Solution for the exam in Econometrics A
B.Sc. in Economics 2010-II

Academic aim:

The aim of the course is to introduce the students to probability theory and statistics. The aim is for the student to be able to:

- understand the most important basic concepts of probability theory such as: probability, simultaneous-, marginal- and conditional probabilities, distribution, density function, independence, means, variance and covariance and apply these ideas on specific problems.
- know the result from the central limit theory.
- know and recognize the most commonly applied discrete and continuous distributions such as: Bernoulli, binomial, Poisson, multinomial, negative binomial, hypergeometric, geometric, uniform, normal, Chi-squared, exponential, gamma, t-, F-distribution and work with these distributions in relation to specific problems.
- understand the most important statistical concepts such as: random sampling, likelihood function, sufficient statistics, the properties and distributions of statistics, estimation, and maximum likelihood estimation and moment estimation, consistency, confidence interval, hypotheses, test statistics, test probability, level of significance, type I and II errors, power functions.
- perform a simple statistical analysis involving estimation, inference and hypothesis test e.g. the comparison of the means in two populations or test of independence for discrete stochastic variable.
- describe the result of his or her own analysis and considerations in a clear and distinct manner

In order for the student to obtain the highest grade possible, the student must demonstrate the mastery of the above-mentioned skills.

1 Question 1

1. The waiting time T is geometric distributed.
2. $\Pr(T > 3) = 1 - \Pr(T \leq 3) = 1 - \left(0.9^0 + 0.9^1 + 0.9^2\right) \cdot 0.1 \approx 1 - 0.271 \approx 0.729$
3.
   (a) $E(T) = \frac{1}{p} = 1/0.1 = 10$ months

   (b) $Var(T) = \frac{1-p}{p^2} = \frac{1-0.1}{0.1^2} = 90$
4. $E(X_A) = E(5,000,000 - T_A \cdot 30,000) = 5,000,000 - 10 \cdot 30,000 = 4,700,000.$

   $E(X_B) = E(4,800,000 - T_B \cdot 30,000) = 4,800,000 - 5 \cdot 30,000 = 4,650,000.$

   Since the expected value of strategy A gives a higher, the couple should choose strategy A.
5. The choice between strategy C and D is similar to the choice between strategy A and B in question 4. Therefore, strategies involving strategy D can never be optimal. Hence, we will only consider the combined strategies {strategy A, strategy C} and {strategy B, strategy C}. Since the strategy is the same after 3 months we only need to compare strategy A and B from a window of 3 months

$$\begin{aligned}
E(X_A|T_A \leq 3) &= \frac{E(X_A|T_A=1)\Pr(T_A=1) + E(X_A|T_A=2)\Pr(T_A=2) + E(X_A|T_A=3)\Pr(T_A=3)}{\Pr(T_A \leq 3)} \\
&= \frac{5,000,000 \cdot \Pr(T_A \leq 3) - 30,000 \cdot \Pr(T_A=1) - 60,000 \cdot \Pr(T_A=2) - 90,000 \cdot \Pr(T_A=3)}{\Pr(T_A \leq 3)} \\
&= 5,000,000 - \frac{30,000 \cdot 0.9^0 \cdot 0.1 + 60,000 \cdot 0.9^1 \cdot 0.1 + 90,000 \cdot 0.9^2 \cdot 0.1}{0.271} \\
&= 5,000,000 - \frac{3000 + 5400 + 7290}{0.271} \\
&= 4,942,103
\end{aligned}$$

$$E(X_C) = 4,200,000$$

$$\begin{aligned}
E(X_{A,C}) &= E(X_A|T_A \leq 3) \cdot \Pr(T_A \leq 3) + E(X_C|T_C > 3) \cdot \Pr(T_A > 3) \\
&= 4,942,103 \cdot 0.271 + 4,200,000 \cdot 0.729 \\
&= 4,401,110
\end{aligned}$$

$$\begin{aligned}
E(X_B|T_B \leq 3) &= \frac{E(X_B|T_B=1)\Pr(T_B=1) + E(X_B|T_B=2)\Pr(T_B=2) + E(X_B|T_B=3)\Pr(T_B=3)}{\Pr(T_B \leq 3)} \\
&= \frac{4,800,000 \cdot \Pr(T_B \leq 3) - 30,000 \cdot \Pr(T_B=1) - 60,000 \cdot \Pr(T_B=2) - 90,000 \cdot \Pr(T_B=3)}{\Pr(T_B \leq 3)} \\
&= 4,800,000 - \frac{30,000 \cdot 0.8^0 \cdot 0.2 + 60,000 \cdot 0.8^1 \cdot 0.2 + 90,000 \cdot 0.8^2 \cdot 0.2}{0.488} \\
&= 4,800,000 - \frac{6000 + 9600 + 11520}{0.488} \\
&= 4,744,426
\end{aligned}$$

$$\begin{aligned}
E(X_{B,C}) &= E(X_B|T_A \leq 3) \cdot \Pr(T_B \leq 3) + E(X_C|T_C > 3) \cdot \Pr(T_B > 3) \\
&= 4,744,426 \cdot 0.488 + 4,200,000 \cdot (1 - 0.488) \\
&= 4,465,680
\end{aligned}$$

It is most profitable for the couple to choose the combination {strategy B, strategy C}.

2 Question 2

1. X follows the hypergeometric distribution.

2.

   (a) $\Pr(X \geq 1|n=5) = 1 - \begin{pmatrix} 5 \\ 0 \end{pmatrix} \cdot 0.05^0 \cdot 0.95^5 = 1 - 0.95^5 = 0.2262.$

   (b) $\Pr(X \geq 1|n=10) = 1 - \begin{pmatrix} 10 \\ 0 \end{pmatrix} \cdot 0.95^{10} = 1 - 0.95^{10} = 0.4013.$

   With a sample of 5 chips the probability of having at least one chip with error is lower than 30 per cent and the firm will not ship the 60 chips to its customer. However, with a sample of 10 chips the probability of at least one errorneous chip is greater than 30 per cent and the 60 chips can be shipped to the customer.

3.

$$
\begin{aligned}
\Pr(X \geq 1|n=5) &= 1 - \frac{\begin{pmatrix} M \\ \sum x_i \end{pmatrix}\begin{pmatrix} N-M \\ n-\sum x_i \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}} \\
&= 1 - \frac{\begin{pmatrix} 3 \\ 0 \end{pmatrix}\begin{pmatrix} 60-3 \\ 5-0 \end{pmatrix}}{\begin{pmatrix} 60 \\ 5 \end{pmatrix}} \\
&= 1 - \frac{\frac{57!}{5!(57-5)!}}{\frac{60!}{5!(60-5)!}} \\
&= 1 - \frac{53 \cdot 54 \cdot 55}{58 \cdot 59 \cdot 60} \\
&\approx 0.2333
\end{aligned}
$$

$$
\begin{aligned}
\Pr(X \geq 1|n=10) &= 1 - \frac{\begin{pmatrix} M \\ \sum x_i \end{pmatrix}\begin{pmatrix} N-M \\ n-\sum x_i \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}} \\
&= 1 - \frac{\begin{pmatrix} 3 \\ 0 \end{pmatrix}\begin{pmatrix} 60-3 \\ 10-0 \end{pmatrix}}{\begin{pmatrix} 60 \\ 10 \end{pmatrix}} \\
&= 1 - \frac{\frac{57!}{10!(57-10)!}}{\frac{60!}{10!(60-10)!}} \\
&\approx 0.4272
\end{aligned}
$$

   Although the probabilities are different, they give rise to the same conclusion as in question 2.

4. When the population (the shipment) is very large compared to the (quality control) sample size, the the hypergeometric variance $n \cdot p \cdot (1-p)\frac{N-n}{N-1}$ is close to the binomial variance as $\frac{N-n}{N-1} \rightarrow 1$ when N becomes large compared to n. Furthermore, the hypergeometric probabilities get closer to the binomial probabilities. The intuition for this is that with a small n and a large N it is unlikely that we sample the same item twice and whether we sample with or without replacement becomes less important.

5.

$$
\begin{aligned}
\Pr(X \geq 12) &= 1 - \Pr(X \leq 11) \\
&\simeq 1 - \Phi\left(\frac{11 + \frac{1}{2} - 500 \cdot 0.03}{\sqrt{500 \cdot 0.03 \cdot 0.97}}\right) \\
&\simeq 1 - \Phi(-0.917) \\
&\simeq 1 - 0.1788 \\
&\simeq 0.8212
\end{aligned}
$$

Since the probability is greater than 60 per cent the firm will ship the $10,000$ chips.

3 Question 3

1. $\hat{\alpha}_0 = \frac{1}{N}\sum_{i=1}^{N} z_{0i}$ where $\hat{\alpha}_0 \sim N\left(\alpha_0, \frac{\sigma_0}{N}\right)$
   $\hat{\sigma}_0^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(z_{0i} - \hat{\alpha}_0\right)^2$
2. $\hat{\alpha}_0 = 75.03$
   $\hat{\sigma}_0^2 = 5.07^2 = 25.705$
3. We use a paired test with null hypothesis $H_0 : \delta = 0$. We can calculate the T statistic as

$$
T = \frac{\bar{D} - E\left(\bar{D}|\delta = 0\right)}{s.e.\left(\bar{D}\right)} = \frac{1.36 - 0}{3.11/\sqrt{48}} = 3.0297
$$

   The p-value is 0.02. The conclusion from the test is that performance pay increases the average math scores.

4.
   (a) Since we have 48 observations we use the standard normal percentile values to compute the 95% confidence interval

$$
1.36 \pm 1.96 \cdot 3.11/\sqrt{48} = \left\{ \begin{array}{l} 0.4802 \\ 2.2398 \end{array} \right.
$$

   such that the 95% confidence interval is given by $[0.4802; 2.2398]$ which implies that with probability 0.95 the population mean is in between 0.4802 and 2.2398.

   (b) The interquartile range of the effect is given by $[-0.37; 3.44]$. This implies that more than 25 per cent of the schools experienced a negative effect. We also observe that the mean is closer to the first quartile than the third quartile such that the distribution is right-skewed. We also notice that the first and third quartiles of 2001 are larger than the first and third quartiles of 2000. This suggests that the schools improving most came from a relative low level in 2000.

5. With 48 observations we should not hope to see something looking exactly as the normal distribution. The distributions looks unimodal, but the right tale is very thin. We could use the sign test or the sign-rank test. These two tests are nonparametric tests since they do not depend on the population distribution. In contrast to the sign test, the sign-rank test takes some account of the magnitudes of deviations. It is not allowed to use the Wilcoxon rank sum test since the samples are dependent.

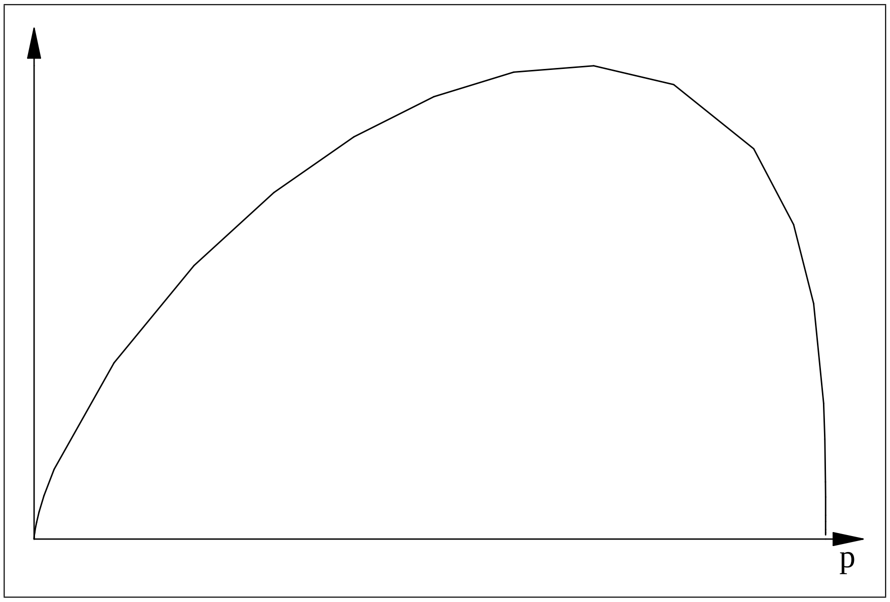
4 Question 4

1. $\hat{p} = \bar{x}_N = \frac{1}{N}\sum_{i=1}^{N} x_i$
   It is a consistent estimator since since $E\left(\bar{x}_N\right) \to p$ and $Var\left(\bar{x}_N\right) = \frac{p(1-p)}{N} \to 0$ as $N \to \infty$.
2. $\hat{p} = \frac{33}{48} = 0.6875$.

3.

$$
\begin{aligned}
L(p) &= \prod_{i=1}^{48} p^{x_i}(1-p)^{(1-x_i)} \\
&= p^{33}(1-p)^{15}
\end{aligned}
$$

The likelihood function is maximized by $\hat{p} = 0.6875$.



4. We can use a χ^2 test of independence. In order to use the χ^2 distribution we need that the minimum expected cell quantity is at least 5.

5.

$$
\begin{aligned}
\chi^2 &= \frac{\left(24-\frac{35\cdot33}{48}\right)^2}{\frac{35\cdot33}{48}} + \frac{\left(11-\frac{35\cdot15}{48}\right)^2}{\frac{35\cdot33}{48}} + \frac{\left(9-\frac{13\cdot33}{48}\right)^2}{\frac{13\cdot33}{48}} + \frac{\left(4-\frac{13\cdot15}{48}\right)^2}{\frac{13\cdot15}{48}} \\
&= \frac{(24-24.0625)^2}{24.0625} + \frac{(11-10.9375)^2}{10.9375} + \frac{(9-8.9375)^2}{8.9375} + \frac{(4-4.0625)^2}{4.0625} \\
&= 0.00192
\end{aligned}
$$

which is χ^2-distributed with 1 degree of freedom. The critical value is on a 5 per cent level is 3.84. Hence, we cannot reject that whether a school benefitted and school type is independent. Therefore, the eperiment was beneficial whether the school is Jewish or Arabic. We notice that the expected cell count for one of the cells is lower than 5 so the conditions for using the test are not met.