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# Written exam
# Macroeconomics C

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**Closed book exam, 3 hours**
**Number of questions:** This exam consists of 2 questions.

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1. Consider the following Ramsey model. The production function of the representative firm is given by

$$Y(t) = AK(t)^{\alpha}L(t)^{1-\alpha}E(t)$$

where the externality $E(t) = K^{AG}(t)^{1-\alpha}$ and $K^{AG}$ is the total amount of capital in the economy. Consumers have utility function

$$u(c(t)) = \frac{c(t)^{1-\sigma}}{1-\sigma}$$

Assume no population growth ($n = 0$), no depreciation of capital ($\delta = 0$), and normalize population to 1. Initial household assets are given by $a_0$. Households supply 1 unit of labor inelastically and earn wage $w(t)$.
Suppose there is a tax policy that subsidizes savings by households: For each unit of assets that the household own at time $t$, the government gives the household a payment of $\phi$ (this is in addition to the rental rate $r(t)$ that the household receives). Assume $\phi$ does not vary over time. The government finances this expenditure with a labor income tax $\tau(t)$.

   (a) Write the household's maximization problem, set up the Hamiltonian,write down the Maximum Principle conditions and transversality condition.

   (b) Use your results from the previous part to show that the differential equations describing the dynamics of $c(t)$ and $a(t)$ are given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}(r(t) + \phi - \rho)$$
$$\dot{a}(t) = (1 - \tau(t))w(t) + (r(t) + \phi)a(t) - c(t)$$

   Give a *brief* interpretation of both equations.

   (c) Write the maximization problem of the firm and solve this problem to obtain solutions for the rental rate $r(t)$ and wage rate $w(t)$.

   Assume the government runs a balanced budget each period. What is its budget constraint?

(d) Use the information from the previous parts to show that the equilibrium differential equations for $c(t)$ and $k(t)$ are given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}(\alpha A + \phi - \rho)$$
$$\dot{k}(t) = Ak(t) - c(t)$$

The social planner problem of this economy is

$$\max \int e^{-\rho t} u(c(t)) dt$$
$$\dot{k}(t) = Ak(t) - c(t)$$

(e) Solve the social planner problem and derive the equilibrium differential equations for $c(t)$ and $k(t)$. Can the subsidy $\phi$ be chosen such that the equilibrium is optimal? If so, what is it, and what is the implied tax rate $\tau(t)$?

2. Consider a competitive economy with an infinite number of identical firms and households. The representative firm maximizes its profits, remunerating labor hours, $h_t$, at the wage rate, $w_t$. Production is carried out by means of the following technology:

$$y_t = z_t h_t^{\alpha} \tag{1}$$

where $y_t$ denotes output and $z_t$ is a technology shock. Households' income is allocated between consumption and equity (i.e., stocks of the representative firm). The representative household maximizes the discounted stream of expected utility:

$$\max_{c_t, h_t, \mu_t} \mathbf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[ \log c_t - \frac{h_t^{1+\nu}}{1+\nu} \right] \right\}, \tag{2}$$

subject to the following constraint:

$$w_t h_t + \mu_{t-1}(d_t + q_t) = c_t + \mu_t q_t. \tag{3}$$

where $\mu_t$ denotes the amount of equity holdings at time $t$ (i.e., the amount of shares), $d_t \equiv y_t - w_t h_t$ are the dividends (profits) rebated by firms to households at time $t$ (these are taken as given by households) and $q_t$ denotes the stock price at time $t$. The parameters ($\beta$, $\alpha$ and $\nu$) are all positive, with $\beta \in [0, 1)$ and $\alpha \in [0, 1]$. Furthermore, households' time endowment is normalized to 1 and the aggregate resource constraint is such that $y_t = c_t$. Given this environment, address the following questions, providing adequate comment to the derivation of each and every result:

(a) Set up the representative firm's and household's optimization problems and derive the necessary first order conditions, respectively.

(b) Characterize the labor demand and supply schedules and prove that the equilibrium wage and hours are $w_t = \alpha^{\frac{a+v}{1+v}} z_t$ and $h_t = \alpha^{\frac{1}{1+v}}$, respectively.

(c) Find the equilibrium value of dividends. Following a positive realization of the technology shock (i.e., $z_t > 0$), equilibrium dividends are negative. True or false? Why?

(d) Starting from the first order condition with respect to $\mu_t$, prove that

$$q_t = \beta \mathbf{E}_t \left[ \frac{z_t \alpha^{\frac{\alpha}{1+v}}}{z_{t+1} \alpha^{\frac{\alpha}{1+v}}} \left( z_{t+1} \alpha^{\frac{\alpha}{1+v}} (1-\alpha) + q_{t+1} \right) \right] \tag{4}$$

Under which value for $\alpha$ does equation (4) reduce to $\frac{q_t}{z_t} = \beta \mathbf{E}_t \left[ \frac{q_{t+1}}{z_{t+1}} \right]$?

**Solution:**

1. (a) The household problem is

$$\max \int e^{-\rho t} u(c(t)) dt$$
$$\dot{a}(t) = (1-\tau(t))w(t) + (r(t)+\phi)a(t) - c(t)$$
$$a(0) = a_0, a(t) \geq -B, c(t) \geq 0 \forall t$$

It's ok if they write $a(t) \geq 0$ implicitly assuming that the only asset will be physical capital. But there should be some terminal asset condition. It's ok if they skip the non-negativity constraint on consumption. The Hamiltonian is

$$H(t,a,c,\lambda) = e^{-\rho t} u(c(t)) + \lambda(t)((1-\tau(t))w(t) + (r(t)+\phi)a(t) - c(t))$$

The Maximum Principle conditions are

$$\frac{\partial H}{\partial c} = e^{-\rho t} u'(c(t)) - \lambda(t) = 0 \tag{5}$$
$$\frac{\partial H}{\partial a} = \lambda(t)(r(t)+\phi) = -\dot{\lambda}(t) \tag{6}$$
$$\frac{\partial H}{\partial \lambda} = (1-\tau(t))w(t) + (r(t)+\phi)a(t) - c(t) = \dot{a}(t) \tag{7}$$

They can write this in either current or present value terms, as long as they are consistent and arrive at the right conclusion. The appropriate transversality condition is

$$\lim_{T\to\infty} a(T)\lambda(T) = 0$$

(b) Start with equation (5):

$$\begin{aligned}
e^{-\rho t}u'(c(t)) &= \lambda(t) \\
-\rho t + \log u'(c(t)) &= \log \lambda(t) \\
-\rho + \frac{\mathrm{d}\log u'(c(t))}{\mathrm{d}u'(c(t))}\frac{\partial u'(c(t))}{\partial c(t)}\frac{\partial c(t)}{\partial t} &= \frac{\mathrm{d}\log \lambda(t)}{\mathrm{d}\lambda(t)}\frac{\partial \lambda(t)}{\partial t} \\
-\rho + \frac{1}{u'(c(t))}u''(c(t))\dot{c} &= \frac{1}{\lambda(t)}\dot{\lambda} \\
\frac{\dot{\lambda}}{\lambda(t)} &= \frac{u''}{u'}\dot{c} - \rho \qquad (8)
\end{aligned}$$

where the second line follows from taking logs, the third from differentiating with respect to time, and the remainder is rearranging. Use (8) in (6):

$$\begin{aligned}
-\frac{\dot{\lambda}}{\lambda(t)} &= r(t) + \phi \\
-\frac{u''}{u'}\dot{c} - \rho &= r(t) + \phi \\
\frac{\dot{c}}{c(t)} &= -\frac{u''}{u'c}(r(t) + \phi - \rho) \\
\frac{\dot{c}}{c(t)} &= \frac{1}{\sigma}(r(t) + \phi - \rho)
\end{aligned}$$

where in the last line we use the functional form for utility that is given in the question. This gives the first differential equation asked for in the question. The other is the budget constraint, (7).

(c) Firms are not directly affected by this policy, nor the externality, so their problem is standard:

$$\max_{K(t),L(t)} AK(t)^{\alpha}L(t)^{1-\alpha}E(t) - r(t)K(t) - w(t)L(t)$$

which yields

$$\begin{aligned}
r(t) &= \alpha AK(t)^{\alpha-1}L(t)^{1-\alpha}E(t) \\
w(t) &= (1-\alpha)AK(t)^{\alpha}L(t)^{-\alpha}E(t)
\end{aligned}$$

The government budget constraint is

$$\phi a(t) = \tau(t)w(t)$$

(d) This follows from algebra as well as the equilibrium conditions $L(t) = L^{AG}(t) = 1, K(t) = K^{AG}(t) = k(t) = a(t)$.

(e) Yes, if $\phi = (1-\alpha)A$ then the Euler equation is the same as the one for the social planner's problem. The implied tax rate is

$$\begin{aligned}\tau(t) &= \frac{k(t)}{w(t)}\phi \\ &= \frac{1}{(1-\alpha)A}(1-\alpha)A \\ &= 1\end{aligned}$$

2. (a) We first set up the Lagrangian for households' optimization:

$$\mathcal{L}_t = \mathbf{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t - \frac{h_t^{1+\nu}}{1+\nu} + \lambda_t \left[ w_t h_t + \mu_{t-1}(d_t + q_t) - c_t - q_t \mu_t \right] \right\} \tag{9}$$

The first order conditions with respect to the choice variables $(c_t, h_t, \mu_t)$ are:

$$\frac{\partial \mathcal{L}_t}{\partial c_t} = 0 \Longrightarrow \frac{1}{c_t} - \lambda_t = 0 \tag{10}$$

$$\frac{\partial \mathcal{L}_t}{\partial h_t} = 0 \Longrightarrow -h_t^{\nu} + \lambda_t w_t = 0 \tag{11}$$

$$\frac{\partial \mathcal{L}_t}{\partial \mu_t} = 0 \Longrightarrow -q_t \lambda_t + \beta \mathbf{E}_t \left[ \lambda_{t+1}(d_{t+1} + q_{t+1}) \right] = 0 \tag{12}$$

Firms' objective is to maximize lifetime discounted profits. However, since there are no dynamic links across periods, this is the same as maximizing profits in every periods. Specifically, the representative firm chooses labor hours so that

$$\max_{h_t} \left( z_t h_t^a - w_t h_t \right) \tag{13}$$

which leads to the necessary condition:

$$\alpha z_t h_t^{a-1} = w_t \tag{14}$$

(b) To characterize the labor market equilibrium, we take (14) as the labor demand function. As to the supply side of the labor market, combine (10) and (11) so as to get:

$$h_t^{\nu} = \frac{w_t}{c_t} \tag{15}$$

We then employ the market clearing condition $y_t = c_t$ and the technology constraint $y = z_t h_t^{\alpha}$ to substitute for $h_t$ and $c_t$ in (15):

$$h_t = \left( \frac{w_t}{z_t} \right)^{\frac{1}{v+\alpha}} \tag{16}$$

To find equilibrium wage and hours, we equalize (14) and (16), obtaining:

$$\begin{aligned} h_t &= \alpha^{\frac{1}{1+v}} && (17) \\ w_t &= \alpha^{\frac{a+v}{1+v}} z_t && (18) \end{aligned}$$

(c) We replace the equilibrium levels of $h_t$ and $w_t$ so as to get dividends in equilibrium:

$$d_t = z_t \alpha^{\frac{\alpha}{1+v}} (1-\alpha) \tag{19}$$

The statement is false given that, at best, $\alpha = 1$, in which case profits are null.

(d) Combine (12) with (10), so as to get the following equilibrium condition for the stock price:

$$q_t = \beta \mathbf{E}_t \left[ \frac{c_t}{c_{t+1}} (d_{t+1} + q_{t+1}) \right] \tag{20}$$

In the latter, we replace $c_t$, $c_{t+1}$ and $d_{t+1}$ with their equilibrium values, obtaining

$$q_t = \beta \mathbf{E}_t \left[ \frac{z_t \alpha^{\frac{\alpha}{1+v}}}{z_{t+1} \alpha^{\frac{\alpha}{1+v}}} \left( z_{t+1} \alpha^{\frac{\alpha}{1+v}} (1-\alpha) + q_{t+1} \right) \right] \tag{21}$$

It is immediate to verify that (21) reduces to $\frac{q_t}{z_t} = \beta \mathbf{E}_t \left[ \frac{q_{t+1}}{z_{t+1}} \right]$ for $\alpha = 1$, i.e. when dividends are zero.