Let us begin by discussing logic. A *statement* is a sentence that asserts something. Examples of statements are the following:

"It is raining."

"The $n \times n$ matrix $A$ has $n$ pivot positions."

"The ground is wet."

"The columns of the $n \times n$ matrix $A$ form a linearly indepnedent set."

"I love my dog."

"I hate vegetables."

Let $P$ be a statement. By this we mean $P$ is a variable representing some assertion. $P$ could be one of the statements listed above, or it could be something else; it is a completely *arbitrary* statement.

Now let $Q$ be another statement. Again, we mean $Q$ is a variable representing some arbitrary assertion. $Q$ could be one of the statements above, it could be the statement "Math sucks.," or it could even be the same statement as $P$ (whatever that is). It is *arbitrary.*

When we say "statement $P$ implies statement $Q$", which we denote by $P \Rightarrow Q$, we mean that the truth of statement $Q$ follows from the truth of statement $P$. For example (now we must *specify* what we want statments $Q$ and $P$ to be in order to formulate an example), suppose

$$P = \text{"It is raining."}$$

and

$$Q = \text{"The ground is wet."}$$

We know from experience that when it is raining, the ground is wet. To put it more formally, if statement $P$ is true, then statement $Q$ is true. Hence we write $P \Rightarrow Q$.

Let us go back to $P$ and $Q$ being arbitrary statments (ie. they no longer are necessarily represent the statements "It is raining." or "The ground is wet."). We say "statement $P$ and statement $Q$ are equivalent," denoted by $P \Leftrightarrow Q$, when both $P \Rightarrow Q$ *and* $Q \Rightarrow P$, that is, when the truth of statement $P$ follows from the truth of statement $Q$ *and* the truth of statement $Q$ follows from the truth of statment $P$.

For example, let us go back to the previous example where we suppose $P =$ "It is raining." and Q = "The ground is wet." Are $P$ and $Q$ equivalent? Well we know $P \Rightarrow Q$, but does $Q \Rightarrow P$? We know from experience there are many reasons besides rain that the ground might be wet (for example, there could be a sprinkler on or someone might have spilled something). Hence we conclude that $Q$ does *not* imply $P$, so that $P$ and $Q$ are *not* equivalent.

Notice in this example, $Q$ certainly *could* imply $P$, but *in general* it does not. Hence $Q \not\Rightarrow P$. Indeed, if the ground is wet, it certainly *could* be the case that it is raining, but one cannot conclude definitively that it is in fact raining just from noticing that the ground is wet.

The invertible matrix theorem lists 18 mathematical statements, labeled as $(1), (2), \ldots, (18)$ and says that these statements are *equivalent.* This means that if you pick any two statements $(i)$ and $(j)$, then $(i) \Leftrightarrow (j)$. Hence, to "prove the equivalence of the statments $(i)$ and $(j)$ from the invertible matrix theorem" is to show that the truth of statement $(i)$ follows from the truth of statement $(j)$, and the truth of statement $(j)$ follows from the truth of statement $(i)$.

Following are some examples.

(1) $\Leftrightarrow$ (2) Assume that statement (1) is true. That is, assume that the $n \times n$ matrix $A$ is invertible. Then by the definition of an invertible matrix, there must exist a matrix $B$ such that $AB = I = BA$. Denote the matrix $B$ as $A^{-1}$.

Let $\mathbf{b}$ be an arbitrary vector in $\mathbb{R}^n$. Notice that the matrix equation $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = A^{-1}\mathbf{b}$. Since $\mathbf{b}$ was arbitrary, this implies that the augmented matrix $[A\ \mathbf{b}]$ does not have a pivot in the last column for any $\mathbf{b}$. Hence either there is a pivot in every column of $A$ or the reduced echolon form of $A$ contains at least one row of all zeros. But the latter cannot occur, or else there would be a $\mathbf{b}$ in $\mathbb{R}^n$ for which the augmented matrix $[A\ \mathbf{b}]$ has a pivot in the last column! Hence $A$ must have a pivot position in every column, which implies $A$ is row equivalent to the identity. But $A$ being row equivalent to the identity is exactly statement (2). We have now shown that (1) $\Rightarrow$ (2).

What is left is to show (2) $\Rightarrow$ (1), so assume (2) is true. That is, suppose that the $n \times n$ matrix $A$ is row equivalent to the identity. By the definition of row equivalence, this means there is a sequence of elementary row operations such that when performed on $A$, one arrives at $I$. Put another way, there is a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that when multiplied in this order on the left of $A$, the product is $I$, ie. $E_1E_2 \cdots E_kA = I$. But elementary matrices are invertible, and the product of invertible matrices is invertible, with the inverse being the product of the inverses in reverse order, ie. $(E_1E_2 \cdots E_k)^{-1} = E_k^{-1} \cdots E_2^{-1}E_1^{-1}$. Multiplying the equation $E_1E_2 \cdots E_kA = I$ on both sides by $(E_1E_2 \cdots E_k)^{-1}$, we thus see $A = (E_1E_2 \cdots E_k)^{-1}$. In particular, the matrix $B = (E_1E_2 \cdots E_k)$ is such that $BA = I = AB$. But this is exactly the requirement that $A$ be invertible. Hence we have now shown (2) $\Rightarrow$ (1), and we are done.

(10) $\Leftrightarrow$ (11) Assume (10) is true. That is, suppose $A$ is an $n \times n$ matrix such that there exists an $n \times n$ matrix $C$ such that $CA = I$. Notice then that $CA - I = 0$. Multiplying both sides of this equation by $C$ on the right, we have $CAC - C = 0$. We can then factor out a $C$ on the left, ie. $C(AC - I) = 0$. By assumption, $C$ cannot be 0; hence $AC - I = 0$. Moving the $I$ to the right hand side of this equation, $AC = I$. Thus, if we take $D = C$, we have shown there is an $n \times n$ matrix $D$ such that $AD = I$. This is exactly statement (11).

Now assume (11) is true. That is, suppose $A$ is an $n \times n$ matrix such that there exists an $n \times n$ matrix $D$ such that $AD = I$. Notice then that $AD - I = 0$. Multiplying both sides of this equation by $D$ on the left, we have $DAD - D = 0$. We can then factor out a $D$ on the right, ie. $(DA - I)D = 0$. By assumption, $D$ cannot be 0; hence $DA - I = 0$. Moving the $I$ to the right hand side of this equation, $DA = I$. Thus, if we take $C = D$, we have shown there is an $n \times n$ matrix $C$ such that $CA = I$. This is exactly statement (10).

(4) $\Leftrightarrow$ (5) Assume (4) is true. Write the product $A\mathbf{x}$ as

$$A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where $\mathbf{a}_i$ is the column vector representing the $i$th column of $A$. It is now clear if $A\mathbf{x} = \mathbf{0}$ only has the trival solution, then the equation $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$ is only true if $x_1 = x_2 = \cdots = x_n = 0$. This is exactly condition (5).

On the other hand, assume (5) is true. Then, writing the vectors corresponding to the columns of the $n \times n$ matrix $A$ as $\mathbf{a}_1, \ldots, \mathbf{a}_n$, the equation $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$ is only true if $x_1 = x_2 = \cdots = x_n = 0$ (this is the definition of linear independence of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$). But we can rewrite the left-hand side of this equation as

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = A\mathbf{x}.$$

It then follows that the only solution to the matrix equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, that is, the trivial solution.

Notice the techniques for proving (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) were pretty different. On the other hand, (10) $\Rightarrow$ (11) and (11) $\Rightarrow$ (10) where almost identical, except with $D$ and $C$ switched around; likewise after showing (4) $\Rightarrow$ (5), it may not even feel necessary to show (5) $\Rightarrow$ (4). But in order to show these statements are *equivalent*, you need to produce some argument in both directions, even if it seems like the arguments are the same (they will always in fact be slightly different, since you are starting with different assumptions!).