Written Exam for M.Sc. in Economics 2010-I

Advanced Microeconomics

18. December 2009

Master course

Solution

1.1

(UMP) is

$$\max_x \quad u_i(x)$$
$$\text{s.t.} \quad p \cdot x \leq p \cdot \omega_i \text{ and } x \in X_i$$

1.2

The set X_i is closed and the set $\{x \in \mathbb{R}^L | p \cdot x \leq p \cdot \omega_i\}$ is closed. Their intersection is non-empty because $\omega_i, p \in \mathbb{R}^L_{++}$ and their intersection is bounded from below by 0 and from above by $(w_i/p_1, \ldots, w_i/p_L)$, so their intersection is non-empty and compact. The utility function is continuous by assumption. Therefore (UMP) has a solution as a continuous function has a maximum on a non-empty compact set.

1.3

A preference relation $\succeq \subset X \times X$ is strictly convex if and only if $(1-\alpha)x + \alpha x' \succ x''$ for all $x, x', x'' \in X$ with $x, x' \succeq x''$ and $x' \neq x$ and $\alpha \in]0, 1[$.

Uniqueness by contradiction: Suppose that x and x' are different two solutions to (UMP), then $0.5x + 0.5x' \in X$ and $p \cdot (0.5x + 0.5x') \leq \max\{p \cdot x, p \cdot x'\}$, so $0.5x + 0.5x'$ is in the budget set. By strict convexity $0.5x + 0.5x' \succ x, x'$. This contradicts that x and x' are two different solutions to (UMP).

1.4

Definition 1 *A* **Walrasian equilibrium** *is a price vector and a list of individual consumption bundles* $(\bar{p}, \bar{x})$ *such that*

- $\bar{x}_i$ *is a solution to (UMP) given* $\bar{p}$ *for all* i.
- $\sum_i \bar{x}_i = \sum_i \omega_i$.

Suppose that

$$\sum_i x_i(\bar{p}, \bar{p} \cdot \omega_i) \; = \; \sum_i \omega_i.$$

Then $\bar{x}_i = x_i(\bar{p}, \bar{p} \cdot \omega_i)$ is a solution to (UMP) given $\bar{p}$ by definition and markets clear by assumption. Therefore $(\bar{p}, \bar{x})$, where $\bar{x}_i = x_i(\bar{p}, \bar{p} \cdot \omega_i)$ for all i, is a Walrasian equilibrium.

1.5

Definition 2 *A* **Pareto optimal allocation** $\bar{x}$ *is a feasible allocation such that there is no other feasible allocation* x *with* $u_i(x_i) \geq u_i(\bar{x}_i)$ *for all* i *and* $u_i(x_i) > u_i(\bar{x}_i)$ *for at least one* i.

Suppose that $(\bar{p}, \bar{x})$ is a Walrasian equilibrium. Then $\bar{p} \in \mathbb{R}^L_{++}$ and $\bar{p} \cdot \bar{x}_i = \bar{p} \cdot \omega_i$ because preferences are strongly monotone. If $u_i(x_i) > u_i(\bar{x}_i)$, then $\bar{p} \cdot x_i > \bar{p} \cdot \bar{x}_i$ because $\bar{x}_i$ is a solution to (UMP). If $u_i(x_i) \geq u_i(\bar{x}_i)$, then $\bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i$. Indeed if $\bar{p} \cdot x_i < \bar{p} \cdot \bar{x}_i$, then $u_i(x_i + \varepsilon \bar{p}) > u_i(\bar{x}_i)$ for $\varepsilon > 0$ by strong monotonicity and $\bar{p} \cdot (x_i + \varepsilon \bar{p}) \leq \bar{p} \cdot \bar{x}_i$ for ε small enough contradicting that $\bar{x}_i$ is a solution to (UMP). All in all if x Pareto dominates $\bar{x}$, then $\sum_i x_i^\ell \neq \sum_i \bar{x}_i^\ell$ for some ℓ becasue $\bar{p} \cdot \sum_i x_i > \bar{p} \cdot \sum_i \bar{x}_i$. Therefore if x Pareto dominates $\bar{x}$, then x is not feasible. Hence $\bar{x}$ is Pareto optimal.

1.6

For $p \in \mathbb{R}^2_{++}$ the demand is $x(p, p_1 + p_2) = ((p_1 + p_2)/p_1, 0)$ and for other p's there is no solution to (UMP) because the lexicographic preference relation is strongly monotone. Therefore $x = (1, 1)$ is not a solution to (UMP) for any p.

2.1

The problem of consumer i is

$$\begin{aligned} \max_{x} \quad & x^1 \ln x^2 \\ \text{s.t.} \quad & p_1 x^1 + p_2 x^2 \leq w_i \text{ and } x \in \mathbb{R}_+ \times \mathbb{R}_{++} \end{aligned}$$

where $w_i = 6p_1 + \pi(p_1, p_2)$ and $\pi(p_1, p_2)$ is the profit of the firm.

The problem of the firm is

$$\begin{aligned} \max_{y} \quad & p_1 y^1 + p_2 y^2 \\ \text{s.t.} \quad & y \in \{z \in \mathbb{R}^2 | z^1 \leq 0 \text{ and } z^2 \leq -5z^1\}. \end{aligned}$$

2.2

A Pareto optimal allocation $(\tilde{x}, \tilde{y})$ can be found by solving

$$\begin{aligned} \max_{(x,y)} \quad & x^1 + \ln x^2 \\ \text{s.t} \quad & \begin{cases} x \in \mathbb{R}_+ \times \mathbb{R}_{++} \\ y \in \{z \in \mathbb{R}^2 | z^1 \leq 0 \text{ and } z^2 \leq -5z^1\} \\ x = \begin{pmatrix} 6 \\ 0 \end{pmatrix} + y. \end{cases} \end{aligned}$$

This problem reduces to

$$\max_{z} (6 - z) + \ln z$$

The solution is $z = 1$. Therefore $(\tilde{x}, \tilde{y})$ with $\tilde{x} = (5, 5)$ and $\tilde{y} = (-1, 5)$ is a Pareto optimal allocation.

2.3

There is a unique Pareto optimal allocation because the utility function represents a strictly convex preference relation and the production set is convex. Therefore according to the first welfare theorem the equilibrium allocation has to be the Pareto optimal allocation found in 2.2. For prices the gradient of the consumer at the consumption bundle can be used. Hence $(\bar{p}, \bar{x}, \bar{y})$, where $\bar{p} = (1, 1/5)$, $\bar{x} = (5, 5)$ and $\bar{y} = (-1, 5)$, is a Walrasian equilibrium.

3.1

Definition 3 *An* **equilibrium with spot markets** *is a price system and an allocation $((\bar{p}_t)_{t\in\mathbb{Z}}, (\bar{x}_t)_{t\in\mathbb{Z}})$ such that there exist $(\bar{m}_t)_{t\in\mathbb{Z}}$ and M such that*

- $(\bar{x}_t, \bar{m}_t)$ *is a solution to*

$$\max_{(x,m)} \quad u_t(x)$$
$$s.t. \quad \begin{cases} \bar{p}_t x^y + m \le \bar{p}_t \omega_t^y \\ \bar{p}_{t+1} x^o \le \bar{p}_{t+1} \omega_t^o + m \\ x \in X, \ \ m \in \mathbb{R} \end{cases}$$

- $\bar{x}_t^y + \bar{x}_{t-1}^o = \omega_t^y + \omega_{t-1}^o$ *and* $\bar{m}_t = M$ *for all* t.

Definition 4 *A* **Walrasian equilibrium** *is a price system and an allocation $((\bar{p}_t)_{t\in\mathbb{Z}}, (\bar{x}_t)_{t\in\mathbb{Z}})$ such that*

- $(\bar{x}_t)$ *is a solution to*

$$\max_{(x,m)} \quad u_t(x)$$
$$s.t. \quad \begin{cases} \bar{p}_t x^y + \bar{p}_{t+1} x^o \le \bar{p}_t \omega_t^y + \bar{p}_{t+1} \omega_t^o \\ x \in X \end{cases}$$

- $\bar{x}_t^y + \bar{x}_{t-1}^o = \omega_t^y + \omega_{t-1}^o$ *for all* t.

3.2

The budget constraints in the definition of equilibrium with spot markets can be rewritten as

$$\bar{p}_t x^y + \bar{p}_{t+1} x^o \leq \bar{p}_t \omega_t^y + \bar{p}_{t+1} \omega_t^o$$

$$\bar{p}_{t+1}(x^o - \omega_t^o) \leq m \leq \bar{p}_t(\omega_t^y - x^y)$$

Therefore if (x, m) is a solution to the consumer problem in case of spot markets, then x is a solution to consumer problem is case of forward markets. Hence equilibria with spots markets are also Walrasian equilibria.

3.3

Definition 5 *An* **ordinarily Pareto optimal** *allocation* $(\tilde{x}_t)_{t\in\mathbb{Z}}$ *is an allocation such that there is no other allocation* (

$_t)_{t\in\mathbb{Z}}$ *with*

- $x_t = \tilde{x}_t$ *for all* $t \leq T_L - 2$ *and* $x_t^y = \tilde{x}_t^y$ *for* $t = T_L - 1$ *for some* T_L*.*
- $u_t(x_t) \geq u_t(\tilde{x}_t)$ *for all* t *with at least one "$>$".*

There are two other forms of Pareto optimality: strong Pareto optimality, where consumption can be changed at all dates, and weak Pareto optimality, where consumption can be changed at finitely many dates. Weak optimality is the relevant notion for the welfare theorem. Ordinarily optimality is relevant for pension-like redistributions of goods.

3.4

The utility function is a CES function and the demand function is

$$x = \left(\frac{p_t^b}{p_t^b + p_{t+1}^b} \cdot \frac{p_t \omega_t^y + p_{t+1}\omega_t^o}{p_t}, \frac{p_{t+1}^b}{p_t^b + p_{t+1}^b} \cdot \frac{p_t \omega_t^y + p_{t+1}\omega_t^o}{p_{t+1}} \right)$$

where $b = (a-1)/a$.

3.5

If $\bar{p}_t = (7/3)^{at}$ for all t, then $((\bar{p}_t)_{t\in\mathbb{Z}}, (\bar{x}_t)_{t\in\mathbb{Z}}$ with $\bar{x}_t = (7,3)$ for all t is an equilibrium. It is not ordinarily Pareto optimal because if $x_{t-1}^o = 5$ and $x_t^y = 5$ from date $t = 0$ and forward, then consumer -1 and forward are better off and no consumer is worse off. Alternatively it can be used that $\sum_{t=0}^{\infty} 1/\bar{p}_t < \infty$.

3.6

Clearly $((\bar{p}_t)_{t\in\mathbb{Z}}, (\bar{x}_t)_{t\in\mathbb{Z}}$ with $\bar{p}_t = 1$ and $\bar{x}_t = (5,5)$ for all t is a monetary equilibrium.