Answer to:

Written Exam for M.Sc. in Economics Summer School 2013

Investment Theory

Master Course

6th January 2013

3 hours closed books exam

Exercise 1.

(a) As stated in the text the example is entry in a market. The cost function is convex indicating that there is decreasing returns to scale. The fixed cost can be rent for buildings and equiment or part of the wages.

(b) The strategies could take the forms:

$$\begin{cases} P < P_H & \Rightarrow \quad \text{wait} \\ P \geq P_H & \Rightarrow \quad \text{invest} \end{cases} \qquad \begin{cases} P \leq P_L & \Rightarrow \quad \text{exit} \\ P > P_L & \Rightarrow \quad \text{continue} \end{cases}$$

For these strategies the relation between $F(P)$ and and $J(P)$ is

$$F(P) = \begin{cases} ? & \text{for } P < P_H \\ J(P) - I & \text{for } P \geq P_H \end{cases} \qquad J(P) = \begin{cases} F(P) - E & \text{for } P \leq P_L \\ ?? & \text{for } P > P_L \end{cases}$$

$F(P)$ and $J(P)$ should satisfy value matching (VM), smooth pasting (SP), $F(P)$ should satisfy "$P \to 0 \Rightarrow F(P) \to 0$ and $J(P)$ should satisfy "no bubbles".

We need to find "?", "??", P_H, P_L, $F(P)$ and $J(P)$.

(c) The profit is found by solving

$$\max_Y PY - \frac{1}{4}Y^2 - H.$$

The first-order condition is

$$P - \frac{1}{2}Y = 0 \text{ or } Y = 2P$$

so the profit is

$$\Pi(P) = P^2 - H.$$

(d) The dividend rate is

$$\frac{\alpha PF'(P) + 0.5\sigma^2 P^2 F''(P) - n(\alpha + \delta)Q}{F(P) - nQ} dt + \frac{\sigma PF'(P) - n\sigma Q}{F(P) - nQ} dz.$$

Here Ito's lemma is used to find how $F(P)$ changes in the time interval dt. For $n = PF'(P)/Q$ the dividend rate is certain, so by no arbitrage the dividend rate is equal to r. Therefore

$$0.5\sigma^2 P^2 F''(P) + (r - \delta)PF'(P) - rF(P) = 0.$$

The mathematical solution is

$$F(P) \;=\; A_1 P^{\beta_1} + A_2 P^{\beta_2}$$

where $\beta_1 > 1$ and $\beta_2 < 0$ are the two solutions to

$$0.5\sigma^2(\beta-1)\beta + (r-\delta)\beta - r \;=\; 0.$$

The economic solution is

$$F(P) \;=\; A_1 P^{\beta_1}$$

because of "$P \to 0 \Rightarrow F(P) \to 0$".

(e) The dividend rate is

$$\frac{P^2 - H + \alpha P J'(P) + 0.5\sigma^2 P^2 J''(P) - n(\alpha+\delta)Q}{J(P) - nQ} dt + \frac{\sigma P J'(P) - n\sigma Q}{J(P) - nQ} dz.$$

For $n = PJ'(P)/Q$ the dividend rate is certain, so by no arbitrage the dividend rate is equal to r. Therefore

$$0.5\sigma^2 P^2 J''(P) + (r-\delta) P J'(P) - rJ(P) + P^2 - H \;=\; 0.$$

A particular solution to the differential equation is

$$J(P) \;=\; \frac{P^2}{2\delta - \sigma^2 - r} + \frac{H}{r}.$$

Therefore the mathematical solution is

$$J(P) \;=\; \frac{P^2}{2\delta - \sigma^2 - r} + \frac{H}{r} + B_1 P^{\beta_1} + B_2 P^{\beta_2}$$

The economic solution is

$$J(P) \;=\; \frac{P^2}{2\delta - \sigma^2 - r} + \frac{H}{r} + B_2 P^{\beta_2}$$

because of "no bubbles".

(f) The assumption needed to ensure that $J(P)$ is increasing in P for P large enough is $2\delta - \sigma^2 - r > 0$.

(g) For $F(P)$: $A_1 P^{\beta_1}$ is the value of the option to invest including the values of future exits and investments in case P is so low that investor should wait with the investment. Hence I expect $A_1 > 0$.

For $J(P)$, $\frac{P^2}{2\delta-\sigma^2-r} + \frac{H}{r}$ is the value of an active firm forever and $B_2 P^{\beta_2}$ is value of the option to exit including the values of future investments and exits. Hence I expect $B_2 > 0$.

(h) The strategies P_H and P_L and the undetermined constants A_1 and B_2 can be found by solving the VM and SP equations:

$$\begin{aligned}
A_1 P_H^{\beta_1} &= \frac{P_H^2}{2\delta - \sigma^2 - r} + \frac{H}{r} + B_2 P_H^{\beta_2} - I \\
\beta_1 A_1 P_H^{\beta_1 - 1} &= \frac{2P_H}{2\delta - \sigma^2 - r} + \beta_2 B_2 P_H^{\beta_2 - 1} \\
\frac{P_L^2}{2\delta - \sigma^2 - r} + \frac{H}{r} + B_2 P_L^{\beta_2} &= A_1 P_L^{\beta_1} - E \\
\frac{2P_L}{2\delta - \sigma^2 - r} + \beta_2 B_2 P_L^{\beta_2 - 1} &= \beta_1 A_1 P_L^{\beta_1 - 1}.
\end{aligned}$$

If it is assumed that $J(P)$ is increasing in P, then it is easily seen that $P_H > P_L$. Therefore there is hysteresis. For $P \in]P_L, P_H[$ it depends on the history of P whether the project is active or not: if P comes from below P_L, then the project isn't active; and, if P comes from above P_H, the project is active.