CORRECTIONAL GUIDE

Written Exam for the B.Sc. or M.Sc. in Economics winter 2013-14

Operations Research

Elective Course

Friday, January 17th

(3-hour open book exam)

The language used in your exam paper must be English or Danish.

This exam question consists of xx pages in total

Part 1 – Linear programming

Q1.1: The model P is an LP model. What characterizes an LP model?

An LP model has an objective function which must be maximized or minimized and which is linear in the decision variables. Constraints can be equality and inequality constraints and are also linear in the decision variables.

Q1.2: Is the presented solution optimal? If it is not optimal, then please continue the Simplex algorithm until optimality is reached.

It is not optimal since the reduced cost of x_3 is negative (in this maximization problem the optimality criterion is non-negative reduced costs). We therefore have to perform a Simplex iteration to bring x_3 into basis. Ratio test shows that this has to happen in the first row:

z	x1	x2	x3	s1	s2	s3	RHS
0	0	0	1	1/2	0	-1/2	1/2
0	1	0	0	0	1/2	-1/2	3/2
0	0	1	0	-1/4	-1/4	1	5/2
1	0	0	0	1/2	1/2	1	16

Q1.3: Set up the dual model to model P. Call it D. What is the optimal solution to model D?

$$\begin{aligned}
&\text{Min } w = 8y_1 + 10y_2 + 7y_3 \\
&\text{s.t.} \quad 1y_1 + 3y_2 + 1y_3 \geq 3 \\
&\qquad\ \ 2y_1 + 2y_2 + 2y_3 \geq 4 \\
&\qquad\ \ 3y_1 + 1y_2 + 1y_3 \geq 3 \\
&\qquad\ \ y_i \geq 0 \qquad (i = 1, 2, 3)
\end{aligned}$$

The optimal solution is $\mathbf{y} = (½, ½, 1)$. This is seen as the reduced costs slack variables in the optimal Simplex tableau for P.

Q1.4: Without solving this new model P2, determine whether the solution for P (with x_4=0) is still optimal. Use the results from Question 1.2 and 1.3

Adding a new decision variable in P (making it P2) corresponds to adding a new constraint in D (making it D2). If the optimal solution from D is still feasible in D2 then the solution from P will still be optimal in P2 (Winston, 4th ed. p. 323).
The new constraint in D2 is $1y_1 + 1y_2 + 2y_3 \geq 2$
Inserting $\mathbf{y} = (½, ½, 1)$ we see that the constraint is satisfied and conclude that the solution from P (with x_4=0) is optimal in P2.

Part 2 – Assignment and Transportation problem

Q2.1: Find a minimum cost assignment in AP
We use the Hungarian method to solve the assignment problem with the following cost matrix:

1	1	2	2	2	4
1	1	2	2	2	4
1	1	2	2	2	4
4	4	1	1	1	2
4	4	1	1	1	2
4	4	1	1	1	2

Subtract lowest from each column:

~~0~~	~~0~~	1	1	1	2
~~0~~	~~0~~	1	1	1	2
~~0~~	~~0~~	1	1	1	2
~~3~~	**~~3~~**	~~0~~	~~0~~	~~0~~	~~0~~
~~3~~	**~~3~~**	~~0~~	~~0~~	~~0~~	~~0~~
~~3~~	**~~3~~**	~~0~~	~~0~~	~~0~~	~~0~~

Since no zero-assignment can be found, we cover all zeros with less than 6 lines in the above. The smallest non-covered cost is 1 so we subtract 1 to all non-covered cells and add 1 to all double covered cells:

0	0	0	0	0	1
0	**0**	0	0	0	1
0	0	**0**	0	0	1
4	4	0	**0**	0	0
4	4	0	0	**0**	0
4	4	0	0	0	**0**

Now several zero-assignments can be found. For instance in the diagonal. Looking at the original cost matrix e find the minimal cost to be 8.

Q2.2: Describe how the TP above can be transformed into the AP above.

The first column describes the demand of the first "agent", which demands 2 units. We split this column in two (since it has the demand of 2) columns, each with a demand of 1, and use the same costs. We then get the following transportation problem:

					supply:
	1	1	2	4	3
	4	4	1	2	3
demand:	1	1	3	1	

This transportation problem is equivalent to the original TP.
By continuing this process with all columns and then with all rows, we get demands and supplies of 1, and the cost matrix of the AP.

Q2.3: Find a feasible solution to the TP by using the minimum cost heuristic.
Does it (in this case) find an optimal solution?

The minimum cost heuristic may end up in two different ways, where the student will locate one of them:

				supply:
	1 X_{11}=2	2 X_{12}=0	4 X_{13}=1	3→ 1→ 1→ X
	4	1 X_{22}=3	2	3→ X
demand:	2 → X	3→ 0→ X	1→ X	

The other could be:

				supply:
	1 X_{11}=2	2	4 X_{13}=1	3→ 1→ X
	4	1 X_{22}=3	2 X_{23}=0	3→ 0→ X
demand:	2 → X	3→ X	1→ X	

The beginnings of the different paths are highlighted above.

In the first case the weight is 9 and in the second case it's also 9.
We know the optimal solution since we solved the AP and got a value of 8.
In either case, the optimal solution is NOT reached.

The student is not asked for an optimal TP solution, but that could be the following:

				supply:
	1 X_{11}=2	2 X_{13}=1	4	3
	4	1 X_{22}=2	2 X_{23}=1	3
demand:	2	3	1	

Part 3 – Integer programming

Q3.1: Find the additional constraint that results from the Cutting Plane Algorithm (do not solve the resulting model)

We notice that all the coefficients in **A** and in **b** (that is the technological coefficients and the right-hand-sides) are integers. Therefore the Cutting Plane Algorithm can be applied.
We next notice that only one integer constraint is violated, namely the one on x_1, which takes the value of 4.5. We therefore extract this constraint from the tableau:

$$x_1 + 3/8\ s_1 - 1/16\ s_2 = 4.5$$

We rewrite this so we have the integer parts on the left, and the fractional parts of the right:

$$x_1 - 1\ s_2 - 4 = 0.5 - 3/8\ s_1 - 15/16\ s_2$$

We know that the left-hand side is Integer and we know that the right-hand side is strictly less than 1. Therefore it must also be less than or equal to 0. This gives us the new constraint:

$$0.5 - 3/8\ s_1 - 15/16\ s_2 \leq 0$$

or: $\boxed{-3/8\ s_1 - 15/16\ s_2 \leq -0.5}$

Q3.2: Find the additional constraints that results from the Branch and Bound Algorithm (do not solve the resulting models)

As before, we notice that only one of the variables is non-integer so there is no choice in branching variable. x_1 takes on the value of 4.5 and in MIP Branch & Bound we will then create two sub-problems by using the fact that either $\boxed{x_1 \leq 4}$ or $\boxed{x_1 \geq 5}$.

Q3.3: The LP relaxed problem has an optimal objective function value of 83.5. What do we know about the optimal objective function value of the IP problem?

Since this was the value of the relaxed problem we know, that the non-relaxed problem cannot be better. So 83.5 is an upper bound on the optimal value of the IP problem.

In fact, since all cost coefficients in the objective function are integers and the decision variables also have to be integer, we know that all integer-feasible solutions must have integer objective function values. And since this integer value cannot exceed 83.5 it can also not exceed 83.