Formulario I3 - EYP1113 2024 - 02

Igualdades

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}; \qquad \sum_{k=x}^{\infty} \phi^k = \frac{\phi^x}{1-\phi} \quad \text{si } |\phi| < 1;$$

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \exp(\lambda); \qquad \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} \phi^x = \frac{1}{(1-\phi)^k} \quad \text{si } 0 < \phi < 1 \text{ y } k \in \mathbb{N}, \qquad \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\,\pi}$$

Propiedades función $\Gamma(\cdot)$ y $B(\cdot,\,\cdot)$

$$(1) \quad \Gamma(k) = \int_0^{\infty} u^{k-1}\, e^{-u}\, du = \texttt{gamma}(k); \quad (2) \quad \Gamma(a+1) = a\,\Gamma(a); \quad (3) \quad \Gamma(n+1) = n!, \quad \text{si } n \in \mathbb{N}_0;$$

$$(4) \quad \Gamma(1/2) = \sqrt{\pi}; \quad (5) \qquad B(q,\,r) = \int_0^{1} x^{q-1}\,(1-x)^{r-1}\, dx; \quad (6) \quad B(q,\,r) = \frac{\Gamma(q)\,\Gamma(r)}{\Gamma(q+r)} = \texttt{beta}(q,r)$$

Distribución Gamma

$$(1) \quad \text{Si } T \sim \text{Gamma}(k,\,\nu), \text{ con } k \in \mathbb{N} \longrightarrow F_T(t) = 1 - \sum_{x=0}^{k-1} \frac{(\nu\,t)^x\, e^{-\nu\,t}}{x!}, \quad (2) \quad \text{Gamma}(1,\,\nu) = \text{Exp}(\nu)$$

Medidas descriptivas

$$\mu_X = \mathsf{E}(X), \quad \sigma_X^2 = \mathsf{E}\left[(X-\mu_X)^2\right], \quad \delta_X = \frac{\sigma_X}{\mu_X}, \quad \theta_X = \frac{\mathsf{E}\left[(X-\mu_X)^3\right]}{\sigma_X^3}, \quad K_X = \frac{\mathsf{E}\left[(X-\mu_X)^4\right]}{\sigma_X^4} - 3$$

$$M_X(t) = \mathsf{E}\left(e^{t\,X}\right), \quad \mathsf{E}[g(X)] = \begin{cases} \displaystyle\sum_{x \in \Theta_X} g(x) \cdot p_X(x) \\ \\ \displaystyle\int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx \end{cases}, \quad \text{Rango} = \text{máx} - \text{mín}, \quad \text{IQR} = x_{75\,\%} - x_{25\,\%}$$

$$\text{Cov}(X,\,Y) = \mathsf{E}[(X-\mu_X)\cdot(Y-\mu_Y)] = \mathsf{E}(X\cdot Y) - \mathsf{E}(X)\cdot\mathsf{E}(Y) \qquad , \qquad \rho = \frac{\text{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

Teorema de Probabilidades Totales

$$p_Y(y) = \sum_{x \in \Theta_X} p_{X,Y}(x,y); \qquad f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$$

$$p_X(x) = \int_{-\infty}^{+\infty} p_{X\,|\,Y=y}(x) \cdot f_Y(y)\, dy; \qquad f_Y(y) = \sum_{x \in \Theta_X} f_{Y\,|\,X=x}(y) \cdot p_X(x)$$

$$\mathsf{E}(X) = \int_{-\infty}^{+\infty} \mathsf{E}(X \mid Y = y) \cdot f_Y(y)\, dy \qquad \mathsf{E}(Y) = \sum_{x \in \Theta_X} \mathsf{E}(Y \mid X = x) \cdot p_X(x)$$

Teoremas de Esperanzas Iteradas

$$\mathsf{E}(Y) = \mathsf{E}[\mathsf{E}(Y \mid X)] \quad \text{y} \quad \mathsf{Var}(Y) = \mathsf{Var}[\mathsf{E}(Y \mid X)] + \mathsf{E}[\mathsf{Var}(Y \mid X)]$$

Transformación

Sea $Y = g(X)$ una función cualquiera, con k raíces:

$$f_Y(y) = \sum_{i=1}^{k} f_X\left(g_i^{-1}(y)\right) \cdot \left|\frac{d}{dy}\, g_i^{-1}(y)\right| \quad \text{o} \quad p_Y(y) = \sum_{i=1}^{k} p_X\left(g_i^{-1}(y)\right)$$

Sea $Z = g(X, Y)$ una función cualquiera:

$$p_Z(z) = \sum_{g(x,y)=z} p_{X,Y}(x, y)$$

Sea $Z = g(X, Y)$ una función invertible para X o Y fijo:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(g^{-1}, y) \left|\frac{\partial}{\partial z} g^{-1}\right| dy = \int_{-\infty}^{\infty} f_{X,Y}(x, g^{-1}) \left|\frac{\partial}{\partial z} g^{-1}\right| dx$$

Suma Normales Independientes

Consideremos X e Y variables aleatorias independientes con distribución Normal$(\mu_X,\, \sigma_X)$ y Normal$(\mu_Y,\, \sigma_Y)$ respectivamente. Si $Z = a + b \cdot X + c \cdot Y$, con a, b y c constantes, entonces

$$Z = a + b \cdot X + c \cdot Y \sim \text{Normal}(\mu,\, \sigma), \quad \mu = a + b \cdot \mu_X + c \cdot \mu_Y \quad \text{y} \quad \sigma = \sqrt{|b|^2 \cdot \sigma_X^2 + |c|^2 \cdot \sigma_Y^2}$$

Distribución Normal Bivariada

$$f_{X,Y}(x, y) = \frac{1}{2\,\pi\,\sigma_X\,\sigma_Y\,\sqrt{1-\rho^2}} \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\,\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]\right\}$$

$$Y \mid X = x \sim \text{Normal}\left(\mu_Y + \frac{\rho\,\sigma_Y}{\sigma_X}\,(x-\mu_X),\, \sigma_Y\,\sqrt{(1-\rho^2)}\right)$$

$$X \sim \text{Normal}(\mu_X,\, \sigma_X) \qquad \text{e} \qquad Y \sim \text{Normal}(\mu_Y,\, \sigma_Y)$$

Teorema del Límite Central

Sean $X_1,\, \ldots,\, X_n$ variables aleatorias independientes e idénticamente distribuidas, entonces

$$Z_n = \frac{\displaystyle\sum_{i=1}^{n} X_i - n \cdot \mu}{\sqrt{n}\,\sigma} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \longrightarrow Z \sim \text{Normal}(0, 1),$$

cuando $n \to \infty$, $\mathsf{E}(X_i) = \mu$ y $\mathsf{Var}(X_i) = \sigma^2$.

Mínimo y Máximo

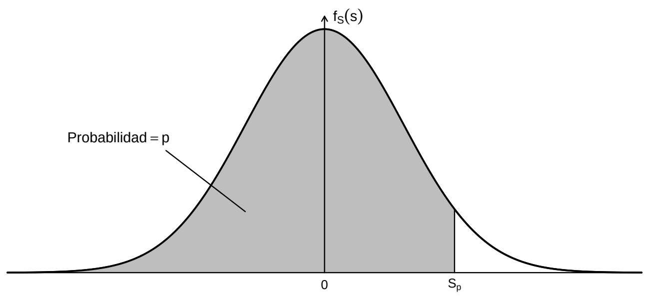
Sean $X_1,\, \ldots,\, X_n$ variables aleatorias continuas independientes con idéntica distribución (f_X y F_X), entonces para:

$$Y_1 = \min\{X_1, \ldots, X_n\} \longrightarrow f_{Y_1} = n\,[1 - F_X(y)]^{n-1}\, f_X(y);\; Y_n \;= \max\{X_1, \ldots, X_n\} \longrightarrow f_{Y_n} = n\,[F_X(y)]^{n-1}\, f_X(y)$$

Mientras que la distribución conjunta entre Y_1 e Y_n está dada por:

$$f_{Y_1,Y_n}(u, v) = n\,(n-1)\,[F_X(v) - F_X(u)]^{n-2}\, f_X(v)\, f_X(u), \qquad u \leq v$$

Tabla Percentiles Distribución Normal Estándar



S_p	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998

Distribución	Densidad de Probabilidad	Θ_X	Parámetros	Esperanza y Varianza
Binomial	$\binom{n}{x} p^x (1-p)^{n-x}$	$x = 0, \ldots, n$	$n,\ p$	$\mu_X = n\,p$ $\sigma_X^2 = n\,p\,(1-p)$ $M(t) = [p\,e^t + (1-p)]^n, \quad t \in \mathbb{R}$
Geométrica	$p\,(1-p)^{x-1}$	$x = 1, 2, \ldots$	p	$\mu_X = 1/p$ $\sigma_X^2 = (1-p)/p^2$ $M(t) = p\,e^t/[1-(1-p)\,e^t], \quad t < -\ln(1-p)$
Binomial-Negativa	$\binom{x-1}{r-1} p^r (1-p)^{x-r}$	$x = r, r+1, \ldots$	$r,\ p$	$\mu_X = r/p$ $\sigma_X^2 = r\,(1-p)/p^2$ $M(t) = \left\{p\,e^t/[1-(1-p)\,e^t]\right\}^r, \quad t < -\ln(1-p)$
Poisson	$\dfrac{(\nu\,t)^x\,e^{-\nu\,t}}{x!}$	$x = 0, 1, \ldots$	ν	$\mu_X = \nu\,t$ $\sigma_X^2 = \nu\,t$ $M(t) = \exp\left[\lambda\left(e^t - 1\right)\right], \quad t \in \mathbb{R}$
Exponencial	$\nu\,e^{-\nu\,x}$	$x \geq 0$	ν	$\mu_X = 1/\nu$ $\sigma_X^2 = 1/\nu^2$ $M(t) = \nu/(\nu - t), \quad t < \nu$
Gamma	$\dfrac{\nu^k}{\Gamma(k)}\,x^{k-1}\,e^{-\nu\,x}$	$x \geq 0$	$k,\ \nu$	$\mu_X = k/\nu$ $\sigma_X^2 = k/\nu^2$ $M(t) = [\nu/(\nu - t)]^k, \quad t < \nu$
Normal	$\dfrac{1}{\sqrt{2\,\pi}\,\sigma}\,\exp\left[-\dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right]$	$-\infty < x < \infty$	$\mu,\ \sigma$	$\mu_X = \mu$ $\sigma_X^2 = \sigma^2$ $M(t) = \exp(\mu\,t + \sigma^2\,t^2/2), \quad t \in \mathbb{R}$
Log-Normal	$\dfrac{1}{\sqrt{2\,\pi}\,(\zeta\,x)}\,\exp\left[-\dfrac{1}{2}\left(\dfrac{\ln\,x - \lambda}{\zeta}\right)^2\right]$	$x \geq 0$	$\lambda,\ \zeta$	$\mu_X = \exp\left(\lambda + \dfrac{1}{2}\,\zeta^2\right)$ $\sigma_X^2 = \mu_X^2\left(e^{\zeta^2} - 1\right)$ $E(X^r) = e^{r\,\lambda}\,M_Z(r\,\zeta)$, con $Z \sim$Normal(0,1)
Uniforme	$\dfrac{1}{(b-a)}$	$a \leq x \leq b$	$a,\ b$	$\mu_X = (a+b)/2$ $\sigma_X^2 = (b-a)^2/12$ $M(t) = [e^{t\,b} - e^{t\,a}]/[t\,(b-a)], \quad t \in \mathbb{R}$
Beta	$\dfrac{1}{B(q,\ r)}\,\dfrac{(x-a)^{q-1}\,(b-x)^{r-1}}{(b-a)^{q+r-1}}$	$a \leq x \leq b$	$q,\ r$	$\mu_X = a + \frac{q}{q+r}\,(b-a)$ $\sigma_X^2 = \frac{q\,r\,(b-a)^2}{(q+r)^2\,(q+r+1)}$
Hipergeométrica	$\dfrac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$	$\max\{0, n+m-N\} \leq x \leq \min\{n, m\}$	$N,\ m,\ n$	$\mu_X = n\,\frac{m}{N}$ $\sigma_X^2 = \left(\frac{N-n}{N-1}\right)\,n\,\frac{m}{N}\,\left(1 - \frac{m}{N}\right)$

Otras distribuciones

- Si $T \sim$ Weibull$(\eta,\, \beta)$, se tiene que

$$F_T(t) = 1 - \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right] \quad f_T(t) = \frac{\beta}{\eta}\,\left(\frac{t}{\eta}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right], \quad t > 0$$

  Con $\beta > 0$, es un parámetro de forma y $\eta > 0$, es un parámetro de escala. Si t_p es el percentil $p \times 100\,\%$, entonces

$$\ln(t_p) = \ln(\eta) + \frac{1}{\beta}\cdot\Phi^{-1}_{\text{Weibull}}(p), \quad \Phi^{-1}_{\text{Weibull}}(p) = \ln[-\ln(1-p)]$$

  Mientras que su m-ésimo momento está dado por

$$E(T^m) = \eta^m\,\Gamma(1 + m/\beta)$$

$$\mu_T = \eta\,\Gamma\left(1 + \frac{1}{\beta}\right), \quad \sigma^2_T = \eta^2\,\left[\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right]$$

- Si $Y \sim$ Logística$(\mu,\, \sigma)$, se tiene que

$$F_Y(y) = \Phi_{\text{Logística}}\left(\frac{y-\mu}{\sigma}\right); \qquad f_Y(y) = \frac{1}{\sigma}\,\phi_{\text{Logística}}\left(\frac{y-\mu}{\sigma}\right), \quad -\infty < y < \infty$$

  donde

$$\Phi_{\text{Logística}}(z) = \frac{\exp(z)}{[1+\exp(z)]} \quad \text{y} \quad \phi_{\text{Logística}}(z) = \frac{\exp(z)}{[1+\exp(z)]^2}$$

  son la función de probabilidad y de densidad de una Logística Estándar. $\mu \in \mathbb{R}$, es un parámetro de localización y $\sigma > 0$, es un parámetro de escala. Si y_p es el percentil $p \times 100\,\%$, entonces

$$y_p = \mu + \sigma\,\Phi^{-1}_{\text{Logística}}(p) \quad \text{con} \quad \Phi^{-1}_{\text{Logística}}(p) = \log\left(\frac{p}{1-p}\right)$$

  Su esperanza y varianza están dadas por: $\mu_Y = \mu$ y $\sigma^2_Y = \frac{\sigma^2\,\pi^2}{3}$.

- Si $T \sim$ Log-Logística$(\mu,\, \sigma)$, se tiene que

$$F_T(t) = \Phi_{\text{Logística}}\left(\frac{\ln(t)-\mu}{\sigma}\right); \quad f_T(t) = \frac{1}{\sigma\,t}\,\phi_{\text{Logística}}\left(\frac{\ln(t)-\mu}{\sigma}\right) \quad t > 0$$

  Donde $\exp(\mu)$, es un parámetro de escala y $\sigma > 0$, es un parámetro de forma. Si t_p es el percentil $p \times 100\,\%$, entonces

$$\ln(t_p) = \mu + \sigma\,\Phi^{-1}_{\text{Logística}}(p)$$

  Para un entero $m > 0$ se tiene que

$$E(T^m) = \exp(m\,\mu)\,\Gamma(1 + m\,\sigma)\,\Gamma(1 - m\,\sigma)$$

  El m-ésimo momento no es finito si $m\,\sigma \geq 1$.

  Para $\sigma < 1$: $\mu_T = \exp(\mu)\,\Gamma(1+\sigma)\,\Gamma(1-\sigma)$

  y para $\sigma < 1/2$: $\sigma^2_T = \exp(2\,\mu)\,\left[\Gamma(1+2\,\sigma)\,\Gamma(1-2\,\sigma) - \Gamma^2(1+\sigma)\,\Gamma^2(1-\sigma)\right]$

- Un variable aleatoria T tiene distribución t-student(ν) si su función de densidad está dada por:

$$f_T(t) = \frac{\Gamma[(\nu+1)/2]}{\sqrt{\pi\,\nu}\,\Gamma(\nu/2)}\,\left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2}, \quad -\infty < t < \infty$$

  - $\mu_T = 0$, para $\nu > 1$.
  - $\sigma^2_T = \frac{\nu}{\nu-2}$, para $\mu > 2$.

- Si $T \sim$ Fisher$(\eta,\, \nu)$, se tiene que

$$f_T(t) = \frac{\Gamma(\frac{\eta+\nu}{2})}{\Gamma(\eta/2)\Gamma(\nu/2)}\,\left(\frac{\eta}{\nu}\right)^{\frac{\eta}{2}}\,\frac{t^{\frac{\eta}{2}-1}}{\left(\frac{\eta}{\nu}\,t+1\right)^{\frac{\eta+\nu}{2}}}, \quad t > 0$$

  - $\mu_T = \frac{\nu}{\nu-2}$, para $\nu > 2$.
  - $\sigma^2_T = \frac{2\,\nu^2\,(\eta+\nu-2)}{\eta\,(\nu-2)^2\,(\nu-4)}$, para $\nu > 4$.