OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF

FAKULTÄT FÜR INFORMATIK

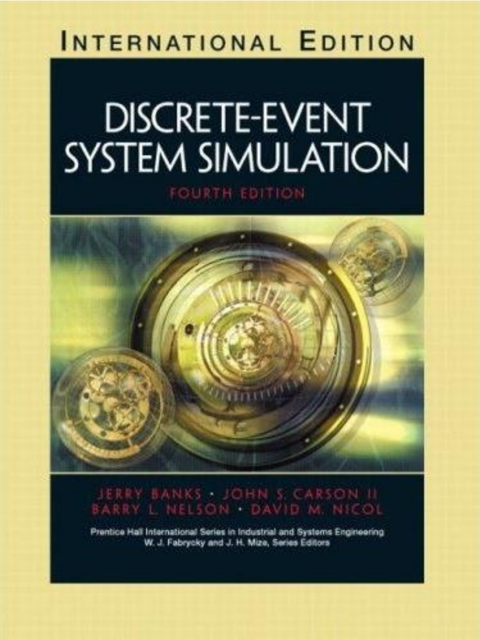
Introduction to Simulation

Output Analysis

Background Reading

Relevant sections of the book:

- 11.1
- 11.2
- 11.3
- 11.4 (parts)
- 11.5 (parts)



A Question

Let X be the random variable

- "Result from throwing a die"

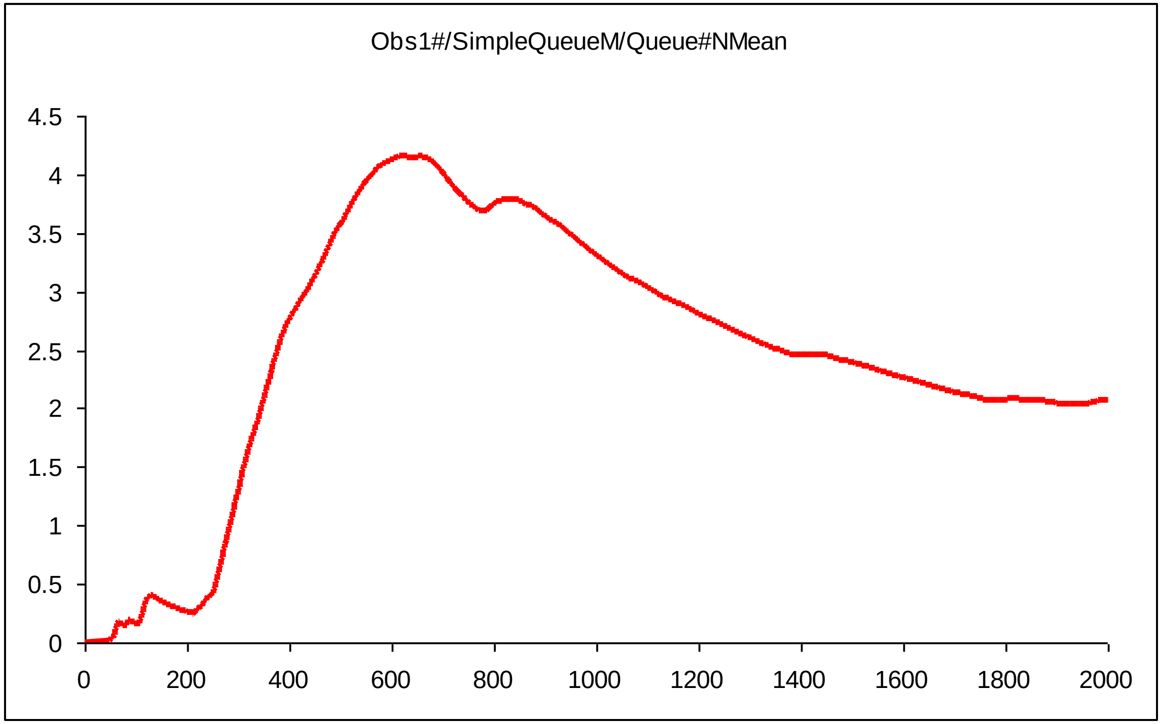
Question: What is the expected value $E(X)$?

Would you throw the die just once and take the result as your answer?

What Are We Doing Wrong?

We have been leaving something out in our simulations

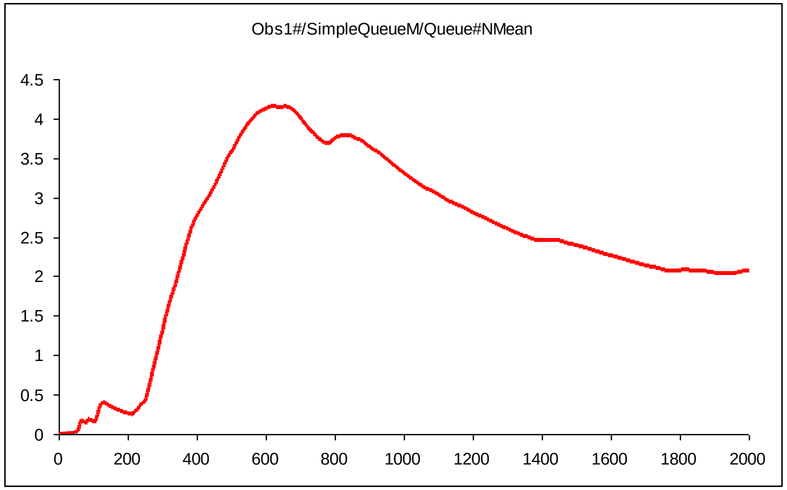
- We have made one observation only
- Example – the average length of the queue in the bank:

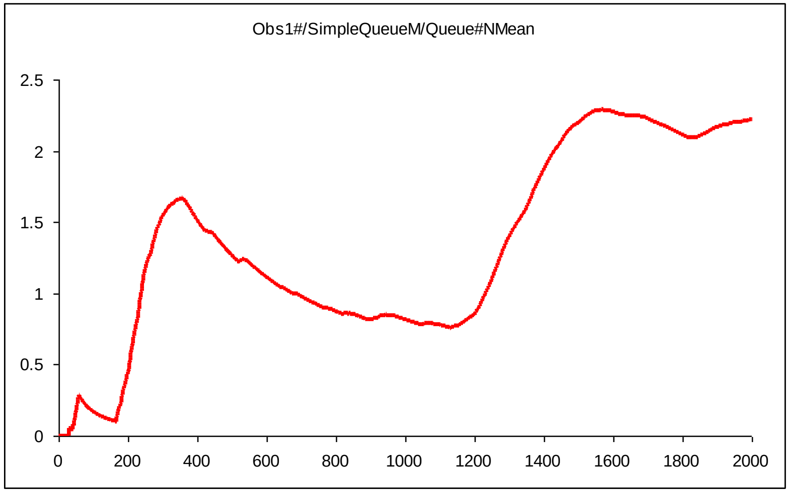


Example

Consider our bank example

- Observe the average queue length $0 < T < 2000$
- Simulate with different sets of random numbers (with the same distribution!)





Example

Consider our bank example

- Observe the queue length at $T = 2000$
- Repeat the simulation using different sets of random numbers

Results:

- 2, 12, 3, 7, 0, 10, 2, 5

Which is the "right" answer?

What Are We Doing Wrong?

Real systems behave randomly

- They contain random variables

Our simulation results are also random

- They depend on random numbers

Running a simulation means taking one sample of a RV

- $\Rightarrow$ We need a more sophisticated approach!

Mean & Sample Variance

Consider a random variable Y

Take a set of observations $Y_i \quad i = 1 \ldots n$

The *Sample Mean* is defined as: $\bar{Y} = \frac{1}{n} \sum Y_i$

The *Sample Variance* is defined as: $S^2 = \frac{1}{n-1} \sum (Y_i - \bar{Y})^2$

Then $S^2 \approx \mathrm{var}(Y)$

(The sample variance approximates the actual variance in Y)

Bias

Given a value θ and an estimator for it $\hat{\theta}$

In general, we may have $E(\hat{\theta}) = \theta + b$
(i.e. the estimator may be biased)

Bias means we have a systematic error

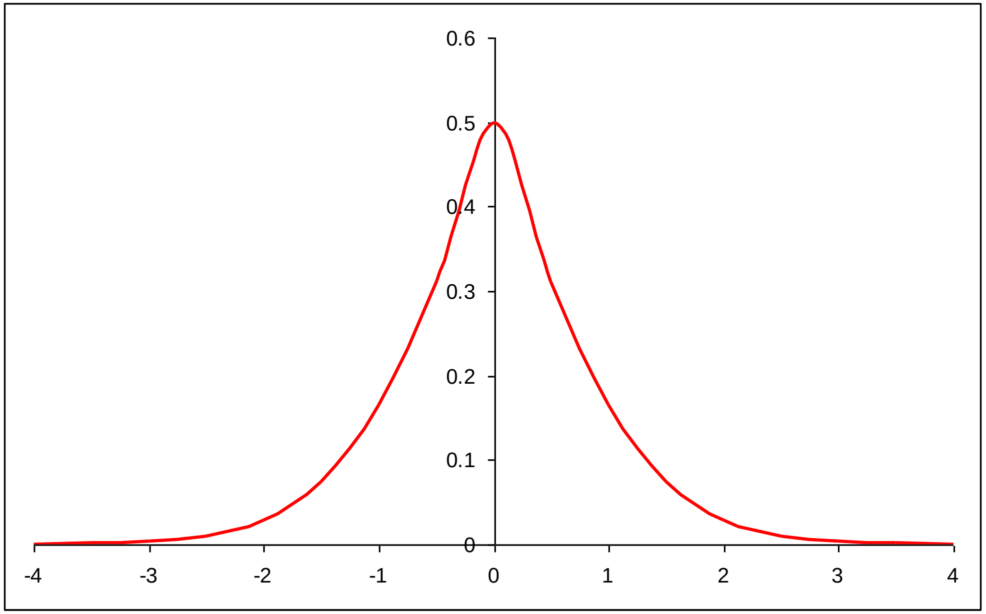
- Our estimate will be too large / too small

Many statistical methods assume (require) that $b = 0$

Student's t–Distribution

Introduced by W. Gosset (a.k.a. "Student")

- Has one parameter f ("degrees of freedom")
- Used for hypothesis testing
- Tables of values are available



W. Gosset
1876-1937

Student's *t*-Distribution

Given:

- A measure of the real system θ
- An estimator $\hat{\theta}$ for θ
- An estimator $\hat{\sigma}(\hat{\theta})$ for $\sigma(\theta)$

If $\theta = E(\hat{\theta})$ then the value

$$t = \frac{\theta - \hat{\theta}}{\hat{\sigma}(\hat{\theta})}$$

is *t*-distributed with n – 1 d.o.f.

Confidence Intervals

Choose a level of significance α

Rearrange the expression

$$0 - t_{\alpha/2,f} \leq \frac{\theta - \hat{\theta}}{\hat{\sigma}(\hat{\theta})} \leq 0 + t_{\alpha/2,f}$$

to obtain the confidence interval

$$\hat{\theta} - \hat{\sigma}(\hat{\theta}) \cdot t_{\alpha/2,f} \leq \theta \leq \hat{\theta} + \hat{\sigma}(\hat{\theta}) \cdot t_{\alpha/2,f}$$

Confidence Intervals

What does the confidence interval mean?

$$\hat{\theta} - \hat{\sigma}(\hat{\theta}) \cdot t_{\alpha/2,f} \leq \ \theta \ \leq \hat{\theta} + \hat{\sigma}(\hat{\theta}) \cdot t_{\alpha/2,f}$$

The value of θ lies in the c.i. with prob. 1−α

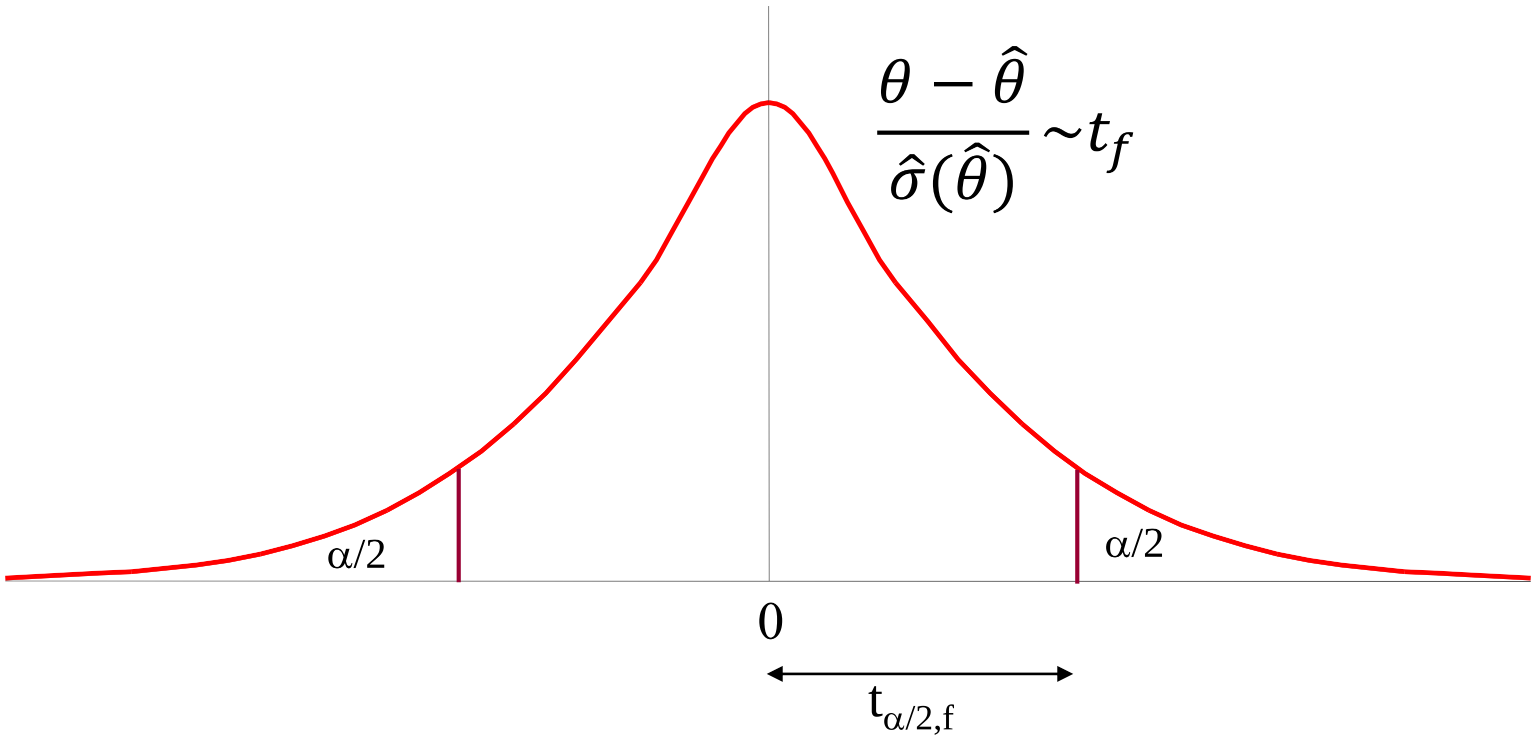
This is the preferred way to present the results of a simulation

Reminder:

- θ is the (theoretical) output of the simulation model
- $\hat{\theta}$ is the result of a (finite) simulation experiment

Confidence Intervals

What are we doing?



Confidence Intervals

How to obtain $\hat{\sigma}(\hat{\theta})$?

Answer: Use the approximation $\frac{S}{\sqrt{n}} \approx \hat{\sigma}(\hat{\theta})$

Confidence Intervals

Improving accuracy:

We have $$\frac{S}{\sqrt{n}} \approx \hat{\sigma}(\hat{\theta})$$

If we wish to halve the width of the the c.i. ...

- … we must use $4n$ samples!

This can imply:

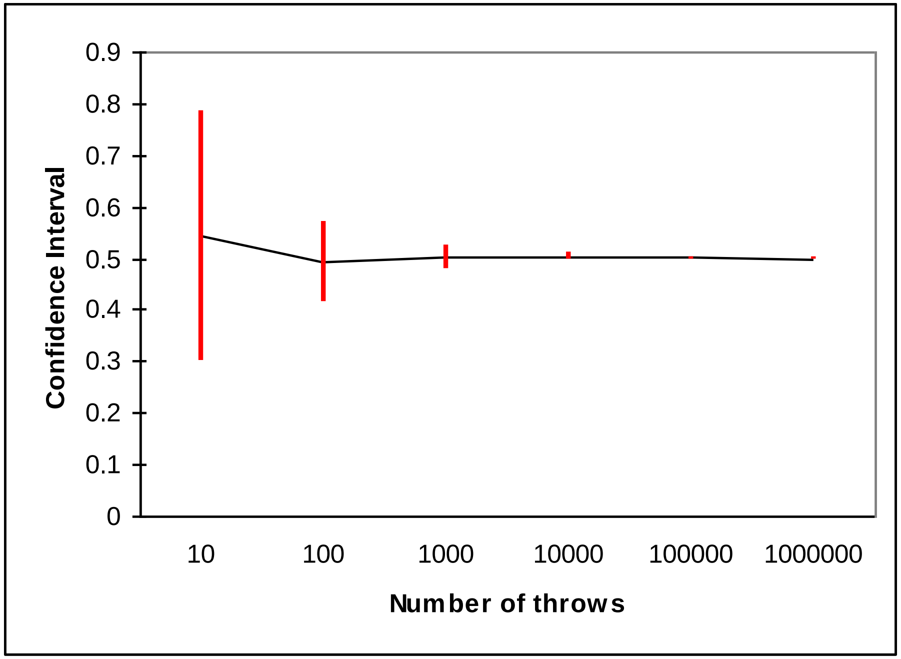
- High accuracy can be expensive in simulation!

Confidence Intervals

Example: Throwing a coin (interpret the result as 0 or 1)

- θ is the (true, theoretical) expected value
- $\hat{\theta}$ is the result from a (finite) experiment

# Throws	Lower	Mean	Upper
10	0.302	0.545	0.788
100	0.416	0.493	0.570
1000	0.479	0.503	0.527
10000	0.496	0.503	0.511
100000	0.499	0.501	0.503
1000000	0.499	0.500	0.501



Confidence Intervals

A confidence interval is only accurate if:

- $\hat{\theta}$ is an unbiased estimator of θ
- $\hat{\sigma}^2(\hat{\theta})$ is an unbiased estimator of $\sigma^2(\hat{\theta})$

If the observations are not independent...

- ... then the estimator will be biased
- … the confidence interval will be shifted

Example

Consider a simple queue

- One arrival stream
- One server

Intervals:

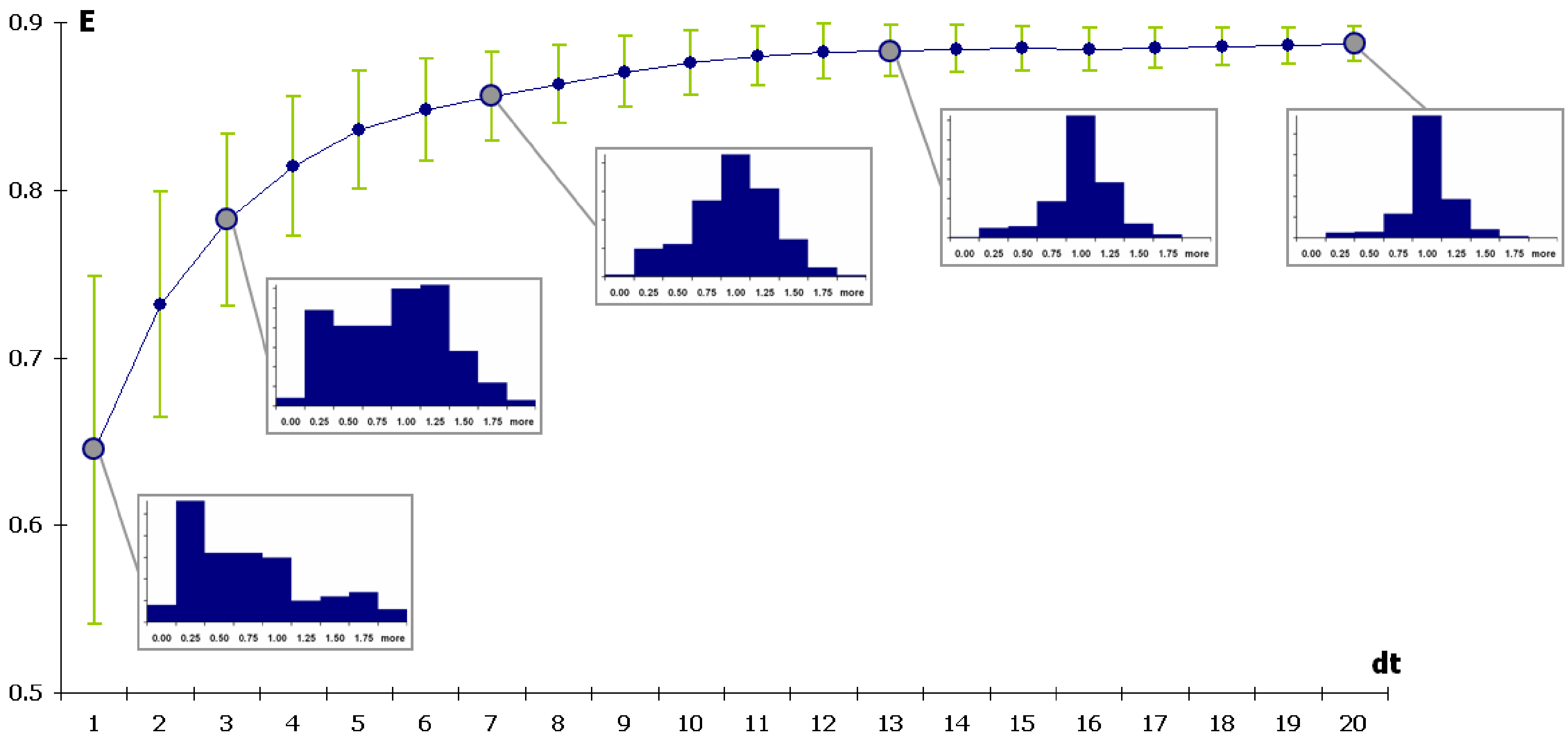
- Arrivals: Exponential distribution, mean = 13s
- Service: Normal distribution, $\mu = 10\text{s}$, $\sigma = 2\text{s}$

How does the queue length behave over time?

Simulation experiment:

- 100 samples (replications), $\alpha = 0.05$

Example



E
0.9
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0.6
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11
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13
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15
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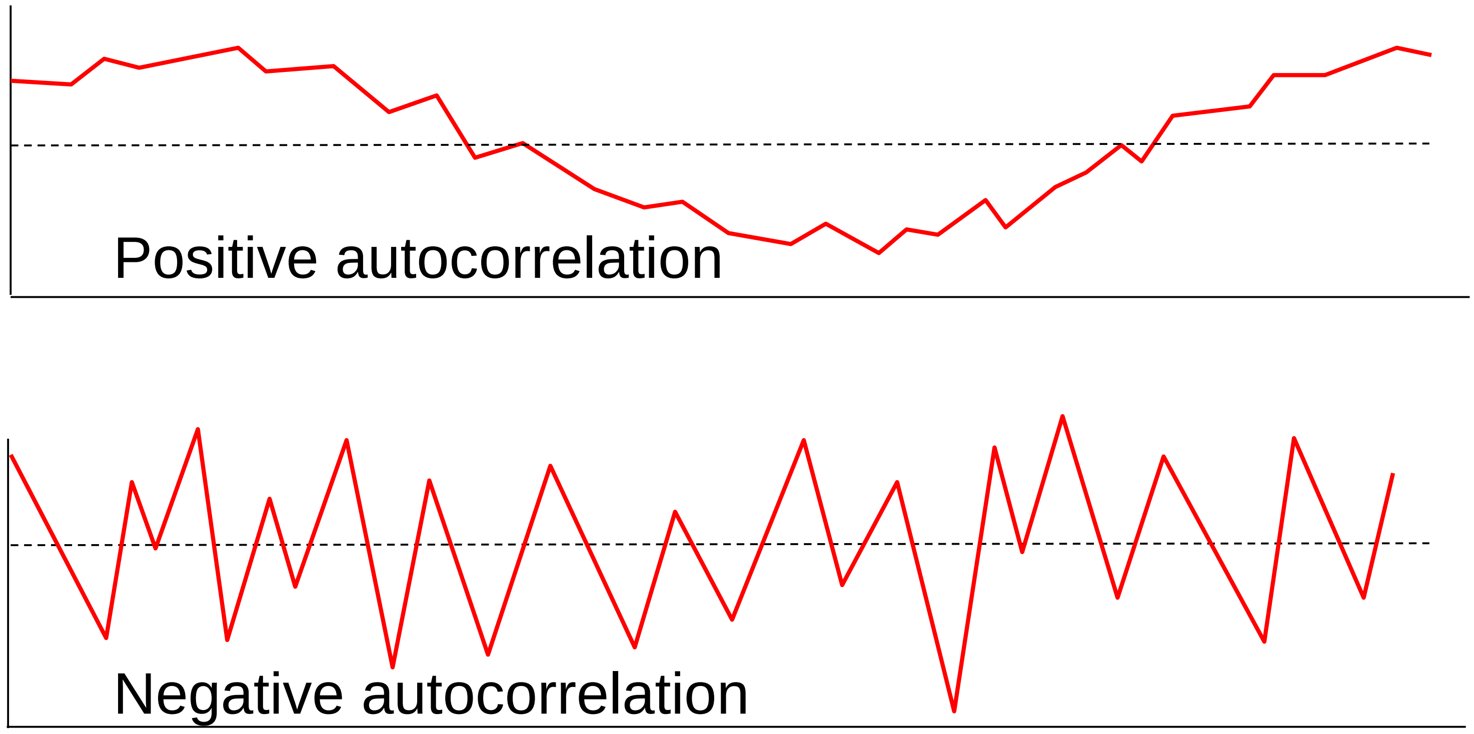
19
20
0.00 0.25 0.50 0.75 1.00 1.25 1.50 1.75 more

Bias due to Autocorrelation

Consider the Y_i from one simulation run (time series):



Bias due to Autocorrelation

Positive autocorrelation ...

- leads to under-estimation of S^2
- leads to over-optimistic confidence intervals
- Example: Most queues

Negative autocorrelation ...

- leads to over-estimation of S^2
- leads to under-optimistic confidence intervals
- Example: Some inventory systems

$\Rightarrow$ Don't use time series for confidence intervals!

Independent Replications

The method of independent replications:

- Run the simulation R times
- Use independent sets of random numbers for each run
- Make the observations $Y_r,\ r = 1 \ldots R$
- Compute $\hat{\theta}$ and S^2 from the Y_r
- Compute a confidence interval from $\hat{\theta}$ and S^2

Independence ensures that $\hat{\theta}$ is unbiased

Terminating Simulations

A terminating simulation is one...

- which runs up to a specified time
- which has known initial conditions
- in which the initial conditions are important

Examples:

- Will the satellite survive for 5 years?
- How full is the bank 2 hours after opening time?

Non-Terminating Simulations

A non-terminating simulation is one...

- which runs for an indefinite period
- in which the steady-state behaviour is of interest
- in which the initial conditions are not important

Examples:

- Any continuously running system
- Computer centre, traffic system, ...

Non-Terminating Simulations

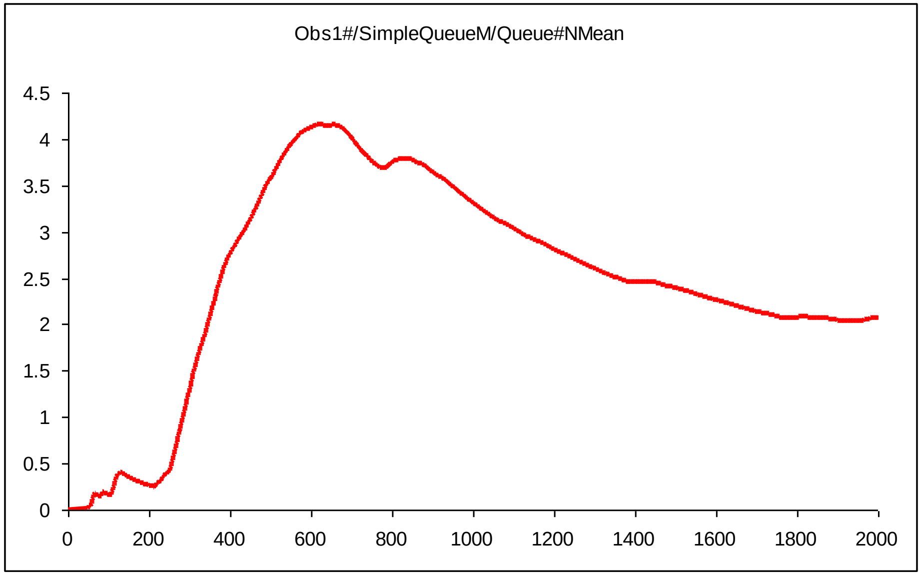
Difficulties with non-terminating simulations:

- Initial bias
- How long to run the simulation?
- Trade off between replication and duration

Initial Bias

In a non-terminating simulation

- We are interested in the steady-state behaviour
- The values at the beginning will usually be untypical ("initial bias")
- Example: Queue in bank (starting empty):

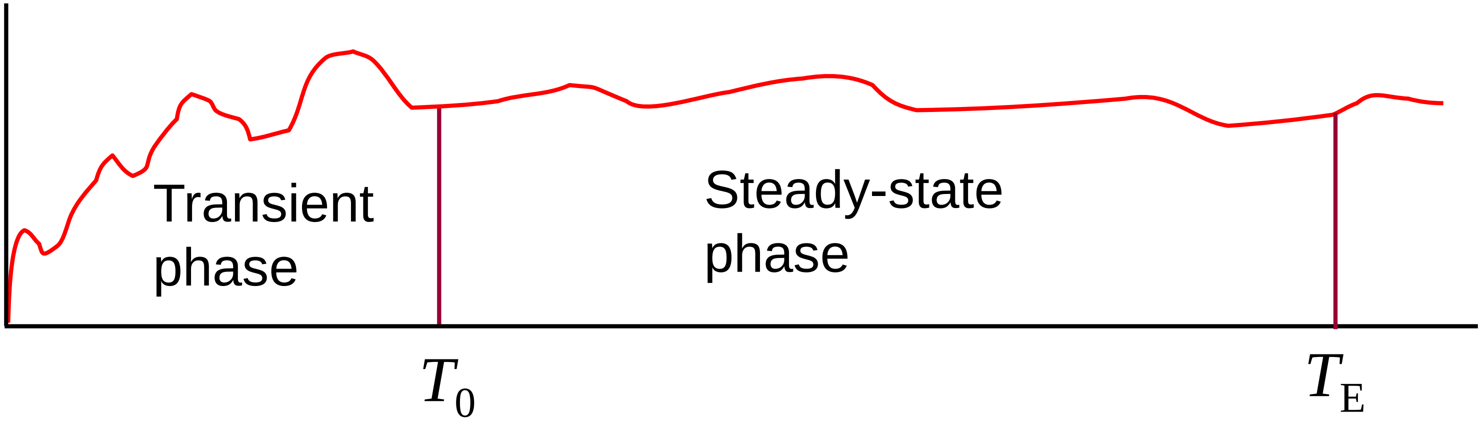


Initial Bias

Solution:

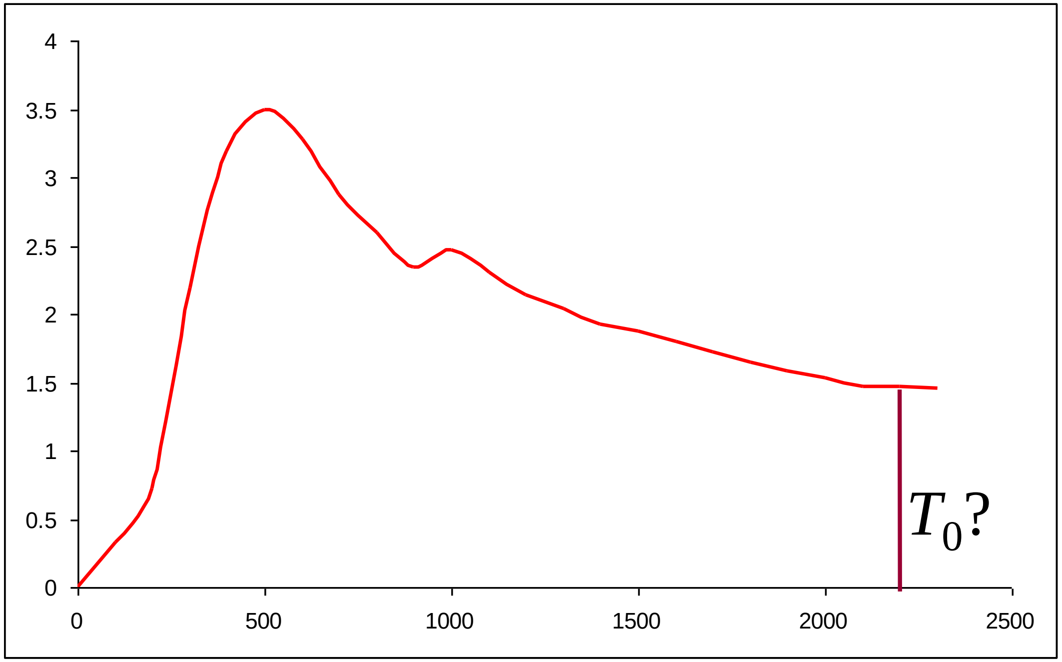
- Delete values from transient phase
- Choose T_0 after transient phase is over
- Observe from T_0 to T_E

How to choose T_0 (and T_E)?



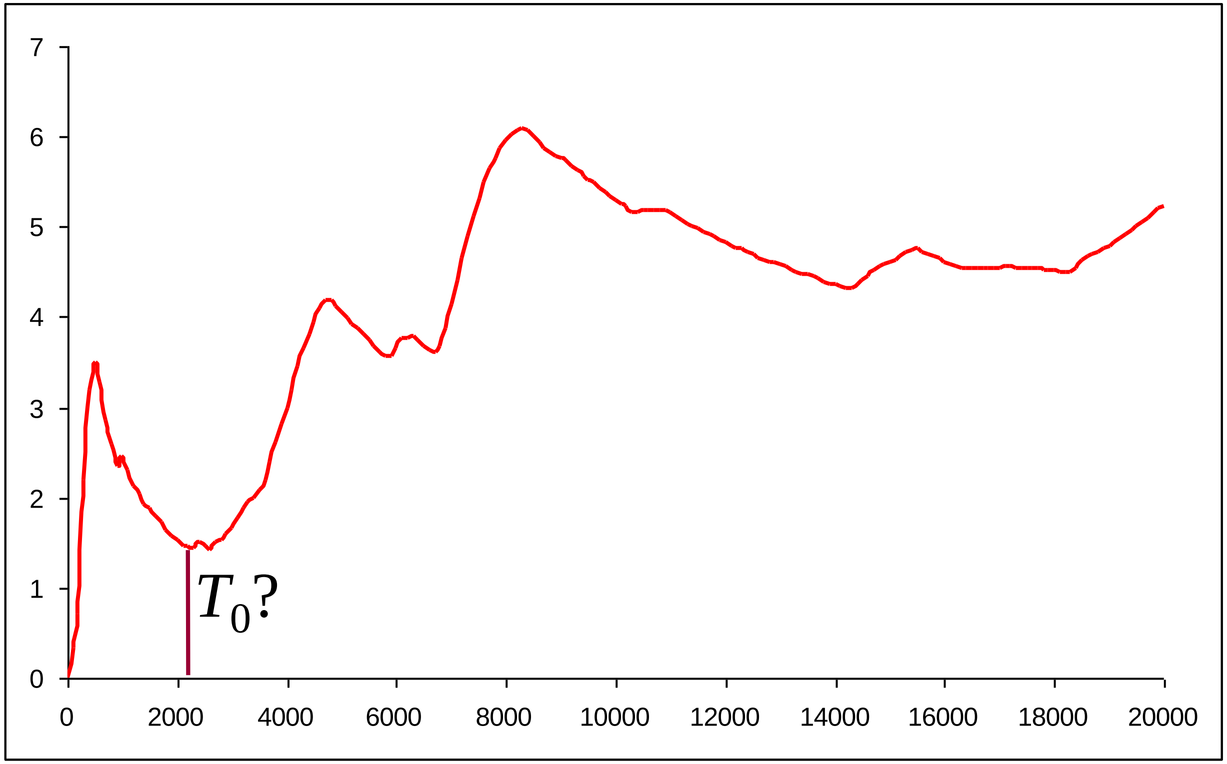
Initial Bias

How long is the transient phase in a non-terminating simulation?



Initial Bias

A difficult question!



Ensemble Averages

Using just one run to find T_0 is dangerous

- Compute ensemble averages

Perform independent replications to obtain

$$Y_{r,i} \quad i = 1 \dots n, \; r = 1 \dots R$$

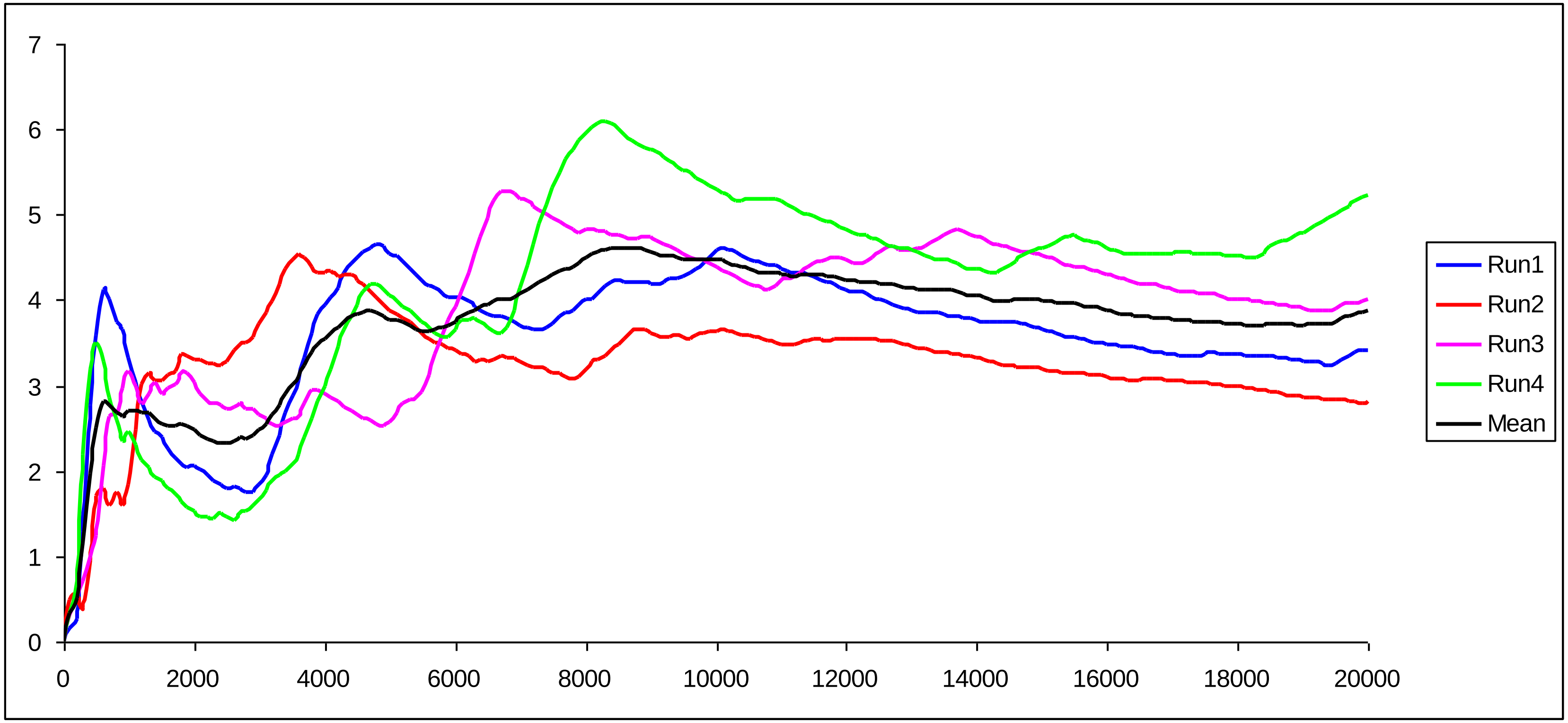
Compute average values across replications

$$Y_i = \frac{1}{R} \sum_{r=1}^{R} Y_{r,i} \quad i = 1 \dots n$$

Test the sequence Y_i for the end of the transient phase

Ensemble Averages

Compute ensemble averages:



Non–Terminating Simulations

Increasing R will make the c.i. narrower

It will not reduce the initial bias, i.e.

- ...we will get a better c.i. around $\theta + b$!

A computing time tradeoff is necessary between

- ...increasing T_E and increasing R

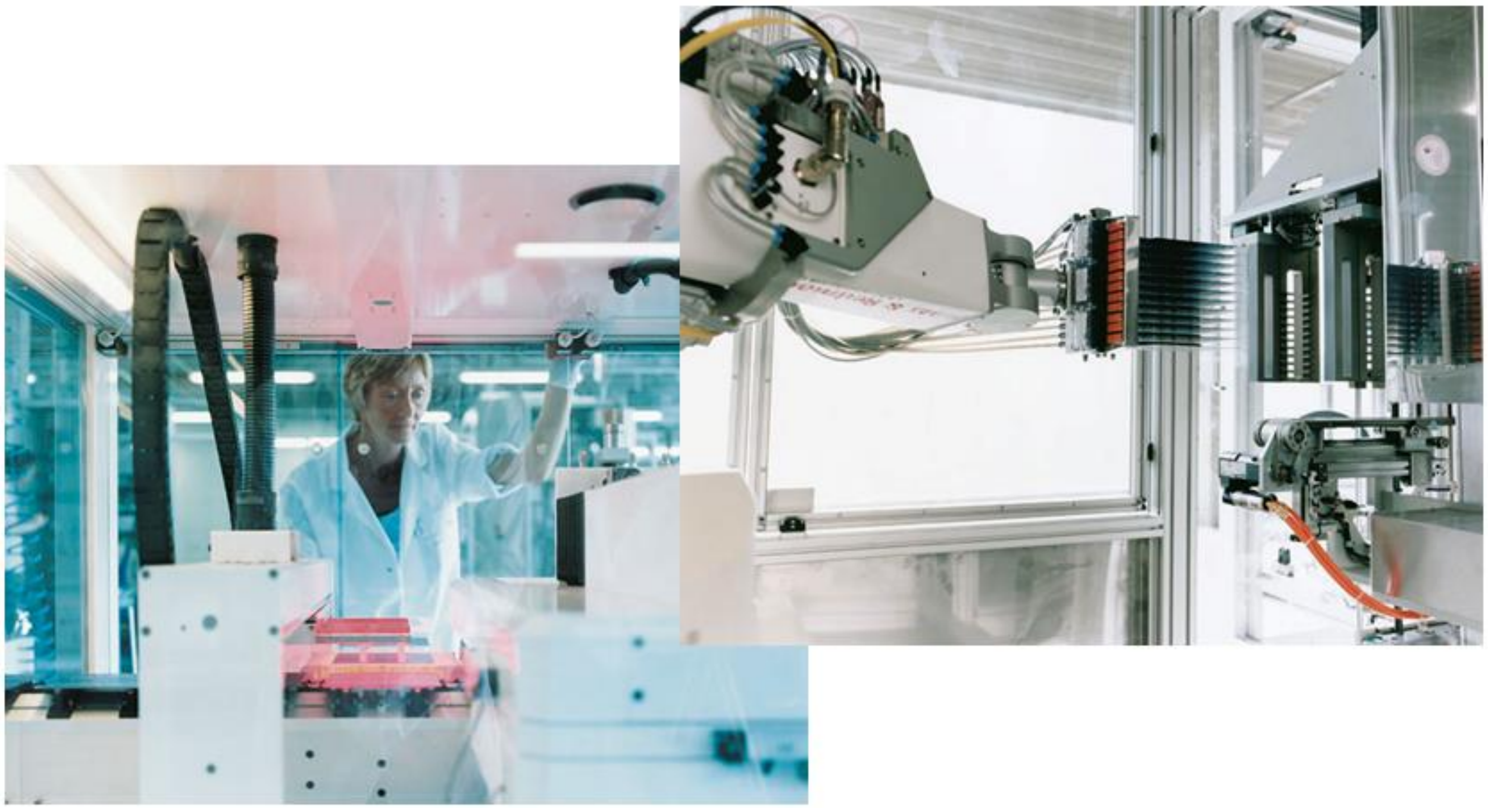
Solution:

- $R > 25$ is not useful

Simulation Project

Utilization of a machine ("Nassbank") at Q-Cells:

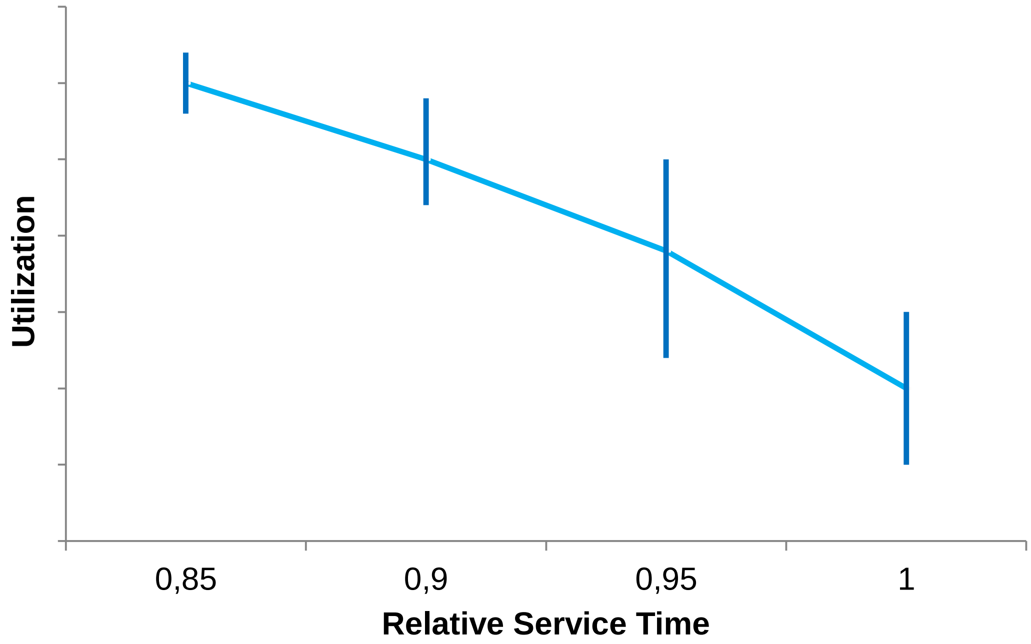
- Test dependency of utilization on service time reduction



Simulation Project

Simulation experiment:

- 25 replications
- 99% confidence interval



Simulation Project

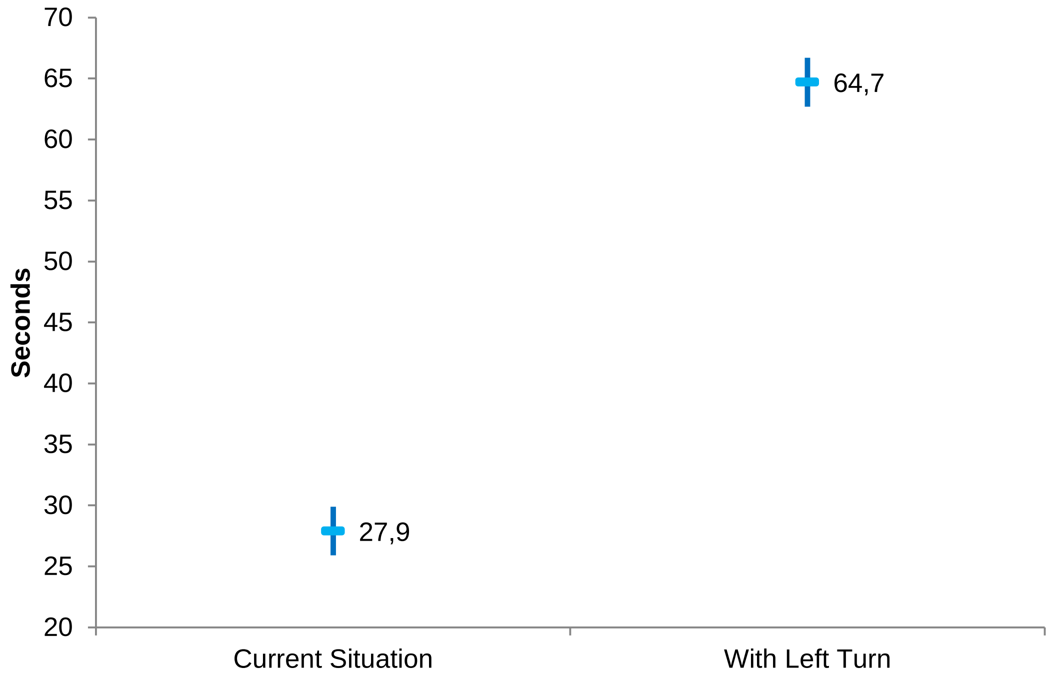
Intersection of Gustav-Adolf-Straße and Bundesstraße 1

- Analyse the effect of allowing left turns to the north

Simulation Project

Simulation experiment for waiting time for westbound traffic:

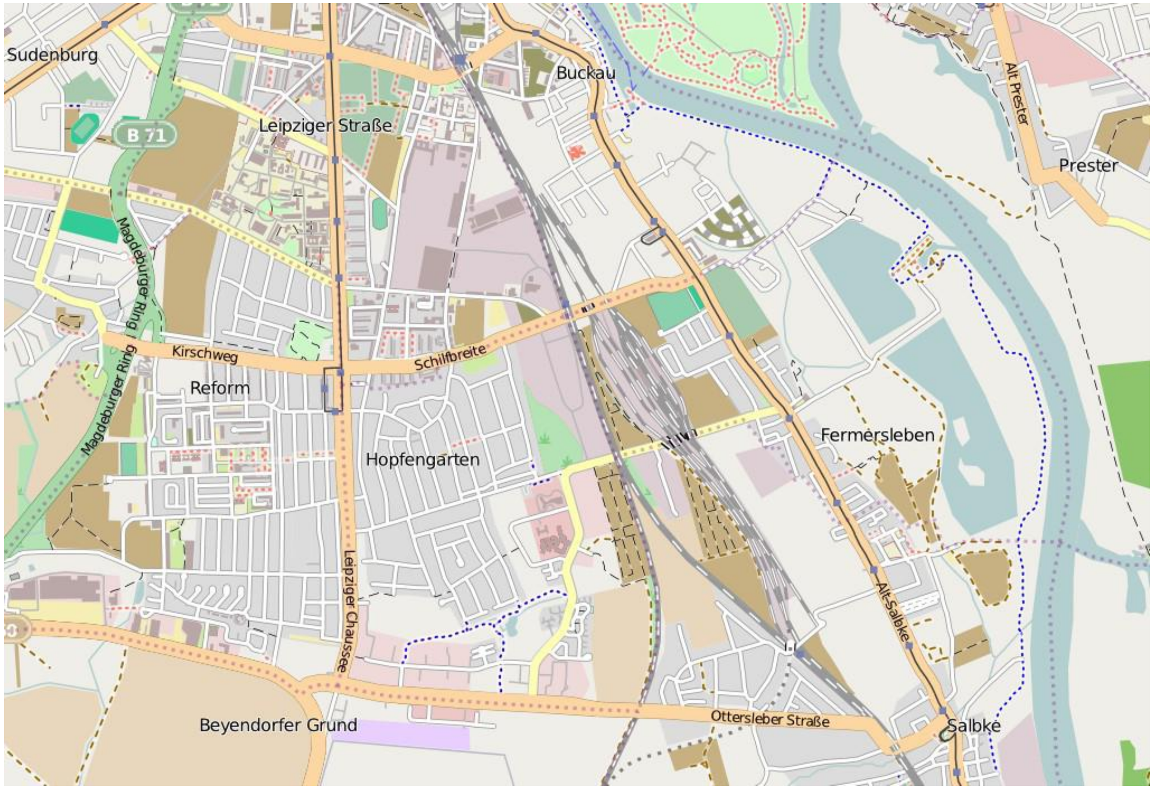
- 50 replications
- 90% confidence interval



Simulation Project

Magdeburg Buckau

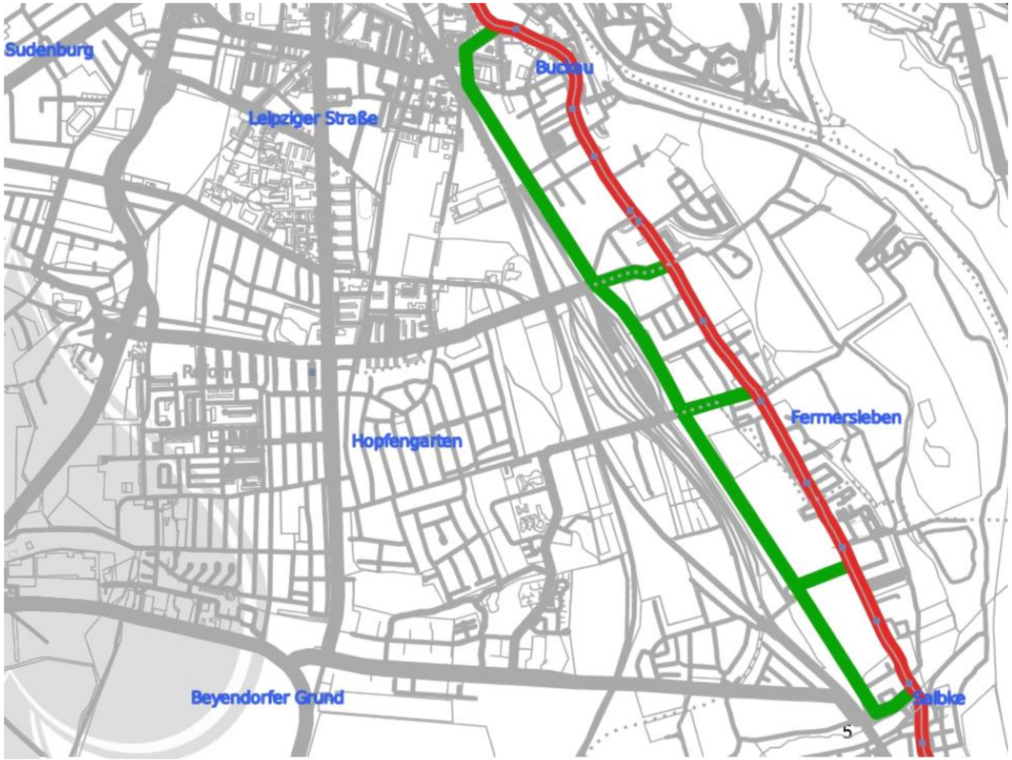
- Analyze the effect of a planned street to disburden an existing one of through traffic when developing new industrial areas



Simulation Project

Magdeburg Buckau

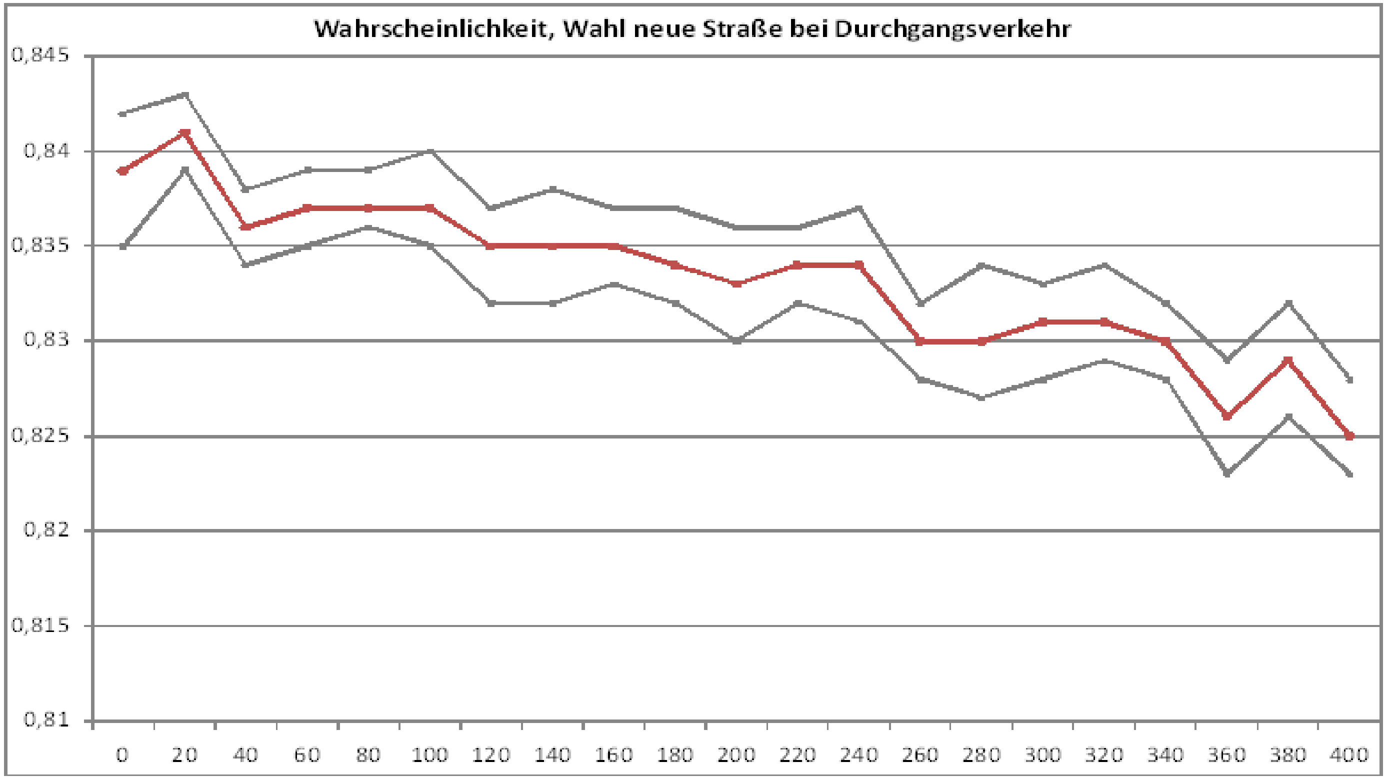
- Analyze the effect of a planned street to disburden an existing one of through traffic when developing new industrial areas



Simulation Project

Probability of choosing the new street when varying the industry density adjacent to the new street:

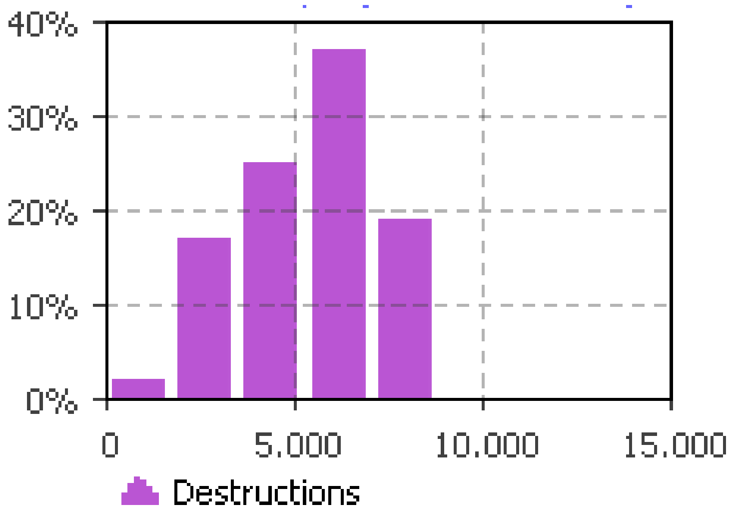
- 10 replications, 90% confidence interval

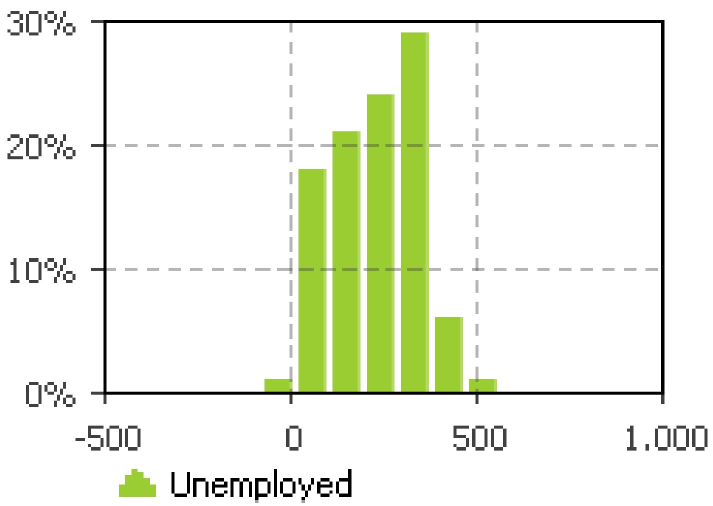


The Sims – Almost Normal Family Life

You have to compute confidence intervals:

- For how long will the father be unemployed on average?
- How much money will be spent on damaged school property?

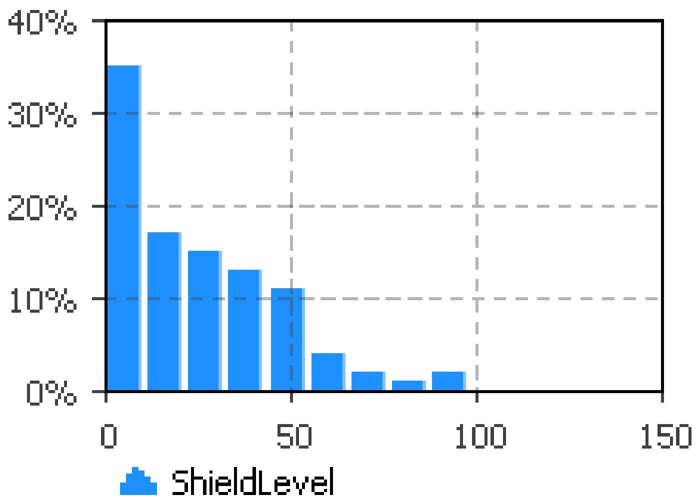


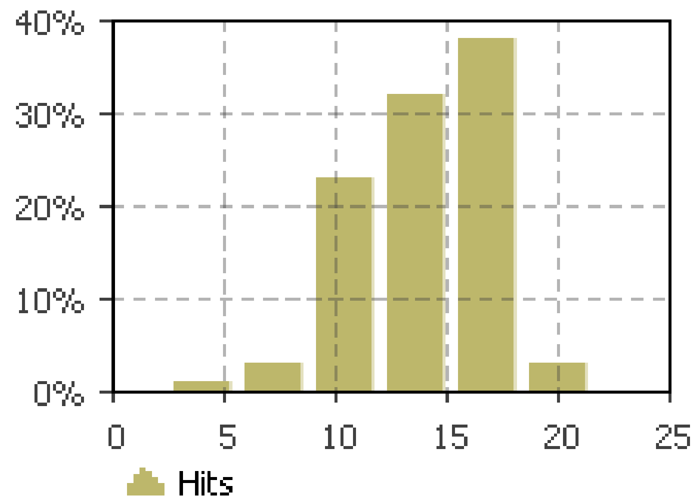


Star Trek – USS Enterprise in Danger

You have to compute confidence intervals:

- What will the shield energy level be after 2 hours?
- How many antimatter particles will hit the shield?





Learning Goals

Learning questions:

- How is the sample variance of a set of random samples defined?
- What is the method of independent replications?
- How is a confidence interval computed?
- What does a confidence interval signify?
- How would you reduce the width of a confidence interval?
- What are terminating and non-terminating simulations?
- What is initial bias? How can it be avoided?
- What is an ensemble average?