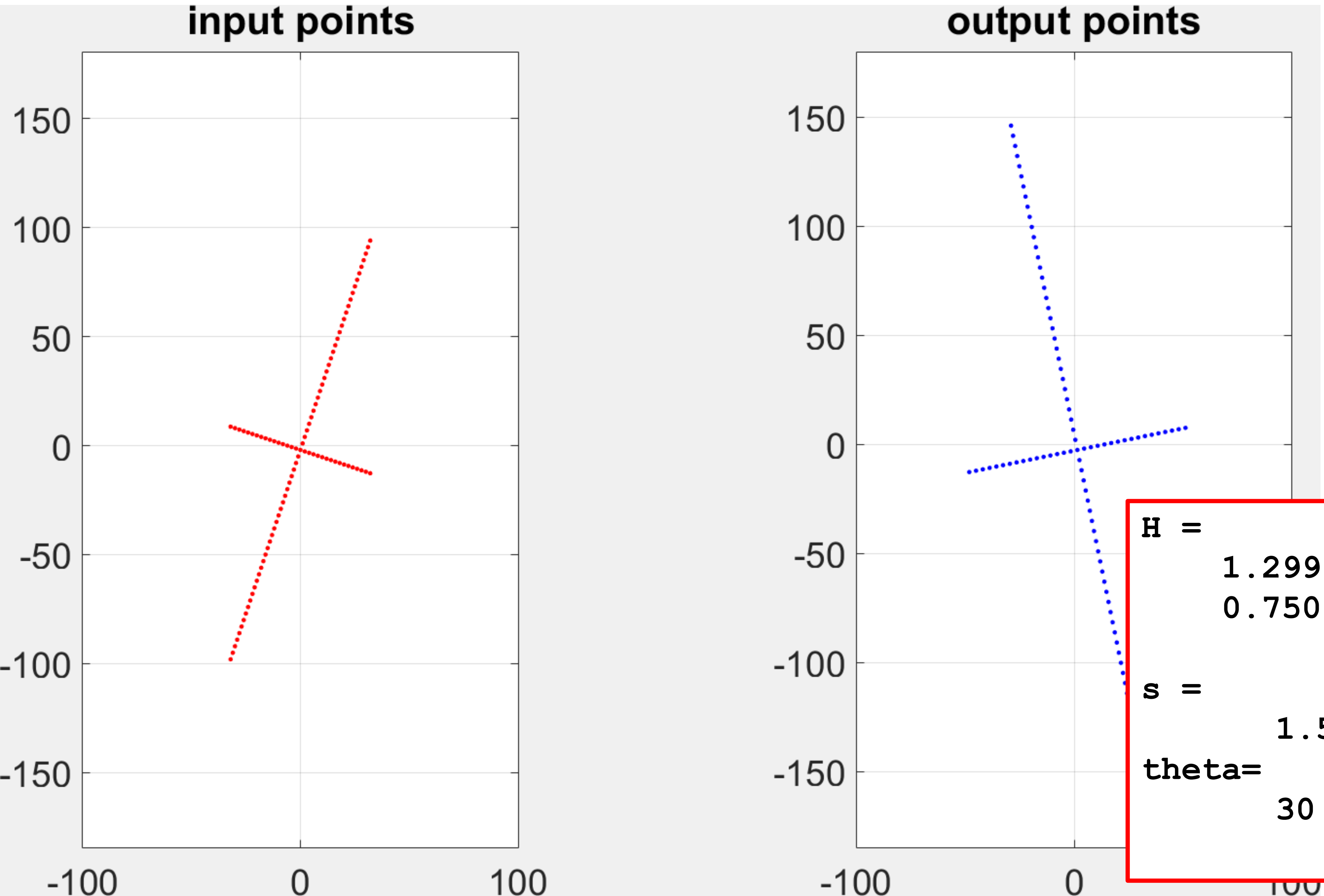
Rmk Isometries do not modify ideal points:

- Ideal points in $\mathbb{P}^2$ remain ideal points (they might change but always remain ideal points in ℓ_∞)
- Finite points in $\mathbb{P}^2$ remain finite points
- $\ell_\infty = H_I^{-\top} \ell_\infty$

Similarities



Similarities




```
H =
     1.2990   -0.7500         0
     0.7500    1.2990         0
          0         0    1.0000
s =
        1.5
theta=
        30
```

Similarities

isotropic scaling

$$\begin{bmatrix} s & & 0 \\ & s & \\ 0 & & 1 \end{bmatrix} H_I$$

Similarities can be written as

$$\boldsymbol{x}' = H_S \boldsymbol{x}$$

where

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and $s \in \mathbb{R}$

Rmk Similarities can be seen as an isometry composed with an isotropic scaling of the axis

Similarities

Similarities can be written as

$$\boldsymbol{x}' = H_S \boldsymbol{x}$$

A more compact representation is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\mathrm{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where

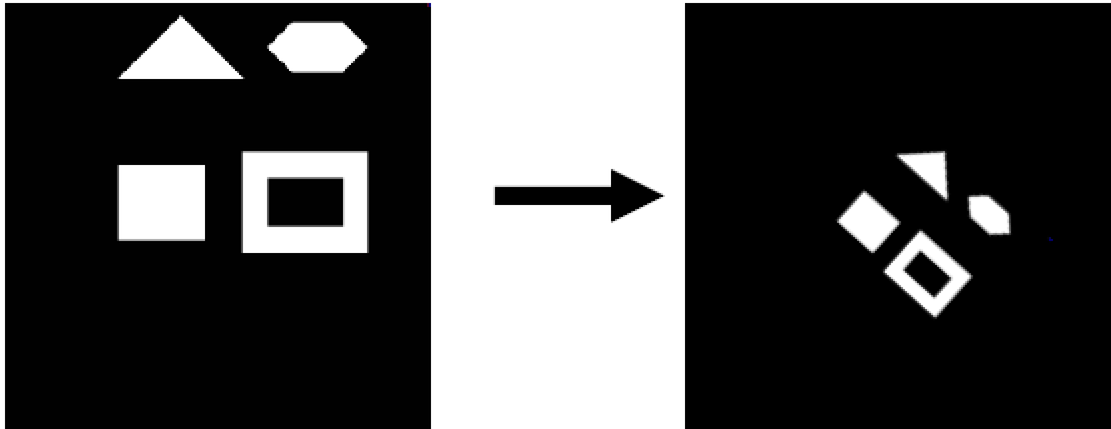
- $s \in \mathbb{R}$ is the scaling factor
- $\mathrm{R} \in O(2)$ is a rotation matrix,
- $\mathbf{t} \in \mathbb{R}^{\mathbf{2}}$ is a translation vector.

Similarities: Invariants

Similarities are also known as *equi-form* transformations

Similarities preserves the angles and the shapes, while not the length and the areas.

The ratio between lengths and areas is preserved since the scaling factor cancels out



Similarities : Properties

Similarities have **four degrees of freedom**: the scaling factor s, the rotation angle θ and the translation vector $[t_x, t_y]$

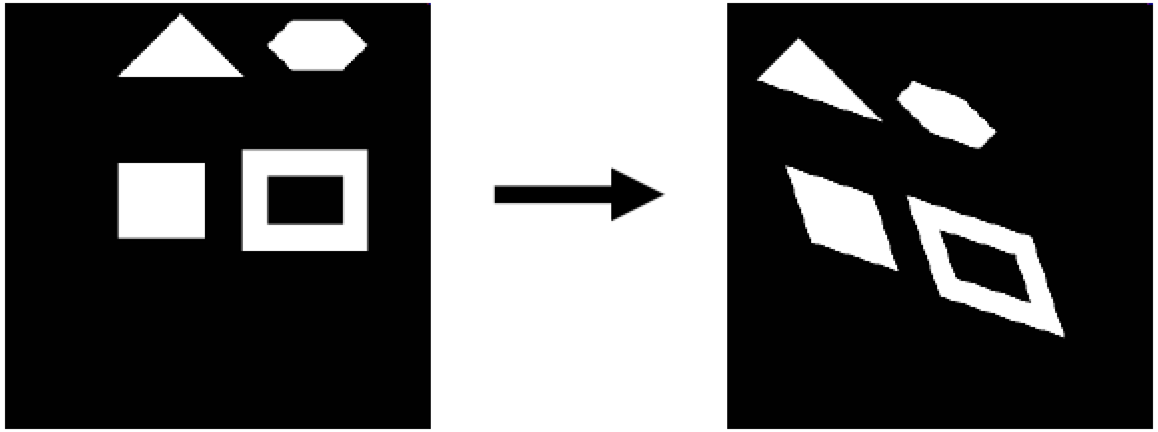
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ \boxed{0 \quad\quad 0} & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rmk $\ell_\infty = H_S^{-\top}\ell_\infty$, thus finite points remain finite points

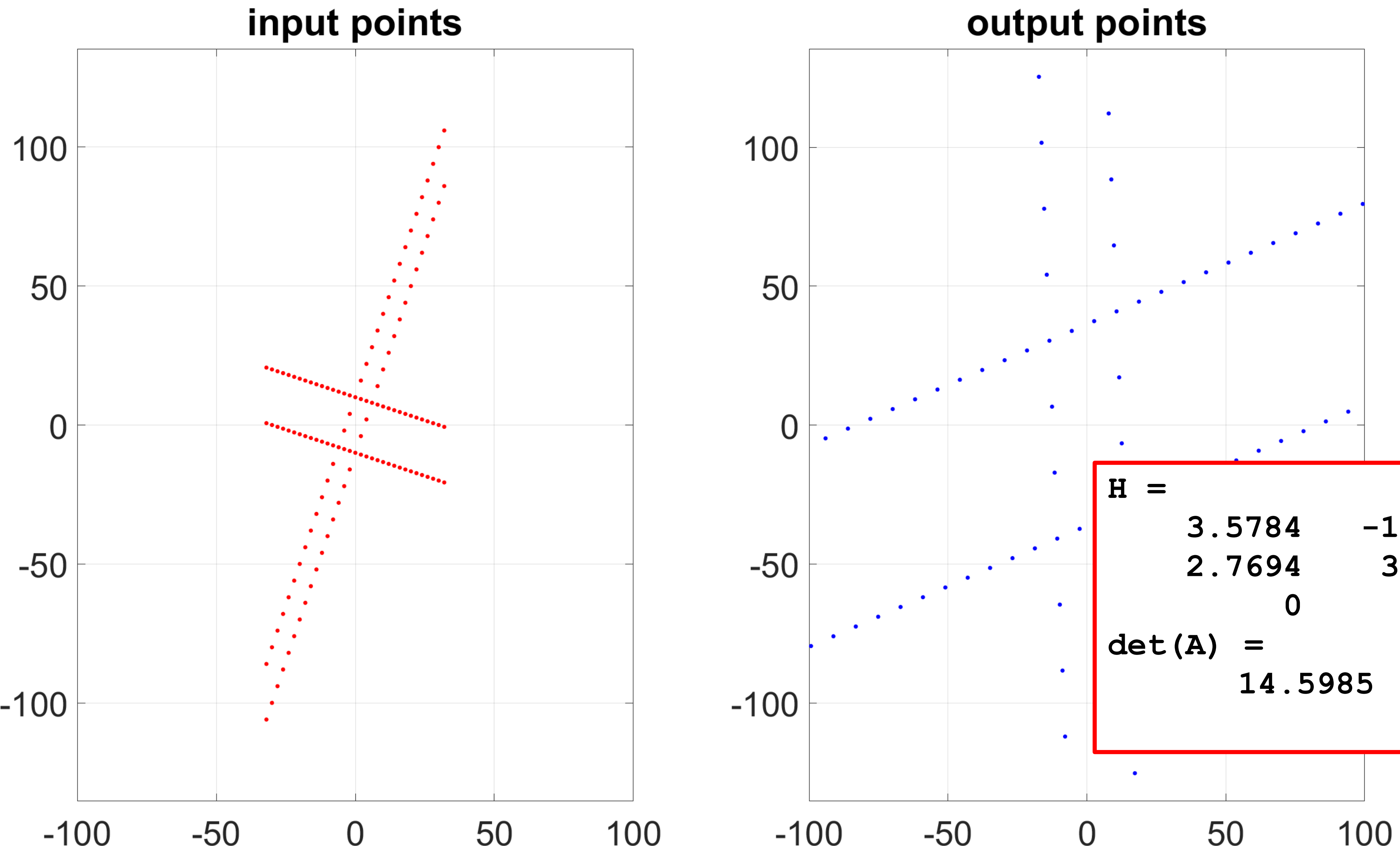
Rmk Similarities can be estimated from two point correspondences

Metric structure means that the structure is preserved up to a scaling

Affinities



Affine Transformation




```
H =
    3.5784   -1.3499         0
    2.7694    3.0349         0
         0         0    1.0000
det(A) =
   14.5985
```

Affinities

Affinities can be written as

$$\boldsymbol{x}' = H_A \boldsymbol{x}$$

where

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & t_x \\ a_{2,1} & a_{2,2} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ is an invertible matrix

Affinities

Affinities can be written as

$$\boldsymbol{x}' = H_A \boldsymbol{x}$$

A more compact representation is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where

- $A \in \mathbb{R}^{2,2}$ is an invertible matrix
- $\mathbf{t} \in \mathbb{R}^{\mathbf{2}}$ is a translation vector.

Rmk Affinities can be seen as a linear, non singular transformation of followed by a translation

Decomposition of Affinities

The linear part of the affine transformation **can be seen as the composition of two fundamental transformations:**

- rotation
- and non-isotropic scaling.

The affine matrix A can always be decomposed

$$A = R(\theta)\, R(-\varphi) \underline{D} R(\varphi)$$

where $R(\theta)$ and $R(\varphi)$ are rotations by θ and φ respectively, and D is

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Anisotropic scaling $\lambda_1 \neq \lambda_2$ in principle

Decomposition of Affinities

Id V is orthogonal

This decomposition can be obtained via $SVD(A)$

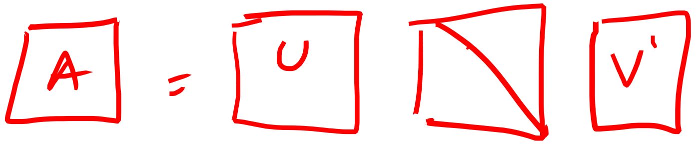
$$A = UDV^\top, \text{being } UU^\top = VV^\top = I_2$$

$$A = U(VV^\top)DV^\top =$$

$$A = (UV^\top)(V^\top DV^\top)$$

$$A = R(\theta)\,R(-\varphi)DR(\varphi)$$

The only “new” geometry, compared to a similarity, is the **non-isotropic scaling.**



Non-isotropic Scaling in Affinities

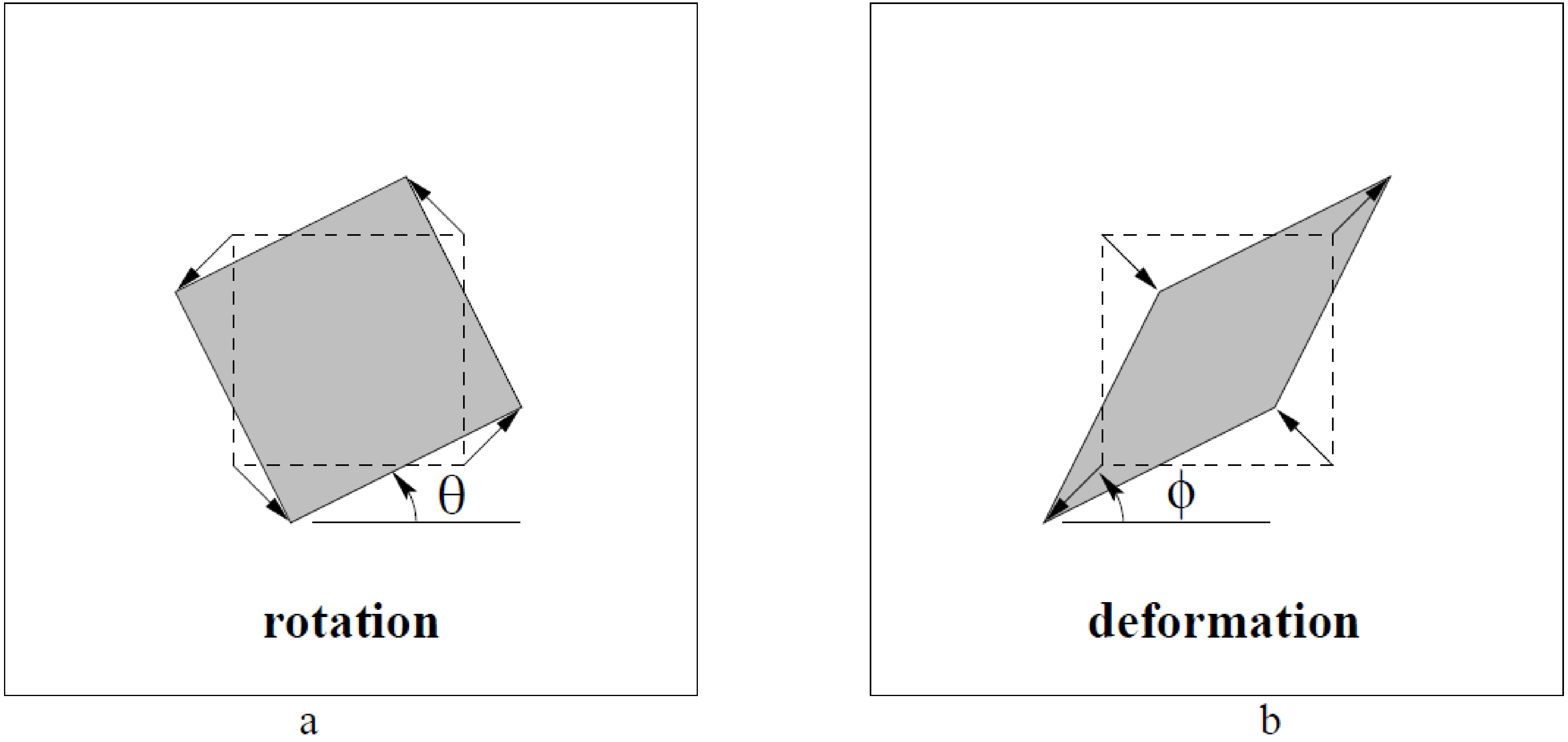


Fig. 2.7. **Distortions arising from a planar affine transformation.** *(a) Rotation by* $\mathtt{R}(\theta)$. *(b) A deformation* $\mathtt{R}(-\phi)\,\mathtt{D}\,\mathtt{R}(\phi)$. *Note, the scaling directions in the deformation are orthogonal.*

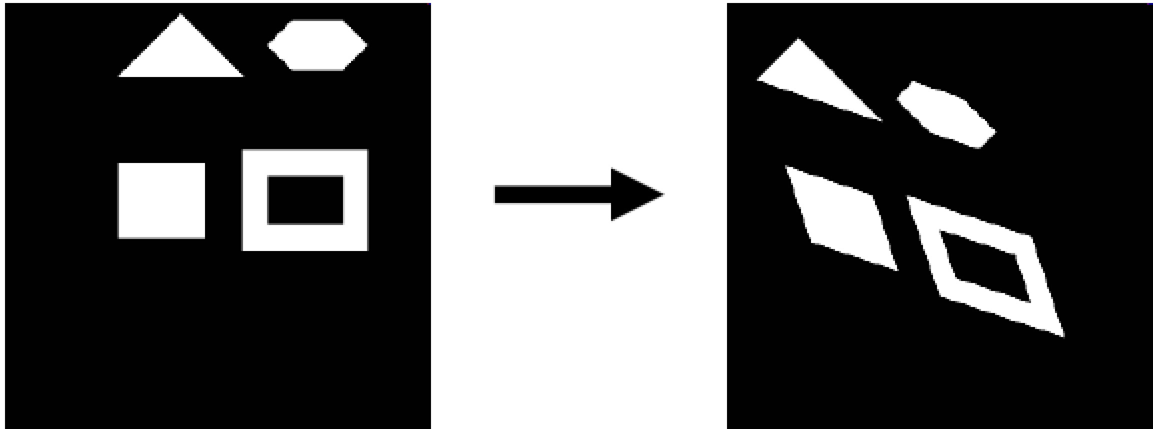
Hartley Zisserman Fig.2.7

Affinities: Invariants

Due to the non-isotropic scaling, the similarity invariants of length ratios and angles between lines are not preserved under an affinity.

Affinities preserves

- Parallel lines, since $\ell_\infty = H_A^{-\top}\ell_\infty$
- Ratio of lengths over parallel lines
- Ratio of areas (since each area is scaled of $\det(A)$, which cancels out)



Affinities: Properties

Similarities have **six degrees of freedom**: the four terms of A and the translation vector $[t_x, t_y]$

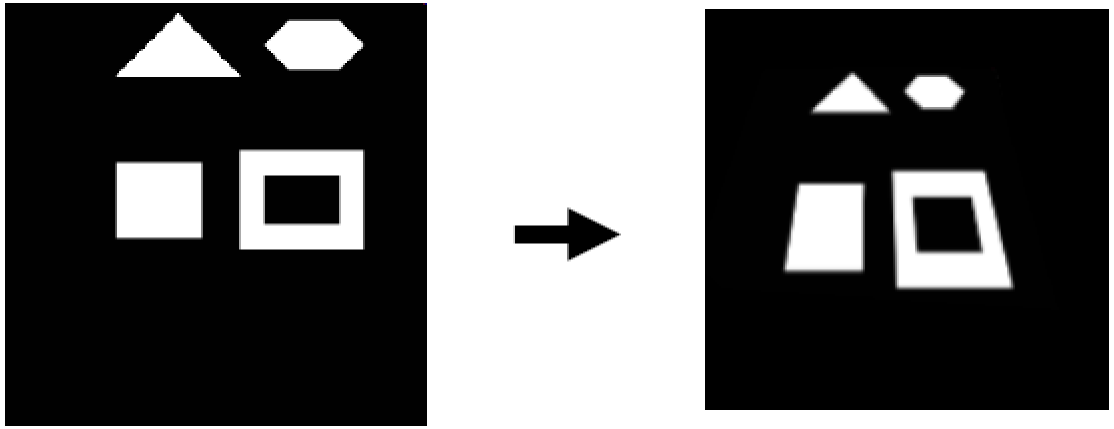
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & t_x \\ a_{2,1} & a_{2,2} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$H_A \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ 0 \end{bmatrix} = \begin{bmatrix} - \\ - \\ 0 \end{bmatrix}$$

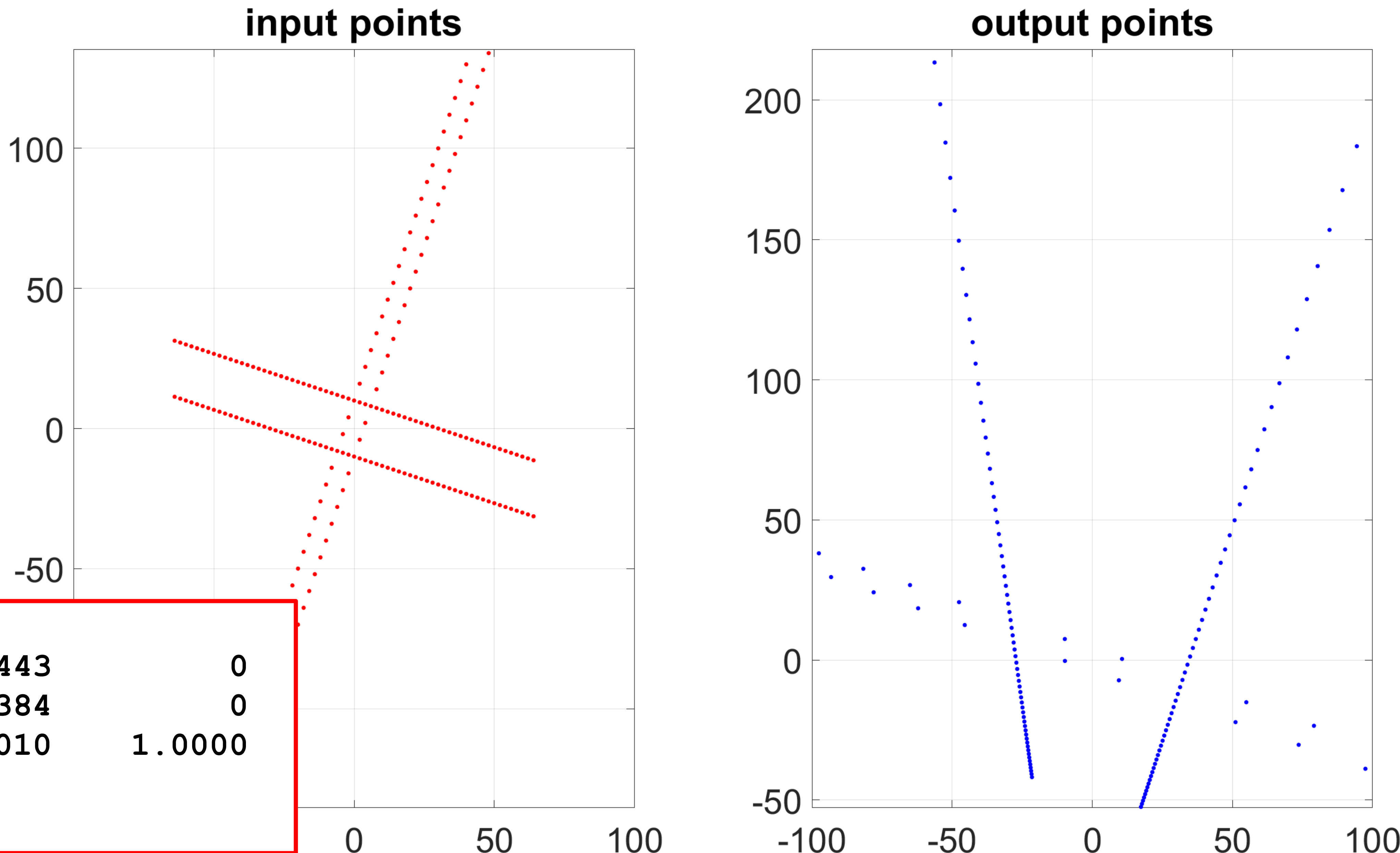
Rmk Similarities can be estimated from four point correspondences

Rmk $\ell_\infty = H_A^{-\top} \ell_\infty$, thus finite points remain finite points

Homographies



Projective Transformation




```
H =
   -1.0689   -2.9443         0
   -0.8095    1.4384         0
    0.0100    0.0010    1.0000
det(A) =
      -3.9209
```

Homographies

Homographies are the most general transformation *in homogeneous coordinates,* and can be written as

$$\boldsymbol{x}' = H_P \boldsymbol{x}$$

namely

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{t} \\ \boxed{\boldsymbol{v}^\top} & v \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= H_P \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ 0 \end{bmatrix} = \begin{bmatrix} - \\ - \\ v_1 \tilde{x} + v_2 \tilde{y} \end{bmatrix}$$

$\cancel{A}$

ℓ_∞

where $\boldsymbol{v} = [v_1; v_2]$ and $v \in \mathbb{R}$ (possibly zero)

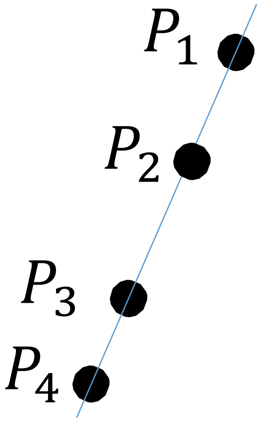
Despite this transformation applies in $\mathbb{P}^2$, it is not always possible to scale H_P to have $v = 1$, as this might be zero.

Homographies: Invariants

Homographies generalize an affine transformation, which is the composition of a general non-singular linear transformation of *inhomogeneous* coordinates and a translation.

Invariants: the *cross ratio* of four collinear points: a ratio of lengths on a line is invariant under affinities, but not under projectivities. However, a ratio of ratios or *cross ratio* of lengths on a line is a projective invariant.

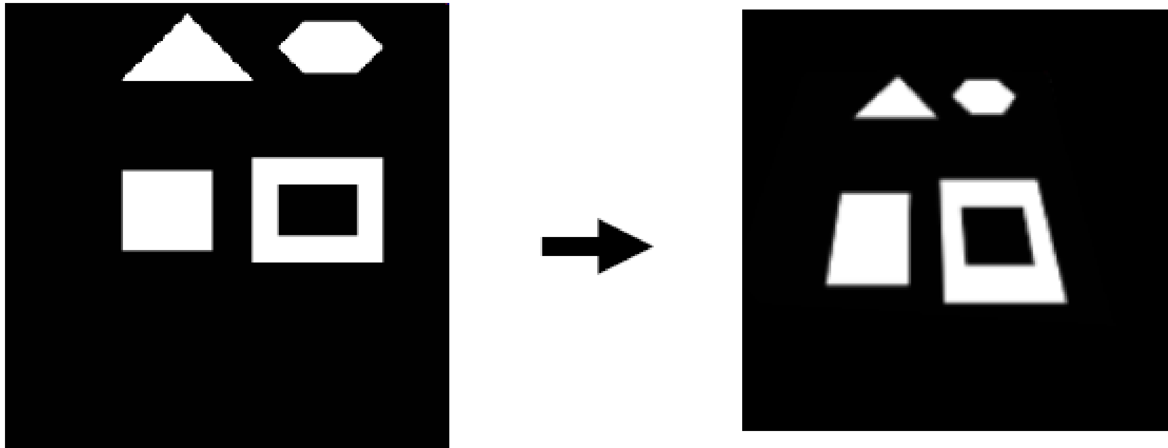
$$\frac{\|P_3 - P_1\| \|P_4 - P_2\|}{\|P_3 - P_2\| \|P_4 - P_1\|}$$



Homographies: Invariants

The matrix has nine elements with only their ratio significant, so the transformation is specified by **eight parameters**

A projective transformation between two planes can be computed from four point correspondences, with no three collinear on either plane.



Homographies: Properties

Similarities have **eight degrees of freedom**: the four terms of A, the two of v, and the translation vector $[t_x, t_y]$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{t} \\ \boxed{\boldsymbol{v}^\top} & v \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rmk Homographies can be estimated from four point correspondences

Rmk $\ell_\infty \neq H_P^{-\top} \ell_\infty$ when $\boldsymbol{v} \neq \mathbf{0}$, thus ideal points might become finite points. For instance given $\boldsymbol{x} = (x_1;\ x_2;\ 0)$, then

$$H_P \boldsymbol{x} = \begin{bmatrix} \cdots \\ \cdots \\ v_1 x + v_2 y + v0 \end{bmatrix}$$

Homographies: Properties

Under a projective transformation ideal points may be mapped to finite points, and consequently ℓ_∞ is mapped to a finite line.

However, if the transformation is an affinity ($\boldsymbol{v} = [0,0]$), then ℓ_∞ is not mapped to a finite line, but remains at infinity.

The image of l_∞ is

$$(H^{-1})^\top \; l_\infty$$

$$\begin{bmatrix} A^{-\top} & \boldsymbol{v} \\ \boxed{\boldsymbol{t}^{-\top} A^{-\top}} & v \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v \end{bmatrix}$$

Affine Trasformation and Line at Infinity

Rmk: l_∞ is not **fixed pointwise** under an affine transformation

Under an affinity a point on l_∞ (i.e., an ideal point) can be mapped to a different point on l_∞. This is the reason why orthogonality is lost.

Affine Rectification

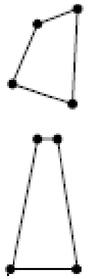
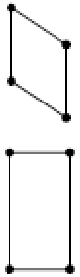
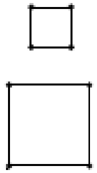
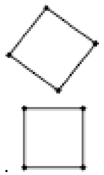
horizon

a b c

$H =$

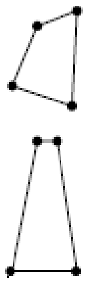
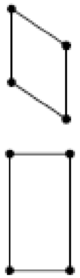
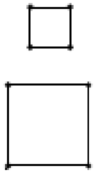
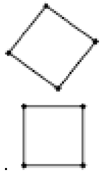
Hartley Zisserman Fig.2.6

Summarizing

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Hartley Zisserman Tab 2.1

Summarizing

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Pa … on co … n- ea … Tl
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

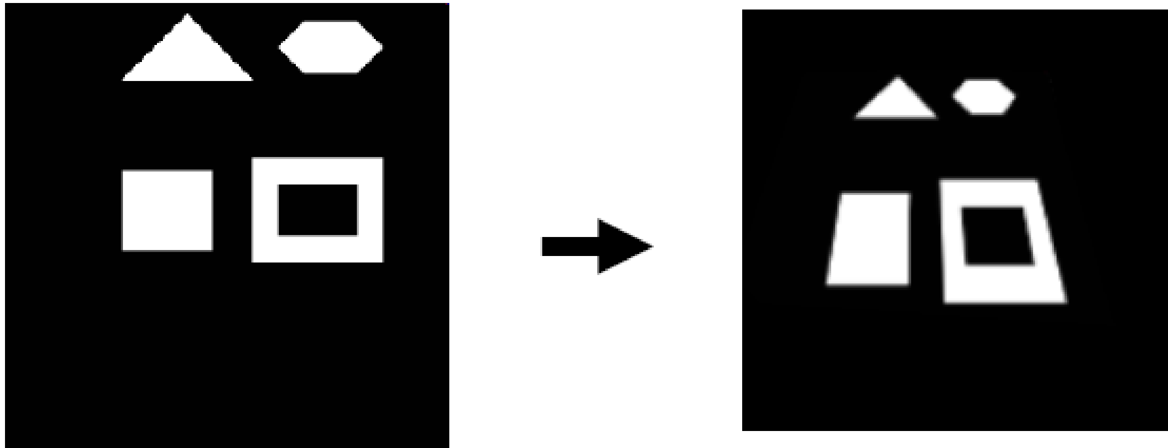
Transformations higher in the table can produce all the actions of the ones below

Hartley Zisserman Tab 2.1

Homographies: Properties

For a given affinity the areas are scaled of $\det(A)$ anywhere on the plane; and the orientation of a transformed line depends only on its initial orientation, not on its position on the plane.

In contrast, area scaling in homographies varies with position (e.g. under perspective a more distant square on the plane has a smaller image than one that is nearer)



Affine Rectification

Idea:

- Identify the ℓ_∞ in the image
- transform the identified ℓ_∞ to its canonical position of $[0; 0; 1]$

Let $l = (l_1; l_2; l_3)$ be the image of the line at the infinity with $l_3 \neq 0$,

A suitable homography which maps l back to $\ell_\infty = [0; 0; 1]$ is

$$H = H_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

Affine Rectification

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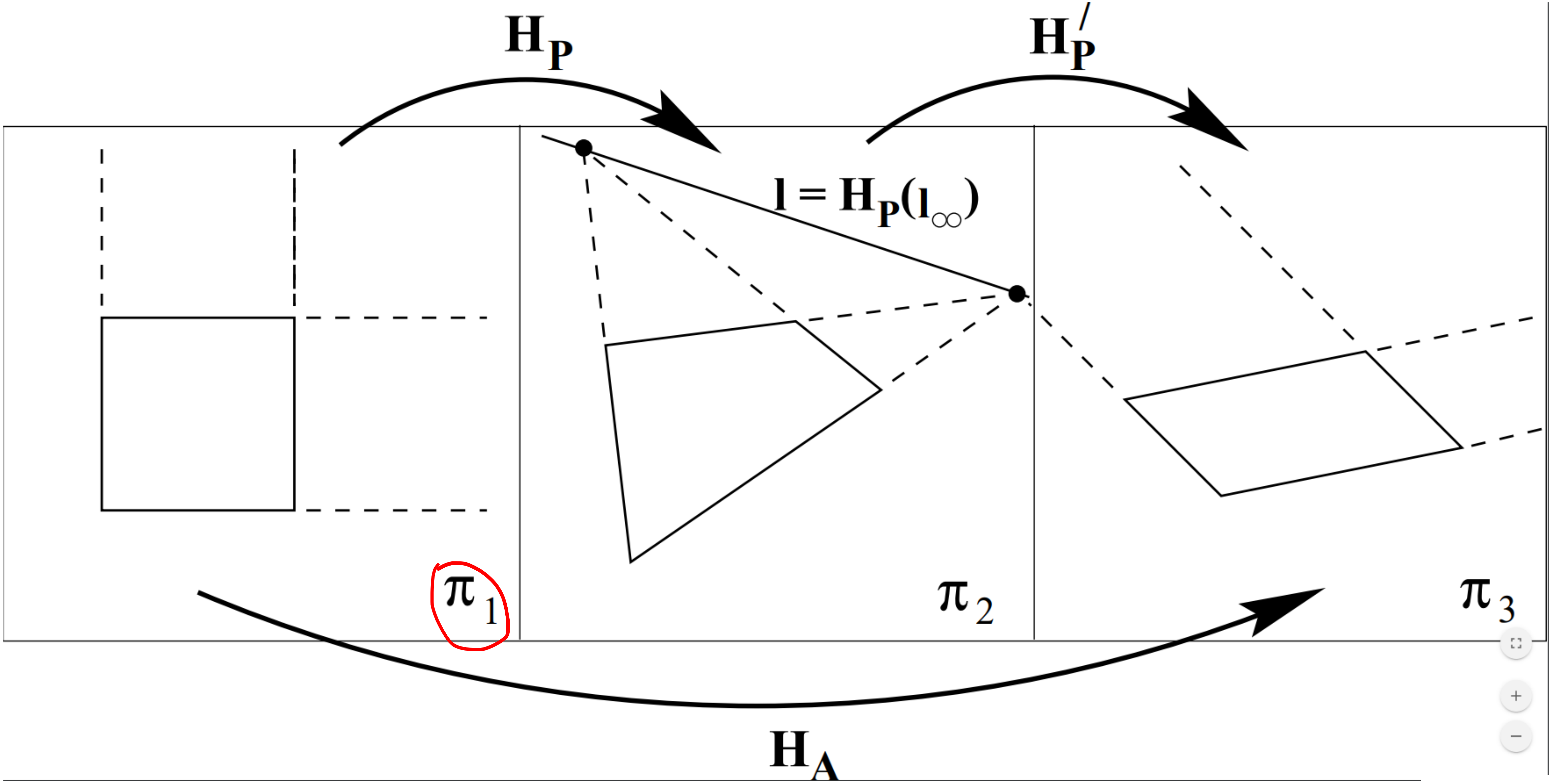
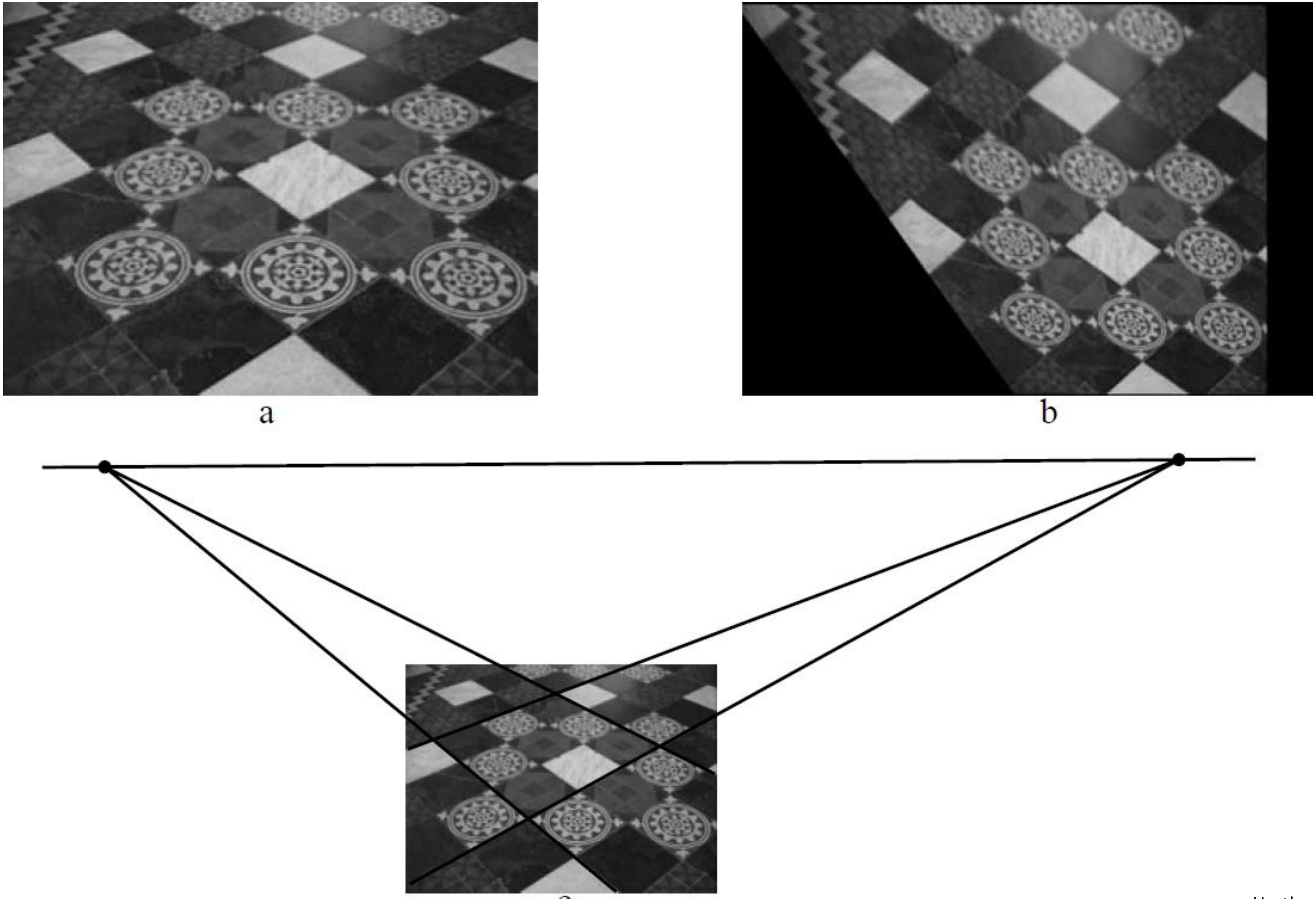
$$H = H_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

Rmk: l can be rescaled to improve conditioning of H

Affine Rectification



Hartley Zisserman Fig.2.12



Hartley Zisserman Fig. 2.13

DLT algorithm

How to estimate homographies and much more…

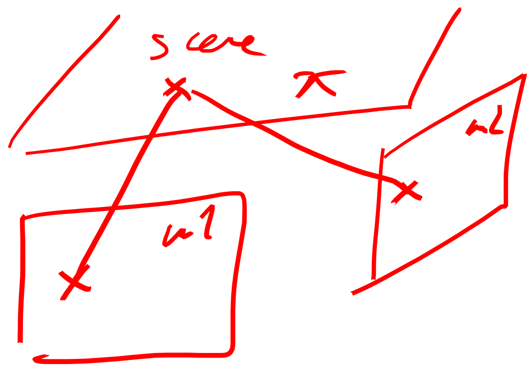
HZ chapter 4

DLT

Direct Linear Trasformation (DLT) algorithm, solves many relevant problems in Computer Vision

- 2D Homography estimation
- Camera projection matrix estimation (projections from 3D to 2D)
- Fundamental matrix computation
- Trifocal tensor computation

Homography estimation



We consider a set of point correspondences

$$\{(\boldsymbol{x}_i', \boldsymbol{x}_i), i = 1, \dots, 4\}$$

belonging to two different images. Our problem is to compute a $H \in \mathbb{R}^{3\times3}$ such that $\boldsymbol{x}_i' = H\boldsymbol{x}_i$ for each i.

$\dim(3)$ $\quad \mathbb{P}^2 \quad \Downarrow$

$x_i \to H x_i = \lambda x_i', \quad \lambda \neq 0$

Rmk if we look at $\boldsymbol{x}$ as a 3d vector the equality $\underset{3\times1}{\boldsymbol{x}_i'} = \underset{3\times3}{H}\underset{3\times1}{\boldsymbol{x}_i}$ **does not hold**

Equality holds in $\mathbb{P}^2$ (but linear systems are solved as in $\mathbb{R}^3$).

What makes the DLT is distinct from standard cases since the left and right sides of the defining equation can differ by an unknown multiplicative factor which is dependent on the number of equations.

DLT for Homography Estimation

$\boldsymbol{x}_i'$ and $\boldsymbol{x}_i$ are actually corresponding through an homography H iff:

$$\boxed{\boldsymbol{x}_i' = \lambda H \boldsymbol{x}_i} \quad \forall \lambda \neq 0$$

(x_i, x_i')

To estimate an homography using standard algebraic tools, we need to

$x_i' = H x_i$

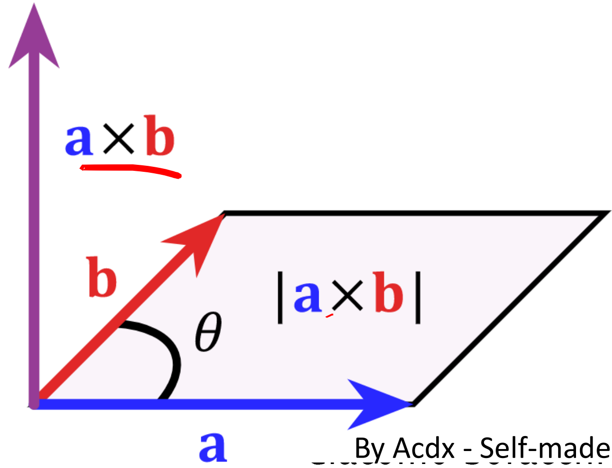
1) bring this constraint in equations of $\mathbb{R}^3$

2) Find out enough matches to identify the unknown parameters

The equation implies that $\boldsymbol{x}_i'$ and $\boldsymbol{x}_i$ have the same direction (possibly different magnitude). Collinearity constraints can be written through the cross product

$$\boldsymbol{x}_i' \times H\boldsymbol{x}_i = \boldsymbol{0}\,, i = 1, \dots, 4$$

where $\boldsymbol{0} = [0; 0; 0]$



DLT for Homography Estimation

Let $\boldsymbol{x_i} = [x_i; y_i; w_i]$ and $H = [\boldsymbol{h}_1^\top; \boldsymbol{h}_2^\top; \boldsymbol{h}_3^\top]$ being $\boldsymbol{h}_1^\top$ the first row…

$$H = \begin{bmatrix} h_1^T \\ h_2^T \\ h_3^T \end{bmatrix}$$

Then the cross product can be written as..

$$\boxed{\boldsymbol{x}_i' \times H\boldsymbol{x_i}} = \begin{pmatrix} y_i'\boldsymbol{h}_3^\top\boldsymbol{x_i} - w_i'\boldsymbol{h}_2^\top\boldsymbol{x_i} \\ w_i'\boldsymbol{h}_1^\top\boldsymbol{x_i} - x_i'\boldsymbol{h}_3^\top\boldsymbol{x_i} \\ x_i'\boldsymbol{h}_2^\top\boldsymbol{x_i} - y_i'\boldsymbol{h}_1^\top\boldsymbol{x_i} \end{pmatrix} = 0$$

So, given one pair of corresponding points, we are given 3 equations…

DLT for Homography Estimation

.. after some linear algebra, this can be expressed in a matrix form as

$$\begin{array}{cc} & \begin{array}{ccc} 3 & \quad\quad 3 & \quad\quad 3 \end{array} \\ \begin{array}{c} -x_i/w_i \cdot \\ -y_i/w_i \cdot \\ \end{array} & \overbrace{\begin{bmatrix} \mathbf{0}^\top & -w'_i \boldsymbol{x}_i^\top & y'_i \boldsymbol{x}_i^\top \\ w'_i \boldsymbol{x}_i^\top & \mathbf{0}^\top & -x_i \boldsymbol{x}_i^\top \\ -y'_i \boldsymbol{x}_i^\top & x'_i \boldsymbol{x}_i^\top & \mathbf{0}^\top \end{bmatrix}}^{3\times 9} \end{array} \overbrace{\begin{bmatrix} \boldsymbol{h_1} \\ \boldsymbol{h_2} \\ \boldsymbol{h_3} \end{bmatrix}}^{9\times 1} = 0$$

$x_i' \times H x_i = 0$

$x_i' \quad \underline{x}_i = (x_i, y_i, w_i)$

This is a 3×9 matrix multiplied times a vector of 9 elements

Rmk The equation is an equation *linear* in the unknown vector

$$\boldsymbol{h} = [\boldsymbol{h_1}\,; \boldsymbol{h_2}\,; \boldsymbol{h_3}\,]$$

Rmk the **three rows of the matrix are linearly dependent** (the third row is the sum of $-x_i/w_i$ times the first row and $-y_i/w_i$ times the second)

DLT for Homography Estimation

Thus, **keep only two rows** giving rise to **an homogeneous linear system**

$x_i' x_i$ $\qquad$ 9 $\qquad$ (x_i', x_i)

$$2 \quad \begin{bmatrix} \mathbf{0}^\top & -w'_i \boldsymbol{x}_i^\top & y'_i \boldsymbol{x}_i^\top \\ w'_i \boldsymbol{x}_i^\top & \mathbf{0}^\top & -x_i \boldsymbol{x}_i^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \boldsymbol{h}_3 \end{bmatrix} = \mathbf{0}$$

$x_i' x_i$ $\qquad$ $\mathbf{0} \leftarrow H$

Which can be written as

$$\underset{8\times 9}{A_i} \underset{9\times 1}{\boldsymbol{h}} = \underset{9\times 1}{\mathbf{0}}, \qquad i = 1, \dots, 4$$

Rmk remember that $\boldsymbol{h}$ has been unrolled row-wise, not column-wise

DLT for Homography Estimation

Stacking 4 point correspondences gives an 8×9 matrix

$$\boxed{A}\boldsymbol{h} = \mathbf{0}$$

Solve it as $\boldsymbol{h} = RNS(A)$ and arbitrarily imposing $||\boldsymbol{h}|| = 1$ since we are not interested in the trivial solution $\boldsymbol{h} = \mathbf{0}$

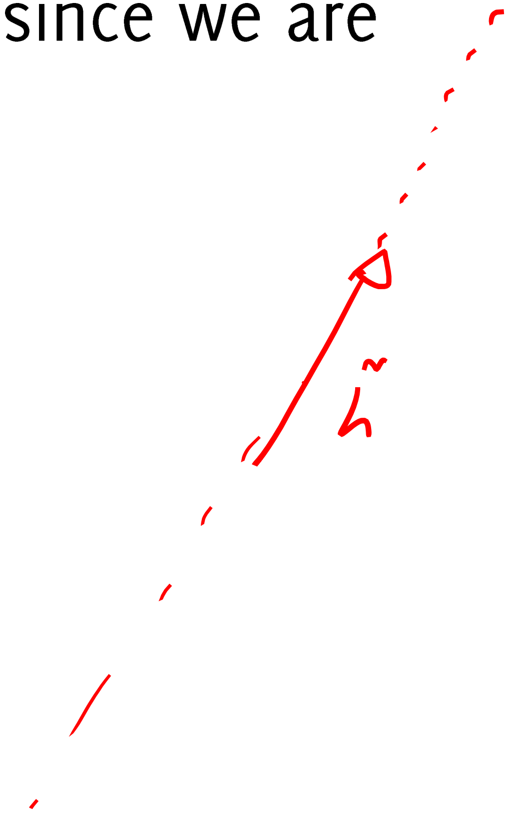
$Ax = b$

$rk([A|b]) = rk(A)$

$rk([A|0]) = rk(A)$

$n - rk(A) = 1$ dimensional affine space for solutions

$$\begin{array}{cc} & 9 \\ 8 & \begin{bmatrix} A_i \end{bmatrix} \end{array} \begin{bmatrix} h \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$



DLT in the overdetermined case

Let us assume $n > 4$ point correspondences are given, such that A is a $2n \times 9$ matrix

$$\underset{2n\times 9}{A}\boldsymbol{h} = \mathbf{0}$$

The system is overdetermined. We are not interested

- in the trivial solution $\boldsymbol{h} = \mathbf{0}$ and
- not even in an exact solution, since typically this does not exist because of noise in the measurements $\boldsymbol{x}_i', \boldsymbol{x}_i$

Thus, impose the constraint $||\boldsymbol{h}|| = 1$ and minimize a cost function

$$\boldsymbol{h}^* = \underset{h}{\operatorname{argmin}} ||A\boldsymbol{h}||_2 \; s.t. \; ||\boldsymbol{h}||_2 = 1$$

DLT in the overdetermined case

The solution of this problem is obtained by

$$\boldsymbol{h}^* = \underset{h}{\operatorname{argmin}} ||A\boldsymbol{h}||_2 \; s.t. \; ||\boldsymbol{h}||_2 = 1$$

The solution is the (unit) eigenvector of $A^T A$ with least eigenvalue.
Equivalently, the solution is the **unit singular vector corresponding to the smallest singular value of $\boldsymbol{A}$**. (See HZ A5.3(p592))

Specifically, if $A = UDV^\top$ with D diagonal with positive diagonal entries, arranged in descending order down the diagonal, then $\boldsymbol{h}$ is the last column of V.

Rmk DLT algorithm minimizes the **residual** $||A\boldsymbol{h}||$, which has to be interpreted as an *algebraic error*

$$\underset{h}{\arg\min} \|Ah\|_2 \quad s.t. \quad \|h\|_2 = 1$$

$A \in \mathbb{R}^{2n \times 9}$

$SVD(A)$

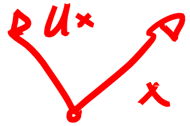
$$\underbrace{A}_{2n \times 9} = \underbrace{U}_{2n \times 2n} \underbrace{D}_{2n \times 9} \underbrace{V'}_{9 \times 9}$$

$$\underset{h}{\arg\min} \|UDV'h\|_2$$

$\|Ux\|_2$

$x \rightarrow Ux$

$\|x\|_2 = \|Ux\|_2$



$$\arg\min \|DV'h\|_2$$

$y = V'h$

$$\underset{y}{\arg\min} \|Dy\|_2 \qquad \|y\|_2 = 1$$

$$\underset{y}{\arg\min} \underbrace{\sum_{i=1}^{n} (d_i y_i)^2}$$

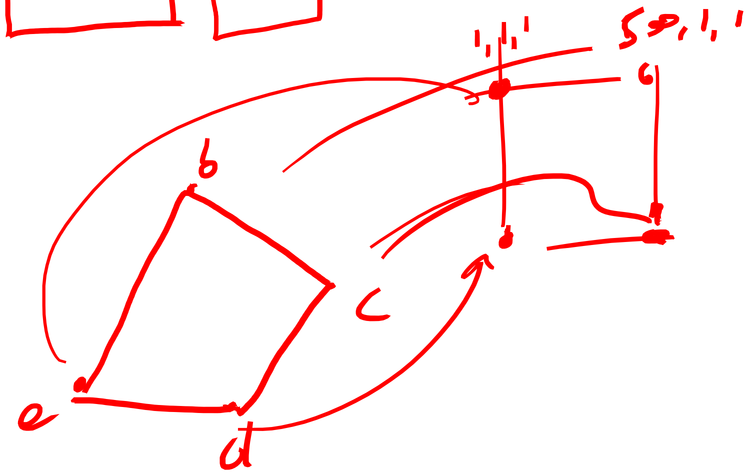
$y = [0,0,0,\dots,1]'$

$d_1 \geq d_2 \geq \dots \geq d_9$

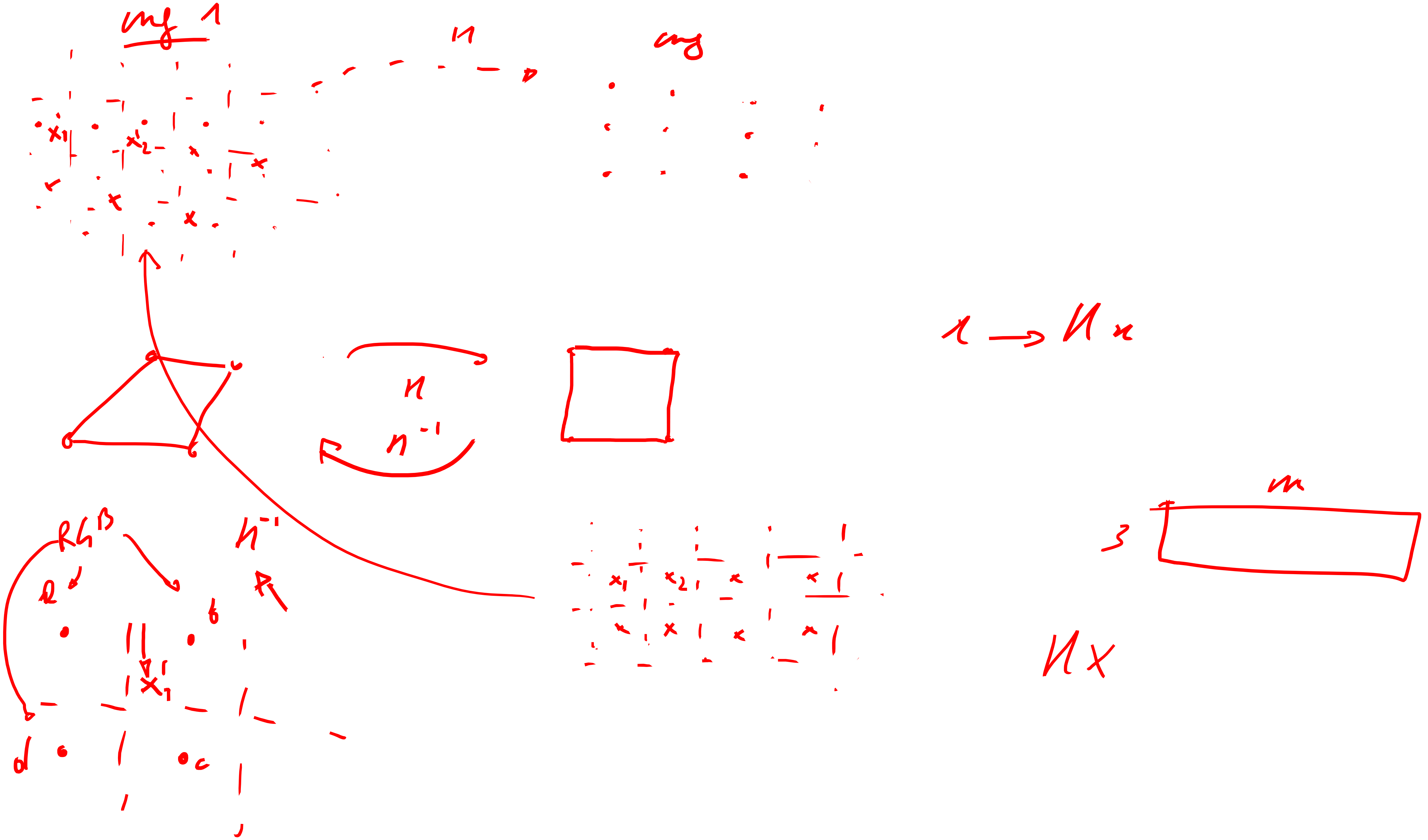
$Vy = h$

$$V \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = h$$

$\boxed{} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$ last column of V



Giacomo Boracchi


img 1
H
img
x → Hx
H
H⁻¹
RGB
a
b
c
d
x₁
x₂
H⁻¹
m
3
Hx

Preconditioning

DLT and the reference system

Are the outcome of DLT independent of the reference system being used to express $\boldsymbol{x}'$ and $\boldsymbol{x}$?

Unfortunately **DLT is not invariant to similarity transformations**.

Therefore, it is necessary to **apply a normalizing transformation** to the data before applying the DLT algorithm.

Normalizing the data makes the DLT invariant to the reference system, as it is always being estimated in a canonical reference

Normalization is also called Pre-conditioning

Preconditioning

Needed because in homogeneous coordinate systems, components typically have very different ranges

- Row and Column indexes ranges in $[0 - 4K]$
- Third component is 1

Define a maping

$$\boxed{x \rightarrow Tx}$$

That brings all the points «arorud the origin» and rescale each component to the same range (say at an average distance $\sqrt{2}$ from the origin)

Preconditioning

$$\boldsymbol{x} \rightarrow T\boldsymbol{x}$$

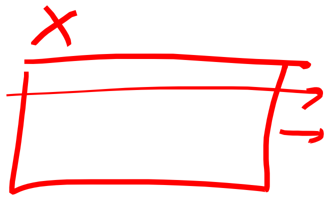
It's a similarity scaling in $\mathbb{P}^2$ (with $\theta = 0$ since we do not need rotations)

$$T = \begin{bmatrix} 1/s & 0 & -t_x/s \\ 0 & 1/s & -t_y/s \\ 0 & 0 & 1 \end{bmatrix}$$

The preconditioning for a set of points X of $\mathbb{P}^2$ is defined as

$$t_x = \text{mean}(X(1,:))$$
$$t_y = \text{mean}(X(2,:))$$

std(X(1,:))



Which brings the barycentre of X to the origin, the scaling is

$$s = \frac{\text{mean}\big(\text{std}(X,2)\big)}{\sqrt{2}}$$

Homography estimation with preconditioning

Estimate the homography between two sets of points X, X'

1. Compute T, T' preconditioning transformation of X, X'
2. Apply transformation

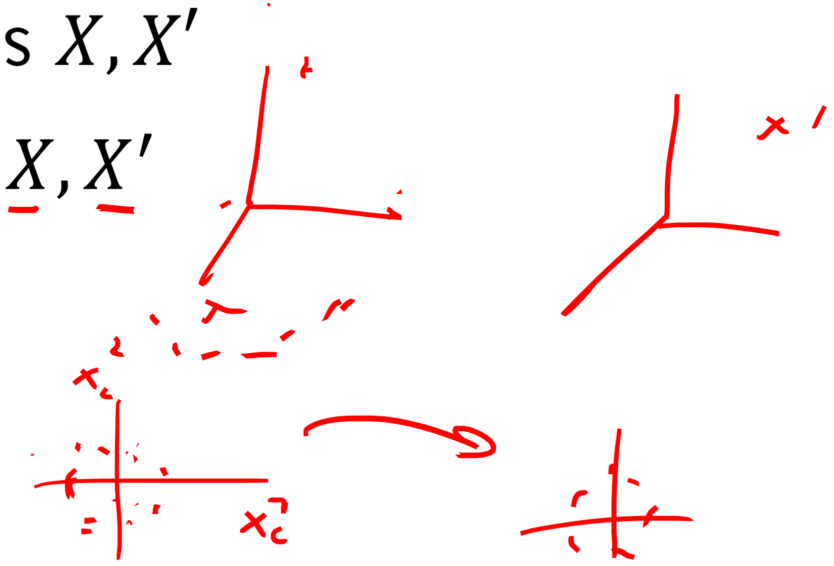
$$Xc = TX, \qquad X_c' = T'X'$$

1. Estimate the homography from X_c and X_c',

$$H = DLT(X_c, X_c')$$

1. Define the transformation

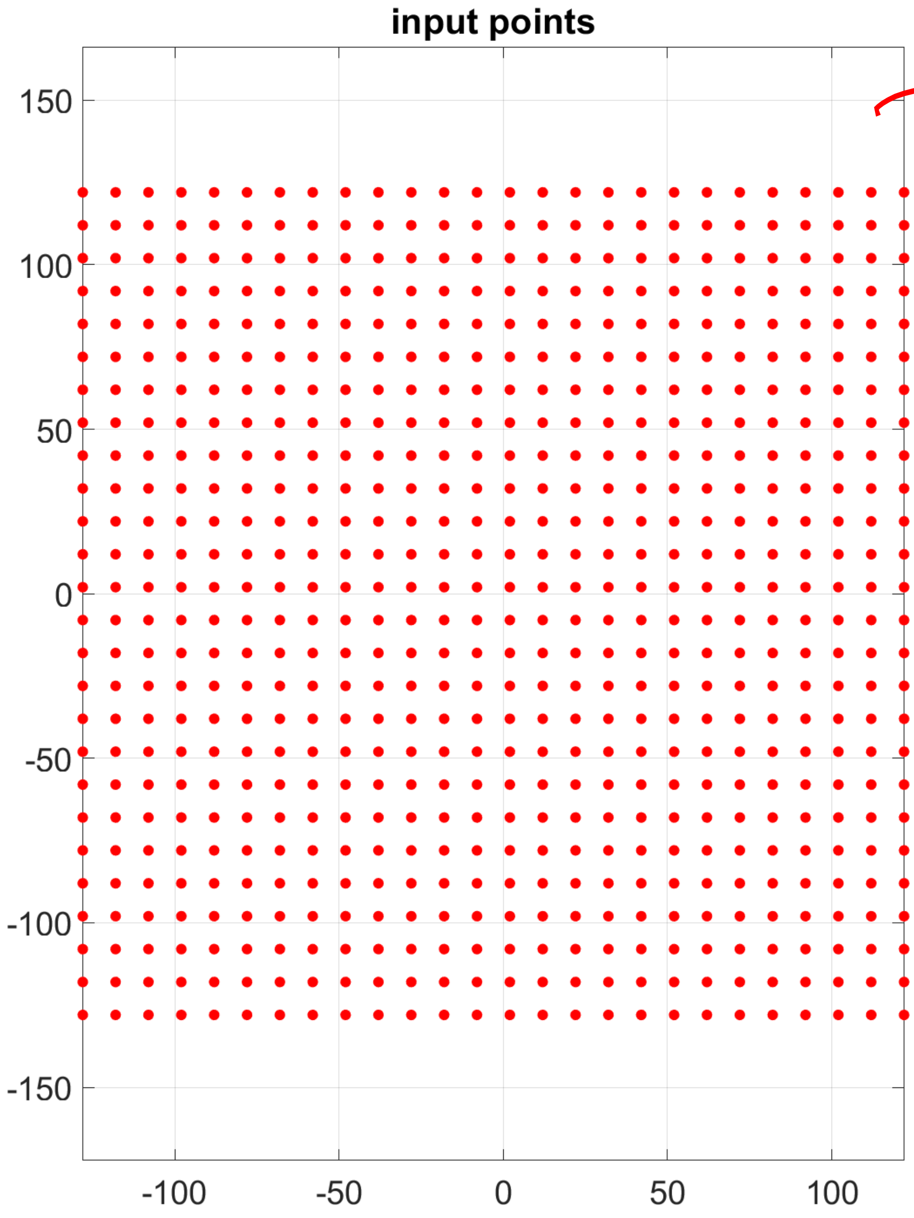
$$H_c = {T'}^{-1} H T$$

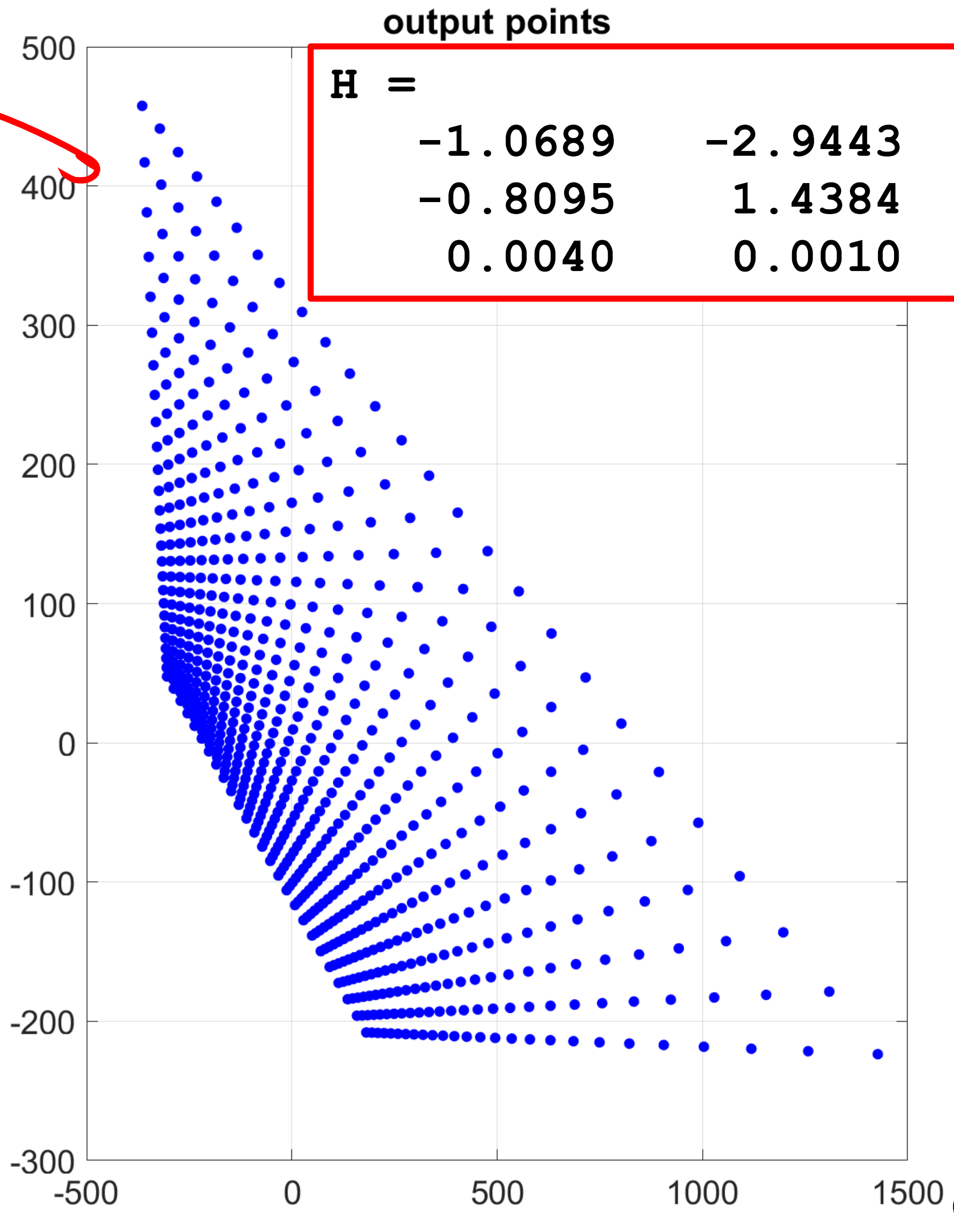


How to apply linear transformations to an image?

…these are transformations between points, not pixel intensities

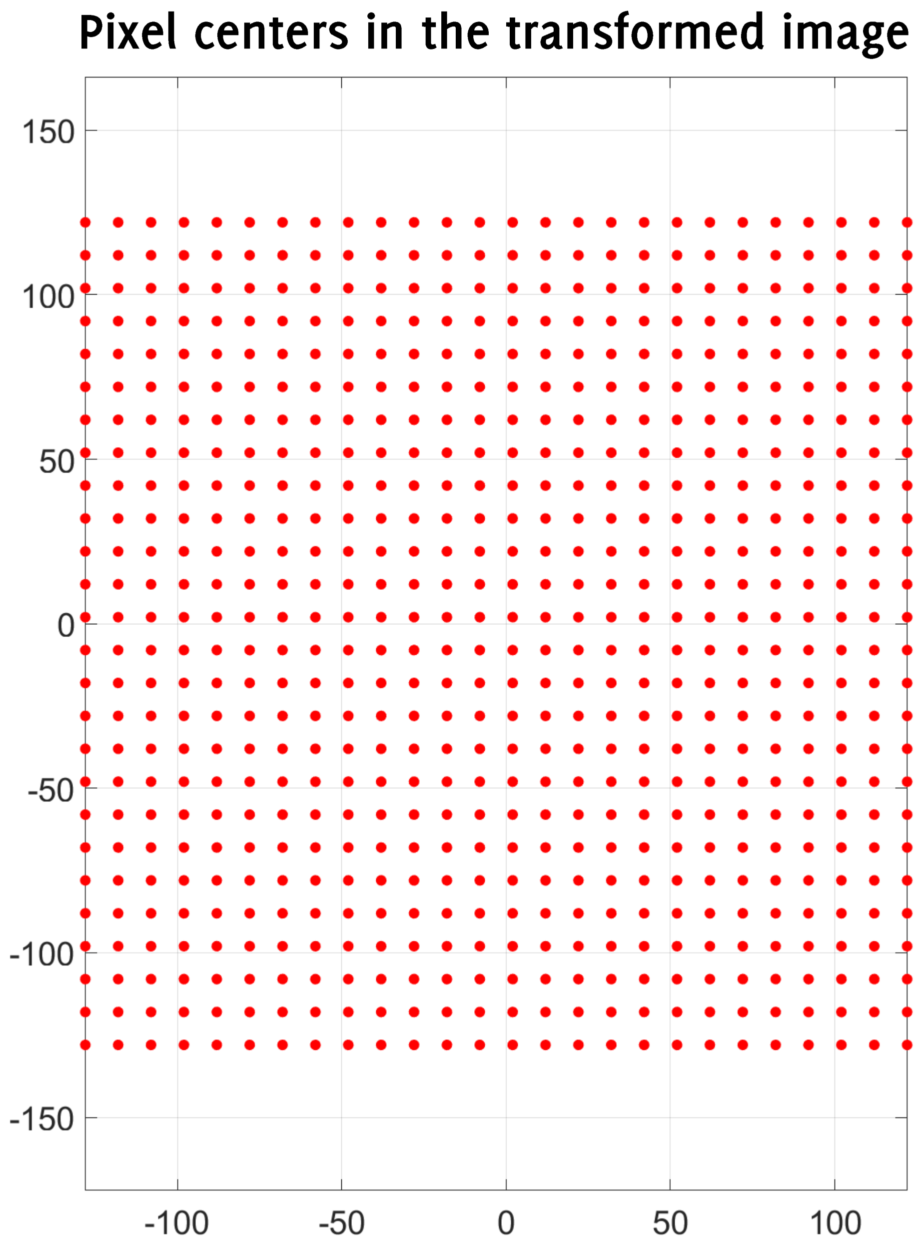
Apply the inverse of the desired transformation on a grid covering the output image





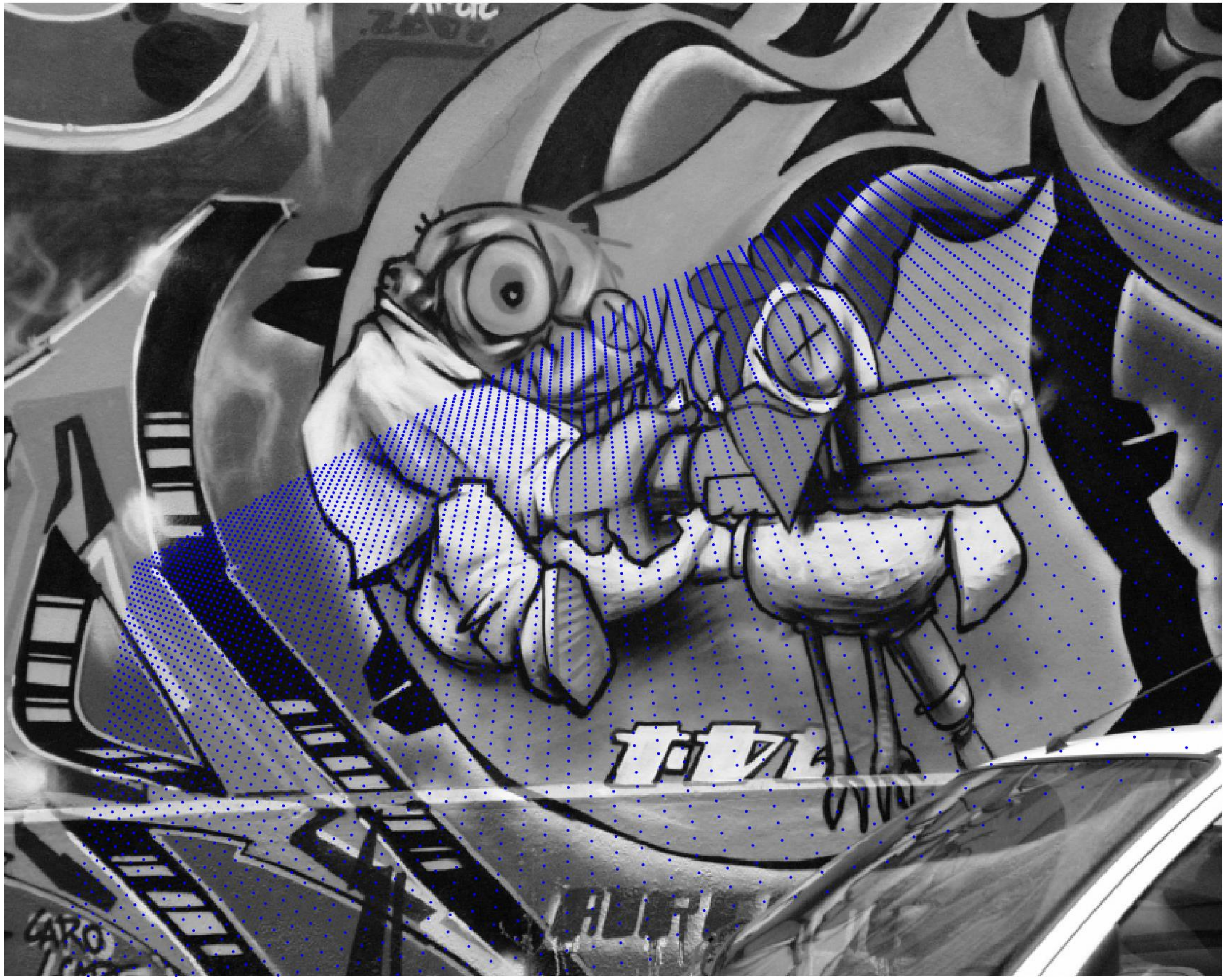
$$H = \begin{matrix} -1.0689 & -2.9443 & 0 \\ -0.8095 & 1.4384 & 0 \\ 0.0040 & 0.0010 & 1.0000 \end{matrix}$$

Here is the grid of the transformed image

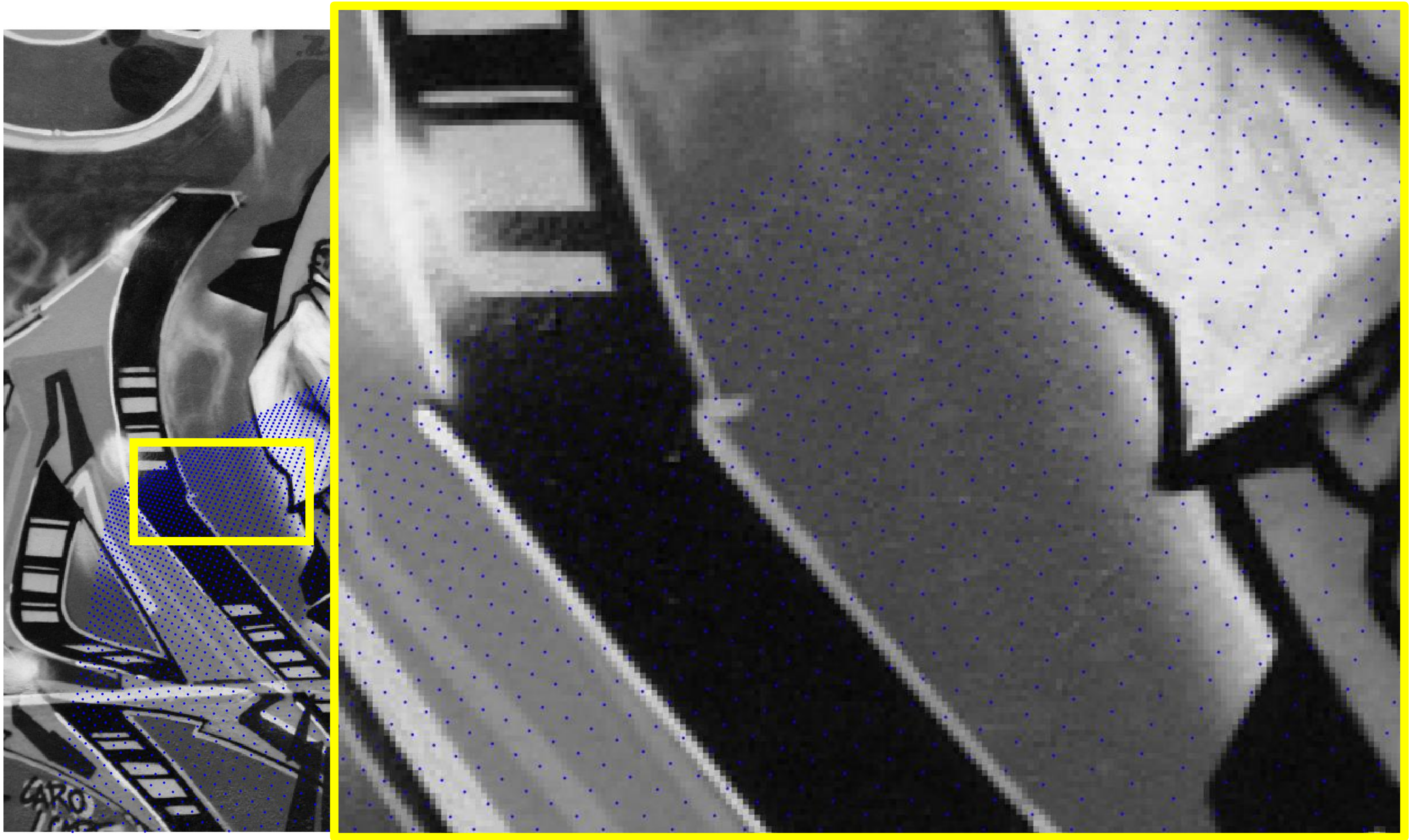


Place the grid transformed by H^{-1} on the input image

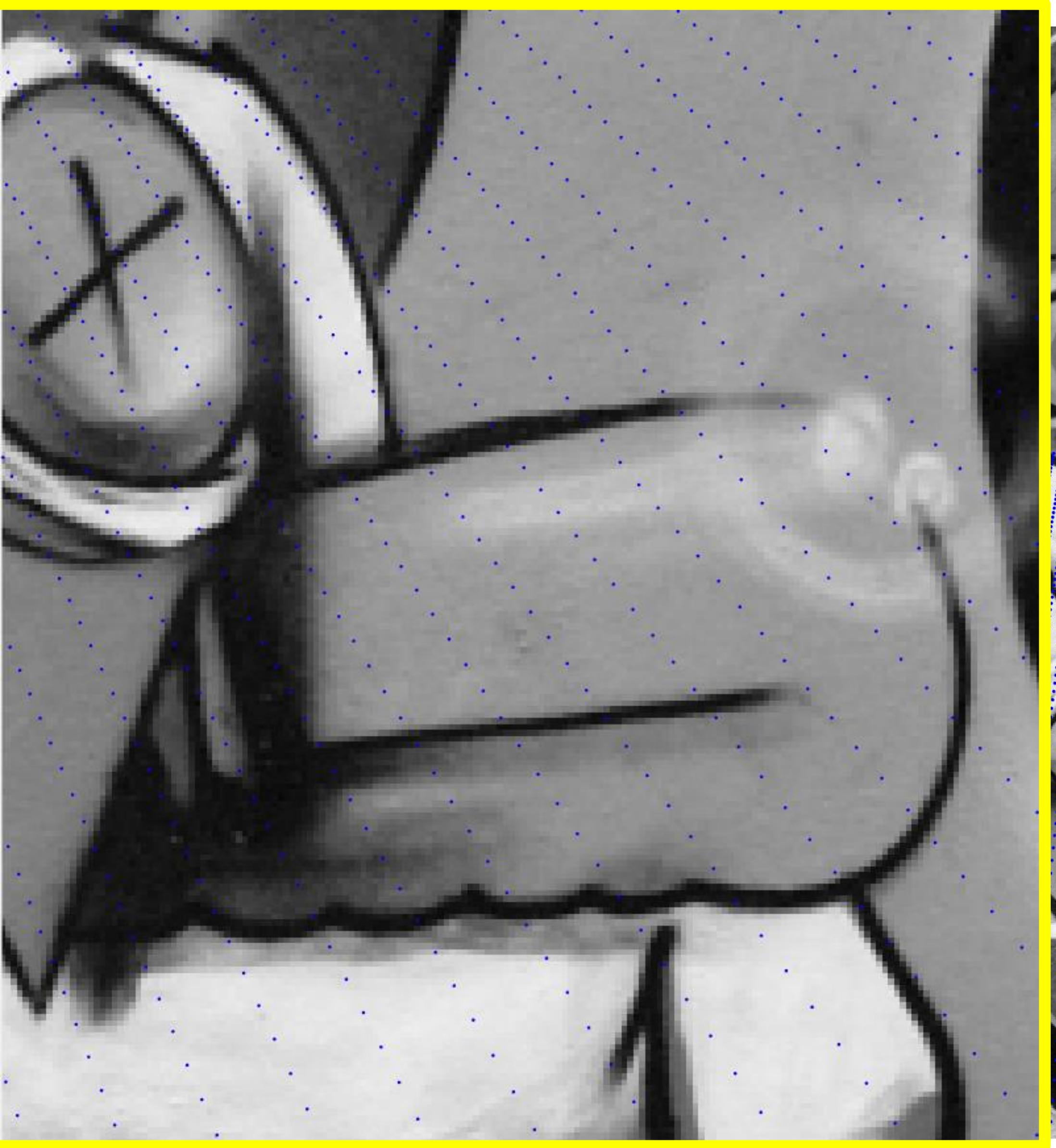
transformed grid

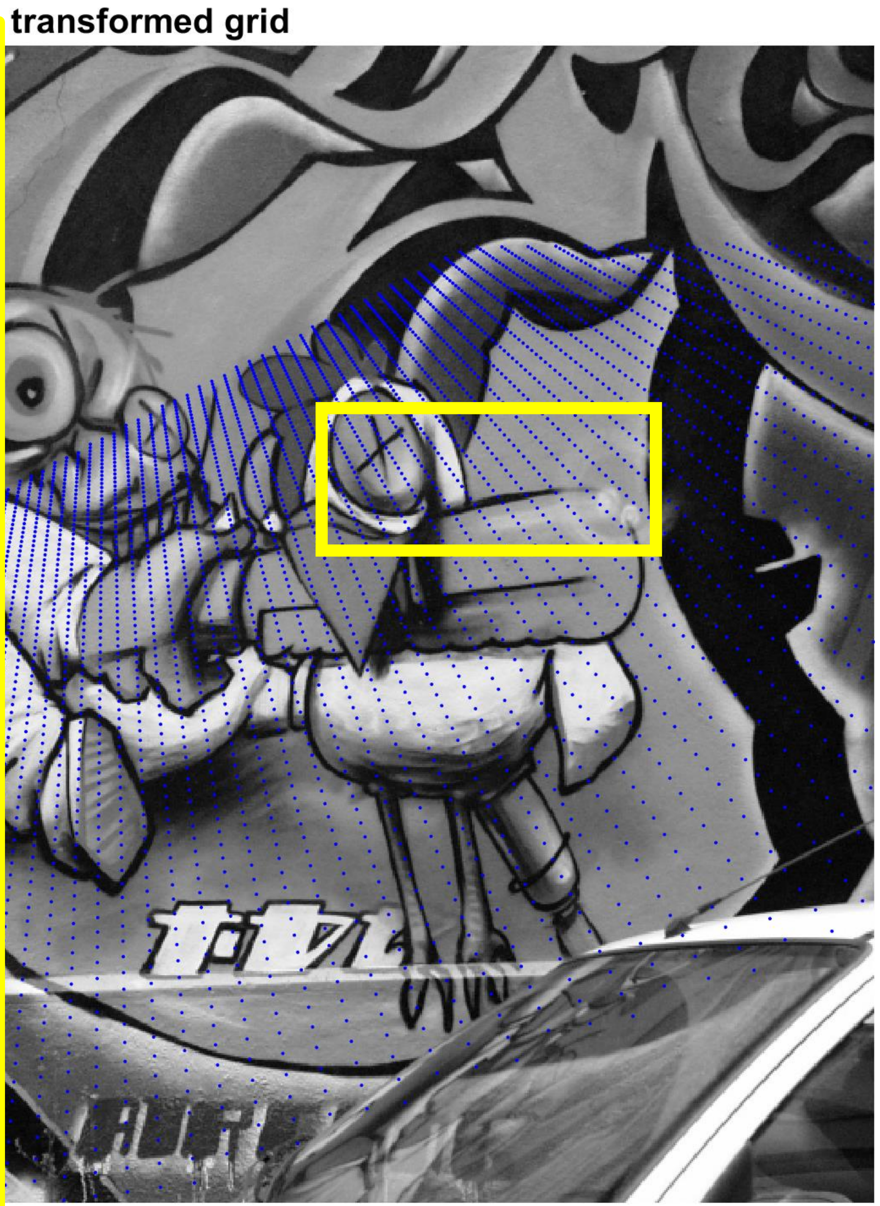


Place the grid transformed by H^{-1} on the image

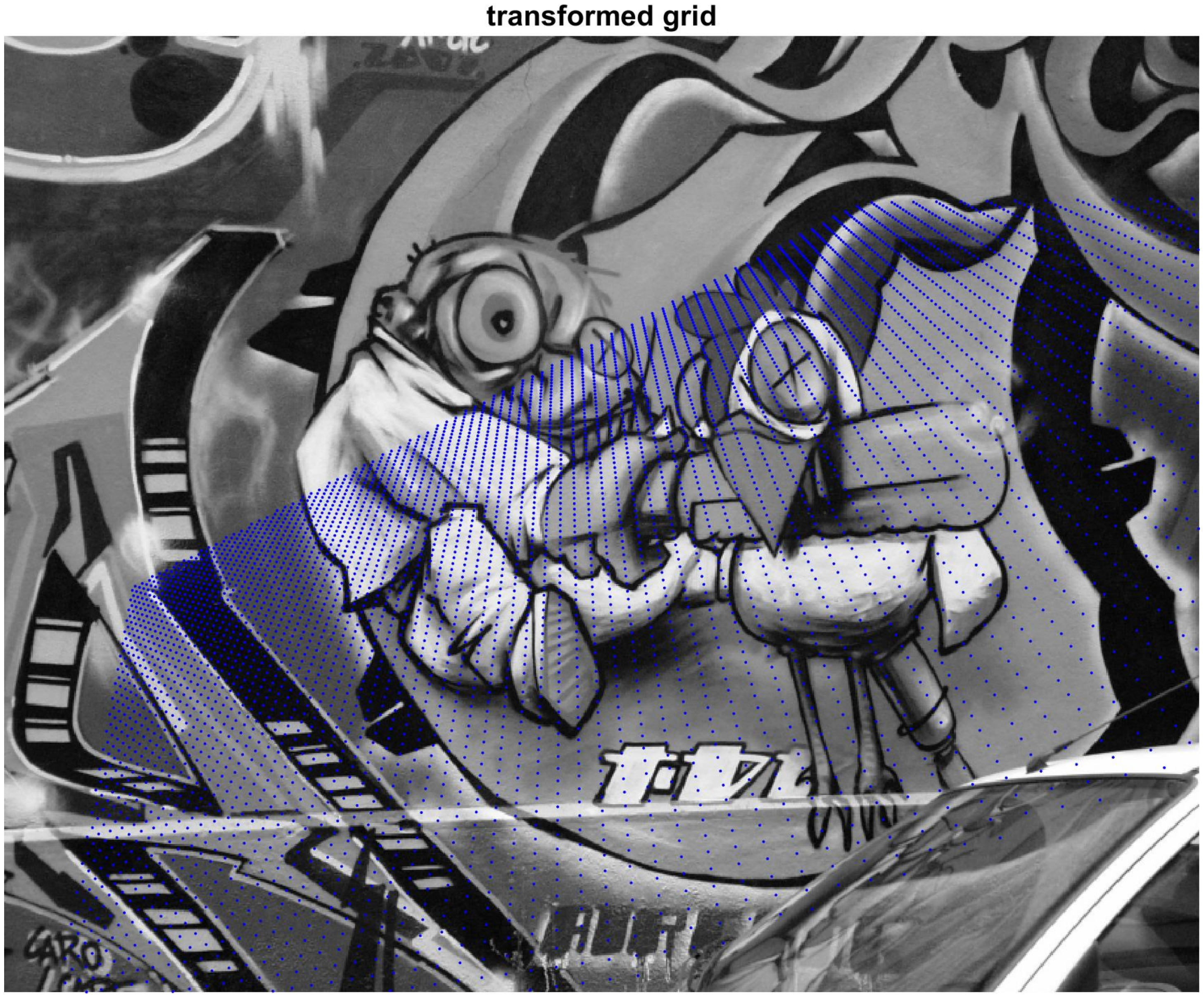


Place the grid transformed by H^{-1} on the image



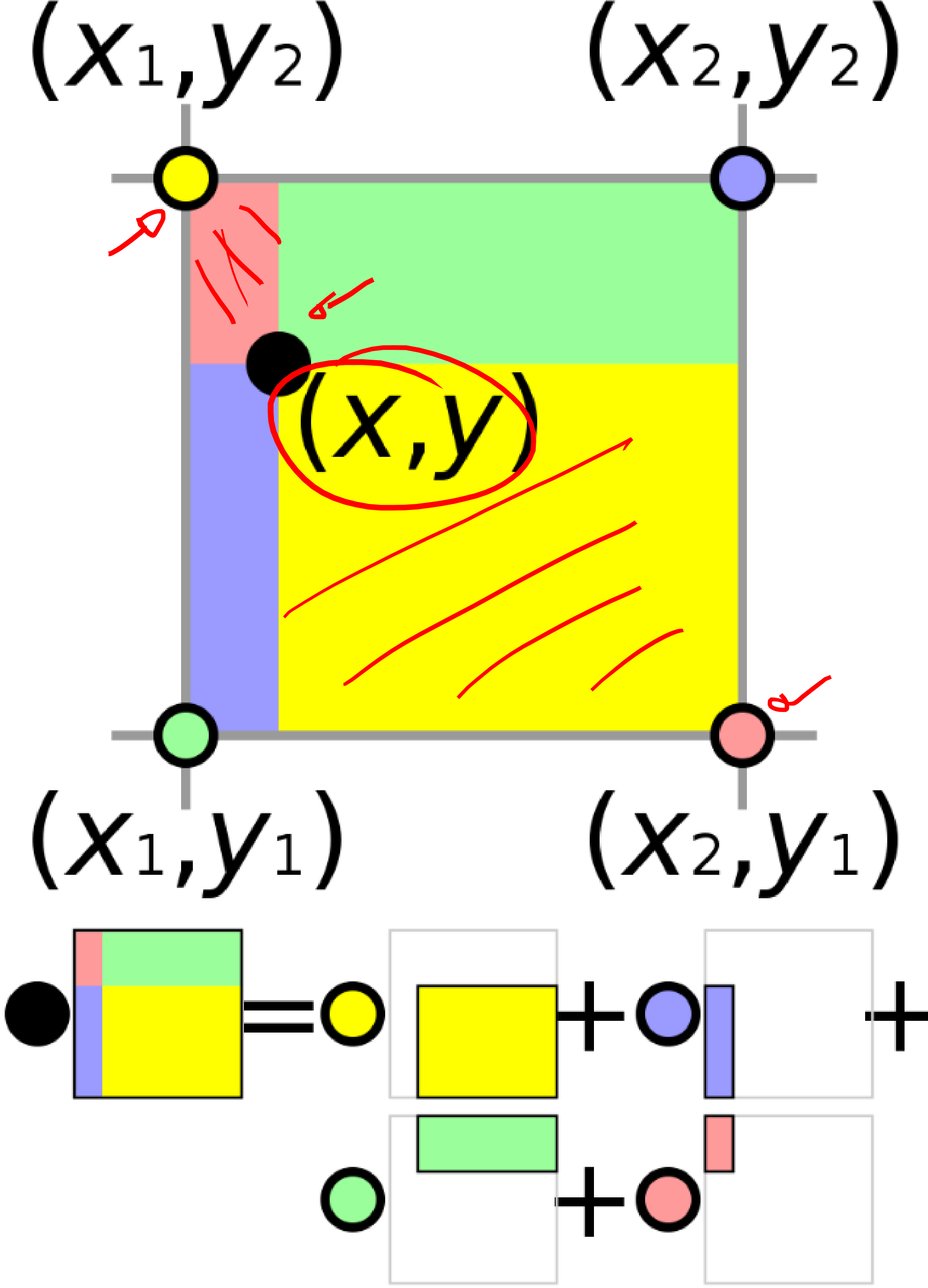


Define image intensity at blue points by means of bilinear interpolation



Bilinear Interpolation

Intensity value over the transformed grid (x, y) are defined by interpolating values of neighbouring pixels $\{(x_i, y_i), i = 1,2\}$. Several interpolation options are viable. Bilinear interpolation is a weighted average with weights proportional to the areas illustrated here.



By Cmglee - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=21409164

Bilinear Interpolation

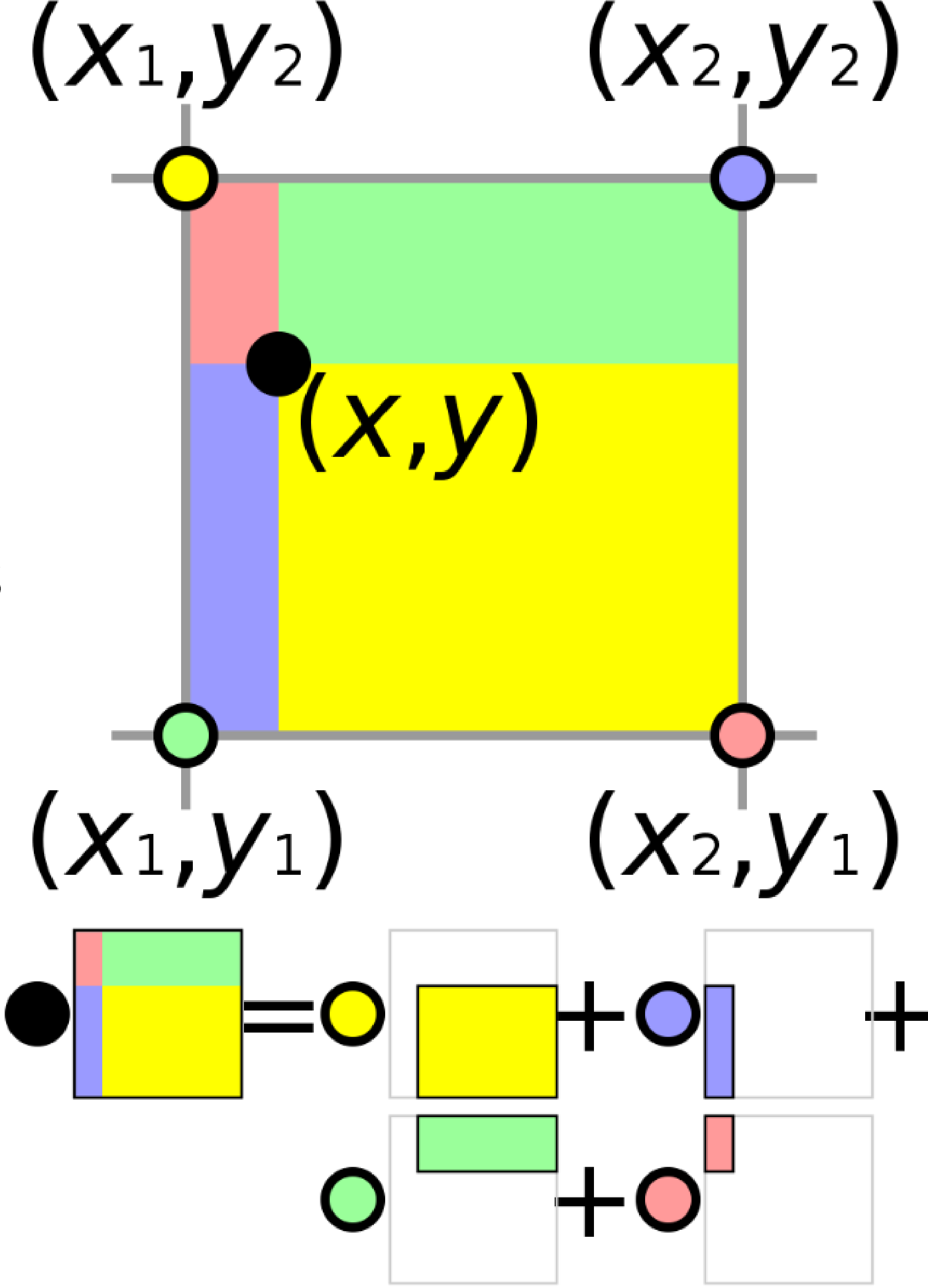
Let (x, y) be the point and $f(x, y)$ the value we want to estimate.

Bilinear interpolation corresponds to

- Perform a 1D-linear interpolation along x axis to estimate $f(x, y_1)$ and $f(x, y_2)$

$$f(x, y_1) = \frac{x_2 - x}{x_2 - x_1} f(x_1, y_1) + \frac{x - x_1}{x_2 - x_1} f(x_2, y_1)$$
$$f(x, y_2) = \frac{x_2 - x}{x_2 - x_1} f(x_1, y_2) + \frac{x - x_1}{x_2 - x_1} f(x_2, y_2)$$

- Perform 1D-linear interpolation along y axis but considering $f(x, y_1)$ and $f(x, y_2)$



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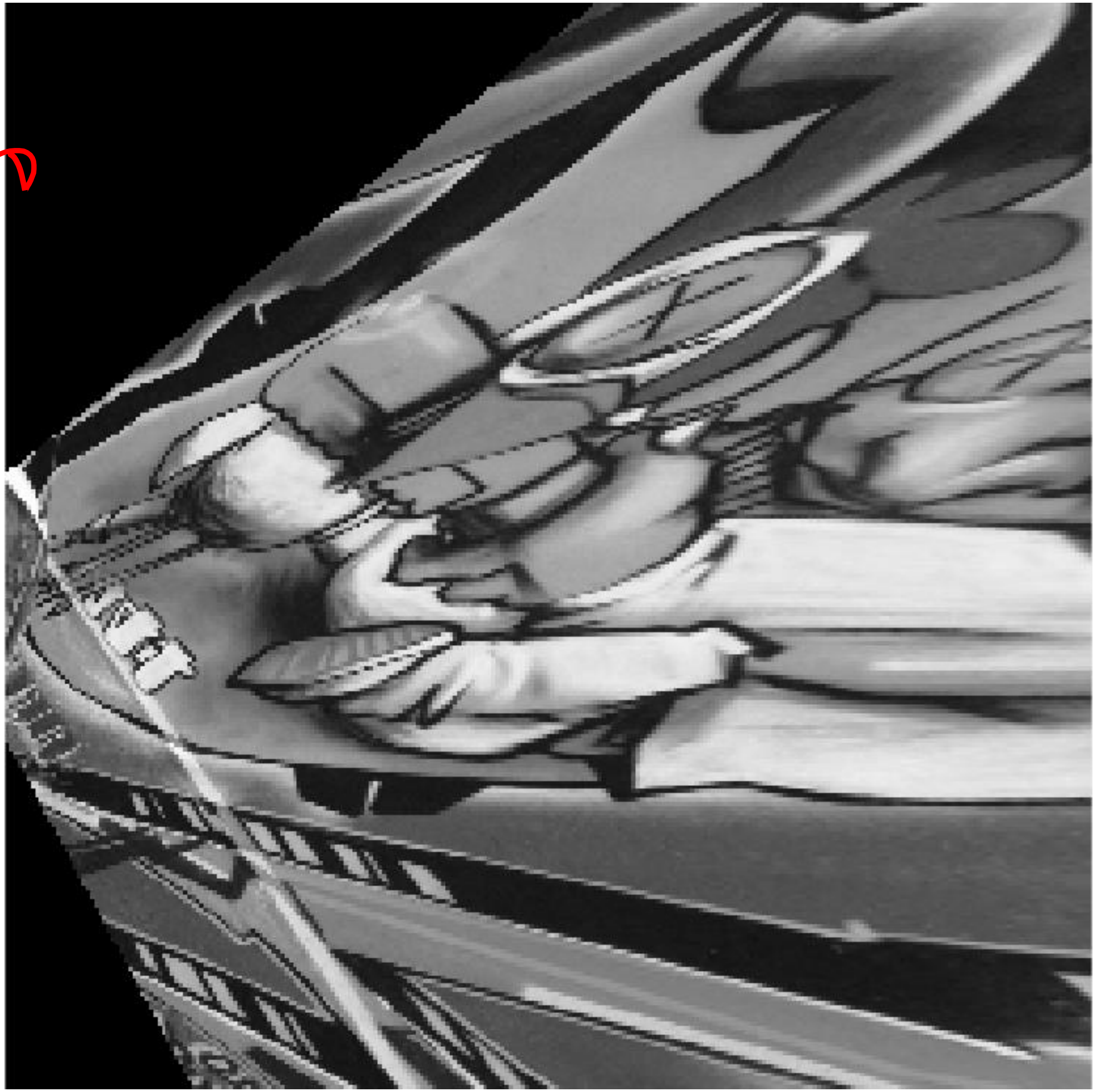
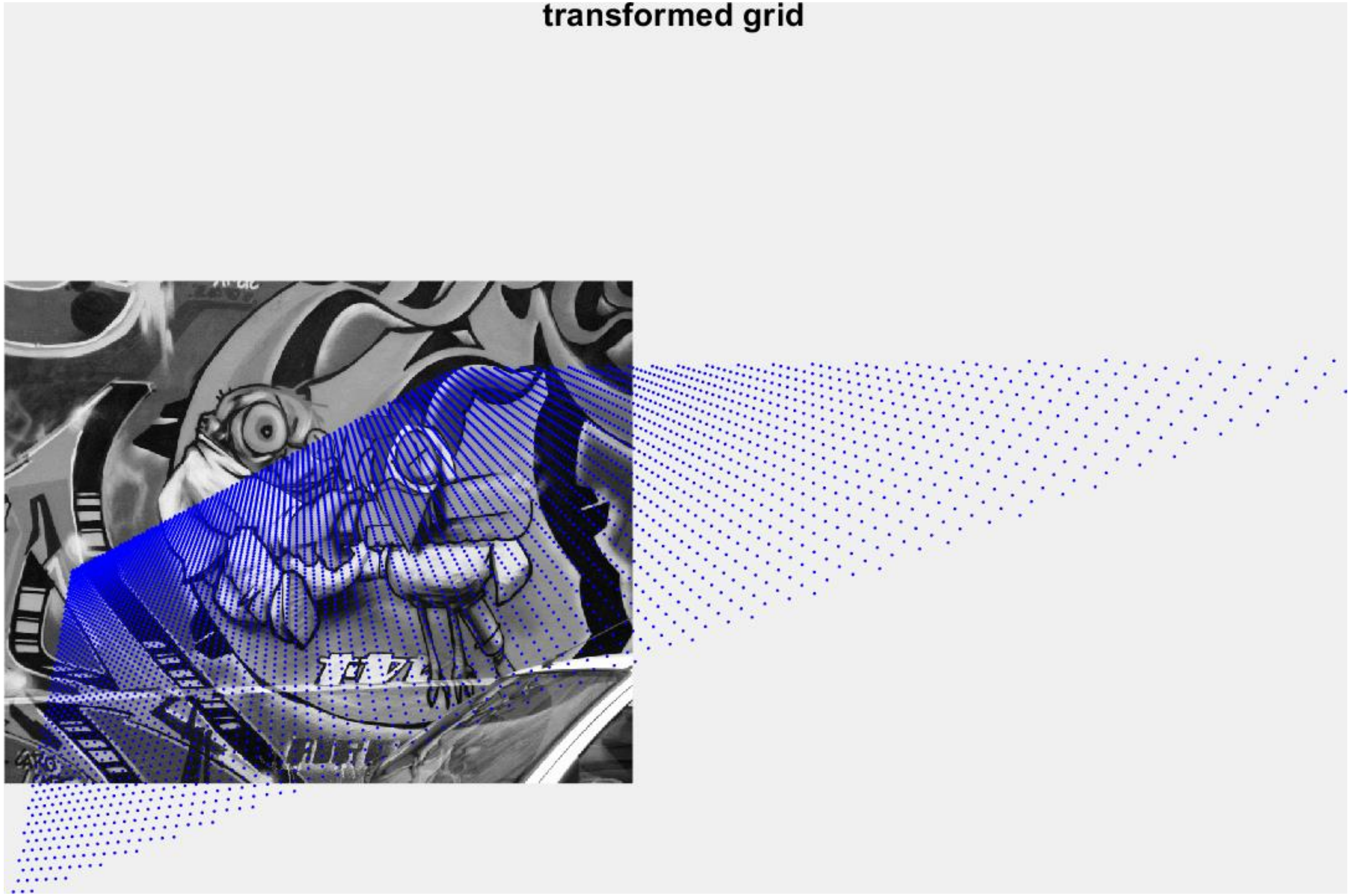
Re-arrange the new estimated values over a regular grid to obtain the transformed image

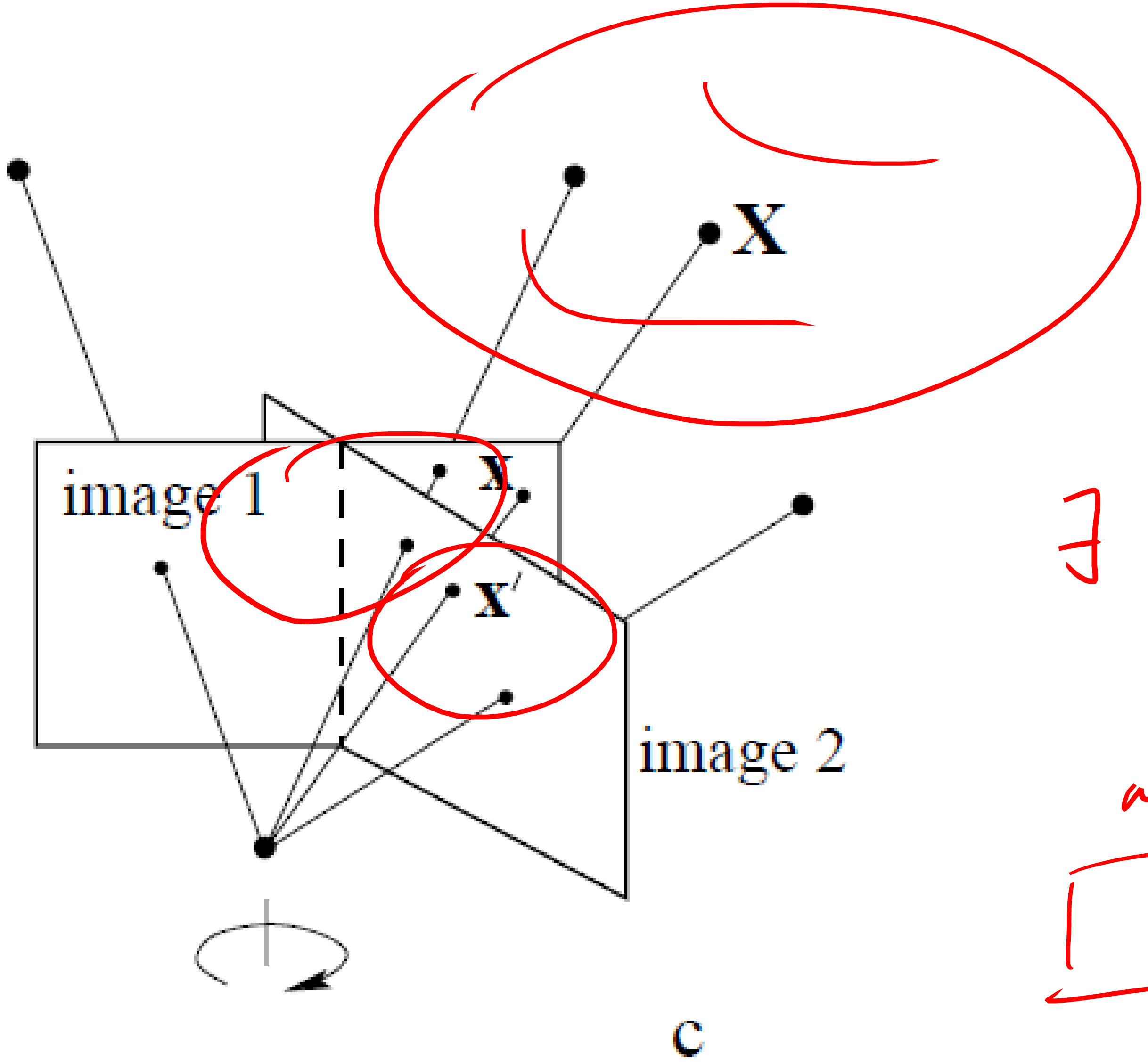


Black pixels are points out of the image grid

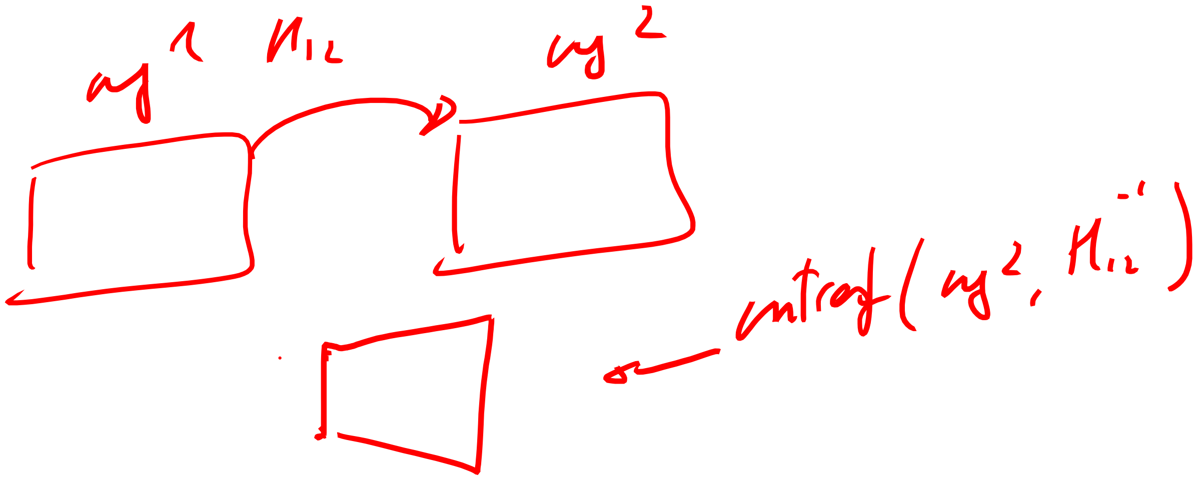


Stitching Example

Other examples where homographies apply



$\exists\, H_{1,2}$ describes point correspondences between img ^{1} & img ^{2}



Hartley Zisserman Fig.2.5