Problem 1. Consider a stock in a binary two-period model with value 100 USD at time $t = 0$. We assume that the value in each node increases by 15 percent in the up-state and decreases with 5 percent in the down-state. The risk free interest rate $r = 5$ percent.

(i) Calculate the risk neutral probability distribution at maturity.

(ii) Calculate the arbitrage free price c of a European call option written on this stock with strike price $K = 100$ USD.

Answer:

The risk neutral probability distribution at maturity is $\big(1/4, 1/2, 1/4\big)$.

The call price $c \simeq 12.08$.

Problem 2. The police is observing speeding cars on a highway. We assume that the speeding cars arrive independently and at random to the checkpoint. The average number of speeding violations is 4 during each thirty minutes period.

(i) What is the distribution of the number of speeding cars observed during a thirty minutes period?

(ii) What is the probability that exactly 3 speeding cars are observed during a thirty minutes period?

(iii) Let Y_1 denote the stochastic variable that measures the time in hours from the moment that the observations at the checkpoint begin until the first (or more) speeding violations are observed. What is the distribution of Y_1.

Answer:

(i) The Poisson distribution with parameter $\lambda = 4$.

(ii) Let X denote the stochastic variable that counts the number of speeding tickets in a two hour period. Then

$$P[X = 3] \sim 0.195.$$

(iii) Y_1 is exponentially distributed with parameter $\mu = 8$.

Problem 3. Consider the Brownian motion $(B_t)_{t\geq 0}$ with its natural filtration

$$\mathcal{F}_t = \sigma\big(B_s \mid 0 \leq s \leq t\big).$$

(i) Show that $B_t - B_s$ is independent of $\mathcal{F}_s$ for $0 \leq s \leq t$.

(ii) Calculate the conditional expectation

$$E\big[(B_t - B_s)^2 \mid \mathcal{F}_s\big]$$

for $0 \leq s \leq t$.

Answer:

(i) The time series $(B_0\,, B_r\,, B_s\,, B_t)$ has independent increments for $0 \leq r \leq s$. It follows that $B_t - B_s$ and B_r are independent for $0 \leq r \leq s$. Since $\mathcal{F}_s$ is generated by the stochastic variables B_r for $0 \leq r \leq s$, it follows that $B_t - B_s$ is independent of $\mathcal{F}_s$.

(ii) Since $B_t - B_s$ is independent of $\mathcal{F}_s$ it follows that also $(B_t - B_s)^2$ is independent of $\mathcal{F}_s$. Therefore,

$$E\big[(B_t - B_s)^2 \mid \mathcal{F}_s\big] = E\big[(B_t - B_s)^2\big] = E\big[B_{t-s}^2\big] = t - s,$$

where we used that Brownian motion has stationary increments.

Problem 4. A traded CO_2-permit has an expiry date $T > 0$ at which the value of the permit drops to zero. The value Y_t of the permit may therefore conveniently be modeled by setting

$$Y_t = \exp X_t\,,$$

where X_t is an Ito process given on differential form

$$dX_t = -\frac{b}{T-t}\,dt + \sigma\,dB_t\,,$$

B_t is the Brownian motion, and b, σ are positive constants.

(i) Integrate dX_t to obtain a formula for X_t.

(ii) Calculate the mean $E[Y_t]$.

(iii) Show that $E[Y_t] \to 0$ for $t \to T$.

(iv) Use Ito's lemma to write the process Y_t on differential form.

Answer:

(i) The expression

$$X_t = X_0 + b\log\left(\frac{T-t}{T}\right) + \sigma B_t$$

is obtained by a straight-forward calculation using the rules for how to integrate an Ito process.

(ii) The expected value is calculated to be

$$E[Y_t] = Y_0\left(\frac{T-t}{T}\right)^b e^{\sigma^2 t/2}.$$

(iii) Thus $E[Y_t] \to 0$ for $t \to T$.

(iv) The differential form of Y_t is given by

$$dY_t = \left(-\frac{b}{T-t} + \frac{\sigma^2}{2}\right) Y_t\, dt + \sigma Y_t\, dB_t$$

for $0 \leq t < T$.