Devoir maison 2 : *Intégration et mesurabilité*

Premier semestre 2022

A rendre le lundi 28 novembre durant le TD. A faire par groupe de 3. Tout recopiage d'une solution sur internet (stackexchange ou autre) sera puni par un 0/20.

Exercise 1. Compute the following limits

1. $\lim_{n\to\infty}\int_1^n \frac{dx}{x^2+1/n}$
2. $\lim_{n\to\infty}\frac{1}{2^n}\int_{\mathbb{R}}(1+\cos x)^n e^{-x^2}dx$
3. $\lim_{n\to\infty}\int_0^{2\pi}\left(1+\frac{ix}{n}\right)^n dx$
4. $\lim_{n\to\infty}\int_1^n \frac{dx}{x+1/n}$.

Exercise 2.

1. Let $(X,\mathcal{F},\mu)$ be a measure space and (M,d) be a metric space. Consider $f_m : X \to \mathbb{C}$ for all $m\in M$, $f : X\to\mathbb{C}$ and $m_0\in M$. Suppose the following :
   - f_m is measurable for all $m\in M$ and f is measurable.
   - For almost every $x\in X$, $\lim_{m\to m_0} f_m(x) = f(x)$.
   - There exists $g : X\to\mathbb{R}_+$ summable and $r>0$ such that for almost every $x\in X$ and for all $m\in M$ such that $d(m,m_0)\le r$, we have $|f_m(x)|\le g(x)$.

   Show that
   $$\lim_{m\to m_0}\int_X f_m(x)d\mu(x) = \int_X f(x)d\mu(x).$$
2. Compute the following limit
   $$\lim_{a\to 0}\int_0^1 \frac{1-e^{ax}}{x}dx$$
   where the convergence $a\to 0$ takes place in $\mathbb{R}$.

Exercise 3. Let $(X,\mathcal{F},\mu)$ be a measure space. Consider $f_n : X\to\mathbb{R}_+$ measurable and non-negative for all $n\in\mathbb{N}$ and $f : X\to\mathbb{R}_+$ measurable and non-negative. Suppose that f_0 is summable and that for all $x\in X$, $(f_n(x))_{n\in\mathbb{N}}$ is non-increasing and converges towards $f(x)$. Show that

$$\lim_{n\to\infty}\int_X f_n(x)d\mu(x) = \int_X f(x)d\mu(x) < +\infty.$$

Is the above equality still true if f_0 is no longer summable ? Justify.

Exercise 4. Let $(X,\mathcal{F},\mu)$ be a measure space and consider a measurable non-negative function $f : X\to\mathbb{R}_+$. Suppose that

$$\liminf_{n\to\infty}\int_X f(x)^n d\mu(x) < +\infty.$$

1. Show that $\mu\left(f^{-1}(]1,+\infty[)\right) = 0$.
2. Show that $\mu\left(f^{-1}(\{1\})\right) < +\infty$.
3. Using the result of Exercise 3 show that

$$\liminf_{n\to\infty}\int_X f(x)^n d\mu(x) = \lim_{n\to\infty}\int_X f(x)^n d\mu(x) = \mu\left(f^{-1}(\{1\})\right).$$

Exercise 5. For all $x \geq 0$, let

$$F(x) = \int_0^{+\infty} e^{-xt} \frac{(1-e^{-t})^2}{t^2} dt.$$

1. Show that F is well-defined and continuous on $[0, +\infty[$.
2. Show that F is of class $\mathcal{C}^2$ on $]0, +\infty[$.
3. Compute the limit of $F(x)$ and $F'(x)$ when x goes to $+\infty$.
4. Compute $F(x)$ for $x > 0$.
5. Compute $F(0)$.

Exercise 6. Let $f \in L^2(\mathbb{R}_+)$ be real-valued and define $F(x) := \int_0^x f(t)dt$ for all $x \geq 0$.

1. Show that $F(x)$ is well-defined for all $x \geq 0$ and that $F(x)x^{-1/2} \longrightarrow 0$ when $x \longrightarrow 0$.
2. Show that $F(x)x^{-1/2} \longrightarrow 0$ when $x \longrightarrow +\infty$.

Exercise 7. Let $g \in L^2(\mathbb{R}_+)$ be real-valued and define $G(x) := \frac{1}{x}\int_0^x g(t)dt$ for all $x > 0$. From the first question of Exercise 6, G is well-defined.

1. Show that for all $x > 0$
$$\left(\int_0^x g(t)dt\right)^2 \leq 2\sqrt{x}\int_0^x \sqrt{t}g(t)^2 dt.$$
2. Deduce that $G \in L^2(\mathbb{R}_+)$ and $||G||_2 \leq 2||g||_2$.

Exercise 8. Let $(X, \mathcal{F})$ be a measure space. Consider $f_n : X \longrightarrow \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \longrightarrow \mathbb{R}$. Suppose that f_n is measurable for all $n \in \mathbb{N}$.

1. Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$. Show that f is measurable.
2. Let μ be a measure on $(X, \mathcal{F})$. Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$ (according to the measure μ). Is it true that f is measurable ? Justify your answer. (Hint : you can admit, without any proof, that there exists a negligible set $N \subset \mathbb{R}$ which is not Borel).
3. We keep the assumptions of the previous question here. Show that there exists a measurable function $g : X \longrightarrow \mathbb{R}$ such that $g(x) = f(x)$ for almost every $x \in X$.

Exercise 9. Let X be a set and consider $f : X \longrightarrow \mathbb{R}$. We define

$$\mathcal{F} := \{f^{-1}(B) : B \subset \mathbb{R} \text{ Borel set}\}.$$

1. Show that $\mathcal{F}$ is a σ-algebra on X.
2. Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be measurable. Show that $h \circ f : (X, \mathcal{F}) \longrightarrow \mathbb{R}$ is measurable.
3. Let $g : (X, \mathcal{F}) \longrightarrow \mathbb{R}$ be measurable. Show that there exists $h : \mathbb{R} \longrightarrow \mathbb{R}$ measurable such that $g = h \circ f$. (Hint : start with the case where g is an indicator function).

Exercise 10. Let $f \in L^1(\mathbb{R})$ be summable on $\mathbb{R}$. Suppose that for every continuous function $\varphi \in \mathcal{C}_c(\mathbb{R})$ with compact support $\int_{\mathbb{R}} f(x)\varphi(x)dx = 0$. Show that $f = 0$ almost everywhere.