Solutions for exam in Auctions, 10 June 2014

The following solutions manual gives thorough answers to all questions. This level of detail is not required to obtain full marks on each question.

Exercise 1: True or false statements (25%)

Question 1a: False.
The exclusion principle states that it is optimal for a seller to set a reserve price above his own valuation $r > x_0$
In the case of all bidders having identical and independently distributed private values one can show through the optimization of expected revenue of the seller with respects to the reserve price that [Krishna, Chapter 2.5]:

$$\boldsymbol{x_0 = r^* - \frac{1}{\lambda(r^*)}}$$

However, in the case of bidders with interdependent values and exemplified with affiliated potentially private values in the book [Krishna, chapter 8.4] the exclusionary principle fails since bidders' information is no longer statistically independent. If this assumption is made and explained an answer of reserve price at sellers valuation is accepted.

Question 1b: False.
The expected revenue of the seller is 5.

With uniform distributed values $X \sim U[a,b]$ where $a = 0, b = 10$ the cumulative distribution function and probability distribution functions are

$$F(x) = \frac{x-a}{b-a} = \frac{x-0}{10-0} = \frac{1}{10}x$$
$$f(x) = \frac{1}{b-a} = \frac{1}{10}$$

The value of the highest competing bid, Y_1 a highest order statistic distributed according to $G(x)$ [see appendix C]

$$G(x) = \left(F(x)\right)^{n-1} = \left(\frac{1}{10}x\right)^{3-1} = \frac{1}{100}x^2$$
$$g(x) = G'(x) = \frac{2}{100}x = \frac{1}{50}x$$

The expected payment of a bidder in a standard auction is

$$m^A(x) = m^0(x) + \int_0^x y g(y) dy$$

It is given that the expected payment with value zero is zero, $m^0(x) = 0$ and inserting the $g(x)$ yields

$$m^A(x) = 0 + \int_0^x y \cdot \left(\frac{1}{50}y\right) dy$$
$$m^A(x) = \frac{1}{50}\left[\frac{1}{3}y^3\right]_0^x = \frac{1}{150}(x^3 - 0^3) = \frac{1}{150}x^3$$

The *ex-ante* expected payment of a bidder is calculated by integrating ("summing") the expected payment, $m^A(x)$, over probability, $f(x)$, the entire value interval $[0, \omega]$

$$E[m^A(x)] = \int_0^\omega m^A(x) f(x) dx = \int_0^{10} \frac{1}{150}x^3 \cdot \left(\frac{1}{10}\right) dx$$
$$= \frac{1}{1500}\left[\frac{1}{4}x^4\right]_0^{10} = \frac{1}{1500 \cdot 4} \cdot (10^4 - 0^4) = \frac{10.000}{6.000} = \frac{10}{6} = \frac{5}{3}$$

The expected revenue to the seller is the *ex-ante* expected payment multiplied with the number of bidders

$$E[R^A] = N \cdot E[m^A(X)] = 3 \cdot \frac{5}{3} = 5$$

Hence the statement is false as the expected revenue is 5 and not 10/3

Question 1c: True. In a Combinatorial Clock Auction (CCA) the bidders can pay more than their individual opportunity costs when the sum of the individual opportunity costs exceeds the joint opportunity costs for a group of bidders.

The pricing in CCA's are determined as socalled "nearest-Vickrey core prices". The core determines a set of payments that support the efficient assignment in the sense that there does not exist an alternative coalition of bidders that has collectively offered the seller more. The "nearest-Vickrey" implies that the point in the core that is closest to the Vickrey prices (i.e. individual opportunity costs) will be chosen. In mathematical terms, "nearest-Vickrey" corresponds to selecting the point in the core that minimises the sum of squared deviations from the Vickrey prices. In some cases, bidders will actually pay their individual opportunity costs (Vickrey prices), in other cases they will pay more than their individual opportunity costs (nearest-Vickrey core prices).

<u>Exercise 2: Copenhagen Real Estate Market (35%)</u>

Question 2a:
Given that bidders on the market have similar preferences one can assume they are identically and independently distributed.

A bidder with risk neutral private values that are identically and independently distributed have a symmetric and increasing equilibrium bidding strategy of:

$$\beta^I(x) = \frac{1}{G(x)} \int_0^x yg(y)dy = E[Y_1 | Y_1 < x]$$

The strategy is equivalent to bidding the highest expected value of the N-1 competing bidders given that the bidder has the highest of all values.

The payoff from this strategy can be broken down in two terms:

$$\Pi(x) = G(x)\big(x - \beta^I(x)\big)$$

- The probability of winning, G(x)
- The net value left to the bidder after paying his bid (first-price auction) in the case he wins, $\big(x - \beta^I(x)\big)$.

Bidding his value increases his probability of obtaining zero net value and equivalent to not winning. A bidder can increase his expected value by lowering his bid below his valuation, i.e. shading his bid, $\beta^I(x) < x$.

Question 2b:
If a bidder is risk-averse the revenue equivalence principle no longer holds and specifically the expected revenue of a first-price auction is greater than that in a second-price auction. Since the REP was satisfied before the expected revenue to the seller of the ideal apartment increases. [Krishna, Chapter 4.1]

The intuition is that a risk-averse seller wishes to buy insurance against the "loss" of not winning the apartment. He "pays" by having a lower expected payoff. In the two terms described in **2a** he puts more emphasis on the probability of winning G(x) than his net value in case he wins $\big(x - \beta^I(x)\big)$.

Question 2c:
In a second-price auction it is optimal to bid your valuation, $\beta^{II}(x) = x$
This holds no matter the risk appetite of the bidder since bidding below or above can only result in outcomes that are "as good" or worse than bidding ones valuation.

Thus the expected revenue of selling apartments on the Copenhagen real estate market is lowered by adopting a second-price auction. [Krishna, Chapter 4.1]

Both the first-price and the second-price auctions are efficient no matter the risk appetite of the bidders on the market, as the bidder with the highest value will bid the most and win the apartment.

Question 2d:
The National Coalition of Concerned Sellers are opposing the adoption of the second-price auction as it –temporarily– lowers the expected revenue to the seller by deflating the price bubble.

Further and more importantly adopting a second-price auction is a mean preserving spread of the expected revenues to the seller in first-price auction in case of independently and identically distributed private values. That is, in expectation a seller will obtain the same revenue (Revenue equivalence) but the variance is greater. A risk-averse seller is **not** interested in this variation of the expected revenue, as it generates uncertainty about the price a seller will get. (chapter 2.4)

Question 2e:
In the case of asymmetric bidders one can show that stochastic weakness in terms of valuations leads to aggression in the first-price bidding strategy (chapter 4.3). Again in two terms described in **2a** a stochastically weak bidder forgoes net value in case he wins (x-beta^I) to increase his chance of winning G(x), since he knows his competing bidders are likely to have a higher valuation than him as they draw values from a stochastically dominating distribution.

With asymmetric bidders no general ranking of the revenues is possible. (chapter 4.3.2)

In terms of efficiency the second-price auction is efficient, since bidders are still bidding their valuations. However the first-price auction is with positive probability inefficient. This happens in the case, where the weak bidder bids aggressively above the shaded bid of a strong bidder but the weak bidders valuation is below the valuation of the strong bidder.

Exercise 3: Multiple object auctions (40%)

Question 3a: It is a weakly dominant strategy to bid truthfully, i.e. $\beta(\boldsymbol{x}) = \boldsymbol{x}$, in a Vickrey auction [proposition 13.1 in Krishna].

To argue why, one can e.g. follow the reasoning in the proof of proposition 13.1 by considering the three cases for bidder i for given bids from its competitors:

1) i bids $\boldsymbol{b}^i \neq \boldsymbol{x}^i$ such that he wins the same number of units as when bidding $\boldsymbol{b}^i = \boldsymbol{x}^i$: In this case the prices paid by bidder i are unaffected and so is the utility.
2) i bids $\boldsymbol{b}^i \neq \boldsymbol{x}^i$ such that he wins a larger number of units, $l^i > k^i$, than when bidding $\boldsymbol{b}^i = \boldsymbol{x}^i$: In this case the prices paid by bidder i are unaffected for the first k^i units. For the remaining units the bidder pays prices that exceed (or at best, equal to) the marginal valuations for these units. Therefore, the utility is lower than (or at best, equal to) the utility when bidding truthfully.
3) i bids $\boldsymbol{b}^i \neq \boldsymbol{x}^i$ such that he wins a lower number of units, $l^i < k^i$, than when bidding $\boldsymbol{b}^i = \boldsymbol{x}^i$: In this case the prices paid by bidder i are unaffected for the first l^i units. For the remaining units, $k < k^i$, the bidder could have won the units at a price that is lower than (or at worse, equal to) the marginal valuations for these units. Therefore, the utility is lower than (or at best, equal to) the utility when bidding truthfully.

Question 3b: When bidders bid truthfully the two bids from the high interval (the interval from 10 to 20) must be the winning bids and the prices paid for these two winning bids must be the two bids from the low interval (the interval from 0 to 10). Because bidders bid truthfully, the prices paid are equal to the two valuations in the low interval. The expected revenue is thus the sum of the expectations of the two valuations in the lower interval.

From this point, one can use an intuitive or a mathematical approach to find the answer.

Intuitive approach: The expected value of a random variable from a uniform distribution is the mid-point of the interval. The expectation of a uniformly distributed random variable in the interval $[0; 10]$ is therefore 5. The expected revenue is equal to the sum of the two random variables, i.e. $E[R^V] = 2 \cdot 5 = 10$.

Mathematical approach:

$$E[R^V] = 2 \cdot E[X_2^i] = 2 \cdot \int_0^{10} x f(x) dx = 2 \cdot \int_0^{10} x \frac{1}{10} dx = \frac{2}{10}\left[\frac{1}{2}x^2\right]_0^{10} = \frac{2}{10}\frac{1}{2}100 = 10$$

Question 3c: According to the multi-unit Revenue Equivalence Principle [proposition 14.1 in Krishna] the expected payments and revenue in two auctions with the same allocation rule are also the same. To use the multi-unit Revenue Equivalence Principle we thus need to verify whether or not the three auctions allocate the two items in the same way.

In the Vickrey auction, bidders bid truthfully and the two units are allocated efficiently to the two highest bids, i.e. one unit to each bidder.

In the Uniform-price auction, bidders bid truthfully (bid=marginal value) for the first unit but bid below valuations for the remaining units (bids < marginal values) [proposition 13.4 in Krishna]. With this strategy the two highest bids will be the two valuations from the high interval. Thus, the two units are allocated efficiently to the two highest bids, i.e. one unit to each bidder, just as in the Vickrey auction.

In the Discriminatory auction, bidders will submit flat demands if the difference in marginal values is small and submit downward sloping demands if the difference is large [proposition 13.6 in Krishna]. One can e.g. infer it in this case by considering the two extreme outcomes:

1) $x_1^i = x_2^i = 10$. In this case the bidder has the same valuation for the two units so there is no reason to differentiate the bids. The bidder will submit a flat demand curve $b_1^i = b_2^i$.
2) $x_1^i = 20, x_2^i = 0$. In this case the bidder has no valuation for the second unit so there is no reason to bid a positive amount on the second unit. The bidder will submit a downward sloping demand curve $b_1^i > b_2^i = 0$.

Generally, a bidder will bid more aggressively on the second unit than on the first unit [page 198 in Krishna]. In this case, one can consider the case of $x_1^i = 10 + \varepsilon$ and $x_2^j = 10$ to see why. The first bid from bidder i is competing with a bid for a unit with a valuation drawn from the lower interval (from 0 to 10), whereas the second bid from bidder j is competing with a bid for a unit with a valuation drawn from the higher interval (from 10 to 20). Without knowing the competitors' realized valuations, bidder i will decrease his probability of winning slightly when decreasing his bid whereas bidder j will decrease his probability of winning significantly when decreasing his bid. Hence, bidder j will bid more aggressively on his second unit than bidder i will bid on his first unit and bidder j will win both units. The two units are therefore not allocated efficiently to the two highest bids.

The conclusion is that one can use the multi-unit Revenue Equivalence Principle to argue that $E[R^V] = E[R^U]$ because the Vickrey and the Uniform-price auctions allocate in the same way. One cannot use the multi-unit Revenue Equivalence Principle to argue that $E[R^V] = E[R^D]$ because the Discriminatory auction does not allocate in the same way.

Question 3d: This question is similar to exercise 14.1 in Krishna [page 210 in Krishna].

In the Vickrey auction the bidders bid truthfully so the three units are allocated efficiently to the three highest bids. In the Discriminatory auction bidders submit flat demand curves ($\beta(x_1, x_2) = \big(\beta(x_2), \beta(x_2)\big)$) because both bidders win one unit with certainty. Any equilibrium of the Discriminatory function with an increasing β will be efficient since a bidder will then only win a second unit if his valuation for the second unit is higher than the competitor's valuation for the second unit. The two auctions allocate efficiently and identically.

Due to the multi-unit Revenue Equivalence Principle and the hint:

$$m^D(x_1, x_2) = m^V(x_1, x_2) = \int_0^{x_2} y f_2(y) dy$$

The expected payment in the Discriminatory auction for a bidder with value x_2 for the second unit is:

$$m^D(x_1, x_2) = \beta(x_2) + F_2(x_2)\beta(x_2)$$

Where the first term is the price paid for the first unit and the second term is the probability of winning the second unit times the price paid for the second unit when submitting a flat demand curve. Inserting the two expressions into each other gives:

$$\beta(x_2) + F_2(x_2)\beta(x_2) = \int_0^{x_2} y f_2(y) dy$$

Isolating $\beta(x_2)$ gives:

$$\beta(x_2) = \frac{1}{1 + F_2(x_2)} \int_0^{x_2} y f_2(y) dy = \frac{1}{1 + \frac{1}{10}x_2} \int_0^{x_2} y \frac{1}{10} dy = \frac{1}{1 + \frac{1}{10}x_2} \frac{1}{10} \left[\frac{1}{2} y^2\right]_0^{x_2} = \frac{(x_2)^2}{20 + 2x_2}$$

An equilibrium bidding strategy in the Discriminatory auction is thus to submit the following bids:

$$\beta(x_1, x_2) = \big(\beta(x_2), \beta(x_2)\big) \text{ where } \beta(x_2) = \frac{(x_2)^2}{20+2x_2}$$