CHAPTER 25

Quadratic equations

In this chapter you will **learn how to**:

- solve quadratic equations by factorising
- solve quadratics by the general formula
- set up and solve problems using quadratics.

You will also be **challenged to**:

- investigate conic sections.

Starter: Solutions of equations

Here are some equations, and some suggested solutions.

Substitute the suggested values into each equation to discover which are correct.

Equation	Suggested solutions
$5x + 3 = 18$	$x = 1, x = 2, x = 3, x = 4, x = 5$
$5x^2 + 4 = 9$	$x = -1, x = 0, x = 1, x = 2, x = 3$
$x^2 = 7x - 10$	$x = 1, x = 2, x = 3, x = 4, x = 5$
$12x - 3 = 45$	$x = 1, x = 2, x = 3, x = 4$
$x^2 = 1$	$x = -2, x = -1, x = 0, x = 1, x = 2$
$x + 4 = 10 - x$	$x = 0, x = 1, x = 2, x = 3, x = 4$
$x(x + 1) = 2$	$x = -2, x = -1, x = 0, x = 1, x = 2$
$x^2 = 36$	$x = -6, x = -3, x = 0, x = 3, x = 6$
$4x^2 = 100$	$x = -5, x = -3, x = 0, x = 1, x = 5$
$x^3 - 6x^2 + 11x - 6 = 0$	$x = 1, x = 2, x = 3, x = 4, x = 5$

Here is an extract from an old mathematics book:

> *Linear equations like $3x + 5 = 21$ have only one solution. Equations containing an x^2 term often have two solutions, however, and equations containing x^3 terms may have as many as three solutions.*

Do your results support this extract?

25.1 Solving quadratic equations – factorising

An equation like $x^2 + 4x + 3 = 0$ is called a **quadratic equation**. Quadratic equations must contain a square term, (such as the x^2 in this example) with no higher power of x, such as x^3. You may be able to spot a solution of a quadratic equation by inspection (i.e. by guesswork), but this is not a reliable method because quadratics may have two solutions. **Factorising** is a method of making sure that all of the solutions to a quadratic equation are found.

EXAMPLE

Solve the equation $x^2 + 4x + 3 = 0$

SOLUTION

$x^2 + 4x + 3 = 0$

$(x + 1)(x + 3) = 0$

If $(x + 1)(x + 3) = 0$ then one of the brackets must be equal to 0.

$x + 1 = 0$ or $x + 3 = 0$

So, $\underline{x = -1}$ or $\underline{x = -3}$

Factorisation can be more difficult, especially if the coefficient of x^2 is greater than 1.

EXAMPLE

Solve the equation $2x^2 - 9x - 5 = 0$

SOLUTION

$2x^2 - 9x - 5 = 0$

$(2x + 1)(x - 5) = 0$

$2x + 1 = 0$ or $x - 5 = 0$

So, $\underline{x = -\frac{1}{2}}$ or $\underline{x = 5}$

Some quadratics contain only two terms, not three. If the constant term at the end is missing, then all you need to do is take out a common factor of x.

EXAMPLE

Solve the equation $10x^2 - 4x = 0$

SOLUTION

$10x^2 - 4x = 0$

$2x(5x - 2) = 0$

$2x = 0$ or $5x - 2 = 0$

So, $\underline{x = 0}$ or $\underline{x = \frac{2}{5}}$

If, instead, the middle term is missing, then you can simply solve to find x^2. Then take the square root of both sides to find x. Remember to allow for both positive and negative answers.

EXAMPLE

Solve the equation $5x^2 - 80 = 0$

SOLUTION

$$5x^2 - 80 = 0$$
$$5x^2 = 80$$
$$x^2 = \frac{80}{5}$$
$$x^2 = 16$$

Square rooting both sides gives:

$\underline{x = 4}$ or $\underline{x = -4}$

Alternatively, by factorising:

$$5x^2 - 80 = 0$$
$$5(x^2 - 16) = 0$$
$$5(x - 4)(x + 4) = 0$$

and so $x = 4$ or $x = -4$

EXERCISE 25.1

Solve each of these quadratic equations by using the factorisation method.

1 $x^2 + 3x + 2 = 0$

2 $x^2 + 6x + 5 = 0$

3 $x^2 + 7x - 8 = 0$

4 $x^2 + x - 2 = 0$

5 $x^2 + 2x - 8 = 0$

6 $x^2 + 4x - 12 = 0$

7 $x^2 - 7x + 12 = 0$

8 $x^2 - 8x + 15 = 0$

9 $x^2 - 2x - 8 = 0$

10 $x^2 - 4x + 4 = 0$

11 $2x^2 + 3x + 1 = 0$

12 $2x^2 + 5x - 3 = 0$

13 $3x^2 + 7x + 2 = 0$

14 $2x^2 + x - 3 = 0$

15 $3x^2 + 8x + 4 = 0$

16 $2x^2 - 9x + 9 = 0$

17 $3x^2 + 8x + 5 = 0$

18 $2x^2 - 9x + 10 = 0$

19 $5x^2 + 26x + 5 = 0$

20 $4x^2 + 4x + 1 = 0$

Here are some more difficult quadratic equations. Solve them by the factorisation method.

21 $6x^2 + x - 1 = 0$

22 $5x^2 - x = 0$

23 $4x^2 - 1 = 0$

24 $3x^2 - 3x = 0$

25 $12x^2 - 7x + 1 = 0$

26 $10x^2 - x = 0$

27 $8x^2 - 10x + 3 = 0$

28 $8x^2 - 11x + 3 = 0$

29 $4x^2 + 12x + 9 = 0$

30 $4x^2 - 9 = 0$

Rearrange these quadratic equations so that the right-hand side is zero. Then solve them, by factorisation.

31 $x^2 - 6x = 7$

32 $x^2 + 40 = 13x$

33 $x^2 + 20x = 7x - 30$

34 $x^2 + 10x = 3x + 44$

35 $2x^2 = 11x + 6$

36 $8 - 23x = 3x^2$

37 $2 = x + 3x^2$

38 $4x^2 = 8x - 3$

39 $6x^2 + 6x = x + 6$

40 $5x^2 + 30 = x^2 + 55$

25.2 Solving quadratic equations – formula

A quadratic equation contains three **coefficients**. For example:

$$x^2 + 4x + 3 = 0$$

has an x^2 coefficient of 1, an x coefficient of 4 and a constant term of 3.

$2x^2 - 4x - 1 = 0$ has an x^2 coefficient of 2, an x coefficient of -4 and a constant term of -1.

Similarly, $4x^2 - 1 = 0$ has an x^2 coefficient of 4, an x coefficient of 0 and a constant term of -1.

There is a formula that can be used to find solutions to a quadratic equation. If $ax^2 + bx + c = 0$ is a quadratic equation, then the solutions are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The sign $\pm$ is read as 'plus or minus'.

You obtain one of the solutions of the quadratic by using $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

and the other one by using $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

The formula method can be applied to a much wider range of quadratic equations than the factorising method. You would normally use the formula if the equation cannot be factorised in an obvious way. The quadratic formula will be given to you in an IGCSE exam, on the formula sheet.

EXAMPLE

Solve the equation $2x^2 - 4x - 1 = 0$. Give your answers to 3 decimal places.

SOLUTION

There is no obvious factorisation, so use the formula.
Inspecting the equation, $a = 2$, $b = -4$ and $c = -1$.
Then substituting these values into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{16 + 8}}{4}$$

$$= \frac{4 \pm \sqrt{24}}{4}$$

$$= 2.224\,744\,871 \text{ or } -0.224\,744\,871$$

$$= \underline{2.225} \text{ or } -\underline{0.225} \text{ (3 d.p.)}$$

If you are asked to solve a quadratic equation in an IGCSE exam, and the number that you calculate under the square root sign is negative, for example $\sqrt{-25}$, then you know you must have made an error.

EXERCISE 25.2

Solve these equations using the quadratic equation formula. Give your answers correct to 3 decimal places.

1 $x^2 + 5x + 2 = 0$

2 $x^2 + 10x + 7 = 0$

3 $2x^2 - 14x + 13 = 0$

4 $2x^2 + 11x - 5 = 0$

5 $x^2 - 7x + 1 = 0$

6 $3x^2 + 2x - 3 = 0$

7 $x^2 + 5x - 1 = 0$

8 $2x^2 - 3x - 4 = 0$

9 $5x^2 - x - 1 = 0$

10 $2x^2 + 9x - 2 = 0$

Rearrange the equations below so that they are in the form $ax^2 + bx + c = 0$. Then solve them using the formula method. Give your answers correct to 3 significant figures.

11 $x^2 + 5x = 7$

12 $2x^2 = 3x + 1$

13 $3x^2 = 5 + 4x$

14 $x^2 + x = 2 - 9x$

15 $11x = 1 - 2x^2$

16 $3x^2 = 12x + 1$

17 $2x = 5x^2 - 4$

18 $21x + 1 = 7x^2$

19 $20x + 4 = 3x - 6x^2$

20 $9x^2 = 2 + x$

25.3 Problems leading to quadratic equations

At IGCSE you may be expected to set up a problem that leads to a solution involving a quadratic equation. You will then need to solve the quadratic equation to complete the problem.

EXAMPLE

A rectangular flower bed measures $2x + 5$ metres by $x + 3$ metres.
It has an area of 45 square metres.

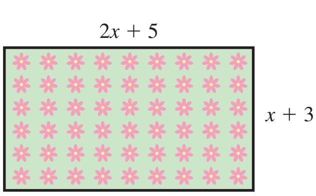
a) Draw a sketch to show this information.

b) Show that x must satisfy the equation $2x^2 + 11x - 30 = 0$.

c) Solve this equation, to find the value of x. Hence find the dimensions of the flower bed.

SOLUTION

a)



b)

$$\begin{aligned}(2x + 5)(x + 3) &= 45\\ 2x^2 + 5x + 6x + 15 &= 45\\ 2x^2 + 11x + 15 &= 45\\ 2x^2 + 11x - 30 &= 0\end{aligned}$$

c) Factorising $2x^2 + 11x - 30 = 0$ gives:

$$(2x + 15)(x - 2) = 0$$

So, $2x + 15 = 0$ or $x - 2 = 0$

Therefore $x = -7\frac{1}{2}$ or $x = 2$

But $x = -7\frac{1}{2}$ will lead to negative dimensions for the flower bed, so it must be rejected.

Therefore $x = 2$

We know that the dimensions of the flower bed are:

$2x + 5$ metres by $x + 3$ metres

Substituting $x = 2$ gives dimensions of 9 metres by 5 metres

EXERCISE 25.3

1 Two whole numbers x and $x + 7$ are multiplied together. The result is 144.

a) Write down an equation in x.

b) Show that this equation can be expressed as $x^2 + 7x - 144 = 0$

c) Solve the equation, to find the values of the two whole numbers (there are two possible sets of answers, and you should give both).

2 A rectangular playing field is x metres wide and $2x - 5$ metres long. Its area is 3000 m^2.

a) Write down an equation in x.

b) Show that this equation can be expressed as $2x^2 - 5x = 3000$

c) Solve the equation, to find the value of x. Hence find the dimensions of the playing field.

3 Hannah and Jamal each thought of a positive whole number. Jamal's number was 3 more than Hannah's number. Let Hannah's number be represented by x.

a) Their two numbers multiply together to make 180. Write down an equation in x.

b) Show that this equation can be expressed as $x^2 + 3x - 180 = 0$

c) Solve the equation, and hence find the numbers that Hannah and Jamal thought of.

4 A square measures x cm along each side, and a rectangle measures x cm by $2x + 1$ cm. The total area of the square and the rectangle is 114 cm^2.

a) Write down an equation in x.

b) Show that this equation can be expressed as $3x^2 + x - 114 = 0$

c) Solve the equation, to find the value of x.

5 A rectangle measures $3x + 1$ cm by $2x + 5$ cm. Two squares, each of side x cm, are removed from it. The remaining shape has an area of 55 cm^2.

a) Express this information as an equation in x.

b) Show that this equation can be expressed as $4x^2 + 17x - 50 = 0$

c) Solve your equation, and hence find the dimensions of the rectangle.

6 A rectangle measures x cm by $2x + 3$ cm. A second rectangle measures $x + 3$ cm by $x + 4$ cm.

a) Write down expressions for the areas of the two rectangles.

Both rectangles have the same area.

b) Write an equation in x.

c) Solve this equation. Hence determine the dimensions of each rectangle.

REVIEW EXERCISE 25

1 **a)** Factorise $x^2 - 6x + 8$. **b)** Solve the equation $x^2 - 6x + 8 = 0$. **[Edexcel]**

2 Solve the equation $(2x - 3)^2 = 100$. **[Edexcel]**

3 Find the solutions of the equation $x^2 - 4x - 1 = 0$.
Give your solutions correct to three decimal places. **[Edexcel]**

4 $(x + 3)(x - 2) = 1$.
a) Show that $x^2 + x - 7 = 0$.
b) Solve the equation $x^2 + x - 7 = 0$.
Give your answers correct to 3 significant figures. **[Edexcel]**

5 The length of a rectangle is $(x + 4)$ cm.
The width is $(x - 3)$ cm.
The area of the rectangle is 78 cm^2.

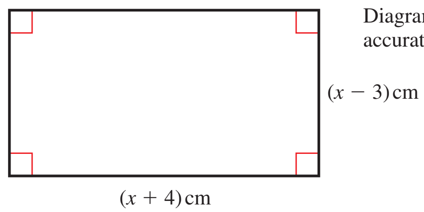


Diagram *not* accurately drawn

a) Use this information to write down an equation in terms of x.
b) **(i)** Show that your equation in part **a)** can be written as $x^2 + x - 90 = 0$.
(ii) Find the values of x which are solutions of the equation $x^2 + x - 90 = 0$.
(iii) Write down the length and the width of the rectangle. **[Edexcel]**

6 AT is a tangent to a circle, centre O. OT $= x$ cm, AT $= (x + 5)$ cm and OA $= (x + 8)$ cm.

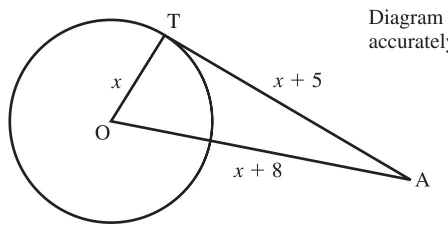
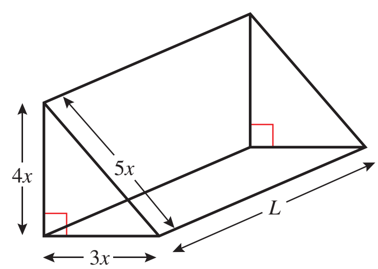


Diagram *not* accurately drawn

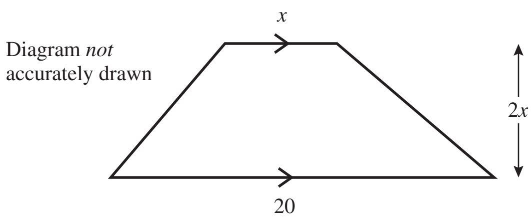
a) Show that $x^2 - 6x - 39 = 0$.
b) Solve the equation $x^2 - 6x - 39 = 0$ to find the radius of the circle.
Give your answer correct to 3 significant figures. **[Edexcel]**

7 The diagram shows a prism.
The cross-section of the prism is a right-angled triangle.
The lengths of the sides of the triangle are $3x$ cm, $4x$ cm and $5x$ cm.
The total length of all the edges of the prism is E cm.



a) Show that the length, L cm, of the prism is given by the formula $L = \frac{1}{3}(E - 24x)$.
The surface area, A cm^2, of the prism is given by the formula $A = 12x^2 + 12Lx$. $E = 98$ cm and $A = 448$ cm.
b) Substitute these values into the formulae of L and A to show that x satisfies the equation $3x^2 - 14x + 16 = 0$. Make the stages in your working clear.
c) Solve the equation $3x^2 - 14x + 16 = 0$. **[Edexcel]**

8 The diagram shows a trapezium.
The measurements on the diagram are in centimetres.
The lengths of the parallel sides are x cm and 20 cm. The height of the trapezium is $2x$ cm.
The area of the trapezium is 400 cm^2.



a) Show that $x^2 + 20x = 400$.
b) Find the value of x. Give your answer correct to 3 decimal places. **[Edexcel]**

9 a) **(i)** Factorise $2x^2 - 35x + 98$.
(ii) Solve the equation $2x^2 - 35x + 98 = 0$.

A bag contains $(n + 7)$ tennis balls. n of the balls are yellow. The other seven balls are white.
John will take a ball at random from the bag. He will look at its colour and then put it back in the bag.

b) **(i)** Write down an expression, in terms of n, for the probability that John will take a white ball.
Bill states that the probability that John will take a white ball is $\frac{2}{5}$.
(ii) Prove that Bill's statement cannot be correct.

After John has put the ball back into the bag, Mary will then take at random a ball from the bag.
She will note its colour.

c) Given that the probability that John and Mary will take balls with different colours is $\frac{4}{9}$, prove that $2n^2 - 35n + 98 = 0$.
d) Using your answer to part **a) ii)**, or otherwise, calculate the probability that John and Mary will both take white balls. **[Edexcel]**

Key points

1. Quadratic equations contain a term in x^2, and are often written in the form $ax^2 + bx + c = 0$. Sometimes a solution may seem obvious, but you should always use formal methods to solve the equation fully since quadratics can have two solutions.
2. If the factors of a quadratic are easy to spot, then the factorising method is best.

   Otherwise, use the quadratic equation formula:

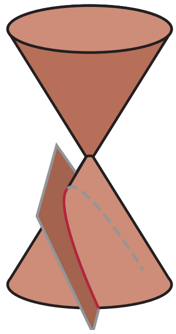
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. If an exam question asks you to solve a quadratic correct to 3 significant figures, this is a clue that the quadratic formula will be required.

Internet Challenge 25

Conic sections

When a quadratic expression is graphed, the result is a distinctive curve called a parabola.



1 The diagram above shows a plane slice through a cone. The slice is parallel to one edge of the cone. What shape is the curve (marked in red) where the plane cuts the cone?

2 The parabola is one of four conic sections. Use the internet to find out the names of the other three.

3 Draw up a set of coordinate axes on squared paper. Then draw some line segments like this:

Join the point (10, 0) to the origin (0, 0)
Join (9, 0) to (0, 1)
Join (8, 0) to (0, 2), etc.

You should see a curve forming inside these lines. Is it a conic section? If so, which one?

4 When a body such as a planet or a comet moves through the Solar System, it traces out a path known as its orbit. The Earth's orbit, for example, is an ellipse. What shapes are the orbits followed by other bodies in the Solar System?

5 Use the internet to find a method for drawing an ellipse using a string and two drawing pins. Then use the method to draw some ellipses. Is this a good method?