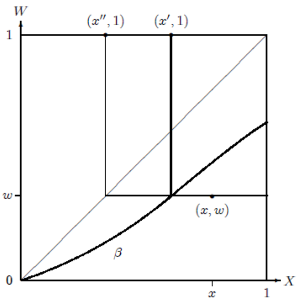
<u>**Solutions for exam in Auctions, 6 June 2013**</u>

The following solutions manual gives thorough answers to all questions. This level of detail is not required to obtain full marks on each question.

<u>Exercise 1: True or false statements (20%)</u>

Question 1a: False. Bidders will change their strategies according to the chosen auction format. In a second-price auction bidders bid truthfully, $\beta^{II}(x) = x$ whereas in a first-price auction bidders will shade their bids $\beta^{I}(x) = \frac{1}{G(x)}\int_0^x yg(y)dy$. Thus the highest bid will be lower in a first price auction.

Question 1b: True. If bidders have low budgets they are more likely to be budget constrained (cannot bid as much as you want to) in a second-price auction, as $\beta^{II}(x) = x > \beta^{I}(x) = \frac{1}{G(x)}\int_0^x yg(y)dy$ – i.e. bidders shade bids in first-price auctions. A budget constrained bidder will bring in less expected revenue, thus a first-price auction will bring in a larger revenue.



The figure shows the bidding strategy for a given valuation and budget. The thin line is derived from the strategy $B^{II}(x, w) = \min\{x, w\}$ and the thick line is derived from the strategy $B^{I}(x, w) = \min\{\beta(x), w\}$.
With a given budget of w bidders with values larger than x''will be constrained in the second-price auction. But only bidders with values larger than x' will be constrained in the first-price auction. Since $x' > x''$ bidders are less likely to be budget constrained in a first-price auction compared to a second-price auction.
From proposition 4.3, we know if bidders are subject to budget constraints, then in a symmetric equilibrium, the expected revenue in a first-price auction is greater than the expected revenue in a second-price auction:

$$E\left[Y_2^{I(N)}\right] > E\left[Y_2^{II(N)}\right]$$

Question 1c: False. Bidder 1 wins 2 units because he has the two highest bids, but bidder 1 pays 26 (14+12), corresponding to the two highest losing bids of the other bidders.

Formally, if bidder *i* wins k^i units in a Vickrey auction he pays $\sum_{k=1}^{k^i} c_{K-k^i+k}^{-i}$ where c^{-i} is the K-vector of competing bids facing bidder *i*. With *K=2* units for sale, the K-vector of competing bids facing bidder 1 is $c^{-i}=(14,12)$. Thus, the price paid by bidder 1 is:

$$\sum_{k=1}^{k^i} c_{K-k^i+k}^{-i} = \sum_{k=1}^{2} c_{2-2+k}^{-i} = c_1^{-i} + c_2^{-i} = 14 + 12 = 26$$

Exercise 2: The Revenue Equivalence Principle (25%)

Question 2a: Yes. We can verify each of the requirements to live up to the Revenue Equivalence Principle in turn:

- *Standard auction*: Yes. The person with the highest bid wins the auction so contribution auction is indeed a standard auction.
- *The expected payment of a bidder with value zero is zero*: Yes. Each bidder pays 50% of its bid so a bidder with value zero can bid zero and pay zero.
- *Independently and identically distributed values*: Yes. The market survey shows that bidders' values are independently and identically distributed.
- *Risk neutral:* Yes. The market survey shows that bidders are risk neutral.

Question 2b: The derivation in this question follows the derivation of the equilibrium strategy of the *all-pay auction* covered in Krishna, section 3.2.1.

According to the Revenue Equivalence Principle, the expected payment per bidder in the contribution auction (CA) is:

$$m^{CA}(x) = \int_0^x y g(y) dy$$

The rules of the contribution auction states that each bidder pays 50% of his bid. Thus, if there exists a symmetric and increasing equilibrium strategy B^{CA} it must be that:

$$B^{CA}(x) = 2 \cdot m^{CA}(x) = 2 \cdot \int_0^x y g(y) dy$$

Since values are uniformly distributed on the interval [0,1000] then

$$F(x) = \frac{x}{1000}$$

$$G(y) = F(y)^{N-1} = \left(\frac{y}{1000}\right)^{N-1}$$

$$g(y) = G'(y) = \frac{(N-1)y^{N-2}}{1000^{N-1}}$$

Therefore, if there exists a symmetric and increasing equilibrium strategy, it must be:

$$B^{CA}(x) = 2 \cdot m^{CA}(x) = 2 \cdot \int_0^x y \left(\frac{(N-1)y^{N-2}}{1000^{N-1}}\right) dy = \frac{2(N-1)}{1000^{N-1}} \int_0^x y^{N-1} dy$$

$$= \frac{2(N-1)}{1000^{N-1}} \left[\frac{1}{N} y^N\right]_0^x = \frac{2}{1000^{N-1}} \cdot \frac{(N-1)}{N} \cdot x^N$$

Question 2c: From the Revenue Equivalence Principle we know that the expected payment per bidder is

$$m^A(x) = \int_0^x y g(y) dy$$

Since there are three bidders in the auction the expected revenue, $E[R^{CA}]$, to the seller must be:

$$E[R^{CA}] = N \cdot E[m^{CA}(x)] = N \cdot \int_0^\omega m^{CA}(x) \cdot f(x) dx = N \cdot \int_0^\omega \frac{1}{2} B^{CA}(x) \cdot f(x) dx$$

This can be rewritten to:

$$E[R^{CA}] = N \cdot \int_0^\omega \frac{1}{2}\left[\frac{2}{1000^{N-1}} \cdot \frac{(N-1)}{N} \cdot x^N\right] \cdot \frac{1}{1000} dx = 3 \cdot \int_0^{1000} \frac{1}{2}\left[\frac{2}{1000^{3-1}} \cdot \frac{(3-1)}{3} \cdot x^3\right] \cdot \frac{1}{1000} dx$$

$$= \frac{2}{1000^{2+1}} \cdot \int_0^{1000} x^3 dx = \frac{2}{1000^3} \cdot \left[\frac{1}{4} x^4\right]_0^{1000} = \frac{2}{1000^3} \cdot \frac{1000^4}{4} = 500$$

The expected revenue is thus 500. Due to the Revenue Equivalence Principle, one could have calculated the expected revenue for another auction that satisfies the Revenue Equivalence Principle, e.g. a first-price sealed-bid auction or a second-price sealed-bid auction, and obtained the same result.

Exercise 3: An auction for oil and gas rights (25%)

Question 3a: This is the mineral rights model known from the textbook, chapter 6. It is an auction where bidders have interdependent values and each bidder draws a signal, $X_i \epsilon [0, \omega]$.
It is reasonable to assume bidders have pure common valuations, i.e. valuations are identical on the form
$V = (X_1, X_2, \dots, X_N)$. Once all signals are revealed, everyone can infer the true value of the oil and gas in the sea zone, *ex-post*. The true value is the same for all companies (pure common valuations), i.e. each company's extraction technology is the same and all can obtain the same selling price on the world energy markets.
It is typically assumed each X_i is an unbiased estimator of V so $E[X_i | V = v] = v$. So each company's search technology is the same.

Question 3b: Symmetric equilibrium strategy in II can be described as:

$$\beta^{II}(x) = v(x, x) = E[V | X_1 = x, Y_1 = x]$$

Thus, for instance, bidder 1 will bid the expectation of the common value given his own signal, $X_1 = x$, and as if the highest competing bidder has the same valuation as himself, $Y_1 = x$. If bidder 1 bids as $\beta^{II}(x)$ and wins with the highest competing bid also being $\beta^{II}(x)$ then bidder 1 would just break even, through updating his valuation to

$$E[V | X_1 = x, Y_1 = x] = v(x, x) = \beta^{II}(x).$$

It should be noted that winner's curse, i.e. winning brings the bad news of having to pay more than the true value, is not an equilibrium feature. Bidders will rationally incorporate this risk of winner's curse by shading their bids below their initial estimates of the true value in the strategy above.

The proof of the strategy is not asked for in the exam question. It exploits the typical second-price argumentation, but it hinges critically on everyone else bidding symmetrically. There are possible asymmetric equilibrium strategies, and thus truth-telling is not a weakly dominant strategy as it was in the private values setting.

Proof of $\beta^{II}(x)$. Krishna, p. 89:
Suppose all other bidders $j \neq 1$ follow the strategy $\beta \equiv \beta^{II}$. Bidder 1's expected payoff when his signal is x and he bids an amount b is

$$\Pi(b, x) = \int_0^{\beta^{-1}(b)} \big(v(x, y) - \beta(y)\big) g(y|x) dy = \int_0^{\beta^{-1}(b)} \big(v(x, y) - v(y, y)\big) g(y|x) dy$$

Where $g(\cdot\, | x)$ is the density of $Y_1 \equiv \max_{i \neq 1} X_i$ conditional on $X_1 = x$
Since v is increasing in the first argument, for all $y < x$ we have that $v(x, y) - v(y, y) > 0$. Similarly for all $y > x$ we have that $v(x, y) - v(y, y) < 0$. Thus Π is maximzed by choosing b so that $\beta^{-1}(b) = x$ or equivalently, by choosing $b = \beta(x)$. The argument is very simple: An integral of a decreasing line (for instance: $-x + 1$) is maximized when the line ends at zero, i.e. $x = 1$.

Question 3c: An open English auction may be a good idea given the interdependent values.

The revenue linkage principle ranks the expected revenue to the seller: $E[R^{ENG}] \geq E[R^{II}] \geq E[R^{I}]$
Thus the expected revenue to the government is higher.

This is due to the fact that an English auction allows for information to be shared between bidders; when a bidder drops out others will infer his signal and update their estimate of the true valuation accordingly. This will allow for more aggressive bidding as bidders will now not have to shade their bids as much in order to avoid winner's curse. However it is important to note it only works for $N > 2$. One bidder has to drop out <u>and</u> the auction has to continue.

The strategy in the English auction can be described as

$$\beta^{ENG} = \begin{cases} \beta^N(x) = u(x, x, \dots x) \\ \beta^k(x, p_{k+1}, \dots, p_N) = u(x, \dots, x, x_{k+1}, \dots, x_N) \end{cases}$$

Where $(p_{k+1}, \dots, p_N)$ are the prices at which the dropped-out bidders dropped out and $(x_{k+1}, \dots, x_N)$ are their inferred values.

Another point can be made towards efficiency. An auction is said to be efficient if the highest valuing bidder will win the auction. With symmetric, interdependent values and affiliated signals English, second-price and first-price auctions all require the same condition to be efficient, namely the single crossing condition:

$$\frac{\partial v_i}{\partial x_i}(\mathbf{x}) \geq \frac{\partial v_j}{\partial x_i}(\mathbf{x}).$$

This condition ensures that the *ex post* values of different bidders will be ordered in the same way as their signals.
Thus the English auction is not better or worse than a second-price auction in terms of efficiency. (Proposition 6.7)

Note however, if the valuations are assumed to be pure common value, then all oil companies will value the oil field equally *ex post*. Thus it is efficient to for any oil company to win in both auction formats.

Exercise 4: A Combinatorial Clock Auction (30%)

Question 4a: The three stages are described below.

In the *Primary or Clock stage* bidders bid for generic lots of spectrum. Bidders bid for one package (i.e. combination of lots) in each round at prices determined by the auctioneer. In each clock round the price is increased for the generic lots for which there is excess demand. This continues until there is no longer excess demand for any generic lots. The purpose of this stage is to encourage price discovery (by having multiple round and observing total demand and price increases) and induce truthful bidding (by having an activity rule, the eligibility point rule).

In the *Suppelementary stage* bidders bid for generic lots of spectrum. Bidders can bid for many different packages at prices determined by themselves, subject to an activity rule (the relative cap rule). There is only one round of sealed bidding. The purpose of this stage is to let bidders express their full preferences (by allowing bidders to bid for many packages at prices of their own choice) and induce truthful bidding (by having an activity rule, the relative cap rule, and having a price rule based on second-price logic).

In the *Assignment stage* bidders bid for specific lots of spectrum. Bidders can bid for assignments for the generic lots of spectrum they have won after the Supplementary stage. There is only one round of sealed bidding. The purpose of this stage is to let bidders express preferences over assignments of the lots won (by allowing bidders to bid for assignments of their generic lots) and induce truthful bidding (by having a price rule based on second-price logic).

Question 4b: In a Combinatorial Clock Auction the winner determination states that the winning combination is the combination of bids that gives the highest total value subject to:

- Condition 1: At most one bid accepted as a winning bid per bidder
- Condition 2: Demand of winning bids does not exceed supply

Since each bidder has only submitted one bid, condition 1 is fulfilled no matter which combination is chosen as the winning combination. Since there are only 14 lots for sale, condition 2 implies that the winning combination can either contain 3 bidders that win 4 lots each or 2 bidders that win 4 lots and two bidders that win 3 lots. We compare these two possible winning combinations:

- Possible winning combination 1: Award 4 lots to bidder 1, 2 and 5. Total value = 100 + 120 + 80 = 300.
- Possible winning combination 2: Award 4 lots to bidder 1 and 2, and award 3 lots to bidder 3 and 4. Total value = 100 + 120 + 90 + 70 = 380.

Since the possible winning combination 2 gives the highest total value subject to the two conditions, this is the winning combination. Therefore, each bidder wins the following number of lots:

- *Bidder 1:* 4 lots
- *Bidder 2:* 4 lots
- *Bidder 3:* 3 lots
- *Bidder 4:* 3 lots
- *Bidder 5:* 0 lots

Question 4c: The pricing rule in a Combinatorial Clock Auction is a generalized second-price rule or a Vickrey-nearest-core price rule. The prices must be sufficiently high that there is no alternative bidder or bidders prepared to pay more than any winning bidder or group of winning bidders.

This means that any winning bidder individually pays at least its individual opportunity costs. It also means that any group of winning bidders jointly pay their joint opportunity costs. If joint opportunity costs for a group of winning bidders exceed the sum of their individual opportunity costs, then the extra costs are split evenly among the winning bidders on top of their individual opportunity costs (this is the principle of minimizing the Euclidean distance, or sum of squared deviations, from the Vickrey prices), subject to the constraint that each bidder cannot pay more than its bid for a package.

Calculation of individual opportunity costs:

- *Bidder 1:* If bidder 1 had not participated in the auction, bidder 5 would have won 4 lots with a bid of 80. Therefore, the individual opportunity cost of bidder 1 is 80.
- *Bidder 2:* Same as for bidder 1. The individual opportunity cost of bidder 2 is 80.
- *Bidder 3:* If bidder 3 had not participated in the auction, bidder 5 would have won 4 lots with a bid of 80 and bidder 4 would have won 0 lots. Therefore, the individual opportunity cost of bidder 3 is 80 – 70 = 10.
- *Bidder 4:* If bidder 4 had not participated in the auction, the 3 lots that bidder 4 is buying would have gone unsold (note that it gives a higher total value to award 3 lots to bidder 3 than to award 4 lots to bidder 5). Therefore, the individual opportunity cost of bidder 4 is zero.
- *Bidder 5:* Since bidder 5 does not win any lots, his individual opportunity costs are zero.

Since there is just one losing bidder (bidder 5), it is straightforward to check whether this bidder would be prepared to pay more than any group of winning bidders (i.e. to check whether the joint opportunity costs for any group of bidders is higher than the sum of the individual opportunity costs). This is clearly the case for the group of bidders consisting of bidder 3 and 4. Bidder 3 and 4 have individual opportunity costs of 10 + 0 = 10 although bidder 5 is willing to pay 80 for a subset of their lots. Thus, bidder 3 and 4 must jointly pay 80. According to the nearest-Vickrey price rule, the extra costs are split evenly on top of their individual opportunity costs such that bidder 3 pays 45 (10+35) and bidder 4 pays 35 (0+35).

The prices paid by each bidder are thus:

- *Bidder 1:* 80
- *Bidder 2:* 80
- *Bidder 3:* 45
- *Bidder 4:* 35
- *Bidder 5:* 0

One can note that it is bidder 3 and 4 are paying different prices although they win the same number of lots. This is perfectly in accordance with the principle of the price rule (and second-price thinking in general); bidder 3 individually prevents bidder 5 from winning any lots and thus has a higher individual opportunity cost and pays a higher price than bidder 4.

One can also note that this case illustrates that the incentives to bid truthfully are not perfect in a Combinatorial Clock Auction. As stated in Cramton (2013), "in the general case, the incentives for straightforward bidding are strong, but not perfect". If bidder 4 had lowered its bid for 3 lots, e.g. to 1, bidder 4 would still have won 3 lots but would only have paid 1 instead of 35 (because each bidder cannot pay more than its own bid). Instead, bidder 3 would have paid a larger share of the joint opportunity cost, in this case 79.