MAA301 Homework 2

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Oct 17, 2022

1 Exercise 1

We start by introducing some easy lemmas

Lemma 1.a. *Consider the map*

$$f : \mathcal{P}(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$$

$$f(A) = (a_n)_{n\geq 0} : a_i = \begin{cases} 1 \text{ } if \text{ } i \in A \\ 0 \text{ } otherwise \end{cases} \qquad \forall i \geq 0$$

f is a bijection.

Lemma 1.b. *For $A \subset \mathbb{N}$, and $(a_n)_{n\geq 0} = f(A)$, then for all $n \geq 0$,*

$$|A \cap \{0, ..., n\}| = \Sigma_{i=0}^{n} a_i$$

Lemma 1.c. *For $A \subset \mathbb{N}$ and $B \subset \mathbb{N}$, $(a_n)_{n\geq 0} := f(A)$ and $(b_n)_{n\geq 0} := f(B)$ then,*

$$f(A \cap B) = (a_n b_n)_{n\geq 0} = \left(\frac{a_n + b_n - |a_n - b_n|}{2}\right)_{n\geq 0}$$

And

$$f(A \cup B) = \left(\frac{a_n + b_n + |a_n - b_n|}{2}\right)_{n\geq 0}$$

And

$$f(A \setminus B) = (|a_n - b_n|)_{n\geq 0} \text{ } if \text{ } B \subset A$$

Lemma 1.d.

$$d(X) = \lim_{n\to\infty} \frac{\Sigma_{i=0}^{n} f(X)_i}{n} \forall X \subset \mathbb{N}$$

1.1 Let $A \subset \mathbb{N}$ be a finite set. Show that A has a density and compute $d(A)$

First we know that for all $n \in \mathbb{N}$, we have that $\emptyset \subseteq A \cap \{0, 1, ..., n\} \subseteq A$.
Thus

$$0 \leq |A \cap \{0, 1, ..., n\}| \leq |A| \leq M$$

for some $M \in \mathbb{N}$ since A is a finite set.
Thus we can easily compute

$$\begin{aligned} d(A) &= \lim_{n\to\infty} \frac{|A \cap \{0, 1, ..., n\}|}{n} \\ &\leq \lim_{n\to\infty} \frac{M}{n} \\ &= 0 \end{aligned}$$

for any finite set.

1.2 Let $A := p\mathbb{N} = \{p\mathbb{N} : n \in \mathbb{N}\}$ with $p \geq 1$. Show that A has a density and compute $d(A)$

We can easily see that $|A \cap \{0, 1, ..., n\}| = \lfloor \frac{n}{p} \rfloor + 1$ since it includes the number 0. Now we just compute:

$$\begin{aligned} d(A) &= \lim_{n\to\infty} \frac{|A \cap \{0, 1, ..., n\}|}{n} \\ &= \lim_{n\to\infty} \frac{\lfloor \frac{n}{p} \rfloor}{n} \\ &= \frac{1}{p} \end{aligned}$$

1.3 (Bonus) Let $1 \leq p_1 < p_2 < ...$ be an increasing sequence of integers that are pairwise co-prime. $A := \{p_1, p_2, ...\}$. Show that $A \in \mathcal{D}_0$.

In this question, for all $m \in \mathbb{N}$, we denote $\pi(m)$ is the number of primes $p \leq m$. It's easy to see that π is a increasing function.

Consider $n > p_3$, since $(p_i)_{i\geq 1}$ is strictly increasing, there exists $k \geq 3$ such that $p_k < n \leq p_{k+1}$. Now let Q be the set of all prime divisors of $p_2 ... p_k$, we will prove that $|Q| \geq k - 1$.

Indeed, assume the contrary that $|Q| < k - 1$, then by the pigeonhole principle, there exists a prime $q \in Q$ and $2 \leq u < v \leq k$ such that $q \mid p_u$ and $q \mid p_v$, then we have that $\gcd(p_u, p_v) \geq q > 1$, contradicting the definition of $(p_i)_{i\geq 0}$.

Thus, we must have $|Q| \geq k - 1$, so $k \leq |Q| + 1 \leq \pi(p_k) + 1 \leq \pi(n) + 1$

On the other hand, we have that

$$\left|(p_i)_{i\geq 1} \cap \{0, ..., n\}\right| = \left|\{p_1, ..., p_k\}\right| = k \leq \pi(n) + 1$$

Hence,

$$\frac{\left|(p_i)_{i\geq 1} \cap \{0, ..., n\}\right|}{n} \leq \frac{\pi(n) + 1}{n}$$

Since n is chosen arbitrarily, we deduce that

$$0 \leq \frac{\left|(p_i)_{i\geq 1} \cap \{0, ..., n\}\right|}{n} \leq \frac{\pi(n) + 1}{n} \ \forall n \geq p_3 \ (1)$$

Using **Prime Number Theorem**, we have that

$$\lim_{n\to\infty} \frac{\pi(n)}{\left\lfloor \frac{n}{\ln(n)} \right\rfloor} = 1$$

So we must have

$$\lim_{n\to\infty} \frac{\pi(n) + 1}{n} = 0 \ (2)$$

From (1), (2) we deduce that $\lim_{n\to\infty} \frac{\left|(p_i)_{i\geq 1} \cap \{0, ..., n\}\right|}{n} = 0$ so $A = (p_i)_{i\geq 1} \in \mathcal{D}_0$

1.4 Find a subset of $\mathbb{N}$ with no density.

For this question, for the simplicity of writing, for $a, b \in \mathbb{N}$ we denote $[a, b]$ as the real interval $[a, b]$ intersection with $\mathbb{N}$. This is defined similarly with (a, b), $[a, b)$, and $(a, b]$.

Let $A := \lim_{k\to\infty} A_k$, We define each A_k for $n \geq 1$ as follows:

$$A_k = A_{k-1} \cup [2^{2k}, 2^{2k+1})$$

and initialize A_0 with:

$$A_0 = \{1\}$$

We get that:

$$A = \{1\} \cup [4,8) \cup [16,32)...$$

We prove inductively that for $k \geq 1$: we have that if $n = 2^{2k+1}$

$$\frac{2}{3} \geq \frac{|A \cap [0,n]|}{n} > \frac{1}{2}$$

but also if $n = 2^{2k} - 1$

$$0 < \frac{|A \cap [0,n]|}{n} \leq \frac{1}{3}$$

<u>Base case</u>:
$k = 1$, Then if $n = 2^{2+1} = 8$ Then $\frac{2}{3} \geq \frac{|A \cap [0,8]|}{8} = \frac{5}{8} > \frac{1}{2}$
and if $n = 2^2 - 1 = 3$ Then $0 < \frac{|A \cap \{0,1,2,3\}|}{3} = \frac{1}{3} \leq \frac{1}{3}$
<u>Inductive case</u>:
Assume the induction holds for all $k \geq 1$,
We first look at the case $n = 2^{2(k+1)+1}$. Since $A \cap [2^{2(k+1)}, 2^{2(k+1)+1}] = [2^{2(k+1)}, 2^{2(k+1)+1})$, by the construction of our set A and so $\frac{|A \cap [2^{2k+2}, 2^{2k+3}]|}{2^{2k+3}} = \frac{1}{2}$
So we have that:

$$\begin{aligned}
\frac{2}{3} &= \frac{1}{6} + \frac{1}{2} \\
&= \frac{2(2^{2k+1})}{3(2^{2k+3})} + \frac{1}{2} \\
&\geq \frac{|A \cap [0, 2^{2k+1}]|}{2^{2k+3}} + \frac{1}{2} \\
&= \frac{|A \cap [0, 2^{2k+1}]| + |A \cap (2^{2k+1}, 2^{2k+2})| + |A \cap [2^{2k+2}, 2^{2k+3}]|}{2^{2k+3}} \\
&= \frac{|A \cap [0, 2^{2k+3}]|}{2^{2k+3}} \\
&= \frac{|A \cap [0, 2^{2k+2})| + |A \cap [2^{2k+2}, 2^{2k+3}]|}{2^{2k+3}} \\
&> \frac{|A \cap [2^{2k+2}, 2^{2k+3}]|}{2^{2k+3}} \\
&= \frac{1}{2}
\end{aligned}$$

In which we used the induction step to get that $|A \cap [0, 2^{2k+2})| \geq |A \cap [0, 2^{2k})| > 0$ and $|A \cap [0, 2^{2k+1}]| \leq \frac{2(2^{2k+1})}{3}$ the construction of A to get $|A \cap (2^{2k+1}, 2^{2k+2})| = 0$.

We now look at the case $n = 2^{2(k+1)} - 1$. Since $A \cap (2^{2k+1}, 2^{2k+2}) = \emptyset$, by the construction of our set A and so $\frac{|A \cap (2^{2k+1}, 2^{2k+2}-1]|}{2^{2k+2}-1} = 0$
Also we have that:

$$\begin{aligned}
0 &< \frac{|A \cap [0, 2^{2k+2} - 1]|}{2^{2k+2} - 1} \\
&= \frac{|A \cap [0, 2^{2k})| + |A \cap [2^{2k}, 2^{2k+1}]| + |A \cap (2^{2k+1}, 2^{2k+2} - 1]|}{2^{2k+2} - 1} \\
&\leq \frac{\frac{1}{3}(2^{2k} - 1) + 2^{2k}}{2^{2k+2} - 1} \\
&= \frac{\frac{1}{3}(2^{2k+2} - 1)}{2^{2k+2} - 1} \\
&= \frac{1}{3}
\end{aligned}$$

In which we used the induction step to get that $|A \cap [0, 2^{2k})| \leq (2^{2k} - 1)\frac{1}{3}$ and also $|A \cap [0, 2^{2k+2} - 1]| \geq |A \cap [0, 2^{2k})| > 0$

Since the induction hold true for $k+1$ and so hypothesis, by the principle of induction, holds true for all $k \geq 1$. When we compute density as the limit of the sequence:

$$\frac{|A \cap \{0, 1, ..., n\}|}{n} = D_n$$

We have a subsequence taking into account the indices $n = 2^{2k+1}$ and $n = 2^{2k} - 1$ for $k \geq 1$ oscillating between two disjoint intervals in fact at least $\frac{1}{6}$ apart. Thus it is easy to see that this sequence is non-cauchy and thus non-convergent.

1.5 Let $\mathcal{B}$ be a Boolean algebra of $\mathbb{N}$ included in $\mathcal{D}$. Show that d is a set additive function on $(\mathbb{N}, \mathcal{B})$.

First of all the intersection of the empty set and any set is again the empty set with the cardinality 0. Thus:

$$d(\emptyset) = \lim_{n\to\infty} \frac{|\emptyset|}{n} = 0$$

Take $A, B \in \mathcal{B} \subset \mathcal{B}$ s.t. they are disjoint. We know that A and B have density.

$$\begin{aligned} d(A \cup B) &= \lim_{n\to\infty} \frac{|(A \cup B) \cap \{0, 1, ..., n\}|}{n} \\ &= \lim_{n\to\infty} \frac{|(A \cap \{0, 1, ..., n\}) \cup (B \cap \{0, 1, ..., n\})|}{n} \\ &= \lim_{n\to\infty} \frac{|(A \cap \{0, 1, ..., n\})|}{n} + \lim_{n\to\infty} \frac{|(B \cap \{0, 1, ..., n\})|}{n} \\ &= d(A) + d(B) \end{aligned}$$

We used the fact that $(A \cap \{0, 1, ..., n\}) \subseteq A$ and $(B \cap \{0, 1, ..., n\}) \subseteq B$ and with A and B disjoint we have:

$$(A \cap \{0, 1, ..., n\}) \cap (B \cap \{0, 1, ..., n\}) = \emptyset$$

which are disjoint too.
Thus d is an additive function on $(\mathbb{N}, \mathcal{B})$

1.6 Is $\mathcal{D}$ a boolean algebra? Justify.

No. We will provide a counter-example.

We consider the 2 following sequences:

$$(a_n)_{n\geq 0} : \begin{cases} a_{2k} = 1, \\ a_{2k+1} = 0 \end{cases} \quad \forall k \geq 0$$

$$(b_n)_{n\geq 0} : (b_{2k}, b_{2k+1}) = \begin{cases} (1, 0) \text{ if } 2 \mid \left\lceil \log_2(2k) \right\rceil \\ (0, 1) \text{ otherwise} \end{cases} \quad \forall k \geq 0$$

Then, we have that

$$(a_{2k}b_{2k}, a_{2k+1}b_{2k+1}) = \begin{cases} (1, 0) \text{ if } 2 \mid \left\lceil \log_2(2k) \right\rceil \\ (0, 0) \text{ otherwise} \end{cases}$$

Hence,

$$\frac{\Sigma_{i=0}^{2^{2k+1}-1} a_i b_i}{2^{2k+1} - 1} = \frac{\Sigma_{i=1}^{k} 2^{2i-1}}{2^{2k+1} - 1} > \frac{\Sigma_{i=1}^{k} 2^{2i-1}}{2^{2k+1}} = \frac{1}{3} - \frac{1}{3.2^{2k}} \geq \frac{5}{16} \ \forall k \geq 2 \qquad (1)$$

And,

$$\frac{\Sigma_{i=0}^{2^{2k+2}-1} a_i b_i}{2^{2k+2}-1} = \frac{\Sigma_{i=1}^{k} 2^{2i-1}}{2^{2k+2}-1} = \frac{2(2^{2k}-1)}{3(2^{2k+2}-1)} < \frac{2^{2k}}{2^{2k+2}} = \frac{1}{4}\ \forall k \geq 2 \qquad (2)$$

From (1), (2) we deduce that $\frac{\Sigma_{i=0}^{n} a_i b_i}{n}$ doesn't converge when $n \to \infty$. (3)

From **Lemma 1.a**, let $A := f^{-1}((a_n)_{n\geq 0})$ and $B := f^{-1}((b_n)_{n\geq 0})$. From **Lemma 1.c**, we deduce that $A \cap B = f^{-1}((a_n b_n)_{n\geq 0})$. From (3) and **Lemma 1.d**, we deduce that $d(A \cap B)$ is not defined so $A \cap B \notin \mathcal{D}$ (4).

On the other hand we have that

$$\frac{\lfloor \frac{n}{2} \rfloor}{n} \leq \frac{\Sigma_{i=0}^{n} a_i}{n} \leq \frac{\lceil \frac{n}{2} \rceil}{n}\ \forall n \geq 0$$

And

$$\frac{\lfloor \frac{n}{2} \rfloor}{n} \leq \frac{\Sigma_{i=0}^{n} b_i}{n} \leq \frac{\lceil \frac{n}{2} \rceil}{n}\ \forall n \geq 0$$

Since $\lim_{n\to\infty} \frac{\lfloor \frac{n}{2} \rfloor}{n} = \lim_{n\to\infty} \frac{\lceil \frac{n}{2} \rceil}{n} = \frac{1}{2}$ then,

$$\lim_{n\to\infty} \frac{\Sigma_{i=0}^{n} a_i}{n} = \lim_{n\to\infty} \frac{\Sigma_{i=0}^{n} b_i}{n} = \frac{1}{2}$$

From **Lemma 1.d**, we deduce that $A, B \in \mathcal{D}^2$ (5).

From (4), (5) we conclude that $\mathcal{D}$ is not a boolean algebra.

1.7 Show that $\mathcal{D}_0 \cup \mathcal{D}_1$ is a boolean algebra

We know $\emptyset \in \mathcal{D}_0$ and $\mathbb{N} \in \mathcal{D}_1$ since

$$|\emptyset \cap \{0, 1, ..., n\}| = |\emptyset| = 0$$

and also

$$|\mathbb{N} \cap \{0, 1, ..., n\}| = |\{0, 1, ..., n\}| = n$$

and by taking the limit as n goes to ∞ of $\frac{0}{n}$ and $\frac{n}{n}$ gives us 0 and 1 respectively.
Hence they are in the union.

Take any element $A \in \mathcal{D}_0 \cup \mathcal{D}_1$.
We can compute similarly to before:

$$\lim_{n\to\infty} \frac{|A \cap \{0, 1, ..., n\}|}{n}$$

exists and we set it equal to a.
Now we compute the complement

$$\begin{aligned} d(A^c) &= \lim_{n\to\infty} \frac{|(N \setminus A) \cap \{0, 1, ..., n\}|}{n} \\ &= \lim_{n\to\infty} \frac{|\{0, ..., n\} \setminus (A \cap \{0, 1, ..., n\})|}{n} \\ &= \lim_{n\to\infty} \frac{|\{0, ..., n\}|}{n} - \lim_{n\to\infty} \frac{|(A \cap \{0, 1, ..., n\})|}{n} \\ &= 1 - a \end{aligned}$$

Now either $A \in \mathcal{D}_0$ and $a = 0$ or $A \in \mathcal{D}_1$ and $a = 1$. Both ways we know that the density of the complement will end up being either 1 or 0. Thus $A^c \in \mathcal{D}_0 \cup \mathcal{D}_1$.

Take $A, B \in \mathcal{D}_0 \cup \mathcal{D}_1$, If $A \in \mathcal{D}_0$ and $B \in \mathcal{D}_0$ then

$$0 \leq d(A \cup B) \leq d(A) + d(B) = 0 + 0 = 0$$

So we must have $d(A \cup B) = 0$, implying that $(A \cup B) \in \mathcal{D}_0 \cup \mathcal{D}_1$ (1)

If $A \in \mathcal{D}_1$ then

$$1 \geq d(A \cup B) \geq d(A) = 1$$

So we must have $d(A \cup B) = 1$, implying that $(A \cup B) \in \mathcal{D}_0 \cup \mathcal{D}_1$ (2)

The case $B \in \mathcal{D}_1$ is similar. Therefore, from (1) and (2), we deduce that $\mathcal{D}_0 \cup \mathcal{D}_1$ is stable under finite union.

And combining complement with finite union gives stability under finite intersection. Therefore, we conclude that $\mathcal{D}_0 \cup \mathcal{D}_1$ is a boolean algebra.

1.8 We define the following relation between subsets of $\mathbb{N}$: for all $A, B \subset \mathbb{N}$, $A \backsim B$ if and only if $A \Delta B \in \mathcal{D}_0$ where $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is the so-called symmetric difference between A and B.

Show that $\backsim$ is an equivalence relation.

We can first obtain **Reflexivity** with the fact that the symmetric difference of a set and itself is the empty set and we've found that density to be 0 in the previous question.

Take $A, B \subset \mathbb{N}$ We have **Symmetry** very easily that

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = (B \setminus A) \cup (A \setminus B) = B \Delta A$$

Thus if $A \Delta B \in \mathcal{D}_0$, we just substitute with equality to get that $B \Delta A \in \mathcal{D}_0$ and vice versa.

To get **Transitivity**, assume $A \backsim B$ and $B \backsim C$. We need to prove that $A \backsim C$.

From **Lemma 1.a**, let $(a_n)_{n \geq 0}, (b_n)_{n \geq 0}, (c_n)_{n \geq 0} := f(A), f(B), f(C)$.

Since $(X \setminus Y) \cup (Y \setminus X) = (X \cup Y) \setminus (X \cap Y)$, and $(X \cap Y) \subset (X \cup Y)$, from **Lemma 1.c**, we deduce that

$$f(A \Delta B), f(B \Delta C), f(C \Delta A) = (|a_n - b_n|)_{n \geq 0}, (|b_n - c_n|)_{n \geq 0}, (|a_n - c_n|)_{n \geq 0}$$

And from **Lemma 1.d**, we deduce that

$$X \backsim Y \text{ iff } \lim_{n \to \infty} \frac{\Sigma_{i=0}^{n} |f(X)_i - f(Y)_i|}{n} = 0$$

On the other hand, for all $n \geq 0$ we have that

$$0 \leq \frac{\Sigma_{i=0}^{n} |a_i - c_i|}{n} \leq \frac{\Sigma_{i=0}^{n} |a_i - b_i|}{n} + \frac{\Sigma_{i=0}^{n} |b_i - c_i|}{n} \qquad (1)$$

Since $A \backsim B$ and $B \backsim C$,

$$\lim_{n \to \infty} \frac{\Sigma_{i=0}^{n} |a_i - b_i|}{n} = \lim_{n \to \infty} \frac{\Sigma_{i=0}^{n} |b_i - c_i|}{n} = 0$$

Hence,

$$\lim_{n \to \infty} \frac{\Sigma_{i=0}^{n} |a_i - b_i|}{n} + \frac{\Sigma_{i=0}^{n} |b_i - c_i|}{n} = 0 \qquad (2)$$

From (1), (2), we deduce that

$$\lim_{n \to \infty} \frac{\Sigma_{i=0}^{n} |a_i - c_i|}{n} = 0$$

Therefore, we have $A \backsim C$. So we conclude that $\backsim$ is an equivalence relation

1.9 For all $A \subset \mathbb{N}$, denote by $Cl(A)$ the equivalence class of A for the relation $\backsim$

Compute $Cl(\emptyset)$ and $Cl(\mathbb{N})$

We first fix $A \subset \mathbb{N}$.
It is easy to see that $\emptyset \Delta A = A$ no matter what the set it is. Thus the density of the symmetric difference is just the density of A. We have that:

$$Cl(\emptyset) = \mathcal{D}_0$$

Now the symmetric difference of $\mathbb{N}$ and any of its subset A is just $\mathbb{N} \setminus A$ or equivalently the complement of A. As we computed previously that the density of the complement of A in terms of the density of A, if it exists, is $1 - d(A)$. Thus any element of the equivalence class has to have density 1.

$$Cl(\mathbb{N}) = \mathcal{D}_1$$

1.10 Let $A \subset \mathbb{N}$, show that $Cl(A^c) = \{B^c : B \in Cl(A)\}$.

We just compute with a little bit of set algebra.

$$\begin{aligned}
Cl(A^c) &= \{B : (A^c \setminus B) \cup (B \setminus A^c) \in \mathcal{D}_0\} \\
&= \{B : (B \cup A)^c \cup (A \cap B) \in \mathcal{D}_0\} \\
&= \{B^c : (B^c \cup A)^c \cup (A \cap B^c) \in \mathcal{D}_0\} \\
&= \{B^c : (B \cap A^c) \cup ((A \cap B^c)^c)^c \in \mathcal{D}_0\} \\
&= \{B^c : (B \cap A^c) \cup (A^c \cup B)^c \in \mathcal{D}_0\} \\
&= \{B^c : (B \setminus A) \cup (A \setminus B) \in \mathcal{D}_0\} \\
&= \{B^c : B \in Cl(A)\}
\end{aligned}$$

1.11 Let $A, A', B, B' \subset \mathbb{N}$ such that $A \backsim A'$ and $B \backsim B'$, prove that $A \cup B \backsim A' \cup B'$.

We have that $A \sim A'$ and $B \sim B'$. From **Lemma 1.c**, we deduce that

$$f(A \Delta A'), f(B \Delta B') = (|a_n - a'_n|)_{n \geq 0}, (|b_n - b'_n|)_{n \geq 0}$$

$$f(A \cup B) = \left(\frac{a_n + b_n + |a_n - b_n|}{2} \right)_{n \geq 0}$$

$$f(A' \cup B') = \left(\frac{a'_n + b'_n + |a'_n - b'_n|}{2} \right)_{n \geq 0}$$

To prove that $(A \cup B) \sim (A' \cup B')$, we use **Lemma 1.d**, from which

$$(A \cup B) \backsim (A' \cup B') \text{ iff } \lim_{n \to \infty} \frac{\Sigma_{i=0}^n |f(A \cup B)_i - f(A' \cup B')_i|}{n} = 0$$

Moreover, because $A \sim A'$ and $B \sim B'$, for all $n \geq 0$ we have that

$$\lim_{n \to \infty} \frac{\Sigma_{i=0}^n |a_i - a'_i|}{n} = 0 \qquad (1)$$

$$\lim_{n \to \infty} \frac{\Sigma_{i=0}^n |b_i - b'_i|}{n} = 0 \qquad (2)$$

On the other hand, we know that

$$|x + y| \leq |x| + |y| \qquad (3)$$

And

$$|x| - |y| \leq |x - y| \qquad (4)$$

Using (1), (2), (3) we can deduce that,

$$|a_i + b_i - a'_i - b'_i| \leq |a_i - a'_i| + |b_i - b'_i| \qquad (5)$$

And from (1), (2), (4), (3) we can deduce that,

$$|a_i - b_i| - |a'_i - b'_i| \leq |(a_i + b_i) - (a'_i - b'_i)| \leq |(a_i - a'_i)| + (|b_i - b'_i)| \qquad (6)$$

Now, from (5) and (6), we can deduce that,

$$|a_i + b_i - a'_i - b'_i + |a_i - b_i| - |a'_i - b'_i|| \leq 2(|a_i - a'_i| + |b_i - b'_i|) \qquad (7)$$

Essentially, after rearranging the terms in (7) we can say that

$$\lim_{n\to\infty} \frac{\Sigma_{i=0}^{n} |f(A \cup B)_i - f(A' \cup B')_i|}{n} \leq \lim_{n\to\infty} \frac{\Sigma_{i=0}^{n} |a_i - a'_i| + |b_i - b'_i|}{n} = 0$$

Hence, we have proved that $(A \cup B) \backsim (A' \cup B')$

1.12 Show that, if $A \in \mathcal{D}_a$ for some $a \in [0, 1]$, then $Cl(A) \subset \mathcal{D}_a$.

We have that $d(A) \in [0, 1]$. Now we take any $B \in Cl(A)$, by definition we have

$$d\left((A \setminus B) \cup (B \setminus A)\right) = 0$$

Since they are disjoint we can use the additivity:

$$d(A \setminus B) + d(B \setminus A) = 0$$

Since the cardinality is non-negative, we can bound each of the two densities $d(A \setminus B)$ and $d(B \setminus A)$ from below using 0. This means that both of the densities are 0 in fact.

We also can decompose sets into disjoint subsets as follows:

$$B = (B \setminus A) \cup (B \cap A)$$

$$A = (A \setminus B) \cup (B \cap A)$$

And thus when we apply the densities we found just above and the additivity property we have from above.

$$\begin{aligned} d(A) &= d(A \setminus B) + d(B \cap A) \\ &= 0 + d(B \cap A) \\ &= d(B \setminus A) + d(B \cap A) \\ &= d(B) \end{aligned}$$

This means that $B \in \mathcal{D}_a$, and thus we have our inclusion.

1.13 Let $\mathcal{B}$ be a Boolean Algebra on $\mathbb{N}$. Show that $\bigcup_{A\in\mathcal{B}} Cl(A)$ is a Boolean Algebra.

Contains $\emptyset$ and $\mathbb{N}$:
Since the boolean algebra $\mathcal{B}$ contains $\emptyset$ and $\mathbb{N}$, we have that $\bigcup_{A\in\mathcal{B}} Cl(A)$ which we denote as S includes $Cl(\emptyset)$ and $Cl(\mathbb{N})$.
In particular $\emptyset \Delta \emptyset = \emptyset$, which has a density of 0 as we computed above. Thus we have that:

$$\emptyset \in Cl(\emptyset) \subset \bigcup_{A\in\mathcal{B}} Cl(A)$$

Additionally, we know that $\mathbb{N} \Delta \mathbb{N} = \emptyset$ which has a density of 0. And so we find:

$$\mathbb{N} \in Cl(\mathbb{N}) \subset \bigcup_{A\in\mathcal{B}} Cl(A)$$

Stable by complement:
If we take any $W \in \bigcup_{A \in \mathcal{B}} Cl(A)$, we know that $W \in Cl(W)$ due to the fact that $W \Delta W = \emptyset$ which has a density of 0.

Thus since equivalence classes are disjoint, $Cl(W)$ has to be in the union, and so we can deduce that W is in $\mathcal{B}$ and hence W^c too since $\mathcal{B}$ is a boolean algebra.

Since the $W^c \in \mathcal{B}$, we find that $Cl(W^c)$ is in the union and also W^c is in $Cl(W^c)$ since $W^c \Delta W^c = \emptyset$. Hence we can write:

$$W^c \in Cl(W^c) \subset \bigcup_{A \in \mathcal{B}} Cl(A)$$

Hence we get stability by passage to the complement.

Stable by Finite Union:
Now if we take any finite union of elements in $W_i \in \bigcup_{A \in \mathcal{B}} Cl(A)$ which we use the index set $[k]$. By similar logic to deducing W was in $\mathcal{B}$, we can deduce that each of the W_i's for $i \in [k]$ are in $\mathcal{B}$.

We use the boolean algebra property of $\mathcal{B}$ to get that $\bigcup_i^k W_i \in \mathcal{B}$. Thus we have that

$$Cl(\bigcup_i^k W_i) \subset \bigcup_{A \in \mathcal{B}} Cl(A)$$

And since

$$\bigcup_i^k W_i \Delta \bigcup_i^k W_i = \emptyset$$

which has a zero density, we can conclude that:

$$\bigcup_i^k W_i \in Cl(\bigcup_i^k W_i) \subset \bigcup_{A \in \mathcal{B}} Cl(A)$$

Hence we get stability by finite union.

Stable by Finite Intersection:
We take apply the complement and finite union stability assumptions and we get stability by finite intersection.

Thus we conclude that $\bigcup_{A \in \mathcal{B}} Cl(A)$ is a boolean algebra.

2 Exercise 2

2.1 Show that $\mathcal{F}_0$ is a filter.

$\mathbb{N} \in \mathcal{F}_0$
Since the complement of $\mathbb{N}$ in $\mathbb{N}$ is $\emptyset$ and is obviously finite we have $\mathbb{N} \in \mathcal{F}_0$
$\emptyset \notin \mathcal{F}_0$
Since $\emptyset^c = \mathbb{N}$ and is not finite, $\emptyset \notin \mathcal{F}_0$.

$\forall A \in \mathcal{F}_0, \forall B \subset \mathbb{N}, A \subset B \implies B \in \mathcal{F}_0$
Take $A \in \mathcal{F}_0$, now take any $B \subset \mathbb{N}$ s.t. $A \subset B$, we have that

$$A \subset B$$

$$B^c \subset A^c$$

Thus since A is in $\mathcal{F}_0$, A^c is finite, B^c its subset is finite too, thus $B \in \mathcal{F}_0$

$\underline{\forall A, B \in \mathcal{F}_0, A \cap B \in \mathcal{F}_0}$
Take A and B in $\mathcal{F}_0$ Thus A^c and B^c are finite.

$$(A \cap B)^c = (A^c \cup B^c)$$

Since a union of two finite sets are finite, we have the that $A \cap B \in \mathcal{F}_0$
Thus $\mathcal{F}_0$ is a filter.

2.2 Let $(\mathcal{F}_i)_{i\geq 1}$ be a sequence of filters such that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for all $i \geq 1$. Show that $\bigcup_{i\geq 1} \mathcal{F}_i$ is a filter.

We define: $\mathcal{F} := \bigcup_{i\geq 1} \mathcal{F}_i$

$\underline{\mathbb{N} \in \mathcal{F}}$
Since $\mathbb{N} \in \mathcal{F}_1 \subset \mathcal{F}$ since $\mathcal{F}_1$ is a filter, we have this property satisfied.

$\underline{\emptyset \notin \mathcal{F}}$
Since $\emptyset \notin \mathcal{F}_i$ for all i since each $\mathcal{F}_i$ is a filter, we have this property satisfied easily for any union of filters. $\emptyset \notin \mathcal{F}$

$\underline{\forall A \in \mathcal{F}, \forall B \subset \mathbb{N}, A \subset B \implies B \in \mathcal{F}}$
Take $A \in \mathcal{F}$, Thus there exists $i \geq 1$, s.t. $A \in \mathcal{F}_i$, since $\mathcal{F}_i$ is a filter, for all $B \subset \mathbb{N}$, $B \in \mathcal{F}_i \subset \mathcal{F}$

$\underline{\forall A, B \in \mathcal{F}, A \cap B \in \mathcal{F}}$
Take A and B in $\mathcal{F}$. Thus there exists $i \geq 1$, s.t. $A \in \mathcal{F}_i$, and there exists $k \geq 1$, s.t. $B \in \mathcal{F}_k$. By the construction of our increasing sequence of filters, A and B are in $\mathcal{F}_{max(i,k)}$ a filter. Thus $A \cap B \in \mathcal{F}_{max(i,k)} \subset \mathcal{F}$

Hence we have that $\mathcal{F}$ is a filter.

2.3 Let $\mathcal{F}$ be an ultra filter and define $\mu_{\mathcal{F}}$ by: for all $A \subset \mathbb{N}$, $\mu_{\mathcal{F}}(A) = 0$ if $A \notin \mathcal{F}$ and $\mu_{\mathcal{F}}(A) = 1$ if $A \in \mathcal{F}$. Show that $\mu_{\mathcal{F}}$ is a set additive function on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

It is easy to verify that since $\emptyset \notin \mathcal{F}$ an ultra filter, $\mu_{\mathcal{F}}(\emptyset) = 0$

Now take $A, B \in \mathcal{P}(\mathbb{N})$ which are disjoint. We have two cases:
<u>Both sets are not in $\mathcal{F}$</u>

$$\mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B) = 0 + 0 = 0$$

But we know that $A^c, B^c \in \mathcal{F}$ since it is an ultra filter and thus $A^c \cap B^c \in \mathcal{F}$, in particular,

$$A \cup B = (A^c \cap B^c)^c$$

By property (ii) and (iv), a set and its complement can not both be in the ultrafilter since the intersection is the empty set and would be in the filter. $A \cup B \notin \mathcal{F}$, thus:

$$\mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B) = 0 = \mu_{\mathcal{F}}(A \cup B)$$

<u>One of the sets are in the ultra filter, WLOG we say it is A:</u>
We can easily use property (iii) since,

$$A \subset A \cup B \subset \mathbb{N}$$

Meaning that $A \cup B \in \mathcal{F}$. Thus:

$$\mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B) = 1 + 0 = 1 = \mu_{\mathcal{F}}(A \cup B)$$

We note that there is no case that two disjoint sets are both in the filter since using (ii) and (iv) again we would have the intersection is an empty set in the filter.

2.4 Let $\mathcal{F}$ be a filter and $A \subset \mathbb{N}$. We define the set $\mathcal{F}_A := \{D \subset \mathbb{N} : \exists B \in \mathcal{F}, A \cap B \subset D\}$. Show that $\mathcal{F} \subset \mathcal{F}_A$ and that $\mathcal{F}_A$ satisfies conditions i, iii, and iv in the definition of filter.

Take $B \in \mathcal{F}$, since: $A \cap B \subseteq B \subset \mathbb{N}$, we have that $B \in \mathcal{F}_A$
Condition i:
Since $\mathbb{N} \in \mathcal{F}$, $A \cap \mathbb{N} = A \subset \mathbb{N}$, thus $\mathbb{N} \in \mathcal{F}_A$
Condition iii:
Take $C \in \mathcal{F}_A$, Now take any $D \subset \mathbb{N}$ s.t. $C \subset D$. Since we know that there is $B \in \mathcal{F}$ s.t. $A \cap B \subset C \subset \mathbb{N}$, We can extend this inclusion to have

$$A \cap B \subset C \subset D \subset \mathbb{N}$$

Thus $D \in \mathcal{F}_A$.
Condition iv:
Take $D_1, D_2 \in \mathcal{F}_A$. Since there is $B_1 \in \mathcal{F}$ s.t. $A \cap B_1 \subset D_1 \subset \mathbb{N}$ and $B_2 \in \mathcal{F}$ s.t. $A \cap B_2 \subset D_2 \subset \mathbb{N}$, We know that there is $B_1 \cap B_2 \in \mathcal{F}$ and we can write:

$$A \cap (B_1 \cap B_2) \subset D_1 \cap D_2 \subset \mathbb{N}$$

Thus $D_1 \cap D_2 \in \mathcal{F}_A$

2.5 Let $\mathcal{F}$ be a filter and $A \subset \mathbb{N}$. Suppose that $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$, show that $\mathcal{F}_A$ is a filter and $\mathcal{F} \neq \mathcal{F}_A$

We just need to prove that ii to get $\mathcal{F}_A$ is a filter and use the proofs above for the other three conditions.
We know that $A \neq \mathbb{N} \in \mathcal{F}$, and also that $A \neq \emptyset$ since $A^c \neq \mathbb{N} \in \mathcal{F}$. Thus A is a non empty strict subset of $\mathbb{N}$.

Take $A \in \mathcal{F}_A$, this is true since $\mathbb{N} \in \mathcal{F}$ and so $A \cap \mathbb{N} = A \subset A\mathbb{N}$, Since $A \notin \mathcal{F}$ we have the inequality.

Now we will prove that $\emptyset \notin \mathcal{F}_A$. Assume the contrary, that $\emptyset \in \mathcal{F}_A$, then there exists $B \in \mathcal{F}$ such that

$$B \cap A = \emptyset$$

Which implies that $B \subset A^c$. Since $\mathcal{F}$ is a filter, we deduce that $A^c \in \mathcal{F}$, which contradicts the assumption of $\mathcal{F}$.

Therefore, $\emptyset \notin \mathcal{F}_A$ so $\mathcal{F}_A$ is a filter.

2.6 Placeholder

We will prove that $\mathcal{F}$, as defined in the question, is a ultrafilter.

Indeed, by contradiction, if $\mathcal{F}$ is maximal but not an ultrafilter, there exists $A \subset \mathbb{N}$ such that $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$ then from **2.4** and **2.5**, we deduce that $\mathcal{F}_A$ is a filter, and $\mathcal{F} \subsetneq \mathcal{F}_A$, which contradicts the fact that $\mathcal{F}$ is maximal for the inclusion. Therefore, we conclude that $\mathcal{F}$ is a ultrafilter, by **2.3** and the fact that it contains $\mathcal{F}_0$, we deduce that $\mu := \mu_F$ is a satisfied set additive function.

2.7 Placeholder

Consider F is the set of all filters including $\mathcal{F}_0$. Since for all chain of filters $(\mathcal{F}_i)_{i\geq 0}$ that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ in F, the chain is bounded by the filter $\mathcal{F}_\infty := \bigcup_{i\geq 1} \mathcal{F}_i$, by Zorn's lemma, there exists a filter $\mathcal{F}$ that is maximal for the inclusion in F. By the definition of F, we have that $\mathcal{F}_0 \subset \mathcal{F}$. Now we will prove that $\mathcal{F}$ is also a maximal filter for the inclusion in the set of all filters. Let $\mathcal{F}'$ is a filter such that $\mathcal{F} \subset \mathcal{F}'$, we have that

$$\mathcal{F}_0 \subset \mathcal{F} \subset \mathcal{F}'$$

Hence, $\mathcal{F}' \in F$. Since $\mathcal{F}$ is a maximal filter in F, we must have $\mathcal{F}' = \mathcal{F}$. Since $\mathcal{F}'$ is arbitrary, we conclude that $\mathcal{F}$ is indeed a maximal filter for inclusion that includes $\mathcal{F}_0$

3 Exercise 3:

3.1 Show that Π_∞ is a Boolean algebra on $\mathbb{R}$

First it is easy to see that $\Pi_n \subset \Pi_{n+1}$.
By induction, if $n = 0$ since we can Take $A = \mathbb{R}$ and open set in Π_0 and for any $B \in \Pi_0$, we have that $\mathbb{R} \cap B = B \in \Pi_1$. Thus $\Pi_0 \in \Pi_1$

Assume this holds true for all $n \geq 0$. We can do similarly for the $n+1$ case since we can take $A = \mathbb{R}$ and open set in $\Pi_0 \subset ... \subset \Pi_{n+1}$ and for any $B \in \Pi_{n+1}$, we have that $\mathbb{R} \cap B = B \in \Pi_{n+2}$. Thus this holds true for $n+1$ and so by the principle of mathematical induction it holds true for all $n \geq 0$

Closed by path to the complement

We perform an induction: for all $n \geq 0$, and for all $A \in \Pi_n$, there exists $m \geq 0$ such that $A^c \in \Pi_m$

Base case is trivial, since any $A \in \Pi_0$ is either open or closed, and its complement is closed or open respectively and thus $A^c \in \Pi_0$.

Inductive case, assume that this holds true for some $n \geq 0$. Let $A \in \Pi_{n+1}$, assume it is composed as an union as follows: $A = A' \cup B$ s.t. $A', B \in \Pi_n$, thus $A^c = A'^c \cap B^c$. We know by our induction hypothesis that there is $m_1, m_2 \geq 0$ s.t. $A'^c \in \Pi_{m_1}$ and $B^c \in \Pi_{m_2}$ thus using what we proved in the beginning $A'^c, B^c \in \Pi_{max(m_1,m_2)}$, and so $A^c \in \Pi_{max(m_1,m_2)+1}$. We can do the same thing for $A = A' \cap B$ substituting the union in place of the intersection in the complement composition and get the same results.

Thus we have that it holds for $n+1$ and by the induction principle, it holds for all $n \geq 0$.

Now taking $A \in \Pi_\infty$, we know that there is $n \geq 0$ s.t. $A \in \Pi_n$ and by above there is $m \geq 0$ s.t. $A^c \in \Pi_m \subset \Pi_\infty$ we have the closure by complement.

Closed by finite Union:

Now take a finite union of N elements we call C_n the sequence of elements in this finite union for $n \in [1, N]$ an index set. Let $i_0 = sup\{i : C_n \in \Pi_i, \forall n \in [1, N]\}$. We know that:

$$C = \bigcup_{n \in [1,N]} C_n$$

is in Π_{i_0+N}, since we can construct C by iterating through n in the finite index set $[1, N]$ of cardinality N.

We start where we know for all $n \in [1, N]$, $C_n \in \Pi_{i_0}$ by the induction hypothesis we proved above. Now we take $\mathbb{R} \in \Pi_0 \subset \Pi_{i_0}$ and $C_1 \in \Pi_{i_0}$, and we get that

$$\mathbb{R} \cup C_1 \in \Pi_{i_0+1}$$

We can continue doing this until we add C_N, and we will get $C \in \Pi_{i_0+N} \subset \Pi_\infty$

Closed by finite Intersection:

We combine finite union and complement to get this property.

Contains $\emptyset$ and $\mathbb{R}$

Finally, $\emptyset$ and $\mathbb{R}$ are open and closed sets thus in $\Pi_0 \subset \Pi_\infty$

3.2 Show that Σ_1 is a Boolean Algebra on $\mathbb{R}$

Contains $\emptyset$ and $\mathbb{R}$

Since $\emptyset$ is both open and closed $\emptyset = \emptyset \cap \emptyset \in \Sigma_0$ and thus $\emptyset \in \Sigma_1$.

We do similarly: $\mathbb{R}$ is both open and closed $\mathbb{R} = \mathbb{R} \cap \mathbb{R} \in \Sigma_0$ and thus $\mathbb{R} \in \Sigma_1$.

Closed by path to complement:

Take $A \in \Sigma_1$, we can write it as $A = \bigcup_{i=1}^{n} A_i$ with $A_i = C_i \cap B_i$ where C_i is closed and B_i is open.

We compute that:

$$
\begin{aligned}
A^c &= \bigcap_{i=1}^{n} (C_i^c \cup B_i^c) \\
&= \bigcup_{J \in \mathcal{P}(n)} \left((\bigcap_{i \in J} B_i^c) \cap (\bigcap_{j \in [1,n] \setminus J} C_j^c) \right)
\end{aligned}
$$

Since B_i^c is closed, the finite intersection is too, and also C_i^c is open and thus the finite intersection too. If $|J| = 0$ or $|J| = n$, one of the sets is the empty set, but is both closed and open. Hence we have a decomposition of a finite union of sets made out of the intersection of a closed and open set. Thus $A^c \in \Sigma_1$

Closed by finite Union:

We take finite union of $C_n \in \Sigma_1$ with $n \in J \subset \mathbb{N}$ index set of the finite union. Each $C_n = \bigcup_{i=1}^{k_n} (A_{i_n} \cap B_{i_n})$ s.t. $k \in \mathbb{N}$ and A_{i_n} is open while B_{i_n} is closed.

$$
C = \bigcup_{n \in J} C_n = \bigcup_{n \in J} \bigcup_{i=1}^{k_n} (A_{i_n} \cap B_{i_n})
$$

Since J is a finite index set and $k_n \in \mathbb{N}$ we can combine and re-index the union have a finite union of $\sum_{n \in J} k_n$ intersections of pairs of open and closed sets which characterizes an element in Σ_1. Thus $C \in \Sigma_1$.

Closed by finite Intersection:

By combining complement and union stability, we get finite intersections stability too.

Hence Σ_1 is a boolean algebra on $\mathbb{R}$.

3.3 Show that $\Sigma_1 = \Pi_\infty$.

We have that $A \in \Pi_1 \forall A \in \Sigma_0$. Consider $X \in \Sigma_1$, then there exists $n \geq 1, A_1, ..., A_n \in \Sigma_0^n$ such that

$$
X = A_1 \cap ... \cap A_n
$$

Since $A_i \in \Pi_1 \forall 1 \leq i \leq n$, we have that

$$
X = A_1 \cap ... \cap A_n \in \Pi_n \subset \Pi_\infty
$$

Thus, $\Sigma_1 \subset \Pi_\infty$. (1)

Now we will prove that $\Pi_n \subset \Sigma_1 \forall n \geq 0$ by induction on n.

Indeed, it's obvious that $\Pi_0 \subset \Sigma_0 \subset \Sigma_1$. Now assume that $\Pi_n \subset \Sigma_1$ for some $n \geq 0$. Consider $X \in \Pi_{n+1}$. Then there exists $A, B \in \Pi_n^2$ such that $X = A \cap B$ or $X = A \cup B$. Using induction hypothesis, we have that $A, B \in \Sigma_1^2$. Combining with the fact that Σ_1 is a Boolean algebra, we deduce that

$$
\begin{cases}
A \cup B \in \Sigma_1 \\
A \cap B \in \Sigma_1
\end{cases}
$$

Thus, we deduce that $X \in \Sigma_1$.

Therefore, by induction we conclude that $\Pi_n \subset \Sigma_1 \forall n \geq 0$, which implies $\Pi_\infty \subset \Sigma_1$ (2)

From (1) and (2), we conclude that $\Pi_\infty = \Sigma_1$

3.4 Show that $\Sigma_1 = \Sigma'_1$

We will prove that for all $X, Y \in \Sigma_0$, $X \setminus Y = U \sqcup V \sqcup W$ for some $U, V, W \in \Sigma_0$.

Indeed, we can write $X = A \cap B$ and $Y = C \cap D$ with A, C open and B, D closed. Then, we have that

$$
(A \cap B) \setminus (C \cap D) = (A \cap B \cap C \cap D^c) \cup (A \cap B \cap C^c \cap D) \cup (A \cap B \cap C^c \cap D^c)
$$

Let $U, V, W := (A \cap B \cap C \cap D^c), (A \cap B \cap C^c \cap D), (A \cap B \cap C^c \cap D^c)$. It's easy to see that $U \cap V = V \cap W = W \cap U = \emptyset$

We also have that $(A \cap C \cap D^c)$, $(A \cap D^c)$ are open and $(B \cap C^c \cap D)$, $(B \cap C^c)$ are closed. Hence, combining with the fact that A, C open and B, D closed, we deduce that $U, V, W \in \Sigma_0$.

Thus, we deduce that $X \setminus Y = U \sqcup V \sqcup W$ for $U, V, W \in \Sigma_0$. (1)

Now back to the problem, it's easy to see that $\Sigma_1' \subset \Sigma_1$. So we only need to prove that $\Sigma_1 \subset \Sigma_1'$.

Indeed, we will prove that for all $n \geq 1$ and for all $A_1, ..., A_n \in \Sigma_0$, there exists $m \geq 1$ and $A_1', ..., A_m' \in \Sigma_0$ such that

$$A_1 \cup ... \cup A_n = A_1' \sqcup ... \sqcup A_m'$$

We will induct on n. It's obvious for the case $n = 1$. Now we assume that the statement is true for some $n \geq 1$, we will prove the statement for $n + 1$. Consider $n + 1$ sets $A_1, ..., A_{n+1} \in \Sigma_0$. Using the induction hypothesis, there exists $m \geq 1$ and $A_1', ..., A_m' \in \Sigma_0$ such that

$$A_1 \cup ... \cup A_n \cup A_{n+1} = A_1' \sqcup ... \sqcup A_m' \cup A_{n+1}$$

We also have that

$$A_1' \sqcup ... \sqcup A_m' \cup A_{n+1} = (A_1' \setminus A_{n+1}) \sqcup ... \sqcup (A_m' \setminus A_{n+1}) \sqcup A_{n+1}$$

Now applying (1), for all $1 \leq i \leq m$, there exists $U_i, V_i, W_i \in \Sigma_0$ such that

$$A_i' \setminus A_{n+1} = U_i \sqcup V_i \sqcup W_i$$

So we have that

$$(A_1' \setminus A_{n+1}) \sqcup ... \sqcup (A_m' \setminus A_{n+1}) \sqcup A_{n+1} = U_1 \sqcup V_1 \sqcup W_1 \sqcup ... \sqcup U_m \sqcup V_m \sqcup W_m \sqcup A_{n+1}$$

Which implies that

$$A_1 \cup ... \cup A_n \cup A_{n+1} = U_1 \sqcup V_1 \sqcup W_1 \sqcup ... \sqcup U_m \sqcup V_m \sqcup W_m \sqcup A_{n+1}$$

So the statement is also true for $n + 1$.

Therefore, by induction we conclude that the statement is true for all $n \geq 1$, implying that $\Sigma_1' \subset \Sigma_1$, so we have $\Sigma_1 = \Sigma_1'$

3.5 Prove that Σ_1 can't contain a set $D \subset R$ such that : $\exists$ a non-empty open interval J such that D is dense in J and for every non-empty open interval $I \subset J, I \cap D \notin \prod_0$.

We assume the contrary, that there exists $D \subset \mathbb{R}$ such that $D \in \Sigma_1$, there exists a non-empty open interval J such that D is dense in J and $I \cap D \notin \Pi_0$ for all non-empty open interval $I \subset J$. Since $D \in \Sigma_1$, we can write

$$D = A_1 \cup ... \cup A_n$$

with $n \in \mathbb{N}$, $A_i \in \Pi_0 \forall 1 \leq i \leq n$.

We will prove that for any such set D, we must have $n > m \forall m \in \mathbb{N}$.

Indeed, if $n = 1$, we have that

$$D \cap J = A_1 \cap J = (C \cap J) \cap D$$

with C is open and D is closed. Since $C \cap J$ is also open, we have $D \cap J = (C \cap J) \cap D \in \Pi_0$, contradicting to our assumption. Thus, $n > 1$.

Now assume that $n > m$ for some $m \geq 1$. We will prove that $n > m + 1$.

Assume that there exists $A_1, ..., A_{m+1} \in \Pi_0$ such that $D = A_1 \cup ... \cup A_{m+1}$ then, If there exists $1 \leq i \leq m+1$ and a non-empty open interval $I \subset J$ such that $A_i \cap I = \emptyset$ then denoting $D' := D \cap I$ we have that

$$D' = \bigcup_{1 \leq k \leq m+1, k \neq i} (A_k \cap I)$$

Since $X \cap I \in \Pi_0$ for all $X \in \Pi_0$, then we deduce that $(A_k \cap I) \in \Pi_0$ $\forall 1 \leq k \leq m+1, k \neq i$, implying there exists $A'_1, ..., A'_m \in \Pi_0^m$ such that

$$D' = A'_1 \cup ... \cup A'_m$$

Since D is dense in J, $I \subset J$, we also have that D' is dense in I, and $D' \cap K = D \cap K \notin \Pi_0$ for all non-empty open interval $K \subset I$. Using the induction hypothesis for D', we must have $m > m$, a contradiction.

Thus, $I \cap A_i \neq \emptyset$ for all $1 \leq i \leq m+1$ and non-empty open interval $I \subset J$.

On the other hand, $\forall 1 \leq i \leq m+1$, we can write

$$A_i = C_i \cap D_i$$

with C_i open and D_i closed. Hence, we can deduce that $D_i \cap I \neq \emptyset$ for all $1 \leq i \leq m+1$ and non-empty open interval $I \subset J$, which implies that D_i is dense in J. Since D_i is closed, we deduce that $\bar{J} \subset D_i$ for all $1 \leq i \leq m+1$.

Thus, $D_i \cap J = J$ $\forall 1 \leq i \leq m+1$

Meaning that

$$D \cap J = \bigcup_{i=1}^{m+1} (C_i \cap J)$$

Since J is open and C_i is open for all $1 \leq i \leq m+1$, we have that $C_i \cap J$ is open for all $1 \leq i \leq m+1$. Thus,

$$D \cap J = \bigcup_{i=1}^{m+1} (C_i \cap J) \text{ is open}$$

Which implies that $D \cap J \in \Pi_0$, contradicting to the definition of D. So we must have $n > m+1$.

Therefore, by induction we deduce that $n > m$ for all $m \in \mathbb{N}$, a contradiction. Thus, we conclude that such set D cannot exist.

3.6 Is Σ_1 equal to the Borel σ-algebra on $\mathbb{R}$?

No. Denote $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$ then we have $\mathbb{Q} \in \mathcal{B}$. However, we can't have $\mathbb{Q} \in \Sigma_1$.

Indeed, assume the contrary, we have that $\mathbb{Q} \in \Sigma_1$ and $\mathbb{Q}$ is dense in $(0,1)$. Using **3.5**, we must have a non-empty open interval $I = (a,b)$ with $0 \leq a < b \leq 1$ such that $J := (\mathbb{Q} \cap I) \in \Sigma_0$, which means that there exists A open and B closed such that $J = A \cap B$. Thus, we have $J \subset B$, so $I \subset [a,b] = \bar{J} \subset \bar{B} = B$ because B is closed. Since $J = A \cap B$, then we have that $J = A \cap I$. Take $q \in J$, then we have that $q \in A$. Since A is open, then $\exists \varepsilon \in (0, \min\{q-a, b-q\})$ such that $(q-\varepsilon, q+\varepsilon) \subset A$. Since the set of irrational numbers is dense in $\mathbb{R}$, $\exists r \in (q-\varepsilon, q+\varepsilon)$ such that $r \notin \mathbb{Q}$. Thus, we have $r \notin J$, but $r \in A$ and $r \in (q-\varepsilon, q+\varepsilon) \subset I$ since $\varepsilon \in (0, \min\{q-a, b-q\})$. So $r \in (A \cap I) = J$, a contradiction. Therefore, our assumption is wrong, so $\mathbb{Q} \notin \Sigma_1$, implying that $\Sigma_1 \neq \mathcal{B}$