Individual Take-Home Exam for the M.Sc. in Economics 2009-II

ADVANCED MACROECONOMETRICS

Proposed Solution

1 Background

The examination considers econometric models for the monetary and financial sector, e.g. money demand and yield curve, etc. The main purpose of the examination is to assess the students ability to use statistical procedures to make inference on the equilibrium structures and the dynamic adjustment properties, and their ability to interpret the results, cf. the stated aims of the course.

The given data set consists of five variables

$$\mathsf{Money},\quad \mathsf{Prices},\quad \mathsf{RIncome},\quad \mathsf{Rm},\quad \text{and}\quad \mathsf{Rb},$$

where Money denotes the nominal broad money stock (M3), Prices denotes the deflator to real income (2000 = 1), RIncome denotes real income (GDP), Rm is the short interest rate (deposit rate in percent *p.a.*), and Rb is the 10 year bond yield in percent *p.a.*

Most of the empirical analysis consider the transformed variables

$$\begin{aligned} m_t &= \log(\mathsf{Money}_t/\mathsf{Prices}_t) = m_t^n - p_t \\ y_t &= \log(\mathsf{RIncome}_t) \\ dp_t &= 4 \cdot \Delta p_t \\ R_{mt} &= \mathsf{Rm}_t/100 \\ R_{bt} &= \mathsf{Rb}_t/100 \end{aligned}$$

and hence the $p = 5$ dimensional vector

$$x_t = (m_t : y_t : dp_t : R_{mt} : R_{bt})',$$

where m_t is the log of the real money, y_t is the log of real income, dp_t is the quarterly inflation rate (scaled to be comparable to the *p.a.* interest rates).

All assignments are based on *different* data sets. There are all simulated from the following data generating process (DGP):

$$
\begin{pmatrix} \Delta m_t \\ \Delta y_t \\ \Delta dp_t \\ \Delta R_{mt} \\ \Delta R_{bt} \end{pmatrix}
=
\begin{pmatrix} -0.13556 & 0 \\ 0.05755 & 0 \\ 0 & -0.36048 \\ 0 & 0.04794 \\ -0.00492 & 0.06807 \end{pmatrix}
\begin{pmatrix} 1 & -1 & 0 & -23.52749 & 23.52749 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}
\begin{pmatrix} m_{t-1} \\ y_{t-1} \\ dp_{t-1} \\ R_{mt-1} \\ R_{bt-1} \end{pmatrix}
$$

$$
+ \begin{pmatrix} -0.02658 & -0.09539 & 0.05474 & 0.57307 & -4.44357 \\ -0.03596 & 0.04181 & -0.13466 & -0.66244 & -3.24159 \\ 0.04375 & -0.07045 & -0.22460 & -2.25845 & 1.38401 \\ -0.00835 & -0.00263 & -0.01289 & 0.03104 & 0.36002 \\ -0.01741 & 0.02185 & -0.02512 & -0.26231 & 0.40971 \end{pmatrix}
\begin{pmatrix} \Delta m_{t-1} \\ \Delta y_{t-1} \\ \Delta dp_{t-1} \\ \Delta R_{mt-1} \\ \Delta R_{bt-1} \end{pmatrix} + \epsilon_t,
$$

with $\epsilon_t \sim N(0, \Omega)$,

$$
\Omega = \frac{1}{1000} \begin{pmatrix} 0.44312 & & & & \\ 0.04509 & 0.15419 & & & \\ -0.06182 & -0.00505 & 0.05541 & & \\ -0.00369 & -0.00087 & 0.00255 & 0.00255 & \\ -0.00588 & 0.00104 & 0.00308 & 0.00040 & 0.00178 \end{pmatrix}.
$$

The levels of the variables are set to reflects potential realizations of the variables for a country like Denmark. 120 observations are generated, covering quarterly data from $1977:1$ to $2006:4$. Outlying observations are drawn randomly with a probability of 3%, and the typical data set will have $3-5$ outliers, all with a magnitude of 5 standard deviations.

The students are informed that the regulations of international capital flows were changed in January 1992, and they should be aware of a potential structural break in the equilibrium relationships. In the generated data, the level of the bond rate increases 1 percent from $1992:1$, which should be reflected in the cointegrating relationships.

For all data sets it is ensured that if the correct outliers are modelled with dummy variables, then the *trace test* for the cointegration rank, when compared to the simulated critical values allowing for a break in the levels in $1992:1$, will correctly suggest a rank of $r=2$, and the true structure of the cointegration space is not rejected by a likelihood ratio (LR) test. It is not important *per se* that the students recover the true DGP, it is more important that they use sound arguments and that they convincingly motivate the choices they make.

The students are informed that the exam paper should not exceed 20 pages plus a maximum of 20 pages of supporting material. As a guiding principle, the sections have the following weights: Section 2 (20%), Section 3 (15%), Section 4 (20%), Section 5 (15%), Section 6 (10%), and Section 7 (20%).

The proposed solution below is based on the data for a tentative exam number 1001 (i.e. `Data1001.xls`).

2 The Statistical Model

The starting point is the vector autoregressive (VAR) model of order k:

$$x_t = \Pi_1 x_{t-1} + \Pi_2 x_{t-2} + ... + \Pi_k x_{t-k} + \mu_0 + \mu_1 t + \phi D_t + \epsilon_t. \tag{2.1}$$

for $t = 1, 2, ..., T$.

[1] The paper should load the variables and construct the transformed variables for the empirical analysis.
Then the paper should show a number of relevant graphs illustrating the time series behavior of the variables and the empirical relevance of the candidates suggested above. As in Figure 1 it could be relevant to: (A) show the logs of the nominal variables, noting that the variables may be I(2). (B) show the real variables to be included in the empirical analysis, noting that they look I(1) from a visual inspection, suggesting cointegration from I(2) to I(1). (C) note that velocity look non-stationary, but that it may cointegrate with the interest rate spread (opportunity cost of holding money). In some data sets they may be a visible jump in the interest rate spread in 1992, indicating a level shift. (D) Show the interest rates and inflation. The non-stationarity of inflation again suggests I(2)-ness of price levels. Also interest rates look non-stationary. (E) real interest rates, which look bordeline stationary.

[2] Now the paper should set up a well specified statistical model for the data. The following issues should be discussed:

[2.1] Since the variables are clearly trending over time, the relevance of a deterministic linear trend cannot be rejected *a priori*. The relevant starting point is therefore the model with a restricted trend term (`CIDRIFT` in CATS).

[2.2] The paper should use information criteria, likelihood ratio tests for the significance of the Π_k matrices as well as misspecification tests for the hypothesis of no autocorrelation to determine the lag length, k of the VAR model.

[2.3] The paper should perform a residual analysis to ensure that the model complies with the assumptions.

[2.4] Outlying observations should be accommodated by the inclusion of dummy variables. The good paper may discuss whether the large shocks, producing level shifts in the non-stationary directions, should also be allowed to change equilibrium means. This can be done by inserting level shifts and testing their exclusion. This is complicated in practice, however, because the conclusion may depend on the cointegration rank, and to determine the cointegration rank we need to specify potential level shifts. The solution should note 1992 : 1 as a potential level shift.

[2.5] Constancy of the parameters according to the results of the available recursive tools.

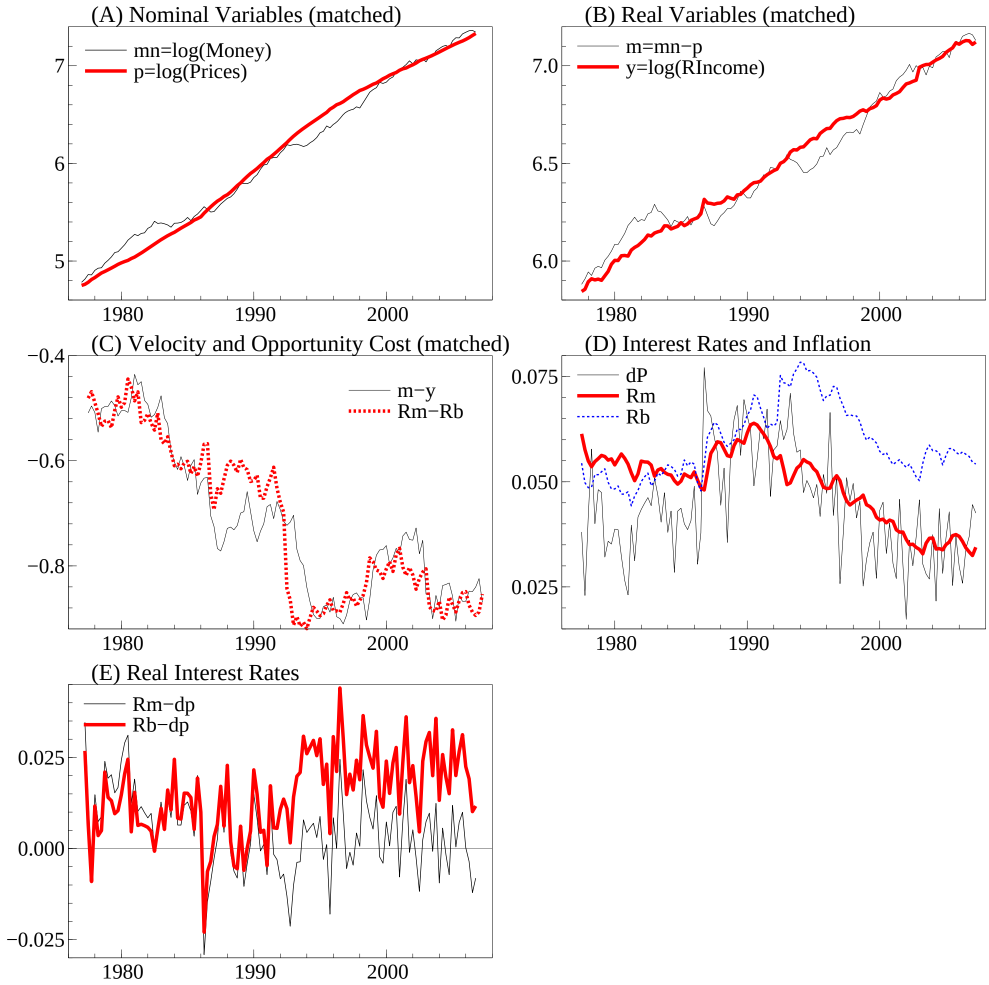


Figure 1: *Data and certain linear combinations.*

In practice it may be necessary to iterate between the steps above before an acceptable model is found but to save space the paper only has to explain the steps and present the final, preferred model.

For the present example, $k = 2$ lags are sufficient to account for the autoregressive nature of the variables. The level shift in $1992:1$ is very significant for all choices of the cointegration rank, and it is included with the option `break=level`. This includes $\phi_0 \Delta DS_t + \alpha\beta_0' DS_{t-1}$ as deterministic terms to the model. In turns out that the level shift also produces an outlier for observation $1992:2$ corresponding to a dummy for ΔDS_{t-1}, which is included manually (with the option `dum`). In addition two more outliers are identified (standardized residual larger than 3.5, say), namely $1986:2$ and $2002:3$. With this deterministic specification, the model seems to be a well specified characterization of the information in the data. There are no signs of parameter non-constancy based on recursive estimation.

[3] The paper should explain that the VAR model is stable if the eigenvalues of the companion matrix are inside the unit circle. The very good paper may illustrate by the solution to the VAR(1) or may refer to the Jordan decomposition to show the

role of the eigenvalues. The paper should comment on the estimated eigenvalues. In the present case:

The Roots of the COMPANION MATRIX // Model: H(5)				
	Real	Imaginary	Modulus	Argument
Root1	0.969	0.000	0.969	0.000
Root2	0.886	−0.117	0.894	−0.131
Root3	0.886	0.117	0.894	0.131
Root4	0.654	0.000	0.654	0.000
Root5	0.445	−0.290	0.531	−0.577
Root6	0.445	0.290	0.531	0.577
Root7	0.019	0.338	0.338	1.515
Root8	0.019	−0.338	0.338	−1.515
Root9	−0.283	−0.039	0.286	−3.006
Root10	−0.283	0.039	0.286	3.006

The point estimates are inside the unit circle, and the potential three unit roots informally suggest a cointegration rank of $r = 2$.

3 The Cointegration Rank

Now consider also the VAR model in error correction form (VECM):

$$\Delta x_t = \Pi x_{t-1} + \Gamma_1 \Delta x_{t-1} + ... + \Gamma_{k-1} \Delta x_{t-k+1} + \mu_0 + \mu_1 t + \phi D_t + \epsilon_t. \tag{3.1}$$

for $t = 1, 2, ..., T$.

[4] The paper should explain how a unit root in the characteristic polynomial is related to the reduced rank of Π. We note that the characteristic polynomial to the VECM based on a VAR(2) is given by

$$A(z) = (1 - z) - \Pi z - \Gamma_1 (1 - z) z.$$

A unit root implies that

$$|A(1)| = | - \Pi| = 0,$$

which again implies that Π is singular. When Π has reduced rank we can write it as

$$\Pi = \alpha \beta',$$

where α and β are $p \times r$ matrices containing the independent columns and rows of Π, respectively.

[5] The paper should present the LR test for the cointegration rank. First note that $r = 0$ corresponds to p I(1) trends and no cointegration, while a rank of p implies that all linear combinations of x_t are stationary, so that $x_t \sim I(0)$. Reduced rank $0 < r < p$ allows cointegration of the I(1) variables. There exist r stationary linear combinations and p non-stationary variables. We denote the cointegrated VAR (CVAR) model as

$$H(r) : \Delta x_t = \alpha\beta' x_{t-1} + \Gamma_1 \Delta x_{t-1} + \mu_0 + \mu_1 t + \phi D_t + \epsilon_t,$$

with α and β unrestricted $p \times r$. In this model the rank is at most r. The unit root restrictions imply the nesting structure

$$H(0) \subset H(1) \subset \cdots \subset H(r) \subset \cdots \subset H(p).$$

Normally the cointegration rank is determined from the LR test for $H(r)$ against the unrestricted VAR:

$$LR(H(r) \mid H(p)) = -2 \left[\log L(H(r)) - \log L(H(p))\right].$$

The model $H(r)$ is estimated by reduced rank regression (RRR). First consider the concentrated regression

$$R_{0t} = \alpha\beta' R_{1t} + \epsilon_t,$$

where R_{0t} and R_{1t} are OLS residuals. RRR amounts to solving the eigenvalue problem

$$\left|\lambda S_{11} - S_{10} S_{00}^{-1} S_{01}\right| = 0,$$

where $S_{ij} = T^{-1} \sum_{t=1}^{T} R_{it} R_{jt}'$. That produces p eigenvectors, $\hat{v}_1, ..., \hat{v}_p$ and p corresponding eigenvalues $1 > \hat{\lambda}_1 > ... > \hat{\lambda}_p \geq 0$. The latter can be interpreted as the squared canonical correlations between $\hat{v}_i' R_{it}$ and R_{0t}. The estimate of β is $\hat{\beta} = (\hat{v}_1 : ... : \hat{v}_r)$, and the maximized value of the likelihood function is given by

$$L_{\max}^{-2/T}(H(r)) = |S_{00}| \prod_{i=1}^{r} \left(1 - \hat{\lambda}_i\right).$$

It follows that the LR statistic can be written as

$$LR(H(r) \mid H(p)) = -T \sum_{i=r+1}^{p} \log(1 - \hat{\lambda}_i),$$

which has a squared Dickey Fuller distribution, which depends on the included deterministic terms.

The good paper notes that the trace test is only *similar* with respect to the coefficients to the deterministic components if the model is balanced in terms of the deterministic behavior, i.e. if the stationary and non-stationary directions share the same deterministic components. This is not the case if μ_1 is unrestricted. From the Granger representation theorem, it hold that

$$x_t = C\sum_{i=1}^{t}(\mu_0+\mu_1 i+\phi D_i+\epsilon_i)+C^*(L)(\mu_0+\mu_1 t+\phi D_t+\epsilon_t)+A,$$

where $C=\beta_\perp(\alpha_\perp'\Gamma\beta_\perp)\alpha_\perp'$, so that $\beta' C$. The non-stationary directions $\beta_\perp' x_t$ have a quadratic trend, $C\mu_1\sum_{i=1}^{t} i$, while this cancels in the stationary directions, $\beta' x_t$. To avoid this we restrict the trend terms so that $\mu_1=\alpha\beta_0'$ and write the model as

$$H^*(r):\Delta x_t=\alpha\begin{pmatrix}\beta\\ \beta_0\end{pmatrix}'\begin{pmatrix}x_{t-1}\\ t\end{pmatrix}+\Gamma_1\Delta x_{t-1}+\mu_0+\phi D_t+\epsilon_t.$$

The rank test now considers the rank of $\Pi^*=(\Pi:\mu_1)$. Similar arguments could be made for potential level shifts, and the final model is

$$H^*(r):\Delta x_t=\alpha\begin{pmatrix}\beta\\ \beta_0\\ \beta_1\end{pmatrix}'\begin{pmatrix}x_{t-1}\\ DS_{t-1}\\ t\end{pmatrix}+\Gamma_1\Delta x_{t-1}+\mu_0+\phi_0\Delta DS_t+\phi D_t+\epsilon_t,$$

where DS is the level shift.

The paper may also start from the unobserved components representation, $x_t = z_t+\tau_0+\tau_1 t+\tau_2 DS_t$, with z_t being cointegrated with no deterministic terms. It may also explain the idea for a univariate DF test, say

$$\Delta x_t=\delta+\pi x_{t-1}+\epsilon_t,$$

where the test for $\pi=0$ has a DF-distribution for $\delta=0$ but a N(0,1) for $\delta\neq 0$. Both of these approaches are excellent.

[6] Finally, the paper should determine the cointegration rank in the empirical application. All available information should be used: (i) The trace test and the Bartlett corrected trace test. The paper should note that in the presence of a level shift the critical values should be simulated. If the dummies are correctly identified, the trace test should indicate $r=2$. For the present data , the trace test statistics are given by

I(1)-ANALYSIS

p-r	r	Eig.Value	Trace	Trace*	Frac95	P-Value	P-Value*
5	0	0.381	134.182	126.499	99.414	0.000	0.000
4	1	0.312	78.128	73.609	72.675	0.017	0.041
3	2	0.164	34.458	32.066	50.106	0.576	0.699
2	3	0.081	13.459	12.344	31.070	0.892	0.933
1	4	0.030	3.540	3.270	14.938	0.922	0.940

(ii) The graphs of the candidate cointegrating relationships. (iii) The location of the eigenvalues of the companion matrix, see above. (iv) The $t-$ratios for coefficients in α, cf. the book, noting however that the critical values are unknown. (v) The recursive trace test.

4 Testing Hypotheses

The students are told to continue with the true cointegration rank of the DGP, $r = 2$.

[7] The paper should estimate the CVAR with $r = 2$. The good paper explains that the unrestricted estimates of the cointegrating relationships are normalized to be conditionally orthogonal with unit length, $\hat{\beta}' S_{11} \hat{\beta} = I_r$, and they are ordered according to the canonical correlations. This is mathematically convenient, but may not be relevant in terms of economic theory and the unrestricted estimates are often difficult to interpret.

For the present DGP the chosen normalization of the eigenvectors will, in general, make it difficult to recognize the structure. For the present case the estimated β' is given by

	β'						
	M	Y	DP	RM	RB	C(1992:01)	TREND
Beta(1)	1.000	−1.521	−9.996	−13.791	27.508	−0.275	0.008
Beta(2)	0.141	−0.123	1.000	−5.155	3.357	−0.037	−0.001

The paper should be careful not to attach to much structural interpretation to the unidentified relationships.

[8] Next the paper should test for long-run exclusion of the variables. This is a subspace restriction formulated as

$$\mathcal{H}_0 : \tilde{\beta}^c = H\varphi,$$

where $\tilde{\beta} = \left(\beta' : \beta_0' : \beta_1'\right)'$ is the augmented cointegration matrix, H is the $(p+2) \times p$ design matrix and φ contains the $(p+1) \times r$ free parameters. To test exclusion of m_t we use

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The LR statistic $LR(\mathcal{H}_0 \mid H(r))$ is $\chi^2(r)$ under the null.

The automatic test for exclusion may be used as long as the set up of the test is explained. For the present data the automatic test produces

TEST OF EXCLUSION									
r	DGF	5% C.V.	M	Y	DP	RM	RB	C(1992:01)	TREND
1	1	3.841	7.111 [0.008]	8.565 [0.003]	9.569 [0.002]	2.111 [0.146]	8.531 [0.003]	6.754 [0.009]	3.034 [0.082]
2	2	5.991	20.242 [0.000]	13.853 [0.001]	31.515 [0.000]	17.110 [0.000]	28.794 [0.000]	15.515 [0.000]	3.361 [0.186]
3	3	7.815	28.534 [0.000]	17.214 [0.001]	42.596 [0.000]	26.874 [0.000]	39.875 [0.000]	25.171 [0.000]	4.268 [0.234]
4	4	9.488	34.865 [0.000]	22.249 [0.000]	48.826 [0.000]	32.137 [0.000]	46.237 [0.000]	28.805 [0.000]	9.598 [0.048]

For the preferred, $r = 2$, exclusion of the trend cannot be rejected. During the lectures we have seen examples where multicollinearity between the trend and a stochastic I(1) variables make it hazardous to exclude the trend, and most students will probably keep the trend a little longer.

[9] Next the paper should test that each of the variables are stationary when corrected for a level shift and a deterministic linear trend. For the variable c_t this is a statement that $m_t + \varphi_1 DS_t + \varphi_2 t$ is stationary with the second relationship unrestricted. The restricted cointegration matrix has the form

$$\mathcal{H}_1 : \tilde{\beta}^c = (H_1\varphi_1 : \varphi_2) = \begin{pmatrix} \varphi_{1.1} & \varphi_{2.1} \\ 0 & \varphi_{2.2} \\ 0 & \varphi_{2.3} \\ 0 & \varphi_{2.4} \\ 0 & \varphi_{2.5} \\ \varphi_{1.2} & \varphi_{2.6} \\ \varphi_{1.3} & \varphi_{2.7} \end{pmatrix}.$$

The LR statistic $LR(\mathcal{H}_1 \mid H(r))$ is distributed as $\chi^2(v)$ with $v = p - r = 3$ in the present case. Again the automatic tests may be used as long as the test is explained. Here they produce:

TEST OF STATIONARITY							
r	DGF	5% C.V.	M	Y	DP	RM	RB
1	4	9.488	49.455 [0.000]	43.496 [0.000]	42.537 [0.000]	50.598 [0.000]	49.947 [0.000]
2	3	7.815	37.206 [0.000]	31.715 [0.000]	31.811 [0.000]	38.769 [0.000]	38.528 [0.000]
3	2	5.991	14.937 [0.001]	9.132 [0.010]	15.793 [0.000]	16.345 [0.000]	16.853 [0.000]
4	1	3.841	4.433 [0.035]	0.440 [0.507]	6.580 [0.010]	6.024 [0.014]	5.936 [0.015]

For $r = 2$, none of the variables should appear trend-stationary.

[10] Finally, the paper should test a number of theoretical candidates outline in Section 1 of the exam. In each case the hypothesis should be formulated as

$$\mathcal{H}_2 : \tilde{\beta}^c = (H_1\varphi_1 : \varphi_2),$$

where the second relationship is unrestricted. As an example, consider the interest rate spread:

$$R_{mt} = R_{bt} + \phi_1 DS_t + \phi_2 t.$$

This corresponds to a formulation

$$\tilde{\beta}^c = (H_1\varphi_1 : \varphi_2) = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_{1.1} \\ \varphi_{1.2} \\ \varphi_{1.3} \end{pmatrix} : \begin{pmatrix} \varphi_{2.1} \\ \varphi_{2.2} \\ \varphi_{2.3} \\ \varphi_{2.4} \\ \varphi_{2.5} \\ \varphi_{2.6} \\ \varphi_{2.7} \end{pmatrix} \right) = \begin{pmatrix} 0 & \varphi_{2.1} \\ 0 & \varphi_{2.2} \\ 0 & \varphi_{2.3} \\ \varphi_{1.1} & \varphi_{2.4} \\ -\varphi_{1.1} & \varphi_{2.5} \\ \varphi_{1.2} & \varphi_{2.6} \\ \varphi_{1.3} & \varphi_{2.7} \end{pmatrix}.$$

The LR statistic, $LR(\mathcal{H}_2 \mid H(r))$, is distributed as a $\chi^2(v)$, where v is the number of restrictions on β_1 minus $r-1=1$, in the present case $v=3$.
For the present data the following results are obtained (where '$**$' and '$*$' indicate inclusion of dummy and trend and dummy, respectively):

Eq.	M	Y	dP	R_m	R_b	DS	TREND	LR	v	p
$(1.1)^{**}$	0	0	0	1	-1	0.011 [3.053]	0.000 [2.427]	30.632	3	[0.000]
$(1.2a)^{**}$	0	0	-1	1	0	-0.004 [-0.769]	0.000 [2.472]	14.442	3	[0.002]
$(1.2b)^{**}$	0	0	-1	0	1	-0.016 [-4.948]	0.000 [1.121]	6.605	3	[0.086]
$(1.2b)^{*}$	0	0	-1	0	1	-0.013 [-7.993]	0	7.751	4	[0.101]
$(1.3)^{**}$	1	-1	0	0	0	0.125 [1.257]	0.001 [0.798]	35.151	3	[0.000]
$(1.4)^{**}$	1	-1.063 [-4.043]	0	-0.227 [-11.071]	0.227 [11.071]	-0.226 [-5.298]	0.001 [0.243]	0.371	1	[0.543]
$(1.4)^{**}$	1	-1	0	-22.888 [-11.548]	22.888 [11.548]	-0.231 [-5.768]	0.000 [0.036]	0.411	2	[0.814]
$(1.4)^{*}$	1	-1	0	-22.934 [-12.730]	22.934 [12.730]	-0.231 [-5.927]	0	0.413	3	[0.938]
$(1.5)^{*}$	0	1	15.521 [9.959]	0	-15.521 [-9.959]	0.202 [3.855]	-0.014 [-19.819]	3.251	2	[0.197]

The results are straightforward: The yield curve is not stationary. The real bond rate is stationary, once corrected for the break, while the real money market rate is not. This suggests a small I(1) component in the money market that is not present in bond rates and inflation; maybe from monetary policy. For the present data, the real bond rate relation may include detrended income, but the sign is positive, inconsistent with an IS curve. Finally, the homogeneous money demand is stationary, with a quite large elasticity of -23 and no deterministic trend.
The good paper comments on the structure, signs, and magnitudes of coefficients. It may also be of interest to look at the adjustment coefficients, to see if a given interpretation is consistent with the error correction.

5 Identification

[11] The paper should explain that a relationship is *identified* if it is unique among the cointegrating relationships, so that no linear combination of the other relationships will look like β_1. Only in this case a firm economic interpretation, and a name, can be attached. Formally, we require that the restrictions we use to structure β_1 is not fulfilled on the other relationships. Using the notation $\beta = (H_1\varphi_1 : H_2\varphi_2)$ we have

$$Rank(R_1'\beta) = Rank(R_1'(H_1\varphi_1 : H_2\varphi_2)) = r - 1,$$

where $R_1 = H_{1\perp}$ is the restriction matrix, $R_1'\beta_1 = 0$. This condition will depend on the estimated parameters in $\varphi_2, ..., \varphi_r$. The condition implies that

$$R_{1.2} = Rank(R_1'H_2) \geq 1.$$

This condition can be checked prior to estimation and is referred to as *generic* or *formal* identification. After estimation we can check the whether the identifying differences between the relationships are significant yielding *empirical* identification. For β_2, we need $Rank(R_2'H_1) \geq 1$.

[12] In the present case it is natural to try to obtain identification with the relationships (1.2b) and (1.4), cf. the testing above. For the present data:

$$\begin{pmatrix} \Delta m_t \\ \Delta y_t \\ \Delta dp_t \\ \Delta R_{mt} \\ \Delta R_{bt} \end{pmatrix} = \begin{pmatrix} \underset{[-5.271]}{-0.196} & \underset{[-1.213]}{-0.397} \\ \underset{[2.722]}{0.062} & \underset{[0.090]}{0.018} \\ \underset{[2.224]}{0.030} & \underset{[-3.737]}{-0.443} \\ \underset{[0.312]}{0.001} & \underset{[2.483]}{0.041} \\ \underset{[-2.307]}{-0.006} & \underset{[4.363]}{0.096} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & \underset{[-12.784]}{-23.494} & \underset{[12.784]}{23.494} & \underset{[-6.095]}{-0.242} & 0 \\ 0 & 0 & 1 & 0 & -1 & \underset{[8.026]}{0.013} & 0 \end{pmatrix} \begin{pmatrix} m_{t-1} \\ y_{t-1} \\ dp_{t-1} \\ R_{mt-1} \\ R_{bt-1} \\ DS_{t-1} \\ t \end{pmatrix} + ...$$

The restrictions are accepted with a LR statistic of 8.662 and a $p-$value of 0.278 in a $\chi^2(7)$.

The paper should economically interpret the relationships as a homogeneous money demand relation and a real interest rate. For this data set the error correction is quite clear, money and income corrects money demand, and excess money do not appear inflationary. The interest rates and inflation corrects the real interest rate.

[13] The recursive analysis should indicate that the system is fairly stable, at least if the correct dummies are identified. The paper should include enough information to substantiate that claim. The good paper could explain that instability in the beginning of the sample (for short sample lengths) are due to a limited number of observations.

6 The Moving Average Representation

[14] The paper should explain the Granger representation

$$x_t = \beta_\perp \left(\alpha'_\perp \Gamma \beta_\perp\right)^{-1} \alpha'_\perp \sum_{i=1}^{t} (\epsilon_i + \phi D_i) + C^*(L)(\epsilon_t + \phi D_t) + \tau_0 + \tau_1 t.$$

The paper should note that $C = \tilde{\beta}_\perp \alpha'_\perp$ has reduced rank $p - r$, that the common stochastic trends are given by $\alpha'_\perp \sum_{i=1}^{t} \epsilon_i$ and that they affect the variables with the loading coefficients $\tilde{\beta}_\perp = \beta_\perp \left(\alpha'_\perp \Gamma \beta_\perp\right)^{-1}$. The paper should note that the shocks $\alpha'_\perp \epsilon_t$ have permanent effects while the shocks $\alpha' \epsilon_t$ have only transitory effects.

For the example here, the joint effect is given by the 5×5 matrix (of rank 3):

The Long-Run Impact Matrix, C					
	M	Y	DP	RM	RB
M	0.326 [2.462]	0.233 [1.673]	−0.775 [−1.652]	19.620 [4.988]	−10.643 [−4.170]
Y	0.187 [2.575]	0.774 [10.159]	−0.297 [−1.155]	−2.215 [−1.029]	0.204 [0.146]
DP	0.002 [0.215]	0.023 [2.004]	0.157 [4.093]	−0.477 [−1.482]	0.935 [4.477]
RM	0.008 [0.904]	−0.000 [−0.023]	0.137 [4.225]	0.452 [1.666]	0.473 [2.688]
RB	0.002 [0.215]	0.023 [2.004]	0.157 [4.093]	−0.477 [−1.482]	0.935 [4.477]

[15] Finally, the paper should explain that a unit vector in $\alpha_\perp$ implies that cumulated shocks to this variable constitute a common trend. We say that the variable is *weakly exogenous*.

A unit vector in $\alpha_\perp$ is implied by a zero row in α, which can be tested using a likelihood ratio test. For the present case we get the automated tests:

TEST OF WEAK EXOGENEITY							
r	DGF	5% C.V.	M	Y	DP	RM	RB
1	1	3.841	0.187 [0.666]	3.113 [0.078]	10.971 [0.001]	3.034 [0.082]	11.457 [0.001]
2	2	5.991	18.856 [0.000]	5.731 [0.057]	11.034 [0.004]	9.382 [0.009]	20.515 [0.000]
3	3	7.815	27.549 [0.000]	13.978 [0.003]	14.759 [0.002]	12.317 [0.006]	21.553 [0.000]
4	4	9.488	33.715 [0.000]	19.649 [0.001]	14.963 [0.005]	16.495 [0.002]	22.837 [0.000]

For $r = 2$, income is borderline. An economic interpretation of that could be that GDP is mainly given from demand in the short run and the current information set does not include enough information to explain variations in demand.

To explain the implication of weak exogeneity, we now impose it jointly with the identification ($\chi^2(9)$ statistic of 15.484 [0.078]) to get the common trends given by

$\alpha'_\perp$					
	M	Y	DP	RM	RB
CT(1)	0.014 [1.354]	0.000 [NA]	0.079 [1.864]	1.000 [NA]	0.000 [NA]
CT(2)	0.000 [NA]	1.000 [NA]	0.000 [NA]	0.000 [NA]	0.000 [NA]
CT(3)	0.004 [0.218]	0.000 [NA]	0.213 [2.706]	0.000 [NA]	1.000 [NA]

One CT is given by shocks to y. The other shocks are given by the interest rates with additions from inflation. Shocks to the money stock look transitory. The corresponding loadings are given by

The Loadings to the Common Trends, $\tilde{\beta}_\perp$:			
	CT1	CT2	CT3
M	22.714 [4.558]	0.259 [1.501]	−12.182 [−4.071]
Y	2.298 [0.916]	0.897 [10.325]	−2.128 [−1.412]
DP	−0.364 [−1.015]	0.025 [2.048]	0.875 [4.063]
RM	0.440 [1.439]	0.000 [0.029]	0.479 [2.609]
RB	−0.364 [−1.015]	0.025 [2.048]	0.875 [4.063]

and they are just a mirror image of the cointegrating relationships.

7 I(2) Analysis

The I(1) data vector considered so far,

$$x_t = (m_t : y_t : dp_t : R_{mt} : R_{bt})',$$

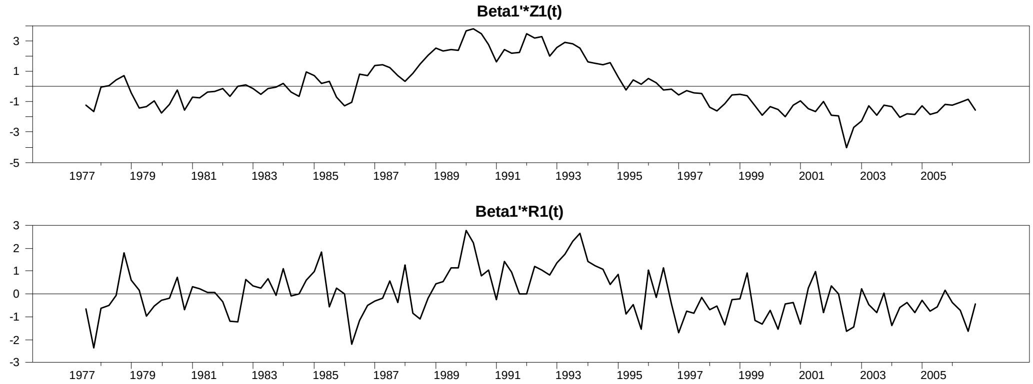
with $m_t = m_t^n - p_t$, is a transformation of a nominal data set:

$$y_t = (m_t^n : p_t : y_t : R_{mt} : R_{bt})'.$$

In this section we consider the original *nominal data* in y_t.

[16] The paper estimate a model for the I(2) data and look for signs of I(2)-ness. These are particularly the following.

[16.1] There is a big difference between $\hat{\beta}' y_t$ and $\hat{\beta}' R_{1t}$. The latter includes the correction for Δy_{t-1} and a big difference suggests that polynomial cointegration, and hence I(2)-ness, is important. In this case we get



where the latter look much more stationary.

[16.2] The Bartlett correction factors are very large. There are not defined for the model having additional unit roots in the characteristic polynomial, indicating that an additional root is very close to unity. In this case we get

I(1)-ANALYSIS							
p-r	r	Eig.Value	Trace	Trace*	Frac95	P-Value	P-Value*
5	0	0.467	151.750	138.503	88.891	0.000	0.000
4	1	0.308	77.443	57.954	64.232	0.002	0.150
3	2	0.162	33.951	24.686	43.493	0.309	0.800
2	3	0.083	13.138	7.605	26.464	0.734	0.981
1	4	0.024	2.914	2.040	12.853	0.890	0.959

In some cases the difference is even more pronounces, and sometimes the corrected statistics are 'NA'.

[16.3] Finally, the number of unit roots do not fit the chosen rank of Π, indicating additional unit roots. In particular, the largest unrestricted root is always close to the unit circle. When it is fixed to $z = 1$, the second largest, jump to the unit circle. In the case here we get the roots with modulus:

$$\begin{array}{rcccccccccc}
r=5: & 0.983 & 0.983 & 0.867 & 0.867 & 0.660 & 0.488 & 0.488 & 0.306 & 0.228 & 0.228 \\
r=4: & 1 & 0.965 & 0.865 & 0.865 & 0.669 & 0.489 & 0.489 & 0.306 & 0.224 & 0.224 \\
r=3: & 1 & 1 & 0.971 & 0.760 & 0.760 & 0.487 & 0.487 & 0.323 & 0.248 & 0.248 \\
r=2: & 1 & 1 & 1 & 0.939 & 0.652 & 0.501 & 0.501 & 0.303 & 0.218 & 0.218 \\
r=1: & 1 & 1 & 1 & 1 & 0.934 & 0.491 & 0.491 & 0.307 & 0.186 & 0.010 \\
r=0: & 1 & 1 & 1 & 1 & 1 & 0.613 & 0.405 & 0.405 & 0.270 & 0.006
\end{array}$$

These signs are clear indications of I(2)-ness in the data.

[17] The paper should explain the based on the parametrization of the unrestricted model

$H(p)$:

$$\Delta^2 y_t = \Pi y_{t-1} - \Gamma \Delta y_{t-1} + \sum_{i=1}^{k-2} \Psi_i \Delta^2 y_{t-k} + \phi D_t + \epsilon_t,$$

the I(1) model $H(r)$ is defined by the reduced rank restriction, $\Pi = \alpha\beta'$, while the I(2) model is defined by the two reduced rank restrictions

$$H(r, s_1) : \Pi = \alpha\beta' \quad \text{and} \quad \alpha'_\perp \Gamma \beta_\perp = \xi\eta'$$

with α, β being $p \times r$ and ξ, η being $(p-r) \times s_1$. It should be noted that the I(2) is a special case of the I(1) model and the nesting structure is given by (here for $p = 4$):

r	Models, $H(r, s_1)$				
0	$H(0,0)$ $\subset$	$H(0,1)$ $\subset$	$H(0,2)$ $\subset$	$H(0,3)$	$\subset$ $H(0)$
					$\cap$
1		$H(1,0)$ $\subset$	$H(1,1)$ $\subset$	$H(1,2)$	$\subset$ $H(1)$
					$\cap$
2			$H(2,0)$ $\subset$	$H(2,1)$	$\subset$ $H(2)$
					$\cap$
3				$H(3,0)$	$\subset$ $H(3)$
					$\cap$
4					$H(4)$
$s_2 = p - r - s_1$	4	3	2	1	0

To determine the cointegration rank we can use standard LR test statistics, noting that asymptotic distribution is non-standard, depending on integrated Brownian motions.

[18] For the data here, we get

Rank Test Statistics							
$p-r$	r	$s_2 = 5$	$s_2 = 4$	$s_2 = 3$	$s_2 = 2$	$s_2 = 1$	$s_2 = 0$
5	0	626.888 [0.000]	478.476 [0.000]	344.666 [0.000]	241.823 [0.000]	178.754 [0.000]	151.750 [0.000]
4	1		312.063 [0.000]	198.612 [0.000]	135.264 [0.000]	81.860 [0.018]	77.443 [0.022]
3	2			128.833 [0.000]	57.104 [0.352]	37.976 [0.597]	33.951 [0.607]
2	3				29.368 [0.827]	15.418 [0.945]	13.138 [0.910]
1	4					5.166 [0.976]	2.914 [0.962]

Since the asymptotic distribution is wrong, it is hard to make conclusions. It seems, however that $r = 2$ with $s_2 = 1$ or $s_2 = 2$. [There is a small bug in CATS related to levels shifts in the I(2) test. Here I have inserted the dummy as a weakly exogenous variable to make the calculations correct. The students cannot be expected to know that and any produced table is fine.]

[19] The students are asked to impose the ranks $(r, s_1) = (2, 2)$, allowing $s_2 = p - r - s_1 = 1$ I(2) trend in the data. The Granger representation in this case may be written as

$$y_t = C_2 \sum_{s=1}^{t} \sum_{i=1}^{s} \epsilon_i + C_1 \sum_{i=1}^{t} \epsilon_i + C_0(L)\epsilon_t + ...,$$

where, in particular,

$$C_2 = \beta_{\perp 2}(\alpha'_{\perp 2}\Theta\beta_{\perp 2})^{-1}\alpha'_{\perp 2}.$$

The double random walks thus affects the variables with an effect proportional with $\beta_{\perp 2}$. For the data here, the estimated version is given by

$$\hat{\beta}_{\perp 2} = \begin{pmatrix} -0.003 \\ 1 \\ 0.421 \\ 0.082 \\ 0.139 \end{pmatrix}.$$

Surprisingly, this suggests that the nominal money stock is not I(2), which is contrary to the graphical inspections. It could reflect that changes in the money stock is very volatile compared to inflation, and the I(2) component is too small to be visible.

[20] The paper should explain the logic of the nominal to real transformation, and how the I(2) trend may cancel in a theoretically relevant linear combination, allowing cointegration from I(2) to I(1). In the present case, one idea could be to let

$$\beta_{\perp 2} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = b, \text{ say}$$

so that $m = m^n - p$ would be integrated of order one, I(1). In the I(2) model it holds that the parameter $\tau_\perp = \beta_{\perp 2}$, where $\tau = (\beta : \beta_{\perp 1})$ is $p \times (r + s_1)$ and contains all the cointegrating relation. The relevant hypothesis can be formulated as

$$\tau = H\phi,$$

with

$$H = b_\perp = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

and the LR statistic will be $\chi^2(r + s_1)$ under the null.

If the restriction is true, the data set can be transformed from $y_t \sim I(2)$ to $x_t \sim I(1)$. $m_t = m_t^n - p_t$ is cointegrating from I(2) to I(1) and $dp_t = 4 \cdot \Delta p_t$ is included to represent the I(2) trend and to allow the polynomially cointegrating relationships to be recovered as standard I(1) cointegrating relationships. If the restriction is true, the transformation and the subsequent I(1) analysis can be performed without loosing information relative to the full I(2) analysis.
For the present case, with $r = s_1 = 2$, we get a LR statistic of 10.578, which produces a borderline tail probability of 0.032 in the $\chi^2(4)$ distribution. The conclusion is that the validity of the homogeneity restriction, which is necessary for the theoretically motivated nominal-to-real transformation that underlies the I(1) analysis above, is borderline.