Written exam for the M.Sc. in Economics, Winter 2013/14

Game theory

Final Exam/Elective Course/Master's Course

21 January 2014

(3 hours, closed book exam)

The exam has 3 pages in total (including cover page).

Explain all your answers!

Question 1: *(rationalizability)* This exercise is a variation of the Bertrand game. Two firms compete by simultaneously setting prices. Prices have to be **natural numbers** (including 0), that is $A_i = \{0, 1, 2, \dots\}$. Consumers buy 100 units from the firm setting the lower price if this price is less or equal to 60. If all posted prices are above 60 consumers buy 0 units. If both firms set the same price (less or equal to 60), consumers buy 50 units from firm 1 and 50 units from firm 2. Each firm has costs of 2 if it sells 100 units and costs of 1 if it sells 50 units (and costs of 0 if it sells 0 units).

If we denote the price of firm i by $a_i \in \mathbb{N}$, the profits of firm i are therefore

$$\pi_i(a_i, a_j) = \begin{cases} 100a_i - 2 & \text{if } a_i < a_j \text{ and } a_i \leq 60 \\ 50a_i - 1 & \text{if } a_i = a_j \text{ and } a_i \leq 60 \\ 0 & \text{else.} \end{cases}$$

Each firm maximizes its own profit.

a) Is setting a price of 0 a rationalizable action?

No, the action $a_i = 0$ is strictly dominated by $a_i = 1$: $\pi(0, a_j) \leq -1$ while $\pi(1, a_j) \geq 0$.

b) Is setting a price of 2 a rationalizable action?

Yes: Let $Z_1 = Z_2 = \{2\}$. $a_i = 2$ is a best response to the belief that firm j plays 2 with probability 1: $\pi_i(2, 2) = 99 > \pi_i(1, 2) = 98$ and $\pi_i(a_i, 2) \leq 0$ for all $a_i \notin \{1, 2\}$.

c) Which actions are rationalizable?

We showed that 2 is rationalizable and also 1 is rationalizable because $a_i = 1$ is a best response to $a_j = 1$. We now show that these are the only rationalizable actions.

Let Z_i be the set of rationalizable actions of firm i. Let $\bar{a}$ be $\max(Z_1 \cup Z_2)$ and let i be the/a firm such that $\bar{a} \in Z_i$. As $\bar{a}$ is rationalizable, there has to exist a belief μ on Z_j such that $\bar{a}$ is a best response given the belief μ. By the definition of $\bar{a}$, we have $a_j \leq \bar{a}$ for all $a_j \in Z_j$. From the previous exercise, we know that $0 \notin Z_j$. The last two statements imply together that $\bar{a} = 2$: No price a' strictly

above 2 is a best response to prices that are at least 1 and less than a'. (If the other firm plays a pure strategy $a_j > 2$, the best response is to undercut its price by 1. If the other firm plays a mixed strategy, the best response is at least 1 lower than the maximum price the other firm plays with positive probability.)
Therefore, $Z_i = \{1, 2\}$ for firm i=1,2.

Question 2: *(extensive form games with incomplete information)* Consider the three-player, extensive form game below.
The dashed line indicates that the two nodes are in the same information set of player 2!

a) Find a *pure strategy* Nash equilibrium. Is the Nash equilibrium subgame perfect? Show that there is (at least) one player for which the Nash equilibrium strategy is not sequentially rational.
Take the strategy profile (b, c, l). This is a Nash equilibrium. As the game itself is its only subgame, it also a subgame perfect Nash equilibrium. It is, however, not sequentially rational as P3 should deviate to d if he was asked to act: Given P1's strategy, this gives him a payoff of 2 instead of 1.

b) Derive a strategy profile that is sequentially rational and where beliefs satisfy Bayes' rule in every information set. (hint: consider mixed strategies)
Let P2 play a mixed strategy that puts weights 1/2 on each of his actions. Then P3 is indifferent between c and d. Let P3 play c with probability 0.4 and d with probability 0.6: This implies that P1 is—given P1's and P3's strategy—indifferent between playing a and b. Let P1 play a with probability 1/1.6=5/8 and b with probability 3/8. This implies that P2's belief (in order to satisfy Bayes' rule) puts probability 1/2 on each of the two nodes inhis information set. Given this belief, P2 is indifferent between playing l and r and his mixed strategy (with which we started) is a best response. As each player is indifferent between his two actions, the strategies are sequentially rational.

Question 3: *(knowledge)* Both Alice and Bob receive an envelope with money. The amount (in $) that Bob receives is an even number between 1 and 100. The amount Alice receives is either 1 more or 1 less than Bob's amount. Each player knows his own amount but not the amount of the other player. This setting is common knowledge among Bob and Alice.

Denoting Alice's amount by A and Bob's amount by B, (A, B) where $B \in \{2, 4, \ldots 100\}$ and $A \in \{B-1, B+1\}$ is a *state*.

Let E be the event "both players have more than 3$", i.e. the set of states where $min\{A, B\} > 3$.

a) Suppose Bob has 6$ and Alice has 5$. Which states are considered possible by Bob? Do Bob and Alice know the event E? Is the event E common knowledge?

   As both know that the difference between their amounts is 1, Bob considers the states (6,5) and (6,7) possible. Alice considers the states (4,5) and (6,5) possible. The event E occurs in all states that are considered possible by any player. Hence, both know the event E. Bob knows E if $B \geq 6$. Alice does not know that $B \geq 6$. Hence, E is not common knowledge in (6,5) as $(6, 5) \notin K_A(K_B(E))$.

b) Now imagine there are 100 visitors who come one by one into the room in which Bob and Alice are. Each of these 100 visitors asks "Does one of you know that he has less money than the other?" and Bob an Alice answer simultaneuously "yes" or "no".

   What is common knowledge among Bob and Alice (apart from the setting) if both answer "no" to the first visitor? In which states and after how many visitors is the event E common knowledge among Alice and Bob? After how many visitors will one of them answer "yes"?

   If Alice had 1$, she would know that she has less than Bob and would answer "yes". If both answer "no", it is, therefore, common knowledge that $A > 1$. Given this, Bob will answer "yes" to the second visitor if he has 2$. Hence, it is common knowledge that $B > 2$ if both asnwer "no" to the first and second visitor. Iterating this reasoning gives that if both answer "no" for n times, it is common knowledge that $min\{A, B\} > n$. Hence, E is common knowledge if both answer "no" 3 times which only happens in states where $min\{A, B\} > 3$. Hence, the player with the lower amount will say "yes" to the $min\{A, B\}$ visitor.