REVISED NOTE 9/3

The goal of our course right now is to solve systems of linear equations. The brute force approach to solving a system of linear equations is by "substitution." This process is best understood by an example.

Consider the system of linear equations

$$x_1 + x_2 + x_3 = 0 \tag{1}$$

$$3x_1 + x_2 - x_3 = -1 \tag{2}$$

$$-x_3 - x_1 = -x_2. \tag{3}$$

Notice we can solve Equation (3) for x_2 by multiplying both sides of the equation by -1 (or by adding x_1, x_2, and x_3 to both sides of the equation, if you prefer to think of it that way); that is, $x_2 = x_3 + x_1$. Now Equation (1) contains the variable x_2 so that we may plug the expression we just found for x_2 into Equation (1) (of course, Equation (2) also contains the variable x_2 so that we could also plug in our expression for x_2 there; it makes no difference which equation you choose to plug into). Equation (1) then becomes $x_1 + x_3 + x_1 + x_3 = 0$; simplifying and dividing by 2, we obtain $x_1 + x_3 = 0$ so that $x_1 = -x_3$. We have now deduced the equations

$$x_2 = x_3 + x_1 \tag{4}$$

$$x_1 = -x_3 \tag{5}$$

from Equations (1) and (3). Plugging the expression for x_1 given by Equation (5) into Equation (4), we see $x_2 = 0$. The only equation we have yet to use is Equation (2). Plugging in 0 for x_2 and $-x_3$ for x_1 in Equation (2), we obtain $-3x_3 - x_3 = -1$; hence, $x_3 = 1/4$. It follows from Equation (5) that $x_1 = -1/4$. Thus a solution to our linear system is given by $x_1 = -1/4$, $x_2 = 0$, $x_3 = 1/4$.

The substitution process we used to obtain this solution is in no way canonical; for example, we could have started by solving Equation (1) for x_1 and continued from there. Try this; you will obtain the same solution. Also, try solving the following linear system with substitution:

$$\begin{aligned} x + y - z &= 6 \\ -2x + 3y - z &= 0 \\ x + 2y + z &= 0. \end{aligned}$$

Given a number c, recall that when we say to "multiply" an equation by c we mean to multiply its left hand side and right hand side by c and form a new equation by setting the results equal. For example, given the equation $x + y = 1$, multiplying this equation by 5 yields the new equation $5x + 5y = 5$. Moreover, recall that when we say to "add" two equations together we mean to add the left hand sides of the equations together and add the right hand sides of the equations together and form a new equation by setting the results equal. For example, given the equations $x + y = 2$ and $3x + 4y = 5$, adding these equations yields the new equation $4x + 5y = 7$.

We can use these operations to solve systems of linear equations in a slightly more clever way (than brute force) by "elimination." Again, we illustrate with an example. Consider the linear system

$$x_1 - 3x_2 + 4x_3 = -4 \tag{6}$$

$$3x_1 - 7x_2 + 17x_3 = -8 \tag{7}$$

$$-4x_1 + 6x_2 - x_3 = 7. \tag{8}$$

Multiplying Equation (6) by 4 we see that $4x_1 - 12x_2 + 16x_3 = -16$. Now add this equation to Equation (8) to eliminate the variable x_1 and obtain the equation $-6x_2 + 15x_3 = -9$. Similarly, multiplying Equation (6) by -3 and adding the resulting equation to Equation (7) eliminates x_1 and gives $2x_2 + 5x_3 = 4$. Hence, from our original system of equations, we deduce the equations

$$-6x_2 + 15x_3 = -9 \tag{9}$$

$$2x_2 + 5x_3 = 4. \tag{10}$$

Now multiply Equation (10) by 3 and add it to Equation (9) to eliminate the variable x_2 and obtain the equation $30x_3 = 3$. Hence we easily see $x_3 = 1/10$; substituting this value of x_3 into Equation (10) yields $x_2 = 7/4$, and finally, substituting these values of x_2 and x_3 into Equation (6) yields $x_1 = 17/20$. Thus, a solution to the given linear system is $x_1 = 17/20$, $x_2 = 7/4$, $x_3 = 1/10$.

Again, this technique is not canonical. We could have started by eliminating x_2 from Equations (7) and (8) and continued from there. Try this to see you get the same solution. Also, solve the following system of equations via elimination for another exercise:

$$\begin{aligned} &x_1 - 3x_3 = 8 \\ &2x_1 + 2x_2 + 9x_3 = 7 \\ &x_2 + 5x_3 = -2. \end{aligned}$$

We will use the correspondence between augmented matrices and systems of linear equations to develop an algorithm for solving systems of linear equations.