Exam Solution Guide

Econometrics II

December 2017

Part 1
House Prices

The Case The goal of this part of the exam is to use cointegration techniques to test the empirical validity of a theoretical long-run relationship between house prices and households' real disposable income relative to the housing stock, the user cost, the lowest possible first-year payments, and the expected future change in the house price.

The Data Graphs of the data and relevant transformations must be shown in the exam. It must be noted that the level of the house prices is clearly non-stationary, but the first-difference appears somewhat stationary.

Econometric Theory The econometric theory must include the following:

(1) A precise definition and interpretation of the models considered and their properties. Specifically, an interpretation of cointegration must be presented along with a presentation of univariate autoregressive (AR) models used to test for unit roots and a single equation cointegration approach based on the Engle-Granger two-step procedure or the autoregressive distributed lag (ADL) and error-correction models (ECM).

(2) A precise description of the estimator used, in particular a precise account of the assumptions used to derive an estimator.

(3) A precise account of the necessary assumptions for consistent estimation and valid inference. This includes a precise definition of the null hypotheses, test statistics, and asymptotic distributions used to test relevant hypotheses.

(4) The theory must be presented precisely and in a logical order with a consistent and correct notation.

Empirical Results The empirical results must include the following:

(1) A presentation and discussion of the relevant empirical results, so that the reader is able to understand the steps carried out in the process as well as the conclusions made.

(2) A description of the model selection process based on a general-to-specific approach, information criteria, or both.

(3) A discussion of whether the assumptions for consistent estimation and valid inference are satisfied for the estimated models. Specifically, this includes misspecification testing, which must be presented and discussed before statistical testing is carried out.

(4) A clear conclusion to the main question and a discussion of the limitations of the approach used to reach the conclusion. Specifically, the conclusion regarding cointegration between the house prices and the disposable income, the user cost, the lowest possible first-year payments, and the expected future change in the house price must be presented and the limitations of the single-equation cointegration approach must be discussed in relation to the conclusion.

Part 2
Volatility Spillovers in International Stock Returns

The Case The goal of this part of the exam is to use GARCH models to test for volatility spillovers from the Nikkei 225 and the SP 500 indexes to the FTSE 100 stock index.

The Data Graphs of the data and relevant transformations must be shown in the exam. It must be noted that there is volatility clustering in the log-returns on the FTSE 100 stock index, and that the log-returns on the FTSE 100, Nikkei 225, and SP 500 indices appear to exhibit high/low volatility during the same periods indicating potential volatility spillovers.

Econometric Theory The econometric theory must include the following:

(1) A precise definition and interpretation of the model considered and its properties.

(2) A precise description of the estimator used, in particular a precise account of the assumptions used to derive an estimator.

(3) A precise account of the necessary assumptions for consistent estimation and valid inference. This includes a precise definition of the null hypotheses, test statistics, and asymptotic distributions used to test relevant hypotheses.

(4) The theory must be presented precisely and in a logical order with a consistent and correct notation.

Empirical Results The empirical results must include the following:

(1) A presentation and discussion of the relevant empirical results, so that the reader is able to understand the steps carried out in the process as well as the conclusions made.

(2) A discussion of whether the assumptions for consistent estimation and valid inference are satisfied for the estimated models.

(3) A precise discussion/judgement of potential misspecification of the model.

(4) A clear conclusion to the main question and a discussion of the limitations of the approach used to reach the conclusion. Specifically, if there are volatility spillovers from the Nikkei 225 and the SP 500 indices.

Part 3
The New Keynesian Phillips Curve

The Case The goal of this part of the exam is to use the generalized method of moments to test if there empirical evidence of three different versions of the New Keynesian Phillips curve.

The Data Graphs of the data and relevant transformations must be shown in the exam.

Econometric Theory The econometric theory must include the following:

(1) A precise definition and interpretation of the model considered and its properties. Specifically, the general method of moments (GMM) used to estimate the parameters of the economic theory.

(2) A precise description of the estimator used, in particular a precise account of the assumptions used to derive an estimator.

(3) A precise account of the necessary assumptions for consistent estimation and valid inference. This includes a precise definition of the null hypotheses, test statistics, and asymptotic distributions used to test relevant hypotheses.

(4) The theory must be presented in a logical order and with a consistent and correct notation.

Empirical Results The empirical results must include the following:

(1) A presentation and discussion of the relevant empirical results, so that the reader is able to understand the steps carried out in the process as well as the conclusions made.

(2) A discussion of whether the assumptions for consistent estimation and valid inference are satisfied for the estimated models.

(3) A robustness analysis of the estimated model.

(4) A clear conclusion to the main question and a discussion of the limitations of the approach used to reach the conclusion. Specifically, if there is empirical evidence of the three different versions of the New Keynesian Phillips curve.

Part 4
Theoretical Problem:

#4.1 Maximum Likelihood Estimation in an AR(1) Model with a Level-Shift

Consider the AR(1) model

$$y_t = \rho \cdot y_{t-1} + \mu \cdot D_t + \epsilon_t, \qquad t = 1, 2, ..., T, \tag{4.1}$$

where $\epsilon_t \sim IIDN(0, \sigma^2)$ and given some initial value y_0. Assume that $|\rho| < 1$. The variable D_t is given by

$$D_t = \mathbf{1}\left(t > \frac{T}{2}\right), \tag{4.2}$$

where $\mathbf{1}(t > \frac{T}{2})$ is an indicator function which takes on the value 1 for $t > \frac{T}{2}$, and 0 otherwise. Assume that T is an even integer.

Question 1

Based on the distributional assumption $\epsilon_t \sim IIDN(0, \sigma^2)$, the log-likelihood function for $y_1, ..., y_T$ conditional on the initial value y_0 and $D_1, ..., D_T$ is given by:

$$\begin{aligned}\ell(\rho, \mu, \sigma^2 | y_0, y_1, ..., y_T, D_1, ..., D_T) &= \sum_{t=1}^{T} \ell_t(\rho, \mu, \sigma^2 | y_t, y_{t-1}, D_t) \\ &= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\sum_{t=1}^{T}\frac{(y_t - \rho y_{t-1} - \mu D_t)^2}{\sigma^2}.\end{aligned} \tag{4.3}$$

To derive the maximum likelihood estimator of μ for a fixed value of ρ, denoted $\widehat{\mu}(\rho)$, we solve

$$\frac{\partial \ell(\rho, \mu, \sigma^2 | y_0, y_1, ..., y_T, D_1, ..., D_T)}{\partial \mu} = 0. \tag{4.4}$$

That yields:

$$\begin{aligned}\sum_{t=1}^{T}\frac{(y_t - \rho y_{t-1} - \widehat{\mu}(\rho) D_t) D_t}{\sigma^2} &= 0 \\ \widehat{\mu}(\rho) \sum_{t=1}^{T} D_t^2 &= \sum_{t=1}^{T} (y_t - \rho y_{t-1}) D_t \\ \widehat{\mu}(\rho) &= \frac{\sum_{t=1}^{T} (y_t - \rho y_{t-1}) D_t}{\sum_{t=1}^{T} D_t^2},\end{aligned} \tag{4.5}$$

It can further be noted that since $\sum_{t=1}^{T} D_t^2 = \sum_{t=1}^{T} D_t = \frac{T}{2}$ and $\sum_{t=1}^{T}(y_t - \rho y_{t-1})D_t = \sum_{t=\frac{T}{2}+1}^{T}(y_t - \rho y_{t-1})$, the expression for $\widehat{\mu}(\rho)$ can be simplified as

$$\widehat{\mu}(\rho) = \frac{\sum_{t=1}^{T}(y_t - \rho y_{t-1})D_t}{\sum_{t=1}^{T} D_t} = \left(\frac{T}{2}\right)^{-1} \sum_{t=\frac{T}{2}+1}^{T} (y_t - \rho y_{t-1}) = \tilde{y} - \rho \tilde{y}_{-1}, \tag{4.6}$$

where $\tilde{y} = (T/2)^{-1} \sum_{\frac{T}{2}+1}^{T} y_t$ and $\tilde{y}_{-1} = (T/2)^{-1} \sum_{\frac{T}{2}+1}^{T} y_{t-1}$ This shows that for a fixed value of ρ, the maximum likelihood estimator of the level-shift coefficient μ is simply the sample average of $y_t - \rho y_{t-1}$ over the subsample after the level-shift has occured.

Question 2

Using the expression for $\widehat{\mu}(\rho)$ in (4.6), the concentrated log-likelihood function ℓ_c is given by

$$\begin{aligned}
\ell_c(\rho, \sigma^2 | y_0, y_1, ..., y_T, D_1, ..., D_T) &= \ell(\rho, \widehat{\mu}(\rho), \sigma^2 | y_0, y_1, ..., y_T, D_1, ..., D_T) \\
&= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\sum_{t=1}^{T} \frac{(y_t - \rho y_{t-1} - \widehat{\mu}(\rho)D_t)^2}{\sigma^2} \\
&= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\sum_{t=1}^{T} \frac{(y_t - \rho y_{t-1} - (\tilde{y} - \rho\tilde{y}_{-1})D_t)^2}{\sigma^2} \\
&= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\sum_{t=1}^{T} \frac{(y_t - \tilde{y}D_t - \rho(y_{t-1} - \tilde{y}_{-1}D_t))^2}{\sigma^2}.
\end{aligned} \tag{4.7}$$

To find the maximum likelihood estimator of ρ, we solve

$$\frac{\partial \ell_c(\rho, \sigma^2 | y_0, y_1, ..., y_T, D_1, ..., D_T)}{\partial \rho} = 0, \tag{4.8}$$

which yields:

$$\begin{aligned}
\sum_{t=1}^{T} \frac{(y_t - \tilde{y}D_t - \widehat{\rho}(y_{t-1} - \tilde{y}_{-1}D_t))(y_{t-1} - \tilde{y}_{-1}D_t)}{\sigma^2} &= 0 \\
\widehat{\rho}\sum_{t=1}^{T}(y_{t-1} - \tilde{y}D_t)^2 &= \sum_{t=1}^{T}(y_t - \tilde{y}D_t)(y_{t-1} - \tilde{y}_{-1}D_t) \\
\widehat{\rho} &= \frac{\sum_{t=1}^{T}(y_t - \tilde{y}D_t)(y_{t-1} - \tilde{y}_{-1}D_t)}{\sum_{t=1}^{T}(y_{t-1} - \tilde{y}_{-1}D_t)^2},
\end{aligned} \tag{4.9}$$

which can also be written out as:

$$\widehat{\rho} = \frac{\sum_{t=1}^{T}(y_t - \tilde{y}D_t)(y_{t-1} - \tilde{y}_{-1}D_t)}{\sum_{t=1}^{T}(y_{t-1} - \tilde{y}_{-1}D_t)^2} = \frac{\sum_{t=1}^{T} y_t y_{t-1} - \frac{2}{T}\sum_{t=\frac{T}{2}+1}^{T} y_t \sum_{t=\frac{T}{2}+1}^{T} y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2 - \frac{2}{T}(\sum_{t=\frac{T}{2}+1}^{T} y_{t-1})^2}. \tag{4.10}$$

It can be noted that by defining $Y_t = y_t - \tilde{y}D_t$ and $Y_{t-1} = y_{t-1} - \tilde{y}_{-1}D_t$, the maximum likelihood estimator can be written as $\widehat{\rho} = (\sum_{t=1}^{T} Y_{t-1}^2)^{-1}(\sum_{t=1}^{T} Y_t Y_{t-1})$, which shows that the maximum likelihood estimator of the autoregressive coefficient has the usual expression but with y_t and y_{t-1} corrected for the level-shift.

#4.2 Stationarity and Forecasting in an ADL Model

Consider the ADL(2,1) model

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \gamma x_t + \epsilon_t, \qquad t = 1, 2, ..., T, \tag{4.11}$$

where $\epsilon_t \sim IID(0, \sigma_y^2)$ and given some initial values y_0 and x_0. Moreover, assume that the process for x_t is given by

$$x_t = \rho x_{t-1} + \eta_t, \tag{4.12}$$

where $|\rho| < 1$ and $\eta_t \sim IID(0, \sigma_x^2)$.

Question 1

It is assumed that $\phi_2 = 0.5$ and that $-1 < \phi_1 < 1$.

It should be noted that x_t is a stationary process with unconditional mean $E(x_t) = 0$ as $|\rho| < 1$. Therefore, the process for y_t is stationary if all characteristic roots have a modulus larger than one, so that they lie outside the unit circle. The values of ϕ_1 for which this condition is satisfied must be derived.

Using the lag-operator, the process for y_t can be written as

$$(1 - \phi_1 L - \phi_2 L^2) y_t = \gamma x_t + \delta + \epsilon_t. \tag{4.13}$$

The characteristic polynomial is given by $\theta(z) = 1 - \phi_1 z - \phi_2 z^2$ and the characteristic roots z, which solve the characteristic equation $\theta(z) = 0$, are given by

$$z = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}. \tag{4.14}$$

With $\phi_2 = 0.5$, the two characteristic roots are given by

$$z = -\phi_1 \pm \sqrt{\phi_1^2 + 2}. \tag{4.15}$$

It should be stated that as $\sqrt{\phi_1^2 + 2} > 0$ the two characteristic roots are real and the process for y_t is stationary if $|z| > 1$, such that the characteristic roots lie outside the unit circle (or, equivalently, the inverse characteristic roots lie inside the unit circle). That yields the inequalities:

$$z = -\phi_1 \pm \sqrt{\phi_1^2 + 2} < -1 \qquad \text{or} \qquad z = -\phi_1 \pm \sqrt{\phi_1^2 + 2} > 1. \tag{4.16}$$

Noting that $\phi_1 < \sqrt{\phi_1^2 + 2}$ for all values of ϕ_1, it holds that $z_1 = -\phi_1 - \sqrt{\phi_1^2 + 2} < 0$ and $z_2 = -\phi_1 + \sqrt{\phi_1^2 + 2} > 0$ for all values of ϕ_1. Therefore, for $\phi_1 \in]-1, 1[$ (such that $-\phi_1 + 1 > 0$ and $1 + \phi_1 > 0$) it is sufficient to solve the inequalities

$$\begin{aligned}
z_1 = -\phi_1 - \sqrt{\phi_1^2 + 2} &< -1 & \text{and} \quad z_2 = -\phi_1 + \sqrt{\phi_1^2 + 2} &> 1 \\
-\phi_1 + 1 &< \sqrt{\phi_1^2 + 2} & \text{and} \quad \sqrt{\phi_1^2 + 2} &> 1 + \phi_1 \\
\phi_1^2 + 1 - 2\phi_1 &< \phi_1^2 + 2 & \text{and} \quad \phi_1^2 + 2 &> 1 + \phi_1^2 + 2\phi_1 \\
-2\phi_1 &< 1 & \text{and} \quad 1 &> 2\phi_1 \\
\phi_1 &> -\frac{1}{2} & \text{and} \quad \frac{1}{2} &> \phi_1.
\end{aligned} \tag{4.17}$$

We conclude that for $\phi_2 = 0.5$ and $|\rho| < 1$, the process for y_t is stationary for $-\frac{1}{2} < \phi_1 < \frac{1}{2}$ and all values of γ.

Alternatively, the companion form of the model can be defined and the stationarity condition derived in terms of the eigenvalues of the companion matrix. The process for y_t is stationary if the eigenvalues of the companion matrix are all less than one, which leads to the same condition for ϕ_1 as above.

Question 2

The forecast of y_{T+1} conditional on the information set $\mathcal{I}_T = \{y_{-\infty}, ..., y_{T-1}, y_T, x_{-\infty}, ..., x_{T-1}, x_T\}$, which we denote $y_{T+1|T}$, is given by

$$\begin{aligned} y_{T+1|T} &= E(y_{T+1}|\mathcal{I}_T) \\ &= E(\delta + \phi_1 y_T + \phi_2 y_{T-1} + \gamma x_{T+1} + \epsilon_{T+1}|\mathcal{I}_T) \\ &= \delta + \phi_1 y_T + \phi_2 y_{T-1} + \gamma E(x_{T+1}|\mathcal{I}_T) + E(\epsilon_{T+1}|\mathcal{I}_T) \\ &= \delta + \phi_1 y_T + \phi_2 y_{T-1} + \gamma E(\rho x_T + \eta_{T+1}|\mathcal{I}_T) + 0 \\ &= \delta + \phi_1 y_T + \phi_2 y_{T-1} + \gamma \rho x_T + E(\eta_{T+1}|\mathcal{I}_T) \\ &= \delta + \phi_1 y_T + \phi_2 y_{T-1} + \gamma \rho x_T, \end{aligned} \tag{4.18}$$

where we have used that y_T, y_{T-1}, and x_T are in the information set $\mathcal{I}_T$ and that $E(\epsilon_{T+i}|\mathcal{I}_T) = E(\eta_{T+i}|\mathcal{I}_T) = 0$ for all $i \geq 1$.

Similarly, we find the forecast of y_{T+1} as

$$\begin{aligned} y_{T+2|T} &= E(y_{T+2}|\mathcal{I}_T) \\ &= E(\delta + \phi_1 y_{T+1} + \phi_2 y_T + \gamma x_{T+2} + \epsilon_{T+2}|\mathcal{I}_T) \\ &= \delta + \phi_1 E(y_{T+1}|\mathcal{I}_T) + \phi_2 y_T + \gamma E(x_{T+2}|\mathcal{I}_T) + E(\epsilon_{T+2}|\mathcal{I}_T) \\ &= \delta + \phi_1(\delta + \phi_1 y_T + \phi_2 y_{T-1} + \gamma\rho x_T) + \phi_2 y_T + \gamma E(\rho^2 x_T + \rho\eta_{T+1} + \eta_{T+2}|\mathcal{I}_T) + 0 \\ &= (1+\phi_1)\delta + (\phi_1^2 + \phi_2) y_T + \phi_1\phi_2 y_{T-1} + \phi_1\gamma\rho x_T + \gamma\rho^2 x_T + \rho E(\eta_{T+1}|\mathcal{I}_T) + E(\eta_{T+2}|\mathcal{I}_T) \\ &= (1+\phi_1)\delta + (\phi_1^2 + \phi_2) y_T + \phi_1\phi_2 y_{T-1} + (\phi_1 + \rho)\gamma\rho x_T. \end{aligned} \tag{4.19}$$

Question 3

Given that the stationarity condition is fullfilled, the forecasts $y_{T+k|T} = E(y_{T+k}|\mathcal{I}_T)$ converge to the unconditional mean of the y_t as the forecast horizon k increases:

$$y_{T+k|T} = E(y_{T+k}|\mathcal{I}_T) \to E(y_t) \quad \text{for } k \to \infty. \tag{4.20}$$

As $E(x_t) = 0$, the unconditional mean of y_t is given by

$$E(y_T) = \frac{\delta}{1-\phi_1-\phi_2} + \frac{\gamma E(x_t)}{1-\phi_1-\phi_2} = \frac{\delta}{1-\phi_1-\phi_2}. \tag{4.21}$$