SOME COMMENTS ON THE MASS-SPRING SYSTEM

Recall that we know how to solve *every* second order linear constant coefficient equation. Indeed, if it is homogeneous, we have the three scenarios corresponding to the different possibilities for the roots of the characteristic equation. On the other hand, if it is nonhomogenous, we can always use variation of parameters to find a particular solution (at least in terms of an integral that we may or may not know how to do); or, if the nonhomogeneous term is an exponential, sine or cosine, polynomial, or any sum or product of functions of this form, we can use undetermined coefficients.

What we showed about a week ago was that a mass-spring system, or more generally any vibrating system that behaves like a mass-spring system, can be modeled with a second order linear constant coefficient equation. In particular, if $x(t)$ represents the distance the mass is from its equilibrium position at time t, then $x(t)$ satisfies the differential equation

$$mx''(t) + \gamma x'(t) + kx(t) = F_e(t), \tag{1}$$

where m is the mass (of the mass), γ is the (positive) proportionality constant for the damping force (the amount of damping force on the mass per unit of speed), k is the (positive) "spring constant" (the amount of force required to stretch the spring one unit of distance), and $F_e(t)$ is some unspecified time-dependent external force.

Hence, given any vibrating system which behaves like a mass-spring system, we can *always* find an expression for the position of the mass as a function of t (just solve the corresponding differential equation!).

When dealing with a vibrating system modeled by Equation (1), we like to use terminology with a physical connotation to talk about the problem. In fact, we are already doing this when we call the coefficients in Equation (1) "mass," "damping coefficient," and "spring constant," when we call the nonhomogenous term an "external force," and when we call the independent variable "time" and the dependent variable "position."

Moreover, we refer to the homogenous solution to Equation (1) as the **transient** solution and a particular solution as a **steady state** solution. This terminology is in line with the fact that for such a system, the homogenous solution will become negligible given enough time (make sure you understand why this is the case) so that after a while, the system is only responding to the external force.

We give special names to several other quantities as well. Consider the case of a system with no damping and no external force ($\gamma = 0$ and $F_e(t) = 0$). Then Equation (1) becomes

$$mx''(t) + kx(t) = 0, \tag{2}$$

and solutions are of the form

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \tag{3}$$

for some constants c_1 and c_2 (dependent on the initial conditions of the system), where $\omega_0 = \sqrt{k/m}$ (make sure you understand why Equation (3) is a general solution for Equation (2)). We call ω_0 the **natural frequency** of the mass-spring system modeled by Equation (1).

Now, notice we can rewrite Equation (3) in terms of a single cosine function. Indeed, suppose $c_1 = R\cos(\delta)$ and $c_2 = R\sin(\delta)$ for some constants R and δ. Then, Equation (3) becomes

$$x(t) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t),$$

and after some manipulation (use the trigonometry identity $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$),

$$x(t) = R\cos(\omega_0 t - \delta). \tag{4}$$

Notice there is nothing mysterious going on here. To get Equation (4), we simply rewrote the *arbitary* numbers c_1 and c_2 in terms of some other *arbitrary* numbers R and δ (albeit in a clever way). Why is this useful? Because we can tell graphically exactly what R and δ represent (on the other hand, it is not clear what c_1 and c_2 tell us). In particular, R tells us the maximum and minimum values for $x(t)$, which we refer to as the **amplitude** of the motion, while δ tells us how far off this cosine wave is from the "standard one" centered on the y-axis, which we refer to as the **phase shift**.

This week, we concentrated on the scenario where we have an external force that is periodic, say of the form $F_e(t) = F_0\cos(\omega t)$. In this case, Equation (1) becomes

$$mx''(t) + \gamma x'(t) + kx(t) = F_0\cos(\omega t). \tag{5}$$

In particular, we analyzed a steady state solution to Equation (5). Since $F_e(t)$ is a cosine function, we can use undetermined coefficients to find a steady state solution; that is, we guess $X(t) = A\cos(\omega t) + B\sin(\omega t)$. Plugging into Equation (5), we have

$$(kA + \gamma\omega B - m\omega^2 A)\cos(\omega t) + (kB - \gamma\omega A - m\omega^2 B)\sin(\omega t) = F_0\cos(\omega t),$$

and equating coefficients of like functions,

$$F_0 = (kA + \gamma\omega B - m\omega^2 A), \quad 0 = (kB - \gamma\omega A - m\omega^2 B).$$

We can now solve for A and B in terms of F_0, k, γ, ω and m:

$$A = \frac{F_0}{k - m\omega^2} - \frac{\gamma\omega F_0}{(k - m\omega^2)^2}\left(\frac{1}{k - m\omega^2 + \frac{\gamma^2\omega^2}{k - m\omega^2}}\right), \tag{6}$$

$$B = \frac{\gamma\omega F_0}{k - m\omega^2}\left(\frac{1}{k - m\omega^2 + \frac{\gamma^2\omega^2}{k - m\omega^2}}\right). \tag{7}$$

Now, write $X(t)$ as a single cosine function just as in Equation 4: letting $A = R\cos(\delta)$ and $B = R\sin(\delta)$, we get $X(t) = R\cos(\omega t - \delta)$. Equating our redefined A and B with Equations (6) and (7) respectively, we can solve for the amplitude and phase shift to obtain exactly the equations we have been writing on the board all week! That is,

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \sin(\delta) = \frac{\gamma\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

(notice we have rewritten $\sqrt{k/m}$ as ω_0).

To be clear, Equation (4) is the general solution to Equation (2), while the $X(t)$ we just found is a *particular* solution to Equation (5).

Now if we solve the equation $R'(\omega) = 0$ for ω, we see that the value of ω for which $R(\omega)$ is at a maximum is given by

(8) $$\omega_{\mathrm{max}} = \omega_0 \sqrt{1 - \frac{\gamma^2}{2mk}},$$

and subsequently that

(9) $$R_{\mathrm{max}} := R(\omega_{\mathrm{max}}) = \frac{F_0}{\gamma\omega_0\sqrt{1 - \frac{\gamma^2}{4mk}}}$$

(verify this as an exercise).