1. Consider the Diamond OLG model. Time is discrete and infinite, there is no population growth. As usual, small letters denote per capita units and large letters denote levels. The representative household in each generation lives for two periods and maximizes the following objective function

$$\log c_{1t} + \frac{1}{1+\rho}\log c_{2t+1}$$

where c_{1t} denotes consumption when young in period t, and c_{2t+1} denotes consumption when old in period $t+1$. He faces the following budget constraint when young

$$c_{1t} + s_t = w_t$$

where w_t is the wage rate (and labor income since he inelastically supplies one unit of labor), and s_t are savings. When old, his budget constraint is

$$c_{2t+1} = (1 + r_{t+1})s_t - \tau_{t+1}$$

where r_{t+1} is the return he earns on his savings. The government imposes a lump sum tax τ_{t+1} on the old and uses the revenue to finance public expenditure. The government runs a balanced budget each period, so its budget constraint is

$$g_t = \tau_t$$

in all periods.

(a) Set up the household problem, derive the first order conditions and the Euler equation. Explain what behavior the Euler equation implies, and why it has to hold at the optimum.

Solution:

Eliminating s_t from the two household budget constraints, we can derive the lifetime budget constraint

$$c_{1t} + \frac{c_{2t+1} - \tau_{t+1}}{1 + r_{t+1}} = w_t$$

The Lagrangian is

$$\mathcal{L} = \log c_{1t} + \frac{1}{1+\rho}\log c_{2t+1} + \lambda\left(-c_{1t} - \frac{c_{2t+1} - \tau_{t+1}}{1 + r_{t+1}} + w_t\right)$$

with first order conditions

$$\begin{aligned}\frac{1}{c_{1t}} &= \lambda \\ \frac{1}{1+\rho}\frac{1}{c_{2t+1}} &= \lambda\frac{1}{1+r_{t+1}}\end{aligned}$$

Eliminating the Lagrange multiplier, we get the Euler equation:

$$\frac{1}{c_{1t}} = \frac{1 + r_{t+1}}{1+\rho}\frac{1}{c_{2t+1}}$$

The Euler equation describes the optimal consumption path of an agent over the course of his life. He chooses consumption in both periods of life such that the marginal utility of an additional unit when young is equal to the discounted present value of the marginal utility of an additional unit when old. If the market interest rate (return to saving) is high relative to the subjective discount factor, then agents prefer higher marginal utility growth over the course of their lives, for example. With the given utility function assumptions, the equation directly describes optimal consumption levels, but this is not key.

If the Euler equation did not hold with equality, agents would not be optimizing. To see this, suppose that marginal utility of consumption when young is smaller than the discounted marginal utility of consumption when old. Then the agent can reduce consumption when young, invest the proceeds and increase consumption by the same amount plus the proceeds when old. The utility cost of this operation is smaller than the gain, by assumption, so the agent can't have been optimizing. At the optimum, therefore, marginal benefit and cost must be equalized.

(b) Derive the individual savings function. Does it depend on the interest rate? Why (not)?

Solution: The individual savings function follows from the Euler equation, with the budget constraints substituted in, and rearranging:

$$\begin{aligned}
\frac{1}{c_{1t}} &= \frac{1+r_{t+1}}{1+\rho}\frac{1}{c_{2t+1}} \\
\frac{1}{w_t - s_t} &= \frac{1+r_{t+1}}{1+\rho}\frac{1}{(1+r_{t+1})s_t - \tau_{t+1}} \\
(1+r_{t+1})s_t - \tau_{t+1} &= \frac{1+r_{t+1}}{1+\rho}(w_t - s_t) \\
s_t\left(1+r_{t+1}+\frac{1+r_{t+1}}{1+\rho}\right) &= \frac{1+r_{t+1}}{1+\rho}w_t + \tau_{t+1}
\end{aligned}$$

$$s_t = \frac{1}{2+\rho}\left(w_t + \frac{1+\rho}{1+r_{t+1}}\tau_{t+1}\right)$$

The savings function here does depend on the interest rate, and is a decreasing function of it. In the model with log utility assumption and no taxes, individual savings do not depend on the interest rate because the income and substitution effect cancel each other out exactly: A higher interest rate implies (i) higher returns to savings, but also (ii) higher gross income from savings. Here the two effects do not cancel out because of the tax that old agents pay. Paying the tax when old everything else equal increases savings when young since the tax reduces lifetime income, and agents prefer to spread this drop out over both periods of their lives. If the interest rate is higher, the discounted present value of the tax payment is lower, so savings required to smooth consumption is lower.

The production side of the economy is standard: Competitive firms produce the consumption good with Cobb Douglas production technology

$$Y_t = K_t^\alpha L_t^{1-\alpha}$$

They hire capital and labor input from households. Assume in addition that government spending is proportional to output in the economy,

$$g_t = \gamma k_t^\alpha$$

and that capital fully depreciates each period, i.e. $\delta = 1$.

(c) Derive an expression for the law of motion for capital in this economy, that is k_{t+1} as a (possibly implicit) function of k_t.

Solution: Capital accumulation: Only the young save, so aggregate capital accumulation is equal to individual savings (recall that here $n = 0$):

$$k_{t+1} = s_t$$

In equilibrium, the government budget constraint holds, so

$$k_{t+1} = \frac{1}{2+\rho}\left(w_t + \frac{1+\rho}{1+r_{t+1}} g_t\right) \tag{1}$$

Since firms are perfectly competitive, inputs are paid their marginal product [derive this from profit maximization]

$$\begin{aligned} r_t &= \alpha k_t^{\alpha-1} - \delta \\ w_t &= (1-\alpha)k_t^\alpha \end{aligned}$$

Using $g_t = \gamma k_t^\alpha$ and $\delta = 1$, note that

$$\frac{g_t}{1+r_{t+1}} = \frac{\gamma k_t^\alpha}{1+\alpha k_{t+1}^{\alpha-1} - \delta} = \frac{\gamma}{\alpha}\frac{k_t^\alpha}{k_{t+1}^{\alpha-1}}$$

Substituting these into (1) we get

$$k_{t+1} = \frac{1}{2+\rho}\left((1-\alpha)k_t^\alpha + (1+\rho)\frac{\gamma}{\alpha}\frac{k_t^\alpha}{k_{t+1}^{\alpha-1}}\right)$$

This is an implicit law of motion for the evolution of capital in this economy.

(d) Show that in steady state an increase in γ increases capital and output in this economy. What is the intuition? How and why would this change if public expenditures were paid for with taxes on the young instead?

Solution: In steady state,

$$\begin{aligned}
k_{SS} &= \frac{1}{2+\rho}\left((1-\alpha)k_{SS}^{\alpha} + (1+\rho)\frac{\gamma}{\alpha}\frac{k_{SS}^{\alpha}}{k_{SS}^{\alpha-1}}\right) \\
k_{SS} &= \frac{1}{2+\rho}\left((1-\alpha)k_{SS}^{\alpha} + (1+\rho)\frac{\gamma}{\alpha}k_{SS}\right) \\
k_{SS} &= \left(\frac{\frac{1-\alpha}{2+\rho}}{1-\frac{1+\rho}{2+\rho}\frac{\gamma}{\alpha}}\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

An increase in public expenditures as a share of GDP (that is, an increase in γ) in steady state leads to an increase in capital and thus output. The increase in γ represents a drop in old age income for households. They want to spread this across their lives to smooth consumption, so they prefer to reduce consumption not just when old, but also when young. The only way to achieve this is by saving more when young - that is, by accumulating more capital. If taxes had instead been levied on the young, the opposite would occur. Households then have relatively less income when young, and smooth this across their lives by saving less. This is the standard case we considered in class.

2. Consider the following model of monetary policy: the government controls inflation directly (i.e. $\pi_t = m_t$, where π_t is the rate of inflation and m_t is the rate of growth of money supply) and its instantaneous loss function is

$$L(\pi_t, x_t) = \frac{1}{2}\left[\pi_t^2 + \lambda\left(x_t - \bar{x}\right)^2\right]$$

where $x_t = \theta_t + \pi_t - \pi_t^e$. The following notation applies

π_t^e: expected rate of inflation
x_t: output level
θ_t: potential output
$\bar{x}$: policy output target

We assume that potential output is stochastic and that its realizations are observed by both the public and the policy maker before expectations are formed by the private sector. Parameter $\lambda > 0$ measures the relative importance of output fluctuations around the target, $\bar{x}$, relative to inflation fluctuations.

(a) Show that the optimal policy under commitment implies $\pi_t^C = 0$ and $x_t^C = \theta_t$ [hint: i) recall that the loss function is quadratic, thus the optimal policy rule is linear and can be guessed to be of the form $\pi_t = \psi + \psi_\theta \theta_t$; ii) recall that the loss to be minimized is the unconditional one].

Solution: Given the linear rule $\pi_t = \psi + \psi_\theta \theta_t$, as well as the fact that θ_t is observed by both the public and the policy maker before expectations are formed, output is determined as follows:

$$x_t = \theta_t + \pi_t - \pi_t^e = \theta_t + \psi + \psi_\theta\theta_t - (\psi + \psi_\theta\theta_t) = \theta_t$$

Thus, the expected loss reads as:

$$\begin{aligned} E\left[L(\pi_t, x_t)\right] &= \frac{1}{2}E\left[\left(\underbrace{\psi + \psi_\theta \theta_t}_{=\pi_t}\right)^2 + \lambda\left(\underbrace{\theta_t}_{=x_t} - \bar{x}\right)^2\right] \\ &= \frac{1}{2}E\left[\psi^2 + 2\psi\psi_\theta\theta_t + \psi_\theta^2\theta_t^2 + \lambda\left(\theta_t^2 - 2\bar{x}\theta_t + \bar{x}^2\right)\right] \\ &= \frac{1}{2}\left[\psi^2 + 2\psi\psi_\theta E\left[\theta_t\right] + \psi_\theta^2 E\left[\theta_t^2\right] + \lambda\left(E\left[\theta_t^2\right] - 2\bar{x}E\left[\theta_t\right] + \bar{x}^2\right)\right] \end{aligned}$$

Taking the first order conditions of $E\left[L(\pi_t, x_t)\right]$ with respect to ψ and ψ_θ we obtain:

$$\begin{aligned} \frac{\partial E\left[L(\pi_t, x_t)\right]}{\partial \psi} &= 0 : \psi + \psi_\theta E\left[\theta_t\right] = 0 \\ \frac{\partial E\left[L(\pi_t, x_t)\right]}{\partial \psi_\theta} &= 0 : \psi E\left[\theta_t\right] + \psi_\theta E\left[\theta_t^2\right] = 0 \end{aligned}$$

Thus, the expected loss is minimized by setting $\psi = \psi_\theta = 0$, which implies $\pi_t^C = 0$ and $x_t^C = \theta_t$.

(b) Show that the optimal policy under discretion implies $\pi_t^D = -\lambda\left(\theta_t - \bar{x}\right)$ and $x_t^D = \theta_t$. The *inflation bias* increases in the target $\bar{x}$: explain why.

Solution: When the central bank conducts a discretionary policy, the inflation rate is chosen after expectations are formed. Hence, the goal of the central bank is to minimize the loss function, i.e. the monetary policy should be ex post optimal. Under this assumption, the problem reads as

$$\min_{\pi_t} \frac{1}{2}\left[\pi_t^2 + \lambda\left(\theta_t + \pi_t - \pi_t^e - \bar{x}\right)^2\right]$$

The first order condition for this problem reads as:

$$\frac{\partial L(\pi_t, x_t)}{\partial \pi_t} = 0 : \pi_t + \lambda\left(\theta_t + \pi_t - \pi_t^e - \bar{x}\right) = 0 \Leftrightarrow \pi_t^D = \frac{\lambda}{1+\lambda}\left(\pi_t^e - \theta_t + \bar{x}\right)$$

Thus, the expected rate of inflation is found by taking expectations:

$$E\left[\left.\pi_t^D\right|\theta_t\right] = \frac{\lambda}{1+\lambda}E_t\left[\pi_t^e - \theta_t + \bar{x}\right] = \frac{\lambda}{1+\lambda}\left(E_t\left[\left.\pi_t^D\right|\theta_t\right] - \theta_t + \bar{x}\right)$$

which implies $E\left[\left.\pi_t^D\right|\theta_t\right] = -\lambda\left(\theta_t - \bar{x}\right)$.
Therefore:

$$\begin{aligned} \pi_t^D &= \frac{\lambda}{1+\lambda}\left(\underbrace{-\lambda\left(\theta_t - \bar{x}\right)}_{=\pi_t^e} - \theta_t + \bar{x}\right) = -\lambda\left(\theta_t - \bar{x}\right) \\ x_t^D &= \theta_t \end{aligned}$$

The excessively high equilibrium inflation associated with the inflation bias problem results from the combination of a lack of commitment and central bank's temptation to temporarily boost the economy beyond its potential level. The latter incentive is embodied by the condition $\bar{x} > \theta$. This makes it clear why raising $\bar{x}$ increases the temptation of the central bank to generate excess inflation in the vain attempt to stimulate real activity.

(c) Now set $\bar{x} = 0$ and assume there are two periods, i.e. $t = 1, 2$. Compute the optimal strategy at time $t = 1$ for a government that is expected to play π_1^C but decides to deviate from the announced strategy (hint: the policy maker takes $\pi_1^e = 0$ as given when minimizing the loss function).

Solution: The problem of the policy maker now reads as

$$\min_{\pi_1} \frac{1}{2}\left[\pi_1^2 + \lambda\left(\theta_1 + \pi_1\right)^2\right]$$

The first order condition for this problem is:

$$\frac{\partial L(\pi_1, x_1)}{\partial \pi_1} = 0 : \pi_1 + \lambda\left(\theta_1 + \pi_1\right) = 0 \Leftrightarrow \pi_1^* = -\frac{\lambda}{1+\lambda}\theta_1$$

which implies

$$x_t^* = \theta_1 + \pi_1^* - \underbrace{\pi_1^e}_{=0} = \theta_1 - \frac{\lambda}{1+\lambda}\theta_1 = \frac{1}{1+\lambda}\theta_1$$

(d) Keep assuming $\bar{x} = 0$. What are the benefits and costs from deviating from commitment at time $t = 1$ and playing discretion at time $t = 2$ (hint: the benefit at $t = 1$ is the difference between the loss under commitment and the loss under the deviation strategy, while the cost at $t = 2$ is computed as the difference between the loss under discretion and the loss under commitment)? Show that for a (gross) rate of growth of potential output $\theta_2/\theta_1 > (1+\lambda)^{-\frac{1}{2}}$ it is ex-post optimal for the government to stick to the commitment rule announced at $t = 1$.

Solution: At time $t = 2$ the central bank plays discretion to accommodate the public's expectations. As suggested, the benefit at $t = 1$ is the difference between the loss under commitment and the loss under the deviation strategy:

$$B\left(\theta_1\right) = L\left(0, \theta_1\right) - L\left(-\frac{\lambda}{1+\lambda}\theta_1, \frac{1}{1+\lambda}\theta_1\right) = \frac{1}{2}\frac{\lambda^2}{1+\lambda}\theta_1^2$$

As to the cost at $t = 2$, this is computed as the difference between the loss under discretion and the loss under commitment:

$$C\left(\theta_2\right) = L\left(-\lambda\theta_2, \theta_2\right) - L\left(0, \theta_2\right) = \frac{1}{2}\lambda^2\theta_2^2$$

Thus, it is convenient to deviate as long as $B(\theta_1) > C(\theta_2)$:

$$\frac{1}{2}\frac{\lambda^2}{1+\lambda}\theta_1^2 > \frac{1}{2}\lambda^2\theta_2^2 \Longleftrightarrow \frac{\theta_2}{\theta_1} > (1+\lambda)^{-\frac{1}{2}}$$