Single-View Geometry

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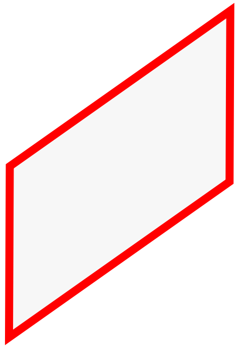
Book: HZ, chapter 2

Points in $\mathbb{P}^2$



In homogeneous coordinates a point on Π corresponds to a triplet

$$\boldsymbol{x} = (x, y, w)^\top$$

We will always consider **column vectors** as these are more convenient in linear algebra operations

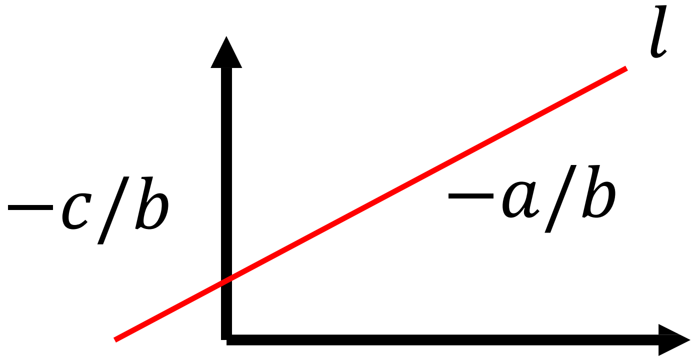
In homogeneous coordinates a point corresponds to a class of equivalence that includes all the scaled versions of $\boldsymbol{x}$

$$\boldsymbol{x} = (x, y, w)^\top = \left(\frac{x}{w}, \frac{y}{w}, 1\right)^\top$$

The latter representation is mapped in $\mathbb{R}^2$ by removing the last coordinate

$$\left(\frac{x}{w}, \frac{y}{w}, 1\right)^\top \rightarrow \left(\frac{x}{w}, \frac{y}{w}\right)^\top \in \mathbb{R}^2$$

Lines in $\mathbb{P}^2$



Thus we can associate a point to each line

$$l \rightarrow [a, b, c]^\top$$

This association is not one-to-one since $\lambda ax + \lambda by + \lambda c = 0$, $\lambda \neq 0$ identifies the same line but with different parameters

$$l \rightarrow [\lambda a, \lambda b, \lambda c]^\top$$

Thus, representations $[\lambda a, \lambda b, \lambda c]^\top$ and $[a, b, c]^\top$ do coincide

Rmk: lines are naturally represented in $\mathbb{P}^2$

Rmk: this is the reason why a line has 3 coefficients but indeed only two degrees of freedom (gradient and intercept)

Rmk: the vector $[0,0,0]^\top$ does not correspond to any line

A model for $\mathbb{P}^2$

Points of $\mathbb{P}^2$ are represented by all the rays of $\mathbb{R}^3$ through the origin, since all

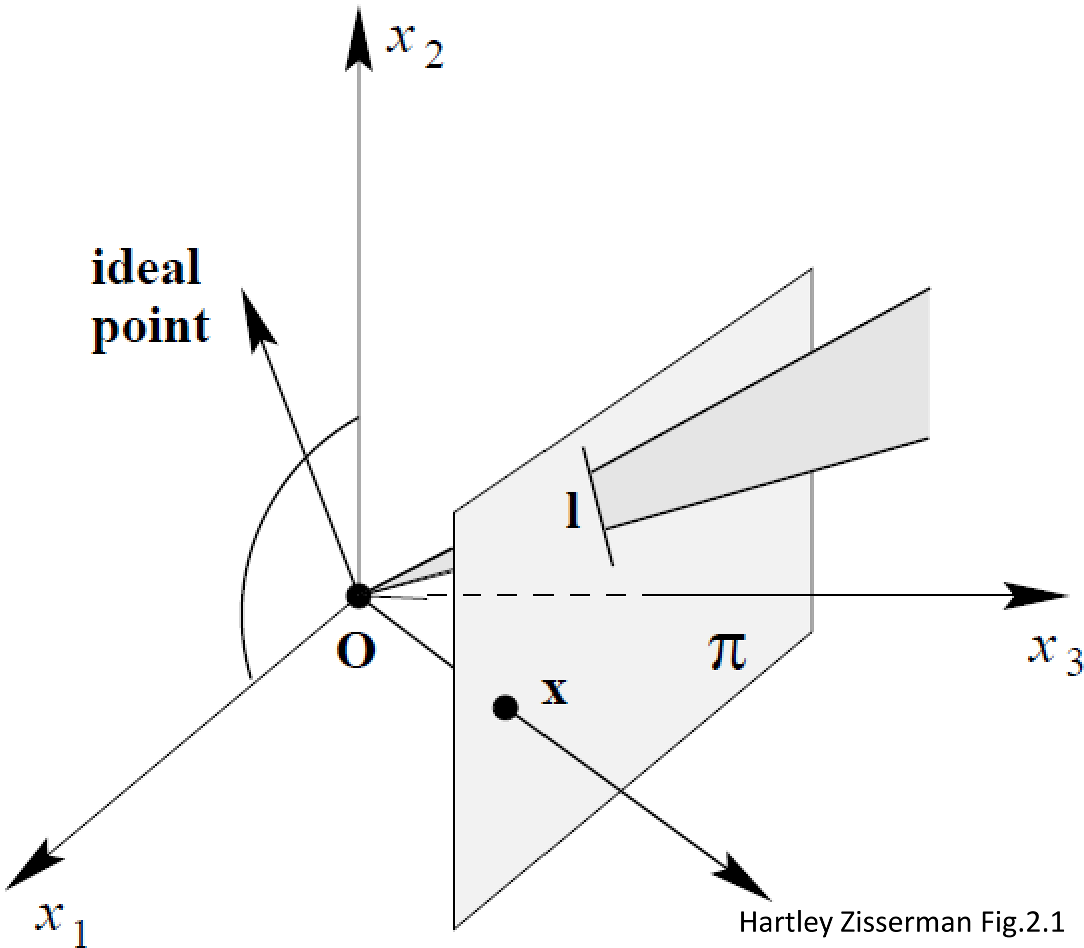
$$\boldsymbol{x} = \lambda[a; b; 1], \forall \lambda$$

corresponds to the same point

Lines of $\mathbb{P}^2$ are represented as planes passing through the origin

The Euclidean plane is the plane $\boldsymbol{x_3} = 1$ and projection to Euclidean coordinates is computing the intersection between the ray and the plane Π, i.e., $\boldsymbol{x_3} = 1$.

Lines lying in the plane $\boldsymbol{x_3} = 0$ represent ideal points (directions) as these do not intersect Π.



Hartley Zisserman Fig.2.1

Incidence relation

A point $\boldsymbol{x} \in \mathbb{R}^2$, $\boldsymbol{x} = (x, y)$ belongs to a line $l = [a, b, c]^\top$ if and only if

$$ax + by + c = 0$$

The above relation can be written as

$$[a, b, c] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

Where $\cdot$ denotes the scalar product. Then

Property (incidence):

A point $\boldsymbol{x} \in \mathbb{P}^2$ lies on the line l if and only if

$$l^\top \boldsymbol{x} = \boldsymbol{x}^\top l = 0$$

Incidence relation

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The above relation can be written as

$$[a, b, c] \cdot \begin{bmatrix} x \\ y \\ \boxed{1} \end{bmatrix} = 0$$

This is the reason why this number i set to 1 when moving from Euclidean to homogeneous coordinates

Where $\cdot$ denotes the scalar product. Then

Property (incidence):

A point $\boldsymbol{x} \in \mathbb{P}^2$ lies on the line l if and only if

$$l^\top \boldsymbol{x} = \boldsymbol{x}^\top l = 0$$

Intersection of lines

Property (intersection of lines):

The intersection of two lines l, m, their intersection $\boldsymbol{x} \in \mathbb{P}^2$ is

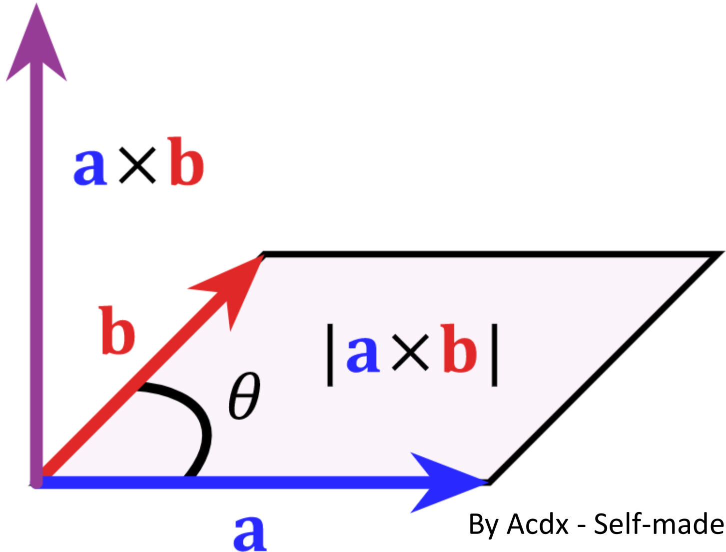
$$\boldsymbol{x} = l \times m$$

where $\times$ denote the **cross product** of two 3-dimensional vectors.

Cross Product

Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ be two vectors, their cross product is a vector $\boldsymbol{a} \times \boldsymbol{b} \in \mathbb{R}^3$

- That is perpendicular to the plane $< \boldsymbol{a}, \boldsymbol{b} >$
- Has orientation of the right-hand rule
- Has lenght proportional to the area of the parallelogram spanned by the vectors, $||\boldsymbol{a} \times \boldsymbol{b}|| = ||\boldsymbol{a}||||\boldsymbol{b}|| \sin \boldsymbol{\theta}$



By Acdx - Self-made

Cross Product

Rmk: the cross product can be also computed as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

being $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ the versors of $\mathbb{R}^3$ and $|\cdot|$ the determinant

Rmk: the cross product is anti-commutative

$$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$$

But this is not an issue when we want to intersect two lines, since the result in the same point of $\mathbb{P}^2$ (equivalence up to a multiplication by -1)

Line joining two points

Property: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{P}^2$, the line l joining $\boldsymbol{x}$ and $\boldsymbol{y}$ is

$$l = \boldsymbol{x} \times \boldsymbol{y}$$

Rmk this can be verified by checking that both $\boldsymbol{x}$ and $\boldsymbol{y}$ belong to $\boldsymbol{x} \times \boldsymbol{y}$ through the incidence equation

Rmk: we have seen that the cross product is anti-commutative

$$\boldsymbol{x} \times \boldsymbol{y} = -\boldsymbol{y} \times \boldsymbol{x}$$

This is not an issue for the resulting lines, since these are intrinsecally equivalent up to a multiplication by a scalar

An interesting property

An interesting property, given two points in homogeneous coordinates

$a = [a_1; a_2; a_3]$ and $b = [b_1; b_2; b_3]$ any linear combination $\lambda a + \mu b$, $\lambda, \mu \in \mathbb{R}$ belongs to the line joining a, b

Proof

Let $l = a \times b$, for which holds $l^\top a = l^\top b = 0$ (incidence equation)

Then, $(\lambda a + \mu b) l^\top = \lambda a l^\top + \mu b l^\top = 0 + 0$

Angles in the Euclidean Plane

In **Euclidean geometry** the angle between two lines is computed from the dot product of their normals. For the lines $l = (l_1; l_2; l_3)$, and $m = (m_1; m_2; m_3)$, the angle θ is such that

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

We will see later interesting properties on projective plane

Matlab Example

Select (a,b,c,d) such that ab and cd are parallel

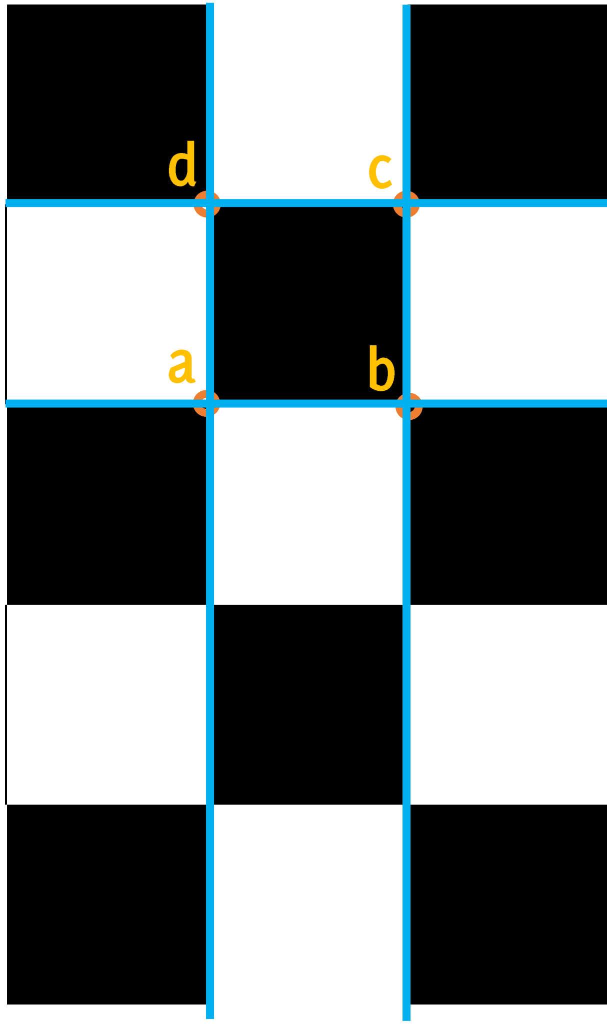
Compute the equations of lines lab, lcd

Check the incidence relation

Define the equations of the first / last row and column

Intersect lab and lcd with the above lines

Compute the angle between these lines



Intersection of parallel Lines

Consider two parallel lines $l = (a, b, c)$ and $m = (a, b, d)$, then their intersection is:

$$l \times m = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a & b & c \\ a & b & d \end{vmatrix} = \boldsymbol{i}(bd - bc) - \boldsymbol{j}(ad - ac) + \boldsymbol{k}(ab - ab)$$

$$l \times m = \begin{bmatrix} bd - bc \\ ac - ad \\ 0 \end{bmatrix} = (d - c) \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

That corresponds to the homogeneous point $\boldsymbol{x} = [b, -a, 0]^\top$

Rmk if we try to move to Euclidean coordinates we get $\left(\frac{b}{0}, \frac{a}{0}\right)$ which goes to infinity.

These are the **ideal points or points at the infinity**

Point at the Infinity

Definition: a point of $\mathbb{P}^2$ with the third coordinate equal zero is a **point at the infinity** (or ideal point)

Rmk: $l = [a, b, c]^\intercal$ passes through the ideal point $l_\infty = [b, -a, 0]^\intercal$

Rmk: ideal points can be seen as sort of *directions*

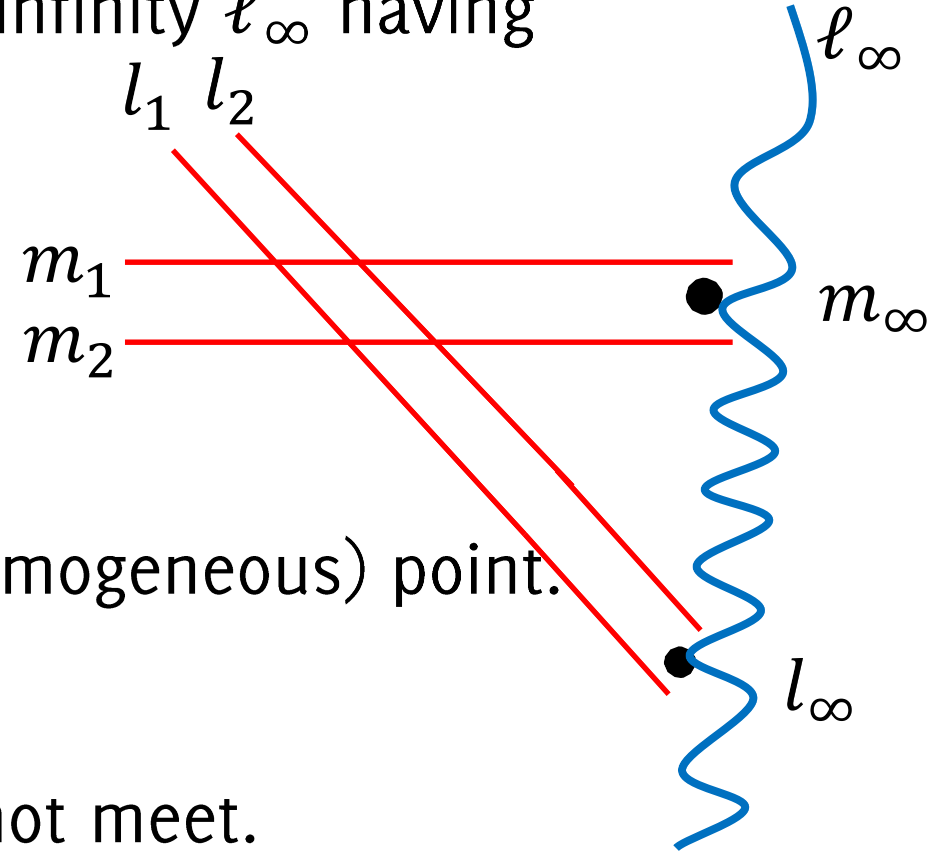
Rmk: $\mathbb{P}^2$ augments $\mathbb{R}^2$ by including *directions*

Rmk: finite points in $\mathbb{P}^2$ are those having the third coordinate $\neq 0$ and these corresponds to $\mathbb{R}^2$ up to a normalization factor

Line at the infinity ℓ_∞

Property: all the ideal points lie in the line at the infinity ℓ_∞ having coordinates

$$\ell_\infty = [0; 0; 1]$$



Rmk: Thanks to ideal points

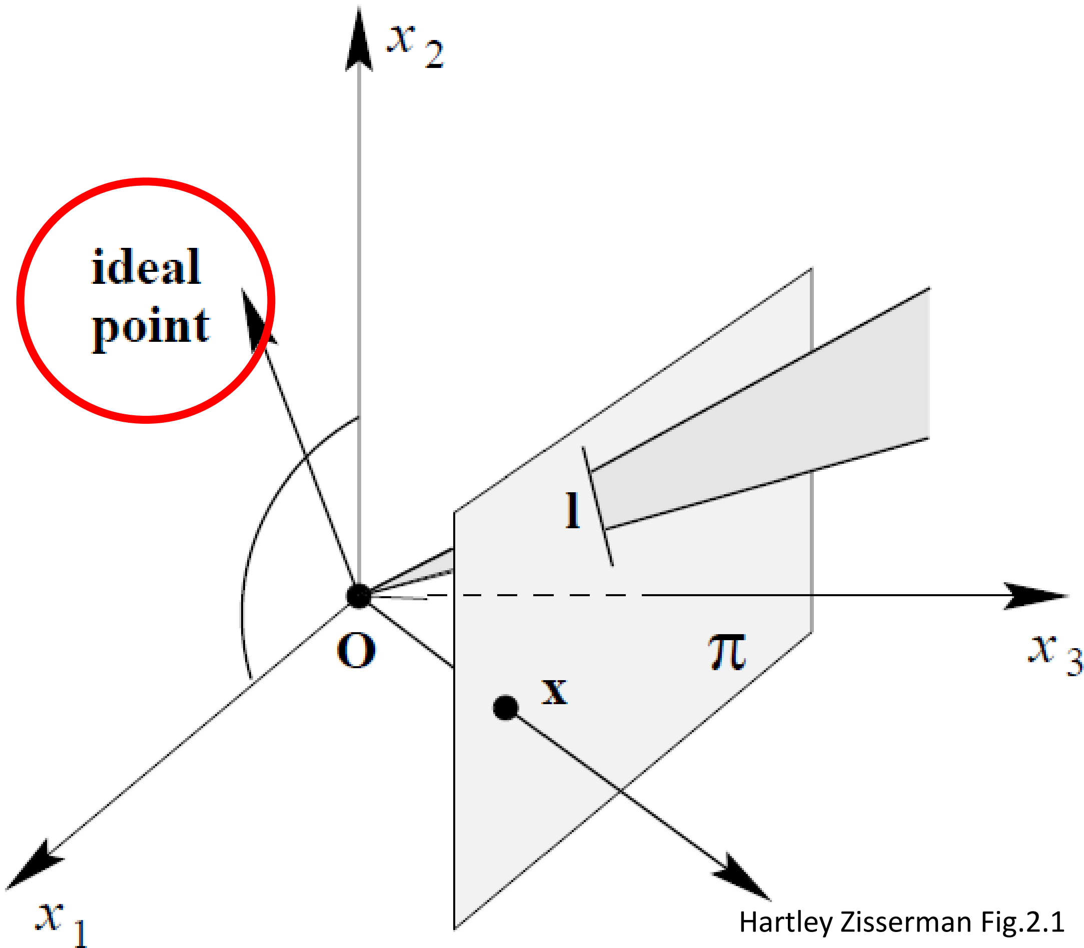
- any pair of lines in $\mathbb{P}^2$ intersect in a single (homogeneous) point.
- any pair of points of $\mathbb{P}^2$ lie a single line

This does not hold in $\mathbb{R}^2$ where parallel lines do not meet.

A model for $\mathbb{P}^2$

Lines lying in the plane $\boldsymbol{x_3} = 0$ represent ideal points (directions) as these do not intersect Π.

These points live in the line at the infinity



Hartley Zisserman Fig.2.1

Example

Isometric cube, draw missing points as the intersection of lines parallel to this and passing through another point

Example

Manually select from this picture three points

Draw all the remaining lines.

This is an orthographic image:
- Lines that are expected to be parallel in the 3D world, are also parelle in the image

Example

Manually select from this picture three points

Draw all the remaining lines.

This is an orthographic image:
- Lines that are expected to be parallel in the 3D world, are also parelle in the image

Point – Line Duality in $\mathbb{P}^2$

Duality principle. *To any theorem of 2-dimensional projective geometry* $(\mathbb{P}^2)$ *there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem.*

Theorem		Dual Theorem
- Point	$\rightarrow$	- Line
- Line	$\rightarrow$	- Point
- Belongs to	$\rightarrow$	- Go through
- Go through	$\rightarrow$	- Belongs to

E.g. The incidence equation, the line passing through two points which has the same formulation of the intersection between lines

Rmk: Note that is it not necessary to prove the dual of a given theorem once the original theorem has been proved

Linear Combinations in $\mathbb{P}^2$

An interesting property, given two points in homogeneous coordinates

$a = [a_1; a_2; a_3]$ and $b = [b_1; b_2; b_3]$ any linear combination $\lambda a + \mu b$, $\lambda, \mu \in \mathbb{R}$ belongs to the line joining a, b

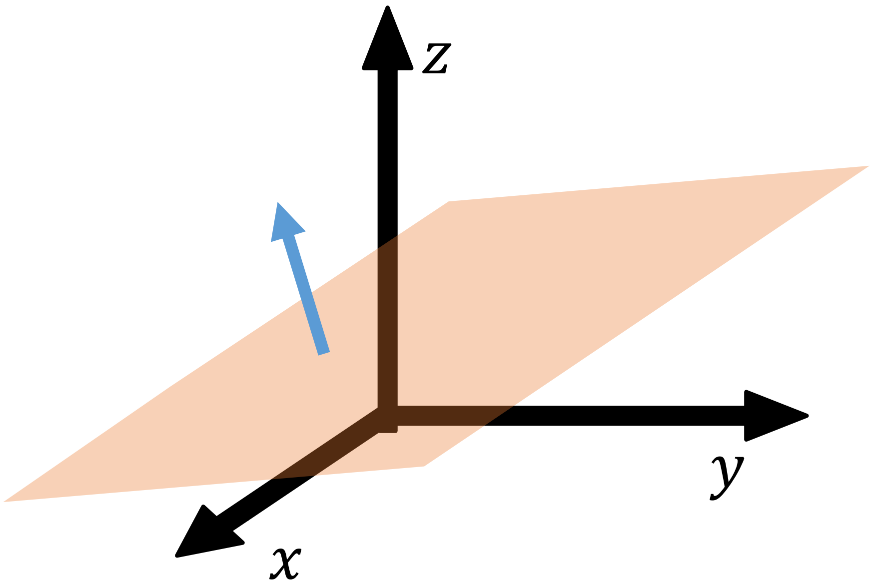
Proof

Let $l = a \times b$, for which holds $l'a = l'b = 0$ (incidence equation)

Then, $(\lambda a + \mu b)l' = \lambda a l' + \mu b l' = 0 + 0$

The Projective Space $\mathbb{P}^3$

Points and Planes in $\mathbb{P}^3$



A point in $\mathbb{P}^3$ is defined as

$$X = [x; y; z; 1]$$

The incidence equation for a plane (i.e. $\boldsymbol{x} \in \pi$) is

$$ax + by + cz + d = 0$$

Which can be written in matrix form as

$$[a, b, c, d] \cdot X = 0$$

And this implies that the plane is identified by a vector in $\mathbb{P}^3$

$$\pi = [a; b; c; d]$$

Rmk planes have 3 degrees of freedom, since their equation holds up to a scaling of a parameter

Rmk planes in $\mathbb{P}^3$ play the same role as lines in $\mathbb{P}^2$

The Plane at infinity

The plane at infinity has the canonical position

$$\pi_\infty = [0; 0; 0; 1]$$

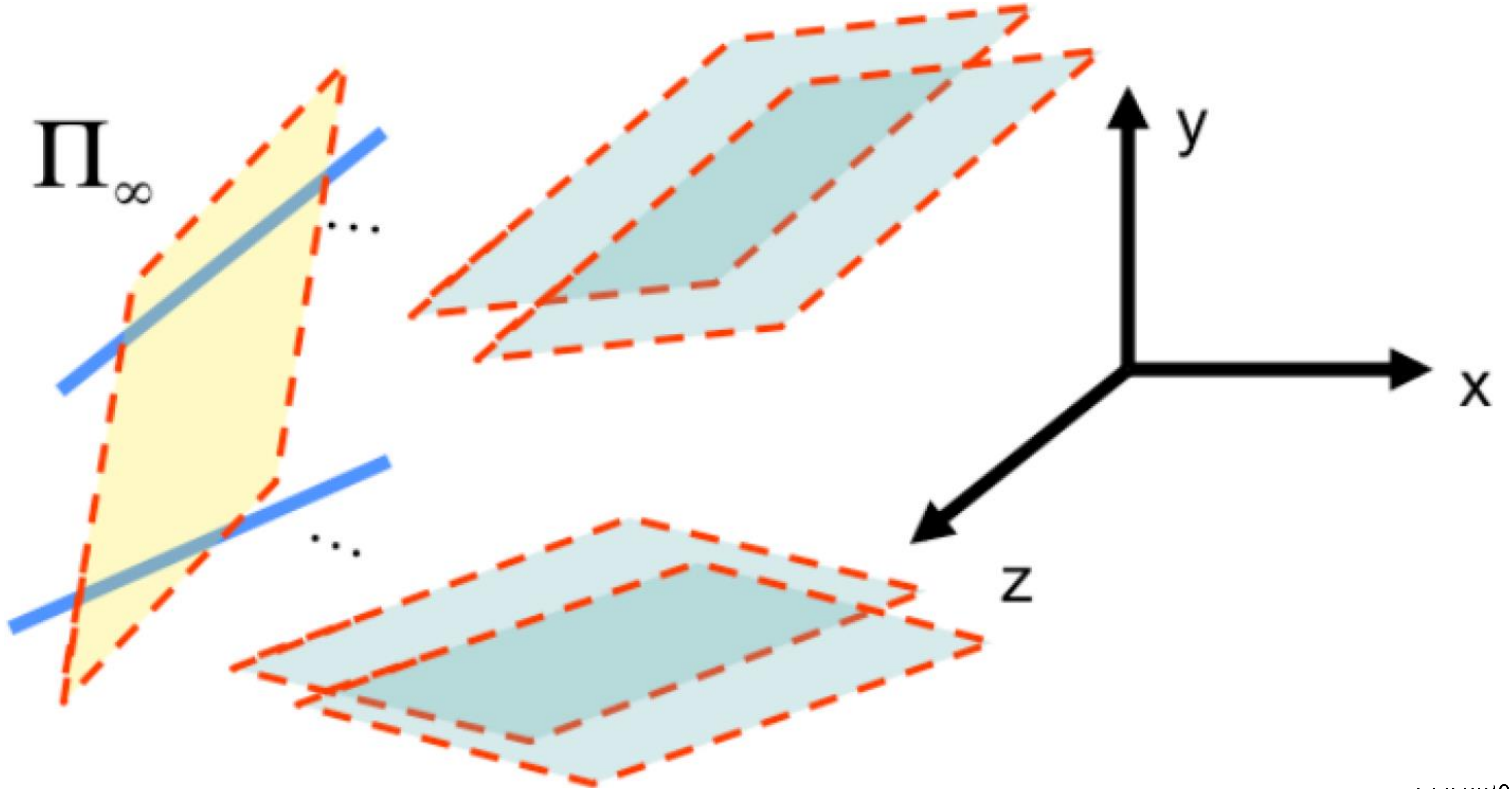
in affine 3D-space.

π_∞contains ideal points (directions) $P_\infty = [x; y; z; 0]$, and enables the identification of affine properties such as parallelism.

In particular:

- Two planes are parallel if and only if their line of intersection is on $\boldsymbol{\pi}_\infty$
- A line is parallel to another line, or to a plane, if their point of intersection is on $\boldsymbol{\pi}_\infty$

The Plane at infinity

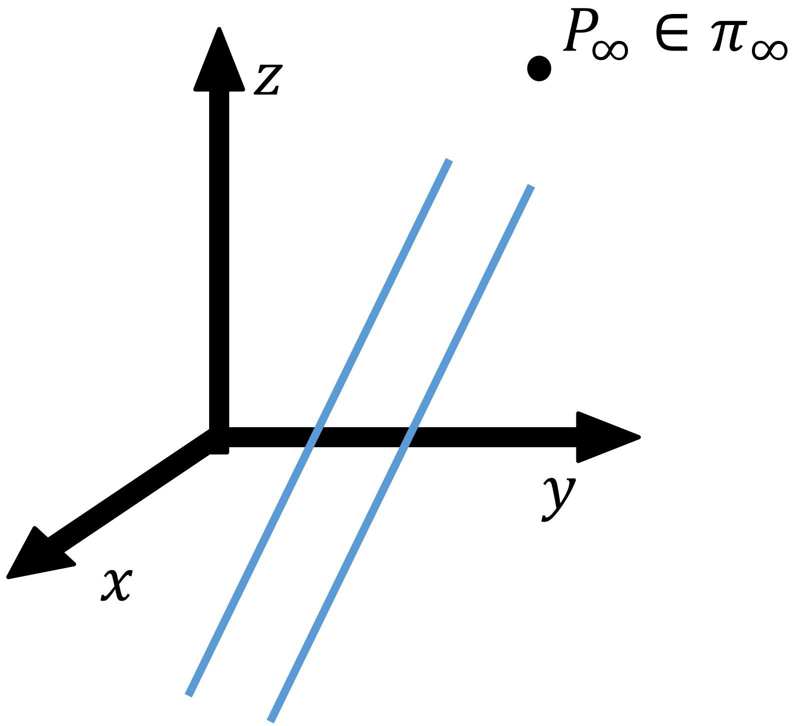


Ideal Points in $\mathbb{P}^3$

Ideal points in $\mathbb{P}^3$ are defined similarly to ideal points in $\mathbb{P}^2$

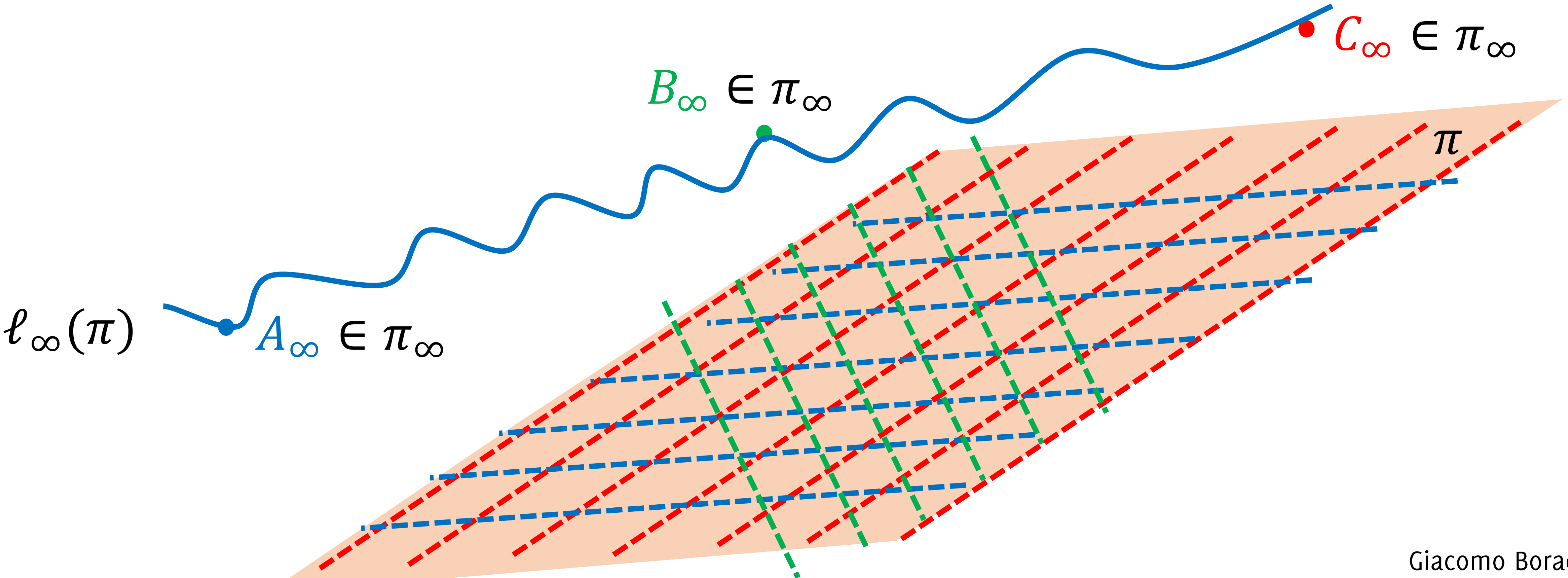
Ideal points are the intersection of parallel lines.

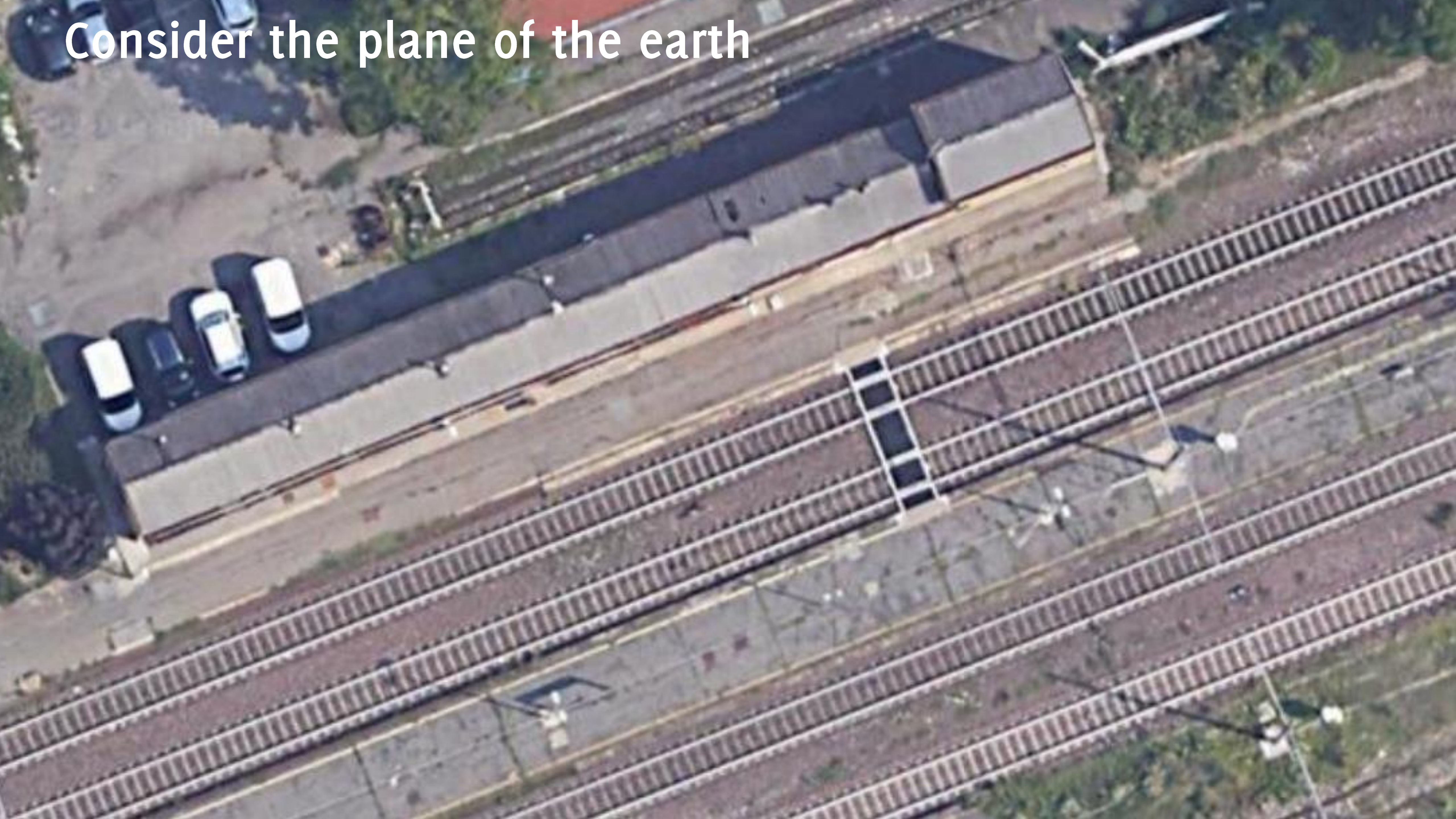
- All the parallel lines in the space intersect in the same ideal point P_∞

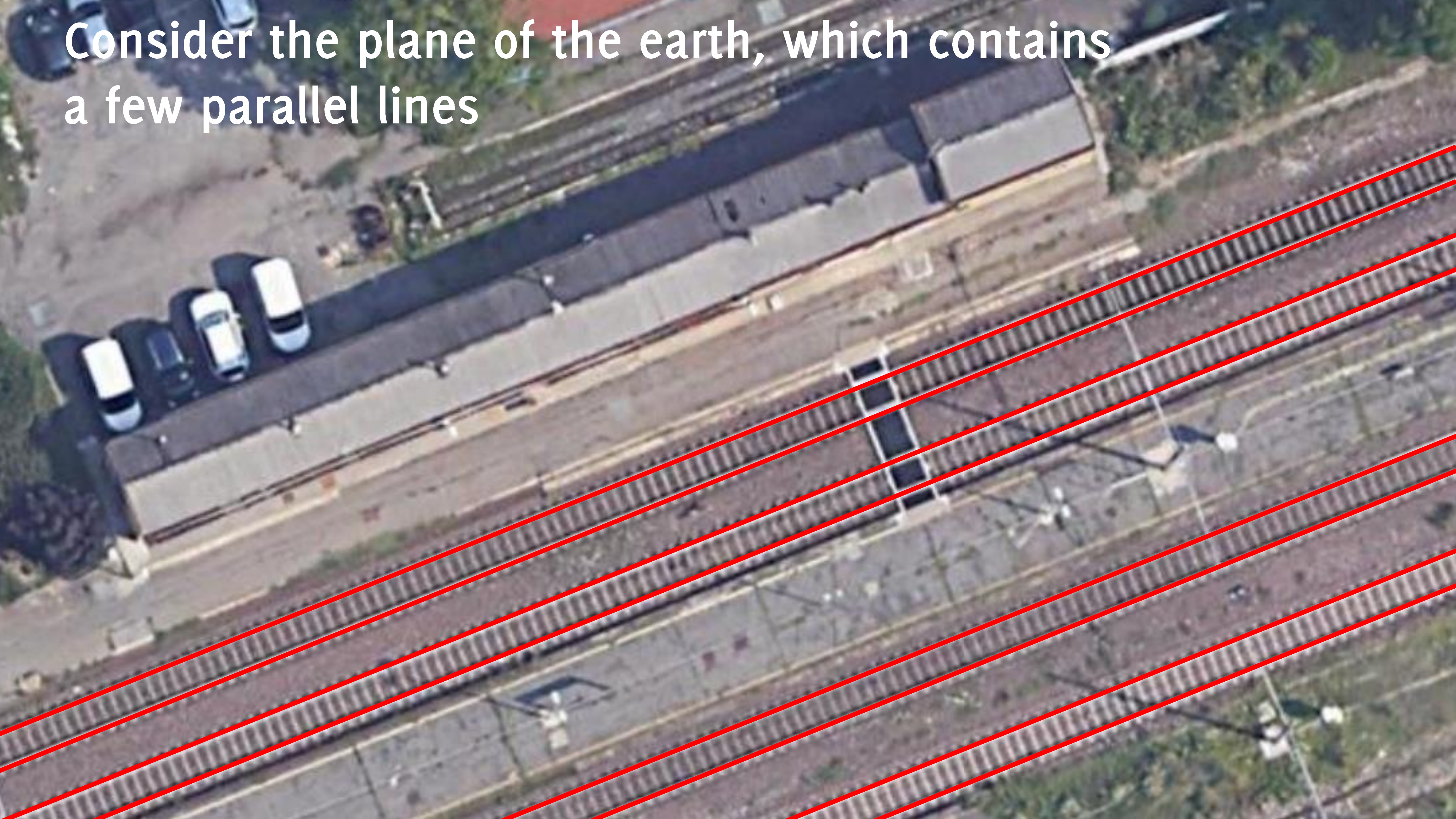


Planes in $\mathbb{P}^3$ and ideal lines

- A plane π contains infinite parallel lines along infinite directions
- Each set of parallel lines intersect in an ideal point (belonging to π_∞)
- All these ideal points lie on the ideal line $\ell_\infty(\pi)$: the line at infinity of π

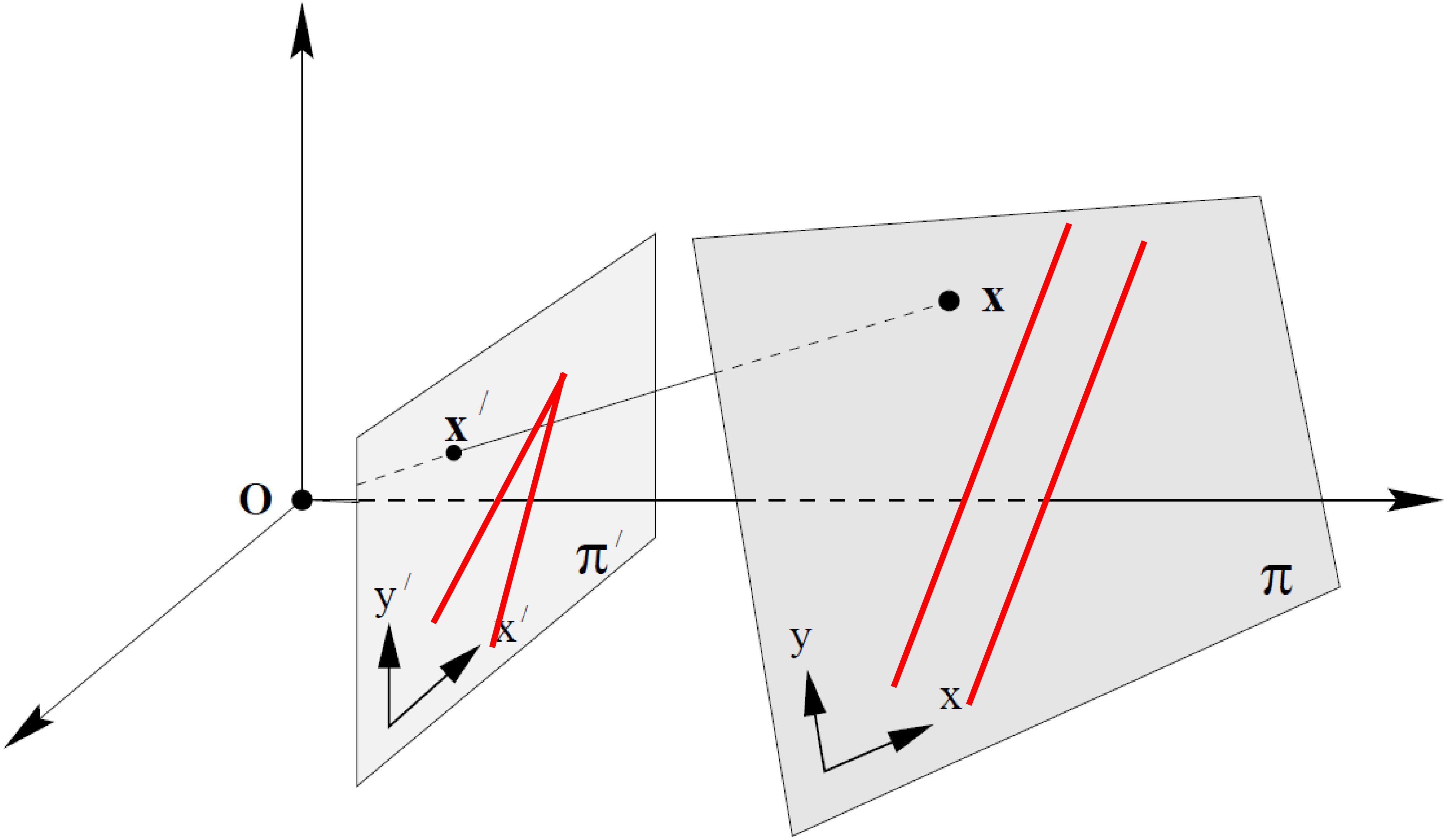



Consider the plane of the earth


Consider the plane of the earth, which contains a few parallel lines


These are not anymore parallel in a photo...
T

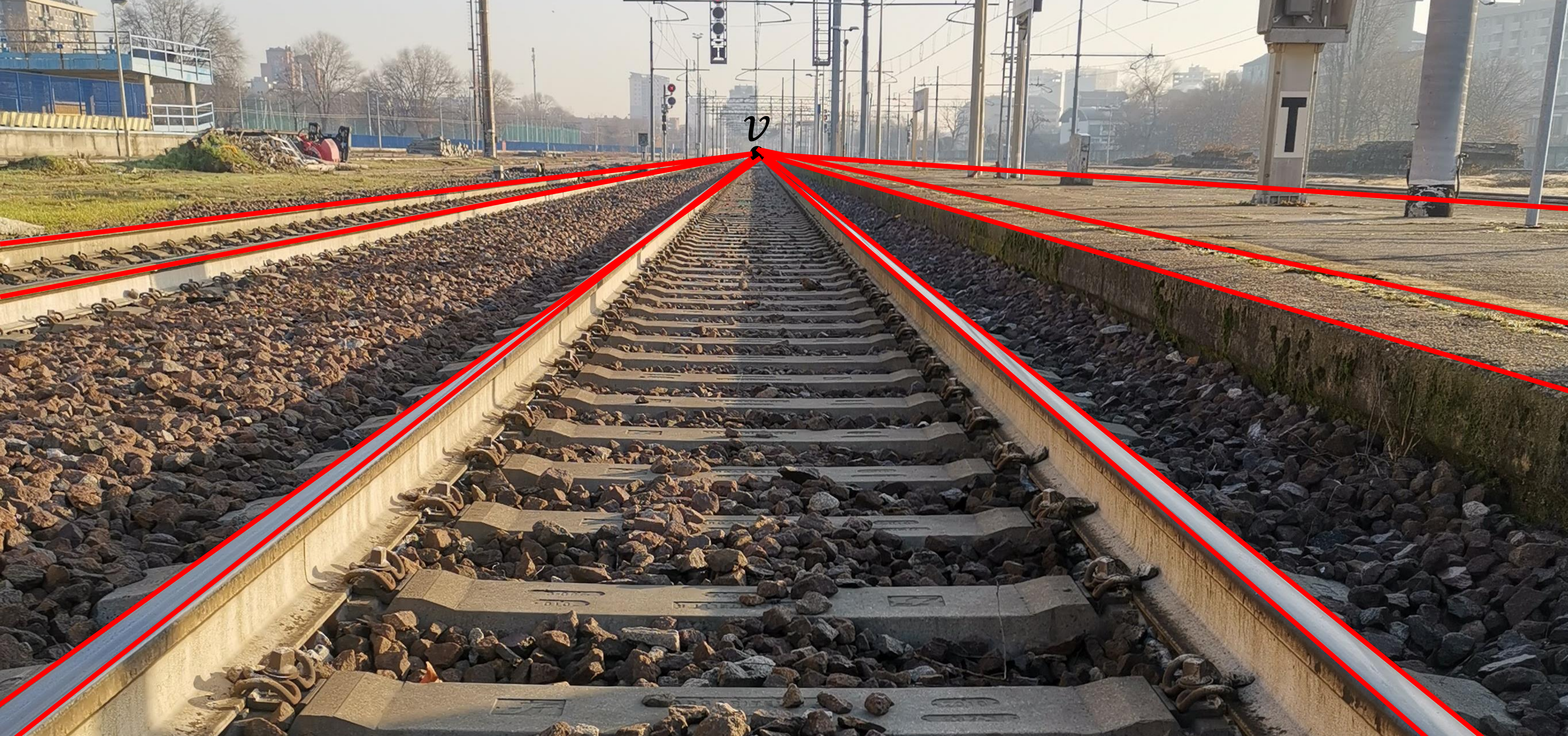
Homography is the mapping among two planes



Hartley Zisserman Fig.

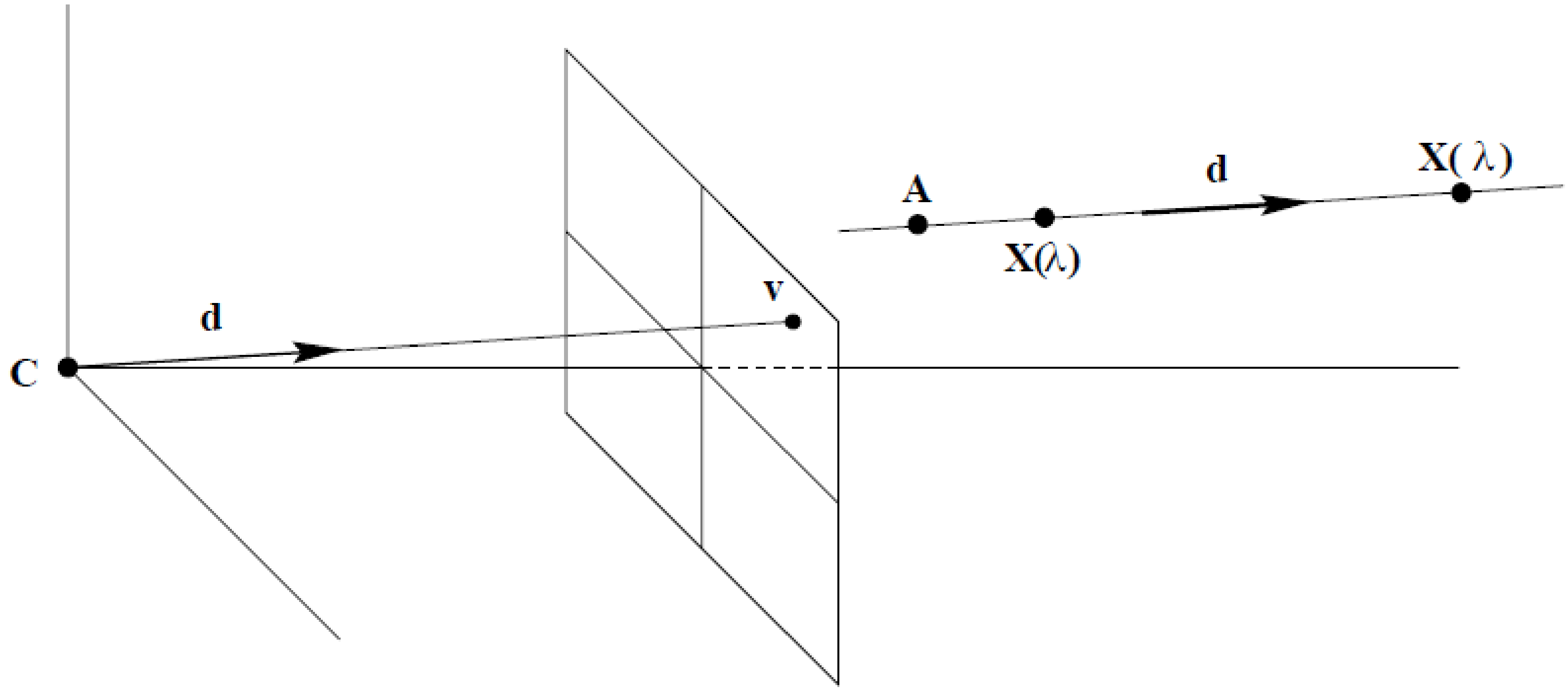
Vanishing point

A set of parallel lines in the world plane intersect in a point in the image, the vanishing point



Vanishing points

Vanishing points are the projection in the image plane of ideal points in the 3D world



Hartley Zisserman Fig.8.14

Horizon

Horizon or Vanishing line for a plane π is the image of the line at the infinity of that plane, $\ell_\infty(\pi)$

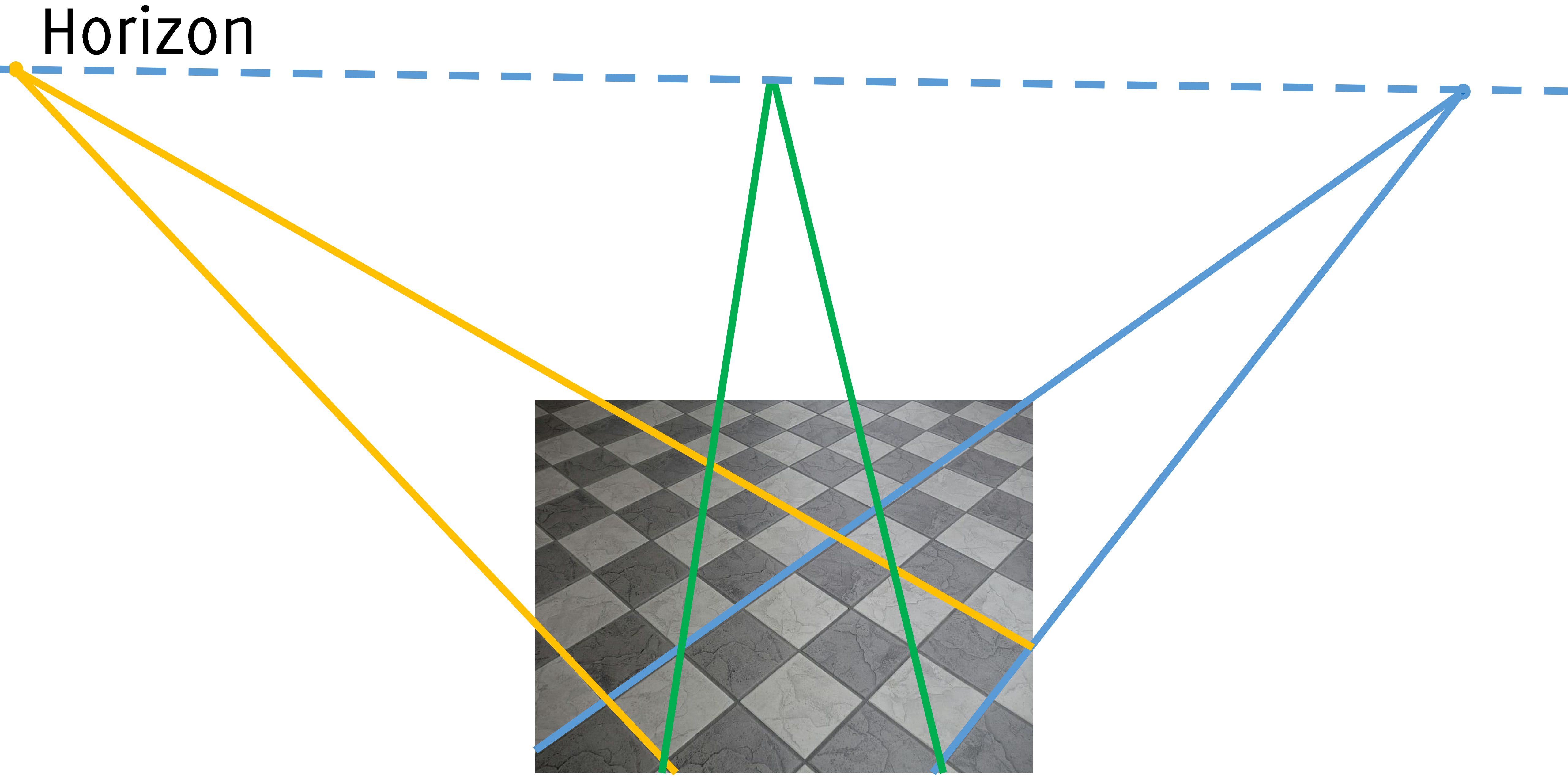
Horizon helps humans to intuitively deduce properties about the image that might not be apparent mathematically.

We can understand when two lines are parallel in the 3D world, since they intersect with the horizon

Example

Select 4 points from this image and draw the horizon

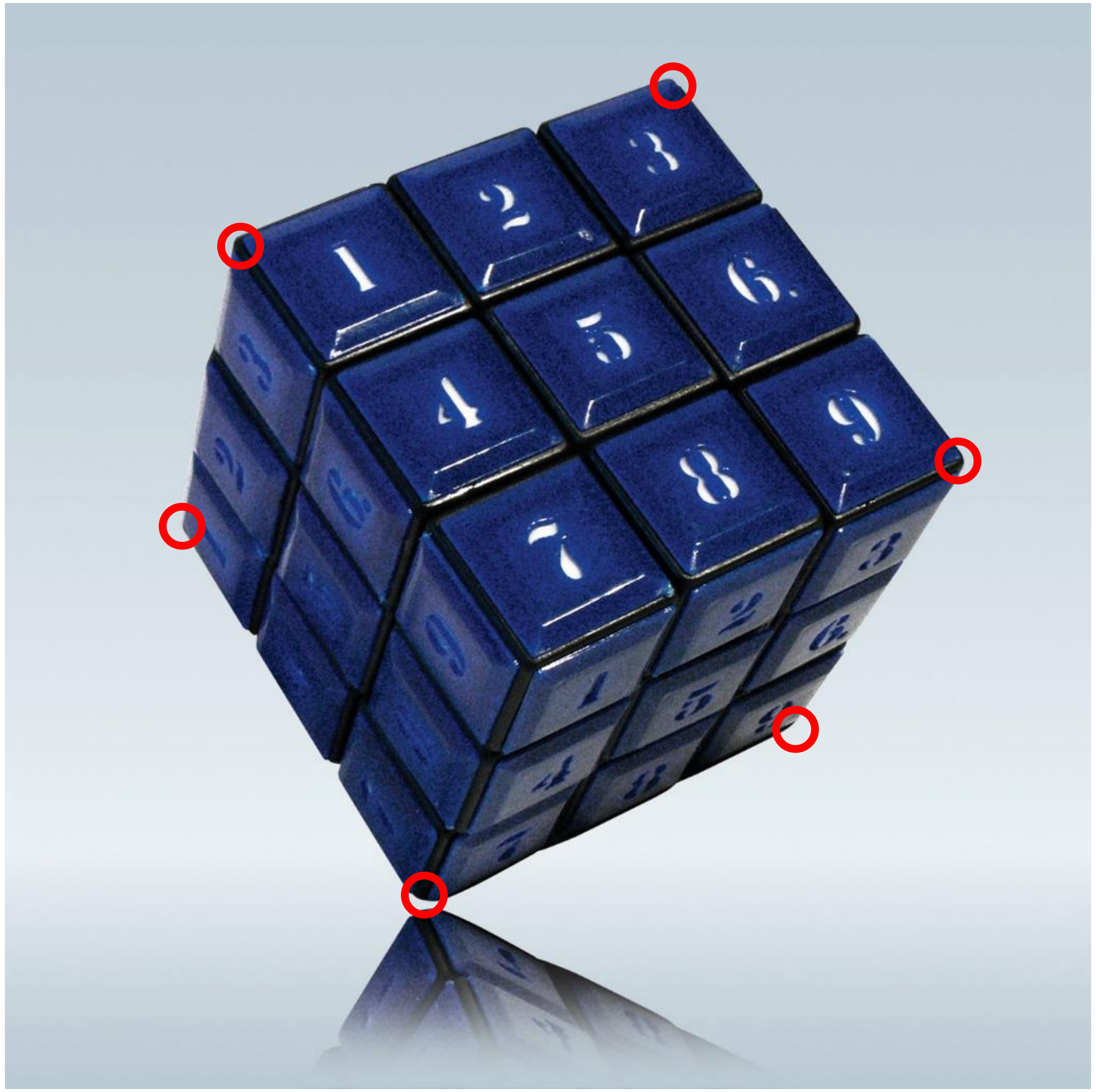
Check that any pair of parallel lines intersect at the horizon


Horizon

Example

Draw all the vertices and edges of this cube, by selecting only 6 the points reported in the figure

Hint: remember that vanishing points are the image of **3D directions**



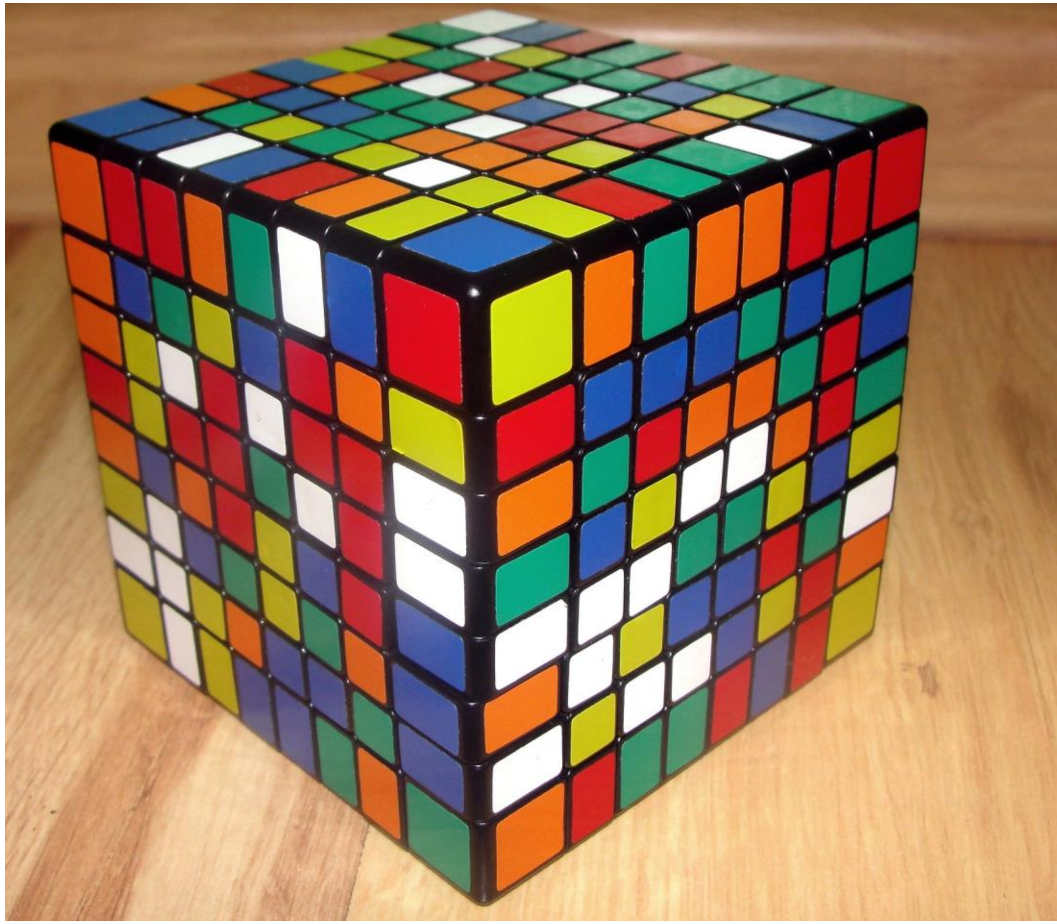
Example

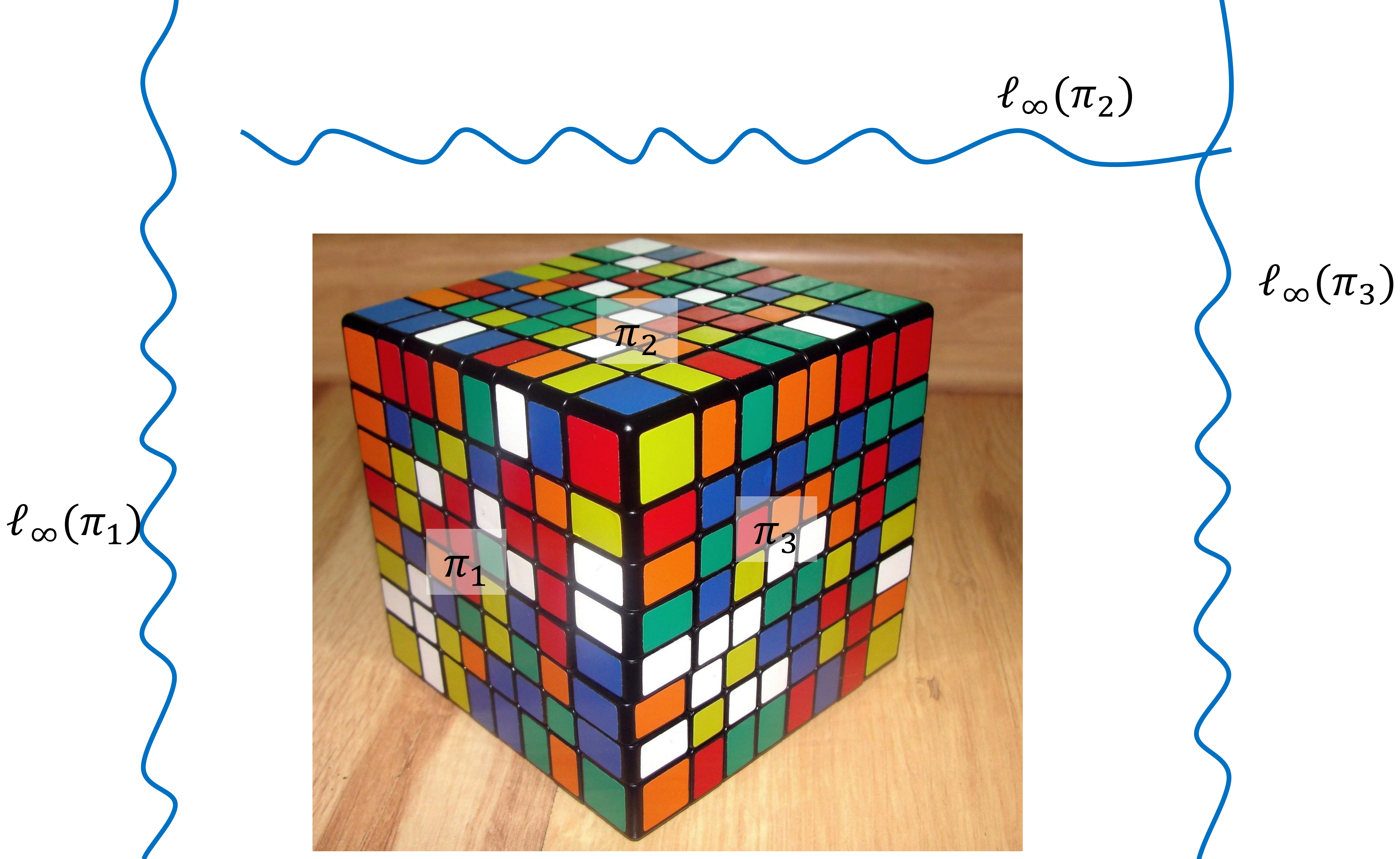
Draw all the vertices and edges of this cube, by selecting only 6 the points reported in the figure

Hint: remember that vanishing points are the image of **3D directions**

Remember!

- The horizon line holds for a single plane of the 3D world
- When the image contains multiple planes, there is an horizon line for each of these
- Ideal points of lines in a plane
  are often not the same of lines on other planes
- Nevertheless, lines having the same 3D direction intersect in the very same vanishing point




$\ell_\infty(\pi_2)$
$\ell_\infty(\pi_3)$
$\ell_\infty(\pi_1)$
π_2
π_3
π_1

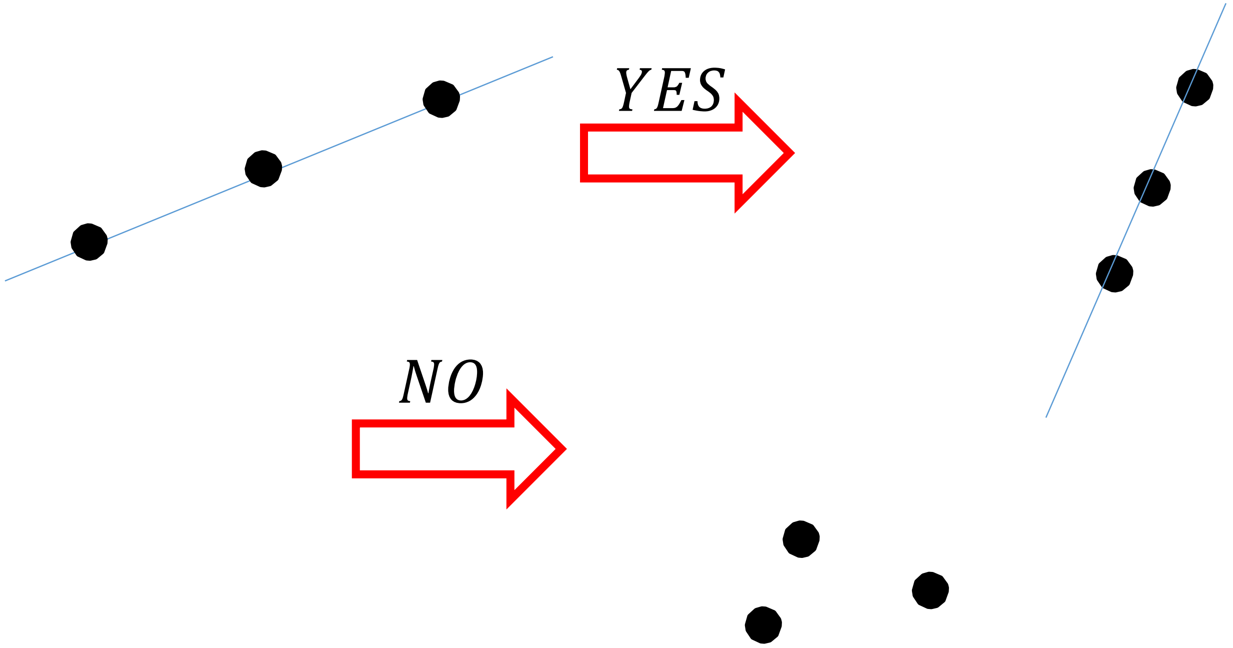
Transformations in $\mathbb{P}^2$

Homographies

The most general transformation we consider in $\mathbb{P}^2$

Definition A *projectivity* is an invertible mapping h from $\mathbb{P}^2$ to itself such that three points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ lie on the same line if and only if $h(\boldsymbol{x}_1)$, $h(\boldsymbol{x}_2)$ and $h(\boldsymbol{x}_3)$ do.

Rmk this property is called **collinearity**



Alternative names

- *Collineation*
- *Projective transformation*
- *Homography*

Homographies

Theorem 2.10. *A mapping* $h: \mathbb{P}^2 \to \mathbb{P}^2$ *is a projectivity if and only if there exists a non-singular* 3×3 *matrix* H *such that for any point in* $\mathbb{P}^2$ *represented by a vector* $\boldsymbol{x}$ *it is true that* $h(\boldsymbol{x}) = H\boldsymbol{x}$

$$H = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{bmatrix}$$

Each and every linear mapping in $\mathbb{P}^2$ is an homography, and only linear mapping are homographies

Homographies and Points in $\mathbb{P}^2$

From theorem follows

$$\boldsymbol{x}' = H\boldsymbol{x}$$

Rmk: if we scale both $\boldsymbol{x}'$ and $\boldsymbol{x}$ by arbitrary factors the relation holds since we are in $\mathbb{P}^2$

$$\boldsymbol{x}' = \lambda H\boldsymbol{x}, \qquad \forall \lambda \in \mathbb{R}\backslash\{0\}$$

Thus, H has 9 entries but only 8 degrees of freedom, since only the ratio between the elements counts. H is said to be an homogeneous matrix.

Homographies and Lines in $\mathbb{P}^2$

An homography transform **each line** $\boldsymbol{l}$ in a line m such that:

$$m = (H^{-1})^{\top} \; l$$

We say that points transform *contravariantly* and lines and conics transform *covariantly*.

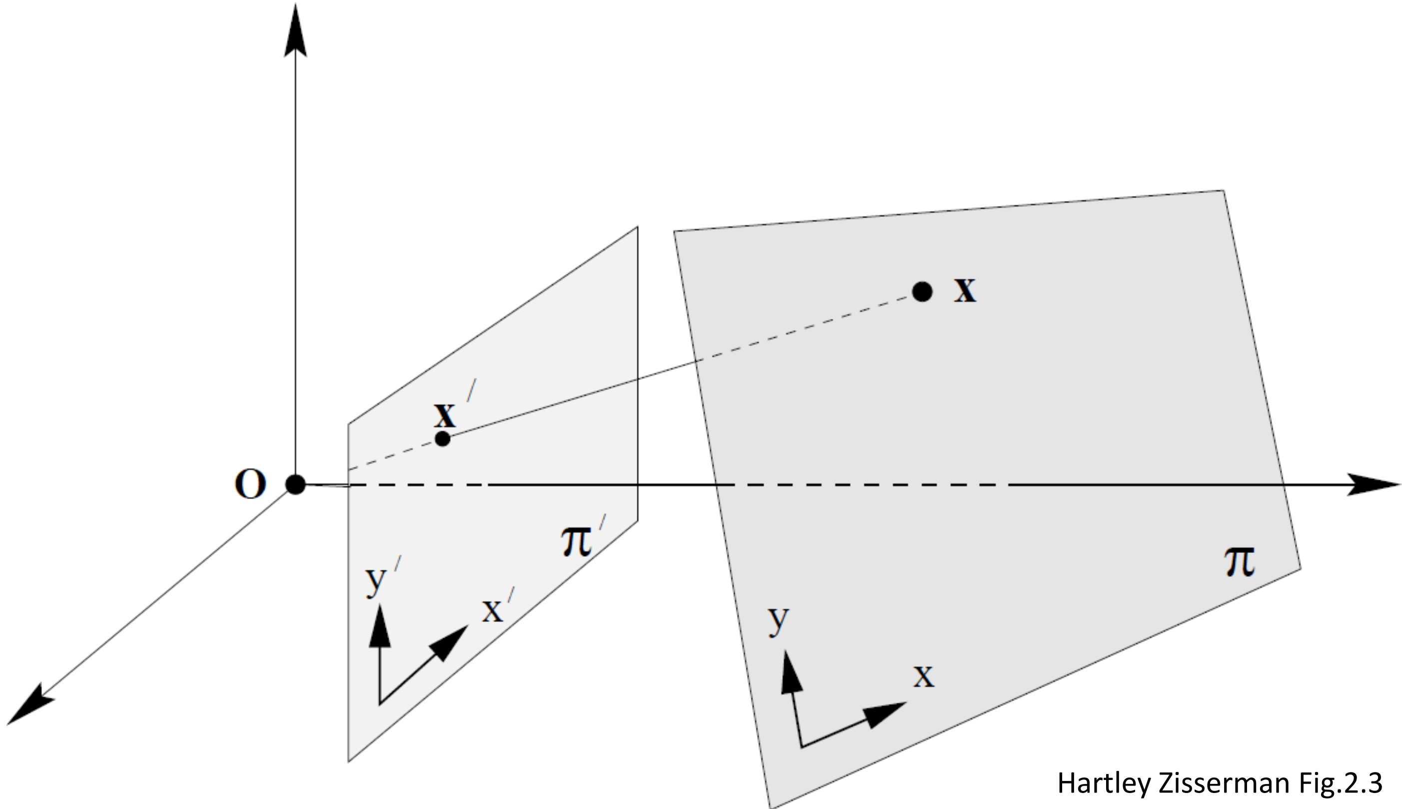
Mapping between planes: homography

Mapping between planes induced by a central projection is an homography, as this preserves collinearity

If a coordinate system is defined in each plane and points are represented in homogeneous coordinates, then the *central projection* mapping may be expressed by

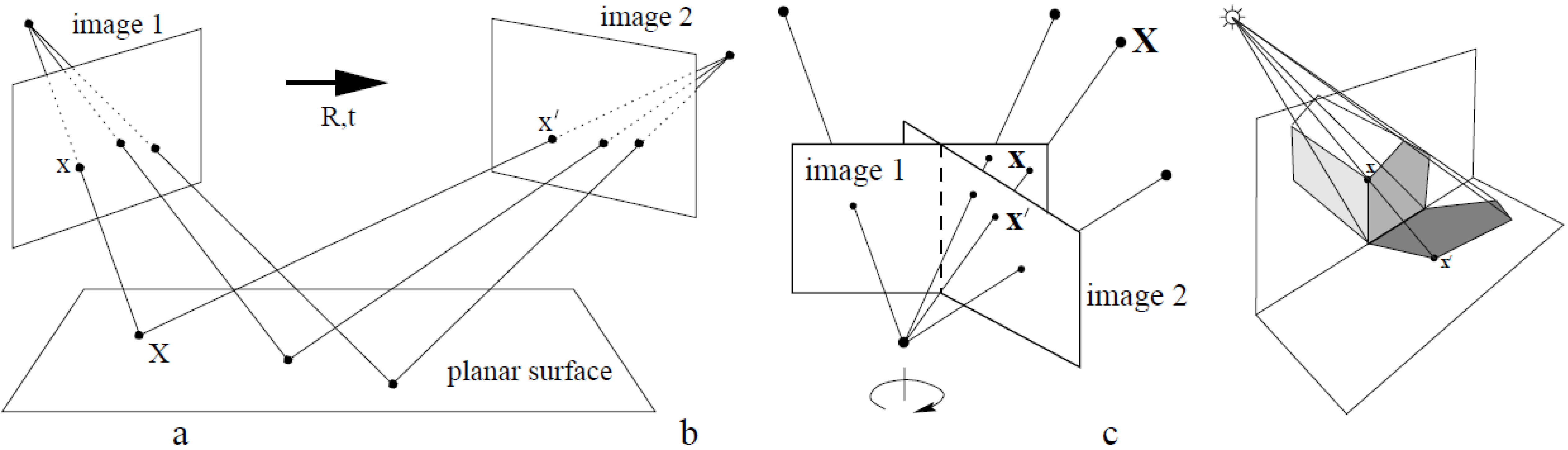
$$\boldsymbol{x}' = H\boldsymbol{x}$$

where H is a non-singular 3×3 matrix



Hartley Zisserman Fig.2.3

Other examples where homographies apply



Hartley Zisserman Fig.2.5

Homographies: Properties

Similarities have **eight degrees of freedom**: the four terms of A and the translation vector $[t_x, t_y]$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{t} \\ \boxed{\boldsymbol{v}^\top} & v \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rmk Homographies can be estimated from four point correspondences

Rmk $\ell_\infty \neq H_P^{-\top} \ell_\infty$ when $\boldsymbol{v} \neq \mathbf{0}$, thus ideal points might become finite points. For instance given $\boldsymbol{x} = (x_1;\ x_2;\ 0)$, then

$$H_P \boldsymbol{x} = \begin{bmatrix} \cdots \\ \cdots \\ v_1 x + v_2 y + v0 \end{bmatrix}$$

Homographies: Properties

Under a projective transformation ideal points may be mapped to finite points, and consequently ℓ_∞ is mapped to a finite line.

However, if the transformation is an affinity ($\boldsymbol{v} = [0, 0]$), then ℓ_∞ is not mapped to a finite line, but remains at infinity.

The image of l_∞ is

$$(H^{-1})^\intercal \ l_\infty$$

$$\begin{bmatrix} A^{-\intercal} & \boldsymbol{v} \\ \boxed{\boldsymbol{t}^{-\intercal} A^{-\intercal}} & v \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v \end{bmatrix}$$

Affine Trasformation and Line at Infinity

Rmk: l_∞ is not **fixed pointwise** under an affine transformation

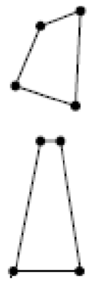
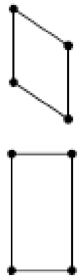
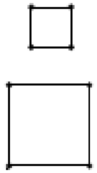
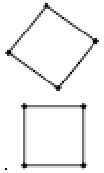
Under an affinity a point on l_∞ (i.e., an ideal point) can be mapped to a different point on l_∞. This is the reason why orthogonality is lost.

Affine Rectification

a b c

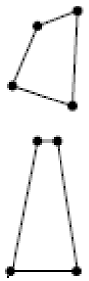
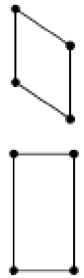
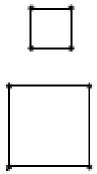
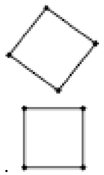
Hartley Zisserman Fig.2.6

Summarizing

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Hartley Zisserman Tab 2.1

Summarizing

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Pa … on co … n- ea Tl
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Transformations higher in the table can produce all the actions of the ones below

Hartley Zisserman Tab 2.1

Affine Rectification

Idea:

- Identify the ℓ_∞ in the image
- transform the identified ℓ_∞ to its canonical position of $[0; 0; 1]$

Let $l = (l_1; l_2; l_3)$ be the image of the line at the infinity with $l_3 \neq 0$,

A suitable homography which maps l back to $\ell_\infty = [0; 0; 1]$ is

$$H = H_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

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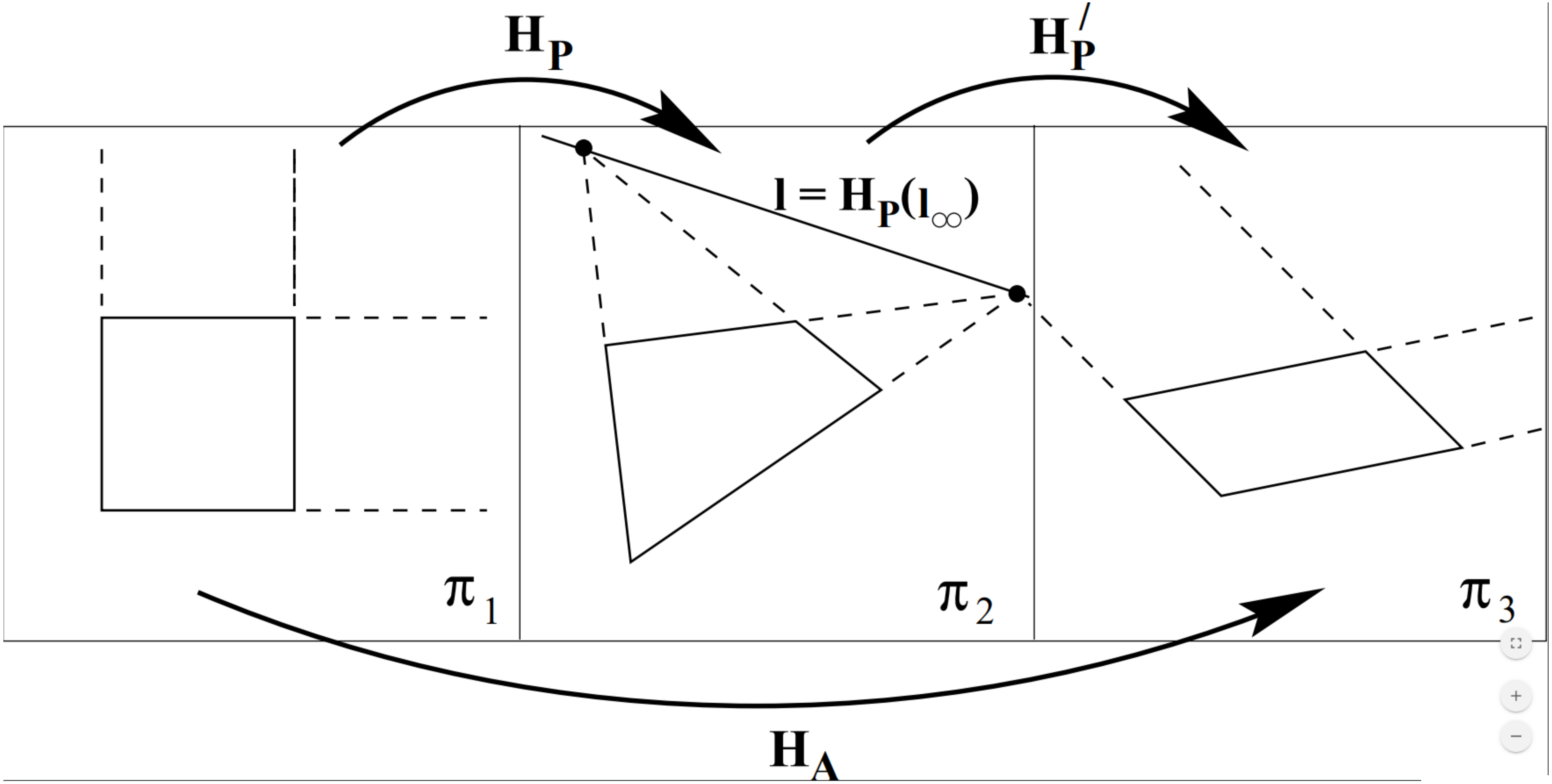
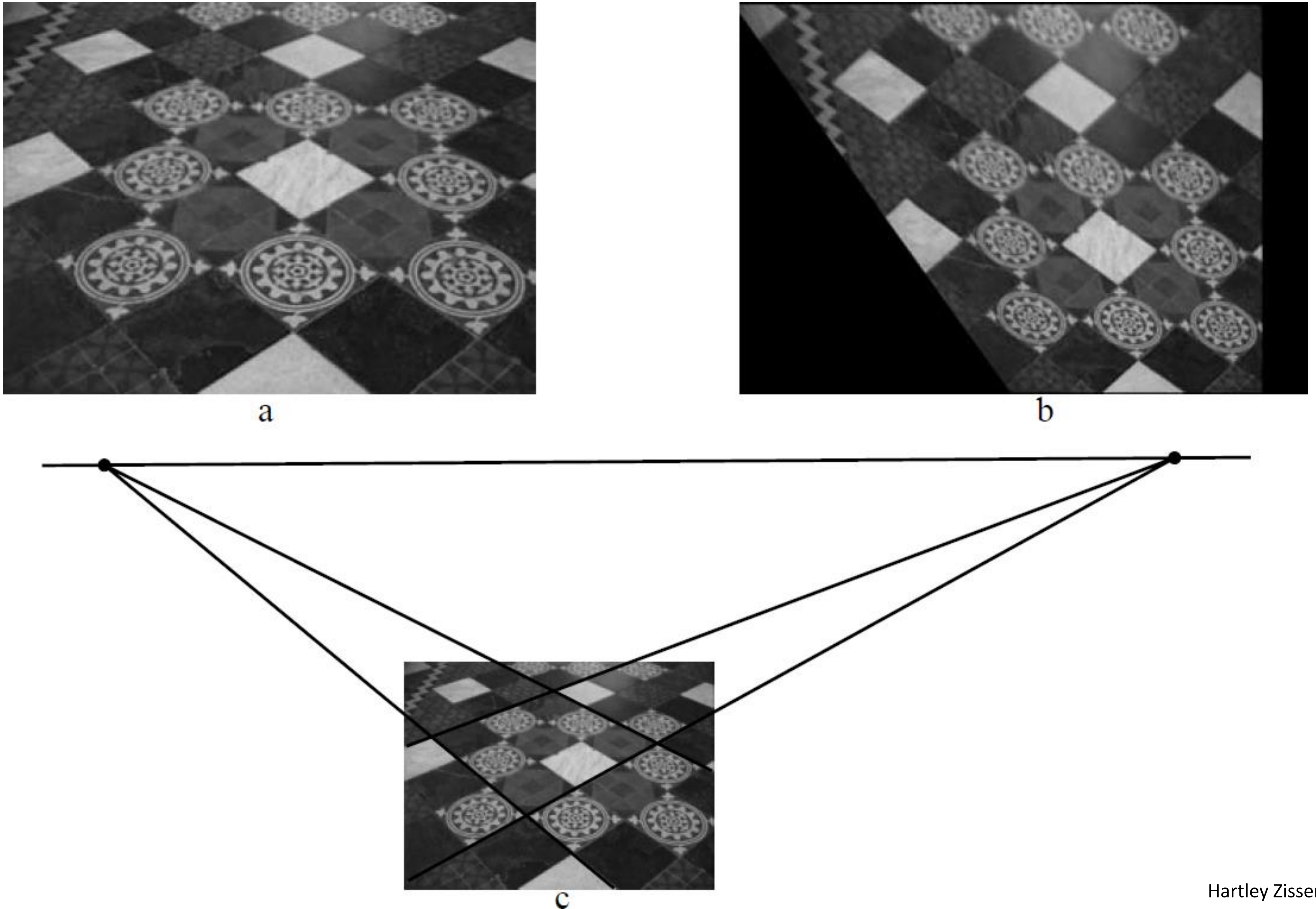
Rmk: l can be rescaled to improve conditioning of H

Affine Rectification



Hartley Zisserman Fig.2.12



Hartley Zisserman Fig. 2.13

Example

Perform Affine Rectification of the following image by selecting only 4 points