CORRECTION GUIDE

Written Exam for the B.Sc. or M.Sc. in Economics autumn 2012-2013

Operations Research

Elective Course

January 23rd, 2013

(3-hour open/closed book exam)

The language used in your exam paper must be English or Danish.

Part 1 – Knapsack problem

Q1.1: Solve the LP relaxed KP instance.

We notice that the variables are already set up with diminishing return to scale with x_1 being the most profitable.
The optimal solution to the LP is then constructed as
x_1 = 1 (rest resource: 10-6=4), x_2 = 4/5 (rest resource: 4 – 4/5*5 = 0), x_3 = 0
v

Q1.2: Use Branch & Bound to solve the KP instance to optimality.

In Q1.1 we already solved the root problem, SUB1.
The variable x_2 = 4/5 is the only non-binary value and we therefore need to branch on x_2
This establishes two new subproblems, which we solve in a FIFO fashion:
SUB2 = SUB1 + "x_2 = 1"
SUB3 = SUB1 + "x_2 = 0"

SUB2: x_2 = 1 (rest resource: 10 – 5 = 5), x_1 = 5/6 (rest resource: 5 – 5/6*6 = 0), x_3 = 0
OBJ = (5 4 3)*(5/6 1 0)^T = 8.16667

SUB3: x_2 = 0 (rest resource: 10 – 0 = 0), x_1 = 1 (rest resource: 10 – 6 = 4), x_3 = 1 (rest: 4 – 4 = 0)
OBJ = (5 4 3)*(1 0 1)^T = 8

SUB3 yields a candidate solution, so LB=8 (from SUB3) and UB=8.16667 (from SUB2)

However, we still need to branch on SUB2 since the OBJ was better than LB. The only possible branching variable is x_1:

SUB4 = SUB2 + "x_1 = 1"
SUB5 = SUB2 + "x_1 = 0"

SUB4: x_2 = 1 (rest resource: 10 – 5 = 5), x_1 = 1 (rest resource: 5 – 6 = – 1) → Infeasible!

SUB5: x_2 = 1 (rest resource: 10 – 5 = 5), x_1 = 0 (rest resource: 5 – 0 = 5), x_3 = 1 (rest: 5 – 4 = 1)
OBJ = (5 4 3)*(0 1 1)^T = 7

We have now branched on all necessary nodes, and even if SUB5 also yielded a candidate solution, it was not better than the already obtained in SUB3.

So the optimal solution is x = (1 0 1) with OBJ = 8

Part 2

Q2.1: Verify that the current solution is a basic feasible solution for the Network Simplex Method.

We must verify that the current solution is feasible and that it is a basic solution.
It is <u>feasible</u> since flow conservation constraints for all nodes are satisfied ("inflow = outflow") when we include the injecting and draining flows and since all limits for min and max flow are abided.
It is a <u>basic solution</u> since the basic arcs (where the flow is sharply within the boundaries, that is, arcs *b, c, d, e* and *h*) – constitute a spanning tree.

Q2.2: Determine whether the current solution is optimal or not using the Network Simplex Method.

We first find the dual variables for all nodes:
$u_1 = 0$ (selected freely), $u_3 = u_1 - c_{13} = 0 - 5 = -5$, $u_4 = u_3 - c_{34} = -5 - 5 = -10$, $u_2 = u_4 + c_{24} = -10 + 7 = -3$, $u_5 = u_2 - c_{24} = -3 - 6 = -9$, $u_6 = u_5 - c_{56} = -9 - 4 = -13$.

We check the optimality criteria by checking the reduced costs on non-basic variables:
On lower bound (optimality criterion $r_{arc} \leq 0$)
$r_a = u_1 - u_2 - c_{12} = 0 - -3 - 4 = -1 \rightarrow$ OK
$r_f = u_3 - u_5 - c_{35} = -5 - -9 - 6 = -2 \rightarrow$ OK
On upper bound (optimality criterion $r_{arc} \geq 0$)
$r_g = u_4 - u_6 - c_{46} = -10 - -13 - 4 = -1 \rightarrow$ NOT OK

Since not all reduced costs have the right sign, we have shown that the current solution is <u>not optimal</u>. [Arc *g* has to enter basis and the flow must be lowered]

Q2.3: Now, disregard the current solution given above. Formulate the LP model that corresponds to the specific network flow problem above. You may use the letters a, b, …, h, *to denote the flows in the arcs (i.e. the decision variables).*

Min	$4a+5b+7c+6d+5e+6f+4g+4h$	
St	$10=a+b$	$2 \leq a \leq 10$
	$a=c+d$	$0 \leq b \leq 10$
	$b=e+f$	$-5 \leq c \leq 5$
	$c+e=g$	$0 \leq d \leq 5$
	$d+f=h$	$5 \leq e \leq 10$
	$g+h=10$	$0 \leq f \leq 5$
	(flow conservation)	$5 \leq g \leq 9$
		$0 \leq h \leq 5$
		(lower and upper bounds)

Part 3

Q3.1: Let K = 7 *and use DP to find the Longest Path from Node* 1 *to Node* 6*.*

Q3.2: With K ≥ 0*, use DP to find the values of* K *for which the Longest Path from* 1 *to* 6 *includes the Arc from Node* 1 *to Node* 2*.*

Since Q3.1 is a special case of Q3.2, we start with the latter.

If we are in State *i* at Stage *t*, we will let the recursion formula $f_t(i)$ denote the length of the longest path from State *i* at Stage *t* to the endpoint.

We apply the recursion formula from right to left:
$f_{t=4}(i=6) = 0$ (we are already there)

$f_{t=3}(i=4) = \max\{c_{46} + f_{t=4}(i=6)\} = K$
$f_{t=3}(i=5) = \max\{c_{56} + f_{t=4}(i=6)\} = 3$

$f_{t=2}(i=2) = \max\{c_{24} + f_{t=3}(i=4) \,;\, c_{25} + f_{t=3}(i=5)\} = \max\{2+K \,;\, 1+3\} = \max\{2+K \,;\, 4\}$
$f_{t=2}(i=3) = \max\{c_{34} + f_{t=3}(i=4) \,;\, c_{35} + f_{t=3}(i=5)\} = \max\{2+K \,;\, 4+3\} = \max\{2+K \,;\, 7\}$

$f_{t=1}(i=1) = \max\{c_{12} + f_{t=2}(i=2) \,;\, c_{13} + f_{t=2}(i=3)\} =$
$\max\{2 + \max\{2+K \,;\, 4\} \,;\, K + \max\{2+K \,;\, 7\}\} =$
$\max\{\underline{\max\{4+K \,;\, 6\}} \,;\, \underline{\underline{\max\{2+2K \,;\, 7+K\}}}\}$

We notice, that the double underlined part always dominate the single underlined part when K ≥ 0 since (7+K) > 4+K and (7+K) > 6. We may also have that 2+2K > 7+K, depending on K, but it remains, that the value of $f_{t=1}(i=1)$ is determined by the value of $c_{13} + f_{t=2}(i=3)$ where the arc from Node 1 to Node 2 is NOT included.

The answer to Q3.2 is therefore, that the arc from Node 1 to Node 2 NEVER is part of the longest path.

The answer to Q3.1 – where K = 7 – is that the longest path is of length max {2+2K ; 7+K} = max { 16 ; 14 } = 16.

Since this was obtained by using 2+2K we find the path Node 1 – Node 3 – Node 4 – Node 6.