Assignment Part A

Multivariate calculus:

$$f(x,y,z) = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$$

To find local Minima, maxima or saddle point, following steps have to be performed.

__step1:__

→ find Gradient Vector

__step2:__

find critical points

step3:

find Hessian vector

step4:

test for Sylvester's criterion for

n×n Hessian.

step 1:

$$\nabla f = \begin{bmatrix} \dfrac{\partial f(x_1, x_2, x_3, \ldots, x_n)}{\partial x_1} \\ \dfrac{\partial f(x_1, x_2, x_3, \ldots, x_n)}{\partial x_2} \\ \vdots \\ \dfrac{\partial f(x_1, x_2, x_3, \ldots, x_n)}{\partial x_n} \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 2x + 6y - 10 \\ 6x + 2y - 3z - 5 \\ -3y + 8z - 21 \end{bmatrix}$$

→ representing this as a system of equations matrix

→ $Ax = b$

$$\rightarrow \begin{bmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 21 \end{bmatrix}$$

→ To find X, lets do LU Decomposition of A.

LU Decomposition of 'A'

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{bmatrix}$$

→ $R_2 - 3R_1$

→ $$U = \begin{bmatrix} 2 & 6 & 0 \\ 0 & -16 & -3 \\ 0 & -3 & 8 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow R_3 - 0R_1$

$$U = \begin{bmatrix} 2 & 6 & 0 \\ 0 & -16 & -3 \\ 0 & -3 & 8 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 - \frac{3}{16} R_2$

$$U = \begin{bmatrix} 2 & 6 & 0 \\ 0 & -16 & -3 \\ 0 & 0 & \frac{137}{16} \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -\frac{3}{16} & 1 \end{bmatrix}$$

$\Rightarrow E_{32} E_{31} E_{21} A = U$

$$\Rightarrow \boxed{A = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} U}$$

$$\Rightarrow L = \begin{pmatrix} 1 & & \\ +3 & 1 & \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & +3/16 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3/16 & 1 \end{pmatrix}$$

$\Rightarrow A = LU$

$\Rightarrow LUx = y$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & \frac{3}{16} & 1 \end{pmatrix} \cdot U = \begin{pmatrix} 10 \\ 5 \\ 21 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & \frac{3}{16} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 21 \end{pmatrix}$$

$\Rightarrow y_1 = 10, \quad y_2 = 5 - 3y_1 = 5 - 30 = -25$

$y_3 = 21 - \frac{3}{16} y_2 = 21 - \frac{3}{16} \times -25$

$= \frac{411}{16}$

$$\Rightarrow \begin{pmatrix} 2 & 6 & 0 \\ 0 & -16 & -3 \\ 0 & 0 & \frac{137}{16} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -25 \\ \frac{411}{16} \end{pmatrix}$$

$$\Rightarrow \boxed{z=3,\quad y=1,\quad x=2}$$

Hence, critical point is $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

Hessian matrix

$$H = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{pmatrix}$$

$H_1 = 2 > 0$ $\Big|$ $H_2 = \begin{bmatrix} 2 & 6 \\ 6 & 2 \end{bmatrix}$

$= 4 - 36 = -34 < 0$

$H_3 = \det[H] =$

$= -274 < 0$

$\boxed{\text{Hence it is a saddle point.}}$