Solution to the Take-home Exam

Theoretical and Empirical Foundations of DSGE Modeling

Summer School, 2016

Question 1 Borrowers maximize their utility under the collateral and the flow-of-funds constraints. The resulting Lagrangian is:

$$\begin{aligned}\mathcal{L}_t^B &= E_t \sum_{t=0}^{\infty} \left(\beta^B\right)^t \left\{ c_t^B - \vartheta_t^B \left[c_t^B + R^B b_{t-1}^B + q_t(k_t^B - k_{t-1}^B) - b_t^B - \alpha_t k_{t-1}^B \right] \right. \qquad (1)\\ & \left. -\varphi_t \left(b_t^B - \omega \frac{q_{t+1} k_t^B}{R^B} \right) \right\},\end{aligned}$$

where ϑ_t^B and φ_t are the multipliers associated with borrowers' budget and collateral constraint, respectively. The first-order conditions are:

$$\frac{\partial \mathcal{L}_t^B}{\partial b_t^B} = 0 \Rightarrow -\beta^B R^B E_t \vartheta_{t+1}^B + \vartheta_t^B - \varphi_t = 0; \qquad (2)$$

$$\frac{\partial \mathcal{L}_t^B}{\partial k_t^B} = 0 \Rightarrow -\vartheta_t^B q_t + \beta^B E_t \left[\vartheta_{t+1}^B q_{t+1}\right] + \beta^B E_t \left[\vartheta_{t+1}^B \alpha_{t+1}\right] + \omega \varphi_t E_t \left[\frac{q_{t+1}}{R^B}\right] = 0. \qquad (3)$$

Condition (2) implies that a marginal decrease in borrowing today expands next period's utility and relaxes the current period's borrowing constraint. As to (3), acquiring an additional unit of capital today allows to expand future consumption not only through the conventional capital gain and dividend channels, but also through the feedback effect of the expected collateral value on the price of capital. As in KM, in the neighborhood of the steady state the collateral constraint turns out to be binding when $\varphi > 0$. This is the case when $R^B < 1/\beta^B$, which is imposed throughout the analysis. As we consider linear preferences (i.e., $\vartheta_t^B = \vartheta^B = 1$), (2) implies $\varphi_t = \varphi = 1 - \beta^B R^B$. As a result, (3) can be rewritten as

$$q_t = \frac{\beta^B R^B + \omega\left(1 - \beta^B R^B\right)}{R^B} E_t q_{t+1} + \beta^B E_t \alpha_{t+1}. \qquad (4)$$

The Lagrangian for lenders' optimization reads as

$$\begin{aligned}\mathcal{L}_t^L &= E_t \sum_{t=0}^{\infty} \left(\beta^L\right)^t \left\{ c_t^L - \vartheta_t^L [c_t^L + b_t^B + q_t(k_t^L - k_{t-1}^L) \right. \qquad (5)\\ & \left. -R^B b_{t-1}^B - \alpha_t (k_{t-1}^L)^{\mu}] \right\},\end{aligned}$$

where ϑ_t^L is the multiplier associated with bankers' budget constraint. The first-order conditions are:

$$\frac{\partial \mathcal{L}_t^L}{\partial b_t^B} = 0 \Rightarrow R^B \beta^L E_t \vartheta_{t+1}^L - \vartheta_t^L = 0; \qquad (6)$$

$$\frac{\partial \mathcal{L}_t^L}{\partial k_t^L} = 0 \Rightarrow -\vartheta_t^L q_t + \beta^L E_t \left[\vartheta_{t+1}^L q_{t+1}\right] + \mu \beta^L E_t \left[\alpha_{t+1} (k_t^L)^{\mu-1}\right] = 0. \qquad (7)$$

As we assume linear preferences, $\vartheta_t^L = \vartheta^L = 1$. Therefore, condition (6) implies $R^B\beta^L = 1$. Finally, from (7) we can retrieve the Euler equation governing bankers' investment:

$$q_t = \beta^L E_t q_{t+1} + \beta^L \mu E_t \left[\alpha_{t+1}(k_t^L)^{\mu-1}\right] \tag{8}$$

Question 2 Recall that $\alpha = 1$. Evaluating (4) in the non-stochastic steady state returns:

$$q = \frac{R^B\beta^B}{\left(1-\beta^B\right)R^B - \omega\left(1-\beta^B R^B\right)}. \tag{9}$$

From (8) we retrieve the marginal product of lenders' capital, as a function of the price of capital:

$$\mu\left(k^L\right)^{\mu-1} = \frac{1-\beta^L}{\beta^L}q, \tag{10}$$

so that Equations (9), (10) and $k^L + k^B = 1$ allow us to pin down both agents' holdings of capital. To check that the marginal product of capital is lower for the lender than for the borrower (for which it is 1), we can write the following inequality:

$$\begin{aligned}
\frac{1-\beta^L}{\beta^L}q &< 1 \Leftrightarrow \\
\frac{1-\beta^L}{\beta^L}\frac{R^B\beta^B}{\left(1-\beta^B\right)R^B - \omega\left(1-R^B\beta^B\right)} &< 1 \Leftrightarrow \\
\left(1-\beta^L\right)R^B\beta^B &< \left(1-\beta^B\right)\beta^L R^B - \omega\beta^L\left(1-R^B\beta^B\right) \Leftrightarrow \\
R^B\beta^B - \beta^L R^B\beta^B &< \beta^L R^B - \beta^B\beta^L R^B - \omega\beta^L\left(1-R^B\beta^B\right) \Leftrightarrow \\
R^B\left(\beta^B-\beta^L\right) &< -\omega\beta^L\left(1-R^B\beta^B\right) \Leftrightarrow \\
\beta^B - \beta^L &< -\frac{\omega\beta^L\left(1-R^B\beta^B\right)}{R^B}.
\end{aligned}$$

We have in steady state that $R^B\beta^L = 1$, which we can use to rewrite the condition as:

$$\begin{aligned}
\beta^B - \beta^L &< -\frac{\omega\beta^L\left(1-R^B\beta^B\right)}{R^B} \Leftrightarrow \\
R^B\beta^B - R^B\beta^L &< -\omega\beta^L\left(1-R^B\beta^B\right) \Leftrightarrow \\
0 &< 1 - R^B\beta^B - \omega\beta^L\left(1-R^B\beta^B\right) \Leftrightarrow \\
0 &< \left(1-\omega\beta^L\right)\left(1-\beta^B R^B\right).
\end{aligned}$$

This inequality is necessarily satisfied given the parameter restrictions. To evaluate the impact of ω, one can write up the difference between marginal productivities and differentiate this with respect to ω. Since the marginal product for the borrower is constant, it suffices to differentiate the marginal product for the lender. It can be easily verified that this partial derivative is positive. The intuition is that a higher LTV ratio leads to a reallocation of capital from the lender to the borrower, driving up the marginal product of capital for the lender, thus reducing the gap between the two marginal products.

Question 3 We impose $\phi \equiv \frac{\beta^B R^B + \omega\left(1-\beta^B R^B\right)}{R^B}$ for notation reasons into borrowers' Euler:

$$q_t = \phi E_t q_{t+1} + \beta^B E_t \alpha_{t+1}.$$

We log-linearize the equation by following these steps:

$$\begin{aligned}
q + q\hat{q}_t &= \phi q + \beta^B \alpha + \phi q E_t \hat{q}_{t+1} + \beta^B \alpha E_t \hat{\alpha}_{t+1} \\
q + q\hat{q}_t &= \underbrace{\phi q + \beta^B \alpha}_{=q} + \phi q E_t \hat{q}_{t+1} + \underbrace{\beta^B \alpha}_{=q(1-\phi)} E_t \hat{\alpha}_{t+1} \\
q\hat{q}_t &= \phi q E_t \hat{q}_{t+1} + (1-\phi)\, q E_t \hat{\alpha}_{t+1} \\
\hat{q}_t &= \phi E_t \hat{q}_{t+1} + (1-\phi)\, E_t \hat{\alpha}_{t+1}
\end{aligned}$$

As for the lenders' Euler:

$$\begin{aligned}
q + q\hat{q}_t &= \beta^L \left[q + \mu\alpha (k^L)^{\mu-1}\right] + \beta^L q E_t \hat{q}_{t+1} \\
&\quad + \beta^L \mu\alpha (k^L)^{\mu-1} E_t \hat{\alpha}_{t+1} + \beta^L \mu\alpha\,(\mu-1)\,(k^L)^{\mu-1} \hat{k}^L_t \\
q + q\hat{q}_t &= \underbrace{\beta^L \left[q + \mu\alpha (k^L)^{\mu-1}\right]}_{=q} + \beta^L q E_t \hat{q}_{t+1} \\
&\quad + \beta^L \mu\alpha (k^L)^{\mu-1} E_t \hat{\alpha}_{t+1} + \beta^L \mu\alpha\,(\mu-1)\,(k^L)^{\mu-1} \hat{k}^L_t \\
q\hat{q}_t &= \beta^L q E_t \hat{q}_{t+1} + \underbrace{\beta^L \mu\alpha (k^L)^{\mu-1}}_{=q\left(1-\beta^L\right)} E_t \hat{\alpha}_{t+1} + \underbrace{\beta^L \mu\alpha (k^L)^{\mu-1}}_{=q\left(1-\beta^L\right)} (\mu-1)\, \hat{k}^L_t \\
\hat{q}_t &= \beta^L E_t \hat{q}_{t+1} + \left(1-\beta^L\right) E_t \hat{\alpha}_{t+1} + \left(1-\beta^L\right)(\mu-1)\, \hat{k}^L_t
\end{aligned}$$

Finally, by log-linearizing the aggregate resource constraint for capital:

$$0 = k^L \hat{k}^L_t + k^B \hat{k}^B_t$$

Thus, lenders' Euler reduces to:

$$\hat{q}_t = \beta^L E_t \hat{q}_{t+1} + \left(1-\beta^L\right) E_t \hat{\alpha}_{t+1} + \frac{1-\beta^L}{\eta} \hat{k}^B_t$$

where $\eta \equiv \frac{1-k^B}{k^B(1-\mu)}$.

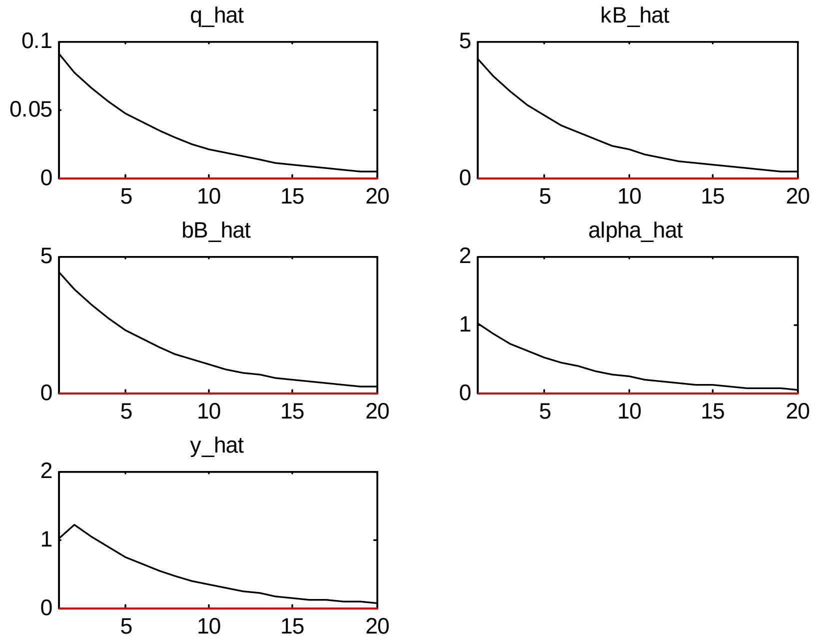
Question 4 By iterating $\hat{q}_t = \phi E_t \hat{q}_{t+1} + (1-\phi)\, E_t \hat{\alpha}_{t+1}$ forward we obtain:

$$\hat{q}_t = \beta^B E_t \sum_{t=0}^{\infty} \phi^t E_t \hat{\alpha}_{t+1} \tag{11}$$

As $\log \alpha_t = \rho \log \alpha_{t-1} + u_t$, it is immediate to show that (11) leads to the following solution:

$$\hat{q}_t = \gamma \hat{\alpha}_t, \tag{12}$$

where $\gamma \equiv \frac{1-\phi}{1-\phi\rho}\rho > 0$.



Thus, we can take the solutions for $\hat{q}_t$ and $E_t\hat{q}_{t+1}$ and plug them into lenders' linearized Euler:

$$\gamma\hat{\alpha}_t = \beta^L\gamma E\hat{\alpha}_{t+1} + \left(1-\beta^L\right)E_t\hat{\alpha}_{t+1} + \frac{1-\beta^L}{\eta}\hat{k}_t^B$$

Finally, since $E\hat{\alpha}_{t+1} = \rho\hat{\alpha}_t$, the following solution can be obtained:

$$\hat{k}_t^B = \nu\hat{\alpha}_t, \tag{13}$$

where $\nu \equiv \frac{\eta}{1-\beta^L}\frac{\left(\beta^L-\phi\right)(1-\rho)\rho}{1-\phi\rho} > 0$.

Question 5 The impulse responses are plotted in the graph below. The productivity variable $\hat{\alpha}_t$ increases as a direct result of the shock. The shock raises the marginal productivity of capital, and more so for the borrower, who therefore increases his stock of capital, while also the price of capital goes up in equilibrium. Through the rise in collateral values, this leads to an increase in the borrowing capacity of the borrower, who therefore borrows more. Finally, the increased capital stock drives up output, albeit in a hump-shaped fashion, where the peak effect on output is observed only one period after the shock itself has hit the economy.

Question 6 The hump-shaped pattern emerges clearly from the graph above. We start from the equation that relates total output to the contemporaneous shock to technology and (lagged) borrowers' capital holdings:

$$\hat{y}_t = \hat{\alpha}_t + \Delta\frac{y^B}{y}\hat{k}_{t-1}^B, \tag{14}$$

where Δ is the productivity gap between borrowers and lenders:

$$\Delta \equiv mpk^B - mpk^L. \tag{15}$$

From Question 4 we know that $\hat{k}_t^B = \nu\hat{\alpha}_t$. Therefore:

$$\hat{y}_t = \hat{\alpha}_t + \Delta\frac{y^B}{y}\nu\hat{\alpha}_{t-1}.$$

Thus, on impact $\hat{y}_0$ moves 1:1 with the contemporaneous shock $\hat{\alpha}_0$. However, after the shock at time t=0 no other shocks occur, so that at t=1 $\hat{y}_1 = \rho\hat{\alpha}_0 + \Delta\frac{y^B}{y}\nu\hat{\alpha}_0$. Since $\rho > 0$, the first term is necessarily positive. Therefore, the following condition is sufficient to ensure that the pass-through of $\hat{\alpha}_0$ onto $\hat{y}_1$ will be greater than the pass-through of $\hat{\alpha}_0$ onto $\hat{y}_0$ (i.e., 1), thus generating a hump-shaped pattern:

$$\Delta\frac{y^B}{y}v > 1.$$

Question 7 The required parameter values to make the model match these two data moments are (rounded to four decimals) $\rho = 0.7488$ and $\sigma_u = 0.0068$. These numbers can be obtained either by changing the parameters manually or by applying the set of codes used for moments-based estimation during the course.

Part 2

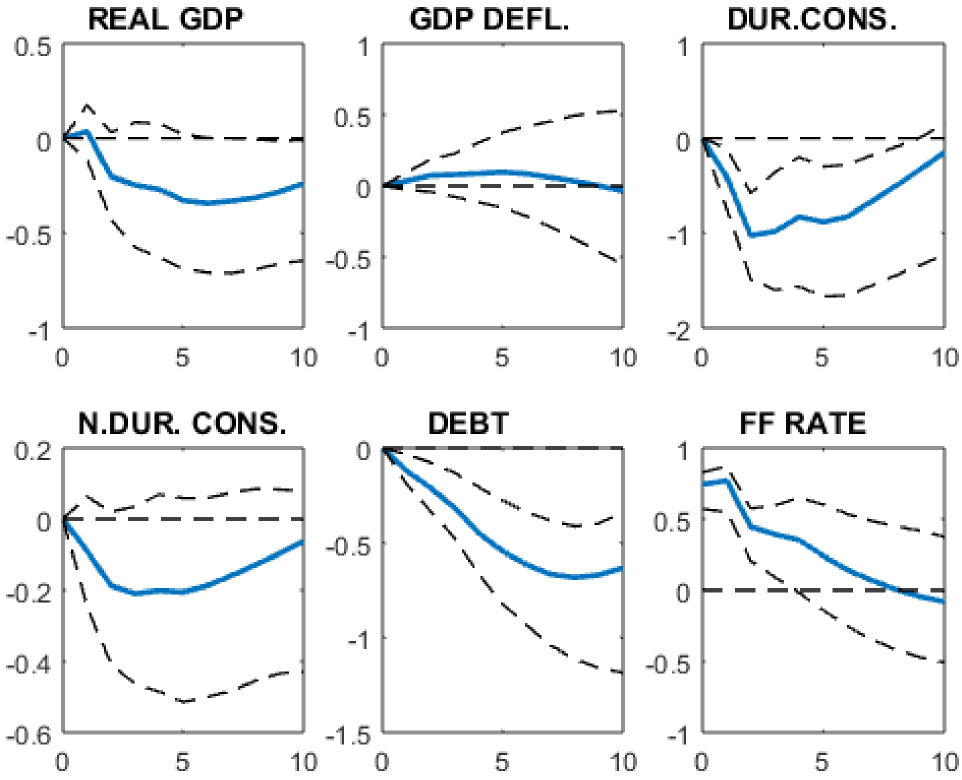
At this stage the students have taken logs of all variables, but the nominal interest rate.

Question 1 The identification in CEE is based on simple ordering of the variables, the implied assumption is that some variables are allowed to respond to movement in the interest rate on impact (so called 'fast moving', e.g. financial variables), whereas most variables react to a monetary policy shock with a lag (so called 'slow moving'). Real variables such as output, consumption and debt are typically considered slow moving variables. It is also usually assumed that prices do not change on impact (e.g. prices are set before the shock is observed).

With the variables in our dataset the federal fund rate is ordered last and the monetary policy shock is identified as the shock hitting the last column of the lower triangular Choleski matrix (computed from the variance covariance matrix of the reduced form residuals of the VAR).

Question 2 The identification of the monetary policy shock is unaffected by the relative order of the variables ordered above the interest rate. In fact, the identification assumption imposes that the impact of the variables ordered above the interest rate is 0, regardless of their relative order.

Question 3



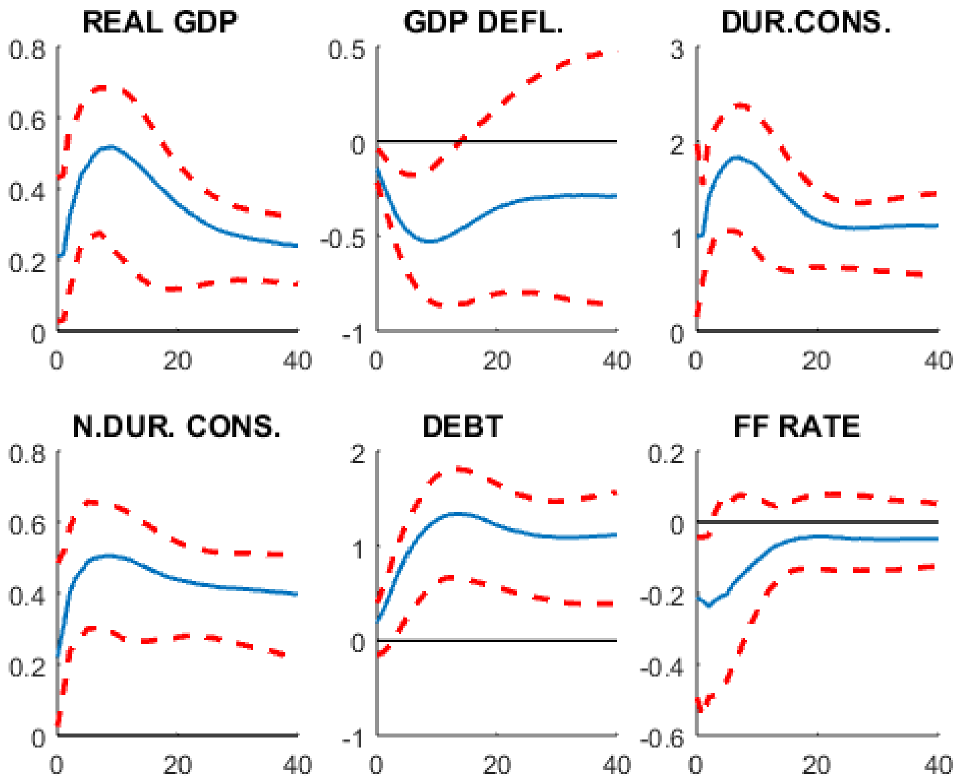
Question 4 A positive monetary policy shock rises the interest rates and contracts output and consumption (in the two sectors), debt is also reduced. The response of durable consumption is stronger than the one of nondurable consumption. The impulse response shows rising prices following a tighter monetary policy stance (price puzzle). This contradicts the standard transmission mechanism of the basic NK model.

Question 5 Debt is a state variables in any DSGE models. The exclusions of state variables from the set of variables in the VAR makes more likely that the VAR is non-fundamental. In such a case the impulse response from the VAR are unlikely to recover the true impulse response to the structural shocks.

Question 6 The pronounced magnitude of durables' response depends on two inherent features of this type of good: first, the demand for durables is for a stock, so that changes in the stock demand translate into much larger fluctuations in the flow demand for newly produced goods; second, potential sources of sectoral price rigidities tend to mitigate the role that changes in the relative price of durables play in insulating the durables sector from shocks (see Erceg and Levin, JME 2006).

Question 7 Standard models have difficulties to rationalize the delayed response of the slow moving variables. In fact it is likely that shocks identified with this identification restrictions are likely to be a weighted average of various structural shocks. For instance if one takes a standard 3 equations NK DSGE model, shocks identified imposing the ordering to the data deliver a monetary policy shock that is a weighted average of the true MP shock and technology shock, with the latter playing a dominant role (see Carlstrom, Fuerst and Paustian, JME 2009).

Technology Shock



Monetary Policy Shock