# MAA301 Homework 2

Yi Yao Tan, Vrushank Agrawal, Hieu Le

Nov 28, 2022

## 1 Exercise

### 1.1 $\lim\limits_{n\longrightarrow\infty} \int_1^n \frac{dx}{x^2+\frac{1}{n}}$

For $x \in (1, \infty)$ the domain.
We define $f_n(x) = \frac{\mathbb{1}_{(1,n]}}{x^2+1/n}$, this defines a piecewise continuous function on the domain which is measurable.
We notice that :

$$\int_1^n \frac{dx}{x^2+\frac{1}{n}} = \int_1^\infty \frac{\mathbb{1}_{(1,n]}dx}{x^2+\frac{1}{n}}$$

We define $f(x) = \frac{1}{x^2}$ which is measurable on the domain as it is continuous and well defined.

- Fix $x \in (1, \infty)$. Take $\epsilon > 0$, if $N = \max(\lceil \frac{1}{\epsilon} \rceil, \lceil x \rceil)$.
  We compute that for all $n \geq N$

$$\begin{aligned}\left|\frac{\mathbb{1}_{(1,n]}}{x^2+\frac{1}{n}} - \frac{1}{x^2}\right| &= \left|\frac{\mathbb{1}_{(1,n]}x^2 - x^2 - \frac{1}{n}}{(x^2+\frac{1}{n})x^2}\right| \\ &= \left|\frac{-\frac{1}{n}}{(x^2+\frac{1}{n})x^2}\right| \\ &\leq \frac{1}{n} \\ &\leq \epsilon\end{aligned}$$

  as $(x^2+\frac{1}{n})x^2 > 1$
  Thus we have that $f_n \to f$ point wise on $x \in (1, \infty)$

- Now since $\forall n \in \mathbb{N}$, $\frac{1}{n} > 0$ and $\frac{1}{x^2} > 0$ on the domain, we have that $0 \leq \left|\frac{\mathbb{1}_{(1,n]}}{x^2+1/n}\right| \leq \frac{1}{x^2}$

Now we integrate

$$\begin{aligned}\int_1^\infty |\frac{1}{x^2}|dx &= \int_1^\infty \frac{1}{x^2} \\ &= [-\frac{1}{x}]_1^\infty \\ &= 1\end{aligned}$$

Hence $f$ is both $L^1$ and we have that the integral evaluates to 1 by DCT.

### 1.2 $\lim\limits_{n\longrightarrow\infty} \frac{1}{2^n} \int_{\mathbb{R}} (1+cos(x))^n e^{-x^2}dx$

- We know that $x \to \left(\frac{1+cos(x)}{2}\right)^n e^{-x^2}$ by composition is a continuous function on $\mathbb{R}$ for each $n \geq 0$ thus are measurable. For $x \in \mathbb{R} \setminus 2k\pi$ with $k \in \mathbb{N}$, we have $0 \leq (1+cos(x)) < 2$. This gives us:

$$\left(\frac{1+cos(x)}{2}\right)^n e^{-x^2} \underset{n\to\infty}{\longrightarrow} 0$$

  pointwise convergence almost everywhere on $\mathbb{R}$. The 0 function is constant and thus measurable.

- We also know that we can bound $0 \leq \left|\left(\frac{1+cos(x)}{2}\right)^n e^{-x^2}\right| \leq e^{-x^2}$ a well known $L^1$ function on $\mathbb{R}$.

By DCT it is immediate that the integral evaluates to 0.

### 1.3 $\underset{n\longrightarrow\infty}{lim} \int_0^{2\pi} \left(1 + \frac{ix}{n}\right)^n dx$

- We know the famous limit that for $x \in (0, 2\pi)$:

$$\underset{n\longrightarrow\infty}{lim} \left(1 + \frac{ix}{n}\right)^n = e^{ix}$$

  We have that $x \rightarrow \left(1 + \frac{ix}{n}\right)^n$ is a continuous function on $x \in (0, 2\pi)$ and thus measurable and the exponential function is also continuous thus measurable.

- For $x \in (0, 2\pi)$, we also can do absolute bounding with $0 \leq |\left(1 + \frac{ix}{n}\right)^n| = |\left(1 + \frac{ix}{n}\right)|^n \leq \left(|1| + |\frac{ix}{n}|\right)^n \leq e^x$. Now we know that $x \rightarrow e^x$ is easily summable on $x \in (0, 2\pi)$.

Hence by DCT we have that

$$\underset{n\longrightarrow\infty}{lim} \int_0^{2\pi} \left(1 + \frac{ix}{n}\right)^n dx = \underset{n\longrightarrow\infty}{lim} \int_0^{2\pi} e^{ix} dx = 0$$

### 1.4 $\underset{n\longrightarrow\infty}{lim} \int_0^{n} \frac{dx}{x+\frac{1}{n}}$

We can simply use fatou's lemma to get non convergence. Fix $x \in (1, +\infty)$.

Similar to the first question we make the same observation that

$$\int_1^n \frac{dx}{x + \frac{1}{n}} = \int_1^\infty \frac{\mathbb{1}_{(1,n]} dx}{x + \frac{1}{n}}$$

Now we define $f_n(x) = \frac{\mathbb{1}_{(1,n]} dx}{x+\frac{1}{n}}$. Since $f_n \xrightarrow{n\longrightarrow\infty} \frac{1}{x}$ pointwise, we have that $\liminf_n f_n = \lim_n f_n$.

Now by fatou:

$$\begin{aligned}
+\infty &= \int_1^\infty \frac{1}{x} dx \\
&= \int_1^\infty \lim_n f_n(x) dx \\
&= \int_1^\infty \liminf_n f_n(x) dx \\
&\leq \liminf_n \int_1^\infty f_n(x) dx \\
&\leq \lim_{n\longrightarrow\infty} \int_1^\infty f_n(x) dx
\end{aligned}$$

Thus the limit can only be $+\infty$.

## 2 Exercise

### 2.1 Show that

$$\lim_{m\rightarrow m_0} \int_X f_m(x) d\mu(x) = \int_X f(x) d\mu(x)$$

Let $U := B(m_0, r)$ then $U$ is an open set of $M$.

Let $m_n$ be an arbitrary sequence of elements of $U$ such that $m_n \to m_0$; set $\phi_n(x) = f_{m_n}(x)$.

For each $n$, the function $\phi_n$ is measurable on $X$, and $\phi_n \to f_{m_n}$ almost everywhere on $X$ as $n \to \infty$.

For almost everywhere $x \in X$, one has $\left|\phi_n(x)\right| \leq g(x)$ for all $n \geq 0$.

More precisely, the set $N_j = \left\{x \in X : \left|\phi_j(x)\right| > g(x)\right\}$ is negligible for each $j \geq 0$; then

$$\left|\phi_n(x)\right| \leq g(x) \text{ for all } x \in X \setminus \bigcup_{j \geq 0} N_j \text{ and } n \geq 0$$

By the dominated convergence theorem

$$\int_X \phi_n(x) d\mu(x) = \int_X f_{m_n}(x) d\mu(x) \to \int_X f_{m_0}(x) d\mu(x) \quad \text{as } m_n \to m_0$$

Since this is true for any sequence $m_n \to m_0$, we conclude that

$$\lim_{m \to m_0} \int_X f_m(x) d\mu(x) = \int_X f_{m_0}(x) d\mu(x)$$

## 2.2 Compute the following limit where the convergence $a \to 0$ takes place in $\mathbb{R}$:

$$\lim_{a \to 0} \int_0^1 \frac{1 - e^{ax}}{x} dx$$

Applying the above result, with $M = (0, 1]$, $X = (0, 1]$, $f_m(x) = \frac{1-e^{mx}}{x}$ and $f(x) = 0$ for all $x \in X$.

It's obvious that $f_m$ and $f$ are measurable for all $m \in M$ (1)

For all $x \in X$, $\lim_{m \to 0} f_m(x) = 0 = f(x)$ (2)

We consider the function $g : x \to \frac{e^x - 1}{x}$ for all $x \in X$.

Since for all $m \in (0, 1]$ and $x \in X$, we have that $e^{mx} > 1$, so

$$\left|f_m(x)\right| = \frac{e^{mx} - 1}{x} \leq \frac{e^x - 1}{x} = g(x)$$

Since $\lim_{x \to 0}(e^x - 1) = \lim_{x \to 0} x = 0$ and $\lim_{x \to 0} \frac{e^x}{1} = 1$, using L'Hopital rule, we deduce that

$$\lim_{x \to 0} g(x) = 1$$

Thus, $g$ is summable on $X$. (3)

From (1), (2), (3), we deduce that $\lim_{m \to 0} \int_X f_m(x) dx = \lim_{a \to 0} \int_0^1 \frac{1-e^{ax}}{x} dx = \int_X f(x) dx = 0$

# 3 Exercise

We have that:
- $f_n$ is measurable for all $n \geq 0$,
- $f_n$ converges to $f$ measurable almost everywhere,
- $\left|f_n\right| \leq f_0$ for all $n \geq 0$, $f_0$ is summable

Thus, by DCT, we deduce that $f$ is summable and

$$\lim_{n \to \infty} \int_X f_n(x) d\mu(x) = \int_X f(x) d\mu(x) < +\infty$$

Let $X = \mathbb{R}$. Take the function $f_n(x) = \mathbb{1}_{\mathbb{R}\setminus(-n,n)}(x)$. This function is clearly non-negative and measurable; and the sequence is non-increasing. The sequence converges to $f(x) = 0$ which is measurable and non-negative too. It is clear that $f_0(x) = \mathbb{1}_{\mathbb{R}}(x)$ is not summable.

$$\begin{aligned}
\infty &= \lim_{n\to\infty} \infty \\
&= \lim_{n\to\infty} \int_{\mathbb{R}\setminus(-n,n)} dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx \\
&\neq \int_{\mathbb{R}} f(x)dx \\
&= \int_{\mathbb{R}} 0dx \\
&= 0
\end{aligned}$$

Thus the equality does not hold true.

# 4 Exercise

## 4.1 Show that $\mu\left(f^{-1}((1,+\infty))\right) = 0$.

Assume by contradiction $\mu\left(f^{-1}((1,+\infty))\right) > 0$.
Let $(M_j)_{j\in J}$ be a partition of $f^{-1}((1,+\infty))$. In particular: $\bigcup_{j\in J} M_j = f^{-1}((1,+\infty))$.
It is clear that $f^{-1}((1,+\infty) = f^{-1}((1,+\infty))^n$
So we have that:

$$\sum_{j\in J} \mu(M_j) = \mu\left(f^{-1}((1,+\infty))\right) > 0$$

and there is $j_0 \in J$ s.t. $\mu(M_{j_0}) > 0$.

We can define a simple function such that $1 < \phi(x) = \underset{x\in M_{j_0}}{min}(f(x)^n) \leq f(x)^n$ on $x \in M_{j_0}$ and $\phi(x) = 0 \leq f(x)^n$ everywhere else. This also means that $0 \leq \phi(x) \leq f(x)^n$ for all $x \in X$. By the definition of the integral of positive measureable functions, the integral of $f^n$ is at least the integral of the simple function of $\phi$ .

$$\begin{aligned}
+\infty &> \liminf_{n\longrightarrow\infty} \int_X f(x)^n d\mu(x) \\
&\geq \liminf_{n\longrightarrow\infty} \int_X \phi(x)d\mu(x) \\
&= \liminf_{n\longrightarrow\infty} \quad \underset{x\in M_{j_0}}{min}(f(x))^n \mu(M_{j_0}) = +\infty
\end{aligned}$$

In this last line we just use the fact that $\underset{x\in M_{j_0}}{min}(f(x)) > 1$ and $\mu(M_{j_0}) > 0$. Which is a contradiction.

## 4.2 Show that $\mu\left(f^{-1}(\{1\})\right) < +\infty$.

By question 1 we find $f^{-1}((1,\infty))$ is a negligble set. Hence integrating over $X$ is same as integrating over $f^{-1}((0,1])$.

Let $(M_j)_{j\in J}$ be a partition of $X$ s.t. there is $j_0 \in J$ s.t. $M_{j_0} = f^{-1}(\{1\})$.

We similarly define a simple function $0 \leq \phi(x) = 1 = f(x)$ for $x \in M_{j_0}$ and $\phi(x) = 0$ everywhere else. Now it is similarly clear that $0 \leq \phi(x) \leq f(x)^n$ for all $x \in X$ and $\phi$ is a simple function. We know that the integral of the positive function $f^n$ is at least the integral of the simple function.

$$\begin{aligned}+\infty &> \liminf_{n\longrightarrow\infty} \int_X f(x)^n d\mu(x)\\ &\geq \liminf_{n\longrightarrow\infty} \int_X \phi(x)^n d\mu(x)\\ &= 1\times\mu(M_{j_0})\\ &= \mu(f^{-1}(\{1\}))\end{aligned}$$

### 4.3 Using Exercise 3 show that

$$\liminf_{n\to\infty} \int_X f(x)^n d\mu(x) = \lim_{n\to\infty} \int_X f(x)^n d\mu(x) = \mu\left(f^{-1}(\{1\})\right)$$

We use the same decomposition and question 1 to integrate over $f^{-1}((0,1])$ instead of $X$ to get:

$$\begin{aligned}\infty &> \liminf_{n\longrightarrow\infty} \int_X f(x)^n d\mu(x)\\ &= \liminf_{n\longrightarrow\infty} \int_{X\setminus f^{-1}((1,+\infty))} f(x)^n d\mu(x)\\ &= \liminf_{n\longrightarrow\infty} \int_{f^{-1}((0,1])} f(x)^n d\mu(x)\end{aligned}$$

It is clear that since $f(x) \in (0,1]$ for the domain of the integral, our function is non-increasing. We see that we have a function that is bounded below, positive, and non-increasing sum; thus liminf is equal to the limit by the monotone convergence theorem.

Assume that $f^n$ is not summable for all $n \in \mathbb{N}$. However, since the liminf over $n \in \mathbb{N}$ is finite we have necessarily there is at least one $N_0 \in \mathbb{N}$ s.t. $f^{N_0}$ is summable. Thus we can apply question 3 to bring the limit into the integral while we start the sequence from $N_0$.

Let $g(x) = \mathbb{1}_{f^{-1}(\{1\})}(x)$ positive measurable function, it is clear that the sequence of functions $f^n(x)$ converges pointwise to g on $x \in f^{-1}((0,1])$

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \int_X f(x)^n d\mu(x) &= \lim_{n\longrightarrow\infty} \int_X f(x)^n d\mu(x)\\ &= \lim_{n\longrightarrow\infty} \int_{f^{-1}((0,1])} f(x)^n d\mu(x)\\ &= \int_{f^{-1}((0,1])} g(x) d\mu(x)\\ &= \int_{f^{-1}((0,1])} \mathbb{1}_{f^{-1}(\{1\})}(x) d\mu(x)\\ &= \mu\left(f^{-1}(\{1\})\right)\end{aligned}$$

We similarly used question 1 to integrate over $f^{-1}((0,1])$ instead of $X$.

## 5 Exercise

### 5.1 Show that F well-defined and continuous on $[0,+\infty)$

- fix $x \geq 0$, for $t \in (0,+\infty)$:

$$t \to e^{-xt}\frac{(1-e^{-t})^2}{t^2}$$

is measurable by composition of continuous function on $(0, +\infty)$

- fix $t \in (0, +\infty)$, we have that function:
$$x \to e^{-xt}\frac{(1-e^{-t})^2}{t^2}$$
is continuous since we have a constant times the exponential function which is continuous.

- For $x \geq 0$ and $t \in (0, +\infty)$, we have that $e^{-xt} \leq 1$ and $0 \leq (1-e^{-t})^2 \leq 1$ since exponential of non-positive numbers are positive but less than 1. We also know that for $1 - e^{-t} \leq t$, since $e^{-u} \leq 1$ and integrating, $\int_0^t e^{-u}du \leq \int_0^t du$ we get our result and hence $\left(\frac{1-e^{-t}}{t}\right)^2 \leq 1$.
With these three inequalities we find that
$$\begin{aligned} e^{-xt}\frac{(1-e^{-t})^2}{t^2}dt &\leq \frac{(1-e^{-t})^2}{t^2}dt \\ &\leq min(1, \frac{1}{t^2}) \end{aligned}$$
$h(t) = min(1, \frac{1}{t^2})$ is in fact integrable, splitting into the two intervals with $h(t) = 1$ on $t \in (0, 1]$ and $h(t) = \frac{1}{t^2}$ on $t \in (1, \infty)$ which are both integrable.

Hence we can conclude by integrals depending on parameter theorem that our function F is well defined and continuous on $x \geq 0$.

## 5.2 Show that F is $C^2$ on $(0, \infty)$

- fix $x > 0$, we have summability since $t \in (0, \infty) \to e^{-xt}\frac{(1-e^{-t})^2}{t^2}$ is a positive function which can be bounded by $h$ as defined in 5.1 is integrable and we reuse the first point in 5.1 to get measurable.

- We compute that the first and second partial derivatives w.r.t. $x$ are
$$\frac{\delta f(x,t)}{\delta x} = -e^{-xt}\frac{(1-e^{-t})^2}{t}$$
and
$$\frac{\delta^2 f(x,t)}{\delta^2 x} = e^{-xt}(1-e^{-t})^2$$
They are defined on $x > 0$ and $t \in (0, +\infty)$ and are obviously continuous as a function of $x$.

- Now fix a constant $\epsilon \in (0, +\infty)$, for $x > \epsilon$ and $t \in (0, +\infty)$. We expand the equation and the first derivative can be absolutely bounded by the function:
$$h_1(t) = min(e^{-\epsilon t} - 2e^{-(\epsilon+1)t} + e^{-(\epsilon+2)t}, \frac{e^{-\epsilon t} - 2e^{-(\epsilon+1)t} + e^{-(\epsilon+2)t}}{t})$$
We expand the equation and we have that the second derivative can be absolutely bounded by the function:
$$h_2(t) = e^{-\epsilon t} - 2e^{-(\epsilon+1)t} + e^{-(\epsilon+2)t}$$
as the $u \to e^{-u}$ is a decreasing function and each part of the sum $e^{-\epsilon t} - 2e^{-(\epsilon+1)t} + e^{-(\epsilon+2)t}$ is absolutely integrable. For $h_1$ we'll have the two intervals of integration– the left side of the minimum function on $(\epsilon, 1]$ and the right $(1, +\infty)$ and are both absolutely integrable. Meanwhile $h_2$ is absolutely integrable for all intervals $(\epsilon, +\infty)$. Since $\epsilon \in (0, +\infty)$, we have the absolute bounding functions for both of the derivatives on the set $(x, t) \in (0, +\infty)^2$.

Thus by the theorem on integrals depending on a parameter giving differentiability gives us that F is $C^2$ on $(0, +\infty)$.

## 5.3 Compute the limit of $F(x)$ and $F'(x)$ when $x$ goes to $+\infty$

- fix $0 \leq x \in \mathbb{N}$, for $t \in (0, +\infty)$:
$$f_x(t) : t \to e^{-xt} \frac{(1-e^{-t})^2}{t^2}$$
forms a sequence of measurable functions $(f_x)_{x\geq 0}$ by the composition of continuous functions on $t \in (0, +\infty)$. This sequence converges to 0 almost everywhere which is a constant function and is measurable.

- We can reuse $h(t) = min(1, \frac{1}{t^2})$ as our summable, dominating function we found in 5.1.

DCT gives that:
$$\lim_{x\longrightarrow\infty} F(x) = \lim_{x\longrightarrow\infty} \int_0^\infty f_x(t)dt = \int_0^\infty 0dt = 0$$

Now:

- fix $0 \leq x \in \mathbb{N}$, for $t \in (0, +\infty)$:
$$f'_x(t) : t \to e^{-xt} \frac{(1-e^{-t})^2}{t}$$
forms a sequence of measurable functions $(f'_x)_{x\geq 0}$ by the composition of continuous functions on $t \in (0, +\infty)$. This sequence converges to 0 almost everywhere which is a constant function and is measurable.

- We can reuse $h_1$ as our summable, dominating function we found in 5.2.

DCT gives that:
$$\lim_{x\longrightarrow\infty} F'(x) = \lim_{x\longrightarrow\infty} \int_0^\infty f'_x(t)dt = \int_0^\infty 0dt = 0$$

## 5.4 Compute $F(x)$ for $x > 0$.

- From 5.2 above, we know that $F''$ exists and is integrable for $x > 0$. Hence,
$$\begin{aligned}
F''(x) &= \int_0^\infty e^{-xt}(1-e^{-t})^2 dt \\
&= \int_0^\infty (e^{-xt} - 2e^{-(x+1)t} + e^{-(x+2)t})dt \\
&= -\frac{e^{-xt}}{x}\bigg|_{t=0}^{t=\infty} + 2\frac{e^{-(x+1)t}}{x+1}\bigg|_{t=0}^{t=\infty} - \frac{e^{-(x+2)t}}{x+2}\bigg|_{t=0}^{t=\infty} \\
&= -\left(0 - \frac{1}{x}\right) + \left(0 - \frac{2}{x+1}\right) - \left(0 - \frac{1}{x+2}\right) \\
&= \frac{1}{x} - \frac{2}{x+1} + \frac{1}{x+2}
\end{aligned}$$

- We can now integrate $F''$ to get,
$$\begin{aligned}
F'(x) &= \int F''(x)dx \\
&= \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{1}{x+2}\right)dx \\
&= \ln(x) - 2\ln(x+1) + \ln(x+2) + C' \\
&= \ln\left(\frac{x(x+2)}{(x+1)^2}\right) + C'
\end{aligned}$$

Now by 5.3 above, we know that $\lim_{x\to\infty} F'(x) = 0$. So,
$$\begin{aligned}
\lim_{x\to\infty} F'(x) &= \lim_{x\to\infty} \ln\left(\frac{x(x+2)}{(x+1)^2}\right) + C' \\
&= 0 + C' = 0 \\
\Longrightarrow\ C' &= 0
\end{aligned}$$

- Finally, we integrate $F'$ to get,

$$
\begin{aligned}
F &= \int F'(x)dx \\
&= \int \ln\left(\frac{x(x+2)}{(x+1)^2}\right)dx \\
&= x\ln\left(\frac{x(x+2)}{(x+1)^2}\right) - \int x\left(\frac{1}{x} - \frac{2}{x+1} + \frac{1}{x+2}\right)dx + C \\
&= x\ln\left(\frac{x(x+2)}{(x+1)^2}\right) - \int \left(\frac{x}{x} - \frac{2(x+1)-2}{x+1} + \frac{x+2-2}{x+2}\right)dx + C \\
&= x\ln\left(\frac{x(x+2)}{(x+1)^2}\right) - \int \left(\frac{2}{x+1} - \frac{2}{x+2}\right)dx + C \\
&= x\ln\left(\frac{x(x+2)}{(x+1)^2}\right) + 2\ln\left(\frac{x+2}{x+1}\right) + C
\end{aligned}
$$

Again by 5.3 above, we know that $\underset{x\to\infty}{lim} F(x) = 0$. So, using L'Hôpital's rule

$$
\begin{aligned}
0 = \lim_{x\to\infty} F(x) &= \lim_{x\to\infty}\left(x\ln\left(\frac{x(x+2)}{(x+1)^2}\right) + 2\ln\left(\frac{x+2}{x+1}\right)\right) + C \\
&= \lim_{x\to\infty}\left(\frac{\ln\left(1 - \frac{1}{(x+1)^2}\right)}{\frac{1}{x}}\right) + 0 + C \\
&= \lim_{x\to\infty} \frac{\frac{2}{(x+1)^3}\cdot\frac{1}{1-\frac{1}{(x+1)^2}}}{-\frac{1}{x^2}} + 0 + C \\
&= \lim_{x\to\infty} \frac{-2x^2}{x^3+3x^2+2} + 0 + C \\
&= \lim_{x\to\infty} \frac{-2}{x+3+\frac{2}{x^2}} + 0 + C \\
&= 0 + 0 + C \\
\implies C = 0
\end{aligned}
$$

and hence, we conclude that for $x > 0$

$$
F(x) = x\ln\left(\frac{x(x+2)}{(x+1)^2}\right) + 2\ln\left(\frac{x+2}{x+1}\right)
$$

### 5.5 Compute $F(0)$.

- Let $F(x) = L_1(x) + L_2(x)$ where

$$
L_1(x) = x\ln\left(\frac{x(x+2)}{(x+1)^2}\right) \quad \text{and} \quad L_2(x) = 2\ln\left(\frac{x+2}{x+1}\right)
$$

.

- It is straightforward that $\lim_{x\to 0^+} L_2(x) = 2\ln 2$.

- For $\lim\limits_{x\to 0^+} L_1(x)$ we can use L'Hôpital's rule to get

$$
\begin{aligned}
\lim_{x\to 0^+} L_1(x) &= \lim_{x\to 0^+} x\ln\left(\frac{x(x+2)}{(x+1)^2}\right)\\
&= \lim_{x\to 0^+} x\ln\left(1-\frac{1}{(x+1)^2}\right)\\
&= \lim_{x\to 0^+} \frac{\ln\left(1-\frac{1}{(x+1)^2}\right)}{\frac{1}{x}}\\
&= \lim_{x\to 0^+} \frac{\frac{2}{(x+1)^3}\cdot\frac{1}{1-\frac{1}{(x+1)^2}}}{-\frac{1}{x^2}}\\
&= \lim_{x\to 0^+} \frac{-2x^2}{(x+1)((x+1)^2-1)}\\
&= \lim_{x\to 0^+} \frac{-2x^2}{x^3+3x^2+2}\\
&= 0
\end{aligned}
$$

- From 5.1 we know that F is continuous in the interval $]0,+\infty[$ and hence

$$
\begin{aligned}
F(0) &= \lim_{x\to 0^+} F(x)\\
&= \lim_{x\to 0^+} (L_1(x)+L_2(x))\\
&= 0 + 2\ln 2
\end{aligned}
$$

  Hence, we have $F(0) = 2\ln 2$

# 6 Exercise

## 6.1 Show that $F(x)$ is well-defined for all $x \geq 0$ and that $F(x)x^{-\frac{1}{2}} \to 0$ when $x \to 0$.

Since $f \in L^2(\mathbb{R}_+)$, then there exists $M > 0$ such that for all $x \geq 0$, we have that

$$\left(\int_0^x f(t)^2 dt\right) < M$$

Using Cauchy-Schwartz inequality, we deduce that for all $x \geq 0$

$$\left(\int_0^x f(t)dt\right)^2 \leq \left(\int_0^x f(t)^2 dt\right)\left(\int_0^x dt\right) < Mx$$

Being equivalent to

$$F(x) < \sqrt{Mx} \quad \forall x \geq 0$$

Thus, $F(x)$ is well-defined for all $x \geq 0$

Let $I_x := \sqrt{\left(\int_0^x f(t)^2 dt\right)}$ for all $x \geq 0$, then we have that $\lim_{x\to 0} I_x = 0$ (1) and

$$|F(x)| \leq I_x\sqrt{x}$$

Hence,

$$\left|F(x)x^{\frac{-1}{2}}\right| \leq I_x \text{ (2)}$$

From (1) and (2), we deduce that

$$\lim_{x\to 0} F(x)x^{\frac{-1}{2}} = 0$$

### 6.2 Show that $F(x)x^{-\frac{1}{2}} \to 0$ when $x \to \infty$.

Let $\varepsilon > 0$. Since $f \in L^2$, there exists $a > 0$ such that

$$\int_a^\infty f(t)^2 dt < \frac{\varepsilon^2}{4}$$

Let $C := \int_0^a |f(t)|dt$, using Cauchy-Schwartz inequality, we deduce that for all $x > a$

$$|F(x)| \leq \int_0^a |f(t)|dt + \int_a^x |f(t)|dt \leq C + \left(\int_a^x f(t)^2 dt\right)^{\frac{1}{2}} (x-a)^{\frac{1}{2}} \leq C + \frac{\varepsilon}{2}(x-a)^{\frac{1}{2}}$$

Since $\lim_{x\to\infty}\left(C + \frac{\varepsilon}{2}(x-a)^{\frac{1}{2}}\right)x^{\frac{-1}{2}} = \frac{\varepsilon}{2}$, we deduce that there exists $x_0 > a$ such that

$$\left(C + \frac{\varepsilon}{2}(x-a)^{\frac{1}{2}}\right)x^{\frac{-1}{2}} < \varepsilon \ \forall x > x_0$$

Then, we deduce that

$$\left|F(x)x^{\frac{-1}{2}}\right| < \varepsilon \ \forall x > x_0$$

Since $\varepsilon$ is arbitrarily chosen, we conclude that $\lim_{x\to\infty}\left|F(x)x^{\frac{-1}{2}}\right| = 0$, equivalent to $\lim_{x\to\infty} F(x)x^{\frac{-1}{2}} = 0$

## 7 Exercise

### 7.1 Show that for all $x > 0$

$$\left(\int_0^x g(t)dt\right)^2 \leq 2\sqrt{x}\int_0^x \sqrt{t}g(t)^2 dt$$

Using Cauchy-Schwartz inequality, we deduce that

$$\left(\int_0^x g(t)dt\right)^2 \leq \left(\int_0^x \left(g(t)t^{\frac{1}{4}}\right)^2 dt\right)\left(\int_0^x t^{\frac{-1}{2}} dt\right) = 2\sqrt{x}\left(\int_0^x \sqrt{t}g(t)^2 dt\right)$$

### 7.2 Deduce that $G \in L^2(\mathbf{R})_+$ and $\|G\|_2 \leq 2\|g\|_2$.

From the first question, combining with Tonelli theorem, we deduce that

$$\begin{aligned}
\|G\|_2^2 = \int_0^\infty G(x)^2 dx &\leq 2\int_0^\infty \frac{1}{x\sqrt{x}}\int_0^x \sqrt{t}g(t)^2 dt dx \\
&= 2\int_0^\infty \int_0^x \frac{\sqrt{t}}{x\sqrt{x}} g(t)^2 dt dx \\
&= 2\int_0^\infty \int_t^\infty \frac{\sqrt{t}}{x\sqrt{x}} g(t)^2 dx dt \\
&= 2\int_0^\infty \sqrt{t}g(t)^2 \int_t^\infty \frac{1}{x\sqrt{x}} dx dt \\
&= 4\int_0^\infty g(t)^2 dt = 4\left\|g\right\|_2^2
\end{aligned}$$

Hence,

$$\|G\|_2 \leq 2\left\|g\right\|_2$$

Which also gives us $G \in L^2(\mathbb{R}_+)$

# 8 Exercise

## 8.1 Suppose that $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in X$. Show that $f$ is measurable.

We use the equivalent definition that $\forall a \in \mathbb{R}$, $\{x : g(x) < a\}$ is measurable giving measurability of $g$ since the sets $(-\infty, a)$ for all $a \in \mathbb{R}$ generate the borel sigma algebra on $\mathbb{R}$.

Fix $a \in \mathbb{R}$ and let $J \subset \mathbb{N}$. Let $g_n$ be a sequence of measurable functions. We have that:

$$\{x : \inf_{n \in J} g_n(x) < a\} = \bigcup_{n \in J} \{x : g_n(x) < a\} \tag{8.1}$$

This gives that $\inf_{n \in J} g_n$ is measurable since $J$ is countable, so we have a countable union of measurable sets which is measurable.

We can do similar to get that $\sup_{n \in J} g_n$ is measurable

$$\{x : \sup_{n \in J} g_n(x) < a\} = \bigcap_{n \in J} \{x : g_n(x) < a\} \tag{8.2}$$

By a countable intersection of measurable sets.
Now if we set $J_0 = \mathbb{N}$ and $J_1 = \{n : n \geq k\}$ for some $k \in \mathbb{N}$, are countable subsets of $\mathbb{N}$.

Since, $f_n$ is pointwise convergent to $f$, we have that for all $x \in X$.

$$\begin{aligned} f(x) &= \lim_{n \longrightarrow \infty} f_n(x) \\ &= \lim_{n \longrightarrow \infty} \inf f_n(x) \\ &= \sup_{k \in J_0} \{\inf_{n \in J_1} f_n(x)\} \end{aligned}$$

The last part by the previous two results: 8.2, 8.1 proven is measurable.

### 8.2

Take $X = \mathbb{R}$ and $\mathcal{F} = Bor(\mathbb{R})$. Take $f_n(x) = 0 \ \forall x \in X$ which are measurable as they are constant functions. Now we take $N \subset \mathbb{R}$ s.t. $N \notin Bor(\mathbb{R})$ and negligible.

We set $f(x) = \mathbb{1}_N(x)$, and we can see that $f_n \xrightarrow{n \longrightarrow \infty} f$ almost everywhere.

However take an open interval, $(0, 2) \in Bor(\mathbb{R})$,

$$f^{-1}((0, 2)) = \mathbb{1}_N^{-1}((0, 2)) = N \notin Bor(\mathbb{R})$$

Hence $f$ is not measurable.

### 8.3

Let $A := \{x \in X | \lim_{n \to \infty} f_n(x) = f(x)\} \subset X$. Since $f_n \to f$ a.e., we deduce that $\mu(A^C) = 0$. We can define:

$$g(x) = \mathbb{1}_A(x) f(x)$$

and

$$g_n(x) = \mathbb{1}_A(x) f_n(x)$$

which is a measurable function by multiplication of $f_n$ and $\mathbb{1}_A$. We see that $g_n \xrightarrow{n \longrightarrow \infty} g$ pointwise, thus by application of question 1 we have that $g$ is measurable.

Now $g(x) = f(x)$ almost everywhere since we could potentially have $g(x) \neq f(x)$ only for $x \in A^C$; but by the definition of $A$, we have that $A^C$ is in a negligible set.

# 9 Exercise

## 9.1 Show that $\mathcal{F}$ is a $\sigma$-algebra on $X$.

- Since $\mathbb{R} \in \mathcal{B}$ the borel sigma algebra. We have that $f^{-1}(\mathbb{R}) = X \in \mathcal{F}$
- Let $B \in \mathcal{B}$ is a borel set. Now we also know $f^{-1}(B)^c = f^{-1}(B^c)$. And since the borel sigma algebra is stable by complement, we have that $B^c \in \mathcal{B}$. Hence we have that $f^{-1}(B)^c = f^{-1}(B^c) \in \mathcal{F}$. This also gives the existence of the empty set in $\mathcal{F}$ combined with the first point.
- We take a countable union of elements in $\mathcal{F}$ with $I$ a countable index set of elements:
$$\bigcup_{i \in I} f^{-1}(B_i) = f^{-1}(\bigcup_{i \in I} B_i)$$
By the sigma algebra property of the Borel sigma algebra, $\bigcup_{i \in I} B_i \in \mathcal{B}$ is a borel set. Thus
$$\bigcup_{i \in I} f^{-1}(B_i) \in \mathcal{F}$$

Hence $\mathcal{F}$ is a sigma algebra on $X$.

## 9.2 Let $h : \mathbb{R} \to \mathbb{R}$ measurable. Show that $h \circ f : (X, \mathcal{F}) \to \mathbb{R}$ is measurable.

Let $\mathcal{B}$ be the borel sigma algebra on the image space $\mathbb{R}$ of $h \circ f$. Now take $A \in \mathcal{B}$.

First we have that $h^{-1}(A) \in \mathcal{B}$, the borel sigma algebra of the codomain of $f$, since $h$ is measurable. Thus this gives us $f^{-1}(h^{-1}(A)) \in \mathcal{F}$. Hence we have that
$$(h \circ f)^{-1}(A) = f^{-1} \circ h^{-1}(A) \in \mathcal{F}$$
Thus $(h \circ f)$ is a measurable function.

## 9.3 Let $g : (X, \mathcal{F}) \to \mathbb{R}$ be measurable. Show that there exists $h : \mathbb{R} \to \mathbb{R}$ measurable such that $g = h \circ f$.

Let $g_i(x) = \mathbb{1}_{M_i}(x)$ for some $M_i \subset X$ a measurable function.

Thus for $B \in Bor(\mathbb{R})$, we have that $g_i^{-1}(B) = \mathbb{1}_{M_i}^{-1}(B) \in \mathcal{F}$ by measurability of $g_i$. This by the defintion of $\mathcal{F}$ means that there is $B' \in Bor(\mathcal{R})$ such that.
$$f^{-1}(B') = \mathbb{1}_{M_i}^{-1}(B)$$
In fact we can find the existence $h_i$ by $h_i(x) = \mathbb{1}_{M_i} \circ f^{-1}(x)$ for all $x \in \mathbb{R}$ and is in fact measurable since $h_i^{-1}(B) = B' \in Bor(\mathbb{R})$. Now it is clear that we have:
$$f^{-1} \circ h_i^{-1}(B) = \mathbb{1}_{M_i}^{-1}(B) = g_i^{-1}(B)$$
Thus we have existence of $h_i$ measurable s.t. $g_i = h_i \circ f$.

Now any simple function can be written in the unique decomposition
$$(g')_k = \sum_{i \in I_k} c_i \mathbb{1}_{M_i} = \sum_{i \in I_k} c_i g_i$$
for $c_i = g(M_i) \neq 0$ where $I_k$ is the index set of a partition for which $(g')_k$ is constant and not equal to 0 on the partition $M_i$. Thus $(h')_k = \sum_{i \in I_k} c_i h_i$.

Now, by Beppo-Levi monotone convergence theorem, we have a two non-decreasing and non-negative sequences of simple functions $(g')_{k \in \mathbb{N}}$ indexed in the same way for simplicity, pointwise converging the measurable to non-negative functions $|g^-|$ and $g^+$ respectively. Then we take the negative of the sequence converging to $|g^-|$ which is also measurable.

The limit of the sum of the two sequences converging to $g^+$ and $-|g^-|$ respectively, will be $g = g^+ - |g^-|$, their corresponding $h$ which satisfies the question will be of some form: $h = \lim_{k \longrightarrow \infty} \sum_{i \in I_k} c_i h_i$ is measurable by result proven in question 8.1 and linear combination composition of measurable functions.

# 10 Exercise

Consider the function $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = \frac{f(x)}{\left|f(x)\right| + 1}\ \forall x \in \mathbb{R}$$

We have that $|g(x)| \leq |f(x)|$ for all $x \in R$. Thus,

$$\int_{\mathbb{R}} |g(x)| dx \leq \int_{\mathbb{R}} |f(x)| dx < \infty$$

Which implies that $g \in L^1(\mathbb{R})$

Since $f \in L^1$, we have that there exists $M > 0$ s.t. $|f| < M$ a.e. then, we have that $|g| < 1 - \frac{1}{M+1}$ a.e.

Since $\lim_{n\to\infty} \sum_{k=n}^{\infty} 2^{-k} = 0$, then there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{k=n_0}^{\infty} 2^{-k} < \frac{1}{M+1}$$

Since $C_C(\mathbb{R})$ is dense in $L^1$, we can take a sequence $(\phi_n)_{n\geq 0}$ converging to $g$ a.e. (1)

We will prove that we can take a subsequence of $(\phi_n)_{n\geq 0}$ that is bounded by a function $h \in L^1$ almost everywhere and $|h| < 1$ a.e. (2)

Indeed, since $\phi_n \to g$ a.e., we can take $(n_k)_{k\geq 0}$ increasing such that

$$\left\|\phi_{n_k} - g\right\|_{L^1(\mathbb{R})} < 2^{-(k+n_0)}$$

Consider $h = |g| + \sum_{k\geq 0} \|\phi_{n_k} - g\|$

It's obvious that $|\phi_{n_k}| \leq h$ a.e. for all $k \geq 0$, and $h \in L^1$, since $|h| < |g| + \frac{1}{M+1}$

We also have that $|h| < |g| + \frac{1}{M+1} < 1 - \frac{1}{M+1} + \frac{1}{M+1} = 1$ a.e.

Thus, we deduce that (2) is true. Using (2), we can assume that, without losing the generality, $(\phi_n)_{n\geq 0}$ is bounded a.e. by a function $h \in L^1$ and $|h| < 1$ a.e.

Now, for all $n \geq 0$, we denote $f_n := f\phi_n$. From (1), we deduce that $(f_n)_{n\geq 0}$ converges to $fg$ a.e. (3)

Since $|\phi_n| \leq h < 1$ a.e., then $|f_n| < |f|$ a.e. and $|f| \in L^1$. (4)

From (3), (4), by DCT, we deduce that

$$0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} \int_{\mathbb{R}} f(x)\phi_n(x) dx = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) dx = \int_{\mathbb{R}} \frac{f(x)^2}{|f(x)| + 1} dx$$

Therefore, we conclude that $\frac{f(x)^2}{|f(x)|+1} = 0$ a.e., being equivalent to $f(x) = 0$ a.e.