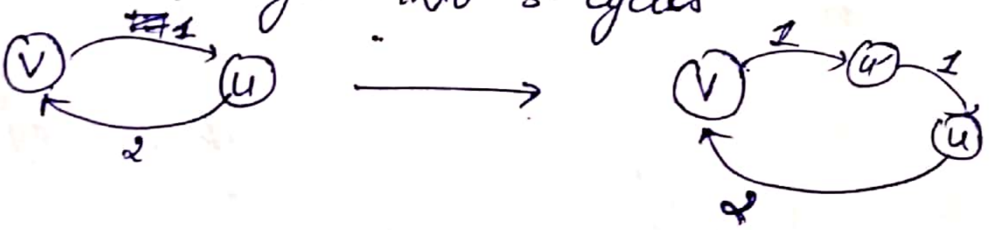
Max-Flow Min-Cut

Flow Networks → $G(V, E)$ directed graphs
2 distinguished vertices: s, t (source, sink).
Each edge $(u, v) \in E$ has a non-negative capacity $c(u, v)$.
If $(u, v) \notin E$ then $c(u, v) = 0$.

Problem Find maximum flow possible from source to sink without exceeding edge capacities & conserving the flow at each node.

2-constraints on flow graphs:
① No self-loops allowed
② Convert 2-cycles into 3-cycles



(Not necessary but makes it easier)

After condition ②, positive flow = net flow.

Flow → A flow on G is f^n $f: V \times V \rightarrow R$ satisfying
(a) capacity constraint: $\forall\ u, v \in V$, $f(u,v) \leq c(u,v)$
(b) Flow conservation: $\forall\ u \in V - \{s, t\}$

$$\sum_{v \in V} f(u, v) = 0$$

(c) Skew Symmetry: $\forall\ u, v \in V$, $f(u,v) = -f(u,v)$

Value of a flow f, ~~[illegible]~~ ↙ implicit summation

$$\boxed{|f| = \sum_{v \in V} f(s,v) = f(s,V)}$$

Some properties

① $f(X, X) = 0$

② $f(X, Y) = -f(Y, X)$

③ $f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$ if $X \cap Y = \phi$

Theorem → $|f| = f(V,t)$

Proof: $|f| = f(s,V)$

$$= \underbrace{f(V, V)}_{0} - f(V-s, V)$$

$$= f(V, V-s)$$

$$= f(V,t) + \underbrace{f(V, V-s-t)}_{0}$$ 0 directly from flow conservation

Hence proved

Cut → A cut (S, T) of a flow network $G(V, E)$ is a partition of V s.t. $s \in S$ & $t \in T$,

If f is a flow on G, then the flow across the cut is $f(S, T)$.

Capacity of a cut $C(S, T) = \sum C(u,v)$ $\forall\, c(u,v) > 0$, $u \in S, v \in T$

Summation of all capacities going from S to T

$$\boxed{f(S,T) \le C(S,T)}$$

↳ Value of any ~~cut~~ flow is bound by capacity of any cut

Another characterization of flow value

Lemma: For any flow f & any cut S, T we have $|f| = f(S,T)$

Proof →

$$f(S,T) = f(S,V) - \underbrace{f(S,S)}_{0}$$

$$= f(S,V)$$

$$= f(s,V) + \underbrace{f(S-s,V)}_{0}$$

0 (due to conservation property on individual vertices)

$$= |f| \text{ (by definition)}$$

Hence proved

Residual Networks →

For a graph $G(V,E)$, the residual network $G_f(V,E_f)$,

$G_f(V,E_f)$ ← strictly positive edge capacities.

$$c_f(u,v) = c(u,v) - f(u,v) \geq 0$$

So, for ex. if $(v,u) \in E$ (original edges), then

$c(v,u) = 0$ (by definition) & $f(v,u) = -f(u,v)$

$\Rightarrow$ $c_f(v,u) = f(u,v)$ ← new edge may appear in residual network.

Observation → Flow can be increased in G by looking for paths from s to t in G_f. ~~If it exists, then~~ If none exists, flow is maximal.

4. Residual network is representing the possible changes we can make to the current flow for each edge.

Ford – Fulkerson Algorithm:

initialize $f(u,v) \leftarrow 0 \quad \forall\, u,v \in V$

while an augmenting path in G_f exists:

decrement f by $c_f(P)$

where $c_f(P) = \min\limits_{(u,v)\in P} c_f(u,v)$ is the residual capacity of the augmenting path P.

Max-Flow Min-Cut Theorem:

~~The~~ Theorem: The following are equivalent:

1. $|f| = c(S,T)$ for some cut (S,T).
2. f is a maximum flow.
3. f admits no augmenting paths.

Proof → To prove 1 ⇒ 2

Since $|f| \le c(S,T)$ for any cut (S,T), hence if $|f| = c(S,T) \Rightarrow f$ is a maxflow as it can't be increased.

To prove 2 ⇒ 3

If there were an augmenting path, we would be able to increase flow value, so f wont be maximum. Hence contradiction. Hence proved.

To prove 3 ⇒ 1

Suppose f admits no augmenting paths

Define $S = \{u \in V : \exists$ a path in G_f from s to $u\}$

& $T = V - S$

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S | T

(S) - path in G_f - - - (u) | (v) path in G_f (t)

there is no edge from u to v in G_f (as t is unreachable from s)

$\Rightarrow \quad G_f(u,v) = 0$

$\Rightarrow \quad c(u,v) - f(u,v) = 0$

$\Rightarrow \quad c(u,v) = f(u,v)$

So any edge in G from u to v is saturated

Similarly, any edge from v to u has 0 flow

$\Rightarrow \quad \sum_{u\in S, v\in V} c(u,v) = \sum_{u \in S, v\in T} f(u,v)$

$\Rightarrow \quad \boxed{f(S,T) = C(S,T)}$ Hence proved

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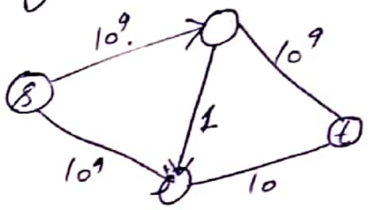
* Ford-Fulkerson can be very slow depending on the paths we pick in G_f. Ex



← This could take $2*10^9$ iterations

Edmond's Karp algorithm → They proved if BFS augmentation is used, we require $O(VE)$ augmentations,
So, we get total complexity $O(VE^2)$

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Edmond's Karp → Tells us to take the shortest available path in ~~E~~ G_f from $s \to t$.

Complexity analysis for Edmond's Karp:

Let $\delta_f(u,v)$ denote shortest path distance in G_f from u to v.

Lemma → $\forall v \in V - \{s,t\}$, shortest distance $\delta_f(s,v)$ in residual network G_f increases monotonically with each augmentation.

Proof → suppose $\exists v \in V - \{s,t\}$ s.t. a flow augmentation from f to f' decreases decreases some shortest path distance. Let v be the vertex with minimum $\delta_{f'}(s,v)$ whose distance decreased due to augmentation.

Let $p = s \rightsquigarrow u \to v$ be a shortest path from from s to v in $G_{f'}$, so, $(u,v) \in E_{f'}$ &

$$\delta_{f'}(s,u) = \delta_{f'}(s,v) - 1$$

The way we chose v, we know that

$$\delta_{f'}(s,u) \geq \delta_f(s,u)$$

Hence $(u,v) \notin E_f$ b/c otherwise $\delta_f(s,v) \leq \delta_f(s,u)+1$

$$\leq \delta_{f'}(s,u)+1$$

$$\leq \delta_{f'}(s,v)$$

Hence $(u,v) \in E_f$ & $(u,v) \notin E_{f'}$

So, augmentation increased flow from v to u.
Since the algorithm only increases flow along shortest paths, shortest path from s to u has (v,u) as the last edge

$$\Rightarrow \quad \delta_f(s,v) = \delta_f(s,u) - 1$$
$$\leq \delta_{f'}(s,u) - 1$$
$$\leq \delta_{f'}(s,v) - 2$$
$$\Rightarrow \quad \delta_{f'}(s,v) > \delta_f(s,v)$$

Hence contradiction, hence proved.

<u>Theorem</u> → Total no. of augmentations performed are $O(VE)$
Lets say an edge (u,v) in G_f is critical on augmenting path p if $c_f(p) = c_f(u,v)$. $\exists$ atleast one such edge on each path.

Let $(u,v) \in E$, since augmenting paths are shortest paths, when (u,v) is critical for the first time,

$$\delta_f(s,v) = \delta_f(s,u) + 1 \quad \text{— (1)}$$

~~Once~~ This edge disappears from G_f after this. It can only reappear when (v,u) appears in an augmenting path. If f' is the flow then,

$$\delta_{f'}(s,u) = \delta_{f'}(s,v) + 1 \quad \text{— (2)}$$

Using the lemma,

$$\delta_{f'}(s,u) = \delta_{f'}(s,v)+1 \geq \delta_f(s,v)+1 = \delta_f(s,u)+2$$

Hence $\delta_f(s,u)$ increases by atleast 2 before it can be the critical edge again. Since $\delta_f(s,u) \leq |V|-2$

$\Rightarrow$ it can be the critical edge atmost $\frac{|V|}{2}$ times.

Since no. of possible edges $= O(E)$

Hence no. of augmented paths iterations $= \boxed{O(VE)}$

Hence proved.

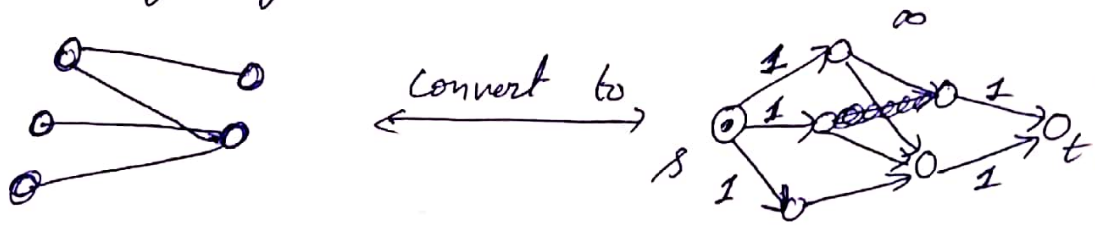
* Now, since the BFS search + augmentation can be done in $O(E)$ time, ~~the~~ the total complexity of the Edmonds Karp algorithm $= \boxed{O(VE^2)}$

Applications → Bipartite Matching

Problem → Find maximum matching in a bipartite graph

Matching → Set of edges with no vertex in common.

ex



max flow with $\text{Cf} \oplus$ == max matching

The complexity here is actually $O(VE)$ rather than $O(VE^2)$ because max-matching in bipartite graph $= O(V)$