1. Consider the following Ramsey model. There is no technological growth, population grows at rate n, capital does not depreciate. As usual, small letters denote per capita variables, large letters variables in levels. Firms operate in perfectly competitive markets and produce the final good using the production technology

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}$$

where $K(t)$ is physical capital, $L(t)$ is labor and $\alpha \in (0,1)$. Firms rent both inputs from households. The infinitely-lived, representative household derives utility from consumption $c(t)$ and discounts the future at rate ρ. His objective is to maximize the discounted present value of lifetime consumption:

$$\int_t^{\infty} e^{-(\rho-n)t} u(c(t)) dt$$

Assume

$$u(c(t)) = \frac{c(t)^{1-\sigma}}{1-\sigma}$$

The government in this economy levies a proportional capital income tax on households, and transfers the revenues lump sum back to them. It runs a balanced budget each period. Households thus face the following budget constraint:

$$\dot{a}(t) = w(t) + (1-\tau) r(t) a(t) - n a(t) - c(t) + s(t)$$

where $a(t)$ denote the household asset holdings, $w(t)$ is labor income, $r(t)$ is the interest rate earned on assets, τ is the capital income tax rate and $s(t)$ the lump-sum transfer. $\dot{a}(t)$ denotes the time derivative of $a(t)$. The government budget constraint is given by

$$s(t) = \tau r(t) a(t)$$

Assume that initial assets in the economy are given, and that the no Ponzi condition holds.

(a) Set up and solve the household problem using optimal control theory. (What are the state, control, and co-state variables? Write down the Hamiltonian, derive the maximum principle conditions.)

Solution: The household problem is

$$\max \int_t^{\infty} e^{-(\rho-n)t} u(c(t)) dt$$

subject to

$$\dot{a}(t) = w(t) + (1-\tau) r(t) a(t) - n a(t) - c(t) + T(t)$$
$$a(0) \text{ given}$$
$$\lim_{t\to\infty} a(t) \exp(-\int_0^t (r(s) - n) ds \geq 0$$

The Hamiltonian is

$$\mathcal{H} = e^{-\rho t} u(c(t)) + \lambda(t) \left[w(t) + (1-\tau) r(t) a(t) - n a(t) - c(t) + s(t) \right]$$

The state is $a(t)$, the control is $c(t)$, the co-state is $\lambda(t)$. The Maximum Principle conditions are

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial c(t)} &= 0 \\
\frac{\partial \mathcal{H}}{\partial \lambda(t)} &= \dot{a}(t) \\
\frac{\partial \mathcal{H}}{\partial a(t)} &= -\dot{\lambda}(t)
\end{aligned}$$

which when evaluating the partial gives

$$\begin{aligned}
\lambda(t) &= e^{-\rho t} u'(c(t)) \\
\dot{a}(t) &= w(t) + (1-\tau) r(t) a(t) - n a(t) - c(t) \\
\dot{\lambda}(t) &= -(1-\tau) r(t) \lambda(t)
\end{aligned}$$

plus the transversality condition

$$\lim_{t\to\infty} \lambda(t) a(t) = 0$$

(b) Derive the Euler condition. Explain in words the what the condition states and why it has to hold at the optimum. How does the capital income tax affect it? It is not sufficient to describe the equations, explain the economics of the effects of the capital tax.

Solution: There are several ways of deriving the Euler equation, this is just one. Start with the first MP condition:

$$\begin{aligned}
e^{-\rho t} u'(c(t)) &= \lambda(t) \\
-\rho t + \log u'(c(t)) &= \log \lambda(t) \\
-\rho + \frac{\mathrm{d} \log u'(c(t))}{\mathrm{d} u'(c(t))} \frac{\partial u'(c(t))}{\partial c(t)} \frac{\partial c(t)}{\partial t} &= \frac{\mathrm{d} \log \lambda(t)}{\mathrm{d} \lambda(t)} \frac{\partial \lambda(t)}{\partial t} \\
-\rho + \frac{1}{u'(c(t))} u''(c(t)) \dot{c} &= \frac{1}{\lambda(t)} \dot{\lambda} \\
\frac{\dot{\lambda}}{\lambda(t)} &= \frac{u''}{u'} \dot{c} - \rho
\end{aligned}$$

Use this equation in the third MP condition:

$$\begin{aligned}
-\frac{\dot{\lambda}}{\lambda(t)} &= (1-\tau) r(t) \\
-\frac{u''}{u'} \dot{c} - \rho &= (1-\tau) r(t) \\
\frac{\dot{c}}{c(t)} &= -\frac{u''}{u' c} ((1-\tau) r(t) - \rho) \\
\frac{\dot{c}}{c(t)} &= \frac{1}{\sigma} ((1-\tau) r(t) - \rho)
\end{aligned}$$

where in the last line we use the functional form for utility that is given in the question. This is the Euler equation.

It describes optimal consumption growth over time. Agents prefer to postpone consumption and thus for consumption to grow over time if the net market return to saving exceeds their subjective discount rate. The extent to which consumption growth responds to changes in the market relatie to the subjective discount is governed by the intertemporal elasticity of substitution σ. Since the capital income tax reduces their return to saving, everything else equal the tax reduces incentives to save, encourages agents to bring consumption forward and lowers optimal consumption growth.

(c) Derive the equilibrium $\dot{c} = 0$ and $\dot{k} = 0$ curves and plot them in the phase diagram. Explain the diagram. How are the curves different from an economy without a capital tax? Again, it is not sufficient to describe the equations, explain the economics.

Solution: The solution to the firm problem implies

$$r(t) = f'(k(t)) = \alpha k(t)^{\alpha-1}$$

and

$$w(t) = (1-\alpha)k(t)^{\alpha}$$

Thus in equilibrium, the Euler equation becomes

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left((1-\tau)f'(k(t)) - \rho\right)$$

We can derive the law of motion for capital by substituting $\dot{k} = \dot{a}$ and the equilibrium prices into the household budget constraint. This yields

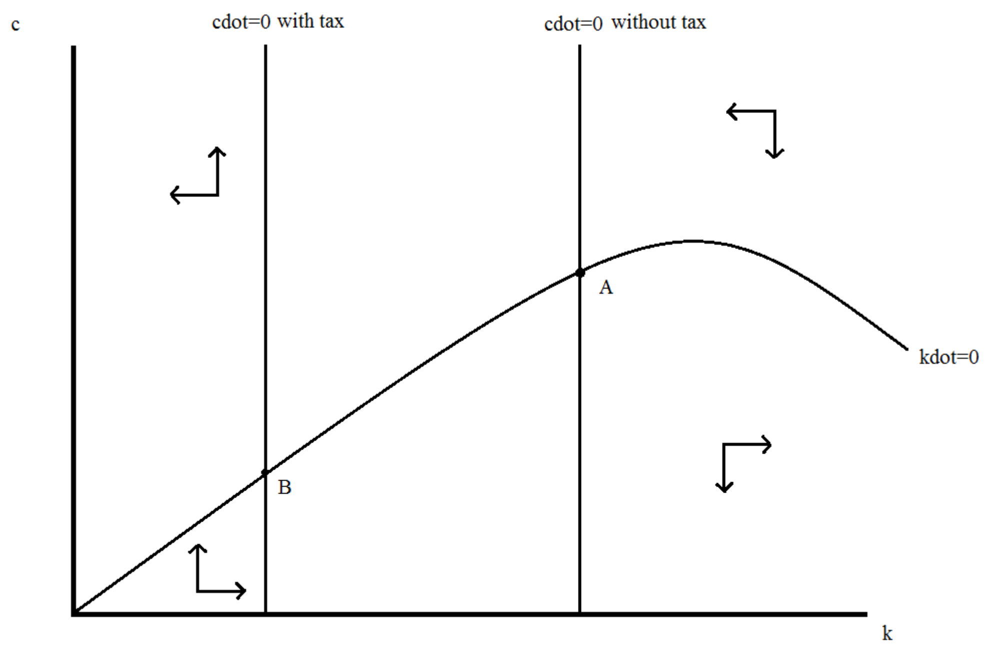
$$\dot{k}(t) = f(k(t)) - nk(t) - c(t)$$

Note that this is not affected by the tax since the tax is purely redistributive. The equilibrium $\dot{c} = 0$ and $\dot{k} = 0$ curves are therefore given by, respectively

$$(1-\tau)f'(k(t)) = \rho$$

and

$$c(t) = f(k(t)) - nk(t)$$



The $\dot{c}$ curve traces out the combinations of consumption levels and capital stocks that are consistent with zero consumption growth. For points to the right of the curve, for example, where capital is higher, the marginal product of capital and thus the interest rate are lower implying lower (negative) consumption growth. The reverse is true for points to the right of the curve. The $\dot{k}$ curve traces out combinations of consumption and capital that are consistent with no change in the capital stock. For points above the curve, consumption is relatively high and thus agents need to run down the capital stock to sustain this level of consumption - the capital stock falls ($\dot{k} < 0$). Conversely, consumption is so low relative to output at points below the curve that capital increases. These dynamics are indicated by the arrows in the diagram. A higher capital tax rate lowers the level of capital that is consistent with zero consumption growth since it lowers the interest rate. In terms of the phase diagram, the $\dot{c}$ curve shifts left.

(d) Suppose the economy is in steady state with $\tau = 0$. In period t_0, the government unexpectedly and permanently raises the capital income tax rate to $\tau > 0$. What happens in the economy on impact, in the transition, in the new steady state, and why? Plot the changes in the phase diagram.

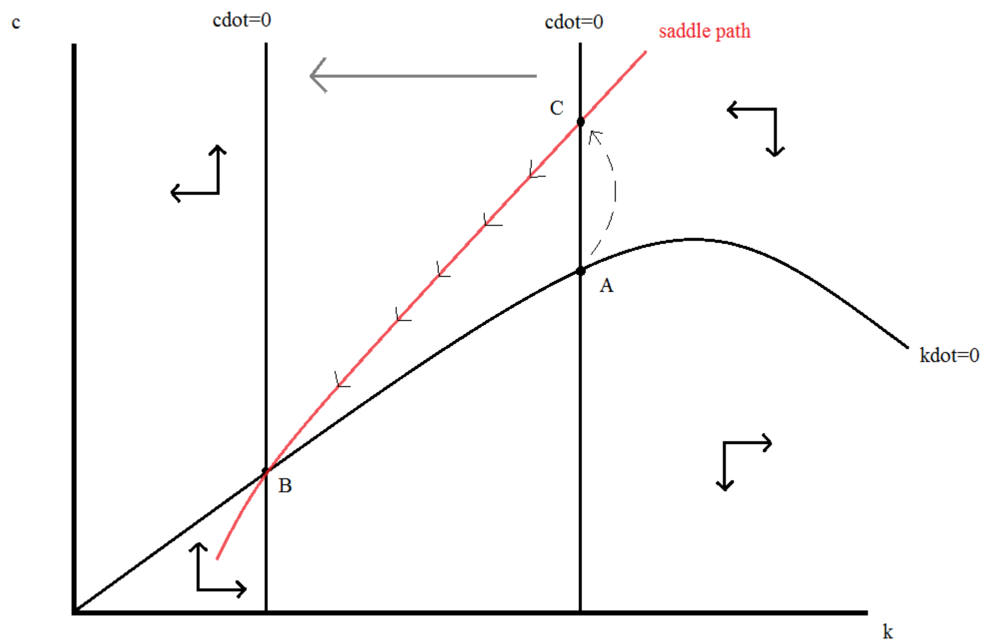
Solution: The higher tax implies a fall in the effective return on capital for households on impact. Once the tax is implemented, the return on capital is too low to be consistent with the previous constant level of consumption. Households have less of an incentive to save.

Without any jumps in consumption this implies negative consumption growth by the Euler equation (at point A in the diagram). This in turn would imply rising capital from the law of motion. Those two combined put the economy on a diverging trajectory (southwest in the phase diagram). In order for the economy to converge to its new steady state, we thus require an upward jump in consumption at the time the capital tax rate is raised - a jump that takes the economy precisely onto the new saddle path (C in the diagram).

In the transition, households dissave and slowly lower the level of consumption towards the new saddle path, consistent with the consumption smoothing behavior implied by the Euler equation. As capital falls, its return rises.

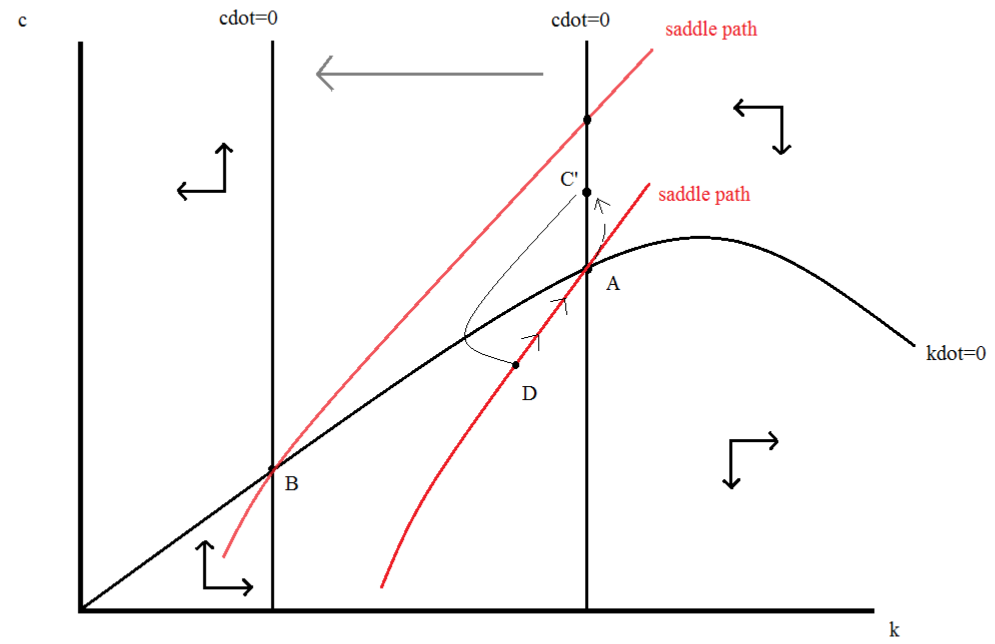
In the new steady state at point B, the levels of capital and consumption must be lower since the effective rate of return is lower than without taxes. The effective after tax interest rate is the same as in the previous steady state without taxes.

Intuitively, the tax hike means it is optimal for households to go on a consumption binge, since the returns to saving just fell, but in the long run to save less, to return to an effective interest rate that is consistent with constant consumption, as implied by household preferences.



(e) Suppose the economy is in steady state with $\tau = 0$. In period t_0, the government unexpectedly and *temporarily* raises the capital income tax rate to $\tau > 0$ until t_1. From t_1 onwards, $\tau = 0$ again. What happens in the economy on impact, in the transition, in the new steady state, and why? Plot the changes in the phase diagram. Compare to the permanent shock case from the previous part.

Solution: Households temporarily face incentives to dissave, but in the long run return to the same level of capital as before (point A). On impact, they thus increase consumption just like in the previous part, but not by as much since the shock is temporary (jump to a point like C). In the transition, consumption falls, and so does capital. Once the level of consumption is sufficiently low, capital starts rising (we cross below the $\dot{k} = 0$ curve). At t_1, the economy hits the new=old saddle path (point D). The interest rate increases, and households once again start saving. Consumption growth is positive until we return to the steady state.



2. The representative agent i maximizes utility

$$U_i = C_i - \frac{1}{\beta} L_i^{\beta}, \qquad \beta > 0$$

subject to the budget constraint

$$PC_i = P_i Y_i$$

where C_i is consumption, L_i labor supply, P the aggregate price level, P_i the price of good i and Y_i the quantity of good i. The production function equals

$$Y_i = L_i^{\alpha}, \quad 0 < \alpha < 1$$

There is monopolistic competition in the goods market. The demand for good i is

$$Y_i = \left(\frac{P_i}{P}\right)^{-\eta} Y$$

where Y denotes aggregate output and $\eta > 1$ is the elasticity of substitution in the demand for differentiated goods. The aggregate demand equation is

$$Y = \frac{M}{P}$$

where M denotes money supply. Agents have rational expectations. Employ the following notation: $x \equiv \ln X$.

(a) After linearizing the first order condition from the utility maximization problem, derive the optimal production y_i^* as a function of the relative price $p_i - p$. Show that the equilibrium (log) aggregate price level equals

$$p = \mu + m$$

where μ is a constant to be found. Show analytically that y increases in η and provide adequate interpretation to this result.

Solution: Substitute the budget constraint, the technology constraint and the demand function into the utility function, so as to get:

$$U_i = \left(\frac{Y_i}{Y}\right)^{-\frac{1}{\eta}} Y_i - \frac{1}{\beta} Y_i^{\frac{\beta}{\alpha}}$$

Maximizing w.r.t. Y_i:

$$\frac{\partial U_i}{\partial Y_i} = 0 \Rightarrow -\frac{1}{\eta}\left(\frac{1}{Y}\right)^{-\frac{1}{\eta}} (Y_i)^{-\frac{1}{\eta}-1} Y_i + \left(\frac{1}{Y}\right)^{-\frac{1}{\eta}} (Y_i)^{-\frac{1}{\eta}} - \frac{1}{\alpha} Y_i^{\frac{\beta-\alpha}{\alpha}} = 0$$

After some manipulation we obtain

$$\left(1 - \frac{1}{\eta}\right)\left(\frac{Y_i}{Y}\right)^{-\frac{1}{\eta}} = \frac{1}{\alpha} Y_i^{\frac{\beta-\alpha}{\alpha}}$$

Which translates into

$$\left(1 - \frac{1}{\eta}\right)\frac{P_i}{P} = \frac{1}{\alpha} Y_i^{\frac{\beta-\alpha}{\alpha}}$$

Taking logs and rearranging to obtain y_i^*:

$$y_i^* = \frac{\alpha}{\beta-\alpha}(p_i - p) + \frac{\alpha}{\beta-\alpha}\left[\ln\left(1 - \frac{1}{\eta}\right) - \ln\left(\frac{1}{\alpha}\right)\right]$$

We aggregate to find y:

$$y = \frac{\alpha}{\beta-\alpha}\ln\left(\alpha\frac{\eta-1}{\eta}\right)$$

Since $y = m - p$:

$$p = m - y = m - \frac{\alpha}{\beta-\alpha}\ln\left(\alpha\frac{\eta-1}{\eta}\right)$$

Therefore

$$\mu = -\frac{\alpha}{\beta-\alpha}\ln\left(\alpha\frac{\eta-1}{\eta}\right)$$

We can compute the following derivative

$$\frac{\partial y}{\partial \eta} = \frac{\alpha}{\beta-\alpha}\frac{1}{\eta(\eta-1)} > 0 \text{ as long as } \beta > \alpha \text{ and } \eta > 1$$

which are imposed by assumption. Interpretation: as the degree of substitutability among the goods traded in the monopolistically competitive market increases, the deadweight loss due to imperfect competition drops, reflecting into higher equilibrium output.

(b) Suppose now that p_i is fixed for 3 periods and that price-setting is staggered, such that 1/3 of the prices are set in period t at the level x_t, 1/3 were set in period $t-1$ at the level x_{t-1}, while a remaining 1/3 were set in $t-2$ at the level x_{t-2}. Thus, the aggregate price level equals

$$p_t = \frac{1}{3}\left(x_t + x_{t-1} + x_{t-2}\right)$$

Assuming certainty equivalence (i.e., $x_t = \frac{1}{3}\left(p^*_{i,t} + \mathbf{E}_t\left[p^*_{i,t+1}\right] + \mathbf{E}_t\left[p^*_{i,t+2}\right]\right)$), show that the equilibrium reset price, x_t, depends on m_t, $\mathbf{E}_t\left[m_{t+1}\right]$ and $\mathbf{E}_t\left[m_{t+2}\right]$.

Solution: Assuming certainty equivalence:

$$x_t = \frac{1}{3}\left(p^*_{i,t} + \mathbf{E}_t\left[p^*_{i,t+1}\right] + \mathbf{E}_t\left[p^*_{i,t+2}\right]\right)$$

Thus

$$\begin{aligned} x_t &= \frac{1}{3}\left(m_t + \mu + \mathbf{E}_t\left[m_{t+1} + \mu\right] + \mathbf{E}_t\left[m_{t+2} + \mu\right]\right) \\ &= \mu + \frac{1}{3}\left(m_t + \mathbf{E}_t\left[m_{t+1}\right] + \mathbf{E}_t\left[m_{t+2}\right]\right) \end{aligned}$$

Clearly, higher (contemporaneous and expected) money supply (m) increases the desired price, thereby x_t.

(c) Suppose that the (log) money supply follows a random walk: $m_t = m_{t-1} + \varepsilon_t$. Show that aggregate price inflation, $\pi_t = p_t - p_{t-1}$, is an MA(2) process taking the following form:

$$\pi_t = \frac{1}{3}\left(\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}\right)$$

Solution: Derive an expression for aggregate price inflation:

$$\begin{aligned} \pi_t &= p_t - p_{t-1} \\ &= \frac{1}{3}\left(x_t + x_{t-1} + x_{t-2}\right) - \frac{1}{3}\left(x_{t-1} + x_{t-2} + x_{t-3}\right) \\ &= \frac{1}{3}x_t - \frac{1}{3}x_{t-3} \\ &= \frac{1}{3}\left(\mu + \frac{1}{3}\left(m_t + \mathbf{E}_t\left[m_{t+1}\right] + \mathbf{E}_t\left[m_{t+2}\right]\right)\right) \\ &\quad -\frac{1}{3}\left(\mu + \frac{1}{3}\left(m_{t-3} + \mathbf{E}_{t-3}\left[m_{t-2}\right] + \mathbf{E}_{t-3}\left[m_{t-1}\right]\right)\right) \\ &= \frac{1}{9}\left(m_t + \mathbf{E}_t\left[m_{t+1}\right] + \mathbf{E}_t\left[m_{t+2}\right]\right) - \frac{1}{9}\left(m_{t-3} + \mathbf{E}_{t-3}\left[m_{t-2}\right] + \mathbf{E}_{t-3}\left[m_{t-1}\right]\right) \end{aligned}$$

Now, use the fact that $m_t = m_{t-1} + \varepsilon_t$, obtaining:

$$
\begin{aligned}
\pi_t &= \frac{1}{9}\left(m_t + \mathbf{E}_t\left[m_t + \varepsilon_{t+1}\right] + \mathbf{E}_t\left[m_{t+1} + \varepsilon_{t+2}\right]\right) \\
&\quad -\frac{1}{9}\left(m_{t-3} + \mathbf{E}_{t-3}\left[m_{t-3} + \varepsilon_{t-2}\right] + \mathbf{E}_{t-3}\left[m_{t-2} + \varepsilon_{t-1}\right]\right) \\
&= \frac{1}{9}\left(m_t + \mathbf{E}_t\left[m_t + \varepsilon_{t+1}\right] + \mathbf{E}_t\left[\underbrace{m_t + \varepsilon_{t+1}}_{=m_{t+1}} + \varepsilon_{t+2}\right]\right) \\
&\quad -\frac{1}{9}\left(m_{t-3} + \mathbf{E}_{t-3}\left[m_{t-3} + \varepsilon_{t-2}\right] + \mathbf{E}_{t-3}\left[\underbrace{m_{t-3} + \varepsilon_{t-2}}_{=m_{t-2}} + \varepsilon_{t-1}\right]\right) \\
&= \frac{1}{3}\left(m_t - m_{t-3}\right) \\
&= \frac{1}{3}\left(\underbrace{m_{t-1} + \varepsilon_t}_{=m_t} - m_{t-3}\right) \\
&= \frac{1}{3}\left(\underbrace{m_{t-2} + \varepsilon_{t-1}}_{=m_t - 1} + \varepsilon_t - m_{t-3}\right) \\
&= \frac{1}{3}\left(\underbrace{m_{t-3} + \varepsilon_{t-2}}_{=m_{t-2}} + \varepsilon_{t-1} + \varepsilon_t - m_{t-3}\right) \\
&\quad \frac{1}{3}\left(\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}\right)
\end{aligned}
$$

So, inflation follows an MA(2) process.