Q2 HW10

Solve the I.V.P.

$$y'' - 3y' + 2y = f(t),\ y(0) = 0 = y'(0), \quad f(t) = \begin{cases} 0, & 0 \le t < 2 \\ 2t - 4, & t \ge 2 \end{cases}$$

with the Laplace transform.

Solution: Begin by taking the Laplace transform of the entire equation as follows:

$$\mathcal{L}[y'' - 3y' + 2y] = \mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[f(t)], \tag{1}$$

where the second equality follows from the linearity of the Laplace transform. Recall from class the expression for the Laplace transform of the nth derivative of a function, or look at Table 6.2.1 in the textbook (line 18); we find

$$\mathcal{L}[y''] = s^2\mathcal{L}[y]$$
$$\mathcal{L}[y'] = s\mathcal{L}[y].$$

As for $\mathcal{L}[f(t)]$, first notice that $f(t)$ is discontinuous. This is an indication that one should attempt to write $f(t)$ in terms of the step function $u_c(t)$, because we know an expression for the Laplace transform of $u_c(t)$ (line 12 in the table). In fact, we know an expression for the Laplace transform of $u_c(t)$ times any function whose domain is shifted by c (line 13). With this in mind, we wish to write $f(t)$ as

$$f(t) = u_c(t)g(t - c)$$

for some constant c and some function g. If we can write $f(t)$ in this form, then by line 13 of the table,

$$\mathcal{L}[f(t)] = e^{-cs}\mathcal{L}[g(t)]. \tag{2}$$

We know that f is 0 for $t < 2$. This suggests that we might want to use the step function with $c = 2$. But can we find a function $g(t)$ so that $g(t - 2) = f(t)$ for $t \ge 2$? Notice that $2t - 4 = 2(t - 2)$! Hence, taking $g(t) = 2t$ we have the desired result; that is, $f(t) = u_2(t)g(t - 2)$. It follows from Equation (2) and line 3 of the table that

$$\mathcal{L}[f(t)] = e^{-2s}\mathcal{L}[2t] = \frac{2e^{-2s}}{s^2}.$$

Thus, solving Equation (1) for $\mathcal{L}[y]$ we have

$$\mathcal{L}[y] = \frac{2e^{-2s}}{s^2(s^2 - 3s + 2)}. \tag{3}$$

Notice that we can factor the quadratic expression $s^2 - 3s + 2 = (s - 1)(s - 2)$ so that

$$\mathcal{L}[y] = 2e^{-2s}\frac{1}{s^2(s - 1)(s - 2)}.$$

Moreover, the Laplace transform table has information about expressions of the form $1/(s - a)$ (line 2) and $1/s^2$ (line 3). So we might be able to get somewhere if we can split the fraction $1/[s^2(s - 1)(s - 2)]$ into a sum of fractions, one having s^2 in the denominator, one having $s - 1$ in the denominator, and one having $s - 2$ in the denominator. Indeed, we would like to find some constants A, B, C and D such that

$$\frac{As + B}{s^2} + \frac{C}{s - 1} + \frac{D}{s - 2} = \frac{1}{s^2(s - 1)(s - 2)}.$$

In particular, we want

$$As^3 - 3As^2 + 2As + Bs^2 - 3Bs + 2B + Cs^3 - 2Cs^2 + Ds^3 - Ds^2 = 1.$$

The resulting system of equations turns out to be

$$A + C + D = 0, \quad -3A + B - 2C - D = 0, \quad 2A - 3B = 0, \quad 2B = 1,$$

with solution

$$A = 3/4, \quad B = 1/2, \quad C = -1, \quad D = 1/4.$$

Equation (3) thus becomes

$$\mathcal{L}[y] = 2e^{-2s}\left(\frac{\frac{3}{4}s + \frac{1}{2}}{s^2} - \frac{1}{s-1} + \frac{\frac{1}{4}}{s-2}\right) = \frac{3}{2}e^{-2s}\frac{1}{s} + e^{-2s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s-1} + \frac{1}{2}e^{-2s}\frac{1}{s-2}.$$

Notice that we can use lines 1, 3 and 2 of the table (in that order) to get

$$\mathcal{L}[y] = \frac{3}{2}e^{-2s}\mathcal{L}[1] + e^{-2s}\mathcal{L}[t] - 2e^{-2s}\mathcal{L}[e^t] + \frac{1}{2}e^{-2s}\mathcal{L}[e^{2t}].$$

Then, appealing to line 13 we have

$$\mathcal{L}[y] = \frac{3}{2}\mathcal{L}[u_2(t)] + \mathcal{L}[u_2(t)(t-2)] - 2\mathcal{L}[u_2(t)e^{t-2}] + \frac{1}{2}\mathcal{L}[u_2(t)e^{2(t-2)}].$$

Finally, we take the inverse Laplace transform of both sides to get the solution to the I.V.P.:

$$y = \frac{3}{2}u_2(t) + u_2(t)(t-2) - 2u_2(t)e^{t-2} + \frac{1}{2}u_2(t)e^{2(t-2)}$$

(note that the homework software wants you to write this solution in terms of $u(t) \equiv u_0(t)$; clearly $u(t-2) = u_2(t)$).