# A tank problem

*Suppose that a tank containing a certain liquid has an outlet near the bottom; let $h(t)$ be the height of the liquid surface above the outlet at time $t$; Torricelli's principle states that the outflow speed $s$ of the liquid at the outlet is equal to the magnitude of the velocity that a particle would have after free falling from rest from a height of $h$; with Torricelli's principle in mind, find an expression for $s$ in terms of $h$.*

To solve the problem, Torricelli's principle says we simply need to calculate the velocity that a particle would have after free falling from rest from a height of $h$. Denote the position of such a particle at time $t$ by $x(t)$. Moreover, let us set a coordinate system with the $\hat{x}$ direction (the "positive" direction) pointing in the opposite direction that the particle falls. The equation of motion of such a particle is then $F_g = ma(t)$, where $F_g = -mg$ is the force on the particle due to gravity, $g$ is the gravitational constant, $m$ is the mass of the particle and $a(t) = x''(t)$ is the acceleration of the particle at time $t$. In particular, we have

$$x''(t) = -g. \tag{1}$$

Hence we have a second order linear constant coefficient nonhomogeneous ordinary differential equation for $x(t)$, with nonhomogeneous term the constant $g$. Furthermore, we have the initial conditions $x(0) = h$ and $x'(0) = 0$. But, noticing that $x'(t)$ is the velocity $v(t)$ of the particle at time $t$, maybe we only have to consider the first order linear separable ordinary differential equation

$$\big(x''(t) = \big)\ v'(t) = -g, \tag{2}$$

for $v(t)$. In fact, solving Equation (2) we have the general solution $v(t) = -gt + k_1$ for some arbitrary constant $k_1$. Then, plugging in the initial condition $(x'(0) =)\ v(0) = 0$, we have the particular solution

$$v(t) = -gt. \tag{3}$$

If we knew how long it takes for the particle to fall a distance $h$, we would be done (assuming we could write this time in terms of $h$). For then, denoting this time by $T$, by Equation (3) we would simply have $s = gT$.

But we do not know how long it takes for the particle to fall a distance $h$. However, we can find out by finding an expression for $x(t)$. Indeed, again denoting this time by $T$, if we have an expression for $x(t)$ then we can solve the equation $x(T) = 0$ for $T$ (because the particle will have fallen a distance $h - x(T) = h$ when $x(T) = 0$). Hopefully this expression for $T$ will be in terms of $h$ (in fact, we have reason to suspect so since we have initial conditions for $x(t)$ involving $h$).

One way to find an expression for $x(t)$ is by solving Equation (1) with initial conditions $x(0) = h, x'(0) = 0$ (another way is to solve the first order linear separable ordinary differential equation $x'(t) = -gt$ got from Equation (3); this way would be easier, but we go through the method of undetermined coefficients for practice). First we solve the homogeneous problem

$$x''(t) = 0.$$

We have characteristic equation $r^2 = 0$; since we have a repeated root $r_1 = 0 = r_2$, our two homogeneous solutions are $e^{0t} = 1$ and $te^{0t} = t$. It follows that the general solution to the homogenous problem is $x_h(t) = c_1 + c_2 t$ for arbitrary constants $c_1$ and $c_2$.

Now we find a particular solution to Equation (1). Since the nonhomogeneous term is a 0th degree polynomial (a constant), we can use the method of undetermined coefficients with initial guess $X(t) = A$ for some 0th degree polynomial $A$. But immediately we notice that the homogeneous solution $x_h(t)$ already includes a constant $c_1$. Hence we guess $X(t) = At + B$, but again we see that $x_h(t)$ already includes (in fact is exactly) a 1st degree polynomial $c_1 + c_2 t$. Thus we finally come to the guess $X(t) = At^2 + Bt + C$. Plugging this guess into Equation (1), we have $2A = -g$ so that $A = -g/2$, with no restrictions on $B$ and $C$; we pick $B = 0 = C$ for simplicity. So the general solution to Equation (1) is

$$x(t) = -\frac{g}{2}t^2 + c_2 t + c_1 \tag{4}$$

(if it seems weird that we can pick $B$ and $C$ however we want, notice that if we leave them unspecified, Equation (4) just becomes $-gt^2/2 + (B + c_2)t + (C + c_1)$ and we can just combine $B$ and $C$ into $c_1$ and $c_2$). Plugging in our initial conditions $h = x(0)$ and $0 = x'(0)$, we solve for $c_1$ and $c_2$ to find

$$x(t) = -\frac{g}{2}t^2 + h. \tag{5}$$

We are almost done. As mentioned earlier, if we denote by $T$ the time it takes for the particle to fall a distance $h$, then Equation (5) yields

$$0 = -\frac{g}{2}T^2 + h,$$

so that

$$T = \sqrt{\frac{2h}{g}}.$$

This is an expression for $T$ in terms of $h$, as we had hoped! Hence, plugging this expression for $T$ into Equation (3), by Torricelli's principle, the outflow speed at the outlet of the tank is $s = \sqrt{2gh}$.

*Assuming the tank itself and the outlet protruding from the tank are right cylinders with radii $R$ and $r$ respectively, write a differential equation for $h(t)$.*

**Solution**: The area of the cross section of the tank is $\pi R^2$. Hence the change in the volume of water in the tank per unit time is

$$\frac{dh(t)}{dt}\pi R^2.$$

Likewise, the area of the cross section of the outlet cylinder is $\pi r^2$. Hence the volume of water leaving the tank per unit time is $\pi r^2 s = \pi r^2 \sqrt{2gh(t)}$. Since the amount of water leaving the tank per unit time must be equal to the change in the amount of water in the tank per unit time (we are not adding anything to the tank, just draining it), we arrive at the first order nonlinear separable ordinary differential equation

$$\frac{dh(t)}{dt} = -\frac{r^2\sqrt{2gh(t)}}{R^2}.$$

*Suppose the tank is filled to height $H$ before it begins to drain. At what time is the tank empty?*

**Solution**: First we solve the above equation to get an implicit expression for $h(t)$:

$$-\sqrt{h(t)} = \frac{\sqrt{2g}r^2t}{2R^2} + C.$$

Using the initial condition $h(0) = H$, we find $C = -\sqrt{H}$. Hence, writing $T$ for the amount of time it takes for the tank to empty, we have

$$0 = \frac{\sqrt{2g}r^2T}{2R^2} - \sqrt{H}$$

so that

$$T = \frac{2R^2}{r^2}\sqrt{\frac{H}{2g}}.$$