Solution Macro C exam

David Tønners

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Problem A

1) Set up the present value Hamiltonian:

$$H = \ln(c(t)) \cdot e^{-(\rho-n)\cdot t} + \lambda(t) \cdot ((r(t) \cdot (1-\tau) - n) \cdot a(t) + w(t) + v(t) - c(t)) \tag{1}$$

The intratemporal first order condition is given by:

$$\frac{\partial H}{\partial c(t)} = \frac{1}{c(t)} \cdot e^{-(\rho-n)\cdot t} - \lambda(t) = 0 \Rightarrow \lambda(t) = \frac{1}{c(t)} \cdot e^{-(\rho-n)\cdot t} \tag{2}$$

Taking logs and differentiating we get the growth rate of $\lambda(t)$:

$$\ln(\lambda(t)) = -\ln(c(t)) - (\rho - n) \cdot t \Rightarrow \tag{3}$$

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \frac{d\ln(\lambda(t))}{dt} = -\frac{\dot{c}(t)}{c(t)} - (\rho - n) \tag{4}$$

The intertemporal first order condition is given by:

$$\frac{\partial H}{\partial a(t)} = -\dot{\lambda}(t) \Rightarrow \tag{5}$$

$$\lambda(t) \cdot (r(t) \cdot (1-\tau) - n) = -\dot{\lambda}(t) \Rightarrow \frac{\dot{\lambda}(t)}{\lambda(t)} = -(r(t) \cdot (1-\tau) - n) \tag{6}$$

Combining (4) and (6) we get:

$$\frac{\dot{c}(t)}{c(t)} + \rho - n = r(t) \cdot (1-\tau) - n \Rightarrow \tag{7}$$

$$\frac{\dot{c}(t)}{c(t)} = r(t) \cdot (1-\tau) - \rho \tag{8}$$

Which is the Keynes Ramsey rule (with log-utility), stating that households choose to let consumption grow over time if and only if the return to postponing consumption (given by the after-tax interest rate) exceeds the cost of postponing consumption (given by the rate of time preference, ρ). In the optimal solution a so-called transversality condition implies that the intertemporal budget constraint (equation (A.2)) holds with an equality siqn since it can never be optimal to leave any ressources.

2) Inserting (A.5) in equation (A.2) (with an equality sign) we get:

$$\int_0^\infty c(0) \cdot e^{\int_0^t (r(s)\cdot(1-\tau)-\rho)\,ds} \cdot e^{-\int_0^t (r(s)\cdot(1-\tau)-n)\,ds}\,dt = a(0) + h(0) \tag{9}$$

The left-hand side of equation (9) can be rewritten using that:

$$e^{\int_0^t (r(s)\cdot(1-\tau)-\rho)\,ds} \cdot e^{-\int_0^t (r(s)\cdot(1-\tau)-n)\,ds} = e^{-(\rho-n)\cdot t} \tag{10}$$

which implies that the left-hand side of (9) can be written as:

$$c(0) \cdot \int_0^\infty e^{-(\rho-n)\cdot t}\,dt = c(0) \cdot \left[\frac{e^{-(\rho-n)\cdot t}}{-(\rho-n)}\right]_0^\infty = c(0) \cdot \frac{0-1}{-(\rho-n)} = \frac{c(0)}{\rho-n} \tag{11}$$

Using (11) in (9)-(10) and multiplying both sides by $\rho - n$ result in equation (A.6). This equation states the level of consumption is determined by initial wealth ($a(0)$), the present value of current and future after-tax labour income (where the discount factor is corrected for growth in the size of the representative houshold) and an "effective" discount rate, given by $\rho - n$. Consumption in any period depend not on current income as such, but on lifetime earnings, reflecting the desire for a stable consumption pattern.

In this special case (with log-utility) the substitution and income effect of a higher interest rate will exactly cancel each other out. However, higher future interest rates still affect consumption by affecting the present value of future after-tax labour income, according to the expression for $h(0)$. Higher future interest rates thus decrease consumption.

3) (A.7) states that firms demand capital untill the marginal product of capital (the right-hand side) equals the real rental rate. (A.8) states that firms demand labour input untill the marginal product of labour equals the real wage rate. (A.9) is an arbitrage equation stating that return to physical capital (given by the rental rate since there is no depreciation) must equal the return to bonds/financial capital (given by the interest rate), since these two kinds of capital are perfect substitutes and they are both taxed at the same rate. Finally (A.10) states that total household wealth must equal the aggregate capital stock, since we consider a closed economy and a balanced public budget (and thus we assume away public debt).
Using $a(t) = k(t)$ in equation (A.3) we get:

$$\dot{k}(t) = (r(t) \cdot (1-\tau) - n) \cdot k(t) + w(t) + v(t) - c(t) \tag{12}$$

Using the public budget constraint (equation (A.11)) we get:

$$\dot{k}(t) = (r(t) - n) \cdot k(t) + w(t) - c(t) \tag{13}$$

Finally using (A.7)-(A.9):

$$\dot{k}(t) = (f'(k(t)) - n) \cdot k(t) + f(k(t)) - k(t) \cdot f'(k(t)) - c(t) \tag{14}$$

implying:

$$\dot{k}(t) = f(k(t)) - c(t) - n \cdot k(t) \tag{15}$$

This equation simply states that saving (the part of output/income which is not consumed) is used for capital accumulation. Population growth tends to reduce the amount of capital per worker, thus a certain amount of investment $(n \cdot k(t))$ is needed simply to keep capital per worker constant. That amount is called replacement investment.

4) Before deriving the phase diagram let's just notice that the steady state is defined by a constant level of consumption per worker and a constant level of capital per worker. From the Keynes-Ramsey rule and (A.7) and (A.9) we see that a constant level of consumption per worker requires:

$$f'(k^*) = \frac{\rho}{1-\tau} \tag{16}$$

If the initial level of the capital stock is below k^* the marginal product is higher that $\rho/(1-\tau)$ (since $f''(k) < 0$), implying that the after-tax interest rate exceeds the rate of time preference. From the Keynes Ramsey rule this implies that consumption per worker is growing over time. The opposite occurs when the initial level of the capital stock exceeds k^*.

From equation (A.12) we see that a constant level of capital per worker implies:

$$c(t) = f(k) - n \cdot k \tag{17}$$

This relationship between consumption per worker and capital per worker is concave, and reaches a maximum for k equal to the golden rule level, defined by:

$$f'\left(k^{GR}\right) = n \tag{18}$$

Since we assume that $\rho > n$ and $\tau \geq 0$ we see that $k^* < k^{GR}$ due to $f''(k) < 0$. That is, ineffeicient overaccumulation (dynamic inefficiency) is not a possibility.

If consumption per worker is initally below the level given by equation (17) saving per worker is high (and importantly: higher than replacement investment), implying that capital per worker is increasing over time. If consumption is initially above the level given by equation (17) the opposite occurs and capital is falling over time.

The resulting phase diagram is showed in figure 1. As the arrows illustrate, the steady state is saddle path stable, i.e. the economy will only converge towards steady state if the economy start out somewhere on the saddle path. At each point in time $k(t)$ is predetermined while $c(t)$ is free to jump. It can never be optimal for households to start out with a consumption level above the saddle path. In that case the capital stock and output would reach zero (in finite time) and consumption would jump to zero. It can never be optimal for households to plan with a discrete change in consumption. Also, it can never be optimal for households to start out with a consumption level below the saddle path. In that case housholds would asymptotically accumulate capital for the only purpose of accumulating capital (since consumption would approach zero over time). Technically this violates the transversality condition. Thus it is only optimal for housholds to choose the consumption level determined by the saddle path, such that the economy actually converges towards the steady state over time.

5) The tax reform reduces the tax rate on the return to capital income. As a result the $\dot{c}(t) = 0$-line moves to the right, while the $\dot{k}(t) = 0$-curve is unaffected, see figure 2.
Immediately at implementation the economy jumps from the initial steady state (at point E_0) to point A. Consumption jumps down, while the capital stock is predetermined in the short run. The discontinous fall in consumption can be explained by the substitution effect of a higher after-tax interest rate which makes saving more attractive. The income effect of a higher after-tax interest rate is neutralized by the fact that the lower tax revenues imply lower household transfers. At point A saving has increased which implies that capital per worker is now growing, as saving per worker now exceeds the amount of investment required to keep $k(t)$ constant over time $(n \cdot k(t))$. This explains why $k(t)$ is increasing over time as the economy moves from A to E_1. At point A the after-tax interest rate has increased above ρ (due to the lower tax rate) implying from the Keynes Ramsey rule that consumption per worker is increasing over time. According to equation (A.6) this can also be explained by the gradual increase in household wealth (see equation (A.10)) and a gradual increase in the present value of future labour income as the economy moves closer to the new steady state where wages are higher. In the new steady state consumption, output per worker per worker and capital per worker are higher than in the initial steady state while the after-tax interest rate is once again equal to ρ. The real wage rate has increased due to the larger amount of capital (which makes the labour force more productive).

Problem B

1) (B.1) is the goods market equilibrium log-linearized around the long run equilibrium. Aggregate demand depends positively on the real exhcange rate (since an increase in the real exchange rate increases international competitiveness and thereby net exports) and negatively on the real interest rate (since a higher real interest rate tends to reduce consumption and in particular investment). z is a shift parameter capturing changes in fiscal policy, world economic activity etc. (B.2) is the money market equilibrium also

formulated in terms of deviations around the long run equilibrium. Higher output increases money demand due to the transactions motive, while a higher interest rate increases the opportunity cost of holding money. It is the nominal interest rate which affects money demand, but it is assumed that expected inflation is equal to zero, such that the real and nominal interest rate coincides. (B.3) is an arbitrage equation stating that domestic and foreign bonds must pay the same rate of return. If the domestic currency is depreciating over time (i.e. if $\dot{e}^n(t) > 0$) there is an additional benefit to holding foreign bonds (besides the interest rate on foreign bonds) which is why the expected (and actual) percentage change in the exchange rate is included in this arbitrage equation. Equation (B.4) is the Philips curve with expected inflation equal to zero. It simply states that a positive output gap triggers demand-pull inflation, e.g. due to higher wage demands as unemployment falls.

Equation (B.5) and (B.6) describes how $y(t)$ and $r(t)$ adjust such as to secure the simultaneous equilibrium in the goods market and in the money market, taking $p(t)$ and $e^n(t)$ as given. An increase in $p(t)$ reduces $y(t)$ directly through lower net exports. The effect on $r(t)$ is however ambigous, as the fall in output tends to reduce money demand and thereby the interest rate, while the lower real money supply tends to increase the interest rate. We see that the first effect dominates when $\beta > 1$ since in this case the effect on net exports is strong. An increase in m reduces the interest rate (due to the higher real money supply) and thereby increases output.

2) In deriving the phase diagram it is usefulll to notice at first that the long run equilibrium is characterized by $\dot{p}(t) = 0$ and $\dot{e}^n(t) = 0$. First from equation (B.4) we see that $\dot{p}(t) = 0$ implies that $y(t) = \bar{y}$. From (B.5) this further implies:

$$e^n(t) = \frac{(\varepsilon \cdot \beta + \epsilon) \cdot p - (\epsilon \cdot m + \varepsilon \cdot z)}{\varepsilon \cdot \beta} \tag{19}$$

which defines an upward sloping line in the $(p(t), e^n(t))$ space. The reason is that an increase in $p(t)$ reduces output, i.e. in order for output to be unchanged e^n must increase in order to stimulate net exports. If the economy is initially located to the right of this line the domestic price level is high and net exports and output are low (i.e. $y(t) < \bar{y}$). From (B.4) this implies $\dot{p}(t) < 0$ such that the domestic price level is falling over time to the right of the line (and conversely increasing to the left of this line). This is the stabilizing element in the Dornbusch model.

From equation (B.3) we see that $\dot{e}^n(t) = 0$ implies $r(t) = r^f$ which from equation (B.6) implies:

$$e^n = \frac{m - z - (1 - \beta) \cdot p(t)}{\beta} \tag{20}$$

The slope of this line depends on whether β is lower than or exceeds 1. When $\beta < 1$ an increase in $p(t)$ increases $r(t)$ whereby the exchange rate must appreciate in order to reduce net exports and thereby output and the interest rate, such that $r(t)$ is kept equal to r^f. If $\beta > 1$ an increase in $p(t)$ on the other hand reduces the interest rate, whereby the exchange rate must depreciate in order to increase output and the interest rate such that $r(t) = r^f$. In this case the slope of the line is lower than the $\dot{p}(t) = 0$-line. In both cases if we start out above this line the domestic interest rate exceeds the foreign interest rate implying that $\dot{e}^n(t) > 0$ from equation (B.3). This is the destabilizing element of the Dornbusch model. The resulting phase diagrams are illustrated in figure 3 and 4. In both cases the long run equilibrium is saddle path stable. When $\beta < 1$ the saddle path is negatively sloped, while the saddle path has a positive slope when $\beta > 1$. We assume that the economy is always on the saddle path. This can e.g. be justified if financial investors anticipate that a central bank will eventually intervene if the exchange rate is either depreciating or appreciating forever.

3) For now $\beta < 1$. The effects are illustrated in figure 5. Immediately at implementation the economy jumps from E_0 to A. In this case there is an initial exchange rate overshooting. There are two reasons for the discountinous increase in e^n. First of all financial investors realize that in the long run the exchange rate will depreciate (since the higher money supply increases the domestic price level one-for-one in the long run which requires a one-for-one increase in e^n in order to keep the real exchange rate unaffected which is required due to the long run monetary neutrality). Financial investors want to benefit from the future depreciation, and thus they sell domestic currency and buy foreign bonds. This behavour immediately makes the exchange rate depreciate. This effect can however not explain the initial overshooting. This is due to the fact that the higher money supply reduces the interest rate in the short run implying a further swith from domestic to foreign bonds by finan-

cial investors, reducing the demand for domestic currency further, whereby the exchange rate depreciates further. Explained in another way: At point A the lower domestic interest rate must be compensated by a following appreciation of the exchange rate (in order for domestic and foreign bonds to yield the same rate of return). Thus the exchange rate initially overshoots in order to make a subsequent appreciation of the exchange possible. At point A the increase in e^n and the fall in the interest rate has increased output such that $y(t) > \bar{y}$. Thus, the domestic price level increases over time due to the excess demand. The gradual increase in the price level gradually increases the domestic interest rate (since $\beta < 1$) which makes investors gradually switch back to domestic bonds, increasing the demand for domestic currency resulting in a gradual appreciation of the exchange rate over time. The gradual increase in $p(t)$ and $r(t)$ and the gradual fall in $e^n(t)$ imply that output falls over time, as the economy converges towards the new steady state where output is once again equal to $\bar{y}$, the domestic interest rate is once again equal to r^f and where $p(t)$ and e^n has increased.

4) Now we consider the case where $\beta > 1$. In this case there is an initial undershooting of the exchange rate, see figure 6. Once again fiancial investors anticipate that the exchange rate will depreciate in the long run and act accordingly, but this time the increase in the money supply actually increases the domestic interest rate. The reason is that the initial increase in e^n has a strong impact on net exports (since $\beta > 1$), and thus increases output and money demand such that the domestic interest rate increases above r^f in this case. This effect dominates the higher real money supply. Thus, there is now a counteracting effect on e^n as investors tend to switch from foreign to domestic bonds (due to the higher domestic interest rate) which tends to increase the demand fore domestic currency and thereby make the exchange rate appreciate. Thus, the exchange rate undershoots in the short run. Explained in another way: Since the domestic interest rate has now increased above r^f this must be compensated by a following depreciation of the exchange rate. Thus, the exchange undershots which makes a subsequent depreciation possible. In spite of the higher interest rate $y(t)$ is once again above $\bar{y}$ at point A (due to the higher real exchange rate). Once again, this triggers an increase in the domestic price level, which now reduces the interest rate over time (as $\beta > 1$). Thus financial investors gradually switch from domestic to foreign bonds implying a gradual depreciation of the exchange rate. In spite of the gradual depreciation and the falling interest rate output

falls over time due to the increase in $p(t)$.

5) Now we consider an anticipated increase in m with $\beta < 1$. The dynamic evolution of the economy must obey the following four properties:
i) $p(t)$ is predeterimend at each point in time and will only adjust gradually over time
ii) e^n is free to jump but if so it must be at time t_0 where the financial investors receive new information
iii) Exactly at time t_1 the economy must be somewhere on the new saddle path
iv) Between time t_0 and time t_1 the dynamic evolution of the economy is guided by the old dynamic system

The effects are illustrated in diagram 7. At implementation the economy jumps from E_0 to A. The exchange rate depreciates which once again reflects that financial investors react to the prospect of a depreciation of the exchange rate in the long run. The money supply hasn't increased yet so the domestic interest rate rises above r^f, due to the increase in output. This dampens the increase in e^n. As output is also above $\bar{y}$ the domestic price level increases over time. The exchange rate depreciates however, in spite of the gradual increase in the interest rate between announcement and implementation. This simply reflects, that the economy moves closer in time to when the domestic interest rate falls. This happen exactly at time t_1 where $r(t)$ falls below r^f due to the increase in the money supply. There is no jump in e^n as the fall in the interest rate is fully anticipated. After implementation $p(t)$ keeps on increasing over time (as $y(t)$ is still above $\bar{y}$) implying that the interest rate increases. This triggers a gradual appreciation of the exchange rate as investors swith from foreign to domestic bonds. Output falls over time due to the increase in $p(t)$ and $r(t)$ and the appreciation of the echange rate.

Figure 1

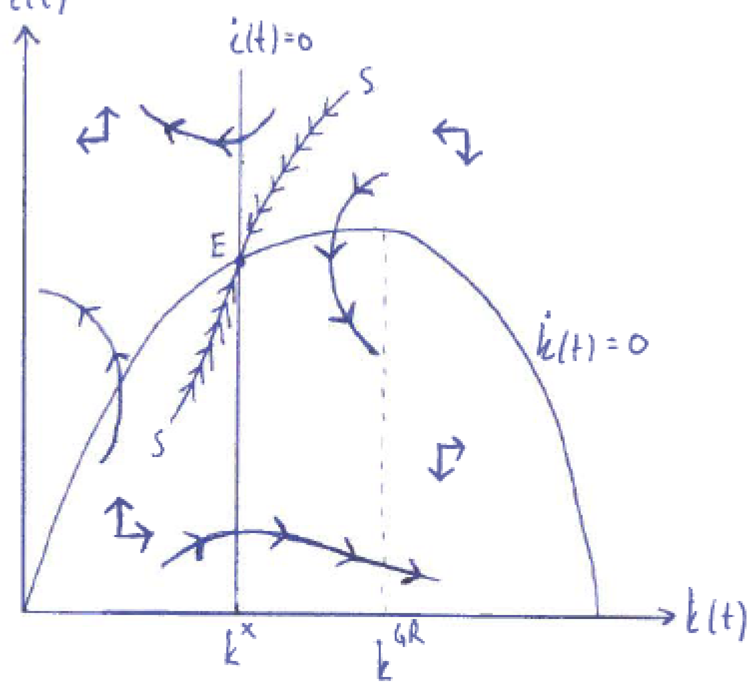


Figure 2

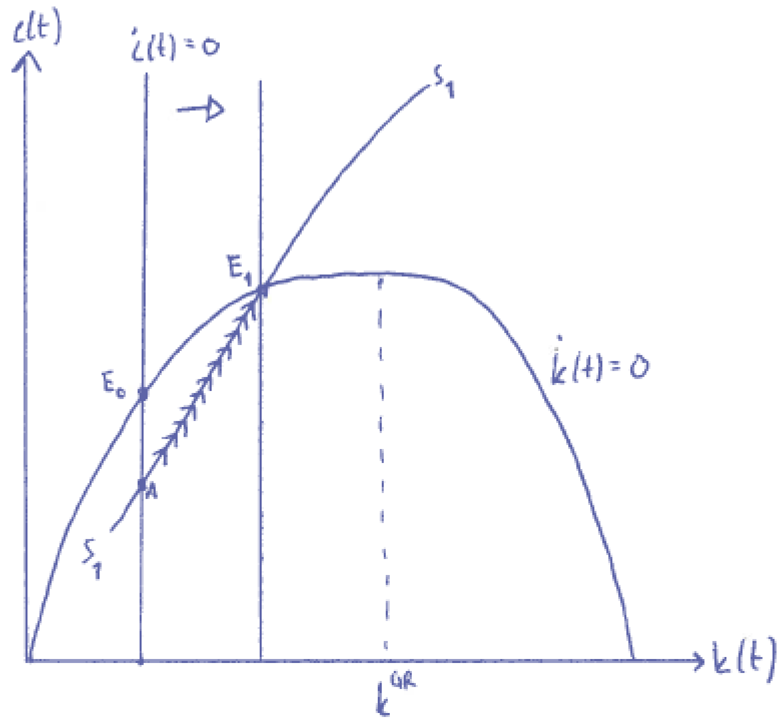
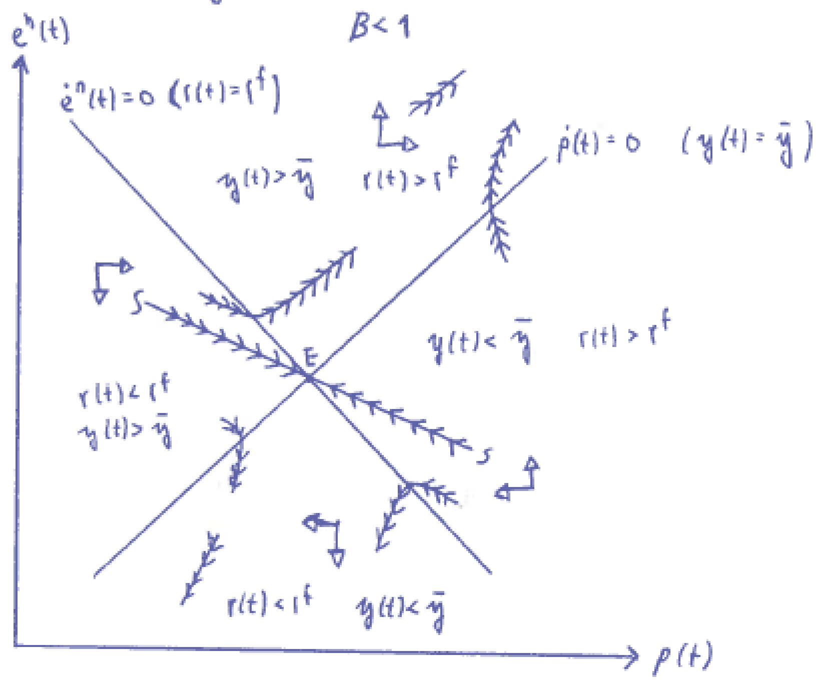


Figure 3



$e^n(t)$
$B < 1$
$\dot{e}^n(t) = 0$ $(r(t) = r^f)$
$\dot{p}(t) = 0$ $(y(t) = \bar{y})$
$y(t) > \bar{y}$
$r(t) > r^f$
S
E
$y(t) < \bar{y}$
$r(t) > r^f$
$r(t) < r^f$
$y(t) > \bar{y}$
S
$r(t) < r^f$
$y(t) < \bar{y}$
$p(t)$

Figure 4

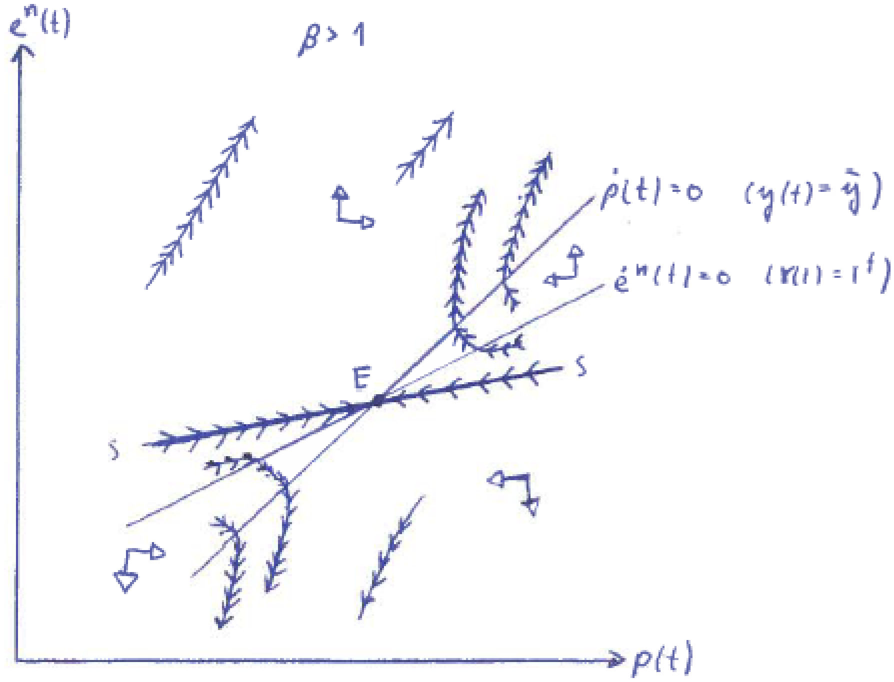


Figure 5

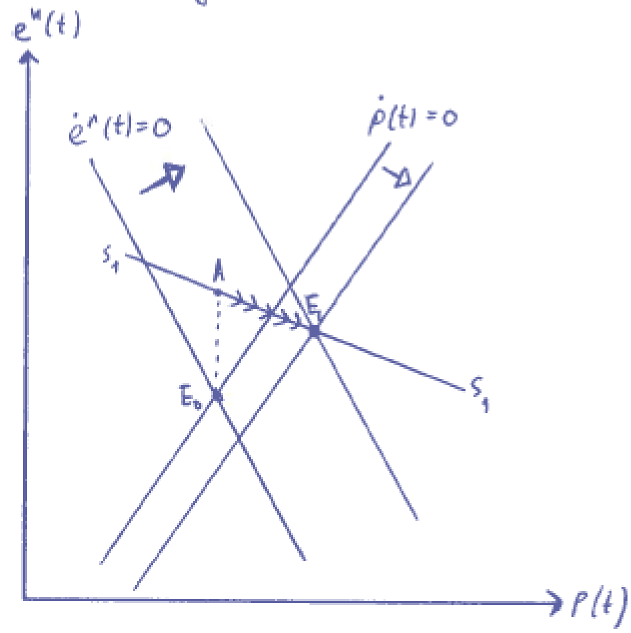


Figure 6

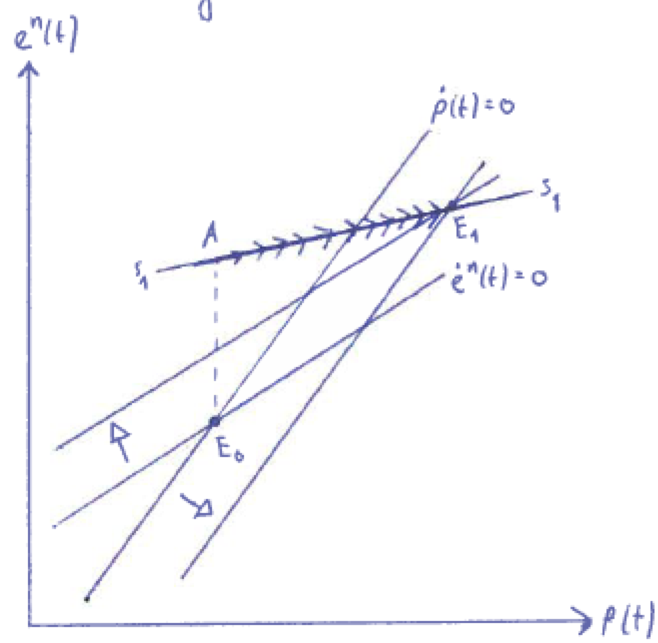


Figure 7