OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Introduction to Simulation

Modelling with ODEs

Workload of Introduction to Simulation

#	Topic	Lecture Attendance	Lecture Revision	Exercise Attendance	Exercise Revision	Assignments	Semester Assignment
1	Introduction	2	1			-	1
2	Modelling ODEs	2	1	2	1	1	3
3	Example ODEs	2	1	2	1	3	-
4	Solving ODEs	2	1	2	1	1	-
5	Discrete Event Simulation	2	1	2	1	2	6
6	Random Variables/ RNGs	2	1	2	1	-	-
7	Input Modelling	2	1	2	1	3	3
8	Output Analysis	2	1	2	1	3	3
9	Petri Nets	2	1	2	1	2	-
10	Markov Models	2	1	2	1	2	-
11	Agent-based Simulation	2	1	2	1	-	**(15)**
12	Validation	2	1	2	1	-	4
13	System Comparison	2	1	2	1	-	6
	Totals	26	13	24	12	17	26(41)

	5CP/6CP	
Total hours used	**118/133**	
Workload for 5CP/6CP	**150/180**	
Hours still available	**32/47**	(For exam preparation, organization, etc.)

Motivation and Content

Why is this topic important?

- Ordinary Differential Equations (ODEs) are very important in all branches of Science and Engineering
- ODEs form the basis for the simulation of almost all continuous phenomena
- Understanding ODEs is essential for understanding natural and technical processes

Content of this lecture:

- Introduce ODEs
- Give some examples of simple ODE models

Continuous Processes

Continuous processes occur everywhere

Here, we are interested in cases with discrete variables

Some examples:

- The spread of a virus
- The motion of the planets orbiting the sun
- The current and voltage in an electrical circuit
- The level of alcohol in my blood on January 1st
- The populations of a predator and its prey (!)

In almost all cases, the relationships between the variables are defined by an ODE

Ordinary Differential Equations

An Ordinary Differential Equation (ODE) describes the rate of change of a quantity y as a function of its current value and the time t :

$$\frac{dy}{dt} = f(y,t)$$

In addition, we usually know the value of y(0):

$$y(0) = y_0$$

Initial Value Problems

What is an Initial Value Problem?

Given an ordinary differential equation

$$\frac{dy}{dt} = f(y,t)$$

and an initial condition

$$y(0) = y_0$$

find

$$y(t)$$

Examples of IVPs

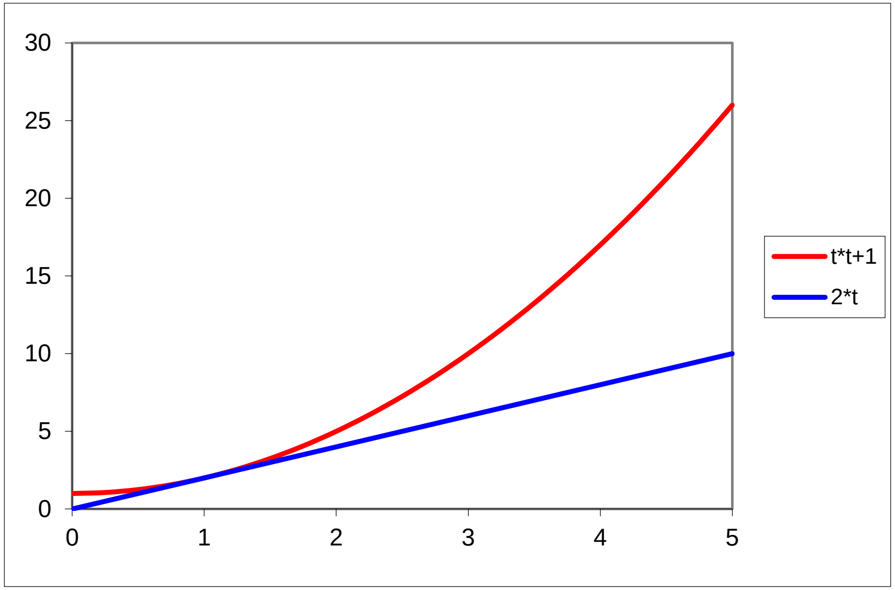
Differential equation and initial condition:

$$\frac{dy}{dt} = 2t$$

$$y(0) = 1$$

Solution:

$$y(t) = t^2 + 1$$

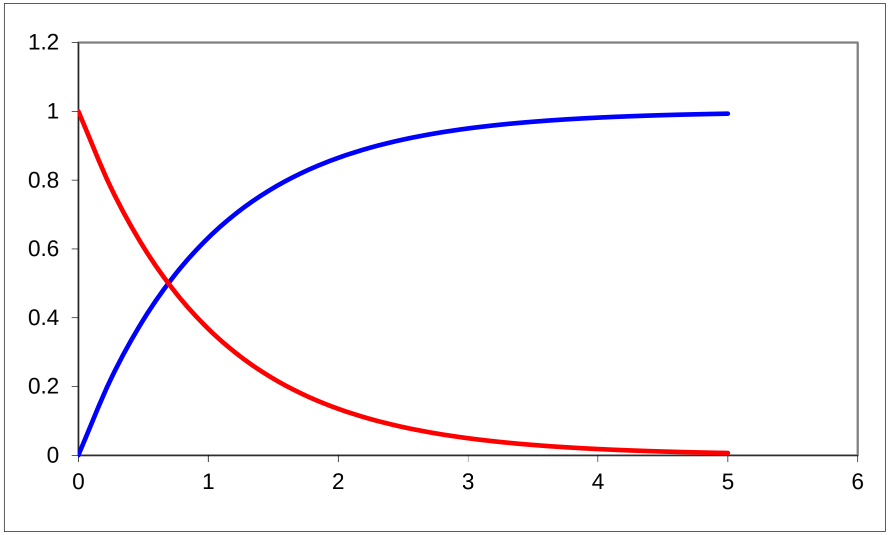


Examples of IVPs

Differential equation and initial condition:

$$\frac{dy}{dt} = \lambda(1 - y)$$

$$y(0) = 0$$



Solution:

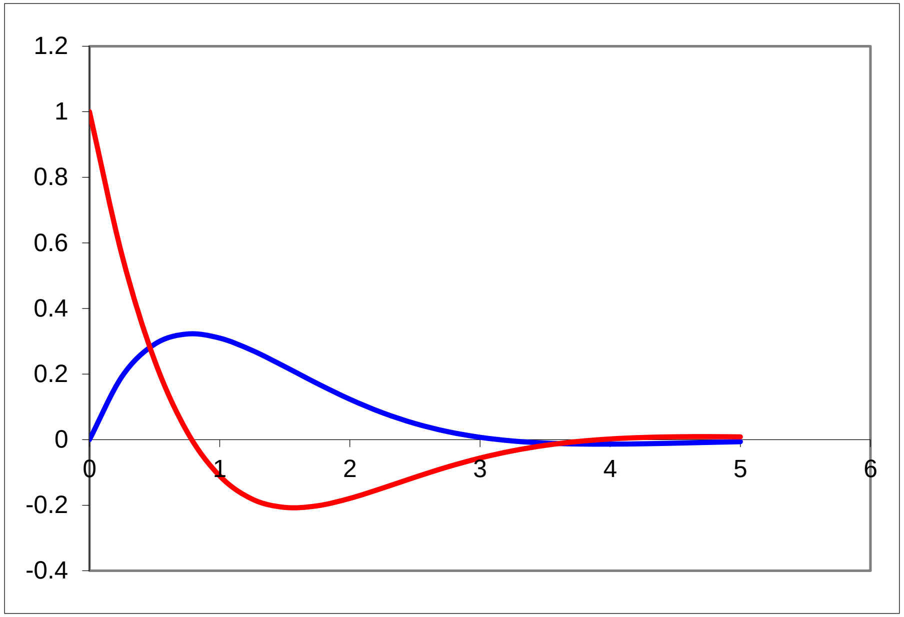
$$y(t) = 1 - e^{-\lambda t}$$

Examples of IVPs

Differential equation and initial condition:

$$\frac{dy}{dt} = e^{-t}\cos(t) - y$$

$$y(0) = 0$$



Solution:

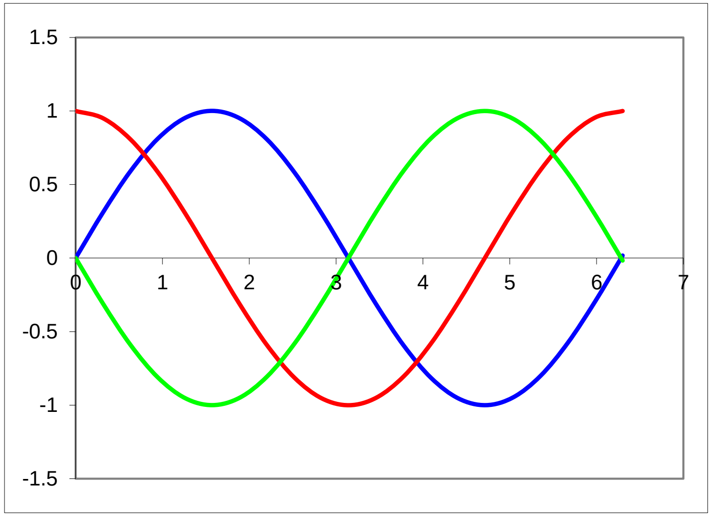
$$y(t) = e^{-t}\sin(t)$$

Examples of IVPs

A system of two differential equations and initial conditions:

$$\frac{du}{dt} = v, \qquad \frac{dv}{dt} = -u$$

$$u(0) = 0, \qquad v(0) = 1$$



Solution:

$$u(t) = \sin(t), \quad v(t) = \cos(t)$$

Solving ODEs

We were able to solve all these examples analytically

- These were special cases!

Usually, there is no analytic solution for systems of ODEs

- We are forced to integrate using numerical methods

The numerical integration of ODEs forms the most important part of continuous simulation

We will look at this next week

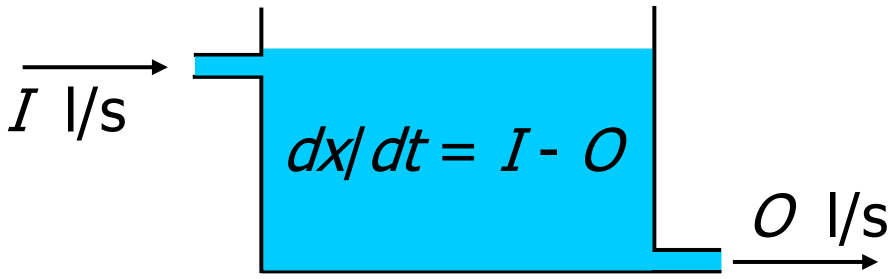
Balance Equations

Most ODEs are balance equations

A balance equation basically says:

$$\text{Change} = \text{Increase} - \text{Decrease}$$

Example: ODE for the amount of water x in a tank:



A Falling Object

Consider dropping an object from a certain height

We will develop a model for the object's location and speed

Variables:

- $p(t)$: distance fallen at time t. (Units [m])
- $v(t)$: velocity of object at time t. (Units [m/s])

Effect of Gravity

The Earth exerts a force F on the object. (Units [N])

Let the mass of the object be m. (Units [kg])

Newton's law ($F = m \cdot a$) states that

- Force = Mass · Acceleration

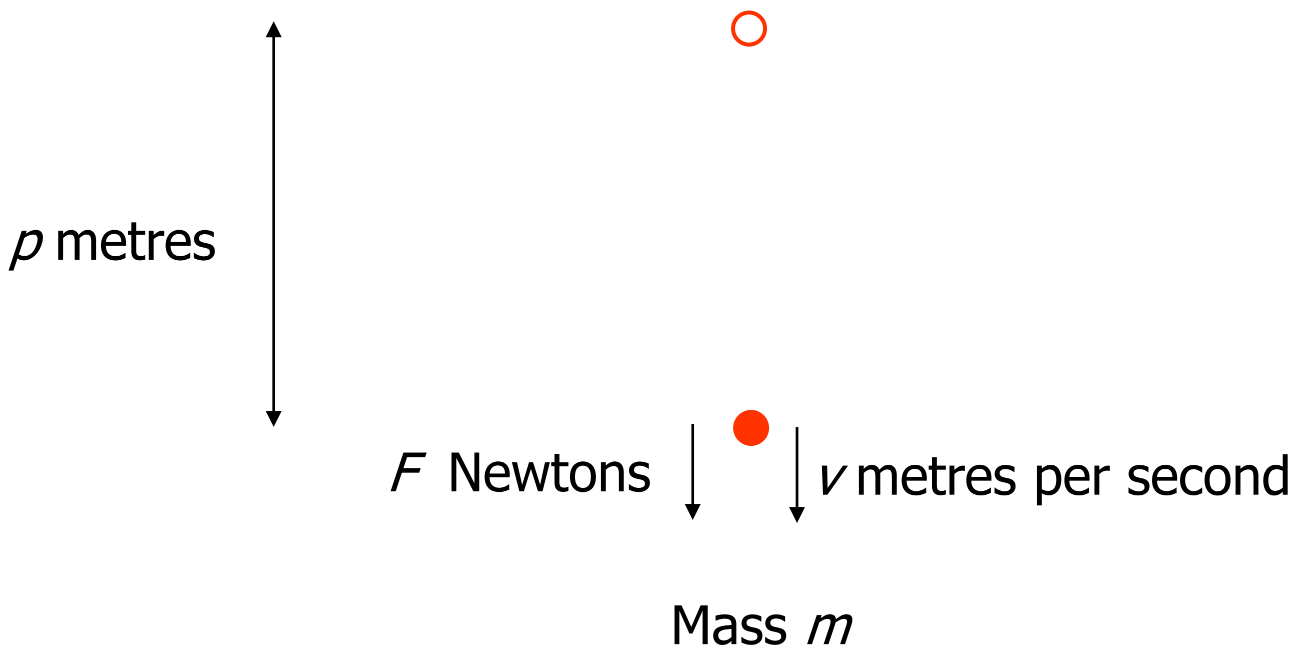
Therefore, $a = F/m$

Parameters:

- a : acceleration due to gravity. (Units [m/s ^{2}])

A Falling Object

The situation at time t:



The Model

Acceleration is the rate of change of velocity:

- $dv/dt = a = 9.81 \ \ [\mathrm{m/s^2}]$

Velocity is the rate of change of position:

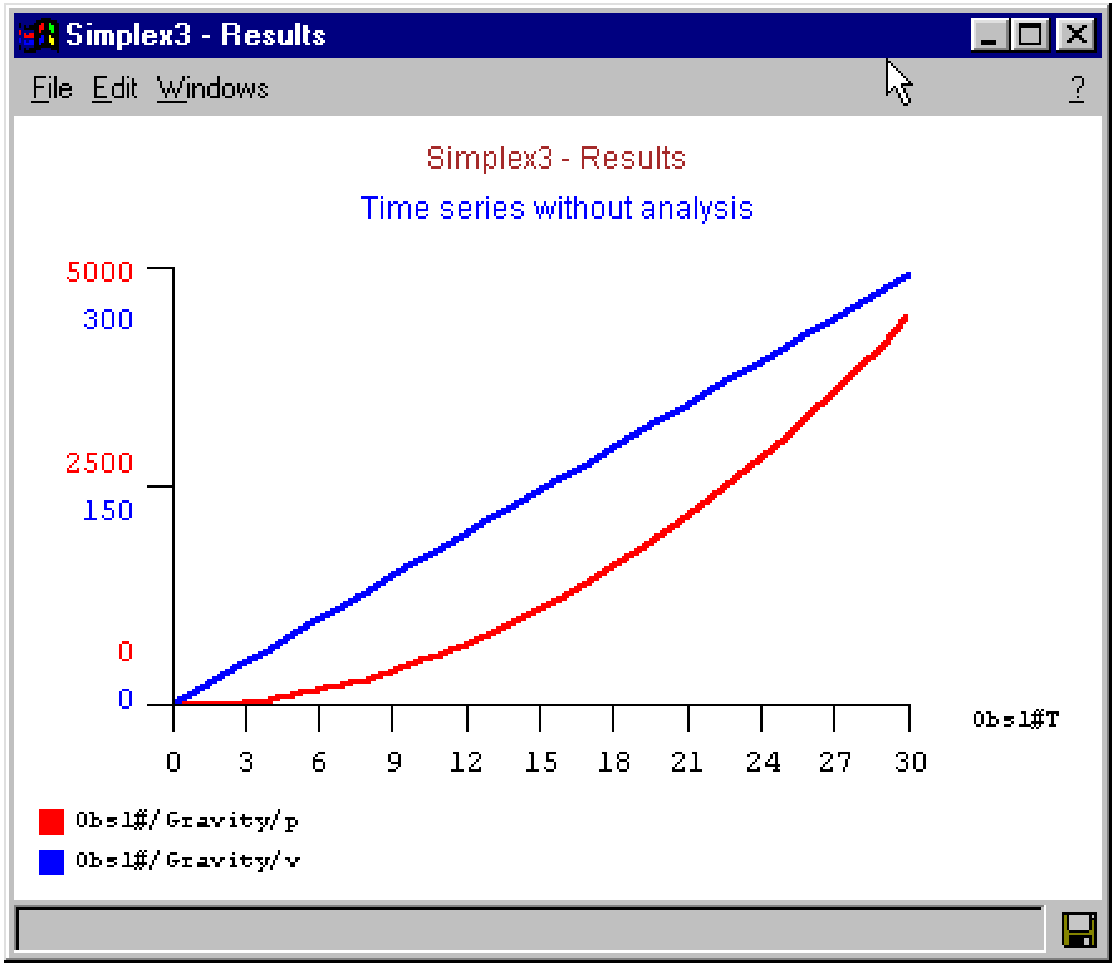
- $dp/dt = v \ [\mathrm{m/s}]$

At time $t = 0$ we define:

- $p(0) = 0 \ \ [\mathrm{m}]$
- $v(0) = 0 \ \ [\mathrm{m/s}]$

Simulation Results

Simulation results:



Wind Resistance

We have assumed that the object is falling in a vacuum

Now let us consider wind resistance

One simple physical model states:

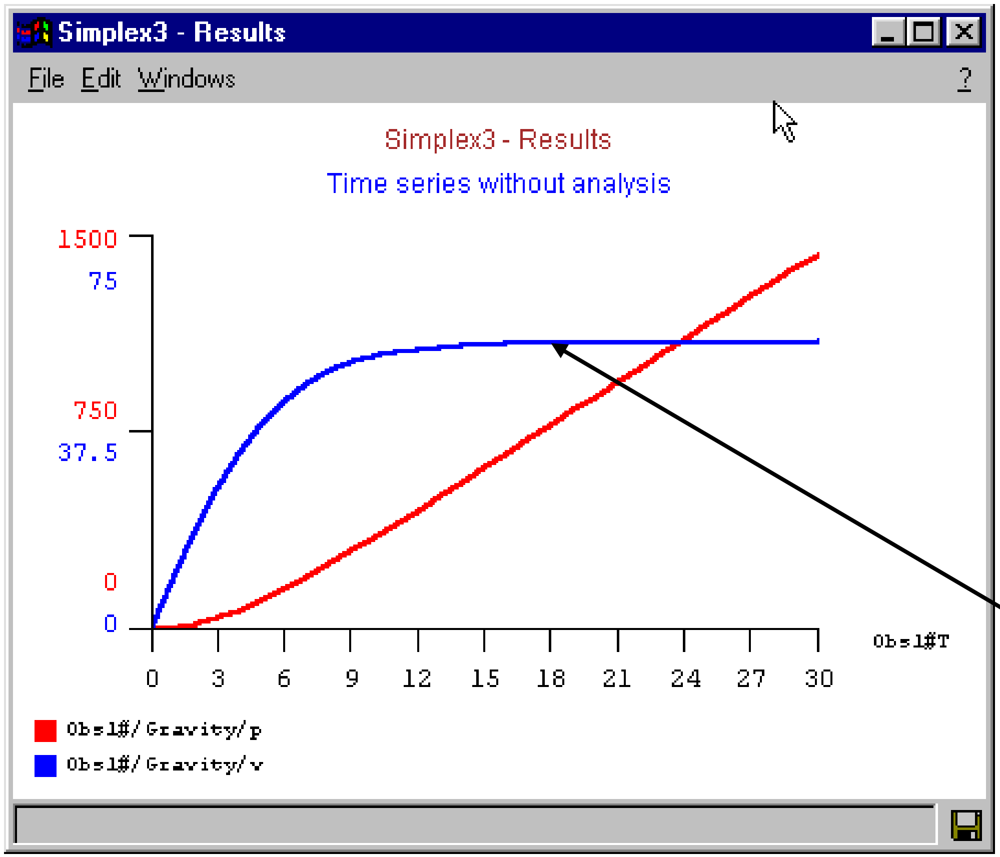
- The force due to wind resistance is proportional to the square of the velocity

We obtain $$\frac{dv}{dt} = a - b \cdot v^2$$

for some constant b with units [1 / m]

Wind Resistance

Result for $b = 0.0033$ (skydiver):



Terminal velocity $\approx 56\ m/s$

Population Biology

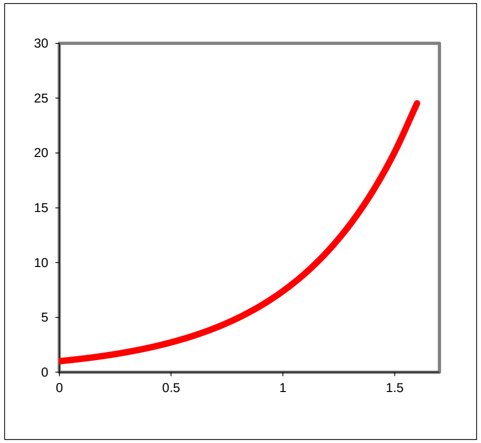
Consider the population x of some animal species

Assume for the moment...

- No deaths
- Infinite space and food supply

The birth rate is proportional to the current population:

$$\frac{dx}{dt} = a \cdot x$$



The solution of this ODE is

$$x = e^{at}$$

We have exponential population growth

Population Biology

Now let us add deaths to the model

The death rate is also proportional to the current population

This gives

$$\frac{dx}{dt} = a \cdot x - b \cdot x = (a - b) \cdot x$$

If $a > b$, we still have exponential population growth

We can just set $c = a - b$ to obtain $$\frac{dx}{dt} = c \cdot x$$

The Logistic Equation

Now assume there is a limited food supply

- As the population increases, food per capita goes down
- This leads to decreasing birth and increasing death rates
- This is called *crowding* or *competition*

One solution is to use the "logistic equation":

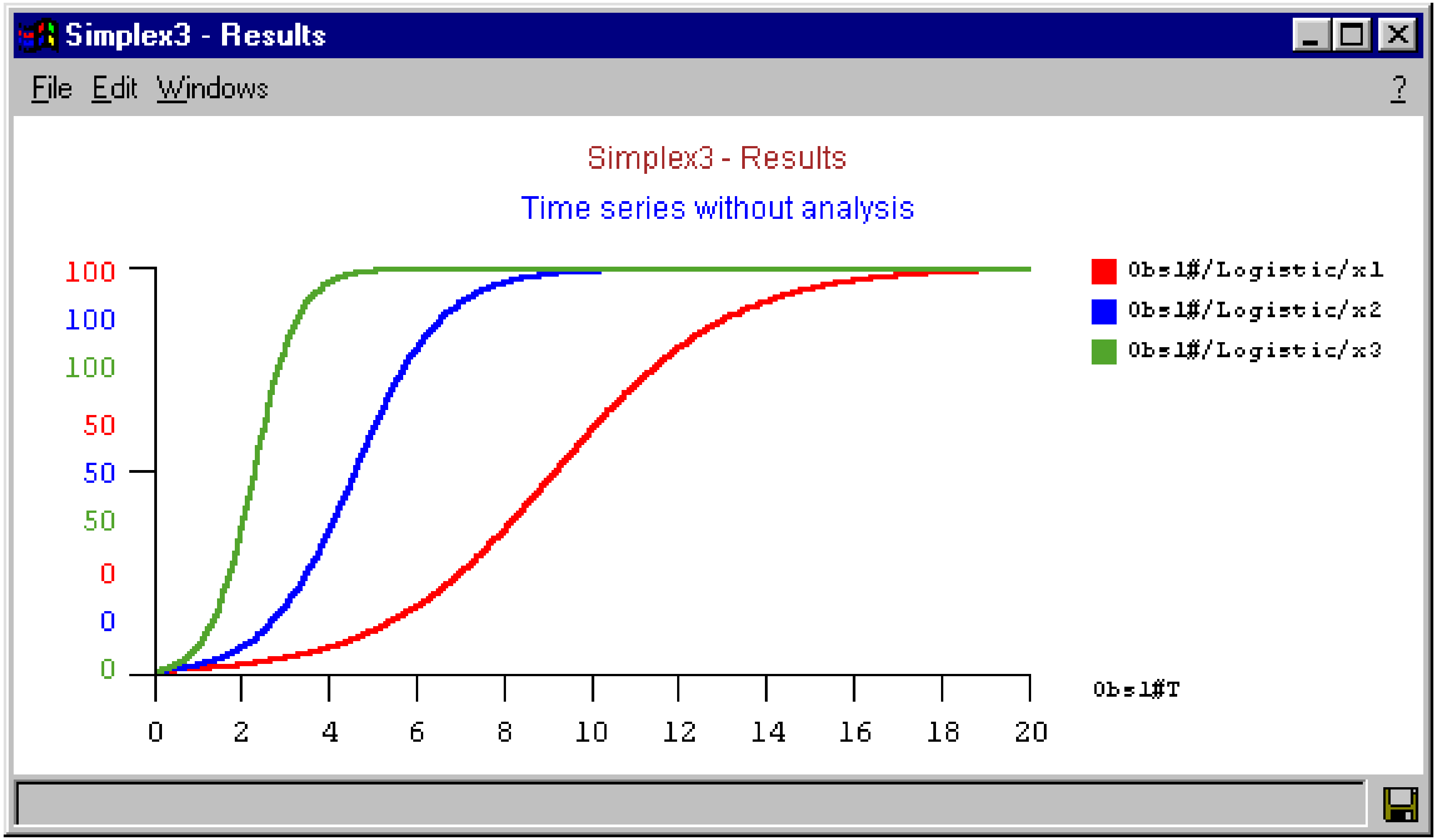
$$\frac{dx}{dt} = c \cdot x - d \cdot x^2$$

We will achieve steady state when $x = c / d$

- This is an assumption which works well in practice

The Logistic Equation

Simulated solutions for $c = 0.5$, 1.0 and 2.0:



Predator–Prey Models

Consider an area in which hares and foxes live

Denote the population of hares by h and of foxes by f

Observations show:

- Foxes must eat hares in order to survive
- Hares have an unlimited supply of food

Without foxes, hares multiply according to $\frac{dh}{dt} = a \cdot h$

Without hares to eat, foxes die according to $\frac{df}{dt} = -b \cdot f$

Lotka–Volterra Equations

More modelling assumptions:

- The probability of a meeting is proportional to $h \cdot f$
- The increase in foxes is $d \cdot h \cdot f$
- The rate at which hares are eaten is $c \cdot h \cdot f$

The resulting equations are:

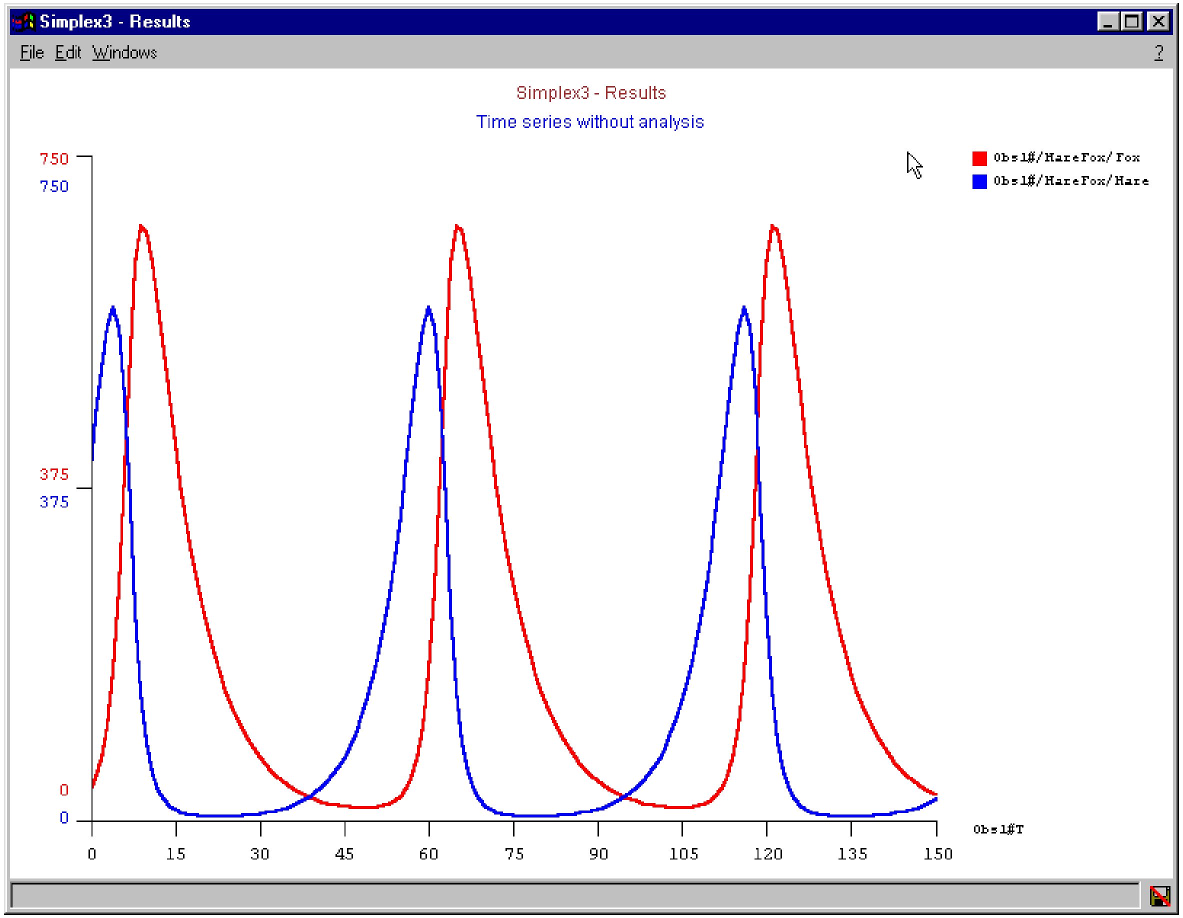
$$\frac{dh}{dt} = a \cdot h - c \cdot h \cdot f \qquad \frac{df}{dt} = -b \cdot f + d \cdot h \cdot f$$

These are the famous Lotka–Volterra equations

Lotka–Volterra Equations

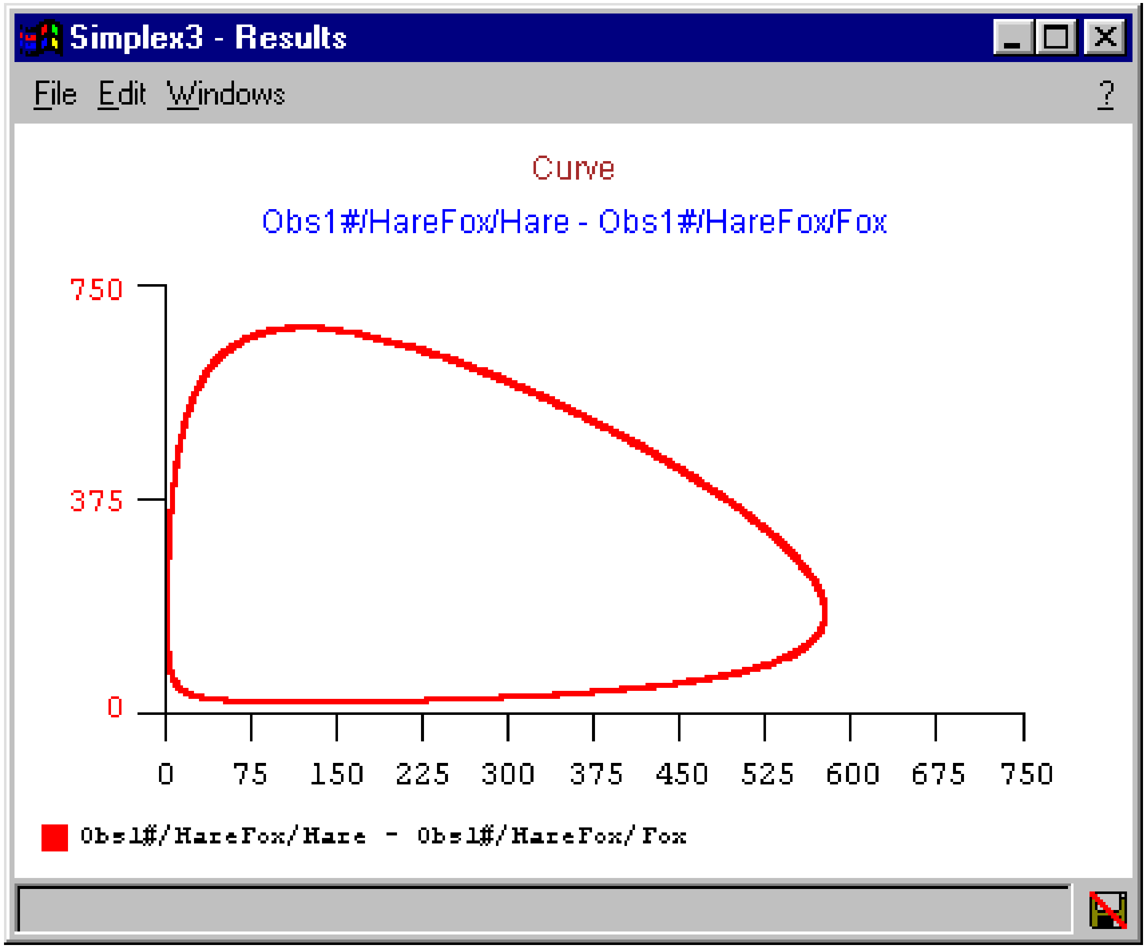
Simulation results (time domain) for parameters

- $h(0) = 400$
- $f(0) = 37$
- $a = 0.175$
- $b = 0.125$
- $c = \mathrm{d} = 0.001$
- $0 < t < 150$



Lotka–Volterra Equations

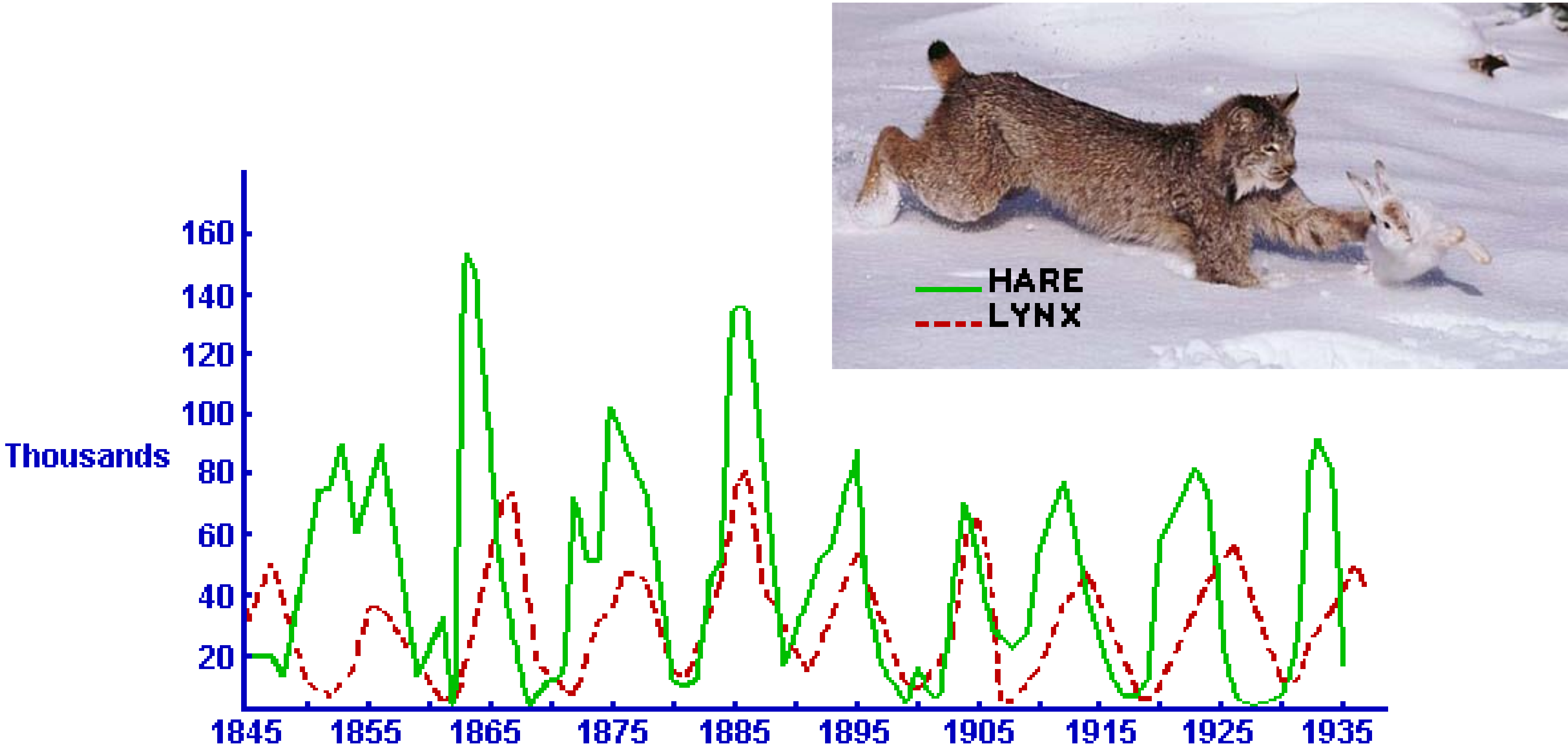
State (phase) space representation: Plot $h(t)$ against $f(t)$



Lynxes and Hares

From the records of the Hudson Bay Company

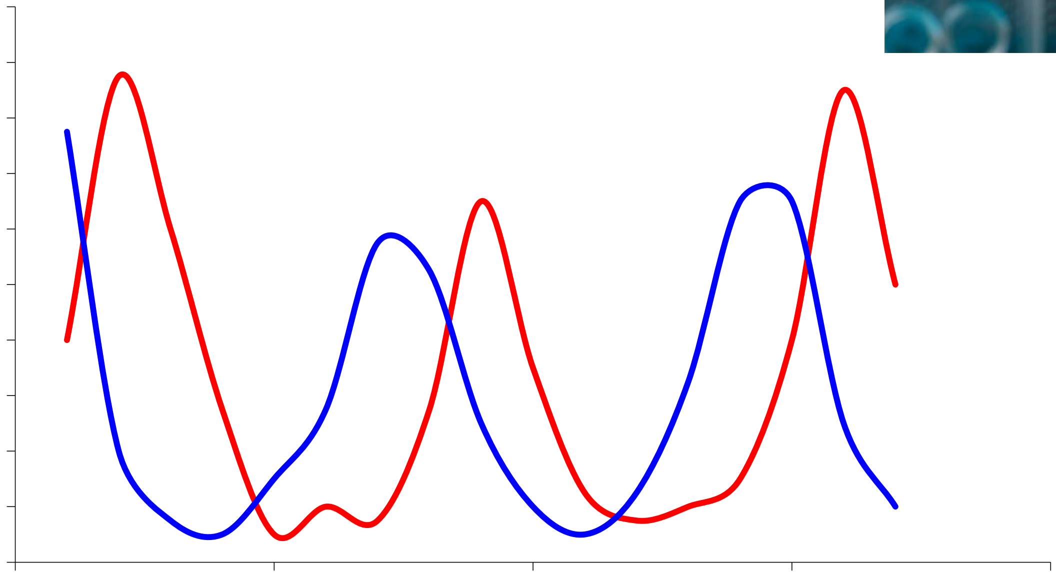
- Number of lynx and hare skins purchased 1845–1935



Another Example

Petri-dish populations of the bacteria

- Paramecium Aurelia (predator)
- Saccharomices Exiguns (prey)



The Gypsy Moth

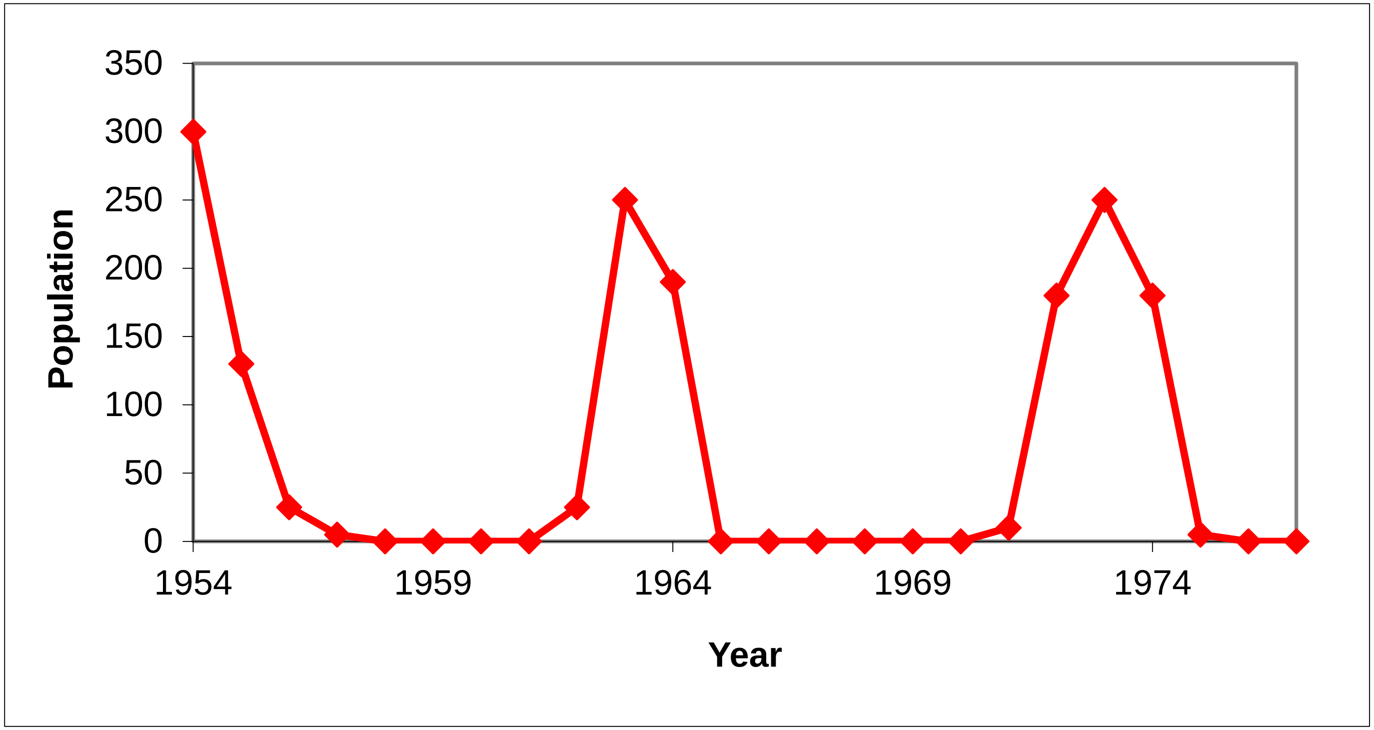
The Gypsy moth is a pest

Its larvae eats the leaves of the larch

This defoliates and often kills the tree

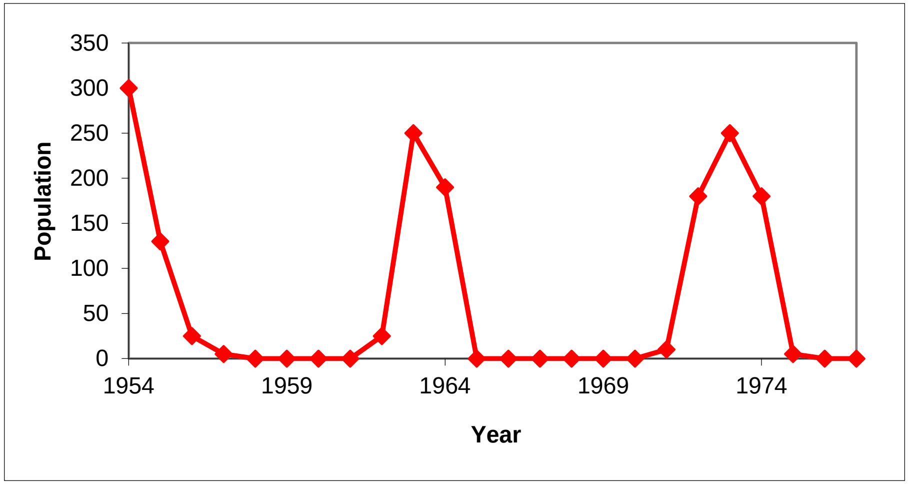
The Gypsy Moth

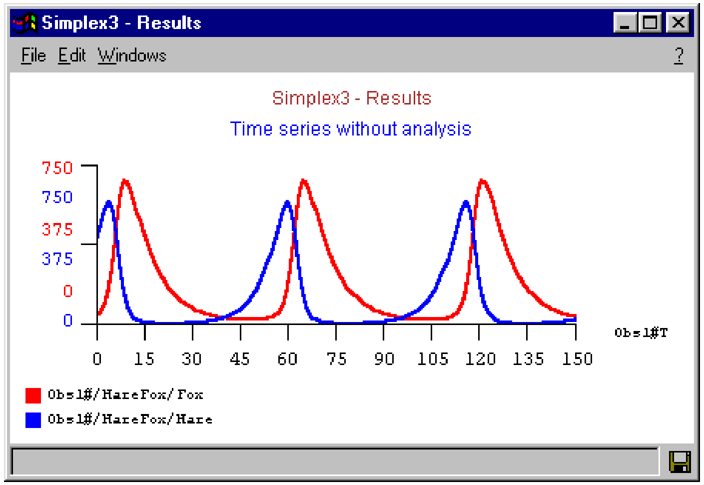
Population of Gypsy moths in the Engadin 1954 – 1976:



The Gypsy Moth

This looks very much like half of the Lotka–Volterra result:





Could the Gypsy moth be part of a predator–prey relationship?

Types of Behaviour

What can the long-term behaviour of an ODE look like?

Explosion:

- One or more values becomes infinite

Steady-state / stationary

- Nothing changes

Periodic

- Behaviour repeats indefinitely

Chaotic

- Finite, non-stationary and non-periodic

Chaos

Chaotic behaviour is a (relatively) new discovery in ODEs

Published by Edward Lorenz in 1963

- Three equations from meteorology
- Discovered by accident

$$\frac{dx}{dt} = a \cdot (y - x)$$

$$\frac{dy}{dt} = -x \cdot z + b \cdot x - y$$

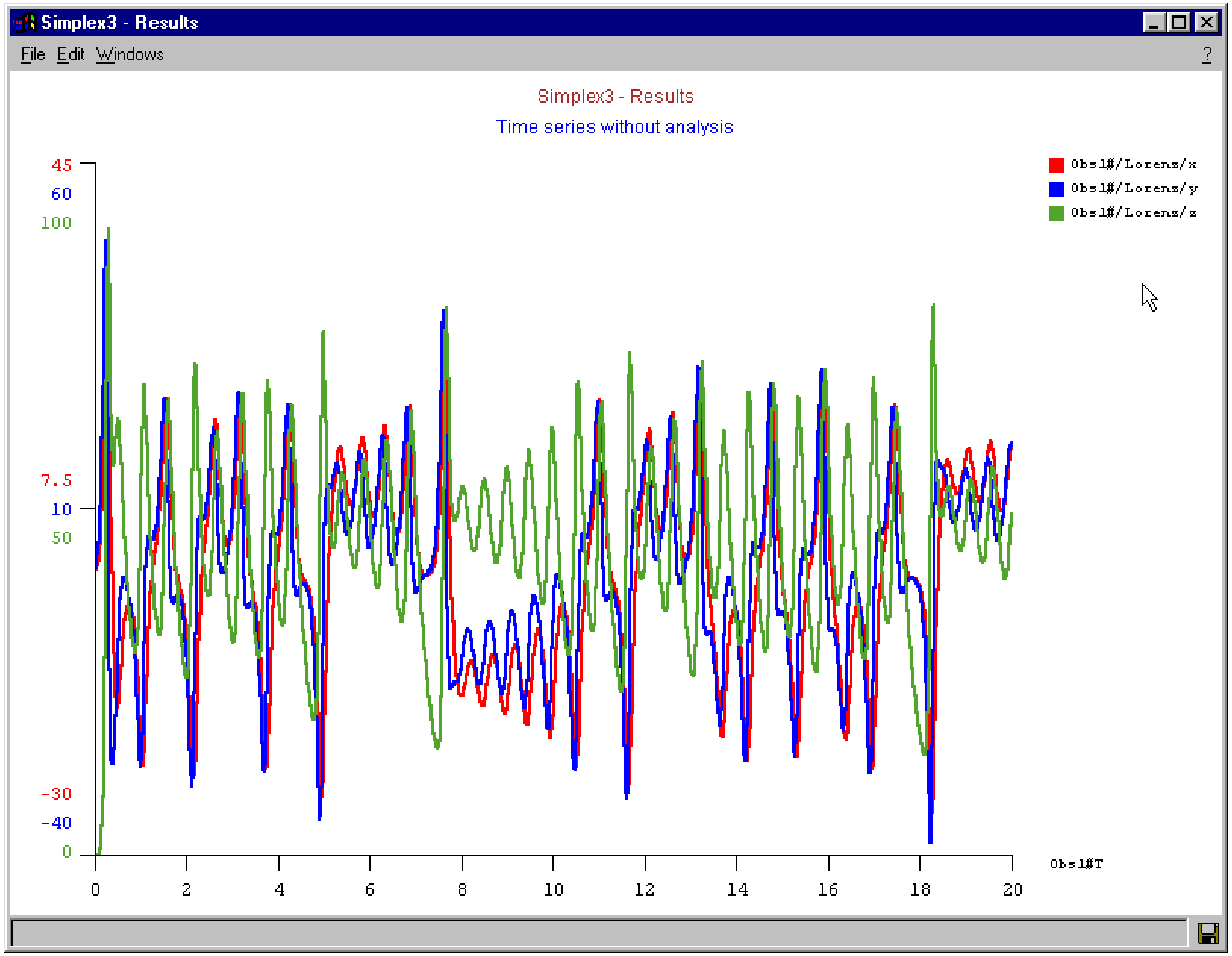
$$\frac{dz}{dt} = x \cdot y - c \cdot z$$

"Chaos" in physics means...

- "Sensitive dependence on initial conditions"
- The "Butterfly Effect"

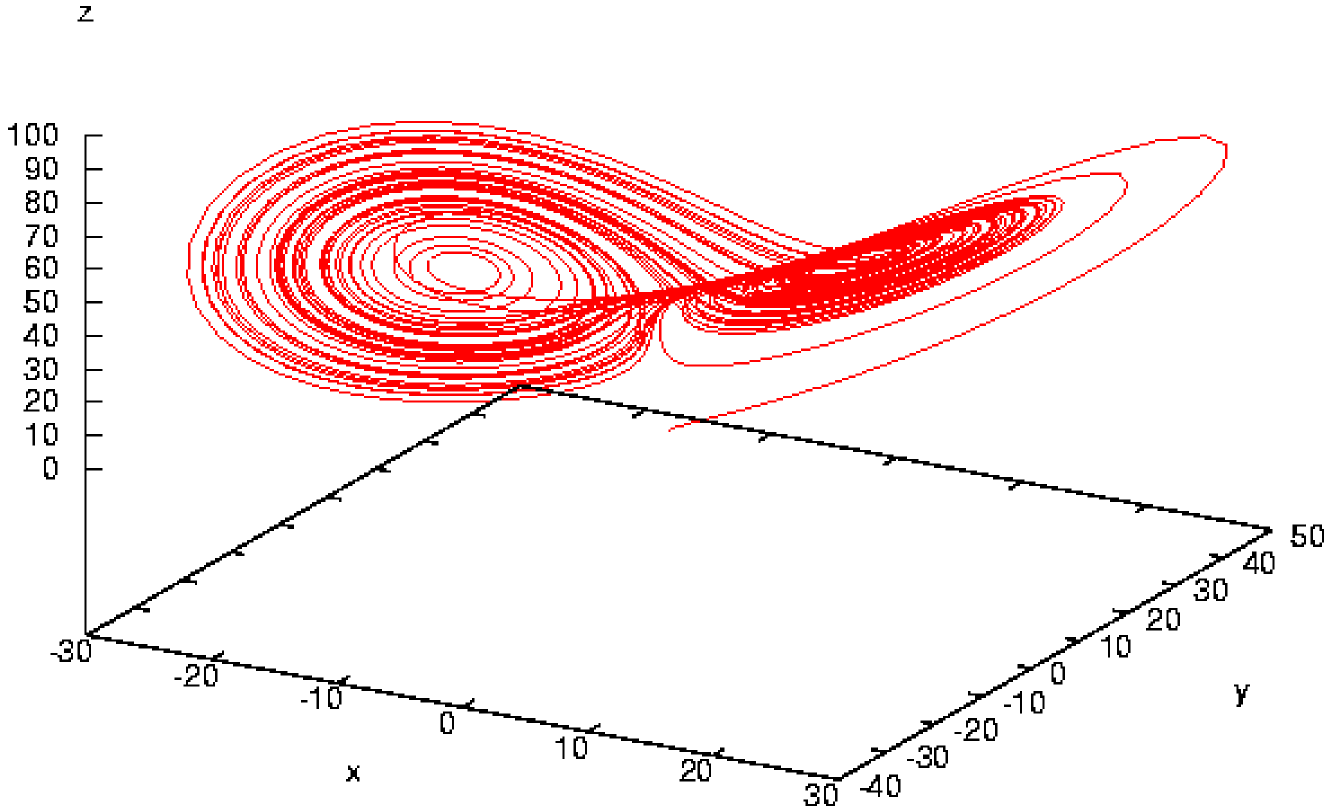
Lorenz's Equations

Solution in the time domain:



The Lorenz Attractor

Simulation result in phase space ("Lorenz attractor"):



Three Species

Now let us build a new three-species predator-prey model:

- Species z preys on species x and y
- There is crowding both within x and y, and between x and y

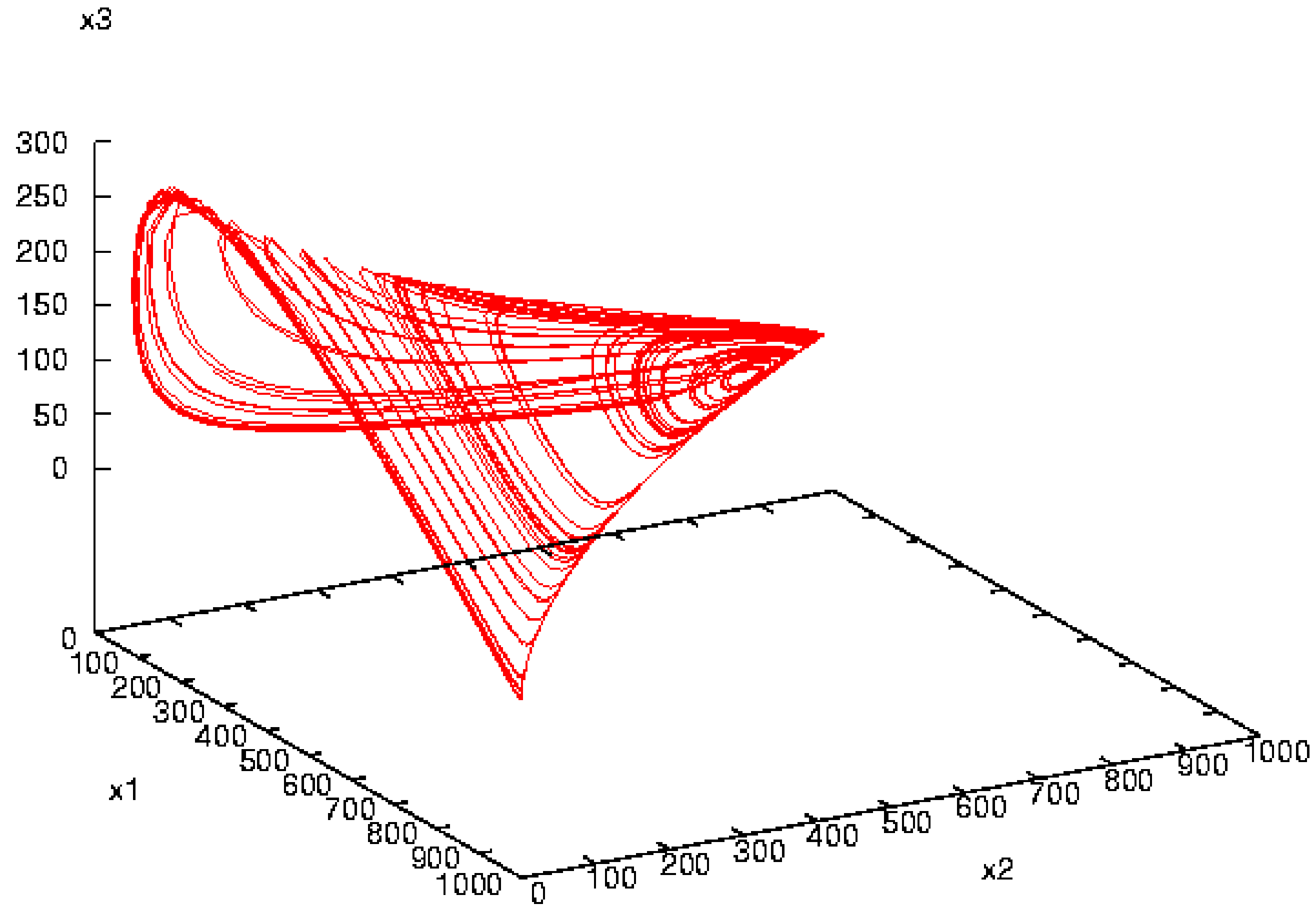
$$\frac{dx}{dt} = x - 0.001x^2 - 0.001xy - 0.01xz$$

$$\frac{dy}{dt} = y - 0.0015xy - 0.001y^2 - 0.001yz$$

$$\frac{dz}{dt} = -z + 0.005xz + 0.0005yz$$

Three Species

The three-species result in 3-D state (phase) space:



The Sims – Almost Normal Family Life

The semester assignment contains three ODEs:

- Mom's mood ("Mom")
- Dad's mood ("Dad")
- Savings account balance ("Savings")

These quantities change continuously:

- *dMom/dt = + c*Dad + c*Son + c*(Savings−500)*
- *dDad/dt = + c*Mom – UnemploymentRate*
- *dSavings/dt = +Income – Expenses (–discrete quantities)*

Star Trek – USS Enterprise in Danger

The semester assignment contains four ODEs:

- The shield level (“Shields”)
- Theta radiation inside the ship (“Radiation”)
- The distance to the rift (“Distance”)
- The current speed of the ship (“Speed”)

These quantities change continuously:

- *dShields/dt = + ChargingRate (–discrete quantities)*
- *dRadiation/dt = + PenetrationRate*
- *dDistance/dt = + Speed*
- *dSpeed/dt = + Acceleration – GravitationalPull*

Learning Goals

Questions to test your knowledge:

- What is an ordinary differential equation? Give an example.
- What is an initial value problem? Give an example with its solution.
- Describe the possible long-term behaviours of an ODE.
- What are the time domain and phase space representations of a function?
- What are the Lotka-Volterra equations? What do they represent? Explain each term.
- Write down the ODEs used in the Stars Wars simulation model. Explain the symbols you have used.