MAA301 — 2022-2023

Homework 1 - Measure and Integration
Due to October 17th, 2022

Exercise 1. *Let $A \subset \mathbb{N}$ be a subset of integers. We say that A has a* density *if the limit*

$$\lim_{n\to\infty} \frac{|A \cap \{0, 1, \ldots, n\}|}{n}$$

exists, where $|\cdot|$ denotes the cardinal of a set. In this case we say that this limit is the density *of A and we denote it by $d(A)$. Set $\mathcal{D} := \{A \subset \mathbb{N} : A \text{ has a density}\}$ and $\mathcal{D}_a := \{A \subset \mathbb{N} : A \text{ has a density and } d(A) = a\}$ for all $a \in [0, 1]$.*

1. *Let $A \subset \mathbb{N}$ be a finite set. Show that A has a density and compute $d(A)$.*
2. *Let $A := p\mathbb{N} = \{pn : n \in \mathbb{N}\}$ with $p \geq 1$. Show that A has a density and compute $d(A)$.*
3. *(Bonus) Let $1 \leq p_1 < p_2 < \cdots$ be an increasing sequence of integers that are pairwise co-prime. Let $A := \{p_1, p_2, \ldots\}$. Show that $A \in \mathcal{D}_0$.*
   (Hint : consider the cases $\sum_{i\geq 1} p_i^{-1} < +\infty$ and $\sum_{i\geq 1} p_i^{-1} = +\infty$).
4. *Find a subset of $\mathbb{N}$ with no density.*
5. *Let $\mathcal{B}$ be a Boolean algebra of $\mathbb{N}$ included in $\mathcal{D}$. Show that d is a set additive function on $(\mathbb{N}, \mathcal{B})$.*
6. *Is $\mathcal{D}$ a Boolean algebra ? Justify.*
7. *Show that $\mathcal{D}_0 \cup \mathcal{D}_1$ is a Boolean algebra.*

We define the following relation between subsets of $\mathbb{N}$: for all $A, B \subset \mathbb{N}$, $A \sim B$ if and only if $A\Delta B \in \mathcal{D}_0$, where $A\Delta B := (A \setminus B) \cup (B \setminus A)$ is the so-called symmetric difference *between A and B.*

8. *Show that $\sim$ is an equivalence relation.*

For all $A \subset \mathbb{N}$, denote by $C\ell(A)$ the equivalence class of A for the relation $\sim$.

9. *Compute $C\ell(\emptyset)$ and $C\ell(\mathbb{N})$.*
10. *Let $A \subset \mathbb{N}$, show that $C\ell(A^c) = \{B^c : B \in C\ell(A)\}$.*
11. *Let $A, A', B, B' \subset \mathbb{N}$ such that $A \sim A'$ and $B \sim B'$, prove that $A \cup B \sim A' \cup B'$.*
12. *Show that, if $A \in \mathcal{D}_a$ for some $a \in [0, 1]$, then $C\ell(A) \subset \mathcal{D}_a$.*
13. *Let $\mathcal{B}$ be a Boolean algebra on $\mathbb{N}$. Show that $\bigcup_{A\in\mathcal{B}} C\ell(A)$ is a Boolean algebra.*

Exercise 2. *Let $\mathbb{N}$ be the set of integers and $\mathcal{P}(\mathbb{N})$ the set of all subsets of $\mathbb{N}$. The goal of this exercise is to build a set additive function μ on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ such that μ takes only the value 0 or 1, $\mu(\{n\}) = 0$ for all $n \in \mathbb{N}$ and $\mu(\mathbb{N}) = 1$. We define the notion of filter : a* filter *is a set $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ such that*

i. $\mathbb{N} \in \mathcal{F}$,

ii. $\emptyset \notin \mathcal{F}$,

iii. $\forall A \in \mathcal{F}, \forall B \subset \mathbb{N},\ A \subset B \Rightarrow B \in \mathcal{F}$,

iv. $\forall A, B \in \mathcal{F},\ A \cap B \in \mathcal{F}$.

Moreover, $\mathcal{F}$ is an ultrafilter *if it is a filter and satisfies the extra condition*

v. $\forall A \subset \mathbb{N},\ A \in \mathcal{F}$ *or* $A^{\mathsf{c}} \in \mathcal{F}$.

Define $\mathcal{F}_0 := \{A \subset \mathbb{N} : A^{\mathsf{c}}$ is finite$\}$.

1. *Show that $\mathcal{F}_0$ is a filter.*
2. *Let $(\mathcal{F}_i)_{i \leq 1}$ be a sequence of filters such that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for all $i \geq 1$. Show that $\bigcup_{i \geq 1} \mathcal{F}_i$ is a filter.*
3. *Let $\mathcal{F}$ be an ultrafilter and define $\mu_{\mathcal{F}}$ by : for all $A \subset \mathbb{N}$, $\mu_{\mathcal{F}}(A) = 0$ if $A \notin \mathcal{F}$ and $\mu_{\mathcal{F}}(A) = 1$ if $A \in \mathcal{F}$. Show that $\mu_{\mathcal{F}}$ is a set additive function on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.*
4. *Let $\mathcal{F}$ be a filter and $A \subset \mathbb{N}$. We define the set $\mathcal{F}_A := \{D \subset \mathbb{N} : \exists B \in \mathcal{F}, A \cap B \subset D\}$. Show that $\mathcal{F} \subset \mathcal{F}_A$ and that $\mathcal{F}_A$ satisfies conditions i, iii and iv in the definition of filter.*
5. *Let $\mathcal{F}$ be a filter and $A \subset \mathbb{N}$. Suppose that $A \notin \mathcal{F}$ and $A^{\mathsf{c}} \notin \mathcal{F}$, show that $\mathcal{F}_A$ is a filter and $\mathcal{F} \neq \mathcal{F}_A$.*
6. *Suppose that there exists a filter $\mathcal{F}$ containing $\mathcal{F}_0$ and such that it is maximal for the inclusion (i.e. if $\mathcal{F}'$ is another filter with $\mathcal{F} \subset \mathcal{F}'$ then $\mathcal{F}' = \mathcal{F}$). Using previous questions, find a set additive function μ on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ such that μ takes only the value 0 or 1, $\mu(\{n\}) = 0$ for all $n \in \mathbb{N}$ and $\mu(\mathbb{N}) = 1$.*
7. *(Bonus) Show that there exists a filter $\mathcal{F}$ containing $\mathcal{F}_0$ and such that it is maximal for the inclusion. (Hint : use Zorn's lemma and question 2).*

Exercise 3. *Let Π_0 be the set of all subsets of $\mathbb{R}$ that are either open or closed. For all $n \geq 0$ we define $\Pi_{n+1} := \{A \cup B : A, B \in \Pi_n\} \cup \{A \cap B : A, B \in \Pi_n\}$. Finally we set $\Pi_\infty := \bigcup_{n \geq 0} \Pi_n$.*

1. *Show that Π_∞ is a Boolean algebra on $\mathbb{R}$.*

Let $\Sigma_0 := \{A \cap B : A, B \subset \mathbb{R},\ A$ open, B closed$\}$ and $\Sigma_1 := \{A_1 \cup \cdots \cup A_n : n \in \mathbb{N}, n \geq 1,\ A_1, \ldots, A_n \in \Sigma_0\}$.

3. *Show that Σ_1 is a Boolean algebra on $\mathbb{R}$.*
4. *Show that $\Sigma_1 = \Pi_\infty$.*

Let $\Sigma_1' := \{A_1 \sqcup \cdots \sqcup A_n : n \in \mathbb{N}, n \geq 1,\ A_1, \ldots, A_n \in \Sigma_0$ are pairwise disjoint$\}$.

6. *Show that $\Sigma_1 = \Sigma_1'$.*
7. *Prove that Σ_1 can't contain a set $D \subset \mathbb{R}$ such that : there exists a non-empty open interval J such that D is dense in J and for every non-empty open interval $I \subset J$, $I \cap D \notin \Pi_0$.*

   (Hint : write $D = A_1 \cup \cdots \cup A_n$ with $A_i \in \Sigma_0$ for all i and do an induction on n).
8. *Is Σ_1 equal to the Borel σ-algebra on $\mathbb{R}$?*