Solve the following initial value problem using the Laplace transform.

$$y^{(4)} - 4y''' + 6y'' - 4y' + y = 0, \quad y(0) = y''(0) = 0,\ y'(0) = y'''(0) = 1.$$

Start by finding the Laplace transform of the equation. We have

$$\mathcal{L}(y^{(4)} - 4y''' + 6y'' - 4y' + y) = \mathcal{L}(0). \tag{1}$$

By the linearity of $\mathcal{L}$, Equation (1) becomes

$$\mathcal{L}(y^{(4)}) - 4\mathcal{L}(y''') + 6\mathcal{L}(y'') - 4\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(0).$$

Clearly $\mathcal{L}(0) = 0$ (for this, go back to the definition of $\mathcal{L}$), and using Line 18 of our Laplace transform table (and subsequently the initial conditions given in the problem), we see

$$\begin{aligned}
\mathcal{L}(y^{(4)}) &= s^4\mathcal{L}(y) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4\mathcal{L}(y) - s^2 - 1\\
\mathcal{L}(y''') &= s^3\mathcal{L}(y) - s^2y(0) - sy'(0) - y''(0) = s^3\mathcal{L}(y) - s\\
\mathcal{L}(y'') &= s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - 1\\
\mathcal{L}(y') &= s\mathcal{L}(y) - y(0) = s\mathcal{L}(y).
\end{aligned}$$

Hence, Equation (1) becomes

$$s^4\mathcal{L}(y) - s^2 - 1 - 4(s^3\mathcal{L}(y) - s) + 6(s^2\mathcal{L}(y) - 1) - 4s\mathcal{L}(y) + \mathcal{L}(y) = 0,$$

and we can solve for $\mathcal{L}(y)$:

$$\mathcal{L}(y) = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1}. \tag{2}$$

Now we must find the inverse transform of the right-hand side of Equation (2). Our table clearly does not contain this expression. But notice

$$s^4 - 4s^3 + 6s^2 - 4s + 1 = (s-1)^4$$

(this of course is not trivial; we had to look it up in class). Hence Equation (2) becomes

$$\mathcal{L}(y) = \frac{s^2 - 4s + 7}{(s-1)^4},$$

which at least looks slightly more managable. Indeed, keeping Line 11 of our table in mind, let us find constants A, B, C, and D such that

$$\frac{s^2 - 4s + 7}{(s-1)^4} = \frac{A}{(s-1)^4} + \frac{B}{(s-1)^3} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)}$$

(*note that it was silly to imagine we could write*

$$\frac{s^2 - 4s + 7}{(s-1)^4} = \frac{A}{(s-1)} + \frac{B}{(s-1)} + \frac{C}{(s-1)} + \frac{D}{(s-1)}$$

since the right-hand side of this equation is just

$$\frac{A + B + C + D}{s-1}\Bigg).$$

Equivalently, let us find constants A, B, C, and D such that

$$\begin{aligned}
s^2 - 4s + 7 &= A + B(s-1) + C(s-1)^2 + D(s-1)^3\\
&= Ds^3 + (C - 3D)s^2 + (B - 2C + 3D)s + (A - B + C - D).
\end{aligned}$$

Then

$$D = 0, \quad C - 3D = 1, \quad B - 2C + 3D = -4, \quad A - B + C - D = 7;$$

that is, $D = 0$, $C = 1$, $B = -2$, and $A = 4$ so that

$$\frac{s^2 - 4s + 7}{(s-1)^4} = \frac{4}{(s-1)^4} - \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2}.$$

It follows that the solution to our initial value problem is (using the linearity of the inverse transform)

$$y = \mathcal{L}^{-1}\left(\frac{4}{(s-1)^4}\right) - \mathcal{L}^{-1}\left(\frac{2}{(s-1)^3}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right).$$

Finally, appealing to Line 11 of our table, we see (for b some real number and again using linearity)

$$\mathcal{L}^{-1}\left(\frac{b}{(s-a)^{n+1}}\right) = \frac{b}{n!}\mathcal{L}^{-1}\left(\frac{n!}{(s-a)^{n+1}}\right) = \frac{b}{n!}t^n e^{at};$$

that is,

$$y = te^t + \frac{1}{2}t^2e^t + \frac{2}{3}t^3e^t.$$