LM Juni 19 vejl. besv.

opg 1

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

1) $N(L)$: $\quad Lx = \underline{0}$

$$L \Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$x_4 = s$, $x_5 = t$

$\underline{x_3 = 0}$ $\qquad x_2 + x_4 = 0$ $\qquad x_1 + x_5 = 0$

$\qquad x_2 = -x_4 = -s$ $\qquad x_1 = -x_5 = -t$

$$N(L): \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\underline{v}_1} + t\underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}_2}, \quad s,t \in \mathbb{R}$$

v_1, v_2 udgør en basis for $N(L)$.

2) L er surjektiv da L har 3 ledende ettaller.

$$5 - \dim N(L) = \dim R(L)$$
$$5 - 2 = 3.$$

3)

$$s \cdot V_1 + t V_2 = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 3 \\ 2 \end{bmatrix} \Leftrightarrow (s,t) = (3,2)$$

som er koordinatene.

4)

$$1 \cdot \underline{V}_1 + 1 \cdot V_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

5)

$$\mathcal{L}X = Y$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & Y_1 \\ 1 & 1 & 0 & 1 & 1 & Y_2 \\ 1 & 0 & 1 & 0 & 1 & Y_3 \end{array}\right] \hookrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & Y_1 \\ 0 & 1 & 0 & 1 & 0 & Y_2 - Y_1 \\ 0 & 0 & 1 & 0 & 0 & Y_3 - Y_1 \end{array}\right]$$

dvs

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 - Y_1 \\ Y_3 - Y_1 \\ 0 \\ 0 \end{bmatrix} + s V_1 + t V_2, \; s,t \in \mathbb{R}.$$

(fra 1)

opg 2

1) $Av_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, dvs $\underline{\underline{\lambda_1 = 1}}$

$Av_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -a+4 \\ 0 \end{bmatrix}$

Hvis $Av_2 = \lambda_2 v_2$ skal $\begin{bmatrix} 0 \\ -a+4 \\ 0 \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Så skal $\underline{\underline{\lambda_2 = 0}}$, hvorfor $\underline{\underline{a = 4}}$.

$Av_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2a+2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$

da $a = 4$. $\begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix} = \lambda_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ giver $\underline{\underline{\lambda_3 = 5}}$.

A er symmetrisk, dvs diagonaliserbar.

2) $f(A) = Q f(D) Q^T$, hvor

$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$, $f(D) = \begin{bmatrix} f(1) & & \\ & f(0) & \\ & & f(5) \end{bmatrix}$

(4)

heraf fås

$$f(A) = \begin{bmatrix} f(1) & 0 & 0 \\ 0 & \frac{1}{5}f(0)+\frac{4}{5}f(5) & -\frac{2}{5}f(0)+\frac{2}{5}f(5) \\ 0 & -\frac{2}{5}f(0)+\frac{2}{5}f(5) & \frac{4}{5}f(0)+\frac{1}{5}f(5) \end{bmatrix}$$

3) $\det e^A = \cancel{e^{f(1)} e^{f(0)} e^{f(5)} = e^{f(1)+f(0)+f(5)}}$

$= e^1 e^0 e^5 = \underline{\underline{e^6}}$

4) $A(v_1+v_2) = Av_1 + Av_2 = 1\cdot v_1 + 0 \cdot v_2 = \underline{\underline{v_1}}$.

Opg 3

1) $\int \sin^2(2x)\cos(3x)\,dx$

$$= \int \left(\frac{e^{i2x}-e^{-i2x}}{2i}\right)^2\left(\frac{e^{i3x}+e^{-i3x}}{2}\right)dx$$

$$= -\frac{1}{8}\int \left(e^{i4x}+e^{-i4x}-2\right)\left(e^{i3x}+e^{-i3x}\right)dx$$

$$= -\frac{1}{8}\int e^{i7x}+e^{ix}+e^{-ix}+e^{-i7x}-2\left(e^{i3x}+e^{-i3x}\right)dx$$

$$= -\frac{1}{4}\int \cos(7x)-2\cos(3x)+\cos(x)\,dx$$

$$= -\frac{1}{4}\left(\frac{1}{7}\sin(7x)-\frac{2}{3}\sin(3x)+\sin(x)\right)+k.$$

2) $w^2 = 3 - i$

$$w^2 = (x+iy)^2 = x^2 - y^2 + i2xy = 3 - i$$

$$x^2 - y^2 = 3 \qquad \text{Så er } y = \frac{-1}{2x}$$

$$2xy = -1$$

og $$x^2 - \left(\frac{-1}{2x}\right)^2 = 3$$

$$x^2 - \frac{1}{4x^2} = 3$$

$$4x^4 - 1 = 12x^2$$

$$4x^4 - 12x^2 - 1 = 0$$

$u = x^2$, så $$u = \frac{12 \overset{+}{(-)} \sqrt{144+16}}{8}$$

$$x^2 = u = \frac{12 + \sqrt{160}}{8} = \frac{3+\sqrt{10}}{2}$$

$$x = \pm\sqrt{\frac{3+\sqrt{10}}{2}}, \text{ dvs}$$

$$w = x + iy = \pm\left(\sqrt{\frac{3+\sqrt{10}}{2}} - i\,\frac{1}{2\sqrt{\frac{3+\sqrt{10}}{2}}}\right)$$

(som kan skrives litt penere).

opg 4

$$\sum_{n=0}^{\infty} (g(x))^n \quad \text{, med } g(x) = x^2(x^2-1)$$

1) $|g(x)| < 1$. $\quad |x^2(x^2-1)| < 1$.

Vi løser $x^2(x^2-1) = 1$:

$$x^4 - x^2 - 1 = 0$$

$$x^2 = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2} \quad (- \text{ forkastes})$$

dvs $x = \pm\sqrt{\frac{1+\sqrt{5}}{2}}$

Vi løser $x^2(x^2-1) = -1$:

$$x^4 - x^2 + 1 = 0$$

<u>ingen løsning.</u>

Da $g(x) = 0$ for $x = 0, 1, -1$

og $g(x) \to \infty$ for $x \to \pm\infty$

ser vi at $|g(x)| < 1$ for

$$x \in \left] -\sqrt{\frac{1+\sqrt{5}}{2}}, \sqrt{\frac{1+\sqrt{5}}{2}} \right[$$

hvor f så er veldefineret,

2) $f(x) = \frac{1}{1-g(x)} = \frac{1}{1-x^2(x^2-1)}$.

3) f har monotoniforhold som g

$g(x) = x^4 - x^2$,

$g'(x) = 4x^3 - 2x$

$g'(x) = 0 \iff x = 0, \pm\sqrt{\frac{1}{2}}$

		$-\sqrt{\frac{1}{2}}$		0		$\sqrt{\frac{1}{2}}$	
f'	$-$	0	$+$	0	$-$	0	$+$
f	$\searrow$	lok min	$\nearrow$	lok max	$\searrow$	lok min	$\nearrow$

Vm: for $x \to \pm\sqrt{\frac{1\pm\sqrt{5}}{2}}$ går $f(x)$ mod ∞

$f(-\sqrt{\frac{1}{2}}) = f(\sqrt{\frac{1}{2}}) = \frac{4}{5}$

$f(0) = 1$

Så er $Vm(f) = [\frac{4}{5}, \infty[$

oplagt ikke injektiv.

5) $f(x) = y \iff x^2(x^2-1) = \frac{y-1}{y}$, dvs

$x^4 - x^2 - \frac{y-1}{y} = 0$

$x^2 = \frac{1 \pm \sqrt{1 + 4(\frac{y-1}{y})}}{2}$ så $x = \pm\sqrt{\frac{1 \pm \sqrt{1+4(\frac{y-1}{y})}}{2}}$.