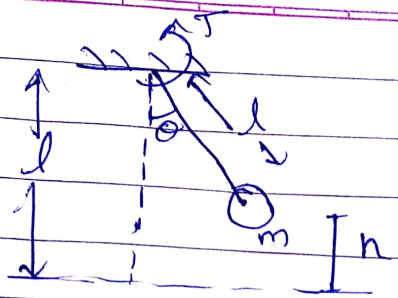
Simple Pendulum



$$X = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\dot{x}_1 = x_2 \quad \text{— (i)}$$

$$P.E = mgh = mg(l - l\cos\theta)$$

$$KE = \frac{1}{2} I w^2 = \frac{1}{2} ml^2 (\dot{\theta})^2$$

$$L = \frac{1}{2} ml^2 (\dot{\theta})^2 - mgl + mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2(\dot{\theta}) \qquad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2 \ddot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = T$$

$$ml^2 \ddot{\theta} + mgl\sin\theta = u\cancel{T}$$

$$\ddot{\theta} = -\frac{\cancel{m}g}{l}\sin\theta + \frac{u\cancel{T}}{ml^2}$$

$$\therefore \dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{u\cancel{T}}{ml^2} \quad \text{— (ii)}$$

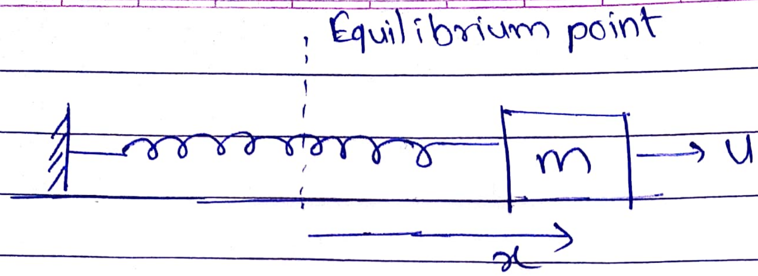
Jacobian of eq^n (i) & (ii) w.r.t $X_{(\pi, 0)}$ will give

$$A = \begin{bmatrix} 0 & 1 \\ +g/l & 0 \end{bmatrix}$$

Jacobian of eq^n (i) & (ii) w.r.t u will give

$$B = \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix}$$

Spring mass system



$$X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\therefore \dot{X}_1 = X_2 \quad \text{— (i)}$$

$$P.E = \frac{1}{2} k x^2$$

$$K.E = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x})^2$$

$$L = K.E - P.E$$
$$= \frac{1}{2} m (\dot{x})^2 - \frac{1}{2} k x^2$$

$$\frac{\partial L}{\partial \dot{x}} = m(\dot{x}) \qquad \frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = u$$

$$m\ddot{x} + kx = u$$

$$\therefore \ddot{x} = -\frac{k}{m} x + \frac{u}{m}$$

$$\therefore \boxed{\dot{x}_2 = -\frac{k}{m} X_1 + \frac{u}{m}} \quad \text{— (ii)}$$

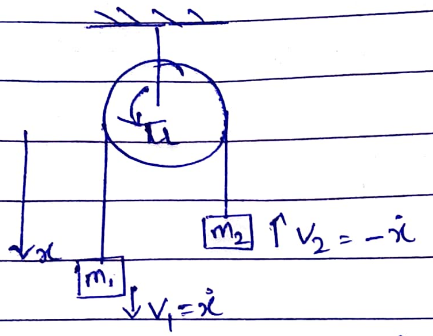
Jacobian of eq^n (i) & (ii) w.r.t X will give

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}$$

Jacobian of eq^n (i) & (ii) w.r.t u will give

$$B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

Simple Pulley



$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x}_1 = x_2 \quad \text{— (i)}$$

$$P.E = m_1 g(l - x) + m_2 g(x) = (m_2 - m_1) g x + m_1 g l$$

$$K.E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2}(m_1 + m_2)(\dot{x})^2$$

$$L = K.E - P.E = \frac{1}{2}(m_1 + m_2)(\dot{x})^2 - (m_2 - m_1) g x \mp m_1 g l$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)(\dot{x}) \qquad \frac{\partial L}{\partial x} = -(m_2 - m_1) g$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_1 + m_2)(\ddot{x})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{u}{r}$$

$$(m_1 + m_2)(\ddot{x}) + (m_2 - m_1) g = \frac{u}{r}$$

$$\therefore \ddot{x} = \frac{(m_1 - m_2) g}{(m_1 + m_2)} + \frac{u}{r(m_1 + m_2)}$$

$$\dot{x}_2 = \frac{(m_1 - m_2) g}{m_1 + m_2} + \frac{u}{r(m_1 + m_2)} \quad \text{— (ii)}$$

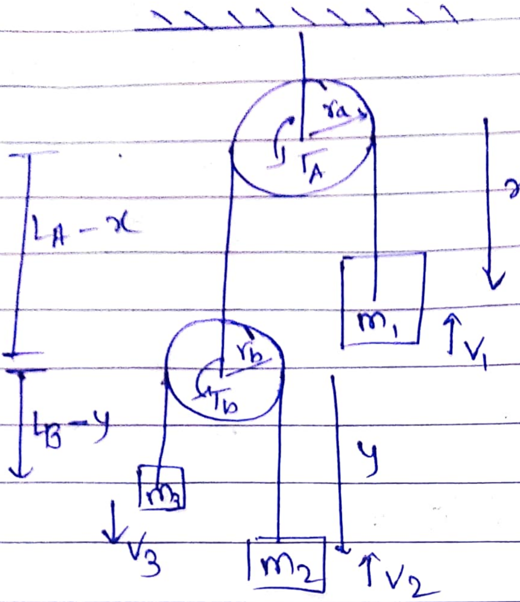
Jacobian of eqⁿ (i) & (ii) w.r.t X will give

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Jacobian of eqⁿ (i) & (ii) w.r.t u will give

$$B = \begin{bmatrix} 0 \\ 1/r(m_1 + m_2) \end{bmatrix}$$

Complex Pulley



Assume
$m_1 < m_2 + m_3$
$m_3 > m_2$

$$X = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix}$$

$\dot{x}_1 = x_2$ — (i)
$\dot{x}_3 = x_4$ — (ii)

$$P.E = m_1 g(-x) + m_2 g(-L_A + x - y) + m_3 g(-L_A + x - L_B + y)$$

$$= (-m_1 + m_2 + m_3)gx + (m_3 - m_2)gy - (m_2 + m_3)L_A g - m_3 L_B g$$

$$K.E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2$$

$$= \frac{1}{2}m_1(\dot{x})^2 + \frac{1}{2}m_2(\dot{x} + \dot{y})^2 + \frac{1}{2}m_3(\dot{x} - \dot{y})^2$$

$L = K.E. - P.E.$

$$\frac{\partial L}{\partial \dot{x}} = m_1\dot{x} + m_2\dot{x} + m_2\dot{y} + m_3\dot{x} - m_3\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_1 + m_2 + m_3)\ddot{x} + (m_2 - m_3)\ddot{y}$$

$$\frac{\partial L}{\partial x} = (-m_1 + m_2 + m_3)g$$

$$\therefore (m_1 + m_2 + m_3)\ddot{x} + (m_2 - m_3)\ddot{y} - (-m_1 + m_2 + m_3)g = \frac{u_1}{r_a}$$

$$\ddot{x} = -\frac{(m_2 - m_3)}{(m_1 + m_2 + m_3)}\ddot{y} + \frac{(-m_1 + m_2 + m_3)g}{m_1 + m_2 + m_3} + \frac{u_1}{r_a(m_1 + m_2 + m_3)} \quad \text{— (iii)}$$

$$\frac{\partial L}{\partial \dot{y}} = m_2\dot{y} + m_2\dot{x} + m_3\dot{y} - m_3\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = (m_2 - m_3)\ddot{x} + (m_2 + m_3)\ddot{y}$$

$$\frac{\partial L}{\partial y} = (m_3 - m_2)g$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \frac{u_2}{r_b}$$

$$(m_2 - m_3)\ddot{x} + (m_2 + m_3)\ddot{y} - (m_3 - m_2)g = \frac{u_2}{r_b}$$

$$\ddot{y} = \frac{(m_3 - m_2)}{(m_2 + m_3)}\ddot{x} + \frac{(m_3 - m_2)g}{(m_2 + m_3)} + \frac{u_2}{r_b(m_2 + m_3)} \quad \text{— (iv)}$$

Putting eq^n (iv) in (iii) we get

$$(m_1 + m_2 + m_3)\ddot{x} + \frac{(m_2 - m_3)^2 \ddot{x}}{(m_2 + m_3)} - \frac{(m_3 - m_2)^2 g}{(m_2 + m_3)} + \frac{u_2(m_2 - m_3)}{r_b(m_2 + m_3)}$$
$$- (-m_1 + m_2 + m_3)g = \frac{u_1}{r_a}$$

$$\therefore \ddot{x} = \frac{-(m_2 - m_3)^2 g + \frac{u_2(m_2 - m_3)}{r_b} + (m_2 - m_3)(-m_1 + m_2 + m_3)g + u_1(m_2 - m_3)}{m_2 r_a (m_1 + 5m_3)} \quad \text{— (v)}$$

Jacobian of eq^n (i), (ii), (iv) and (v) w.r.t X gives

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Jacobian of eq^n (i), (ii), (iv) & (v) w.r.t $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ gives

$$B = \begin{bmatrix} 0 & 0 \\ \dfrac{m_2 - m_3}{m_2 r_a (m_1 + 5m_3)} & \dfrac{m_2 - m_3}{m_2 \cdot r_a \cdot r_b (m_1 + 5m_3)} \\ 0 & 0 \\ \dfrac{(m_2 - m_3)^2}{m_2 r_a (m_1 - 5m_3)(m_2 + m_3)} & \dfrac{\frac{1}{r_b} + \frac{(m_2 - m_3)^2}{m_2 r_a r_b (m_1 + 5m_3)}}{m_2 + m_3} \end{bmatrix}$$