OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF

FAKULTÄT FÜR INFORMATIK

Introduction to Simulation

Examples of ODEs

Motivation and Content

Why is this topic important?

- To convince you that building ODE models is not difficult

Content of this lecture:

- Show some simple ODE models from different fields
- Use Excel to carry out the simulations
  (Quick, but not very elegant!)
- Show the use of AnyLogic for solving ODEs.

The Coffee Problem

An early morning situation:

- You want to drink a cup of (white) coffee
- The (hot) coffee is already in the cup
- You still need to add (cold) milk
- You need to go to the bathroom first
- You would like your coffee to be as hot as possible

Question:

- Is it better to pour in the milk before going to the bathroom, or after you return?

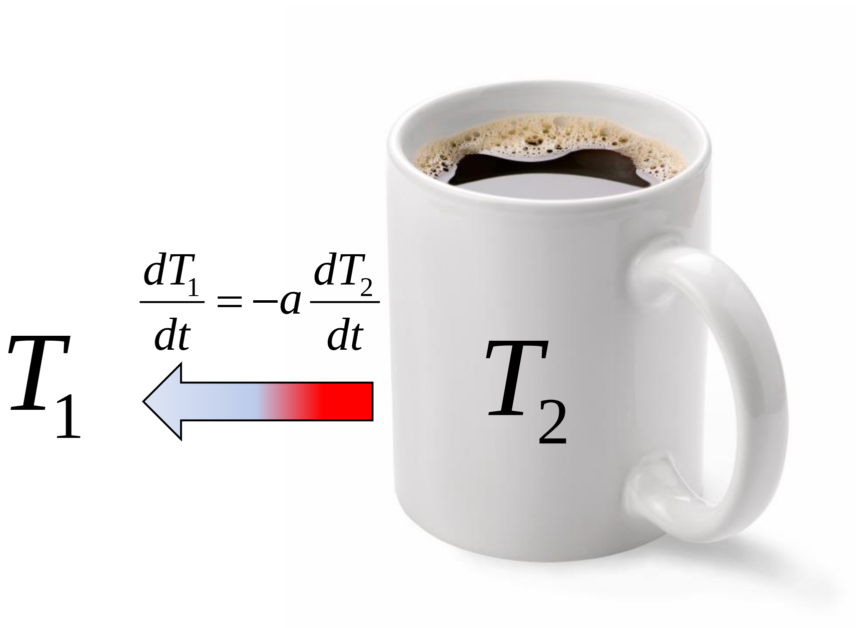
The Coffee Problem

What does Physics (Fourier's Law) tell us?

When heat energy diffuses from one body to another:

- The rate of heat flow is proportional
  to the temperature difference

$$\frac{dT_1}{dt} = -a\frac{dT_2}{dt} = b\cdot\left(T_2 - T_1\right)$$

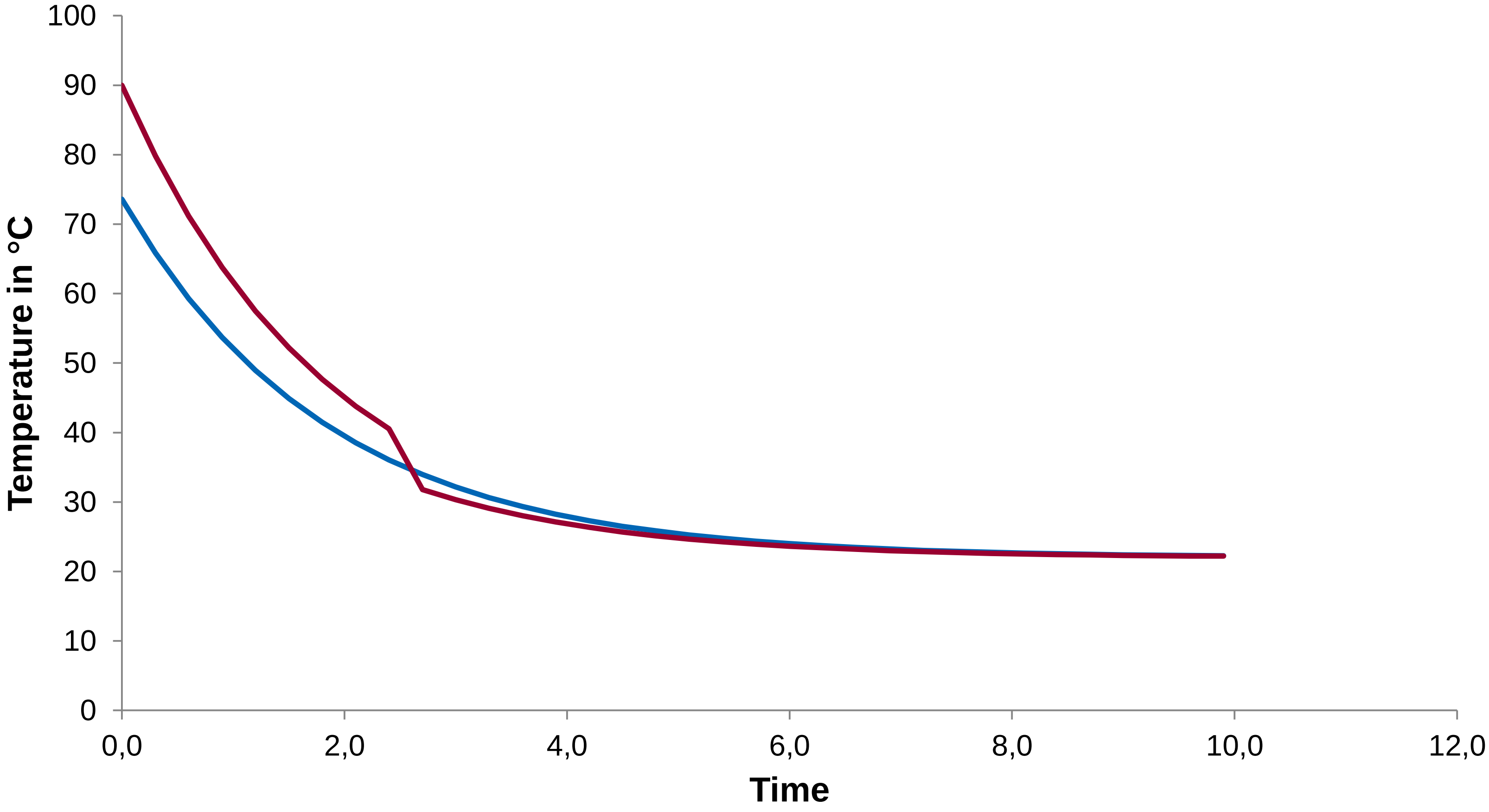


The Coffee Problem

Simulate using the following parameters:

▪ h:	0.3 Minutes
▪ Initial temperature of coffee:	90 Degrees Celsius
▪ Temperature of surroundings:	22 Degrees Celsius
▪ Temperature of milk:	8 Degrees Celsius
▪ Volume of coffee:	0.2 Liters
▪ Volume of milk:	0.05 Liters
▪ Heat transport constant:	0.5 per Minute

The Coffee Problem



Euler's Method

Euler's Method is the simplest numerical integration method

We have the ODE

$$\frac{dy}{dt} = f(y,t)$$

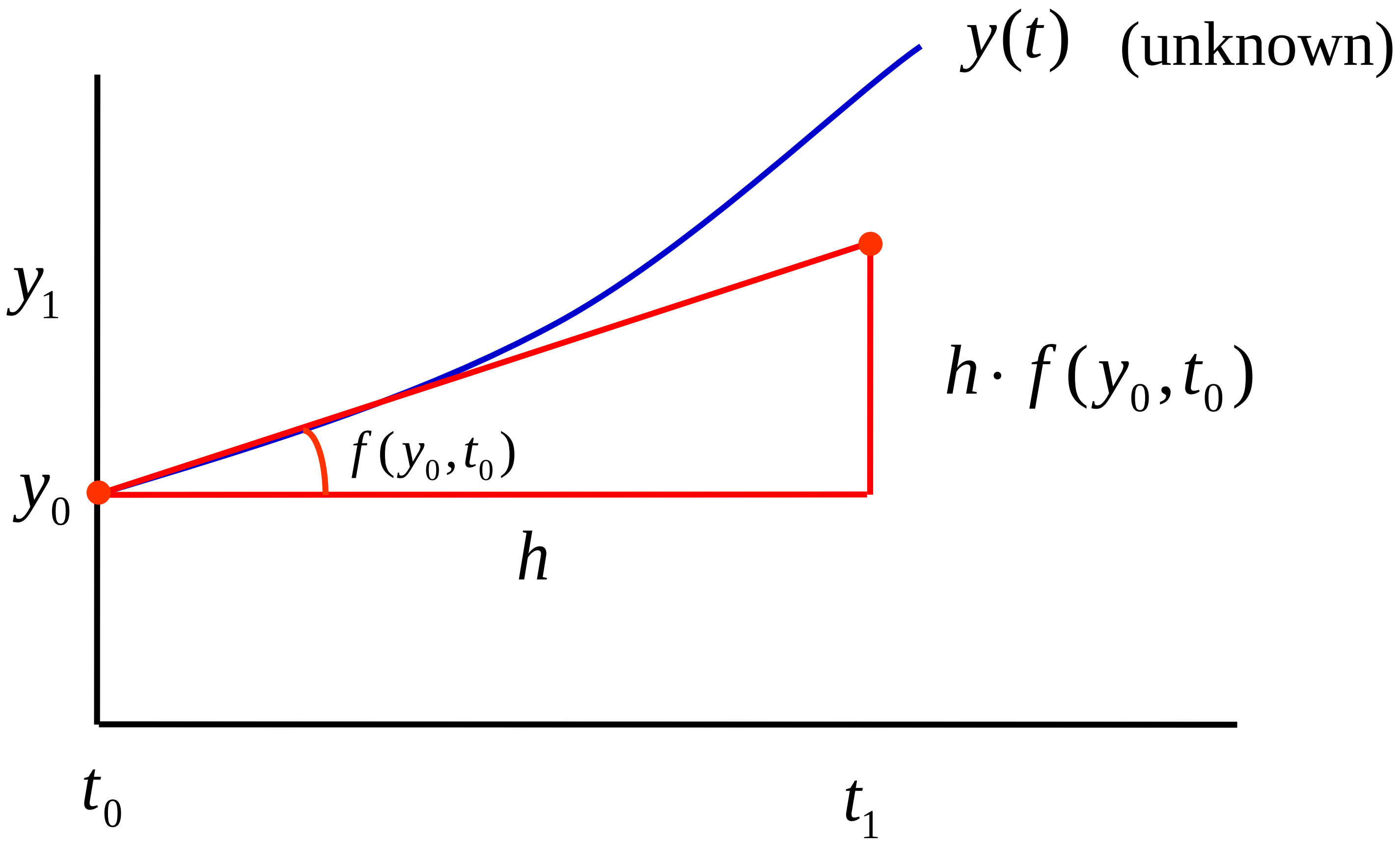
Approach:

- Choose a time increment of size h
- Indicate the number of a time step by k

Then Euler's method is

$$y_{k+1} = y_k + h \cdot f(y_k, t_k)$$

Euler's Method



Euler's Method

Example

$$\frac{dy}{dt} = y + t, \quad y(0) = 1$$

$$y_0 = y(0) = 1$$

$$y_1 = y_0 + h \cdot (y_0 + t_0)$$

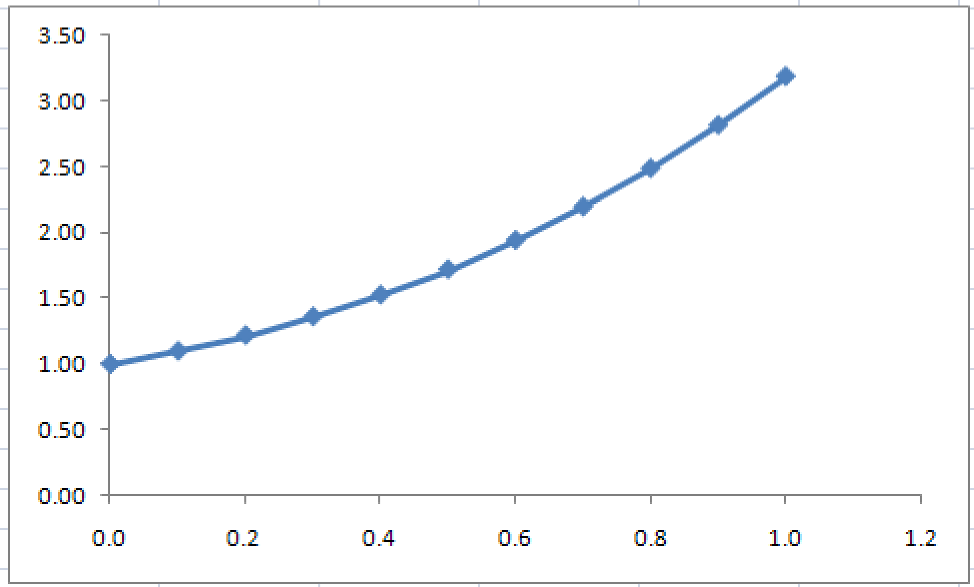
$$y_2 = y_1 + h \cdot (y_1 + t_1)$$

$$y_3 = \ldots$$

Euler's Method

C6 f_x =C5+0.1*(C5+B5)

	A	B	C
4	k	t_k	y_k
5	0	0.0	1.00
6	1	0.1	1.10
7	2	0.2	1.22
8	3	0.3	1.36
9	4	0.4	1.53
10	5	0.5	1.72
11	6	0.6	1.94
12	7	0.7	2.20
13	8	0.8	2.49
14	9	0.9	2.82
15	10	1.0	3.19

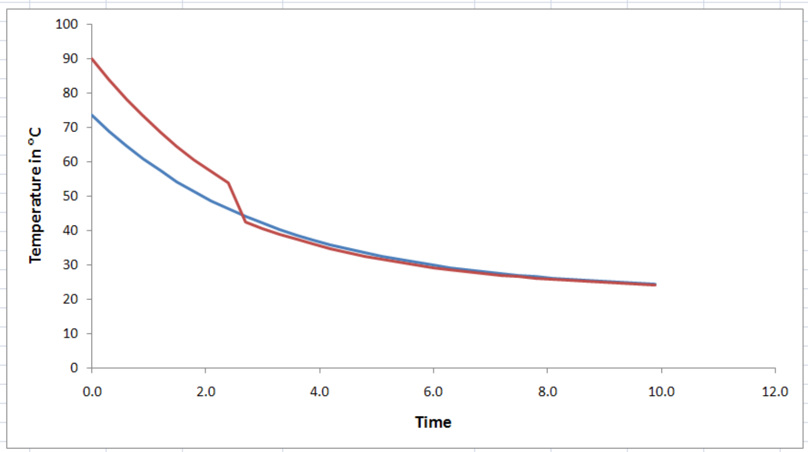


The Coffee Problem Again

Coffee_ODE.xlsx - Microsoft Excel

Time	Add Milk Immediately: Temperature	Add Milk Later: New temperature before milk	Add Milk Later: Add milk now!	Add Milk Later: Temperature
0.0	73.60			90.00
0.3	68.96	83.88	0	83.88
0.6	64.73	78.31	0	78.31
0.9	60.88	73.24	0	73.24
1.2	57.38	68.63	0	68.63
1.5	54.20	64.43	0	64.43
1.8	51.30	60.62	0	60.62
2.1	48.66	57.14	0	57.14
2.4	46.27	53.98	0	53.98
2.7	44.08	51.10	1	42.48
3.0	42.09	40.64	0	40.64
3.3	40.29	38.96	0	38.96
3.6	38.64	37.43	0	37.43
3.9	37.14	36.04	0	36.04
4.2	35.78	34.78	0	34.78
4.5	34.54	33.63	0	33.63
4.8	33.41	32.58	0	32.58
5.1	32.38	31.63	0	31.63
5.4	31.45	30.76	0	30.76
5.7	30.60	29.98	0	29.98
6.0	29.82	29.26	0	29.26
6.3	29.12	28.60	0	28.60
6.6	28.48	28.01	0	28.01
6.9	27.90	27.47	0	27.47
7.2	27.37	26.98	0	26.98
7.5	26.88	26.53	0	26.53
7.8	26.44	26.12	0	26.12
8.1	26.04	25.75	0	25.75
8.4	25.68	25.41	0	25.41
8.7	25.35	25.11	0	25.11
9.0	25.05	24.83	0	24.83
9.3	24.77	24.57	0	24.57
9.6	24.52	24.34	0	24.34
9.9	24.30	24.13	0	24.13

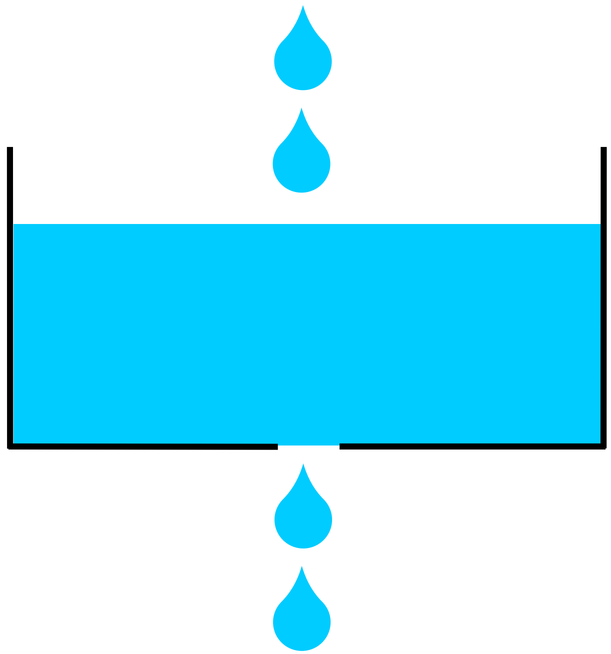
Variable	Value	Units	Meaning
dt	0.3	Minutes	Size of time step
T_coffee	90	Degrees Celsius	Initial temperature of coffee
T_room	22	Degrees Celsius	Temperature of surroundings
T_milk	8	Degrees Celsius	Temperature of milk
V_coffee	0.2	Liters	Volume of coffee
V_milk	0.05	Liters	Volume of milk
c	0.3	per Minute	Heat transport constant



The Leaky Bucket

We are filling a bucket with a hole in the bottom

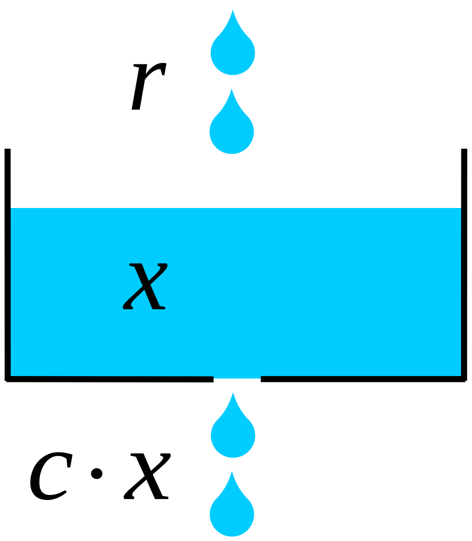
- The fill rate is a constant
- The emptying rate is proportional to the water pressure
- Assumption: the bucket has a constant cross-sectional area



The Leaky Bucket

Variables and parameters:

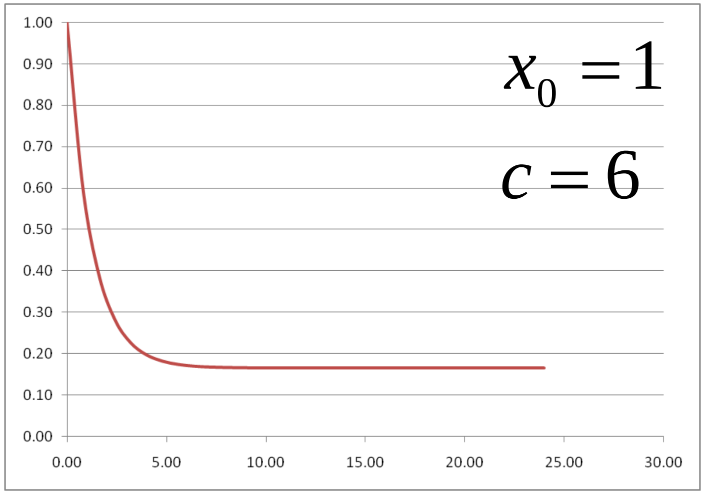
- Volume of water in bucket: x(t)
- Rate of filling: r
- Constant: c

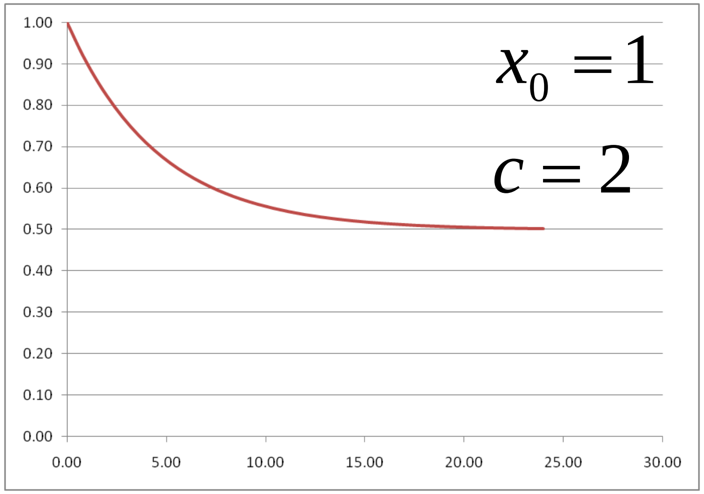


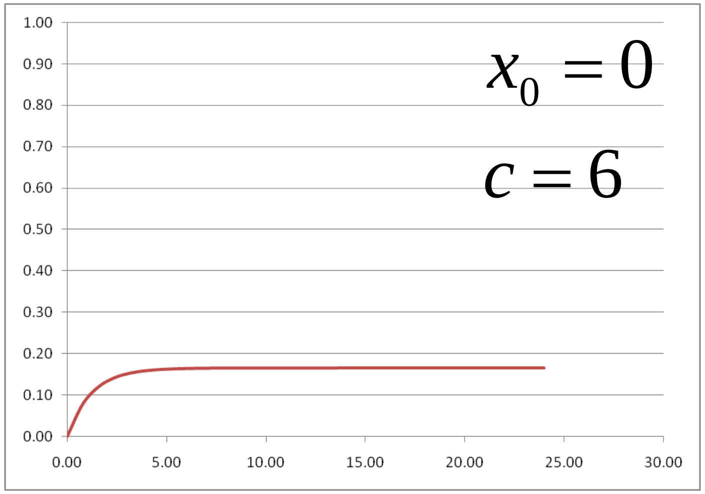
Equation:

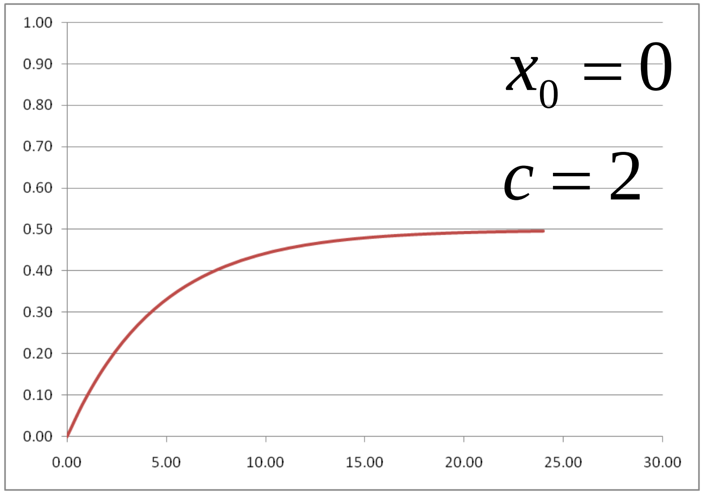
$$\frac{dx}{dt} = r - c \cdot x$$

The Leaky Bucket





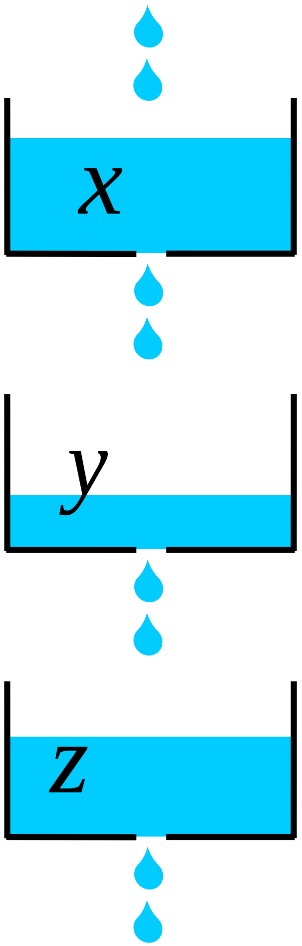




Three Leaky Buckets

Now add two more buckets

- Volumes of water in bucket: $x(t), y(t), z(t)$
- Constants: c_1, c_2, c_3



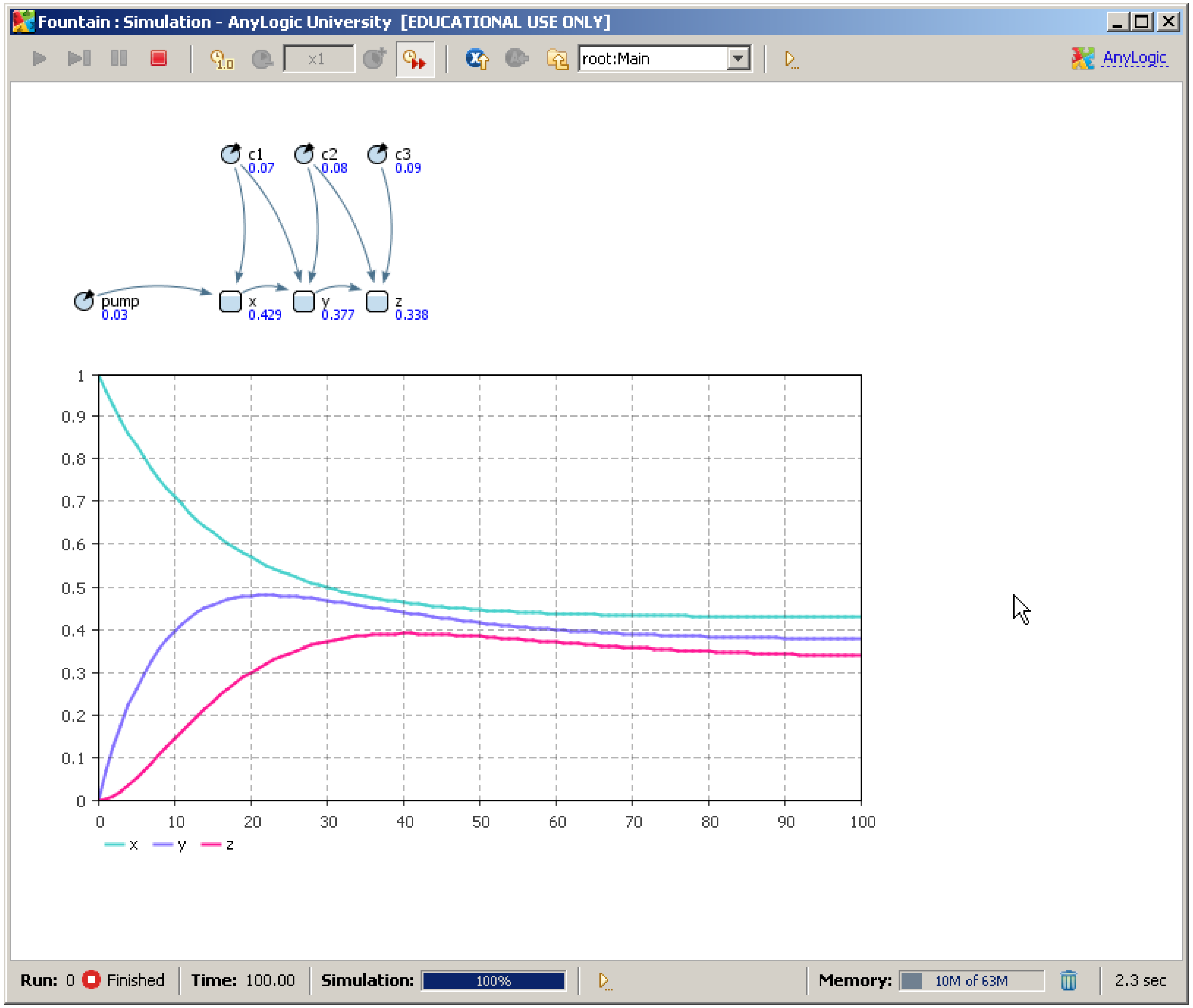
Equations:

$$\frac{dx}{dt} = r - c_1 \cdot x$$

$$\frac{dy}{dt} = c_1 \cdot x - c_2 \cdot y$$

$$\frac{dz}{dt} = c_2 \cdot y - c_3 \cdot z$$

Three Leaky Buckets



Queues

Queues occur in many fields

Logistics

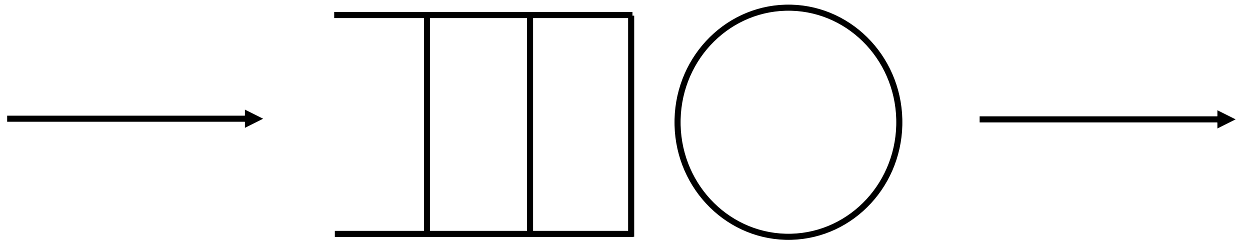
Everyday Life

Computer Networks

Queues

An (abstract) queue has the following components:

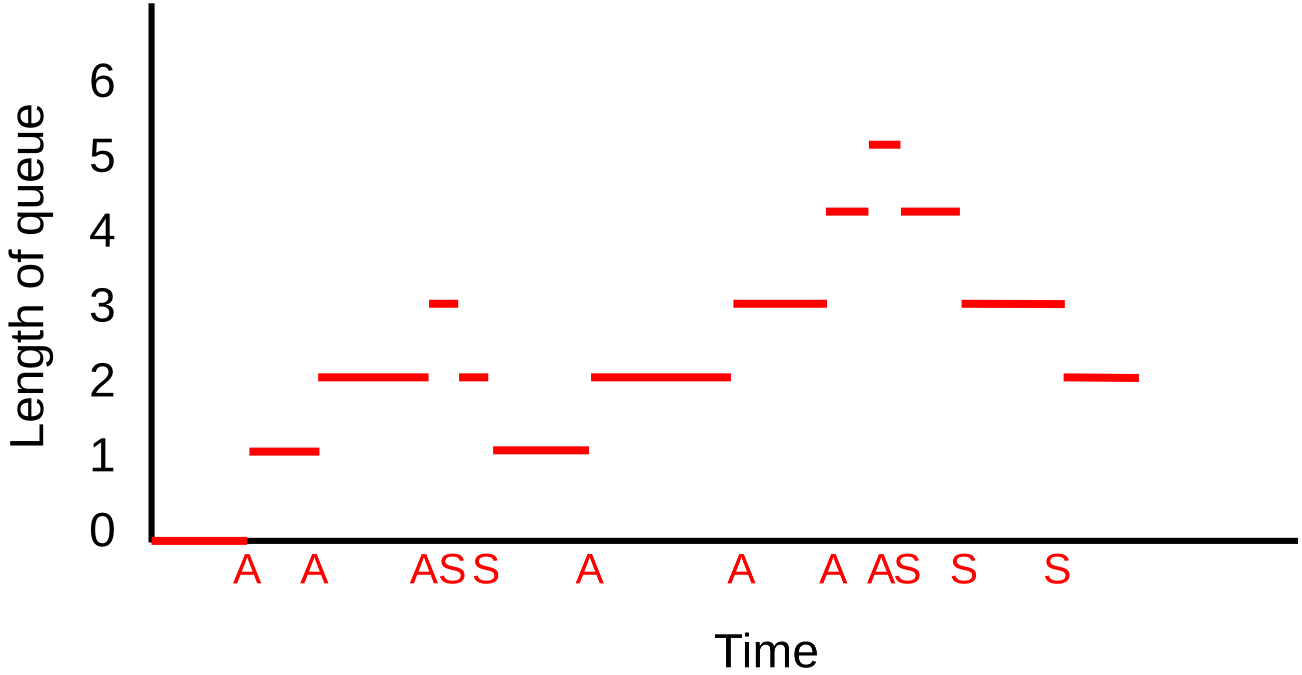
- Customers
- Queue
- Server
- Arrival process
- Service process



Queues

Times between arrivals and services are usually random

- They are described using a random variable



Queues

Queues are an important modelling tool

- They help to design many kinds of systems

Typical questions of queues:

- What is the probability that the queue will become full?
- What proportion of the time is the server busy?
- What is the probability that the queue will become empty?
- How full is the queue at time t?
- What is the average queue length?

Example Queue

We now consider a special case:

- Arrival and service times are exponentially distributed
- (We will see in a few weeks what this means)

Standard view:

- Observe length of queue
- This changes at random discrete points in time

(Only!) in this case we can make an alternative simulation

- Observe probabilities for number of customers in queue
- These change in continuous time

Example Queue

The variables of interest:

- The probability of there being i customers in the queue at time t:

$$\pi_i(t)$$

Assume the maximum queue length is 2

- (Customers arriving at a full queue are turned away)

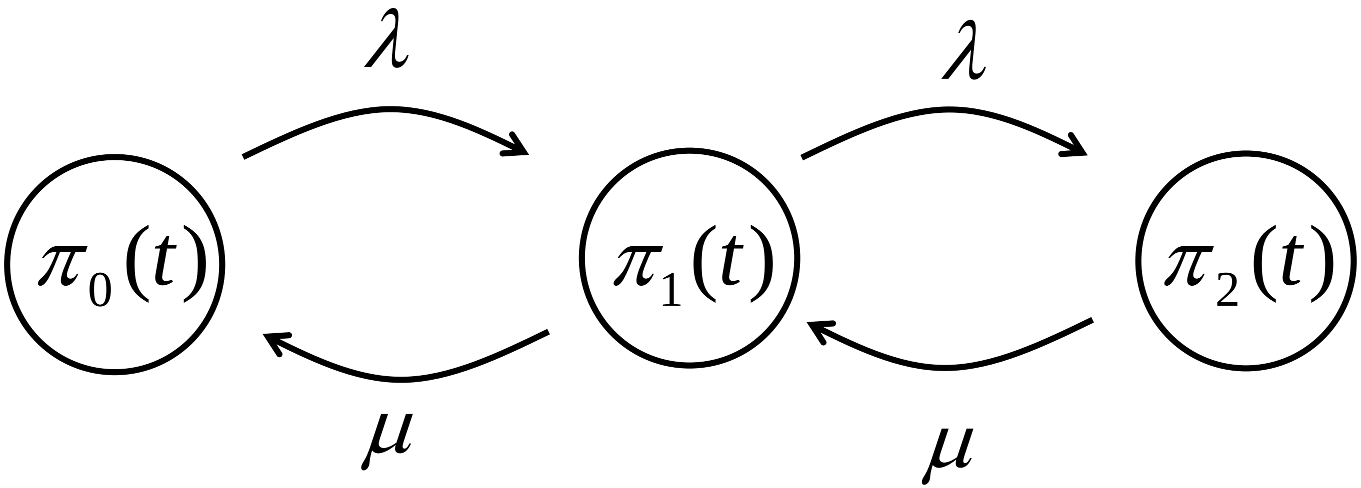
We then have three variables

$$\pi_0(t) \qquad \pi_1(t) \qquad \pi_2(t)$$

Example Queue

Probability flows …

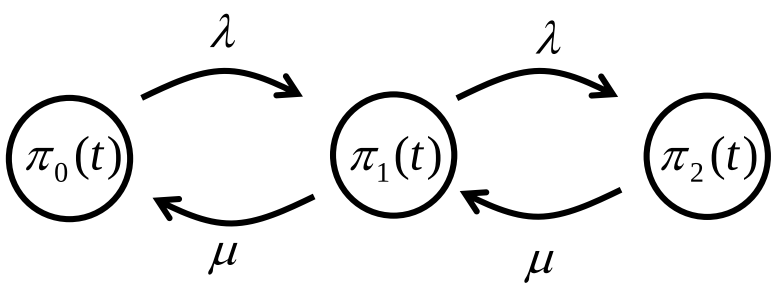
- at rate λ for arrivals
- at rate μ for service completions



Example Queue

Balance equation

- Rate of change = inflow – outflow



$$\frac{d\pi_0}{dt} = \mu \cdot \pi_1 - \lambda \cdot \pi_0$$

$$\frac{d\pi_1}{dt} = \lambda \cdot \pi_0 + \mu \cdot \pi_2 - (\lambda + \mu) \cdot \pi_1$$

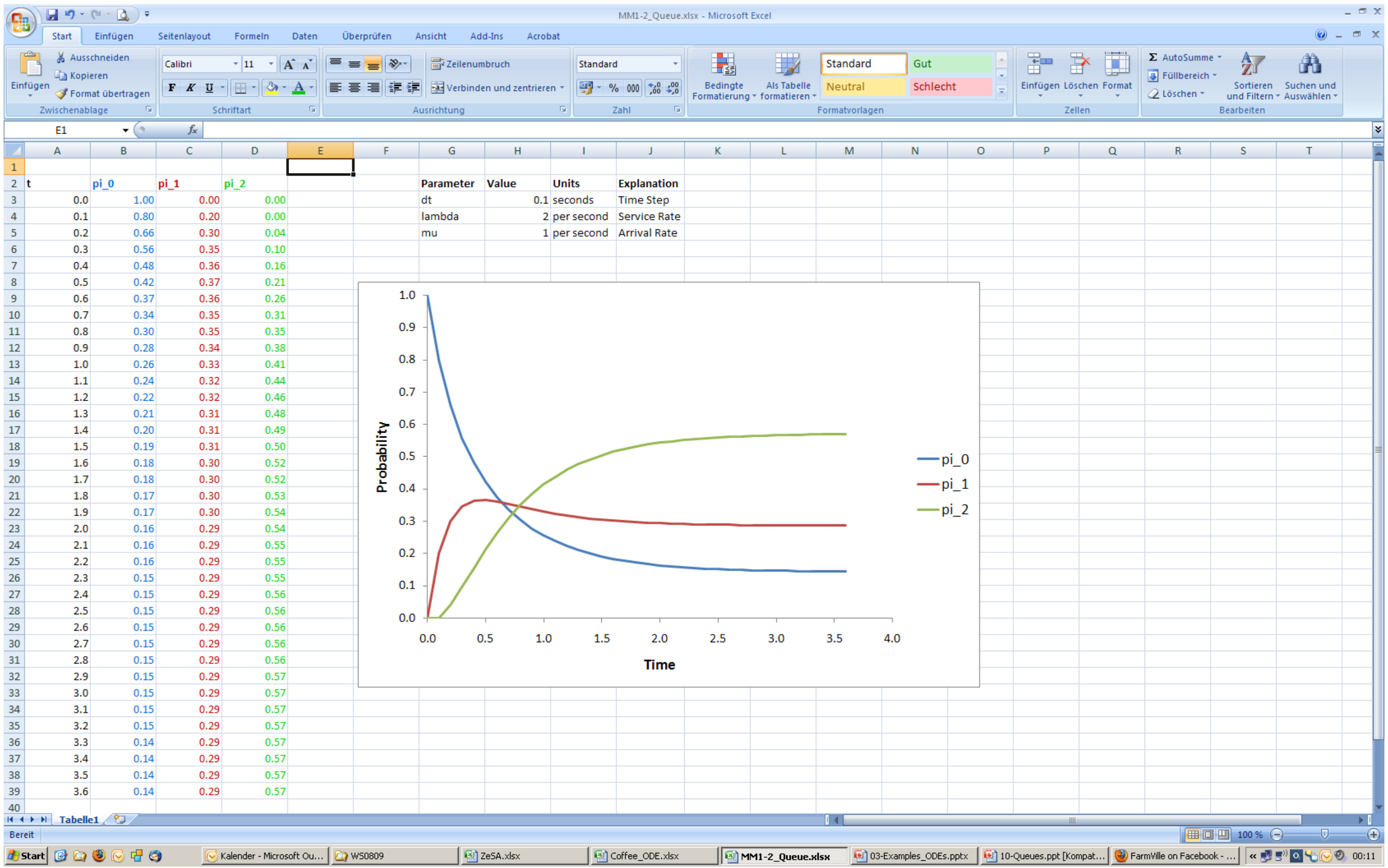
$$\frac{d\pi_2}{dt} = \lambda \cdot \pi_1 - \mu \cdot \pi_2$$

Example Queue

Using Euler's method we obtain at each discrete time step:

$$\pi_{0,k+1} = \pi_{0,k} + h \cdot \left(\mu \cdot \pi_{2,k} - \lambda \cdot \pi_{0,k} \right)$$
$$\pi_{1,k+1} = \pi_{1,k} + h \cdot \left(\lambda \cdot \pi_{0,k} + \mu \cdot \pi_{2,k} - (\lambda + \mu) \cdot \pi_{1,k} \right)$$
$$\pi_{2,k+1} = \pi_{2,k} + h \cdot \left(\lambda \cdot \pi_{1,k} - \mu \cdot \pi_{2,k} \right)$$

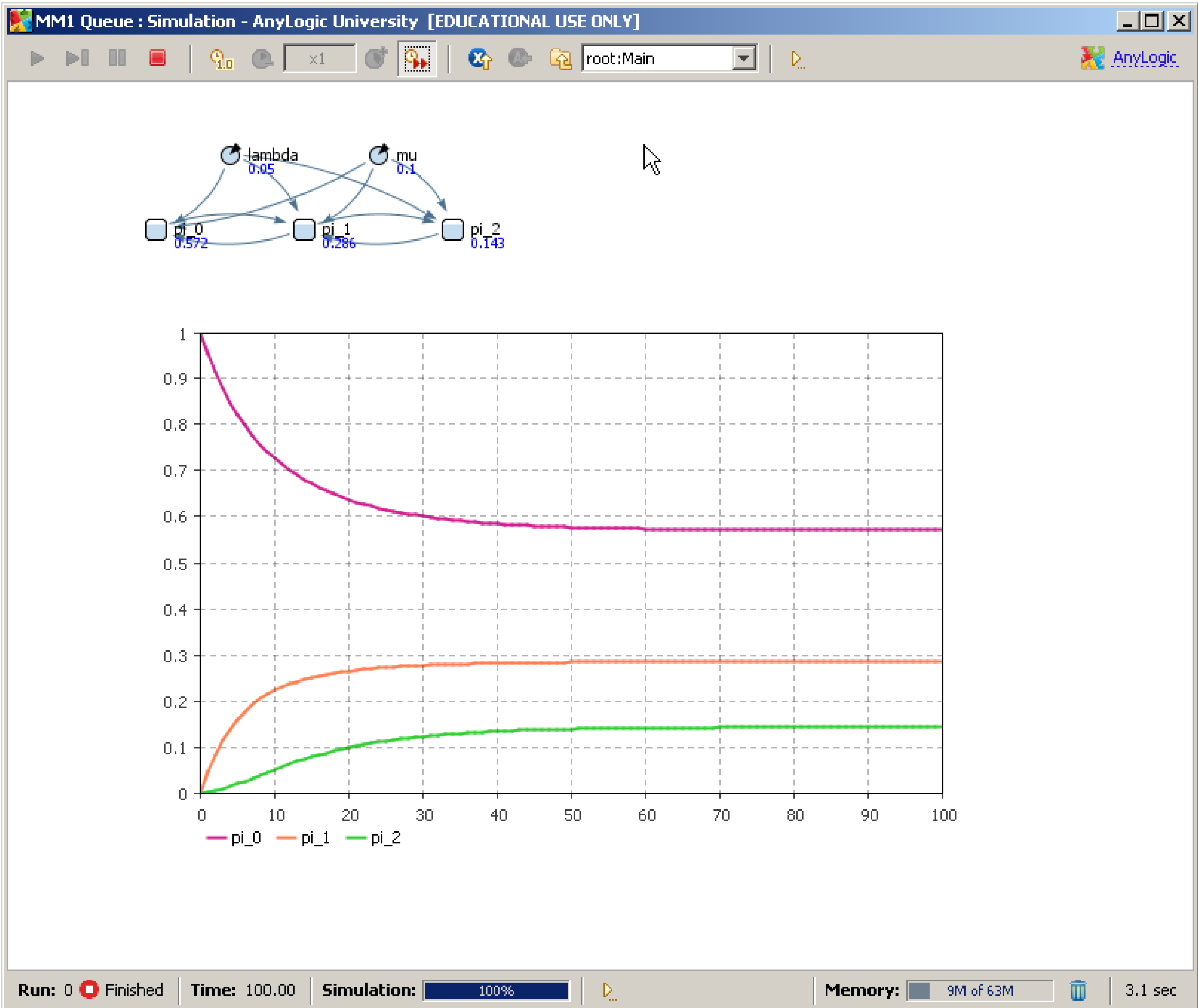
Example Queue



t	pi_0	pi_1	pi_2
0.0	1.00	0.00	0.00
0.1	0.80	0.20	0.00
0.2	0.66	0.30	0.04
0.3	0.56	0.35	0.10
0.4	0.48	0.36	0.16
0.5	0.42	0.37	0.21
0.6	0.37	0.36	0.26
0.7	0.34	0.35	0.31
0.8	0.30	0.35	0.35
0.9	0.28	0.34	0.38
1.0	0.26	0.33	0.41
1.1	0.24	0.32	0.44
1.2	0.22	0.32	0.46
1.3	0.21	0.31	0.48
1.4	0.20	0.31	0.49
1.5	0.19	0.31	0.50
1.6	0.18	0.30	0.52
1.7	0.18	0.30	0.52
1.8	0.17	0.30	0.53
1.9	0.17	0.30	0.54
2.0	0.16	0.29	0.54
2.1	0.16	0.29	0.55
2.2	0.16	0.29	0.55
2.3	0.15	0.29	0.55
2.4	0.15	0.29	0.56
2.5	0.15	0.29	0.56
2.6	0.15	0.29	0.56
2.7	0.15	0.29	0.56
2.8	0.15	0.29	0.56
2.9	0.15	0.29	0.57
3.0	0.15	0.29	0.57
3.1	0.15	0.29	0.57
3.2	0.15	0.29	0.57
3.3	0.14	0.29	0.57
3.4	0.14	0.29	0.57
3.5	0.14	0.29	0.57
3.6	0.14	0.29	0.57

Parameter	Value	Units	Explanation
dt	0.1	seconds	Time Step
lambda	2	per second	Service Rate
mu	1	per second	Arrival Rate

Example Queue



Lorenz's Chaotic Equations

Edward Lorenz (1963)

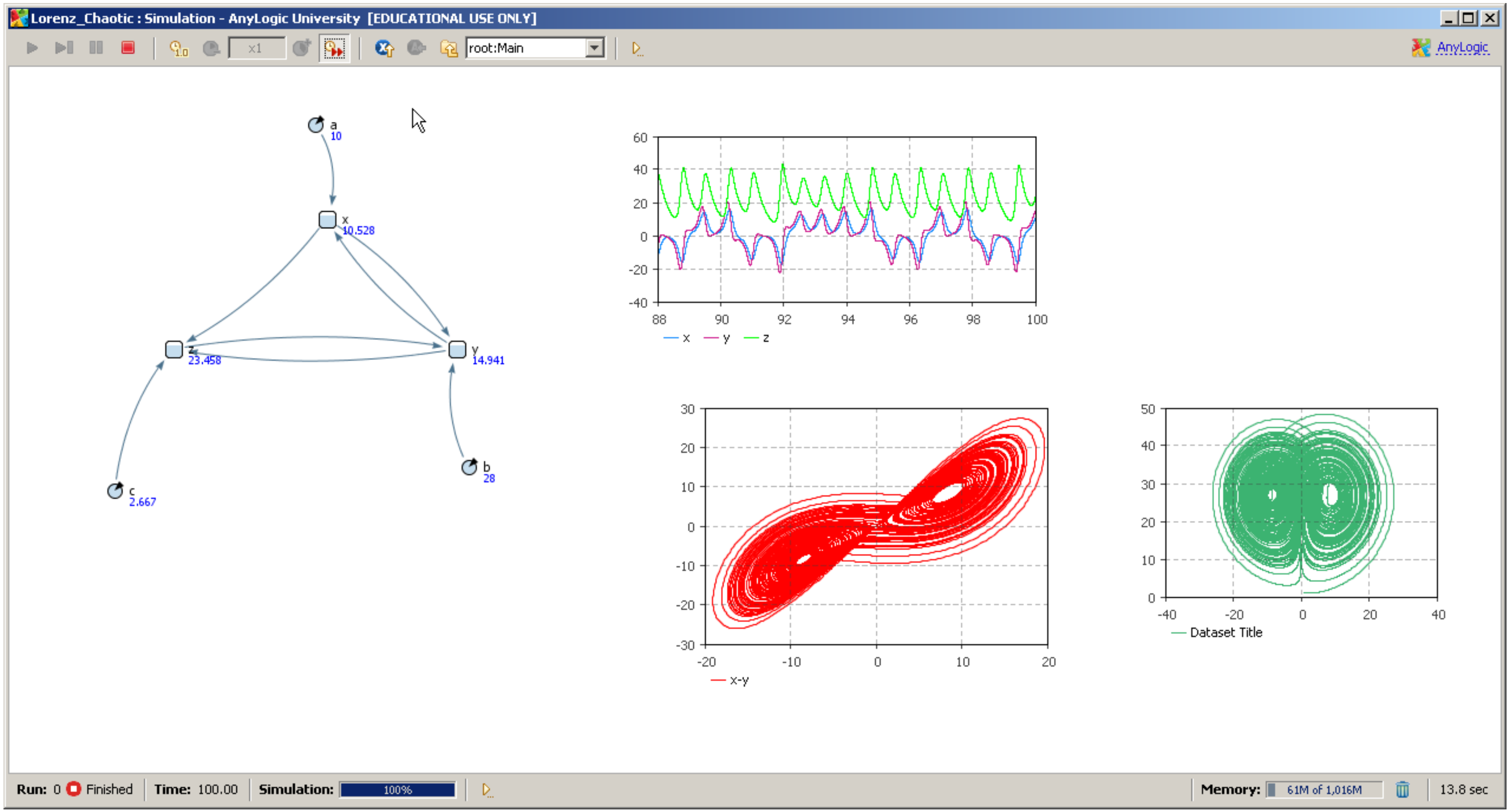
- Three equations from meteorology

$$\frac{dx}{dt} = a \cdot (y - x)$$

$$\frac{dy}{dt} = -x \cdot z + b \cdot x - y$$

$$\frac{dz}{dt} = x \cdot y - c \cdot z$$

Lorenz's Chaotic Equations



Three Species Model

Three-species predator-prey model:

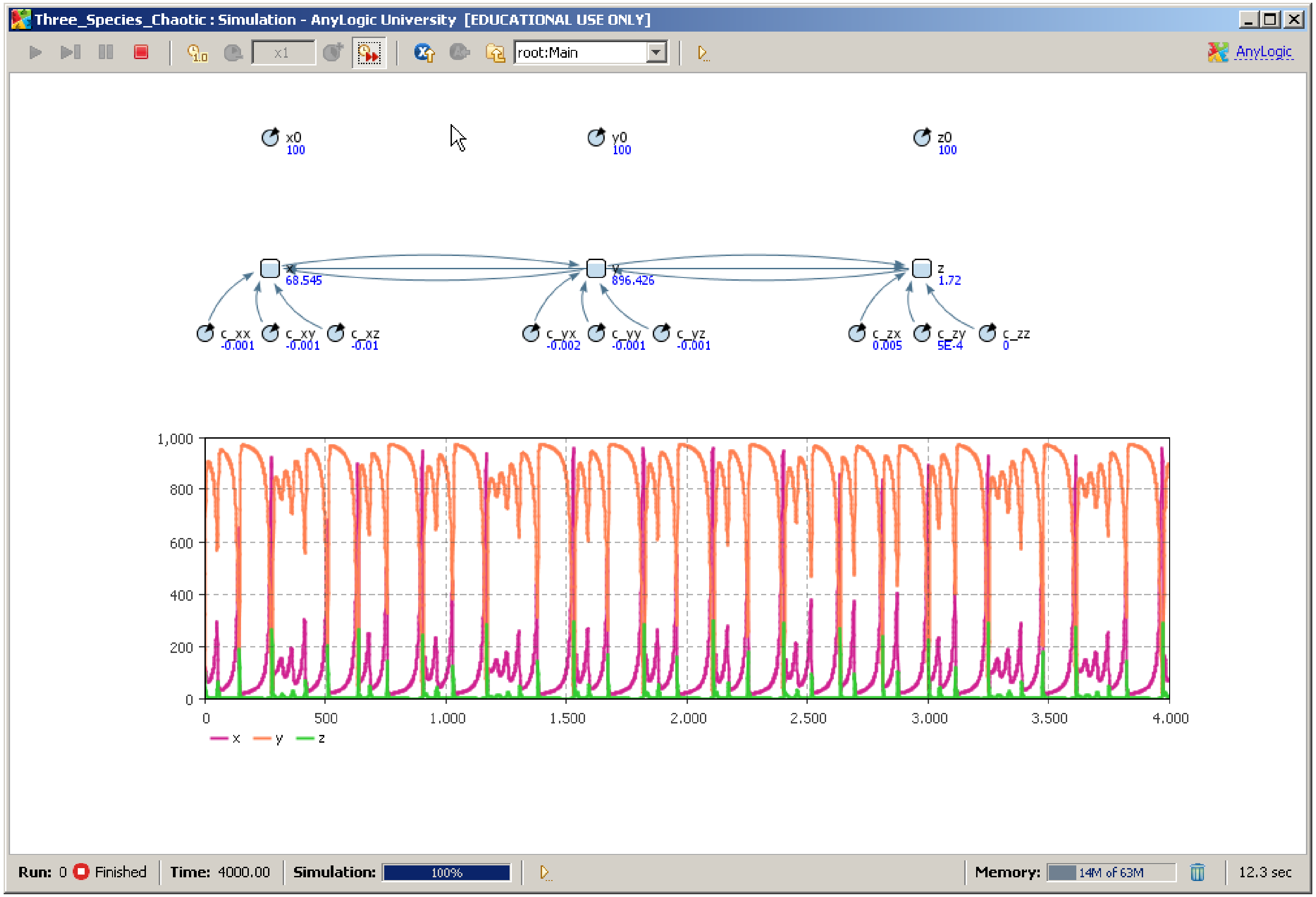
- Species z preys on species x and y
- There is crowding both within x and y, and between x and y

$$\frac{dx}{dt} = x - 0.001x^2 - 0.001xy - 0.01xz$$

$$\frac{dy}{dt} = y - 0.0015xy - 0.001y^2 - 0.001yz$$

$$\frac{dz}{dt} = -z + 0.005xz + 0.0005yz$$

Chaotic Three-Species Model



Learning Goals

Questions to test your knowledge:

- Given a textual description of a system, can you build an ODE model of it?
- Given an ODE (system), write down the equations used by Euler's method to perform the simulation