Corrections for:

Written Exam for M.Sc. in Economics Summer School 2013

Investment Theory

Master Course

19th August 2013

3 hours closed books exam

Question 1.

(a) The project could be to enter a market or whatever. The market can be in a good state Y or in a bad state X. The state of the market is influenced by macroeconomic factors modelled with the jump processes. P could be a revenue index for being in the market and CX and C_Y could be the cost of producing the good. The actual revenue is XP in the bad state and YP in the good state.

(b) The entry strategies could be cut-off strategies:

$$\text{For state } X \quad \begin{cases} P < P_X^* & \Rightarrow \quad \text{Wait} \\ P \geq P_X^* & \Rightarrow \quad \text{Invest} \end{cases}$$

$$\text{For state } Y \quad \begin{cases} P < P_Y^* & \Rightarrow \quad \text{Wait} \\ P \geq P_Y^* & \Rightarrow \quad \text{Invest} \end{cases}$$

Therefore

$$F_X(P) = \begin{cases} ? & \text{for } P < P_X^* \\ V_X(P) - I & \text{for } P \geq P_X^* \end{cases}$$

$$F_Y(P) = \begin{cases} ? & \text{for } P < P_Y^* \\ V_Y(P) - I & \text{for } P \geq P_Y^* \end{cases}$$

Moreover the functions should satisfy: "no bubbles", "$P \to 0 \Rightarrow H(P) \to 0$", value matching and smooth pasting.

I expect $XP_X^* > C_X$ and $YP_Y^* > C_Y$, because the project should only be started when dividends are positive.

(c) The Bellman equations are

$$\rho V_X(P) = XP - C_X + \frac{1}{dt}E(dV_X(P))$$

$$\rho V_Y(P) = YP - C_Y + \frac{1}{dt}E(dV_Y(P))$$

where according to Ito's lemma

$$\begin{aligned} dV_X(P) &= (\alpha P V_X'(P) + \frac{1}{2}\sigma^2 P^2 V_X''(P))dt + \alpha P V_X'(P)dz \\ &\quad +(V_Y(P) - V_X(P))dq_X \end{aligned}$$

$$\begin{aligned} dV_Y(P) &= (\alpha P V_Y'(P) + \frac{1}{2}\sigma^2 P^2 V_Y''(P))dt + \alpha P V_Y'(P)dz \\ &\quad +(V_X(P) - V_Y(P))dq_Y \end{aligned}$$

with

$$dq_X = \begin{cases} 1 & \text{with probability } \lambda_{XY} dt \\ 0 & \text{with probability } 1 - \lambda_{XY} dt \end{cases}$$

$$dq_Y = \begin{cases} 1 & \text{with probability } \lambda_{YX} dt \\ 0 & \text{with probability } 1 - \lambda_{YX} dt. \end{cases}$$

Therefore

$$\frac{1}{dt}E(dV_X(P)) = \alpha P V_X'(P) + \frac{1}{2}\sigma^2 P^2 V_X''(P) + \lambda_{XY}(V_Y(P) - V_X(P))$$

$$\frac{1}{dt}E(dV_Y(P)) = \alpha P V_Y'(P) + \frac{1}{2}\sigma^2 P^2 V_Y''(P) + \lambda_{YX}(V_X(P) - V_Y(P))$$

Hence the Bellman equations become

$$\frac{1}{2}\sigma^2 P^2 V_X''(P) + \alpha P V_X'(P) - (\rho + \lambda_{XY})V_X(P) + XP - C_X + \lambda_{XY} V_Y(P) = 0$$

$$\frac{1}{2}\sigma^2 P^2 V_Y''(P) + \alpha P V_Y'(P) - (\rho + \lambda_{YX})V_Y(P) + YP - C_Y + \lambda_{YX} V_X(P) = 0.$$

(d) The two Bellman equations have to be solved simultanously. I guess that $V_X(P) = a_X P + b_X$ and $V_Y(P) = a_Y P + b_Y$. For the guesses the Bellman equations become

$$(-(\rho - \alpha + \lambda_{XY})a_X + \lambda_{XY} a_Y + X)P + (-(\rho + \lambda_{XY})b_X + \lambda_{XY} b_Y - C_X) = 0$$

$$(\lambda_{YX} a_X - (\rho - \alpha + \lambda_{YX})a_Y + Y)P + (\lambda_{YX} b_X - (\rho + \lambda_{YX})b_Y - C_Y) = 0$$

Therefore

$$V_X(P) = \frac{(\rho - \alpha + \lambda_{YX})X + \lambda_{XY} Y}{(\rho - \alpha)(\rho - \alpha + \lambda_{XY} + \lambda_{YX})} P - \frac{(\rho + \lambda_{YX})C_X + \lambda_{XY} C_Y}{\rho(\rho + \lambda_{XY} + \lambda_{YX})}$$

$$V_Y(P) = \frac{\lambda_{YX} X + (\rho - \alpha + \lambda_{XY})Y}{(\rho - \alpha)(\rho - \alpha + \lambda_{XY} + \lambda_{YX})} P - \frac{\lambda_{YX} C_X + (\rho + \lambda_{XY})C_Y}{\rho(\rho + \lambda_{XY} + \lambda_{YX})}$$

so $a_Y > a_X > 0$ and $b_Y < b_X < 0$. Hence for $P \to 0$, $V_X(P) > V_Y(P)$, and for $P \to \infty$, $V_Y(P) > V_X(P)$. Moreover there is a unique value of P such that $V_X(P) = V_Y(P)$ because both V_X and V_Y are linear.

The value of an active project in both states depend on the revenue index and the cost in both states as well as the probabilities of switching between the two states. The values $XP/(\rho - \alpha) - C_X/\rho$ and $YP/(\rho - \alpha) - C_Y/\rho$ are the expected net present values of having an active project in state X and state Y. $V_X(P)$ and $V_Y(P)$ are convex combinations of these two values where the weights depends on the transition probabilities between the two states.

(e) The Bellman equations are

$$\begin{aligned}
\rho F_X(P) &= \frac{1}{dt}E(dF_X(P)) \\
\rho F_Y(P) &= \frac{1}{dt}E(dF_Y(P))
\end{aligned}$$

where according to Ito's lemma

$$\begin{aligned}
dF_X(P) &= (\alpha P F_X'(P) + \frac{1}{2}\sigma^2 P^2 F_X''(P))dt + \alpha P F_X'(P)dz \\
&\quad + (F_Y(P) - F_X(P))dq_X \\
dF_Y(P) &= (\alpha P F_Y'(P) + \frac{1}{2}\sigma^2 P^2 F_Y''(P))dt + \alpha P F_Y'(P)dz \\
&\quad + (F_X(P) - F_Y(P))dq_Y
\end{aligned}$$

with

$$\begin{aligned}
dq_X &= \begin{cases} 1 & \text{with probability } \lambda_{XY}dt \\ 0 & \text{with probability } 1 - \lambda_{XY}dt \end{cases} \\
dq_Y &= \begin{cases} 1 & \text{with probability } \lambda_{YX}dt \\ 0 & \text{with probability } 1 - \lambda_{YX}dt. \end{cases}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{dt}E(dF_X(P)) &= \alpha P F_X'(P) + \frac{1}{2}\sigma^2 P^2 F_X''(P) + \lambda_{XY}(F_Y(P) - F_X(P)) \\
\frac{1}{dt}E(dF_Y(P)) &= \alpha P F_Y'(P) + \frac{1}{2}\sigma^2 P^2 F_Y''(P) + \lambda_{YX}(F_X(P) - F_Y(P))
\end{aligned}$$

Hence the Bellman equations become

$$\begin{aligned}
\frac{1}{2}\sigma^2 P^2 F_X''(P) + \alpha P F_X'(P) - (\rho + \lambda_{XY})F_X(P) + \lambda_{XY}F_Y(P) &= 0 \\
\frac{1}{2}\sigma^2 P^2 F_Y''(P) + \alpha P F_Y'(P) - (\rho + \lambda_{YX})F_Y(P) + \lambda_{YX}F_X(P) &= 0.
\end{aligned}$$

Subtracting the two Bellman equations gives

$$\frac{1}{2}\sigma^2 P^2 F''(P) + \alpha P F'(P) - (\rho + \lambda_{XY} + \lambda_{YX})F(P) = 0$$

(f) The mathematical solution to the equation in F is

$$F(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2}$$

where A_1 and A_2 are constants and $\beta_1 > 1$ and $\beta_2 < 0$ are solutions to

$$\frac{1}{2}\sigma^2(\beta-1)\beta + \alpha\beta - (\rho + \lambda_{XY} + \lambda_{YX}) \;=\; 0.$$

The economic solution is

$$F(P) \;=\; A_1 P^{\beta_1}$$

because otherwise $P \to 0$ implies $|F_X(P)| \to \infty$ or $|F_Y(P)| \to \infty$. Using this in the Bellman equation for F_X gives

$$\frac{1}{2}\sigma^2 P^2 F_X''(P) + \alpha P F_X'(P) - \rho F_X(P) - \lambda_{XY} A_1 P^{\beta_1} \;=\; 0$$

Hence the mathematical solution is

$$F_X(P) \;=\; B_1 P^{\gamma_1} + B_2 P^{\gamma_2} + \frac{\lambda_{XY}}{\lambda_{XY}+\lambda_{YX}} A_1 P^{\beta_1}$$

where $\gamma_1 > 1$ and $\gamma_2 < 0$ are solutions to

$$\frac{1}{2}\sigma^2(\gamma-1)\gamma + \alpha\gamma - \rho \;=\; 0.$$

The economic solution is

$$F_X(P) \;=\; B_1 P^{\gamma_1} + \frac{\lambda_{XY}}{\lambda_{XY}+\lambda_{YX}} A_1 P^{\beta_1}$$

because otherwise $P \to 0$ implies $|F_X(P)| \to \infty$. Therefore

$$F_Y(P) \;=\; B_1 P^{\gamma_1} - \frac{\lambda_{YX}}{\lambda_{XY}+\lambda_{YX}} A_1 P^{\beta_1}$$

(g) The optimal strategies as well as the undetermined constants can be found by finding a solution to the two times two equations coming from value matching and smooth pasting:

$$B_1(P_X^*)^{\gamma_1} + \frac{\lambda_{XY}}{\lambda_{XY}+\lambda_{YX}} A_1 (P_X^*)^{\beta_1} - I \;=\; a_X P_X^* + b_X$$

$$\gamma_1 B_1(P_X^*)^{\gamma_1-1} + \beta_1 \frac{\lambda_{XY}}{\lambda_{XY}+\lambda_{YX}} A_1 (P_X^*)^{\beta_1-1} \;=\; a_X$$

$$B_1(P_Y^*)^{\gamma_1} - \frac{\lambda_{YX}}{\lambda_{XY}+\lambda_{YX}} A_1 (P_Y^*)^{\beta_1} - I \;=\; a_Y P_Y^* + b_Y$$

$$\gamma_1 B_1(P_Y^*)^{\gamma_1-1} - \beta_1 \frac{\lambda_{YX}}{\lambda_{XY}+\lambda_{YX}} A_1 (P_Y^*)^{\beta_1-1} \;=\; a_Y.$$

There are four unknowns: P_X^*, P_Y^*, A_1, B_1. The equations are not linear, so probably numerical methods have to be used to find a solution. Since $F_X(P) - F_Y(P) = A_1 P^{\beta_1}$, either $F_X(P) > F_Y(P)$ for all P or $F_X(P) < F_Y(P)$ for all P forgetting that $A_1 = 0$ is a possibility.

(h) A relevant option for the example of entry into a market could be exit from the market. Suppose that the exit cost is $E > 0$. Then the strategy for the exit option could be a cut-off strategy:

$$\text{For state } X \quad \begin{cases} P \leq P_X^E & \Rightarrow \text{ Exit} \\ P > P_X^E & \Rightarrow \text{ Continue} \end{cases}$$

$$\text{For state } Y \quad \begin{cases} P \leq P_Y^E & \Rightarrow \text{ Exit} \\ P > P_Y^E & \Rightarrow \text{ Continue} \end{cases}$$

Therefore

$$V_X(P) = \begin{cases} -E & \text{for } P \leq P_X^E \\ ? & \text{for } P > P_X^E \end{cases}$$

$$V_Y(P) = \begin{cases} -E & \text{for } P \leq P_Y^E \\ ? & \text{for } P > P_Y^E \end{cases}$$

The Bellman equations for $V_X(P)$ and $V_Y(P)$ would not change so the solutions found in (d) would still be solutions. However additional terms reflecting the values of the options to exit would have to be added. An approach similar to the approach used for the option to invest could have to be used to find the additional terms. The values of the additional terms should go to zero as P goes to infinity. Undetermined coefficients would be have to be determined by use of value matching and smooth pasting for $V_X(P)$ and $V_Y(P)$.