Written exam for the M.Sc. in Economics, Winter 2012/13

Game theory

Final Exam/Elective Course/Master's Course

(3 hours, closed book exam)

22 January 2013

Question 1

In this exercise we model penalty shooting in soccer. There are two players: The goalkeeper G and the penalty taker P. We neglect the possibility of shooting (or standing) in the middle and assume that both have to choose a side: either "right" (R) or "left" (L). If both choose opposite sides, a goal results with probability 1 if P chooses R and with probability 0.9 if P chooses L. If both choose L, the scoring probability is 0.3. If both choose R, the scoring probability is 0.5.
The objective of P is to maximize the expected probability of a goal and the objective of G is to minimize the expected probability of a goal.

> *Question 1 and course description: This exercise tests (i) the ability to assess a game, (ii) knowledge of its formal representation, (iii) knowledge of solution the concepts Nash equilibrium, subgame perfect Nash equilibrium and Bayesian Nash equilibrium, (iv) understanding how the structure of the game affects the solution, (v) knowledge of how to approach games under uncertainty.*

a. Model this situation in two ways:

- Assume that both players move simultaneously. Model the situation as a strategic game and determine its Nash equilibrium/equilibria.
- Now G is very quick. Assume that he can observe which side P chooses and moves *after* P. Model this situation as extensive form game and determine a subgame perfect Nash equilibrium of this game.

  - Model: either the game table

	L	R
L	0.3,-0.3	0.9,-0.9
R	1,-1	0.5, -0.5

    or the formal decription of the game: set of players $N = \{P, G\}$; action sets $A_1 = A_2 = \{R, L\}$ and utility functions as in the table above
    Nash equilibrium: no pure strategy NE exists; the mixed NE is determined by the indifference conditions

$$0.3\alpha + 0.9 - 0.9\alpha = \alpha + 0.5 - 0.5\alpha \qquad (1)$$
$$0.3\beta + 1 - \beta = 0.9\beta + 0.5 - 0.5\beta \qquad (2)$$

    which gives P1 mixes over (L, R) $(5/11, 6/11)$ and P2 mixes $(4/11, 7/11)$
  - Model description: In the first stage P chooses R or L. In the second stage G chooses R or L. (alternative: game tree)
    The best response of G is to pick the same side as P, i.e. L in the subgame after P chooses L and R after P chooses R. Given this, it

is best for P to choose his strong side R. As we argued backwards, this NE is subgame perfect.

b. For the extensive form game of the previous subquestion:

- Write down the corresponding normal form game.
- Is there a Nash equilibrium (possibly mixed) which is not subgame perfect? Explain your answer briefly.

	LL	RL	LR	RR
L	0.3,-0.3	0.9,-0.9	0.3,-0.3	0.9,-0.9
R	1,-1	1,-1	0.5,-0.5	0.5,-0.5

where the strategy KM for G means take action K if P chooses L and action M if P1 choses R. A NE that is not subgame perfect is the following: P plays R, G plays RR with probability $\varepsilon > 0$ and LR with probability $1 - \varepsilon$. For ε small enough R is a best response by P and G plays his best response on the equilibrium path. However, in the subgame after L G's response is not optimal.

c. Now we go back to the simultaneous move game. G knows that P has shot his last 3 penalties to the left. G suspects that P might be "crazy" i.e. that P's objective might not be maximizing the scoring probability but simply to shoot left.

Assume G assigns probability 1/5 to the possibility that P is "crazy" and 4/5 to the possibility that P is a scoring-probability-maximizer as above. This belief is common knowledge. Derive a Bayesian Nash equilibrium in this game. (note: the previous three penalties are not part of the model!)

Briefly explain how the Bayesian Nash equilibrium of this game depends on the probability that P is "crazy", i.e. what happens for values different from 1/5.

G and his payoff is as in (a). Action spaces as in (a). P has the two types "crazy" and "scoring probabiltiy maximizer". The prior is that the crazy type has probability λ (=1/5). Payoffs of the crazy type are 1 after play $(L, \cdot)$ and 0 after play $(R, \cdot)$. Payoffs of "scoring probabiltiy maximizer" type as in (a).

A crazy type will choose L in every equilibrium. For λ high (in fact $\lambda > 5/11$), there is a pure strategy NE where G goes L and a non-crazy P shoots R. For smaller λ there is a mixed strategy equilibrium similar to the one above: G mixes as above but a non-crazy P has to put more probability than above on R to keep G indifferent (to make up for the crazy types). To make G indifferent we need

$$0.3*(\lambda+(1-\lambda)*\alpha)+(1-\lambda)*(1-\alpha) = 0.9*(\lambda+(1-\lambda)*\alpha)+0.5*(1-\lambda)*(1-\alpha).$$

For $\lambda = 1/5$ a non-crazy P has to put probability $7/22$ on L to satisfy the indifference condition.

Question 2

Consider the "hat game" we had in the lecture on knowledge. Here is a brief summary of the setting as a reminder:

> There are N players in a room and each wears a hat. The color of the hat is either black or white. Each player can see the color of the other hats but does not see the color of his own hat. An outside observer asks: "Do you know the color of your hat?" If a player knows, he raises his hand (and all other players can see who raises his hand). Then the outside observer asks the same question again and again players who know raise their hand. This is repeated several rounds.

Deviating from the case we had in the lecture, the outside observer announces the following before the first round: "At least one of you wears a black hat and at least one of you wears a white head." This announcement is true and it is common knowledge among the players that the outside observer tells the truth.

> *Question 2 and course description: This question tests the familiarity with the concept of common knowledge and its formalization.*

a. For now assume that $N = 5$ and assume that three players wear white hats and two wear black hats. Consider the event E: "at least two players wear a white hat". Is E known by all players? Is E common knowledge? Give and briefly explain the answer to these two questions at the following two points in the game

   - immediately after the announcement of the outside observer
   - after the first round.

   In both cases, the event is known by all players as every player sees at least two players with a white hat. E is not common knowledge after the announcement: Say players 1 to 3 have white hats, i.e. the state is (W, W, W, B, B). P1 knows E whenever at least two of the players 2 to 5 have white hats. P2 knows that P1 knows E whenever two of the players 3 to 5 have white hats. Hence, the state (W, W, W, B, B) is not in $K_2(K_1(E))$. Therefore, E cannot be common knowledge in state (W, W, W, B, B) after the initial announcement.
   After round 1, E is common knowledge: If only one player was wearing a white hat, he would have raised his hand in round 1. As no player did (and this is common knowledge), E is common knowledge.

b. How many rounds will it at most take until the first player(s) raise their hand? (this question is for general N and general distributions of white/black hats)

Does your answer differ from the solution we obtained in the setting of the lecture (where the outside observer only announced "At least one of you wears a black hat.")? Why (not)?

> Let j be the color worn by less people and N_j be the number of people with hats of color j. Then the N_jth round is the first where someone raises his hand: If no one raised his hand until the kth round, then it is common knowledge in round k that at least k people wear white and at least k people wear black. Hence, in round N_j each j wearer sees $N_j - 1$ hats of color j but knows that at least N_j hats of color j are around. So, he knows that he is wearing j.
>
> Consequently, it takes longest when the number of black and white hats are equal (or differ by 1 if N is odd). Rounding down for odd N, it takes at most $N/2$ rounds.
>
> This is different from the lecture setting: There it takes at most N rounds (in case all hats are black). The additional information that the outside observe gives leads to a faster diffusion of knowledge.

Question 3

There are three players (1,2,3) on a committee. They have to decide between three alternatives (a,b,c). There are only two possible preference profiles (R and R'):

$$
\begin{aligned}
R_1 &= R'_1 = a \succ b \succ c \\
R_2 &= b \succ c \succ a \\
R'_2 &= b \succ a \succ c \\
R_3 &= c \succ a \succ b \\
R'_3 &= a \succ c \succ b
\end{aligned}
$$

The social planner wants to implement the following choice function f:

$$f(R) = b \qquad f(R') = a$$

Question 3 and course description: This question tests the knowledge of implementation of decision rules in the solution concepts Nash equilibrium and subgame perfect Nash equilibrium.

a. Is the choice function f Nash equilibrium implementable? Either give an example game implementing it or show that it is not Nash equilibrium implementable.

> f violates monotonicity: going from R' to R, b does not improve in any players' preference ordering (but is now selected by the choice

rule). Monotonicity is a necessary condition for Nash equilibrium implementability. Hence, f is not NE implementable.

b. Is the choice function f subgame perfect Nash equilibrium implementable? Either give an example game or show that it is not.

yes: stage 1: P1 choses "b" or "pass"; if "pass" there is a stage 2 in which P3 chooses between a or c. Under R', P3 chooses a and therefore P1 will choose "pass". Under R, P3 chooses c and P1 will choose b as he preferes b to c.