Written Exam for M.Sc. in Economics

Investment Theory

15. August 2011

Master course

Corrections

Exercise 1.

(a) P can be interpreted as the revenue and C can be interpreted as the cost. The option to expand is the possibility to expand capacity from one unit to one plus K units. An example could be a firm with capacity one producing whatever (beer, concrete, computer games...) building a new plant with capacity K.

(b) The strategy could take the form:

$$\begin{cases} P < P^* & \Rightarrow \quad \text{wait} \\ P \geq P^* & \Rightarrow \quad \text{expand} \end{cases} \qquad V(P) \;=\; \begin{cases} ? & \text{for } P < P^* \\ W(P) - J & \text{for } P \geq P^* \end{cases}$$

$V(P)$ should satisfy value matching (VM), smooth pasting (SP) and "$P \to 0 \Rightarrow V(P) \to$ fundamental value of firm". $W(P)$ should satisfy "$P \to 0 \Rightarrow W(P) \to$ fundamental value of firm" and "no bubbles".

We need to find "?", P^*, $V(P)$ and $W(P)$.

(c) For $V(P)$ in case $P < P^*$ consider a portfolio consisting of the firm and $-n$ units of the portfolio Q. The dividend rate of the portfolio is

$$\frac{P - C + \alpha PV'(P) + 0.5\sigma^2 P^2 V''(P) - n(\alpha + \delta)Q}{V(P) - nQ} dt + \frac{\sigma PV'(P) - n\sigma Q}{V(P) - nQ} dz.$$

Ito's Lemma is used to find $dV(P)$. If $n = PV'(P)/Q$, then there is no uncertainty in the dividend rate of the portfolio. Therefore the dividend rate has to be equal to the interest rate r, because otherwise there are arbitrage possibilities. Hence

$$0.5\sigma^2 P^2 V''(P) + (r - \delta)PV'(P) - rV(P) + P - C \;=\; 0.$$

For $W(P)$ consider a portfolio consisting of the firm after the expansion and $-n$ units of the portfolio Q. The dividend rate of the portfolio is

$$\frac{(1 + K)(P - C) + \alpha PW'(P) + 0.5\sigma^2 P^2 W''(P) - n(\alpha + \delta)Q}{W(P) - nQ} dt + \frac{\sigma PW'(P) - n\sigma Q}{W(P) - nQ} dz.$$

Ito's Lemma is used to find $dW(P)$. If $n = PW'(P)/Q$, then there is no uncertainty in the dividend rate of the portfolio. Therefore the dividend rate has to be equal to the interest rate r, because otherwise there are arbitrage possibilities. Hence

$$0.5\sigma^2 P^2 W''(P) + (r-\delta)PW'(P) - rW(P) + (1+K)(P-C) \;=\; 0.$$

(d) Solutions to the differential equation consist of linear combinations of two independent solutions to the homogenous equation and one solution to the equation. Both parts of the solutions can be found be guessing and verifying that the guesses are correct. Thus

$$W(P) \;=\; B_1 P^{\beta_1} + B_2 P^{\beta_2} + (1+K)\left(\frac{P}{\delta} - \frac{C}{r}\right)$$

where $B_1, B_2 \in \mathbb{R}$ and $\beta_1 > 1$ and $\beta_2 < 0$ are solutions to

$$0.5\sigma^2(\beta-1)\beta + (r-\delta)\beta - r \;=\; 0.$$

By evaluating this polynomium at $\beta = 0$ and $\beta = 1$ it can be seen that there is one root β_1 with $\beta_1 > 1$, because $\delta > 0$, and one root β_2 with $\beta_2 < 0$, because $r > 0$.

Since $W(P)$ should satisfy $P \to 0 \Rightarrow W(P) \to$ fundamental value of firm and "no bubbles", the solution is

$$W(P) \;=\; (1+K)\left(\frac{P}{\delta} - \frac{C}{r}\right).$$

(e) Solutions to the differential equation consist of linear combinations of two independent solutions to the homogenous equation and one solution to the equation. Both parts of the solutions can be found be guessing and verifying that the guesses are correct. Thus

$$V(P) \;=\; A_1 P^{\beta_1} + A_2 P^{\beta_2} + \frac{P}{\delta} - \frac{C}{r}$$

where $A_1, A_2 \in \mathbb{R}$.

Since $V(P)$ should satisfy $P \to 0 \Rightarrow V(P) \to$ fundamental value of firm, the solution is

$$V(P) \;=\; \frac{P}{\delta} - \frac{C}{r} + A_1 P^{\beta_1}.$$

(f) For $V(P)$, P/δ is the value of getting P forever given the process P follows, $-C/r$ is the value of paying C forever given C is fixed and $A_1 P^{\beta_1}$ is the value of the option to expand. Hence it is expected that $A_1 > 0$. The value of the option to expand is increasing in P because $\beta_1 > 1$.

For $W(P)$, KP/δ is the value of getting KP forever and $-KC/r$ is the value of paying KC forever.

(g) (VM) is equivalent to

$$\frac{P^*}{\delta} - \frac{C}{r} + A_1 P^{*\beta_1} \;=\; (1+K)\left(\frac{P^*}{\delta} - \frac{C}{r}\right) - J.$$

(SP) is equivalent to

$$\frac{1}{\delta} + \beta_1 A_1 P^{*\beta_1 - 1} \;=\; \frac{1+K}{\delta}.$$

By rearranging (VM) and (SP) and dividing them with each other it is found that

$$P^* \;=\; \frac{\beta_1}{\beta_1 - 1}\delta\left(\frac{C}{r} + \frac{J}{K}\right).$$

(h) It is found that

$$\frac{\partial P^*}{\partial r} \;=\; -\frac{\partial\beta_1/\partial r}{(\beta_1-1)^2}\delta\left(\frac{C}{r} + \frac{J}{K}\right) - \frac{\beta_1}{\beta_1-1}\delta\frac{C}{r^2}$$

By an application of the implicit function theorem to

$$0.5\sigma^2(\beta-1)\beta + (r-\delta)\beta - r \;=\; 0.$$

at $\beta = \beta_1$ it is found that

$$\frac{\partial\beta_1}{\partial r} \;=\; -\frac{\beta_1 - 1}{0.5\sigma^2(2\beta_1 - 1) + (r-\delta)}$$

so $\partial\beta_1/\partial r > 0$ because

$$\begin{aligned}
0.5\sigma^2(2\beta_1 - 1) + (r - \delta) &= \frac{1}{\beta_1}\left(0.5\sigma^2(\beta_1 - 1)\beta_1 + (r - \delta)\beta_1 + 0.5\sigma^2\beta_1^2\right) \\
&= \frac{1}{\beta_1}(r + 0.5\sigma^2\beta_1^2).
\end{aligned}$$

Therefore $\partial P^*/\partial r < 0$. The expression for $\partial P^*/\partial r$ is found by plugging the expression for $\partial\beta_1/\partial r$ into the expression for $\partial P^*/\partial r$.