MATH/STAT395: Probability II Spring 2020

Homework 1

Due **April 15th, 2020** by 11:59pm

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Upload your answers to the questions below to Canvas in a PDF file. All answers require a **clear and complete** mathematical explanation unless specified differently. An answer without explanation/derivation/proof will not be given credits. One exercise chosen at random will be graded and the rest will be given points for completion.

Exercise 1 Expectation and variance computations of classic random variables

See the lecture note for the expressions of the expectation and the variance

1. Derive the proof of the expressions of the expectation and variance of $X \sim \text{Geom}(p)$

   Hint for expectation: Denote $g(x) = \frac{1}{1-x}$, then for $0 < x < 1$, $g(x) = \sum_{k=0}^{+\infty} x^k$ and $g'(x) = \sum_{k=0}^{+\infty} kx^{k-1}$

   Hint for variance: Decompose $\mathbb{E}[X^2] = \mathbb{E}[X] + \mathbb{E}[X(X-1)]$ and use that for $0 < x < 1$, $g''(x) = \sum_{k=0}^{+\infty} k(k-1)x^{k-2}$

2. Derive the proof of the expressions of the expectation and the variance of $X \sim \text{Poisson}(\lambda)$

   Hint for variance: Decompose $\mathbb{E}[X^2] = \mathbb{E}[X] + \mathbb{E}[X(X-1)]$

3. Derive the proof of the expression of the expectation and the variance of $X \sim \text{Unif}([a, b])$

4. Derive the proof of the expressions of the expectations and the variance of $X \sim \mathcal{N}(\mu, \sigma^2)$ from the expression of the p.d.f.

5. Derive the proof of the expression of the expectation and variance of $X \sim \text{Exp}(\lambda)$

6. *(Optional)* Derive the proof of the expressions of the expectation and the variance of $X \sim \text{Gamma}(r, \lambda)$

   Hint: Prove that for any $\alpha > 0$, $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$

7. *(Optional)* Derive the proof of the expression of the expectation of a hypergeometric random variable
   Hint: A hypergeometric r.v. X can be written $X = I_1 + \ldots + I_n$ where I_i is the indicator random variable that the i^{th} pick from the set belongs to the set of items A. As shown later in the course, the random variables I_i are exchangeable such that they all have the same distribution. Use this decomposition to compute the expectation.

Exercise 2 Modelization

1. On average people have 6 partners in their life. What is the probability that you will have exactly one partner in your life?

   Hint: Model it as a Poisson variable

2. It's been 30 min since my date hasn't answered my text message. On average she/he answers after 2min. If she/he does not answer now in the next 30 minutes, knowing that I already waited 30 minutes, I'm going to break up. What is the probability that I break up?

   Hint: Model it as an exponential variable

3. *(Optional)* Three of my friends schedule to play a game at 9pm remotely. They are never on time and I always have to wait. The average number of my friends coming in a time interval $[a, b]$ is proportional to the size of this time interval, namely the number of friends coming during $[a, b]$ can be modeled as $X \sim \text{Poisson}(\lambda[a, b])$ with $\lambda = 1$. The number of friends coming in two disjoint time intervals $[a, b]$ $[c, d]$, with $[a, b] \cap [c, d] = \emptyset$, are independent.
   (a) What is the distribution of the time before one of my friend comes?
   (b) What is the distribution of the time before two of my friend comes?
   (c) What is the probability that I wait at most 15min before all my friends are there?
   (d) Which distribution do you recognize? How can it be generalized to n friends?

Exercise 3 Medians

1. Show that the median of continuous random variable with positive p.d.f. is uniquely defined
2. Exhibit an example of a continuous random variable for which the median is not uniquely defined
3. I have a date at a restaurant. On average people are on time for dates with a variance of 5min. How much should I arrive earlier to be sure at 95% that I am there before my date?

   Hint: Model it as a Gaussian, use calculators of the error function that you can find on the web to compute the appropriate quantile after an adequate change of variables.

Exercise 4 Useful Lemma

1. Prove the following lemma

   Lemma 1. *Let X be a non-negative r.v. on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that its expectation is defined and denote F its c.d.f.*

   (a) If X is a discrete random variable then $\mathbb{E}[X] = \sum_{t=0}^{+\infty}(1 - F(t))$
   (b) If X is a continuous random variable then $\mathbb{E}[X] = \int_0^{+\infty}(1 - F(t))dt$

Exercise 5 Cauchy Distribution

Choose a point uniformly at random from $\{(x, y) : x > 0, x^2 + y^2 < 1\}$. Let S be the slope of the line through the chosen point and the origin.

1. Find the c.d.f. of S
2. Find the p.d.f. of S

Exercise 6 Undefined moments

1. Give an example of a random variable whose expectation is not defined (check first that it is indeed a random variable)
2. Let $k \in \mathbb{N}$. Give an example of a a random variable such that its k^{th} moment is defined but not its $k + 1$ moment.