Written Exam for M.Sc. in Economics 2012

Investment Theory

20. August 2012

Master course

Answers

Exercise 1.

1.a P could be the price of some good and C could be the cost of producing the good. The project is then to buy a machine or whatever that can produce either K units of the good or L units of the good. It is costly to alter scale. The machine can die or the market for the good can break down.

The project could be extraction of some natural resource where it is costly to adjust the rate of extraction. The project could die because there is no more left to extract.

1.b For all three real options the strategies could be cut-off strategies.

$$\begin{cases} P < P^* & \Rightarrow \quad \text{Wait} \\ P \geq P^* & \Rightarrow \quad \text{Invest} \end{cases}$$

$$\begin{cases} P < P_D & \Rightarrow \quad \text{Continue} \\ P \geq P_D & \Rightarrow \quad \text{Alter from } K(P-C) \text{ to } L(P-C) \end{cases}$$

$$\begin{cases} P \leq P_U & \Rightarrow \quad \text{Alter from } L(P-C) \text{ to } K(P-C) \\ P > P_U & \Rightarrow \quad \text{Continue} \end{cases}$$

Clearly $P^* > C$ so I expect the investor to start the project with scale L. Therefore

$$\begin{aligned} F(P) &= \begin{cases} ? & \text{for } P < P^* \\ V_L(P) - I & \text{for } P \geq P^* \end{cases} \\ V_K(P) &= \begin{cases} ? & \text{for } P < P_U \\ V_L(P) - X & \text{for } P \geq P_U \end{cases} \\ V_L(P) &= \begin{cases} V_K(P) - Y & \text{for } P \leq P_D \\ ? & \text{for } P > P_D \end{cases} \end{aligned}$$

Moreover the functions should satisfy: "no bubbles", "$P \to 0 \Rightarrow H(P) \to 0$", value matching and smooth pasting.

I expect $P^* > C$, because the project should only be started when the dividend is positive. I expect $P_U > C > P_D$, because the project should only be scaled down when dividends are negative and scaled up when dividends are positive. Hence there is hysteresis so for $P_D < P < P_U$ the scale of the project depends on the history of P.

1.c The Bellman equations are

$$\begin{aligned}\rho V_K(P) &= K(P-C) + \frac{1}{dt}E(dV_K(P)) \\ \rho V_L(P) &= L(P-C) + \frac{1}{dt}E(dV_L(P))\end{aligned}$$

By use of Ito's lemma these equations become

$$\begin{aligned}\frac{1}{2}\sigma^2 P^2 V_K''(P) + \alpha P V_K'(P) - (\rho+\lambda)V_K(P) + K(P-C) &= 0 \\ \frac{1}{2}\sigma^2 P^2 V_L''(P) + \alpha P V_L'(P) - (\rho+\lambda)V_L(P) + L(P-C) &= 0.\end{aligned}$$

1.d The mathematical solutions to the Bellman equations are

$$\begin{aligned}V_K(P) &= D_1 P^{\beta_1} + D_2 P^{\beta_2} + \frac{K}{\rho+\lambda-\alpha}P - \frac{K}{\rho+\lambda}C \\ V_L(P) &= E_1 P^{\beta_1} + E_2 P^{\beta_2} + \frac{L}{\rho+\lambda-\alpha}P - \frac{L}{\rho+\lambda}C\end{aligned}$$

where $\beta_1 > 1$ and $\beta_2 < 0$ are solutions to

$$\frac{1}{2}\sigma^2(\beta-1)\beta + \alpha\beta - (\rho+\lambda) = 0.$$

$D_2 = 0$ because of "$P \to 0 \Rightarrow H(P) \to 0$" and $E_1 = 0$ because of "no bubbles", so the economic solutions are

$$\begin{aligned}V_K(P) &= \frac{K}{\rho+\lambda-\alpha}P - \frac{K}{\rho+\lambda}C + D_1 P^{\beta_1} \\ V_L(P) &= \frac{L}{\rho+\lambda-\alpha}P - \frac{L}{\rho+\lambda}C + E_2 P^{\beta_2}.\end{aligned}$$

1.e For V_K: $\frac{K}{\rho+\lambda-\alpha}P$ is the NPV of getting KP forever taking into account that P is stochastic and the project can die; $-\frac{K}{\rho+\lambda}C$ is the NPV of paying KC forever taking into account that the project can die; and, $D_1 P^{\beta_1}$ is the value of being able to alter scale. The value of being able to alter scale should be increasing in P, because the value of increasing the scale should be increasing in P. Hence I expect $D_1 > 0$.

For V_L: $\frac{L}{\rho+\lambda-\alpha}P$ is the NPV of getting LP forever taking into account that P is stochastic and the project can die; $-\frac{L}{\rho+\lambda}C$ is the NPV of paying LC forever taking into account that the project can die; and, $E_2 P^{\beta_2}$ is the value of being able to alter scale. The value of being able to alter scale should be decreasing in P, because the value of decreasing the scale should be decreasing in P. Hence I expect $E_2 > 0$.

1.f The optimal strategies as well as the undetermined constants can be found by finding a solution to the four equations coming from value matching and smooth pasting:

$$\begin{aligned}
\frac{K}{\rho+\lambda-\alpha}P_U - \frac{K}{\rho+\lambda}C + D_1 P_U^{\beta_1} &= \frac{L}{\rho+\lambda-\alpha}P_U - \frac{L}{\rho+\lambda}C + E_2 P_U^{\beta_2} - X \\
\frac{K}{\rho+\lambda-\alpha} + \beta_1 D_1 P_U^{\beta_1-1} &= \frac{L}{\rho+\lambda-\alpha} + \beta_2 E_2 P_U^{\beta_2-1} \\
\frac{L}{\rho+\lambda-\alpha}P_D - \frac{L}{\rho+\lambda}C + E_2 P_D^{\beta_2} &= \frac{K}{\rho+\lambda-\alpha}P_D - \frac{K}{\rho+\lambda}C + D_1 P_D^{\beta_1} - Y \\
\frac{L}{\rho+\lambda-\alpha} + \beta_2 E_2 P_D^{\beta_2-1} &= \frac{K}{\rho+\lambda-\alpha} + \beta_1 D_1 P_D^{\beta_1-1}.
\end{aligned}$$

Four equations, four unknowns: P_U, P_D, D_1 and E_2. However, the equations are not linear. Note that $P_U \neq P_D$. In the book by Dixit & Pindyck (p. 218) it is mentioned that there is a unique solution with $P_U > P_D > 0$ and $D_1, E_2 > 0$. As mentioned above there is hysteresis so for $P_D < P < P_U$ the scale of the project depends on the history of P.

A perfect answer should include some formal analysis of P_U and P_D. However far less than what follows here is needed for a perfect answer. Let G be defined by

$$G(P) = \frac{L-K}{\rho+\lambda-\alpha}P - \frac{L-K}{\rho+\lambda}C - D_1 P^{\beta_1} + E_2 P^{\beta_2}.$$

Then the four value matching and smooth pasting equations become

$$\begin{aligned} G(P_U) &= X & \qquad G(P_D) &= -Y \\ G'(P_U) &= 0 & \qquad G'(P_D) &= 0. \end{aligned}$$

Moreover

$$\begin{aligned} G'(P) &= \frac{L-K}{\rho+\lambda-\alpha} - \beta_1 D_1 P^{\beta_1 - 1} + \beta_2 E_2 P^{\beta_2 - 1} \\ G''(P) &= -(\beta_1 - 1)\beta_1 D_1 P^{\beta_1 - 2} + (\beta_2 - 1)\beta_2 E_2 P^{\beta_2 - 2}. \end{aligned}$$

Therefore $G''(P) = 0$ if and only if $(\beta_1 - 1)\beta_1 D_1 P^{\beta_1} = (\beta_2 - 1)\beta_2 E_2 P^{\beta^2}$. So there is a unique $\bar{P}$ such that $G''(\bar{P}) = 0$. Hence there are two and only two P's such that $G'(P) = 0$, namely P_U and P_D, one is smaller than $\bar{P}$ and one is larger than $\bar{P}$. So $P_D < \bar{P} < P_U$. Clearly $G''(P) > 0$ for $P < \bar{P}$ and $G''(P) < 0$ for $P > \bar{P}$, so $G''(P_D) > 0$ and $G''(P_U) < 0$.

Subtracting the differential equation for V_K from the differential equation for V_L gives

$$\frac{1}{2}\sigma^2 P^2 G''(P) + \alpha P G'(P) - (\rho+\lambda)G(P) + (L-K)(P-C) = 0.$$

Using the four equations in the differential equation for G gives

$$\begin{aligned} \frac{1}{2}\sigma^2 P_D^2 G''(P_D) + (\rho+\lambda)Y + (L-K)(P_D - C) &= 0 \\ \frac{1}{2}\sigma^2 P_U^2 G''(P_U) - (\rho+\lambda)X + (L-K)(P_U - C) &= 0. \end{aligned}$$

Therefore

$$P_D = C - \frac{\rho+\lambda}{L-K}Y - \frac{1}{2(L-K)}\sigma^2 P_D^2 G''(P_D) < C - \frac{\rho+\lambda}{L-K}Y$$

$$P_U = C + \frac{\rho+\lambda}{L-K}X - \frac{1}{2(L-K)}\sigma^2 P_U^2 G''(P_U) > C + \frac{\rho+\lambda}{L-K}X.$$

This shows that P_D can be negative in contradiction with Dixit & Pindyck.

1.g The Bellman equation is

$$\rho F(P) = \frac{1}{dt}E(dF(P)).$$

Using Ito's Lemma the Bellman equation becomes

$$\frac{1}{2}\sigma^2 P^2 F''(P) + \alpha P F'(P) - \rho F(P) = 0.$$

The mathetatical solution is

$$F(P) = A_1 P^{\gamma_1} + A_2 P^{\gamma_2}$$

where $\gamma_1 > 1$ and $\gamma_2 < 0$ solve

$$\frac{1}{2}\sigma^2(\gamma-1)\gamma + \alpha\gamma - \rho = 0$$

and A_1 and A_2 are constants. The economic solution is

$$F(P) = A_1 P^{\gamma_1}$$

because of "$P \to 0 \Rightarrow H(P) \to 0$".

1.h The investor will only start the project in case $P > C$ so the dividend is positive. Therefore the investor will start the project with scale L. The optimal strategy can be found by considering value matching and smooth pasting for F and V_L

$$A_1(P^*)^{\gamma_1} = \frac{L}{\rho+\lambda-\alpha}P^* - \frac{L}{\rho+\lambda}C + E_2(P^*)^{\beta_2} - I$$

$$\gamma_1 A_1(P^*)^{\gamma_1-1} = \frac{L}{\rho+\lambda-\alpha} + \beta_2 E_2(P^*)^{\beta_2-1}.$$

There are two equations and two unknowns: P^* and A_1.