Written Exam for M.Sc. in Economics 2012

Investment Theory

3. January 2013

Master course

Answers

Exercise 1.

1.a P could be the revenue. The revenue can be negative because P follows an ABM. The project could to extract some natural resource. I is the cost of building a mine. The project can die because the mine collapses. E is the cost of cleaning up. Perhaps it would be more natural to have two different exit costs: one in case the mine has collapsed; and, one in case the is closed down.

It is an empirical question whether P follows an ABM.

1.b For both real options the strategies could be cut-off strategies.

$$\begin{cases} P < P_S & \Rightarrow \text{ Wait} \\ P \geq P_S & \Rightarrow \text{ Start} \end{cases}$$
$$\begin{cases} P \leq P_E & \Rightarrow \text{ Exit} \\ P > P_E & \Rightarrow \text{ Continue} \end{cases}$$

Therefore

$$F(P) = \begin{cases} ? & \text{for } P < P_S \\ V(P) - I & \text{for } P \geq P_S \end{cases}$$
$$V(P) = \begin{cases} F(P) - E & \text{for } P \leq P_E \\ ? & \text{for } P \geq P_E \end{cases}$$

Moreover the functions should satisfy: "no bubbles", "$P \to -\infty \Rightarrow H(P) \to 0$", value matching and smooth pasting.

I expect $P_S > 0 > P_E$, because the project should only be started when the dividend is positive and the proejct should only be stopped when the dividend is negative.

1.c Consider a portfolio consisting of one unit of the option to invest and minus n units of the asset. The dividend rate is

$$\frac{\alpha F'(P) + \frac{1}{2}\sigma^2 F''(P) - n\gamma Q}{F(P) - nQ} dt + \frac{\sigma F'(P) - n\tau Q}{F(P) - nQ} dz$$

For $n = \sigma F'(P)/(\tau Q)$ there is no risk. Therefore for $n = \sigma F'(P)/(\tau Q)$

$$\frac{\alpha F'(P) + \frac{1}{2}\sigma^2 F''(P) - n\gamma Q}{F(P) - nQ} dt \;=\; r$$

because of no-arbitrage. Rearranging the equation results in the following differential equation

$$\frac{1}{2}\sigma^2 F''(P) + (\alpha + (r-\gamma)\frac{\sigma}{\tau})F'(P) - rF(P) \;=\; 0.$$

1.d The mathematical solution to the differential equation is

$$F(P) \;=\; A_1 e^{\beta_1 P} + A_2 e^{\beta_2 P}$$

where $\beta_1 > 0$ and $\beta_2 < 0$ are solutions to

$$\frac{1}{2}\sigma^2\beta^2 + (\alpha + (r-\gamma)\frac{\sigma}{\tau})\beta - r \;=\; 0.$$

$A_2 = 0$ because of "$P \to -\infty \Rightarrow H(P) \to 0$", so the economic solution is

$$F(P) \;=\; A_1 e^{\beta_1 P}.$$

This solution is relevant for $P < P_S$.

1.e Consider a portfolio consisting of one unit of the active project and minus n units of the asset. The dividend rate is

$$\frac{P + \alpha V'(P) + \frac{1}{2}\sigma^2 V''(P) - n\gamma Q}{V(P) - nQ} dt + \frac{\sigma V'(P) - n\tau Q}{V(P) - nQ} dz$$

For $n = \sigma V'(P)/(\tau Q)$ there is no risk. Therefore for $n = \sigma V'(P)/(\tau Q)$

$$\frac{P + \alpha V'(P) + \frac{1}{2}\sigma^2 V''(P) - n\gamma Q}{V(P) - nQ} dt = r$$

because of no-arbitrage. Rearranging the equation results in the following differential equation

$$\frac{1}{2}\sigma^2 V''(P) + (\alpha + (r-\gamma)\frac{\sigma}{\tau})V'(P) - rV(P) + P = 0.$$

1.f The mathematical solution to the differential equation is

$$V(P) = \frac{1}{r}P - \frac{\alpha + (r-\gamma)\frac{\sigma}{\tau}}{r^2} + B_1 e^{\beta_1 P} + B_2 e^{\beta_2 P}$$

where $\beta_1 > 0$ and $\beta_2 < 0$ are solutions to

$$\frac{1}{2}\sigma^2\beta^2 + (\alpha + (r-\gamma)\frac{\sigma}{\tau})\beta - r = 0.$$

$B_1 = 0$ because of "no bubbles", so the economic solution is

$$V(P) = \frac{1}{r}P - \frac{\alpha + (r-\gamma)\frac{\sigma}{\tau}}{r^2} + B_2 e^{\beta_2 P}$$

This solution is relevant for $P > P_E$.

1.g $\frac{1}{r}P$ is the value of getting P (P being fixed) forever. $\frac{\alpha+(r-\gamma)\frac{\sigma}{\tau}}{r^2}$ is the value of future changes in P. $B_2 e^{\beta_2 P}$ is the value of being able to stop the project.

1.h The optimal strategy can be found by considering value matching and smooth pasting for F and V

$$\begin{cases} A_1 e^{\beta_1 P_S} &= \dfrac{1}{r} P_S - \dfrac{\alpha + (r-\gamma)\frac{\sigma}{\tau}}{r^2} + B_2 e^{\beta_2 P_S} - I \\ \\ \beta_1 A_1 e^{\beta_1 P_S} &= \dfrac{1}{r} + \beta_2 B_2 e^{\beta_2 P_S} \end{cases}$$

$$\begin{cases} \dfrac{1}{r} P_E - \dfrac{\alpha + (r-\gamma)\frac{\sigma}{\tau}}{r^2} + B_2 e^{\beta_2 P_E} &= A_1 e^{\beta_1 P_E} - E \\ \\ \dfrac{1}{r} + \beta_2 B_2 e^{\beta_2 P_E} &= \beta_1 A_1 e^{\beta_1 P_E} \end{cases}$$

There are four equations and four unknowns: P_S, P_E, A_1 and B_2.