Let us recall some terminology. Let V be a subset of $\mathbb{R}^m$. We say that V is a (vector) *subspace* of $\mathbb{R}^m$ if

1. $\mathbf{0}$ is contained in V
2. $\mathbf{x}+\mathbf{y}$ is contained in V when $\mathbf{x}$ and $\mathbf{y}$ are contained in V
3. $c\mathbf{x}$ is contained in V when $\mathbf{x}$ is contained in V and c is some scalar (some number in $\mathbb{R}$).

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be another subset of $\mathbb{R}^m$. The *span* of S, denoted by span(S), is the set of all linear combinations of the members of S, ie.

$$\text{span}(S) = \{\text{all vectors of the form } c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \text{ for scalars } c_1, \ldots, c_n\}.$$

The *column space* of an $m \times n$ matrix $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, denoted by Col$(A)$, is span$\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, ie. the span of the set consisting of the vectors defined by the columns of A. The *null space* of A, denoted by Nul(A), is the set of all vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0}$; we showed in class that Nul(A) is actually a subspace of $\mathbb{R}^n$. We say that S *spans* the subspace V if span$(S) = V$. Finally, a *basis* for the subspace V is a set of linearly independent vectors that spans V.

Consider the matrix

$$A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5], \quad \mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 8 \\ 7 \\ 9 \\ 9 \end{bmatrix}, \ \mathbf{a}_4 = \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}, \ \mathbf{a}_5 = \begin{bmatrix} -7 \\ 4 \\ 5 \\ -2 \end{bmatrix}. \tag{1}$$

How does one determine a basis for Col(A)? One can easily come up with sets that either span Col(A) or are linearly independent; indeed, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ spans Col(A) and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is linearly independent. But it is not clear if the former is linearly independent or if the latter spans Col(A).

One way to find a basis for Col(A) is by building one out of the columns of A. Consider the set $S_1 = \{\mathbf{a}_1\}$. Clearly this set is linearly independent; hence, if it spans Col(A) we are done. But notice $\mathbf{a}_2$ is not a linear combination of vectors in S_1 (ie. it is not a scalar multiple of $\mathbf{a}_1$). That is, S_1 is not a basis for Col(A). Now try the set $S_2 = \{\mathbf{a}_1, \mathbf{a}_2\}$. We know it is linearly independent, but does it span Col(A)? Start by checking if $\mathbf{a}_3$ is a linear combination of the members of S_2, ie. if the linear system

$$\begin{bmatrix} 1 & 4 \\ -1 & 2 \\ -2 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 9 \\ 9 \end{bmatrix}$$

has any solutions. The augmented matrix of the system has reduced echelon form

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

as the last column is not a pivot column, solutions exist. We now know $\mathbf{a}_3$ is in span(S_2).

Now check if $\mathbf{a}_4$ is in span(S_2), ie. if the linear system

$$\begin{bmatrix} 1 & 4 \\ -1 & 2 \\ -2 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}$$

has any solutions. The augmented matrix of the system has reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the last column is a pivot column, solutions do not exist, and we see $\mathbf{a}_4$ is not in span(S_2); that is, S_2 is not a basis for Col(A).

But now we know $S_3 = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly independent (why?), and its span contains $\mathbf{a}_3$. Let us check if $\mathbf{a}_5$ is in span(S_3). The augmented matrix of the system

$$\begin{bmatrix} 1 & 4 & -3 \\ -1 & 2 & 3 \\ -2 & 2 & 5 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 5 \\ -2 \end{bmatrix}$$

has reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

That is, $\mathbf{a}_5$ is in span(S_3). We have now shown that S_3 is a basis for Col(A), ie. that S_3 is a linearly independent set that spans Col(A).

This is the most natural process for finding a basis for the column space of a matrix. However, the book points out a much more clever way to do it. In particular, they state that the pivot columns of a matrix form a basis for its column space. Indeed, noticing that A has the following reduced echelon form

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 1 & 2.5 & 0 & -0.5 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2}$$

this fact leads us immediately to what we found above, ie. that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for Col(A).

Why do the pivot columns of a matrix form a basis for its column space? First of all, the pivot columns of a matrix always form a linearly independent set. This is because the pivot columns in the reduced echelon form of the matrix are just standard basis vectors, which we know to be linearly independent, and the linear dependence relations between the columns of a matrix and the columns of any row equivalent matrix are the same.

Indeed, one easily checks that if an $m \times n$ matrix has k pivot columns for $1 \le k \le m$, then these columns are given by the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k$ in $\mathbb{R}^m$. One now also sees why the span of the pivot columns of this arbitrary $m \times n$ matrix equal its column space: since the rows without a pivot correspond to rows of all zeros in the reduced echelon form, a nonpivot column can only have nonzero entries in the first k rows in the reduced echelon form. But clearly every vector with nonzero entries only in the first k rows can be written as a linear combination of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k$.

Consider our matrix A in (1) and its reduced echelon form in (2) for example. It has 3 pivot columns and the pivot columns are exactly the standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in $\mathbb{R}^4$. Moreover, looking at the reduced echelon form, only the first three rows of the nonpivot columns have the possibility of being nonzero. We can thus write these nonpivot columns of the reduced echelon form as a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Again, since row equivalent matrices have the same linear dependencies among their columns, we must have that the nonpivot columns $\mathbf{a}_3$ and $\mathbf{a}_5$ of A can be written as a linear combination of the pivot columns $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_4$.

Related to finding a basis for the column space of a matrix is finding a basis for the null space of a matrix. Let us look at the particular matrix A defined in (1). By (2), solutions to the homogenous problem $A\mathbf{x} = \mathbf{0}$ are given by

$$\mathbf{x} = \begin{bmatrix} 2x_3 - 7x_5 \\ -2.5x_3 + 0.5x_5 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix}; \tag{3}$$

that is,

$$\mathbf{x} = x_3 \begin{bmatrix} 2 \\ -2.5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 0.5 \\ 0 \\ -4 \\ 1 \end{bmatrix}. \tag{4}$$

Hence, writing

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2.5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -7 \\ 0.5 \\ 0 \\ -4 \\ 1 \end{bmatrix},$$

clearly $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathrm{Nul}(A)$. But are $\mathbf{v}_1$ and $\mathbf{v}_2$ linearly independent? In fact they must be since, by Equation (3), the only way for $\mathbf{x} = \mathbf{0}$ is for x_3 and x_5 to be 0; looking at Equation 4, this is equivalent to saying the only coefficients satisfying the equation $\mathbf{0} = x_3\mathbf{v}_1 + x_5\mathbf{v}_2$ are $x_3 = x_5 = 0$.

One can use this same process to determine a basis for $\mathrm{Nul}(A)$ for any matrix A.