Rettevejledning til

Eksamen på Økonomistudiet 2011-I

Makro A og Macro A, 2. årsprøve

Efterårssemestret 2010

(Tre-timers prøve uden hjælpemidler)

Målbeskrivelse:

Faget videreudvikler langsigtsdelen af Økonomiske Principper 2, Makro.

I Makro A opstilles og analyseres alternative formelle modeller til forståelse af de langsigtede, trendmæssige tendenser i de vigtigste makroøkonomiske variable, såsom aggregeret indkomst og forbrug (per capita), indkomstfordeling, realløn og realrente, nettofordringsposition overfor udlandet, teknologisk niveau og produktivitet samt ledighed. I sammenhæng hermed præsenteres empirisk materiale under anvendelse af simple statistiske metoder.

Faget bygger op til Makro B ved at beskrive det forankringspunkt, økonomiens fluktuationer foregår omkring. Det bygger også op til Makro C ved at omfatte de mest fundamentale versioner af de langsigtsmodeller, som også indgår i Makro C.

De studerende skal lære de vigtigste såkaldte stiliserede empiriske fakta om økonomisk vækst og strukturel ledighed at kende, og de skal kende til og forstå den række af økonomisk teoretiske modeller, som i kurset inddrages til forklaring af disse fakta og til forståelse af økonomiens trendmæssige udvikling i det hele taget.

En vigtig kundskab, der begyndende skal erhverves i dette kursus, er selvstændig opstilling og analyse af formelle, makroøkonomiske modeller, som af type er som kendt fra faget, men som kan være variationer heraf. Der vil typisk være tale om modeller, som er formulerede som, eller er tæt på at være formulerede som, egentlige generelle ligevægtsmodeller. En del af denne kundskab består i en verbal formidling af en forståelse af modellernes egenskaber.

En anden vigtig kundskab er at kunne koble teori og empiri, så empirisk materiale kan tilvejebringes og analyseres på en måde, der er afklarende i forhold til teorien. Igen er verbal formidling af de konklusioner, der kan drages ud af samspillet mellem teori og empiri, en vigtig del af den beskrevne færdighed.

De typer af modeller, der skal kunne analyseres, omfatter modeller for lukkede såvel som for åbne økonomier, statiske såvel som dynamiske modeller, dynamiske modeller med såvel diskret tid som kontinuert tid. Modellerne skal både kunne analyseres generelt og ved numerisk simulation (sidstnævnte dog kun af ikke-stokastiske dynamiske modeller i diskret tid).

De studerende skal opnå færdigheder i at foretage økonomiske analyser i de typer af modeller, faget beskæftiger sig med, herunder analyser af strukturelle, økonomisk politiske indgreb og formidle analysens indsigter.

Topkarakteren 12 opnås, når de beskrevne færdigheder mestres til en sådan fuldkommenhed, at den studerende er blevet i stand til selvstændigt at analysere nye (fx økonomisk politiske) problemstillinger ved egen opstilling og analyse af varianter af de fra kurset kendte modeller under inddragelse og analyse af relevant empiri og afgive absolut fyldestgørende verbal forklaring af de opnåede analyseresultater.

Problem 1.

1.1. In endogenous growth theory there can be (strictly) positive growth in income per worker and in technology without an assumption of exogenous technological progress like $A_{t+1} = (1+g)A_t$ with g strictly positive and exogenous. There are two types of endogenous growth theory. One is based on positive productive externalities where the technological level, as a spill over, is tied to the aggregate use of (e.g.) capital, $A_t = K_t^{\phi}$, the other is based on explicit formulation of R&D where technological progress (ideas) is produced in an explicit production process using labour and existing technology as inputs, $A_{t+1} - A_t = \rho A_t^{\phi} L_{At}^{\lambda}$.

Semi-endogenous growth is defined by the feature that growth in the labour force is required for sustaining long-run growth in income per worker, and (truly) endogenous growth by this not being required. The first arises in the models mentioned when $0 < \phi < 1$, the other when $\phi = 1$.

1.2 In the R&D based approach existing technology is productive in the creation of new technology, $\phi > 0$. If the elasticity ϕ is less than one, then (only) semi-endogenous growth occurs. The intuition is that a larger and larger input of labour in the R&D sector is needed to maintain a constant growth rate of the technological level, as evidenced by $(A_{t+1} - A_t)/A_t = \rho A_t^{\phi-1} L_{At}^{\lambda}$, where $A_t^{\phi-1}$ must be decreasing. With a given research share, this can only arise if the labour force grows.

If, on the other hand, existing technology is so productive in the creation of new technology that the elasticity ϕ is one, then even without an increase in labour input a constant and strictly positive growth rate of technology can be sustained, as evidenced by $(A_{t+1} - A_t)/A_t = \rho L_{At}^{\lambda}$. Therefore growth of the labour force is not required.

1.3 Endogenous growth, be it "semi" or "truly", has the advantage over exogenous growth theory of delivering an explanation of technological progress at all. Some arguments that can be mentioned in favour or disfavour of each (found in the textbook's Chapter 8, Section 8.4):

The knife edge argument: $\phi = 1$ is a zero probability event, which speaks against truly endogenous growth. However, $\phi = 1$ should be seen as an approximation for ϕ close to, but below one, which has low, but not zero probability.

Scale effects: Under endogenous growth the growth rate of technology and of income per worker is typically larger the larger the (constant) labour force is, and if the labour force grows over time the economic growth rate increases too. These features are highly implausible and speak against endogenous growth theory. It can be remedied (in the model based on productive externalities by letting ther externality go from K_t/L_t rarther than from K_t), but then at the cost of labour not being productive at all in aggregate output, which is also highly implausible. Under semi-endogenous growth there is a milder, but still quite implausible, scale effect from the size of the labour force to the level of income per worker. The scale effects thus seems to speak in favour of exogenous growth theory all together.

Empirics concerning population growth and economic growth: Seems to speak against semi-endogenous growth, since a negative relationship is found, but this could be reversed causality or spurious correlation.

Empirics concerning income convergence: Seems to speak in favour of exogenous or semi-endogenous growth theory. However, endogenous growth can easily be made compatible with convergence.

Empirics concerning economic growth rates and investment rates: A rather robust, positive long-run relationship between growth rates of income per worker or of technology on the one side and investment rates on the other is found empirically. This speaks for endogenous growth theory that typically finds such a connection, while exogenous and semi-endogenous growth theory do not.

Problem 2.

The model repeated from the problem set:

$$\begin{aligned} Y_t &= K_t^{\alpha}\left(A_t L_t\right)^{\beta} X^{\kappa} E_t^{\varepsilon} && (1)\\ K_{t+1} &= sY_t + (1-\delta) K_t && (2)\\ L_{t+1} &= (1+n) L_t && (3)\\ A_{t+1} &= (1+g) A_t && (4)\\ R_{t+1} &= R_t - E_t && (5)\\ E_t &= s_E R_t && (6) \end{aligned}$$

2.1 If $K_t^{\alpha}\left(A_t L_t\right)^{\beta} X^{\kappa} E_t^{\varepsilon} = Y_t$, then:

$$\begin{aligned} \left(\lambda K_t\right)^{\alpha}\left(A_t \lambda L_t\right)^{\beta}\left(\lambda X\right)^{\kappa}\left(\lambda E_t\right)^{\varepsilon} &= \lambda^{\alpha+\beta+\kappa+\varepsilon} K_t^{\alpha}\left(A_t L_t\right)^{\beta} X^{\kappa} E_t^{\varepsilon}\\ &= \lambda K_t^{\alpha}\left(A_t L_t\right)^{\beta} X^{\kappa} E_t^{\varepsilon}\\ &= \lambda Y_t \end{aligned}$$

This shows that multiplying the inputs of capital, labour, land and energy by a factor implies that output is multiplied by the same factor, that is, constant returns.

The replication argument: there must be (close to) constant returns to the rival inputs since one can double the output by conducting the same production process twice.

2.2 Dividing on both sides of (1) by L_t gives:

$$\begin{aligned} \frac{Y_t}{L_t} &= \frac{K_t^{\alpha}\left(A_t L_t\right)^{\beta} X^{\kappa} E_t^{\varepsilon}}{L_t^{\alpha+\beta+\kappa+\varepsilon}} = \frac{K_t^{\alpha}\left(A_t L_t\right)^{\beta} X^{\kappa} E_t^{\varepsilon}}{L_t^{\alpha} L_t^{\beta} L_t^{\kappa} L_t^{\varepsilon}}\\ &= \left(\frac{K_t}{L_t}\right)^{\alpha}\left(A_t\right)^{\beta}\left(\frac{X}{L_t}\right)^{\kappa}\left(\frac{E_t}{L_t}\right)^{\varepsilon} \Leftrightarrow\\ y_t &= k_t^{\alpha} A_t^{\beta} x_t^{\kappa} e_t^{\varepsilon} && (7) \end{aligned}$$

Writing the latter for period $t-1$ as well, taking the (natural) log on both sides and subtracting gives:

$$\ln y_t - \ln y_{t-1} = \alpha\left(\ln k_t - \ln k_{t-1}\right) + \beta\left(\ln A_t - \ln A_{t-1}\right) + \kappa\left(\ln x_t - \ln x_{t-1}\right) + \varepsilon\left(\ln e_t - \ln e_{t-1}\right).$$

Inserting simply the definitions of g_t^y etc. gives:

$$g_t^y = \alpha g_t^k + \beta g_t^A + \kappa g_t^x + \varepsilon g_t^e \qquad (8)$$

2.3 Note that $z_t \equiv K_t/Y_t = k_t/y_t$. If z_t is constant, then $k_t/y_t = k_{t-1}/y_{t-1}$, from which follows $\ln k_t - \ln k_{t-1} = \ln y_t - \ln y_{t-1}$, or $g_t^y = g_t^k$. This holds exactly although g_t^y and g_t^k are the approximate growth rates. In (8) we may write g_t^y for g_t^k. (It is OK simply to say, that if k_t/y_t is constant, then k_t and y_t must be growing by the same rate and then it must be that $g_t^k \approx g_t^y$).

The exact and constant growth rate of A_t is g (Equation (4)), while the approximate growth rate is $g_t^A = \ln A_t - \ln A_{t-1}$. Hence one must have, $g_t^A \approx g$ for all t. This is OK, but a more formal way would be: It follows from (4) that $A_t/A_{t-1} = 1 + g$. Taking log on both sides gives $\ln A_t - \ln A_{t-1} = \ln(1+g)$. For g small, $\ln(1+g) \approx g$ is a good approximation. Hence, $\ln A_t - \ln A_{t-1} \equiv g_t^A \approx g$. This can also be used in (8).

Now, $g_t^x = \ln x_t - \ln x_{t-1} = \ln(X/L_t) - \ln(X/L_{t-1}) = \ln X - \ln X - (\ln L_t - \ln L_{t-1}) \equiv -g_t^L$. By the same reasoning as for g_t^A, one has $g_t^L \approx n$. Hence, $g_t^x \approx -n$.

Finally, $g_t^e = \ln e_t - \ln e_{t-1} = \ln(E_t/L_t) - \ln(E_{t-1}/L_{t-1}) = (\ln E_t - \ln E_{t-1}) - (\ln L_t - \ln L_{t-1}) \approx (\ln E_t - \ln E_{t-1}) - n$. From Equation (6), $\ln E_t - \ln E_{t-1} = \ln(s_E R_t) - \ln(s_E R_{t-1}) = \ln R_t - \ln R_{t-1} \equiv g_t^R$. From (5) and (6), $R_t = R_{t-1} - E_{t-1} = R_{t-1} - s_E R_{t-1} = (1 - s_E) R_{t-1}$. Hence, the exact growth rate of R_t is $-s_E$, and by the reasoning also used above, $g_t^R \approx -s_E$. Collecting gives: $g_t^e \approx -s_E - n$.

By inserting all of the above in (8) one gets:

$$g_t^y \approx \alpha g_t^y + \beta g + \kappa(-n) + \varepsilon(-n - s_E) \Leftrightarrow$$

$$(1-\alpha)\, g_t^y \approx \beta g - \kappa n - \varepsilon(n + s_E) \Leftrightarrow$$

$$g_t^y \approx \frac{\beta g - \kappa n - \varepsilon(n + s_E)}{1-\alpha} \Leftrightarrow$$

$$g_t^y \approx \frac{\beta}{\beta+\kappa+\varepsilon} g - \frac{\kappa}{\beta+\kappa+\varepsilon} n - \frac{\varepsilon}{\beta+\kappa+\varepsilon} (n + s_E) \equiv g^y, \tag{9}$$

where it was used that $1 - \alpha = \beta + \kappa + \varepsilon$ for the last equation. Since all elements on the right hand side are constant, g_t^y must be approximately constant, $g_t^y = g^y$.

Intuitively, if labour input L_t increases by a certain rate, even if capital input did keep pace (grew at the same rate, as it does not), the increased amount af labour and capital would be used together with (press on) a fixed amount of land and an even deacrasing input of energy. The law of diminishing returns would therefore tend to imply that output cannot keep pace with labour input causing a negative influence of n and s_E on growth in the long run, as evidenced by (9). The same equation shopws how technological growth can be a counteracting factor.

2.4 Critical remarks: The economies are open, but here considered as separate closed economies. The observed correlation could be reversed causality or spurious correlation.

Disregarding these objections the significant negative slope is in accordance with the model and Equation (9). Plausible parameter values could be $\alpha \approx 0.2$, $\beta \approx 0.6$, $\kappa \approx \varepsilon \approx 0.1$, for instance. Equation (9) would then be:

$$g^y \approx 0.75g - 0.25n - 0.125 s_E. \tag{9'}$$

The negative influence of n, the slope of -0.25, is not in contradiction to Figure 1, when uncertainty is taken into account. The standard error of 0.14 means that the 95 % confidence interval for the estimated slope goes from 0.22 to 0.78.

2.5 This question has been included since the technique should be used again for the more difficult CES case in Question 2.8.

Writing Equation (7) for period t and period $t-1$, and dividing the first by the second gives:

$$\frac{y_t}{y_{t-1}} = \left(\frac{k_t}{k_{t-1}}\right)^{\alpha} \left(\frac{A_t}{A_{t-1}}\right)^{\beta} \left(\frac{x_t}{x_{t-1}}\right)^{\kappa} \left(\frac{e_t}{e_{t-1}}\right)^{\varepsilon}.$$

Here, $y_t/y_{t-1} = f_t^y$ etc., and under balanced growth, $f_t^y = f_t^k$. Furthermore, $A_t/A_{t-1} = 1+g$ (Equation 4), and

$$\frac{x_t}{x_{t-1}} = \frac{X/L_t}{X/L_{t-1}} = \frac{L_{t-1}}{L_t} = \frac{1}{1+n},$$

the last equality from Equation (3), and

$$\frac{e_t}{e_{t-1}} = \frac{E_t/L_t}{E_{t-1}/L_{t-1}} = \frac{s_E R_t}{s_E R_{t-1}} \frac{L_{t-1}}{L_t} = \frac{1-s_E}{1+n},$$

where for the latter equality it was used again that $R_t = (1-s_E)R_{t-1}$, as derived in Question 2.3. Inserting all of this in the first equation above gives:

$$f_t^y = \left(f_t^y\right)^{\alpha} (1+g)^{\beta} \left(\frac{1}{1+n}\right)^{\kappa} \left(\frac{1-s_e}{1+n}\right)^{\varepsilon} \Leftrightarrow$$

$$\left(f_t^y\right)^{1-\alpha} = (1+g)^{\beta} \left(\frac{1}{1+n}\right)^{\kappa} \left(\frac{1-s_e}{1+n}\right)^{\varepsilon} \Leftrightarrow$$

$$f_t^y = (1+g)^{\frac{\beta}{\beta+\kappa+\varepsilon}} \left(\frac{1}{1+n}\right)^{\frac{\kappa}{\beta+\kappa+\varepsilon}} \left(\frac{1-s_e}{1+n}\right)^{\frac{\varepsilon}{\beta+\kappa+\varepsilon}} \equiv f^y, \tag{10}$$

showing that f_t^y is a constant, $f_t^y \equiv f^y$, and thereby demonstrating (10). Taking log on both sides, remembering that $f_t^y = y_t/y_{t-1}$, gives:

$$\ln y_t - \ln y_{t-1} = \frac{\beta}{\beta+\kappa+\varepsilon} \ln(1+g) - \frac{\kappa}{\beta+\kappa+\varepsilon} \ln(1+n) + \frac{\varepsilon}{\beta+\kappa+\varepsilon} \left[\ln(1-s_E) - \ln(1+n)\right].$$

Using $g_t^y = \ln y_t - \ln y_{t-1}$ and the approximation mentioned in the hint repeatedly gives:

$$g_t^y = \frac{\beta}{\beta+\kappa+\varepsilon} g - \frac{\kappa}{\beta+\kappa+\varepsilon} n - \frac{\varepsilon}{\beta+\kappa+\varepsilon} \left[n + s_E\right],$$

which is (9). Hence the exact expression for the growth factor in (10) is in accordance with the approximate expression for the approximate growth rate in (9).

2.6 Using (7), one gets:

$$z_t = \frac{k_t}{y_t} = \frac{k_t}{k_t^{\alpha} A_t^{\beta} x_t^{\kappa} e_t^{\varepsilon}} = k_t^{1-\alpha} A_t^{-\beta} x_t^{-\kappa} e_t^{-\varepsilon}.$$

Leading one period and using definitions and model equations:

$$\begin{aligned}
z_{t+1} &= k_{t+1}^{1-\alpha}A_{t+1}^{-\beta}x_{t+1}^{-\kappa}e_{t+1}^{-\varepsilon}\\
&= \left(\frac{K_{t+1}}{L_{t+1}}\right)^{1-\alpha}\left[(1+g)A_t\right]^{-\beta}\left(\frac{X}{L_{t+1}}\right)^{-\kappa}\left(\frac{E_{t+1}}{L_{t+1}}\right)^{-\varepsilon}\\
&= \left(\frac{sY_t+(1-\delta)K_t}{(1+n)L_t}\right)^{1-\alpha}\left[(1+g)A_t\right]^{-\beta}\left(\frac{X}{(1+n)L_t}\right)^{-\kappa}\left(\frac{s_E R_{t+1}}{(1+n)L_t}\right)^{-\varepsilon}\\
&= \left(\frac{1}{1+n}\right)^{1-\alpha-\kappa-\varepsilon}\left(\frac{1}{1+g}\right)^{\beta}(sy_t+(1-\delta)k_t)^{1-\alpha}A_t^{-\beta}x_t^{-\kappa}\left(\frac{s_E(1-s_E)R_t}{L_t}\right)^{-\varepsilon}\\
&= \left(\frac{1}{(1+n)(1+g)}\right)^{\beta}(1-s_E)^{-\varepsilon}(sy_t+(1-\delta)k_t)^{1-\alpha}A_t^{-\beta}x_t^{-\kappa}\left(\frac{E_t}{L_t}\right)^{-\varepsilon}\\
&= \left(\frac{1}{(1+n)(1+g)}\right)^{\beta}(1-s_E)^{-\varepsilon}\left(s\frac{y_t}{k_t}+(1-\delta)\right)^{1-\alpha}\overbrace{k_t^{1-\alpha}A_t^{-\beta}x_t^{-\kappa}e_t^{-\varepsilon}}^{=z_t}\\
&= \left(\frac{1}{(1+n)(1+g)}\right)^{\beta}(1-s_E)^{-\varepsilon}\left(\frac{s}{z_t}+(1-\delta)\right)^{1-\alpha}z_t\\
&= \left(\frac{1}{(1+n)(1+g)}\right)^{\beta}(1-s_E)^{-\varepsilon}(s+(1-\delta)z_t)^{1-\alpha}z_t^{\alpha} \qquad (11)
\end{aligned}$$

2.7 First we find the steady state capital-output ratio. Setting $z_t = z_{t+1} = z$ in (11) gives:

$$z^{1-\alpha} = \left(\frac{1}{(1+n)(1+g)}\right)^{\beta}(1-s_E)^{-\varepsilon}(s+(1-\delta)z)^{1-\alpha} \Leftrightarrow$$

$$z = \left(\frac{1}{(1+n)(1+g)}\right)^{\frac{\beta}{1-\alpha}}(1-s_E)^{-\frac{\varepsilon}{1-\alpha}}(s+(1-\delta)z) \Leftrightarrow$$

$$z\left(1-\left(\frac{1}{(1+n)(1+g)}\right)^{\frac{\beta}{\beta+\kappa+\varepsilon}}(1-s_E)^{-\frac{\varepsilon}{\beta+\kappa+\varepsilon}}(1-\delta)\right) = \left(\frac{1}{(1+n)(1+g)}\right)^{\frac{\beta}{\beta+\kappa+\varepsilon}}(1-s_E)^{-\frac{\varepsilon}{\beta+\kappa+\varepsilon}}s \Leftrightarrow$$

$$z\frac{[(1+n)(1+g)]^{\frac{\beta}{\beta+\kappa+\varepsilon}}-(1-s_E)^{-\frac{\varepsilon}{\beta+\kappa+\varepsilon}}(1-\delta)}{[(1+n)(1+g)]^{\frac{\beta}{\beta+\kappa+\varepsilon}}} = \left(\frac{1}{(1+n)(1+g)}\right)^{\frac{\beta}{\beta+\kappa+\varepsilon}}(1-s_E)^{-\frac{\varepsilon}{\beta+\kappa+\varepsilon}}s \Leftrightarrow$$

$$z = \frac{(1-s_E)^{-\frac{\varepsilon}{\beta+\kappa+\varepsilon}}}{[(1+n)(1+g)]^{\frac{\beta}{\beta+\kappa+\varepsilon}}-(1-s_E)^{-\frac{\varepsilon}{\beta+\kappa+\varepsilon}}(1-\delta)}s \Leftrightarrow$$

$$z = \frac{1}{[(1+n)(1+g)]^{\frac{\beta}{\beta+\kappa+\varepsilon}}(1-s_E)^{\frac{\varepsilon}{\beta+\kappa+\varepsilon}}-(1-\delta)}s \equiv z^* \qquad (12)$$

Now, for convergence of z_t to this z^*, it suffices to show the following properties of the transition curve defined by (11):

1. For $z_t = 0$ one has $z_{t+1} = 0$, that is, it passes through (0,0). Follows directly from (11).
2. The curve is everywhere strictly increasing. Follows directly from (11).
3. There is a unique, strictly positive intersection between the transition curve and the 45°-line. This was shown above, since the intersection is the unique $z^* > 0$.
4. The slope of the transition curve at zero is strictly positive: Differentiating (11) gives:

$$\frac{dz_{t+1}}{dz_t} = \left(\frac{1}{(1+n)(1+g)}\right)^{\beta}(1-s_E)^{-\varepsilon}\left[(1-\alpha)(s+(1-\delta)z_t)^{-\alpha}(1-\delta)z_t^{\alpha}+(s+(1-\delta)z_t)^{1-\alpha}\alpha z_t^{\alpha-1}\right].$$

This can be written more nicely, but the stated expression already reveals that the slope dz_{t+1}/dz_t goes to infinity as z_t goes to zero because of the term $z_t^{\alpha-1}$ most far to the right.

From these properties it follows by "stair case iteration" in a transition diagram that z_t converges to z^* from any strictly positive initial value.

2.8 Considerations around (9) or (10) - or perhaps more directly from (9') - support growth optimism at least for developed countries where population growth has come under control. Plausible parameter values on annual basis could perhaps be $g \approx 0.024$ (from growth accounting exercises), $n \approx 0.01$ (a high value for a developed country) and $s_E \approx 0.005$. In that case (9') would be (in per cent):

$$g^y \cdot 100 \approx 1.8 - 0.25 - 0.0625\ \% = 1.5\ \%.$$

This is a decent long-run growth rate of income per worker.

However, the above formulas were all derived under the assumption of a Cobb-Douglas production function, which involves the particular unity elasticity of substitution between inputs. Obviously it is fundamental for the conclusion with respect to growth optimism or pessimism how strongly technology (or technology augmented labour) can substitute for the long-run infinitely scarce natural resources. The unity elasticity is particular.

(The above considerations are (should be) well-known from the text book. The below is not standard stuff, however, and if a student is able to include it, it is very well done).

For the more general CES case,

$$Y_t = \left[\alpha K_t^{\frac{\sigma-1}{\sigma}} + \beta \left(A_t L_t\right)^{\frac{\sigma-1}{\sigma}} + \kappa X^{\frac{\sigma-1}{\sigma}} + \varepsilon E_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{1'}$$

we get by steps similar to those taken in Question 2.5 above:

$$y_t = \left[\alpha k_t^{\frac{\sigma-1}{\sigma}} + \beta A_t^{\frac{\sigma-1}{\sigma}} + \kappa x_t^{\frac{\sigma-1}{\sigma}} + \varepsilon e_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \Rightarrow$$

$$\frac{y_t}{y_{t-1}} = \frac{\left[\alpha k_t^{\frac{\sigma-1}{\sigma}} + \beta A_t^{\frac{\sigma-1}{\sigma}} + \kappa x_t^{\frac{\sigma-1}{\sigma}} + \varepsilon e_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}}{\left[\alpha k_{t-1}^{\frac{\sigma-1}{\sigma}} + \beta A_{t-1}^{\frac{\sigma-1}{\sigma}} + \kappa x_{t-1}^{\frac{\sigma-1}{\sigma}} + \varepsilon e_{t-1}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}} \Rightarrow$$

$$\begin{aligned}\left(\frac{y_t}{y_{t-1}}\right)^{\frac{\sigma-1}{\sigma}} = &\frac{\alpha k_{t-1}^{\frac{\sigma-1}{\sigma}}}{\alpha k_{t-1}^{\frac{\sigma-1}{\sigma}} + \beta A_{t-1}^{\frac{\sigma-1}{\sigma}} + \kappa x_{t-1}^{\frac{\sigma-1}{\sigma}} + \varepsilon e_{t-1}^{\frac{\sigma-1}{\sigma}}}\left(\frac{k_t}{k_{t-1}}\right)^{\frac{\sigma-1}{\sigma}} \\ &+\frac{\beta A_{t-1}^{\frac{\sigma-1}{\sigma}}}{\alpha k_{t-1}^{\frac{\sigma-1}{\sigma}} + \beta A_{t-1}^{\frac{\sigma-1}{\sigma}} + \kappa x_{t-1}^{\frac{\sigma-1}{\sigma}} + \varepsilon e_{t-1}^{\frac{\sigma-1}{\sigma}}}\left(\frac{A_t}{A_{t-1}}\right)^{\frac{\sigma-1}{\sigma}} \\ &+\frac{\kappa x_{t-1}^{\frac{\sigma-1}{\sigma}}}{\alpha k_{t-1}^{\frac{\sigma-1}{\sigma}} + \beta A_{t-1}^{\frac{\sigma-1}{\sigma}} + \kappa x_{t-1}^{\frac{\sigma-1}{\sigma}} + \varepsilon e_{t-1}^{\frac{\sigma-1}{\sigma}}}\left(\frac{x_t}{x_{t-1}}\right)^{\frac{\sigma-1}{\sigma}} \\ &+\frac{\varepsilon e_{t-1}^{\frac{\sigma-1}{\sigma}}}{\alpha k_{t-1}^{\frac{\sigma-1}{\sigma}} + \beta A_{t-1}^{\frac{\sigma-1}{\sigma}} + \kappa x_{t-1}^{\frac{\sigma-1}{\sigma}} + \varepsilon e_{t-1}^{\frac{\sigma-1}{\sigma}}}\left(\frac{e_t}{e_{t-1}}\right)^{\frac{\sigma-1}{\sigma}}\end{aligned}$$

with $\frac{y_{t+1}}{y_t} = \frac{k_{t+1}}{k_t}$ (balanced growth with constant capital-output ratio) $\equiv f_t$:

$$f_t^{\frac{\sigma-1}{\sigma}} = \frac{\beta A_{t-1}^{\frac{\sigma-1}{\sigma}}}{\beta A_{t-1}^{\frac{\sigma-1}{\sigma}} + \kappa x_{t-1}^{\frac{\sigma-1}{\sigma}} + \varepsilon e_{t-1}^{\frac{\sigma-1}{\sigma}}}\left(1+g\right)^{\frac{\sigma-1}{\sigma}}$$

$$+\frac{\kappa x_{t-1}^{\frac{\sigma-1}{\sigma}}}{\beta A_{t-1}^{\frac{\sigma-1}{\sigma}}+\kappa x_{t-1}^{\frac{\sigma-1}{\sigma}}+\varepsilon e_{t-1}^{\frac{\sigma-1}{\sigma}}}\left(\frac{1}{1+n}\right)^{\frac{\sigma-1}{\sigma}}$$

$$+\frac{\varepsilon e_{t-1}^{\frac{\sigma-1}{\sigma}}}{\beta A_{t-1}^{\frac{\sigma-1}{\sigma}}+\kappa x_{t-1}^{\frac{\sigma-1}{\sigma}}+\varepsilon e_{t-1}^{\frac{\sigma-1}{\sigma}}}\left(\frac{1-s_E}{1+n}\right)^{\frac{\sigma-1}{\sigma}}$$

Since $g > 0$, we have $A_t \to \infty$ for $t \to \infty$, and since $n > 0$ we have $x_t \to 0$, and since $s_E > 0$ we have $e_t \to 0$. Therefore:

$$f_t \to f = \begin{cases} \frac{1-s_E}{1+n} \text{ for } \sigma < 1 \\ \\ 1+g \text{ for } \sigma > 1 \end{cases}$$

For $\sigma < 1$, even the slightest below, the growth factor approaches a level below one in the long run, that is, there is negative growth in income per capita independently of the values of g and n etc. Technology cannot, in this case, substitute for the eventually infinitely scarce natural resources.

The Cobb-Douglas degree of substitution is indeed crucial for the optimistic conclusion. However, optimism is perhaps more justified by the consideration that as a certain resource becomes very scarce its price will go to very high levels making a substituting product very profitable. In that way, “capitalism creates” the required substitution. This is, of course, a discussion that is not in any way settled.