LM Juni 2018 Løsninger

Opg 1

Matricen er $L = \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 \end{bmatrix}$

1)

$LX = \overline{0}$

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & -1 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ -R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 1 & 1 \end{bmatrix}$$

Med $x_3 = r$, $x_4 = s$, $x_5 = t$

fås $x_2 = 2r - s - t$, $x_1 = -r + t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad r, s, t \in \mathbb{R}$$

De tre søjlevektorer udgør en basis for $N(L)$

2) De to første søjler i matricen er lin. uafh. og udgør dermed en basis for $R(L)$.

Da er $R(L) = \mathbb{R}^2$, hvorfor L er surjektiv.

(Man kan da bare bruge standardbasen i $\mathbb{R}^2$ som basis.)

$5 - 3 = 2$ er dim. sætn.

3) Vi skal løse $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = (2,-3,1,2,3)$

dvs

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 2 \\ 2 & -1 & -1 & -3 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

dvs $(\alpha_1, \alpha_2, \alpha_3) = \underline{\underline{(1, 2, 3)}}$ er koordinatene

4) Vektorene er

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -a+c \\ 2a-b-c \\ a \\ b \\ c \end{bmatrix}}}$$

5) $Lx = y$:

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 0 & y_1 \\ 1 & 0 & 1 & 0 & -1 & y_2 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 0 & y_1 \\ 0 & -1 & 2 & -1 & -1 & y_2 - y_1 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ -R_2 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & y_2 \\ 0 & 1 & -2 & 1 & 1 & y_1 - y_2 \end{array}\right].$$ Da får

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_1 - y_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad r, s, t \in \mathbb{R}$$

Opg 2.

1) Da egenvektorerne er indbyrdes ortogonale er A symmetrisk.

2) $A(v_1+v_2) = Av_1 + Av_2 = 2v_1 - 2v_2$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

3) $\det A = 2 \cdot (-2) \cdot 1 = -4 \neq 0$, derfor regulær.

4) $A^{-1}(v_1+v_2) = A^{-1}v_1 + A^{-1}v_2 = \frac{1}{2}v_1 - \frac{1}{2}v_2$

$$= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$$

5) $\det(e^{P(A)}e^{A}) = \det(e^{P(A)}) \cdot \det(e^{A}) = \underline{\underline{e^{14}}}$, da

$P(2) = 4+2+1 = 7$, $P(-2) = 4-2+1 = 3$, $P(1) = 1+1+1 = 3$

$\det(e^{P(A)}) = e^7 \cdot e^3 \cdot e^3 = e^{13}$

$\det(e^{A}) = e^2 e^{-2} e^1 = e$.

6) $e^{P(A)}v_1 = e^7 v_1 = \begin{bmatrix} e^7 \\ e^7 \\ e^7 \end{bmatrix}$.

Opg 3

1) $$\int \cos^3(ax)\,dx = \int \left(\frac{e^{iax}+e^{-iax}}{2}\right)^3 dx$$

$$= \frac{1}{8}\int \left(e^{i2ax}+e^{-i2ax}+2\right)\left(e^{iax}+e^{-iax}\right)dx$$

$$= \frac{1}{8}\int e^{i3ax}+e^{iax}+e^{-iax}+e^{-i3ax}+2\left(e^{iax}+e^{-iax}\right)dx$$

$$= \frac{1}{8}\int \left(e^{i3ax}+e^{-i3ax}\right)+3\left(e^{iax}+e^{-iax}\right)dx$$

$$= \frac{1}{4}\int \cos(3ax)+3\cos(ax)\,dx\,.$$

For $a \neq 0$:

$$= \frac{1}{4}\left(\frac{1}{3a}\sin(3ax)+\frac{3}{a}\sin(ax)\right)+k\,.$$

For $a=0$ er integralet Voplagt $\underline{\underline{x+k}}$.

2) $$\frac{1}{z^2} = \frac{1+i}{2} \Leftrightarrow z^2 = \frac{2}{1+i} = \frac{2(1-i)}{(1+i)(1-i)} = 1-i$$

$z^2 = 1-i$ løses ved at skrive $z = x+iy$, så

$z^2 = x^2 - y^2 + i2xy = 1-i$, så

$$\left.\begin{aligned} x^2-y^2 &= 1 \\ 2xy &= -1 \end{aligned}\right\}$$ Heraf $y = \frac{-1}{2x}$ (da $x \neq 0$ og $y \neq 0$) så $x^2 - \left(\frac{-1}{2x}\right)^2 = 1$. $x^2 - \frac{1}{4x^2} = 1$

$4x^4 - 4x^2 - 1 = 0$. $u = x^2$, $u = \frac{4 \pm \sqrt{16+16}}{8}$

$(-)$ forkastes $x^2 = \frac{4+\sqrt{32}}{8} = \frac{1+\sqrt{2}}{2}$, $x = \pm\sqrt{\frac{1+\sqrt{2}}{2}}$

$$z = x+iy = \pm\left(\sqrt{\frac{1+\sqrt{2}}{2}} + i\,\frac{-1}{2\sqrt{\frac{1+\sqrt{2}}{2}}}\right).$$ (Kan skrives pænere!)

Opg 4

1) Type $\sum (g(x))^n$ med $g(x) = e^{(x^3-4x)}$.

Konvergent for $\left| e^{(x^3-4x)} \right| < 1$, dvs

$$e^{x^3-4x} < 1$$

$$\Updownarrow$$

$$x^3-4x < 0 \quad (*)$$

Da $x^3-4x = x(x^2-4) = x(x-2)(x+2) = 0$ for $x \in \{0,-2,2\}$, ses $(*)$ at være opfyldt for $x \in]-\infty;-2[\cup]0,2[= M$.

2) Summen er da $f(x) = \dfrac{1}{1-e^{(x^3-4x)}}$, $x \in M$.

3) f og g har samme monotoniforhold.

$$g'(x) = \left(e^{(x^3-4x)}\right)' = e^{(x^3-4x)} \cdot (3x^2-4)$$

Dvs. $g'(x) = 0 \Leftrightarrow 3x^2-4 = 0 \Leftrightarrow x = \pm \frac{2}{\sqrt{3}}$.

Det er kun $\frac{2}{\sqrt{3}}$ der ligger i M.

x		-2		0		$\frac{2}{\sqrt{3}}$		2
f'	+				−	0	+	
f	↗				↘	lok min	↗	

4) f har lok. min i $x = \frac{2}{\sqrt{3}}$, med værdi

$$f\left(\frac{2}{\sqrt{3}}\right) = \frac{1}{1 - e^{(2/\sqrt{3})^3 - 4(2/\sqrt{3})}}.$$

Her er $\left(\frac{2}{\sqrt{3}}\right)^3 - 4\left(\frac{2}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\left(\frac{4}{3} - 4\right) = \frac{-16}{3\sqrt{3}}$

Så er $f\left(\frac{2}{\sqrt{3}}\right) = \frac{1}{1 - e^{-16/3\sqrt{3}}} > 1$.

For $x \to -\infty$ vil $f(x) \to 1$

For $x \to -2^-$ vil $f(x) \to \infty$

For $x \to 0^+$ vil $f(x) \to \infty$

For $x \to 2^-$ vil $f(x) \to \infty$.

Heraf ses, at $Vm(f) =]1, \infty[$.

Da f har et lok. min. i et indre punkt i M, er f ikke injektiv.

5) $f(x) = y$. Bliver en 3-gradsligning

Så sp. 5 droppes!