NOTES ON PARTIAL FRACTION DECOMPOSITION

Fundamental theorem of algebra. Let n be a natural number, and let p be a polynomial of degree n. Recall that when we say we are "working over the complex numbers," we allow p to have roots that are complex numbers, but when we say we are "working over the reals," we only consider the real roots of p. For example, over the complex numbers, the polynomial $p(z) = z^2 + 1$ clearly has roots i and $-i$, but over the reals it has no roots.

Over the complex numbers, an nth degree polynomial has exactly n roots, including multiplicity, by the fundamental theorem of algebra. As a consequence, when working over the complex numbers, the polynomial $p(z) = z^n + a_{n-1}z^{n-1} \cdots + a_1 z + a_0$ can be written as a product of monomials $p(z) = (z - r_1) \cdots (z - r_n)$, where $r_1, \ldots, r_n$ are the roots of p. Furthermore, letting $r_{l_1}, \ldots, r_{l_k}$ be the distinct roots of p with multiplicities $j_1, \ldots, j_k$ respectively, we can write $p(z) = (z - r_{l_1})^{j_1} \cdots (z - r_{l_k})^{j_k}$ (note $j_1 + \cdots + j_k = n$).

For example, consider the polynomial $p(z) = z^4 - 4z^3 + 5z^2 - 4z + 4$. The roots of p are $2, 2, i, -i$ so that $p(z) = (z-2)(z-2)(z-i)(z+i)$; moreover, since the root 2 appears with multiplicity 2, $p(z) = (z-2)^2(z-i)(z+i)$.

On the other hand, when working over the real numbers, an nth degree polynomial may have less than n roots (in fact, the example in the first paragraph shows it may even have no roots). Consider the previous example $p(x) = x^4 - 4x^3 + 5x^2 - 4x + 4$; over the reals, p has roots $2, 2$. But $(x-2)(x-2) = x^2 - 4x + 4$ is certainly not $p(x)$; this shows that over the reals, we may not be able to write a polynomial as a product of monomials.

In this way, despite complex numbers perhaps seeming difficult to work with, they are actually much nicer than real numbers. By adding this symbol i to our set of possible numbers, we somehow make a more complete set of numbers.

However, while we can't quite write any polynomial over the reals as a product of monomials, it turns out we can write any one as a product of monomials *and quadratics* (this is a corollary of the fundamental theorem of algebra); that is, with $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ having distinct roots $r_1, \ldots, r_k$ of multiplicity $j_1, \ldots, j_k$ respectively, we can write

$$p(x) = (x - r_1)^{j_1} \cdots (x - r_k)^{j_k} (x^2 + b_1 x + c_1)^{l_1} \cdots (x_2 + b_\nu x + c_\nu)^{l_\nu}$$

for some constants $b_1, \ldots, b_\nu, c_1, \ldots, c_\nu$ with $b_s^2 - 4c_s < 0$ for each $1 \leq s \leq \nu$ (note $j_1 + \cdots + j_k + 2(l_1 + \cdots + l_\nu) = n$). For example, with $p(x) = x^4 - 4x^3 + 5x^2 - 4x + 4$ as before, we have $p(x) = (x-2)^2(x^2+1)$.

Implications for partial fraction decomposition. Suppose we are working over the complex numbers, and consider the rational function $p(z)/q(z)$, where $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and $q(z) = z^m + b_{m-1}z^{m-1} + \cdots + b_1 z + b_0$ for $n < m$ natural numbers and $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}$ some real coefficients. We can rewrite $q(z)$ as

$$q(z) = (z - r_1) \cdots (z - r_m) = (z - r_{l_1})^{j_1} \cdots (z - r_{l_k})^{j_k},$$

where $r_1, \ldots, r_m$ are the roots of q and $r_{l_1}, \ldots, r_{l_k}$ are the distinct roots of q with multiplicities $j_1, \ldots, j_k$ respectively. Hence

$$\frac{p(z)}{q(z)} = \frac{p(z)}{(z - r_{l_1})^{j_1} \cdots (z - r_{l_k})^{j_k}},$$

and it turns out that the following equation gives the form of the partial fraction decomposition *over the complex numbers* for $p(z)/q(z)$

$$(1) \quad \frac{p(z)}{q(z)} = \frac{A_{1,1}}{(z - r_{l_1})} + \frac{A_{1,2}}{(z - r_{l_1})^2} + \cdots + \frac{A_{1,j_1}}{(z - r_{l_1})^{j_1}} + \frac{A_{2,1}}{(z - r_{l_2})} + \frac{A_{22}}{(z - r_{l_2})^2} + \cdots + \frac{A_{2,j_2}}{(z - r_{l_2})^{j_2}} + \cdots + \frac{A_{k,1}}{(z - r_{l_k})} + \frac{A_{k,2}}{(z - r_{l_k})^2} + \cdots + \frac{A_{k,j_k}}{(z - r_{l_k})^{j_k}}$$

(this can be proved using some classical results from algebra).

For example if $q(z) = z^4 - 4z^3 + 5z^2 - 4z + 4 = (z - 2)^2(z - i)(z + i)$, then

$$\frac{p(z)}{q(z)} = \frac{A}{(z - 2)} + \frac{B}{(z - 2)^2} + \frac{C}{(z - i)} + \frac{D}{(z + i)}$$

Now suppose we are working over the real numbers, and consider again the polynomials $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $q(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$. The partial fraction decomposition of $p(x)/q(x)$ *over the reals* will in general not be the same as that over the complex numbers since we cannot in general factor the denominator into monomials. However, we can use the result about factoring into monomials and quadratics to our advantage.

Indeed, suppose $q(x)$ factors as

$$q(x) = (x - r_1)^{j_1} \cdots (x - r_k)^{j_k} (x^2 + c_1 x + d_1)^{l_1} \cdots (x_2 + c_\nu x + d_\nu)^{l_\nu}$$

for some constants $c_1, \ldots, c_\nu, d_1, \ldots, d_\nu$ with $c_s^2 - 4d_s < 0$ for each $1 \le s \le \nu$ and $r_1, \ldots, r_k$ the distinct roots of q of multiplicity $j_1, \ldots, j_k$ respectively. Then

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x - r_1)^{j_1} \cdots (x - r_k)^{j_k} (x^2 + c_1 x + d_1)^{l_1} \cdots (x_2 + c_\nu x + d_\nu)^{l_\nu}}$$

and the following equation gives the form of the partial fraction decomposition *over the real numbers* for $p(x)/q(x)$

$$(2) \quad \frac{p(x)}{q(x)} = \frac{A_{1,1}}{(x - r_1)} + \cdots + \frac{A_{1,j_1}}{(x - r_1)^{j_1}} + \cdots + \frac{A_{k,1}}{(x - r_k)} + \cdots + \frac{A_{k,j_k}}{(x - r_k)^{j_k}} + \frac{B_{1,1}x + C_{1,1}}{(x^2 + c_1 x + d_1)} + \cdots + \frac{B_{1,l_1}x + C_{1,l_1}}{(x^2 + c_1 x + d_1)^{l_1}} + \cdots + \frac{B_{\nu,1}x + C_{\nu,1}}{(x^2 + c_\nu x + d_\nu)} + \cdots + \frac{B_{\nu,l_\nu}x + C_{\nu,l_\nu}}{(x^2 + c_\nu x + d_\nu)^{l_\nu}}$$

(again, this follows from results in algebra).

For example, if $q(x) = x^4 - 4x^3 + 5x^2 - 4x + 4 = (x - 2)^2(x^2 + 1)$, then

$$\frac{p(x)}{q(x)} = \frac{A}{(z - 2)} + \frac{B}{(z - 2)^2} + \frac{Cs + D}{(x^2 + 1)}.$$