The Competence Description in Micro 3 says:

Game Theory has become a central analytic tool in much economic theory, e.g. within industrial organization, macroeconomics, international economics, labor economics, public economics and political economics.

The course aims at giving the student knowledge of game theory, non-cooperative as well as cooperative, and its applications in economic models.

The student who successfully completed the course will learn the basic game theory and will be enabled to work further with advanced game theory. The student will also learn how economic problems, involving strategic situations, can be modeled using game theory, as well as how these models are solved. The course intention is thus, that the student through this becomes able to work with modern economic theory, for instance within the areas of within industrial organization, macroeconomics, international economics, labor economics, public economics and political economics.

In the process of the course the student will learn about

- *Static games with complete information*
- *Static games with incomplete information*
- *Dynamic games with complete information*
- *Dynamic games with incomplete information*
- *Basic cooperative game theory.*

For each of these classes of games, the student should know and understand the theory, and learn how to model and analyze some important economic issues within the respective game framework.

More specifically, the students should know the theory and be able to work with both normal and extensive form games. They should know, understand and be able to apply the concepts of dominant strategies, iterative elimination of dominant strategies, as well as mixed strategies. The students should know the central equilibrium concepts in non-cooperative game theory, such as Nash Equilibrium and further refinements: Subgame-Perfect Nash Equilibrium, Bayesian Nash Equilibrium, Perfect Bayesian Equilibrium. They should understand why these concepts are central and when they are used, and be able to apply the relevant equilibrium and solution concepts.

Furthermore, the students should acquire knowledge about a number of special games and the particular issues associated with them, such as repeated games (including infinitely repeated games), auctions and signaling games.

The students should also understand and be able to apply the solution concepts of cooperative game theory, such as the core and the Shapley value. Furthermore, the students should also learn the basics of bargaining theory.

To obtain a top mark in the course the student must be able excel in all of the areas listed above.

In view of this, the grading of the exam should take as a point of departure, the short description of the solutions below

MICRO 3 EXAM AUGUST 2009
QUESTIONS WITH SHORT ANSWERS
(The answers in this solution are often short/indicative, a good exercise should argue for these answers)

1. (a) Find *all* Nash equilibria in the following game

	L	R
T	$3,3$	$1,6$
B	$2,4$	$5,1$

Solution: There are no PSNE. The mixed eq can be determined as follows: assume that all pure strategies are played with positive probability and assign p as the probability that player 1 plays T and q as the probability that player 2 plays L.

		q	1-q
		L	R
r	T	$3,3$	$1,6$
1-r	B	$2,4$	$5,1$

Row player is indifferent between playing T and B iff

$$\begin{aligned} 3q+(1-q) &= 2q+5(1-q) \Leftrightarrow \\ q &= 4/5. \end{aligned}$$

Row player's best response is

$$BR_1(q)=r^*(q)\begin{cases} =1 \text{ if } q>4/5 \text{ (strategy T)} \\ \in[0,1] \text{ if } q=4/5 \text{ (any combination of } T \text{ and } B) \\ =0 \text{ if } q<4/5 \text{ (strategy B)} \end{cases}$$

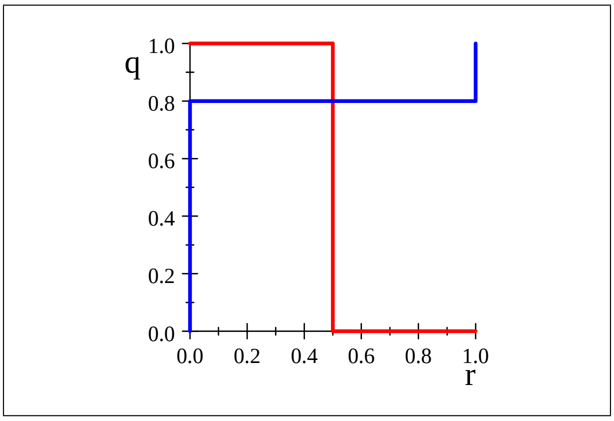
Column player is indifferent between playing L and R iff

$$3r+4(1-r)=6r+(1-r),$$

that is, if the row player is mixing with the weight $r=1/2$. Column player's best response is

$$BR_2(r)=q^*(r)\begin{cases} =0 \text{ if } r>1/2 \text{ (strategy R)} \\ \in[0,1] \text{ if } r=1/2 \text{ (any combination of } L \text{ and } R) \\ =1 \text{ if } r<1/2 \text{ (strategy L)} \end{cases}$$

The intersection of BRs is (the BR of Player 1 is in blue, and the BR of player 2 is in red)



Therefore, the mixed strategy equilibrium is $[(1/2,1/2)(4/5,1/5)]$, i.e. the row player plays T with prob 1/2, and the column player plays L with prob 4/5.

(b) Solve the following game by eliminating strictly dominated strategies. Could there be a Nash equilibrium in this game, in which Player 1 mixes between all three strategies s_1, s_2 and s_2 with positive weights? If yes, find one. If no, explain why not.

	t_1	t_2	t_3
s_1	1,4	4,3	5,2
s_2	1,3	2,2	4,4
s_3	2,0	3,4	5,3

Solution: Elimination iteration: s_3 dominates s_2, then t_2 dominates t_3, no further steps possible. Solution

	t_1	t_2
s_1	1,4	4,3
s_3	2,0	3,4

As Nash equilibrium strategies survive at any step of iterated elimination procedure, and strategy s_2 is eliminated in the first step, there could NOT be a NE in this game, in which Player 1 mixes between all three strategies s_1, s_2 and s_2 with positive weights.

(c) Consider the extensive-form game represented by the game tree on Figure 1:

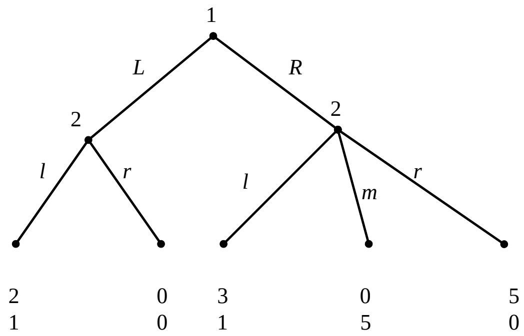


Figure 1.

i. How many subgames are in this game? Find all subgame perfect Nash equilibria.
Solution: 2 subgames not including the game itself. SPNE: $\{L, lm\}$

ii. Rewrite this game in normal form and find all pure-strategy Nash equilibria.
Solution:

	ll	lm	lr	rl	rm	rr
L	$2,\underline{1}$	$\underline{2},\underline{1}$	$2,\underline{1}$	$0,0$	$\underline{0},0$	$0,0$
R	$\underline{3},1$	$0,\underline{5}$	$\underline{5},0$	$\underline{3},1$	$\underline{0},\underline{5}$	$\underline{5},0$

There are two NE: $\{L, lm\}$ and $\{R, rm\}$.

iii. Could you have a game which has more SPNE than NE? If yes, provide an example. If no, explain.
Solution: No, as all SPNE are NE by definition.

2. Two students are working together on a project. When students 1 and 2 choose effort levels e_1 and e_2, $e_i \in [0, 1]$, the probability that the project is successfully completed is equal to

$$\frac{e_1 e_2 + e_1 + e_2}{3}.$$

The disutility of effort for student i, $i = 1, 2$, is given by $\frac{1}{2}e_i^2$. Further, student i values the completed project at A_i utils. That is, student 1's payoff in the game is

$$U_1(e_1, e_2) = A_1 \frac{e_1 e_2 + e_1 + e_2}{3} - \frac{1}{2}(e_1)^2,$$

and student 2's payoff is

$$U_2(e_1, e_2) = A_2 \frac{e_1 e_2 + e_1 + e_2}{3} - \frac{1}{2}(e_2)^2.$$

Assume that students choose their effort levels simultaneously and non-cooperatively.

(a) Assume that each student values the project at $A_i = 1$, $i = 1, 2$, and this is common knowledge. Find the best response functions of both students and determine the Nash equilibrium effort levels e_1^*, e_2^*.
Solution: Student 1 solves

$$\max_{e_1} \frac{e_1 e_2 + e_1 + e_2}{3} - \frac{1}{2}(e_1)^2.$$

FOC is

$$\frac{e_2 + 1}{3} - e_1 = 0,$$

which yields the BR function

$$BR_1 = e_1(e_2) = \frac{e_2 + 1}{3}. \tag{1}$$

Similarly, student 2 solves

$$\max_{e_2} \frac{e_1 e_2 + e_1 + e_2}{3} - \frac{1}{2}(e_1)^2.$$

which yields her BR function

$$BR_2 = e_2(e_1) = \frac{e_1 + 1}{3}. \tag{2}$$

In NE both students play best responses, so we solve the system

$$\begin{aligned} e_1^* &= \frac{e_2^* + 1}{3} \\ e_2^* &= \frac{e_1^* + 1}{3} \end{aligned}$$

which yields the Nash equilibrium effort levels

$$e_1^* = e_2^* = 1/2.$$

(b) Assume that student 1 values the project at $A_1 = 1$, while student 2 does not value the project at all, so that $A_2 = 0$. Again, this is common knowledge. Find the best response functions of both students and determine the Nash equilibrium effort levels e_1^{**}, e_2^{**}.

Solution: Student 1 still solves

$$\max_{e_1} \frac{e_1 e_2 + e_1 + e_2}{3} - \frac{1}{2}(e_1)^2.$$

so her BR is given by expression (1).
Student 2 solves

$$\max_{e_2} -\frac{1}{2}(e_1)^2$$

As the effort should be non-negative, $e_2 \in [0, 1]$, student 2 will choose zero effort no matter how much effort is put in by student 1. That is, her BR function is

$$BR_2 = e_2(e_1) = 0. \tag{3}$$

In NE both students play best responses, so we solve the system

$$\begin{aligned} e_1^{**} &= \frac{e_2^{**} + 1}{3} \\ e_2^* &= 0 \end{aligned}$$

which yields the Nash equilibrium effort levels

$$e_1^* = \frac{1}{3},\ e_2^* = 0.$$

(c) Finally, assume that student 1 still values the project at $A_1 = 1$ and this is common knowledge. The value of the project to student 2, A_2, is however known only to student 2 herself. The only thing student 1 knows about A_2 is that $A_2 = 1$ with probability p, and 0 with probability $(1 - p)$.

i. What is the best response function of the student 2 with $A_2 = 0$, $e_2^L(e_1)$? What is the best response function of the student 2 with $A_2 = 1$, $e_2^H(e_1)$?

Solution: Student 2 does not face any informational asymmetries. Therefore the best response function of the student 2 with $A_2 = 0$, $e_2^L(e_1)$ is given by expression (3), and the best response function of the student 2 with $A_2 = 1$, $e_2^H(e_1)$ is given by expression (2)

$$e_2^L(e_1) = 0 \tag{4}$$

$$e_2^H(e_1) = \frac{e_1 + 1}{3} \tag{5}$$

ii. What is the best response function of student 1 $e_1(e_2^L, e_2^H)$?
Solution: Student 1 solves

$$\max_{e_1} p\frac{e_1 e_2^H + e_1 + e_2^H}{3} + (1-p)\frac{e_1 e_2^L + e_1 + e_2^L}{3} - \frac{1}{2}(e_1)^2 .$$

The FOC of this maximization function is

$$p\frac{e_2^H + 1}{3} + (1-p)\frac{e_2^L + 1}{3} - e_1 = 0,$$

which is yields the following BR of Player 1

$$e_1(e_2^L, e_2^H) = p\frac{e_2^H + 1}{3} + (1-p)\frac{e_2^L + 1}{3} \tag{6}$$

iii. Find the Bayes-Nash equilibrium of this game. How does it depend on p? Interpret (a comparison to your results in (a) and (b) may be useful).
Solution: To find the Bayes-Nash equilibrium we need to solve the system of equations (4), (5) and (6)

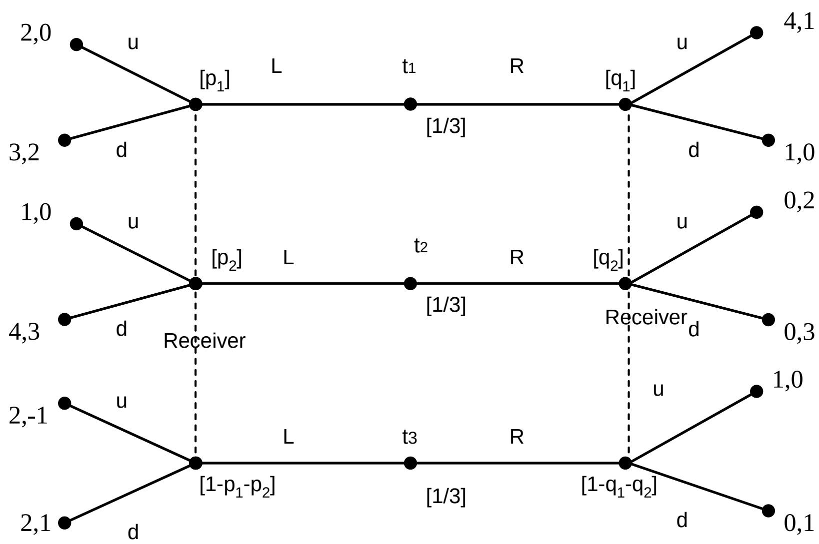
$$\begin{aligned} e_2^L &= 0 \\ e_2^H &= \frac{e_1 + 1}{3} \\ e_1 &= p\frac{e_2^H + 1}{3} + (1-p)\frac{e_2^L + 1}{3} \end{aligned}$$

This system's solution is

$$\begin{aligned} e_1 &= \frac{3+p}{9-p} \\ e_2^L &= 0 \\ e_2^H &= \frac{4}{9-p} \end{aligned}$$

As p increases, the effort levels of Players 1 and the high type of Player 2 also increase. This is because Player 1's and Player 2's efforts are compliments. Therefore Player 1 is happy to contribute more effort as long as there is a high chance that Player 2 contributes as well (i.e. that Player 2 is of high type). The high type of Player 2 also appreciates more effort from Player 1 and responds accordingly.

3. Consider the following signalling game with three types of sender, which are equally probable (the nature move is not shown on the picture):



Find a pooling Perfect Bayesian equilibrium in which all senders choose L.

(a) **Solution:** If all senders choose L, SR 3 implies that

$$p_1 = p_2 = 1 - p_1 - p_2 = 1/3.$$

In the left information set the payoff of the Receiver from choosing u is then

$$0 * 1/3 + 0 * 1/3 + (-1) * 1/3 = -1/3,$$

the payoff from choosing d is

$$2 * 1/3 + 3 * 1/3 + 1 * 1/3 = 2.$$

SR2 then implies that the Receiver chooses d in the left information set. If the Receiver chooses u in the right information set, it cannot be an equilibrium, as type 1 of the Sender will then deviate from L. Therefore, in order to have an equilibrium we should have the Receiver choose d in the right info set. The payoff of the Receiver from choosing u in the right set is

$$1 * q_1 + 2 * q_2 + 0 * (1 - q_1 - q_2) = q_1 + 2q_2,$$

the payoff from choosing d is

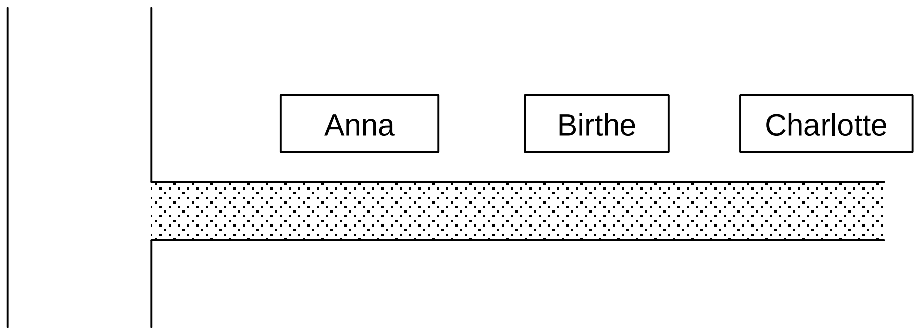
$$0 * q_1 + 3 * q_2 + 1 * (1 - q_1 - q_2) = 1 - q_1 + 2q_2.$$

For Receiver to choose d we should have

$$\begin{aligned} 1 - q_1 + 2q_2 &> q_1 + 2q_2 \Leftrightarrow \\ q_1 &< 1/2 \end{aligned}$$

Therefore, there are many PBE in this case, all of the following type:
$(LLL, dd, (1/3, 1/3, 1/3), (q_1, q_2, 1 - q_1 - q_2))$ such that $q_2 < 1/2$.

4. Anna, Birthe and Charlotte have houses located along a small road. They need to repair the road from the intersection with a bigger road to their houses. Anna's house is the closest to the intersection, and Charlotte's house is the furthest away. More precisely, Anna's house is 100 m away from the intersection, Birthe's house is 200 m away from the intersection and Charlotte's house is 300 m away. To repair 100 m of road costs DKK 1000. They are discussing how to divide the repair costs.



(a) Think of this situation as of coalitional game with transferable utilities. Write down the value of each coalition.

Solution: Denote Anna by A, Birthe by B and Charlotte by C. Let us talk of coalition values as of costs. Then

$$\begin{aligned} v(ABC) &= v(AC) = v(BC) = v(C) = 3000 \\ v(AB) &= v(B) = 2000 \\ v(A) &= 1000 \end{aligned}$$

(b) Assume that they decided to divide the costs according to the Shapley value. What will each of them pay?

Solution: One way to solve it is to relate this situation to the runway building example and remember that in that case the users of each part of the runway (the road in our case) are sharing the costs. Then the cost of the first part of the road, to Anna, is shared between all 3 neighbors, the AB segment cost is shared between Birthe and Charlotte, and the BC segment cost is paid by Charlotte only. Therefore, the respective Shapley values are

$$\begin{aligned} Sh(A) &= 1000/3 = 1000 * 1/3 \approx 333,3, \\ Sh(B) &= 1000/3 + 1000/2 \approx 833,33, \\ Sh(C) &= 1000/3 + 1000/2 + 1000 \approx 1833,33. \end{aligned}$$

Alternatively, one can use the definition of the Shapley value. Then there are 6 possible orderings:

$$ABC, ACB, BAC, BCA, CAB, CBA.$$

Let's find the marginal contributions of Anna to there orderings. Her marginal contribution to the ordering ABC is

$$m(ABC, A) = v(A) - v(\{\varnothing\}) = 1000.$$

Similarly,

$$\begin{aligned} m(ACB, A) &= v(A) - v(\{\varnothing\}) = 1000, \\ m(BAC, A) &= v(BA) - v(B) = 2000 - 2000 = 0, \\ m(BCA, A) &= v(BCA) - v(BC) = 3000 - 3000 = 0, \\ m(CAB, A) &= v(CA) - v(C) = 3000 - 3000 = 0 \\ m(CBA, A) &= v(CBA) - v(BC) = 3000 - 3000 = 0. \end{aligned}$$

Therefore, the Shapley Value of Anna is

$$Sh(A) = \frac{1}{6}(1000 + 1000 + 0 + 0 + 0 + 0) = 1000/3$$

Let's find the marginal contributions of Birthe to there orderings

$$\begin{aligned} m(ABC, B) &= v(AB) - v(A) = 1000, \\ m(ACB, B) &= v(ACB) - v(AC) = 0, \\ m(BAC, A) &= v(B) - v(\varnothing) = 2000, \\ m(BCA, A) &= v(B) - v(\varnothing) = 2000, \\ m(CAB, A) &= v(CAB) - v(CA) = 0 \\ m(CBA, A) &= v(CBA) - v(C) = 0. \end{aligned}$$

Therefore, the Shapley Value of Birthe is

$$Sh(C) = \frac{1}{6}(1000 + 0 + 2000 + 2000 + 0 + 0) =\approx 833,33$$

As Shapley Value is efficient, the Shapley Value for Charlotte is

$$Sh(F) = v(ABC) - Sh(A) - Sh(B) \approx 1833,33.$$