LM 2020 Januar Vejl. løsn.

Opg 1

1) Klart at $v_1, v_2 \in U$ da $V \subseteq U$, så $\text{span}\{v_1, v_2\} = V$ er et UR. af U, udspændt af 2 lin. uafh. vektorer, så $\dim(V) = 2$.

2)
$$L(u_3 - u_4) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

3) Vi løser:
$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4 = 0 \cdot v_1 - 1 v_2$$
Da $-v_2 = -u_1 - u_3 + u_4$ er koordinaterne
$$\underline{\underline{(-1, 0, -1, 1)}}$$

4) $Lx = 0$
$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$
$x_3 = s$, $x_4 = t$ fri

$x_2 - x_3 = 0 \rightarrow x_2 = x_3 = s$

$x_1 + x_3 + x_4 = 0 \rightarrow x_1 = -s - t$

$$N(L): \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \underbrace{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{w_1} + t \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{w_2}, \quad s, t \in \mathbb{R}.$$

$N(L) = \text{span}\{w_1, w_2\}$

og w_1, w_2 er en basis

Ej injektiv da $N(L) \neq \{\underline{0}\}$.

Dim sæt. $4-2=2$, så $\dim R(L) = 2$.

5) Klart at $L(-3,2,2,1) = \underline{0}$

Vi løser

$$\alpha_1 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

, dvs $(\alpha_1, \alpha_2) = \underline{\underline{(2,1)}}$ er koordinaterne.

6) $LX = v_1 - v_2 \quad (= (\cancel{1}, -1))$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \cancel{1} \\ 1 & 1 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \cancel{1} \\ 0 & 1 & -1 & 0 & -2 \end{array}\right]$$

dvs

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + s\,w_1 + t\,w_2 \quad (\text{fra sp. } 4).$$

Opg 2

1) v_1, v_2, v_3 skal være indb. ortogonale.

Med $v_2 = (1,-1,0)$ er $v_1 \perp v_2$

Vælges $v_3 = (1,1,-2)$ er $v_2 \perp v_3$ og $v_3 \perp v_1$.

Men der er mange muligheder.

2) $D_A = \begin{bmatrix} -1 & & \\ & 2 & \\ & & 1 \end{bmatrix}$ så $D_{A+A^2} = \begin{bmatrix} 0 & & \\ & 6 & \\ & & 2 \end{bmatrix}$ så $0, 6, 2$ er egenv.

Ej invertibel.

3) $N(A+A^2) = \underline{\underline{\text{span}\{v_1\}}}$, egenrummet hørende til 0.

4) $\dim R(A+A^2) = 2 \quad (= 3-1$ iflg. dim. sætn.$)$

5) $e^{(A+A^2)}(v_1+v_2+v_3) = e^0 v_1 + e^6 v_2 + e^2 v_3$

Opg 3

$$\int \cos^2((a-b)x)\sin(2bx)\,dx$$

$$= \frac{1}{8i}\int \left(e^{i2(a-b)x} + e^{-i2(a-b)x} + 2\right)\left(e^{i2bx} - e^{-i2bx}\right)dx$$

$$= \frac{1}{8i}\int e^{i2ax} - e^{i2(a-2b)x} + e^{-i2(a-2b)x} - e^{-i2ax} + 2\left(e^{i2bx} - e^{-i2bx}\right)dx$$

$$= \frac{1}{4}\int \sin(2ax) - \sin(2(a-2b)x) + 2\sin(2bx)\,dx$$

$$= \frac{1}{4}\left(-\frac{1}{2a}\cos(2ax) + \frac{1}{2(a-2b)}\cos(2(a-2b)x) - \frac{1}{b}\cos(2bx)\right) + k$$

for $a,\ a-2b,\ b \neq 0$.

Da $\sin(0) = 0$ forsvinder stamfunktionen hvis et af tilfældende optræder (opsamles i k).

$z^2 = t + it$, $t > 0$.

$w = z = x + iy$ giver

$$x^2 - y^2 = t$$
$$2xy = t \quad (\neq 0)$$

Så $y = \frac{t}{2x}$.

$$x^2 - \left(\frac{t}{2x}\right)^2 = t$$

$$x^2 - \frac{t^2}{4x^2} = t$$

$$4x^4 - 4tx^2 - t^2 = 0$$

$u = x^2 > 0$ giver

$$u = \frac{4t \pm \sqrt{16t^2 + 16t^2}}{8}$$

. Da $t > 0$ og $u > 0$ skal $-$ forkastes.

$$u = x^2 = \frac{4t + \sqrt{32t^2}}{8}$$

. Men $\sqrt{t^2} = |t| = t$ da $t > 0$

Så $x^2 = \frac{1}{2}t + \sqrt{\frac{1}{2}}\,t = \left(\frac{1}{2} + \sqrt{\frac{1}{2}}\right)t$

$$x = \pm\sqrt{\left(\tfrac{1}{2} + \sqrt{\tfrac{1}{2}}\right)t}$$

Da $y = \frac{t}{2x}$ fås

$$z = x + iy = \pm\left(\sqrt{\left(\tfrac{1}{2} + \sqrt{\tfrac{1}{2}}\right)t} + i\,\frac{t}{2\sqrt{\left(\frac{1}{2} + \sqrt{\frac{1}{2}}\right)t}}\right)$$

<u>Opg 5</u> $\quad g(x) = \dfrac{x^3}{x^2 - x}$. Klart at $\underline{x \neq 0}$.

Vi løser $|g(x)| < 1$

For $\underline{x \neq 0}$ er $g(x) = \dfrac{x^2}{x-1}$. Husk at $\underline{x \neq 0}$.

$g(x) = 1$ har ingen løsning.

$g(x) = -1$ har løsning $x = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$f(x) = \dfrac{1}{1 - g(x)}$ er veldefinert for

$$x \in \left]-\tfrac{1}{2} - \tfrac{\sqrt{5}}{2}, 0\right[\cup \left]0, -\tfrac{1}{2} + \tfrac{\sqrt{5}}{2}\right[$$

$$f(x) = \frac{1}{1 - \frac{x^2}{x-1}}$$

2) <u>Monotoni</u>

$g(x) = x^2(x-1)^{-1}$

$g'(x) = 2x(x-1)^{-1} - x^2(x-1)^{-2} = 0$, ~~kun for $x \neq 0$~~ ~~Schnitte duer~~.

$$\frac{2x}{x-1} - \frac{x^2}{(x-1)^2} = 0$$

$\Updownarrow$

$$\frac{2x(x-1)}{(x-1)^2} - \frac{x^2}{(x-1)^2} = 0$$

$\Updownarrow$

$$2x(x-1) - x^2 = 0$$

$\Updownarrow$

$$x^2 - 2x = 0$$

$x = \begin{cases} 0 \\ 2 \end{cases}$, begge uden for def. mgd.

	$-\frac{1}{2}-\frac{\sqrt{5}}{2}$		0		$-\frac{1}{2}+\frac{\sqrt{5}}{2}$		2
$g'(x)$		+		−			
f		↗		↘			

4)

Værdimængden

for $x \to -\frac{1}{2} - \frac{\sqrt{5}}{2}^{+}$ vil $f(x) \to$ ~~$\frac{\sqrt{5}}{2}$~~ $\frac{1}{2}$

for $x \to 0^{-/+}$ vil $f(x) \to 1$

for $x \to -\frac{1}{2} + \frac{\sqrt{5}}{2}$ vil $f(x) \to \frac{1}{2}$

Dvs $Vm(f) = \left]\frac{1}{2}, \infty\right[$ Ej injektiv

5)

$$f(x) = y$$

$$\frac{1}{1 - \frac{x^2}{x-1}} = y, \quad y \in Vm(f)$$

$$\frac{1}{y} = 1 - \frac{x^2}{x-1} = \frac{x-1-x^2}{x-1}$$

$$\Updownarrow$$

$$x - 1 = -yx^2 + yx - y$$

$$\Updownarrow$$

$$yx^2 + (1-y)x - (1-y) = 0$$

$$x = \frac{y - 1 \pm \sqrt{(1-y)^2 + 4y(1-y)}}{2y}$$

$= \frac{-1 \pm \sqrt{1 + {4y}/{1-y}}}{\frac{2y}{1-y}}$ evt. (ej nødv.)