Exam Solution Guide

Econometrics II

June 2018

Part 1
Forecasting the Price of Owner-Occupied Apartments

The Case The goal of this part of the exam is to estimate a univariate autoregressive (AR) model and use the model to forecast out-of-sample (log) changes in a price index for owner-occupied apartments in Denmark.

The Data Graphs of the data and relevant transformations must be shown in the exam. It must be noted that the level of the house prices is clearly non-stationary, but the first-difference appears somewhat stationary.

Econometric Theory The econometric theory must include the following:

(1) A precise definition and interpretation of the model considered and its properties. Specifically, a univariate autoregressive (AR) model must be presented. Furthermore, a precise definition of the stationarity condition, the out-of-sample forecasts, and the forecast variances must be given.

(2) A precise description of the estimator used, in particular a precise account of the assumptions used to derive the estimator. Specifically, the method of moments (MM) or the maximum likelihood (ML) estimators can be used.

(3) A precise account of the necessary assumptions for consistent estimation and valid inference. This includes a precise definition of the null hypotheses, test statistics, and asymptotic distributions used to test relevant hypotheses.

(4) The theory must be presented precisely and in a logical order with a consistent and correct notation.

Empirical Results The empirical results must include the following:

(1) A presentation and discussion of the relevant empirical results, so that the reader is able to understand the steps carried out in the process as well as the conclusions made.

(2) A description of the model selection process based on a general-to-specific approach, information criteria, or both.

(3) A discussion of whether the assumptions for consistent estimation and valid inference are satisfied for the estimated models. Specifically, this includes misspecification testing, which must be presented and discussed before statistical testing is carried out.

(4) A clear conclusion to the main question and a discussion of the limitations of the approach used to reach the conclusion. Specifically, the conclusion regarding the out-of-sample forecasts and the forecast variance must be presented and the limitations of the estimated models must be discussed in relation to the forecasts.

Part 2
Interest Rate Pass-Through

The Case The goal of this part of the exam is to use cointegration techniques to estimate the short-run and long-run pass-through from Danmarks Nationalbank's interest rate on certificates of deposits to banks' average deposit rate on loans to non-financial corporations. Furthermore, the hypothesis of full long-run interest rate pass-through must be tested.

The Data Graphs of the data and relevant transformations must be shown in the exam. It must be noted that the level of the interest rates are clearly non-stationary, but the first-differences appear somewhat stationary and they seem to move together over time indicating cointegration.

Econometric Theory The econometric theory must include the following:

(1) A precise definition and interpretation of the models considered and their properties. Specifically, an interpretation of cointegration must be presented along with a presentation of univariate autoregressive (AR) models used to test for unit roots and a single equation cointegration approach based on the Engle-Granger two-step procedure or the autoregressive distributed lag (ADL) and error-correction models (ECM).

(2) A precise description of the estimator used, in particular a precise account of the assumptions used to derive an estimator.

(3) A precise account of the necessary assumptions for consistent estimation and valid inference. This includes a precise definition of the null hypotheses, test statistics, and asymptotic distributions used to test relevant hypotheses.

(4) The theory must be presented precisely and in a logical order with a consistent and correct notation.

Empirical Results The empirical results must include the following:

(1) A presentation and discussion of the relevant empirical results, so that the reader is able to understand the steps carried out in the process as well as the conclusions made.

(2) A description of the model selection process based on a general-to-specific approach, information criteria, or both.

(3) A discussion of whether the assumptions for consistent estimation and valid inference are satisfied for the estimated models. Specifically, this includes misspecification testing, which must be presented and discussed before statistical testing is carried out.

(4) A clear conclusion to the main question and a discussion of the limitations of the approach used to reach the conclusion. Specifically, the conclusion regarding cointegration between the interest rates must be presented and the limitations of the single-equation cointegration approach must be discussed in relation to the conclusion.

Part 3
Volatility of Excess Stock Returns

The Case The goal of this part of the exam is to estimate a GARCH model extended with GARCH-in-mean, a threshold effect, and the risk-free interest rate entering the conditional variance for the return series for the Standard & Poor's 500 Stock Index. The empirical results must be compared to the results in Glosten *et al.* (1993).

The Data Graphs of the data and relevant transformations must be shown in the exam. It must be noted that there is volatility clustering in the log-returns.

Econometric Theory The econometric theory must include the following:

(1) A precise definition and interpretation of the extended GARCH model considered and its properties. Specifically, an interpretation of volatility clustering in the model, the stationarity condition, and the interpretation of the three relevant extensions of the classic GARCH model must be explained.

(2) A precise description of the estimator used, in particular a precise account of the assumptions used to derive an estimator.

(3) A precise account of the necessary assumptions for consistent estimation and valid inference. This includes a precise definition of the null hypotheses, test statistics, and asymptotic distributions used to test relevant hypotheses.

(4) The theory must be presented precisely and in a logical order with a consistent and correct notation.

Empirical Results The empirical results must include the following:

(1) A presentation and discussion of the relevant empirical results, so that the reader is able to understand the steps carried out in the process as well as the conclusions made.

(2) A description of the model selection process based on a general-to-specific approach, information criteria, or both.

(3) A discussion of whether the assumptions for consistent estimation and valid inference are satisfied for the estimated models. Specifically, this includes misspecification testing, which must be presented and discussed before statistical testing is carried out.

(4) A clear conclusion to the main question and a discussion of the limitations of the approach used to reach the conclusion. Specifically, the empirical results must be compared to those in Glosten *et al.* (1993).

Part 4
Theoretical Problem:

#4.1 The Asymptotic Variance of the OLS Estimator

Consider the processes for z_t:

$$z_t = \gamma + \phi z_{t-1} + \epsilon_t, \quad t = 1, 2, ..., T, \tag{4.1}$$

where $\epsilon_t \sim IID(0, \sigma^2)$, the initial value z_0 is given, and the parameter ϕ satisfies $|\phi| < 1$. Define the true parameters $\theta = (\gamma, \phi)'$.

The OLS estimator $\widehat{\theta} = (\widehat{\gamma}, \widehat{\phi})'$ is asymptotically normally distributed,

$$\sqrt{T}(\widehat{\theta} - \theta) \overset{d}{\to} N(0, V) \quad \text{for } T \to \infty, \tag{4.2}$$

where V is the asymptotic covariance matrix.

Question 1
Derive the asymptotic covariance matrix V as a function of the parameter values γ, ϕ, and σ^2.

Defining $x_t = (1, z_{t-1})'$, the model in (4.1) can be written as:

$$z_t = x_t'\theta + \epsilon_t. \tag{4.3}$$

The method of moments/OLS estimator is given by:

$$\widehat{\theta} = \left(\sum_{t=1}^{T} x_t x_t' \right)^{-1} \sum_{t=1}^{T} x_t z_t. \tag{4.4}$$

By decomposing the estimator into the true parameters, θ, and a sampling error,

$$\widehat{\theta} = \theta + \left(\sum_{t=1}^{T} x_t x_t' \right)^{-1} \sum_{t=1}^{T} x_t \epsilon_t, \tag{4.5}$$

it can be shown that

$$\sqrt{T}(\widehat{\theta} - \theta) = \left(\frac{1}{T} \sum_{t=1}^{T} x_t x_t' \right)^{-1} \frac{\sqrt{T}}{T} \sum_{t=1}^{T} x_t \epsilon_t \to N(0, \sigma^2 E(x_t x_t')^{-1}), \tag{4.6}$$

given that the stationarity condition holds ($|\phi| < 1$) and $\epsilon_t \sim IID(0, \sigma^2)$.

Next, we express the asymptotic variance $V = \sigma^2 E(x_t x_t')^{-1}$ as a function of the parameter values γ, ϕ, and σ^2. We find:

$$E(x_t x_t') = E\left(\begin{pmatrix} 1 \\ z_{t-1} \end{pmatrix}\begin{pmatrix} 1 & z_{t-1} \end{pmatrix}\right) = E\begin{pmatrix} 1 & z_{t-1} \\ z_{t-1} & z_{t-1}^2 \end{pmatrix} = \begin{pmatrix} 1 & E(z_{t-1}) \\ E(z_{t-1}) & E(z_{t-1}^2). \end{pmatrix} \tag{4.7}$$

Given stationarity of z_t, the infinite moving-average representation is given by:

$$z_t = \sum_{i=1}^{\infty} \phi^i \gamma + \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}, \tag{4.8}$$

and the unconditional mean and variance are given by $\mu = E(z_t) = \frac{\gamma}{1-\phi}$ and $V(z_t) = E((z_t - E(z_t))^2) = \frac{\sigma^2}{1-\phi^2}$. Using the moving average representation for z_{t-1} and that $\epsilon_t \sim IID(0, \sigma^2)$, such that $E(\epsilon_t) = 0$, $E(\epsilon_t^2) = \sigma^2$, and $E(\epsilon_t \epsilon_s) = 0$ for all $t \neq s$, we find:

$$\begin{aligned} E(z_{t-1}^2) &= E\left(\left(\sum_{i=1}^{\infty} \phi^i \gamma + \sum_{i=0}^{\infty} \phi^i \epsilon_{t-1-i}\right)^2\right) \\ &= \mu^2 + E\left(\left(\sum_{i=0}^{\infty} \phi^i \epsilon_{t-1-i}\right)^2\right) \\ &= \mu^2 + E\left(\left(\sum_{i=0}^{\infty} \phi^{2i} \sigma^2\right)^2\right) \\ &= \left(\frac{\gamma}{1-\phi}\right)^2 + \frac{\sigma^2}{1-\phi^2}. \end{aligned} \tag{4.9}$$

Consequently, the asymptotic variance V is given by:

$$V = \sigma^2 E(x_t x_t')^{-1} = \sigma^2 \begin{pmatrix} 1 & E(z_{t-1}) \\ E(z_{t-1}) & E(z_{t-1}^2) \end{pmatrix}^{-1} = \sigma^2 \begin{pmatrix} 1 & \frac{\gamma}{1-\phi} \\ \frac{\gamma}{1-\phi} & \left(\frac{\gamma}{1-\phi}\right)^2 + \frac{\sigma^2}{1-\phi^2} \end{pmatrix}^{-1}. \tag{4.10}$$

Question 2

Explain briefly *what the result in* (4.2) *can be used for.*

We use the result of asymptotic normality to construct confidence intervals and test hypotheses on the estimated parameters based on asymptotic inference. In particular, asymptotic normality implies what we often refer to as *standard inference*, which means that the asymptotic distribution of various tests are the standard distributions known from cross-sectional data. For example, the t-tests for the null hypotheses $\gamma = 0$ or $\phi = 0$ asymptocially follow a standard normal distribution under the null.

#4.2 Forecasting

Consider the model for s_t:

$$\Delta s_t = \phi_1 \Delta s_{t-1} + \phi_2 \Delta s_{t-2} + \epsilon_t, \tag{4.11}$$

for $t = 1, 2, ..., T$, where $\epsilon_t \sim iidN(0, \sigma^2)$ and the initial values are given. Define the information set $\mathcal{I}_t = (s_t, s_{t-1}, ...)$.

Question 1

Derive the forecast $s_{T+2|T} = E(s_{T+2}|\mathcal{I}_T)$.

It follows from the model that s_{T+1} and s_{T+2} can be written as:

$$s_{T+1} = s_T + \phi_1 \Delta s_T + \phi_2 \Delta s_{T-1} + \epsilon_{T+1} \tag{4.12}$$

$$s_{T+2} = s_{T+1} + \phi_1 \Delta s_{T+1} + \phi_2 \Delta s_T + \epsilon_{T+2}. \tag{4.13}$$

We derive the forecast of s_{T+2} as:

$$\begin{aligned}
s_{T+2|T} &= E(s_{T+2}|\mathcal{I}_T) \\
&= E(s_{T+1} + \phi_1 \Delta s_{T+1} + \phi_2 \Delta s_T + \epsilon_{T+2}|\mathcal{I}_T) \\
&= E((s_T + \phi_1 \Delta s_T + \phi_2 \Delta s_{T-1} + \epsilon_{T+1}) + \phi_1(\phi_1 \Delta s_T + \phi_2 \Delta s_{T-1} + \epsilon_{T+1}) + \phi_2 \Delta s_T + \epsilon_{T+2}|\mathcal{I}_T) \\
&= E(s_T + (\phi_1 + \phi_1^2 + \phi_2)\Delta s_T + (\phi_2 + \phi_1\phi_2)\Delta s_{T-1} + (1 + \phi_1)\epsilon_{T+1} + \epsilon_{T+2}|\mathcal{I}_T) \\
&= E(s_T|\mathcal{I}_T) + (\phi_1 + \phi_1^2 + \phi_2)E(\Delta s_T|\mathcal{I}_T) + (\phi_2 + \phi_1\phi_2)E(\Delta s_{T-1}|\mathcal{I}_T) \\
&\qquad + (1 + \phi_1)E(\epsilon_{T+1}|\mathcal{I}_T) + E(\epsilon_{T+2}|\mathcal{I}_T) \\
&= s_T + (\phi_1 + \phi_1^2 + \phi_2)\Delta s_T + (\phi_2 + \phi_1\phi_2)\Delta s_{T-1}.
\end{aligned} \tag{4.14}$$

The forecast can also be written as:

$$s_{T+2|T} = (1 + \phi_1 + \phi_1^2 + \phi_2)s_T + (\phi_1\phi_2 - \phi_1 - \phi_1^2)s_{T-1} - (\phi_2 + \phi_1\phi_2)s_{T-2}. \tag{4.15}$$

Question 2

Now assume that $\phi_2 = 0.5$ and that $\Delta s_T > 0$ and $\Delta s_{T-1} = -\Delta s_T$. For which values of ϕ_1 is the forecast $s_{T+2|T}$ greater than s_T?

The forecast $s_{T+2|T}$ is greater than s_T if:

$$s_T + (\phi_1 + \phi_1^2 + \phi_2)\Delta s_T + (\phi_2 + \phi_1\phi_2)\Delta s_{T-1} > s_T, \tag{4.16}$$

which is equivalent to:

$$(\phi_1 + \phi_1^2 + \phi_2)\Delta s_T + (\phi_2 + \phi_1\phi_2)\Delta s_{T-1} > 0. \tag{4.17}$$

Using $\phi_2 = 0.5$ and $\Delta s_{T-1} = -\Delta s_T$ yields the inequality:

$$(\phi_1 + \phi_1^2 + 0.5)\Delta s_T + (0.5 + 0.5 \cdot \phi_1)(-\Delta s_T) = (\phi_1^2 + 0.5 \cdot \phi_1)\Delta s_T > 0. \tag{4.18}$$

As $\Delta s_T > 0$, this inequality is satisfied when

$$\phi_1^2 + 0.5 \cdot \phi_1 > 0. \tag{4.19}$$

We find the roots of this second-order polynomial to be -0.5 and 0, so that (4.19) is satisfied for $\phi_1 < -0.5$ or $\phi_1 > 0$. We conclude that given the assumptions specified, the forecast $s_{T+2|T}$ is greater than s_T for $\phi_1 < -0.5$ or $\phi_1 > 0$.

#4.3 Moment Conditions in a Non-Linear Time Series Model

Consider the non-linear time series model:

$$r_t = \rho r_{t-1} + \left(\sqrt{\omega + \alpha r_{t-1}^2}\right) z_t, \qquad z_t \sim IID(0,1), \tag{4.20}$$

for $t = 1, 2, ..., T$ and where $-1 < \rho < 1$, $\omega > 0$, and $\alpha \geq 0$. Define the available information set $\mathcal{I}_t = (r_0, r_1, r_2, ..., r_t)$.

Question 1

Derive the conditional expectation $E(r_t|\mathcal{I}_{t-1})$ and the conditional variance $V(r_t|\mathcal{I}_{t-1}) = E\left((r_t - E(r_t|\mathcal{I}_{t-1}))^2|\mathcal{I}_{t-1}\right)$.

Using that r_{t-1} is included in the information set $\mathcal{I}_{t-1}$ and that $E(z_t|\mathcal{I}_{t-1}) = 0$, we derive the conditional expectation as:

$$\begin{aligned}
E(r_t|\mathcal{I}_{t-1}) &= E\left(\rho r_{t-1} + \left(\sqrt{\omega + \alpha r_{t-1}^2}\right) z_t \middle| \mathcal{I}_{t-1}\right) \\
&= E(\rho r_{t-1}|\mathcal{I}_{t-1}) + E\left(\left(\sqrt{\omega + \alpha r_{t-1}^2}\right) z_t \middle| \mathcal{I}_{t-1}\right) \\
&= \rho r_{t-1} + \left(\sqrt{\omega + \alpha r_{t-1}^2}\right) E\left(z_t|\mathcal{I}_{t-1}\right) \\
&= \rho r_{t-1} + \left(\sqrt{\omega + \alpha r_{t-1}^2}\right) \cdot 0 \\
&= \rho r_{t-1}.
\end{aligned} \tag{4.21}$$

Similarly, we use $E(z_t^2|\mathcal{I}_{t-1}) = 1$ and result above to derive the conditional variance:

$$\begin{aligned}
V(r_t|\mathcal{I}_{t-1}) &= E\left((r_t - E(r_t|\mathcal{I}_{t-1}))^2|\mathcal{I}_{t-1}\right) \\
&= E\left(\left(\rho r_{t-1} + \left(\sqrt{\omega + \alpha r_{t-1}^2}\right) z_t - \rho r_{t-1}\right)^2 \middle| \mathcal{I}_{t-1}\right) \\
&= E\left(\left(\left(\sqrt{\omega + \alpha r_{t-1}^2}\right) z_t\right)^2 \middle| \mathcal{I}_{t-1}\right) \\
&= E\left(\left(\sqrt{\omega + \alpha r_{t-1}^2}\right)^2 z_t^2 \middle| \mathcal{I}_{t-1}\right) \\
&= \left(\omega + \alpha r_{t-1}^2\right) E\left(z_t^2|\mathcal{I}_{t-1}\right) \\
&= \left(\omega + \alpha r_{t-1}^2\right) \cdot 1 \\
&= \omega + \alpha r_{t-1}^2.
\end{aligned} \tag{4.22}$$

Question 2

Specify a set of moment conditions which can be used to estimate the parameters $\theta = (\rho, \omega, \alpha)'$ by generalized method of moments (GMM) and explain why the moment conditions are valid.

[Hint: You can use the results from Question 1 to derive such moment conditions.]

To estimate the $K = 3$ parameters $\theta = (\rho, \omega, \alpha)'$ by GMM, we need a minimum of $R = 3$ valid moment conditions.

The result in (4.21) implies:

$$E(r_{t-1} - \rho r_{t-1} | \mathcal{I}_{t-1}) = 0. \tag{4.23}$$

For a set of instruments x_{t-1} included in the information set $\mathcal{I}_{t-1}$, i.e. $x_{t-1} \in \mathcal{I}_{t-1}$, the conditional expectation in (4.23) implies the moment conditions:

$$E((r_{t-1} - \rho r_{t-1}) x_{t-1}) = 0. \tag{4.24}$$

The validity of these moment conditions can be proved using the law of iterated expectations and the result in (4.23):

$$\begin{aligned} E((r_{t-1} - \rho r_{t-1}) x_{t-1}) &= E(E((r_{t-1} - \rho r_{t-1}) x_{t-1} | \mathcal{I}_{t-1})) & (4.25) \\ &= E(E(r_{t-1} - \rho r_{t-1} | \mathcal{I}_{t-1}) x_{t-1}) = E(0 \cdot x_{t-1}) = 0. & (4.26) \end{aligned}$$

Similarly, (4.22) implies the conditional expectation,

$$E((r_t - \rho r_{t-1})^2 - \omega - \alpha r_{t-1}^2 | \mathcal{I}_{t-1}) = 0, \tag{4.27}$$

from which we can derive the moment conditions:

$$E(((r_t - \rho r_{t-1})^2 - \omega - \alpha r_{t-1}^2) x_{t-1}) = 0. \tag{4.28}$$

The proof of the validity of these moments is equivalent to the one above. Natural candidates for the instruments could be $x_{t-1} = (1, r_{t-1}, r_{t-2})'$. That would give a total of $R = 6$ valid moment conditions, which is sufficient to estimate the $K = 3$ parameters in θ by GMM.