Written Exam for M.Sc. in Economics 2010-I

Advanced Microeconomics

23. February 2010

Master course

Answers

1.1

The problem of consumer i is

$$\begin{aligned} \max_{x} \quad & \ln x^1 + \ln x^2 \\ \text{s.t.} \quad & p_1 x^1 + p_2 x^2 \leq w_i \end{aligned}$$

where $w_i = 10p_i + 0.5\pi(p_1, p_2)$ and $\pi(p_1, p_2)$ is the profit of the firm. The solution is $x^1 = (0.5w_i/p_1, 0.5w_i/p_2)$.

The problem of the firm is

$$\begin{aligned} \max_{y} \quad & p_1 y^1 + p_2 y^2 \\ \text{s.t.} \quad & y \in Y. \end{aligned}$$

1.2

The problem of the firm can be restated as

$$\max_{y^1} p_1 y^1 + p_2 \frac{y^1}{y^1 - 1}.$$

The first-order condition is

$$p_1 - p_2 \frac{1}{y^1 - 1}^{2} \;=\; 0.$$

The solution is

$$y^1 = 1 - \left(\frac{p_2}{p_1}\right)^{1/2} \text{ and } y^2 = 1 - \left(\frac{p_1}{p_2}\right)^{1/2}$$

and the profit is

$$\pi(p_1, p_2) = ((p_1)^{1/2} - (p_2)^{1/2})^2.$$

1.3

Let $(\bar{p}, \bar{x}_1, \bar{x}_2, \bar{y})$ be defined by $\bar{p} = (1,1)$, $\bar{x}_2 = \bar{x}_1 = (5,5)$ and $\bar{y} = (0,0)$. Then it is a Walrasian equilibrium.

2.1

(UMP) is

$$\begin{aligned} \max_x \quad & u_i(x) \\ \text{s.t.} \quad & p \cdot x \leq p \cdot \omega_i + \sum_j \theta_{ij} \pi_i(p) \text{ and } x \in X_i \end{aligned}$$

2.2

The set X_i is closed and the set $\{x \in \mathbb{R}^L | p \cdot x \leq w_i\}$ is closed. Their intersection is bounded from below by 0 and from above by $(w_i/p_1, \ldots, w_i/p_L)$, so their intersection is compact. The utility function is continuous by assumption. Therefore (UMP) has a solution.

2.3

Let $x, v \in X_i$ with $v^\ell > x^\ell$ for all ℓ, then $u_i(v) - u_i(x) = b \cdot (v - x) > 0$ because $v - x \in R^L_{++}$ and $b \in \mathbb{R}^L_+$ with $v - x, b \neq 0$. Therefore the preference relation is monotone.

Let $x, x', x'' \in X_i$ with $u_i(x) = b \cdot x, u_i(x') = b \cdot x' \geq u_i(x'')$, then $u_i((1 - \lambda)x + \lambda x') = b \cdot ((1 - \lambda)x + \lambda x') \geq b \cdot x'' = u_i(x'')$. Therefore the preference relation is convex.

2.4

Definition 1 *A Walrasian equilibrium is a price vector, a consumption bundle and a production plan* $(\bar{p}, \bar{x}, \bar{y})$ *such that*

- $\bar{x}$ *is a solution to*
$$\max_{x} \quad u(x)$$
$$s.t. \quad \bar{p} \cdot x \leq \bar{p} \cdot \omega + \bar{p} \cdot \bar{y} \text{ and } x \in X.$$
- $\bar{y}$ *is a solution to*
$$\max_{y} \quad \bar{p} \cdot y$$
$$s.t. \quad y \in Y.$$
- $\bar{x}_1 + \bar{x}_2 = \omega_1 + \omega_2$.

The illustration is the usual diagram.

2.5

Definition 2 *A Pareto optimal allocation is feasible allocation* $(\tilde{x}, \tilde{y})$ *such that there is no other feasible allocation* (x, y) *with* $u_i(x_i) \geq u_i(\tilde{x}_i)$ *for all* i *and "*$>$*" for at least one* i.

In order to prove that if $(\bar{p}, (\bar{x}, \bar{y}))$ is a Walrasian equilibrium, then $(\bar{x}, \bar{y})$ is a Pareto optimal allocation suppose that $(\bar{x}, \bar{y})$ is not Pareto optimal. Then there exists another feasible allocation (x, y) such that $u_i(x_i) \geq u_i(\bar{x}_i)$ for all i with at least one "$>$". Firstly $u_i(x_i) > u_i(\bar{x}_i)$ implies that $\bar{p} \cdot x_i > \bar{p} \cdot \bar{x}_i$ because $\bar{x}_i$ is a solution to (UMP) given $\bar{p}$. Secondly $u_i(x_i) \geq u_i(x_i)$ implies that $\bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i$ because if $\bar{p} \cdot x_i < \bar{p} \cdot \bar{x}_i$, then there exists x_i' such that $u_i(x_i') > u_i(\bar{x}_i)$ and $\bar{p} \cdot x_i' \leq \bar{p} \cdot \bar{x}_i$ because the utility function represents a monotonic preference relation and this contradicts that $\bar{x}_i$ is a solution to (UMP) given $\bar{p}$. Thirdly $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{y}_j$ for all j because $\bar{y}_j$ is a solution to (PMP) given $\bar{p}$. Summing up

$$\bar{p} \cdot \sum_i x_i > \bar{p} \cdot \sum_i \bar{x}_i = \bar{p} \cdot (\sum_i \omega_i + \sum_j \bar{y}_j) \geq \bar{p} \cdot (\sum_i \omega_i + \sum_j \bar{y}_j).$$

This contradicts that (x, y) is feasible. Hence $(\bar{x}, \bar{y})$ is Pareto optimal.

2.6

For $y \in Y$ the profit of the firm is $p \cdot y$. If y is a solution to (PMP), then $a^1 y^1 = \ldots = a^{L-1} y^{L-1}$ and $y^L = -a^1 y^1$ and the profit of y is $(-p_1/a^1 + \ldots - p_{L-1}/a^{L-1} + p_L) y^L$. Therefore if $p_L > p_1/a^1 + \ldots + p_{L-1}/a^{L-1}$, then there is no solution to (PMP).

3.1

Definition 3 *A strongly Pareto optimal allocation $(x_t)_{t\in\mathbb{Z}}$ is an allocation such that there is no other allocation $(x'_t)_{t\in\mathbb{Z}}$ with $u_t(x'_t) \geq u_t(x'_t)$ for all t and "$>$" for at least one t.*

There are two other forms of Pareto optimality: ordinarily Pareto optimality, where consumption from some date and backward is not changed; and, weak Pareto optimality, where consumption from date and backward and from some other date and forward is not changed.

If consumers at some date consider whether the outcome is efficient or not, then strong Pareto optimality is not relevant. Depending on whether arrangements into the infinite future are possible or not one of the two other notions of optimality is relevant.

3.2

Definition 4 *An* **equilibrium with spot markets** *is a price system and an allocation $((\bar{p}_t)_{t\in\mathbb{Z}}, (\bar{x}_t)_{t\in\mathbb{Z}})$ such that there exist $(\bar{m}_t)_{t\in\mathbb{Z}}$ and M such that*

- $(\bar{x}_t, \bar{m}_t)$ *is a solution to*

$$\max_{(x,m)} \quad u_t(x)$$
$$s.t. \quad \begin{cases} \bar{p}_t x^y + m \leq \bar{p}_t \omega_t^y \\ \bar{p}_{t+1} x^o \leq \bar{p}_{t+1} \omega_t^o + m \\ x \in X, \ m \in \mathbb{R} \end{cases}$$

- $\bar{x}_t^y + \bar{x}_{t-1}^o = \omega_t^y + \omega_{t-1}^o$ *and* $\bar{m}_t = M$ *for all* t.

Definition 5 *A* **Walrasian equilibrium** *is a price system and an allocation* $((\bar{p}_t)_{t\in\mathbb{Z}}, (\bar{x}_t)_{t\in\mathbb{Z}})$ *such that*

- $(\bar{x}_t)$ *is a solution to*

$$\max_{(x,m)} \quad u_t(x)$$
$$s.t. \quad \begin{cases} \bar{p}_t x^y + \bar{p}_{t+1} x^o \leq \bar{p}_t \omega_t^y + \bar{p}_{t+1} \omega_t^o \\ x \in X \end{cases}$$

- $\bar{x}_t^y + \bar{x}_{t-1}^o = \omega_t^y + \omega_{t-1}^o$ *for all* t.

3.3

The budget constraints in the definition of equilibrium with spot markets can be rewritten as

$$\bar{p}_t x^y + \bar{p}_{t+1} x^o \leq \bar{p}_t \omega_t^y + \bar{p}_{t+1} \omega_t^o$$
$$\bar{p}_{t+1}(x^o - \omega_t^o) \leq m \leq \bar{p}_t(\omega_t^y - x^y)$$

Therefore if (x, m) is a solution to the consumer problem in case of spot markets, then x is a solution to consumer problem is case of forward markets. Hence equilibria with spots markets are also Walrasian equilibria.

3.4

The allocation $(x'_t)_{t\in\mathbb{Z}}$ where $x'_t = (0, 15)$ for all t is preferred by all consumers.

3.5

Consider the allocation $(x_t)_{t\in\mathbb{Z}}$ where $x_t = (0, 15)$ for all t and a reallocation $(a_t)_{t\in\mathbb{Z}}$ with $u_t(x^y_t - a_t, x^o_t + a_{t+1}) \geq u_t(x^y_t, x^o_t)$. If $a_t < 0$, then $a_{t-k} \leq 2^k a_t$ for all $k \geq 0$ because the marginal rate of substitution is constant and equal to $1/2$, so there exists $k \geq 0$ such that $a_{t-k} < -15$. If $a_t > 0$ then it is not a reallocation because $x_t + (-a_t, a_{t+1}) \in X = \mathbb{R}^2_+$. Hence there is no reallocation but the trivial reallocation where $a_t = 0$ for all t with $u_t(x^y_t - a_t, x^o_t + a_{t+1}) \geq u_t(x^y_t, x^o_t)$ for all t.

3.6

Let $((\bar{p}_t)_{t\in\mathbb{Z}}, (\bar{x}_t)_{t\in\mathbb{Z}})$ be defined by $\bar{p}_t = 1$ and $\bar{x}_t = (0, 15)$ for all t. Then it is an equilibrium because $\bar{x}_t$ is the solution to the problem of the consumer and market clears with savings and the stock of money being equal to four.