THE TRACE SIMPLEX OF A NONCOMMUTATIVE VILLADSEN ALGEBRA

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Abstract. We define a nonsimple "noncommutative" Villadsen algebra and show that its simplex of tracial states is the Poulsen simplex.

1. Introduction

Let B be a unital C*-algebra obtained from an inductive system $(B_i, \psi_i)_{i\in\mathbb{N}}$ of unital C*-algebras with injective connecting maps. Recall that the trace simplex (the simplex of tracial *states*) of B, denoted by $T(B)$, is the limit of the affine projective system

$$T(B_1) \xleftarrow{\psi_1^*} T(B_2) \xleftarrow{\psi_2^*} T(B_3) \xleftarrow{\psi_3^*} \cdots,$$

where for $\tau \in T(B_{i+1})$, $\psi_i^*(\tau) = \tau \circ \psi_i$. We may represent an element $\tau \in T(B)$ as a sequence $(\tau_i)_{i\in\mathbb{N}}$, where $\tau_i \in T(B_i)$ and $\psi_i^*(\tau_{i+1}) = \tau_i$.

Now let C_0 be a unital C*-algebra, let $n_0 \in \mathbb{N}$, let $A_0 = M_{n_0}(C_0)$, and let $(n_i)_{i\in\mathbb{N}}$ be a sequence of natural numbers. Moreover, for each $i \in \mathbb{N}$, let

$$C_i = \underbrace{C_{i-1} \otimes_{\alpha_i} \cdots \otimes_{\alpha_i} C_{i-1}}_{n_i} \quad \text{and} \quad A_i = C_i \otimes M_{n_0\cdots n_i} \ (\cong M_{n_0\cdots n_i}(C_i)),$$

where the completion is taken with respect to the C*-norm α_i. Denote by A the limit of the inductive system $(A_i, \phi_i)_{i\in\mathbb{N}}$, where the seed for the i-th stage map $\phi_i\colon C_i \to M_{n_{i+1}}(C_{i+1})$ is defined by

$$c \mapsto \operatorname{diag}\big(\underbrace{c \otimes_{\alpha_{i+1}} 1 \otimes_{\alpha_{i+1}} \cdots \otimes_{\alpha_{i+1}} 1}_{n_{i+1}}, \underbrace{1 \otimes_{\alpha_{i+1}} c \otimes_{\alpha_{i+1}} 1 \otimes_{\alpha_{i+1}} \cdots \otimes_{\alpha_{i+1}} 1}_{n_{i+1}}, \ldots, \underbrace{1 \otimes_{\alpha_{i+1}} \cdots \otimes_{\alpha_{i+1}} 1 \otimes_{\alpha_{i+1}} c}_{n_{i+1}}\big).$$

Date: November 9, 2025.

Fix $i \in \mathbb{N}$. For $c \in C_i$, let $\mathbf{c}_0 = c$ and inductively define over $j \in \mathbb{N}$ the element $\mathbf{c}_j \in M_{n_{i+1}\cdots n_{i+j}}(C_{i+j})$ as

$$\mathbf{c}_j = \operatorname{diag}\big(\underbrace{\mathbf{c}_{j-1} \otimes_{\alpha_{i+j}} \mathbf{1}_{j-1} \otimes_{\alpha_{i+j}} \cdots \otimes_{\alpha_{i+j}} \mathbf{1}_{j-1}}_{n_{i+j}}, \underbrace{\mathbf{1}_{j-1} \otimes_{\alpha_{i+j}} \mathbf{c}_{j-1} \otimes_{\alpha_{i+j}} \mathbf{1}_{j-1} \otimes_{\alpha_{i+j}} \cdots \otimes_{\alpha_{i+j}} \mathbf{1}_{j-1}}_{n_{i+j}}, \dots, \underbrace{\mathbf{1}_{j-1} \otimes_{\alpha_{i+j}} \cdots \otimes_{\alpha_{i+j}} \mathbf{1}_{j-1} \otimes_{\alpha_{i+j}} \mathbf{c}_{j-1}}_{n_{i+j}}\big),$$

where $\mathbf{1}_{j-1}$ is the identity of C_{i+j-1}. Then we can write the seed of the composed map $\phi_{i,i+j-1} = \phi_{i+j-1} \circ \cdots \circ \phi_i \colon C_i \to M_{n_{i+1}\cdots n_{i+j}}(C_{i+j})$ as $c \mapsto \mathbf{c}_j$ (notice $\phi_{i,i} = \phi_i$); moreover, suppressing the notation indicating the C*-norm, we have

$$\phi_{i,i+j-1}(c) = \operatorname{diag}\big(\underbrace{c \otimes 1 \otimes \cdots \otimes 1}_{n_{i+1}\cdots n_{i+j}}, \underbrace{1 \otimes c \otimes 1 \otimes \cdots \otimes 1}_{n_{i+1}\cdots n_{i+j}}, \dots, \underbrace{1 \otimes \cdots \otimes 1 \otimes c}_{n_{i+1},\cdots n_{i+j}}\big),$$

where 1 is the identity of C_i.

The C*-algebra A is an example of a "noncommutative" Villadsen algebra. Moreover, it is simple if and only if C_0 is simple. The trace simplex of A is given by the limit of the projective system

$$T(A_1) \xleftarrow{\phi_1^*} T(A_2) \xleftarrow{\phi_2^*} T(A_3) \xleftarrow{\phi_3^*} \cdots,$$

where

$$\phi_i^*(\tau \otimes \mathrm{tr}) = \frac{1}{n_{i+1}}(\tau^{(1)} \otimes \mathrm{tr} + \cdots + \tau^{(n_{i+1})} \otimes \mathrm{tr}), \quad \tau \in T(C_{i+1})$$

and where, for $c \in C_i$,

$$\tau^{(k)}(c) = \tau(\underbrace{1 \otimes_{\alpha_{i+1}} \cdots \otimes_{\alpha_{i+1}} 1}_{j-1} \otimes_{\alpha_{i+1}} c \otimes_{\alpha_{i+1}} \underbrace{1 \otimes_{\alpha_{i+1}} \cdots \otimes_{\alpha_{i+1}} 1}_{n_{i+1}-j}), \quad 1 \le k \le n_{i+1}. \tag{1}$$

It follows that $T(A)$ is homeomorphic to the limit of the projective system

$$T(C_1) \xleftarrow{\theta_1} T(C_2) \xleftarrow{\theta_2} T(C_3) \xleftarrow{\theta_3} \cdots,$$

where

$$\theta_i(\tau) = \frac{1}{n_{i+1}}\big(\tau^{(1)} + \cdots + \tau^{(n_{i+1})}\big), \quad \tau \in T(C_{i+1})$$

with $\tau^{(k)}$ defined by Equation (1) above. The composed map is then, for any $j \in \mathbb{N}$,

$$\theta_{i,i+j-1}(\tau) = (\theta_i \circ \cdots \circ \theta_{i+j-1})(\tau) = \frac{1}{n_{i+1}\cdots n_{i+j}}\big(\tau^{(1)} + \cdots + \tau^{(n_{i+1}\cdots n_{i+j})}\big), \quad \tau \in T(C_{i+j}) \tag{2}$$

where, suppressing the C*-norm notation (as we will do for the remainder of the paper),

$$\tau^{(k)}(c) = \tau(\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes c \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n_{i+1}\cdots n_{i+j}-k}), \quad c \in C_i$$

for $1 \leq k \leq n_{i+1} \cdots n_{i+j}$ (note that $\theta_{i,i} = \theta_i$). We then identify an element $\tau \in T(A)$ with a sequence $(\tau_i)_{i\in\mathbb{N}}$, where $\tau_i \in T(C_i)$ and $\theta_i(\tau_{i+1}) = \tau_i$.

2. Preliminary results

Before we prove our main theorem, that $T(A)$ is the Poulsen simplex, we list a few preliminary results.

Theorem 1 ([1], Theorem 2.1)**.** *Let B be a unital C*-algebra and $n \in \mathbb{N}$. For extreme points $\tau^{(1)}, \ldots, \tau^{(n)} \in T(B)$, the product trace $\tau^{(1)} \otimes \cdots \otimes \tau^{(n)} \in T(\underbrace{B \otimes \cdots \otimes B}_{n})$ is extreme.*

Recall that for a unital C*-algebra B, $\mu \in T(B)$ has a base of neighborhoods consisting of sets of the form $\{\nu : |\mu(b) - \nu(b)| < \epsilon,\ b \in \mathcal{F}\}$, where $\epsilon > 0$ and $\mathcal{F} \subseteq B$ is finite.

Theorem 2 (Krein–Milman)**.** *Let B be a unital C*-algebra, let $\mu \in T(B)$, and let $\mathcal{N}$ be a basic neighborhood of μ. Then there is a number $N \in \mathbb{N}$ such that for all $n \geq N$, there exist extreme points $\tau^{(1)}, \ldots, \tau^{(n)} \in T(B)$ such that $n^{-1}(\tau^{(1)} + \cdots + \tau^{(n)}) \in \mathcal{N}$.*

Following is a simple observation.

Lemma 1. *Let $\tau = (\tau_i)_{i\in\mathbb{N}} \in T(A)$. If τ_i is extreme for sufficiently large i, then τ is extreme.*

3. Main result

Theorem 3. *The trace simplex of A is the Poulsen simplex.*

Proof. It is sufficient to show that the extreme points of $T(A)$ form a dense subset of $T(A)$.

Let $\mu = (\mu_i)_{i\in\mathbb{N}} \in T(A)$. Choose a basic neighborhood $\mathcal{N} = \mathcal{N}(\epsilon, \{a_1, \ldots, a_m\})$ of μ. Fix $i_0 \in \mathbb{N}$ such that there is a set $\{b_1, \ldots, b_m\} \subseteq A_{i_0}$ so that $\|a_k - b_k\| < \epsilon/3$ for each $1 \leq k \leq m$. By Theorem 2, there is a number $j_0 \in \mathbb{N}$ such that, for each $1 \leq k \leq m$,

$$\left| \frac{1}{n_{i_0+1} \cdots n_{i_0+j_0}} \big(\tau^{(1)} + \cdots + \tau^{(n_{i_0+1}\cdots n_{i_0+j_0})}\big)(b_k) - \mu_{i_0}(b_k) \right| < \epsilon/3 \tag{3}$$

for some extreme points $\tau^{(1)}, \ldots, \tau^{(n_{i_0+1}\cdots n_{i_0+j_0})} \in T(A_{i_0})$.

Identifying $\tau^{(1)}, \ldots, \tau^{(n_{i_0+1}\cdots n_{i_0+j_0})}$ with the corresponding extreme points in $T(C_{i_0})$, consider the trace $\nu = \tau^{(1)} \otimes \cdots \otimes \tau^{(n_{i_0+1}\cdots n_{i_0+j_0})} \in T(C_{i_0+j_0})$; it is extreme by Theorem 1. Now consider the trace $\tau = (\tau_i)_{i\in\mathbb{N}} \in T(A)$ for which $\tau_{i_0+j_0} = \nu$; that is,

$$\tau = \big(\theta_{1,i_0+j_0-1}(\nu), \ldots, \theta_{i_0+j_0-2,i_0+j_0-1}(\nu), \theta_{i_0+j_0-1,i_0+j_0-1}(\nu), \nu, \tau_{i_0+j_0+1}, \ldots\big).$$

Letting $k \in \mathbb{N}$ and considering the trace

$$\underbrace{\nu \otimes \cdots \otimes \nu}_{n_{i_0+j_0+1} \cdots n_{i_0+j_0+k}} \in T(C_{i_0+j_0+k}),$$

it follows directly from Equation (2) that

$$\theta_{i_0+j_0, i_0+j_0+k-1}\big(\underbrace{\nu \otimes \cdots \otimes \nu}_{n_{i_0+j_0+1} \cdots n_{i_0+j_0+k}}\big) = \frac{1}{n_{i_0+j_0+1} \cdots n_{i_0+j_0+k}}\big(\underbrace{\nu + \cdots + \nu}_{n_{i_0+j_0+1} \cdots n_{i_0+j_0+k}}\big) = \nu$$

so that

$$\tau_{i_0+j_0+k} = \underbrace{\nu \otimes \cdots \otimes \nu}_{n_{i_0+j_0+1} \cdots n_{i_0+j_0+k}}.$$

Thus, by Lemma 1, we see that τ is extreme.

If we can show that $\tau \in \mathcal{N}$, we will be done. But notice that, again by Equation (2),

$$\tau_{i_0} = \theta_{i_0, i_0+j_0-1}(\nu) = \frac{1}{n_{i_0+1} \cdots n_{i_0+j_0}}\big(\tau^{(1)} + \cdots + \tau^{(n_{i_0+1} \cdots n_{i_0+j_0})}\big).$$

Hence, for each $1 \leq k \leq m$,

$$\begin{aligned} |\tau(a_k) - \mu(a_k)| &\leq |\tau(a_k) - \tau(b_k)| + |\tau(b_k) - \mu(b_k)| + |\mu(b_k) - \mu(a_k)| \\ &\leq 2\|a_k - b_k\| + |\tau_{i_0}(b_k) - \mu_{i_0}(b_k)| \\ &= 2\|a_k - b_k\| + \left| \frac{1}{n_{i_0+1} \cdots n_{i_0+j_0}}\big(\tau^{(1)} + \cdots + \tau^{(n_{i_0+1} \cdots n_{i_0+j_0})}\big)(b_k) - \mu_{i_0}(b_k) \right|, \end{aligned}$$

and applying Equation (3) to this inequality, we see $|\tau(a_k) - \mu(a_k)| < \epsilon$. That is, $\tau \in \mathcal{N}$.

□

References

[1] C. Ivanescu and D. Kučerovský, *Traces and Pedersen ideals of tensor products of nonunital* C*-*algebras*, New York J. Math. **25** (2019), 423–450, ISSN: 1076-9803, MR: 3982248.