Written Exam for the B.Sc. in Economics Summer 2010

Macro A

2nd Final Exam

Date:

(3-hour closed book exam)

Solution Manual

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by “eksamen på dansk” in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students’ self-service system.

Exercise 1

Consider the following economy:

$$Y_t = K_t^{\alpha}\left(A_t L_t\right)^{\beta} X_t^{\kappa}, \quad \alpha+\beta+\kappa=1$$

$$K_{t+1} = sY_t + (1-\delta)K_t, \qquad K_0 \text{ given}$$

$$L_{t+1} = \left(1+n\right)L_t, \quad L_0 \text{ given}$$

$$A_{t+1} = \left(1+g\right)A_t, \quad A_0 \text{ given}$$

Y_t is national output, K_t is the capital stock, A_t is a productivity parameter which grows at rate *g*, L_t is the population size which grows at rate *n*. *s* is the savings rate and δ is the depreciation rate.

a. Explain why a steady state in which K_t / L_t stays constant cannot exist if *g=0*.

Answer: If K_t / L_t stays constant, K_t must grow at rate *n*. When this is the case, total output must grow at a rate smaller than *n*. This is due to the fact that land is not reproducible and cannot grow at a positive rate. Hence, there are diminishing returns to capital and labour. As such, output per capita and also savings per capita will decline. Thus, replacement investments, which are necessary to keep K_t / L_t constant, are not feasible.

b. Assume a balanced growth path exists in which the ratio $z_t = K_t / Y_t$ is constant. Use the production function to show that the growth rate of per-capita income $y_t = Y_t / L_t$ is

$$g_t^y \cong \frac{\beta g - \kappa n}{1-\alpha} = \frac{\beta g - \kappa n}{\beta + \kappa}.$$

Explain why the growth may eventually become negative.

Answer: When $z_t = K_t / Y_t$ is constant, the growth rate of K_t / L_t equals the growth rate of Y_t / L_t. We first write

$$Y_t = K_t^{\alpha} \left(A_t L_t \right)^{\beta} X_t^{\kappa}, \quad \alpha + \beta + \kappa = 1$$

in per-capita form

$$y_t = k_t^{\alpha} A_t^{\beta} x_t^{\kappa}$$

where small letters denote per-capita values. Now, the growth rate of y_t can be written as

$$g^y = \alpha g^k + \beta g - \kappa n$$

Since, as explained above, $g^y = g^k$ and $1 - \alpha = \beta + \kappa$, we have

$$g_t^y \cong \frac{\beta g - \kappa n}{1-\alpha} = \frac{\beta g - \kappa n}{\beta + \kappa}.$$

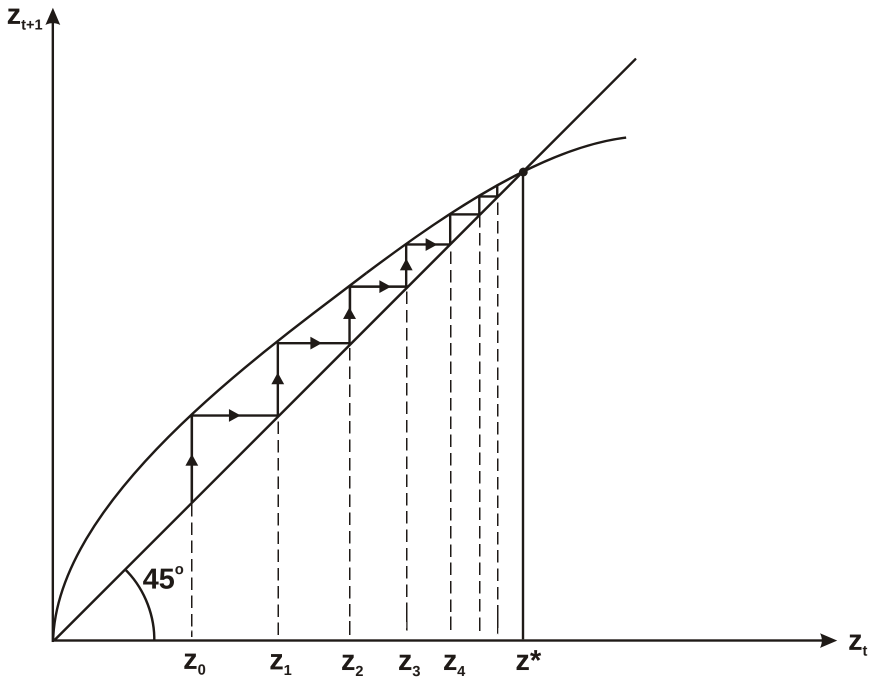
The growth rate of per-capita income may become negative when population growth is sufficiently high. In this case, the negative impact of population growth on per-capita income outweighs the positive effect of technological progress.

The question now is whether the economy converges to a balanced growth path in which $z_t = K_t / Y_t$ stays constant. Consider that the law of motion of the ratio $z_t = K_t / Y_t$ is given by

$$z_{t+1} = \left(\frac{1}{(1+n)(1+g)} \right)^{\beta} \left[s + (1-\delta) z_t \right]^{1-\alpha} z_t^{\alpha}.$$

c. Show graphically whether the economy converges to a balanced growth path.

Answer: The graphical illustration of the law of motion is as follows:



Starting at z_0, the next period's value is z_1 followed by z_2 and so on. The values come arbitrarily close to the value along a balanced growth path z^*. Hence, the economy converges to a balanced growth path.

The following regression equation is frequently used to estimate the impact of the economies' structural characteristics (such as the savings rate and the population growth rate) and the availability of land per worker on GDP per worker. The GDP data is taken from the year 2000 and the land per worker data from the year 1994. The savings rate and the population growth rate are average annual values over the period 1960-2000.

$$\ln y_{00}^i \cong \gamma_0 + \gamma_1 \left[\ln s^i - \ln\left(n^i + 0.075\right) \right] + \gamma_2 \ln\left(\frac{X^i}{L_{94}^i} \right).$$

The regression results are $\gamma_0 = 2.71, \gamma_1 = 1.35$, and $\gamma_2 = 0.38$.

d. Show analytically whether the sign of the estimated coefficients is compatible with a balanced growth path of the economy.

Answer: Inserting $z_t = z_{t+1}$ into the law of motion and solving for the balanced growth path value of z gives

$$z = z^* \equiv \frac{s}{\left[(1+n)(1+g)\right]^{\frac{\beta}{\beta+\kappa}} - (1-\delta)} > 0.$$

A higher savings rate, s, increases z^* while a higher population growth rate, n, lowers z^*. Both theoretical predictions are in line with the empirical finding since $\gamma_1 > 0$. A higher land endowment (per capita) increases output per capita. The theoretical prediction follows from the per-capita production function

$$y_t = k_t^{\alpha} A_t^{\beta} x_t^{\kappa} .$$

Thus, the estimated coefficient should be positive which holds true ($\gamma_2 > 0$).

Exercise 2

Consider the following model of a two-sector economy:

$$Y_t = K_t^{\alpha} \left(A_t L_{Yt}\right)^{1-\alpha}$$
$$A_{t+1} - A_t = \rho A_t^{\phi} L_{At}^{\lambda}$$
$$K_{t+1} = sY_t + \left(1-\delta\right) K_t$$
$$L_{Yt} + L_{At} = L_t$$
$$L_{At} = s_R L_t .$$
$$L_t = L$$

The notation is as in exercise 1. s_R is the fraction of population which works in the research and development (R&D) sector of the economy. L_{Yt} is the number of workers in the final goods sector and the residual $L_{At} = L_t - L_{Yt}$ works in the R&D sector.

Assume that $\phi = \lambda = 1$.

a. Determine the growth rate of the productivity term A_t as a function of total population. Which feature does the growth rate exhibit? Explain whether this model can be classified as an endogenous growth model.

Answer: Inserting $\phi = \lambda = 1$ into

$$A_{t+1} - A_t = \rho A_t^{\phi} L_{At}^{\lambda}$$

and dividing by A_t yields

$$\left(A_{t+1} - A_t\right) / A_t = \rho L_{At} .$$

Since population growth is zero and $L_{At} = s_R L_t$, the growth rate of the productivity term is constant. The model can be classified as an endogenous growth model because the growth rate of technological progress is related to the fundamentals of the economy, i.e. to $L_{At} = s_R L_t$. Note, the share of the population s_R which works in the R&D sector is policy determined.

b. Derive the steady state of the economy. Write the law of motion for the capital intensity $\tilde{k}_t = K_t /(A_t L)$ and show graphically whether the economy converges to this steady state.

Answer: Inserting $L_{Yt} = (1 - s_R) L_t$ into the production function gives

$$Y_t = K_t^{\alpha} \left(A_t L_{Yt} \right)^{1-\alpha} = K_t^{\alpha} \left(A_t \left(1 - s_R \right) L_t \right)^{1-\alpha} \quad \Leftrightarrow \quad \tilde{y}_t = \tilde{k}_t^{\alpha} \left(1 - s_R \right)^{1-\alpha} .$$

Dividing both sides of

$$K_{t+1} = sY_t + \left(1 - \delta\right) K_t$$

by $A_{t+1} L_{t+1}$ and inserting $\tilde{y}_t$ gives

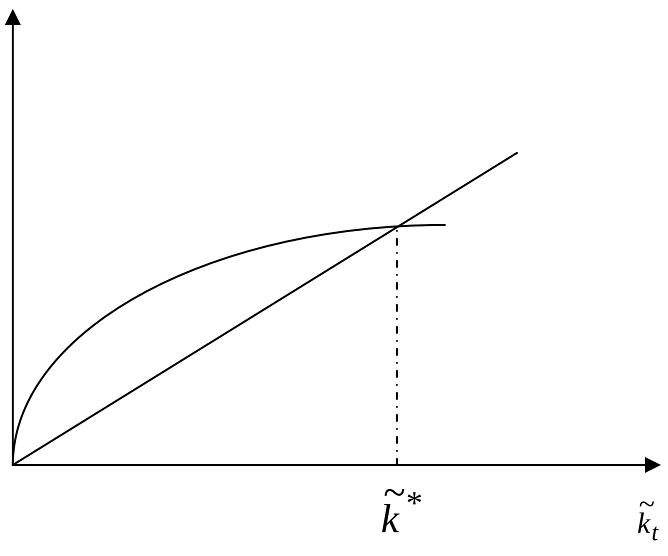
$$\tilde{k}_{t+1} = \left(\frac{1}{1+g} \right) \left(s \tilde{k}_t^{\alpha} (1 - s_R)^{1-\alpha} + (1-\delta) \tilde{k}_t \right)$$

Since the growth rate of technological progress is constant ($g_t = g$), the equation fully describes the law of motion of the economy.

To graphically illustrate the law of motion, we subtract $\tilde{k}_t$ from both sides of the law of motion:

$$\tilde{k}_{t+1} - \tilde{k}_t = \left(\frac{1}{1+g} \right) \left(s \tilde{k}_t^{\alpha} (1 - s_R)^{1-\alpha} - (g + \delta) \tilde{k}_t \right).$$

The following graph depicts convergence of the economy where the straight line depicts the term $(g + \delta) \tilde{k}_t$ and the concave curve depicts the term $s \tilde{k}_t^{\alpha} (1 - s_R)^{1-\alpha}$. When both lines intersect, the change in $\tilde{k}_t$ is zero. Thus, $\tilde{k}_t = \tilde{k}_t^*$.



c. Use the capital accumulation equation and denote the growth rate of the productivity term A_t as *g*. Show that the steady state value of $\tilde{k}_t = K_t /(A_t L)$ is

$$\tilde{k}^* = \left(\frac{s}{g-\delta} \right)^{\frac{1}{1-\alpha}} (1 - s_R).$$

Answer: The steady state value of $\tilde{k}_t = K_t /(A_t L)$ can be derived by setting $\tilde{k}_{t+1} = \tilde{k}_t$ and then solving the law of motion. This yields

$$\tilde{k}^* = \left(\frac{s}{g+\delta} \right)^{\frac{1}{1-\alpha}} (1 - s_R).$$

NOTE: There was a mistake in the equation given in the exam. The denominator should read $g+\delta$ rather than $g-\delta$. Grading has been generous at this point.

d. How does the steady state level of $\tilde{k}_t$ relate to the fraction of the population which works in the R&D sector? Provide an economic explanation.

Answer: The steady state level of $\tilde{k}_t$ decreases in s_R because the number of people who are employed in the production sector drops. The effect is captured by the term (1-s_R). Also, a higher s_R increases *g* and, thereby, lowers the steady state value of $\tilde{k}_t$ further. The intuition for the latter effect is that a higher *g* increases the amount of savings necessary to keep $\tilde{k}_t$ constant (replacement investments). Thus, less saving is available to increase $\tilde{k}_t$.

e. Now, assume the government wants to increase s_R. Explain verbally with the help of the steady state level of per-capita income $y_t = Y_t / L_t$ which effects the government has to balance when setting s_R so as to maximize y_t. A formal derivation of the optimal s_R is not necessary.

Answer: Per capita income $y_t = Y_t / L_t$ can be written as follows:

$$y_t = A_t \tilde{k}_t^{\alpha} (1 - s_R)^{1-\alpha}.$$

A higher s_R has three effects on per capita income. First, as explained above, it lowers $\tilde{k}_t$. Then, there is also a negative effect on $y_t = Y_t / L_t$ since less labour is used in production. This effect is captured by the term $(1 - s_R)^{1-\alpha}$. But, there is also a positive effect of a higher s_R. More resources which are spent on R&D (as measured by s_R) increase the growth rate of A_t and, hence, also the level of A_t in the production function. This increases per capita income. The government has to balance these effects when choosing s_R.

Now assume that $0 < \phi < 1$ and $\lambda = 1$.

f. Compute first the ratio of the growth rate of the productivity term A_t g_{t+1}/g_t and from there the law of motion for the growth rate. Does there exist a steady state growth rate, i.e. a situation in which $g_{t+1}/g_t = 1$? Illustrate your finding with the help of a figure.

Answer: Start from

$$A_{t+1} - A_t = \rho A_t^{\phi} L_{At}^{\lambda}$$

and divide by A_t to get

$$g_t = \rho (A_t)^{\phi-1} L_{At}.$$

The term g_{t+1}/g_t can be written as $g_{t+1}/g_t = (A_{t+1}/A_t)^{\phi-1}$. Since $A_{t+1}/A_t = 1 + g_t$, we get

$$g_{t+1} = g_t (1+g_t)^{\phi-1}.$$

Setting $g_{t+1}/g_t = 1$ and solving for the growth rate in steady state, we get $g_{se} = 0$. The only growth rate which is compatible with $g_{t+1}/g_t = 1$ is a zero growth rate.

To graphically illustrate the dynamics, it is useful to first get an impression of the slope and curvature of the curve $g_{t+1}(g_t)$ which is implied by $g_{t+1} = g_t(1+g_t)^{\phi-1}$. We first note that

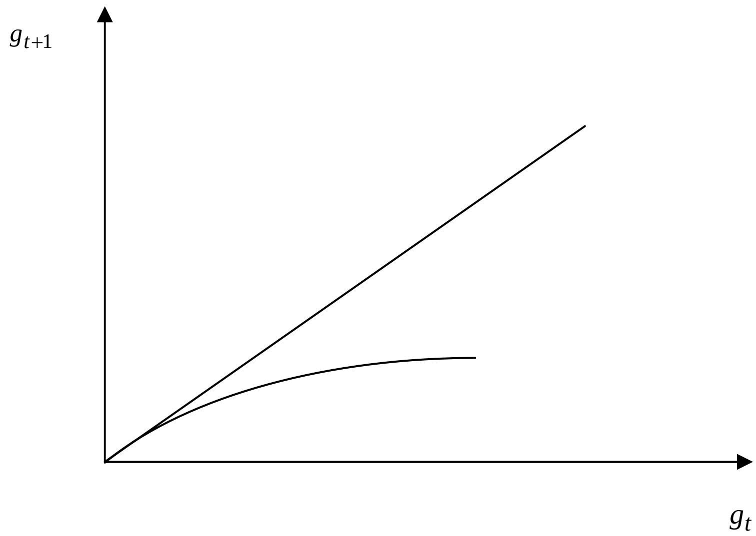
$$dg_{t+1}/dg_t = (1+g_t)^{\phi-1} + g_t(\phi-1)(1+g_t)^{\phi-2}$$

and

$$d^2 g_{t+1}/d(g_t)^2 = (\phi-1)(1+g_t)^{\phi-2} + (\phi-1)(1+g_t)^{\phi-2} + g_t(\phi-1)(\phi-2)(1+g_t)^{\phi-3} < 0.$$

The first derivative is equal to unity at $(g_{t+1} = 0, g_t = 0)$. The second derivative is negative since $2 > -(\phi-2)g_t/(1+g_t)$. Hence, drawn in (g_{t+1}, g_t)-space, the function is strictly concave.

These two findings imply that, when evaluated at $g_{t+1} > 0, g_t > 0$, the slope of the function is smaller than unity. The dynamics can be illustrated as follows. The straight line in the figure below depicts the 45° line (slope=1) and the concave curve depicts the function $g_{t+1}(g_t)$ as implied by $g_{t+1} = g_t(1+g_t)^{\phi-1}$. For any positive initial value of the growth rate, the value of the growth rate always converges to 0.


g_{t+1}
g_t