Written Exam for M.Sc. in Economics

Winter 2010/2011

Advanced Microeconomics

20. December 2010

Master course

3 hours written exam with closed books

Solution

1.1

For p and y (UMP) for consumer i is

$$\begin{aligned} \max_{x_i} \quad & (x_i^1)^{a_i}(x_i^2)^{b_i} \\ \text{s.t.} \quad & p \cdot x_i \leq p \cdot \omega_i + \delta_i p \cdot y \text{ and } x_i \in X_i \end{aligned}$$

For $p \in \mathbb{R}^2_{++}$ and $p \cdot y \geq -p \cdot \omega_i$ the demand function of consumer i is

$$x_i(p, p \cdot \omega_i + \delta_i p \cdot y) \;=\; \left(\frac{a_i}{a_i + b_i} \frac{p \cdot \omega_i + \delta_i p \cdot y}{p_1}, \frac{b_i}{a_i + b_i} \frac{p \cdot \omega_i + \delta_i p \cdot y}{p_2} \right).$$

1.2

The problem of the firm (PMP) is

$$\begin{aligned} \max_{y} \quad & p \cdot y \\ \text{s.t.} \quad & y \in \{z \in \mathbb{R}^2 | z^1 \leq 0 \text{ and } z^2 \leq -z^1\} \end{aligned}$$

There is constant returns to scale. For $p \in \mathbb{R}^2_{++}$ the supply correspondence is

$$y(p) \;=\; \begin{cases} (0,0) & \text{for } p_1 > p_2 \\ \{(-t,t) \in \mathbb{R}^2 | t \geq 0\} & \text{for } p_1 = p_2 \\ \emptyset & \text{for } p_1 < p_2. \end{cases}$$

1.3

Definition 1 *A* **Walrasian equilibrium** *is a price vector, consumption bundles for the two consumers and a production plan* $(\bar{p}, \bar{x}_1, \bar{x}_2, \bar{y})$ *such that*

- *$\bar{x}_i$ is a solution to (UMP) given $\bar{p}$ and $\bar{y}$ for both i.*
- *$\bar{y}$ is a solution to (PMP) given $\bar{p}$.*
- *$\sum_i \bar{x}_i = \sum_i \omega_i + \bar{y}$.*

1.4

Suppose that $(\bar{p}, \bar{x}_1, \bar{x}_2)$ is a Walrasian equilibrium for the pure exchange part of economy. If $\bar{p}_1 \geq \bar{p}_2$, then $(\bar{p}, \bar{x}_1, \bar{x}_2, \bar{y})$, where $\bar{y} = 0$, is a Walrasian equilibrium for the economy. $(\bar{p}, \bar{x}_1, \bar{x}_2)$ is a Walrasian equilibrium for the pure exchange part of economy if and only if

$$\frac{a_1}{a_1 + b_1} \frac{\bar{p} \cdot \omega_1}{\bar{p}_1} + \frac{a_2}{a_2 + b_2} \frac{\bar{p} \cdot \omega_2}{\bar{p}_1} = \omega_1^1 + \omega_2^1$$

and this is equivalent to

$$\frac{\bar{p}_1}{\bar{p}_2} = \frac{\frac{a_1}{a_1 + b_1}\omega_1^2 + \frac{a_2}{a_2 + b_2}\omega_2^2}{\frac{b_1}{a_1 + b_1}\omega_1^1 + \frac{b_2}{a_2 + b_2}\omega_2^1}.$$

Therefore the righthand side is larger than or equal to one if and only if the economy has a Walrasian equilibrium $(\bar{p}, \bar{x}_1, \bar{x}_2, \bar{y})$ with $\bar{y} = 0$.

2.1

Since the preference relation $\succeq_i$ is rational and continuous for every consumer i there exists a continuous utility function u_i that represents $\succeq_i$. Hence (UMP) for consumer i is

$$\begin{aligned} \max_{x_i} \quad & u_i(x_i) \\ \text{s.t.} \quad & p \cdot x_i \leq p \cdot \omega_i \text{ and } x_i \in \mathbb{R}_+^L. \end{aligned}$$

The budget set is closed because it is the intersection of $\{x \in \mathbb{R}^L | p \cdot x \leq p \cdot \omega_i\}$ and R^L_+ and both these sets are closed. For $p \in \mathbb{R}^L_{++}$ the budget set is bounded because it is bounded from below by 0 and from above by $(p \cdot \omega_i/p_1, \ldots, p \cdot \omega_i/p_L)$. All in all the budget set is compact. There exists a solution to (UMP) because u_i is continuous and the budget set is compact.

Suppose that x and x' both are solutions to (UMP) and $x' \neq x$. Then for all $t \in]0,1[$, $(1-t)x + tx'$ is in the budget set and $(1-t)x + tx' \succ_i x, x'$, because $\succeq_i$ is strictly convex. This contradicts that x and x' both are solutions to (UMP). Therefore there is a unique solution.

2.2

Let $p_\ell = 0$ for some ℓ. Suppose that x is a solution to (UMP). Then $x + e_\ell \succ_i x$, where the ℓ'th coordinate of e_ℓ is one and all other coordinates are zero, because $\succeq_i$ is strongly monotone. This contradicts that x is a solution to (UMP), because $x + e_\ell$ is in the budget too.

2.3

Definition 2 *An* **allocation** *is a list of consumption bundles* $(x_1, \ldots, x_I)$ *such that* $x_i \in \mathbb{R}^L_+$ *for all* i*.*

Definition 3 *A* **Walrasian equilibrium** *is a price vector and an allocation* $(\bar{p}, \bar{x}_1, \ldots, \bar{x}_I)$ *such that*

- $\bar{x}_i$ *is a solution to (UMP) given* $\bar{p}$*.*
- $\sum_i \bar{x}_i = \sum_i \omega_i$*.*

2.4

Suppose that $(\bar{p}, \bar{x}_1, \dots \bar{x}_I)$ is a Walrasian equilibrium. Then $\bar{p} \cdot \bar{x}_i = \bar{p} \cdot \omega_i$ and $x_i \succ_i \bar{x}_i$ implies that $\bar{p} \cdot x_i > \bar{p} \cdot \bar{x}_i$. Thus if $x_i \succ_i \bar{x}_i$ for all $i \in C$, then $\bar{p} \cdot \sum_{i \in C} x_i > \bar{p} \cdot \sum_{i \in C} \omega_i$. Therefore $\sum_{i \in C} x_i \neq \sum_{i \in C} \omega_i$.

3.1

The utility function is somewhat special since $(\bar{v}, \bar{v})$ is minimum. Answering the exercise with either $u(x^y, x^o) = (\bar{v} - x^y)^2 + (\bar{v} - x^o)^2$ or $u(x^y, x^o) = -(\bar{v} - x^y)^2 - (\bar{v} - x^o)^2$ is perfectly fine. Below the exercise is answered with $u(x^y, x^o) = -(\bar{v} - x^y)^2 - (\bar{v} - x^o)^2$.

Indifference curves are the intersection of circles with center at $(\bar{v}, \bar{v})$ and $\mathbb{R}^2_+$. The assumption $\omega^y, \omega^o \leq \bar{v}$ implies that ω is below the bliss point. The assumption $(\bar{v} - \omega^y)^2 + (\bar{v} - \omega^o)^2 \leq \bar{v}^2$ implies that ω on a circle that is contained in $\mathbb{R}^2_+$.

3.2

The problem of consumer t is

$$\max_{x,m} \quad -(\bar{v} - x^y)^2 - (\bar{v} - x^o)^2$$

$$\text{s.t.} \quad \begin{cases} p_t x^y + m \leq p_t \omega^y \\ p_{t+1} x^o \leq p_{t+1} \omega^o + m \\ x \in \mathbb{R}^2_+ \text{ and } m \in \mathbb{R} \end{cases}$$

The problem can be solved by using the second budget constraint to eliminate m in the first budget constraint, then maximizing utility using the modified budget constraint and finally finding m by use of the first or second budget constraint.

The first-order conditions are

$$\begin{aligned}
2(\bar{v} - x^y) - \lambda p_t &= 0 \\
2(\bar{v} - x^o) - \lambda p_{t+1} &= 0 \\
p_t x^y + p_{t+1} x^o &= p_t \omega^y + p_{t+1} \omega^o.
\end{aligned}$$

The demand function is

$$F((p_t, p_{t+1}), (p_t, p_{t+1}) \cdot \omega) = (\bar{v} - \lambda p_t, \bar{v} - \lambda p_{t+1})$$

where

$$\lambda = \frac{p_t(\bar{v} - \omega^y) + p_{t+1}(\bar{v} - \omega^o)}{p_t^2 + p_{t+1}^2}.$$

Let $x_t(p_t, p_{t+1}) = ((p_t\omega^y + p_{t+1}\omega^o)/p_t, (p_t\omega^y + p_{t+1}\omega^o)/p_{t+1})$, then for the other utility function the demand is

$$F((p_t, p_{t+1}), (p_t, p_{t+1}) \cdot \omega) = \arg \max_{x \in \{(0,0), (x_t^y(p_t, p_{t+1}), 0), (0, x_t^o(p_t, p_{t+1}))\}} (\bar{v} - x^y)^2 + (\bar{v} - x^o)^2$$

3.3

Definition 4 *An* **ordinarily Pareto optimal** *allocation* $(\tilde{x}_t)_{t \in \mathbb{Z}}$ *is an allocation such that there is no other allocation* $(x_t)_{t \in \mathbb{Z}}$ *with*

- $x_t = \tilde{x}_t$ *for all* $t \leq T_L - 2$ *and* $x_t^y = \tilde{x}_t^y$ *for* $t = T_L - 1$ *for some* T_L.
- $u_t(x_t) \geq u_t(\tilde{x}_t)$ *for all* t *with at least one "$>$".*

Consider the allocation $(x_t)_{t \in \mathbb{Z}}$ where $x_t = (\omega^y + \omega^o, 0)$ for all t. Then $(a_t)_{t \in \mathbb{Z}}$ with $a_t = 0$ for $t < 0$ and $a_t = \varepsilon$ for $t \geq 0$ is an improving reallocation for $\varepsilon > 0$ sufficiently small. This can be shown by considering the derivative of $-(\bar{v} - \omega^y - \omega^o + \varepsilon)^2 - (\bar{v} - \varepsilon)^2$ with respect to ε.

For the other utility function consider an allocation, where x_t is the solution to

$$\begin{aligned} \min_x \quad & (\bar{v} - x^y)^2 + (\bar{v} - x^o)^2 \\ \text{s.t.} \quad & x^y + x^o = \omega^y + \omega^o \text{ and } x \in \mathbb{R}^2_+ \end{aligned}$$

for all t. Then clearly it is not ordinarily Pareto optimal.

3.4

If $\bar{p}_t = 1$ for all t and $\bar{x}_t = ((\omega^y + \omega^o)/2, (\omega^y + \omega^o)/2)$ for all t, then $((\bar{p}_t)_{t\in\mathbb{Z}}, (\bar{x}_t)_{t\in\mathbb{Z}})$ is an equilibrium.

For the other utility function additional assumptions are needed to ensure the existence of an equilibrium.