The Competence Description in Micro 3 says:

Game Theory has become a central analytic tool in much economic theory, e.g. within industrial organization, macroeconomics, international economics, labor economics, public economics and political economics.

The course aims at giving the student knowledge of game theory, non-cooperative as well as cooperative, and its applications in economic models.

The student who successfully completed the course will learn the basic game theory and will be enabled to work further with advanced game theory. The student will also learn how economic problems, involving strategic situations, can be modeled using game theory, as well as how these models are solved. The course intention is thus, that the student through this becomes able to work with modern economic theory, for instance within the areas of within industrial organization, macroeconomics, international economics, labor economics, public economics and political economics.

In the process of the course the student will learn about

- *Static games with complete information*
- *Static games with incomplete information*
- *Dynamic games with complete information*
- *Dynamic games with incomplete information*
- *Basic cooperative game theory.*

For each of these classes of games, the student should know and understand the theory, and learn how to model and analyze some important economic issues within the respective game framework.

More specifically, the students should know the theory and be able to work with both normal and extensive form games. They should know, understand and be able to apply the concepts of dominant strategies, iterative elimination of dominant strategies, as well as mixed strategies. The students should know the central equilibrium concepts in non-cooperative game theory, such as Nash Equilibrium and further refinements: Subgame-Perfect Nash Equilibrium, Bayesian Nash Equilibrium, Perfect Bayesian Equilibrium. They should understand why these concepts are central and when they are used, and be able to apply the relevant equilibrium and solution concepts.

Furthermore, the students should acquire knowledge about a number of special games and the particular issues associated with them, such as repeated games (including infinitely repeated games), auctions and signaling games.

The students should also understand and be able to apply the solution concepts of cooperative game theory, such as the core and the Shapley value. Furthermore, the students should also learn the basics of bargaining theory.

To obtain a top mark in the course the student must be able excel in all of the areas listed above.

In view of this, the grading of the exam should take as a point of departure, the short description of the solutions below

MICRO 3 EXAM JUNE 2009
QUESTIONS WITH SHORT ANSWERS
(The answers in this solution are often short/indicative, a good exercise should argue for these answers)

1. (a) Find *all* Nash equilibria in the following game

	L	R
T	$2,3$	$5,2$
B	$3,2$	$1,3$

Solution: There are no PSNE. The mixed eq can be determined as follows: assume that all pure strategies are played with positive probability and assign p as the probability that player 1 plays T and q as the probability that player 2 plays L.

		q	1-q
		L	R
r	T	$2,3$	$5,2$
1-r	B	$3,2$	$1,3$

Row player is indifferent between playing T and B iff

$$\begin{aligned} 2q+5(1-q) &= 3q+(1-q) \Leftrightarrow \\ q &= 4/5. \end{aligned}$$

Row player's best response is

$$BR_1(q)=r^*(q)\begin{cases} =0 \text{ if } q>4/5 \text{ (strategy B)} \\ \in[0,1] \text{ if } q=4/5 \text{ (any combination of } T \text{ and } B) \\ =1 \text{ if } q<4/5 \text{ (strategy T)} \end{cases}$$

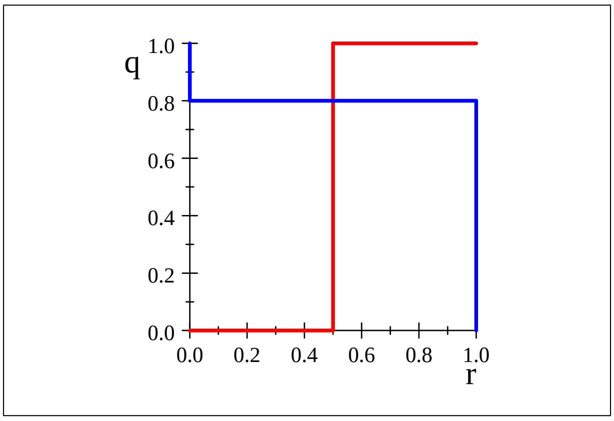
Column player is indifferent between playing L and R iff

$$3r+2(1-r)=2r+3(1-r),$$

that is, if the row player is mixing with the weight $r=1/2$. Column player's best response is

$$BR_2(r)=q^*(r)\begin{cases} =1 \text{ if } r>1/2 \text{ (strategy L)} \\ \in[0,1] \text{ if } r=1/2 \text{ (any combination of } L \text{ and } R) \\ =0 \text{ if } r<1/2 \text{ (strategy R)} \end{cases}$$

The intersection of BRs is (the BR of Player 1 is in blue, and the BR of player 2 is in red)



Therefore, the mixed strategy equilibrium is $[(1/2, 1/2)(4/5, 1/5)]$, i.e. the row player plays T with prob 1/2, and the column player plays L with prob 4/5.

(b) Solve the following game by eliminating strictly dominated strategies. Find all Nash equilibria of this game.

	t_1	t_2	t_3
s_1	$5, 2$	$1, 3$	$7, 2$
s_2	$5, 2$	$2, 4$	$2, 1$
s_3	$4, 1$	$1, 0$	$6, -1$

Solution: Elimination iteration: t_2 dominates t_3, then s_2 dominates s_3, then t_2 dominates t_1, then s_2 dominates s_1. Solution $\{s_2, t_2\}$. As Nash equilibrium strategies survive at any step of iterated elimination procedure, there is a unique NE in this game, i.e. the strategy profile $\{s_2, t_2\}$ that survives the iterated elimination.

(c) Consider the extensive-form game represented by the game tree on Figure 1:

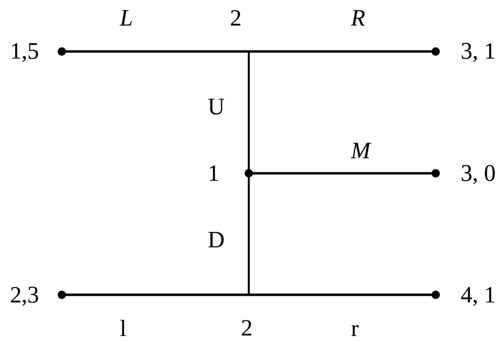


Figure 1

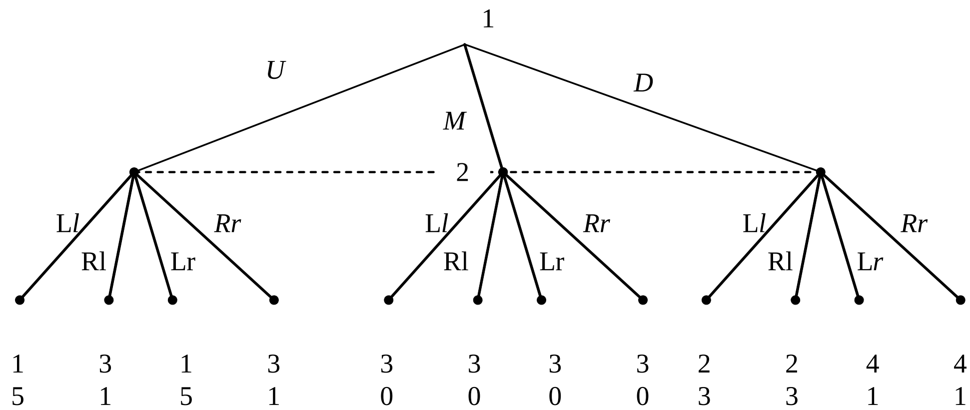
i. How many subgames are in this game? Find all subgame perfect Nash equilibria.
Solution: 2 subgames not including the game itself. SPNE is (M, Ll).

ii. Rewrite this game in normal form and find all pure-strategy Nash equilibria.
Solution:

	Ll	Rl	Lr	Rr
U	1,$\underline{5}$	$\underline{3}$, 1	1,$\underline{5}$	3, 1
M	$\underline{3}$,$\underline{0}$	$\underline{3}$,$\underline{0}$	3,$\underline{0}$	3,$\underline{0}$
D	2,$\underline{3}$	2,$\underline{3}$	$\underline{4}$, 1	$\underline{4}$, 1

There are two pure-strategy NE in this game, (M, Ll) and (M, Rl)

iii. Suggest an extensive form game that corresponds to the normal form you presented in ii) but differs from Figure 1. Comment.
Solution: For example,



The extensive form representation of the game may contain more information than the normal form, e.g., on the order of moves, etc. Therefore there could be multiple extensive form games that correspond to a single normal form game.

2. Two regions of a country $i = 1, 2$ are allowed to set their own local tax, t_1 and t_2 respectively. The tax is lump sum and is imposed on the citizens living in the respective region. The citizens "vote by foot" by moving from the region with higher taxes to the region with lower taxes. Their mobility is however imperfect (because moving is associated with some costs, like getting a new job, a housing, etc.). More precisely, given t_1 and t_2, the share of people settling in region 1 is

$$s_1 = \frac{1}{2} + a(t_2 - t_1),$$

and the share of people settling in region 2 is

$$s_2 = \frac{1}{2} + a(t_1 - t_2),$$

where parameter $a > 0$ reflects the sensitivity of people to the difference in taxation (you may assume that a is sufficiently low so that both s_1 and $s_2 \in [0, 1]$). Both regions maximize their total tax collection less the quadratic cost of collecting taxes

$$U_i = t_i * s_i - \frac{t_i^2}{2}, \quad i = 1, 2.$$

(a) Assume that the regions set their tax levels simultaneously and non-cooperatively. Find the Nash equilibrium tax levels t_1^{NE} and t_2^{NE} in both regions. How do equilibrium tax levels depend on a? Provide an economic intuition to your answer.

Solution: Region 1 chooses its tax level to solve

$$\max_{t_1} t_1 * s_1 - \frac{t_1^2}{2} = \max_{t_1} t_1 \left(\frac{1}{2} + a(t_2 - t_1) \right) - \frac{t_1^2}{2}.$$

The FOC is

$$\frac{1}{2} + a(t_2 - t_1) - at_1 - t_1 = 0,$$

which yields the BR of region 1 to the tax level in the region 2

$$t_1(t_2) = BR_1(t_2) = \frac{1 + 2at_2}{2 + 4a}. \tag{1}$$

Similarly, the BR of region 2 is

$$t_2(t_1) = BR_2(t_1) = \frac{1 + 2at_1}{2 + 4a}. \tag{2}$$

Solving the system of equations (1) and (2) together yields

$$t_1^{NE} = t_2^{NE} = \frac{1}{2(1 + a)}.$$

The larger is a the lower are the tax levels in both regions. That is, the more sensitive are the citizens to the difference in taxes, i.e. the more they are willing to move to a "tax haven", the stronger is the tax competition between the two regions and the lower are the resulting tax levels.

(b) Find the levels of taxes t_1^{so} and t_2^{so} that maximize the joint payoff of the two regions. Comment on the difference between your results in a) and b).

Solution: The joint payoff of both regions is maximized over t_1 and t_2

$$\begin{aligned}\max_{t_1,t_2} U_1 + U_2 &= \max_{t_1,t_2} \left(t_1 * s_1 - \frac{t_1^2}{2}\right) + \left(t_2 * s_2 - \frac{t_2^2}{2}\right) \\ &= \max_{t_1,t_2} \left(t_1\left(\frac{1}{2} + a(t_2 - t_1)\right) - \frac{t_1^2}{2}\right) + \left(t_2\left(\frac{1}{2} + a(t_1 - t_2)\right) - \frac{t_2^2}{2}\right)\end{aligned}$$

The FOC of this maximization problem is a system of two equations

$$\begin{aligned}\text{wrt to } t_1 \quad &: \quad \frac{1}{2} + a(t_2 - t_1) - at_1 - t_1 + at_2 = 0, \\ \text{wrt to } t_2 \quad &: \quad at_1 + \frac{1}{2} + a(t_1 - t_2) - at_2 - t_2 = 0,\end{aligned}$$

Solving this system (for example, by noticing that the equations in it are symmetric) yields

$$t_1^{so} = t_2^{so} = \frac{1}{2}.$$

So the tax levels that maximize the joint payoff of the two regions are above the tax levels in NE for any $a > 0$. The reason is that lower tax in region j causes the citizens move from i to j, which lowers the tax base (and payoff) in i. In the Nash equilibrium in a) the regions do not account for this effect, and the tax competition presses the taxes "too much" down. This effect is however accounted for in b) - by maximizing joint payoff the regions internalize the negative externality which is imposed on region i by lowering the tax in region j.

From now on set $a = 1$ and assume that the tax setting game between the regions is repeated for infinitely many periods. The regions discount future at the rate of $\delta = 8/11$.

(c) Suggest a subgame-perfect equilibrium of this infinite game that would allow the regions to support the tax levels t_i^{so}, $i = 1, 2$ in each period (do not forget to prove that your suggested equilibrium is subgame-perfect).

Solution: For example, consider the following (grim trigger) strategy profile:
Normal phase: In the very first period set taxes equal to t_i^{so}. If in the previous period both regions have chosen t_i^{so}, $i = 1, 2$, choose t_i^{so}. If in the previous period at least one region has chosen anything different from t_i^{so}, $i = 1, 2$, enter a punishment phase.
Punishment phase: Set tax rate equal to t_i^{NE} $i = 1, 2$ forever, no matter what your opponent does.
Let's show that this is a SPNE. Start with the normal phase. If both regions stick to this strategy, each of them gets a per-period payoff of

$$U_i^{so} = t_i^{so} * s_i^{so} - \frac{(t_i^{so})^2}{2} = \left(\frac{1}{2}\left(\frac{1}{2} + 1 * (\frac{1}{2} - \frac{1}{2})\right) - \frac{(\frac{1}{2})^2}{2}\right) = \frac{1}{8},$$

which results in net present value of future discounted payoffs being equal to

$$\left(\frac{1}{8}\right)(1 + \delta + \delta^2 + ...) = \frac{1}{8(1-\delta)} = \frac{1}{8(1 - 8/11)} = \frac{11}{24}.$$

If instead, one of them, say, region 1, chooses to deviate, then its best one-period deviation is its best response to $t_2^{so} = 1/2$. By using equation (1) with $a = 1$ we have

$$t_1^d = BR_1(1/2) = \frac{1 + 2 * \frac{1}{2}}{2 + 4} = \frac{2}{6} = \frac{1}{3}. \tag{3}$$

so that the payoff of region 1 this period is

$$U_1^d = t_1^d * \underbrace{\left[\frac{1}{2} + (t_2^{so} - t_1^d)\right]}_{s_1} - \frac{\left(t_1^d\right)^2}{2} = \frac{1}{3}\left(\frac{1}{2} + (\frac{1}{2} - \frac{1}{3})\right) - \frac{\left(\frac{1}{3}\right)^2}{2} = \frac{1}{6}$$

The next period the game reverts to the eternal Nash equilibrium with taxes

$$t_1^{NE} = t_2^{NE} = \frac{1}{2(1+1)} = \frac{1}{4}$$

and the payoff of

$$U_i^{NE} = t_i^{NE} * s_i^{NE} - \frac{\left(t_i^{NE}\right)^2}{2} = \frac{1}{4} * \frac{1}{2} - \frac{\left(\frac{1}{4}\right)^2}{2} = \frac{3}{32}.$$

Therefore, the discounted payoff of region 1 from deviation is

$$\frac{1}{6} + \frac{3}{32}\delta\left(1 + \delta + \delta^2 + ...\right) = \frac{1}{6} + \frac{3}{32}\frac{\delta}{1-\delta} = \frac{1}{6} + \frac{3}{32} * \frac{8/11}{3/11} = \frac{1}{6} + \frac{1}{4} = \frac{10}{24}.$$

As

$$\frac{11}{24} > \frac{10}{24},$$

region 1 would not deviate. The same is true for the second region in the normal phase. Note that none of the regions wants to deviate in the punishment phase as they play stage game NE. Therefore the proposed strategy profile is SPNE.

3. Anna and Bo are involved in a car accident. The accident happened in such a way that there could be only two possible cases: either only Anna is guilty in the accident, or both Bo and Anna are guilty. Anna knows exactly if she is the only one guilty or if it is their joint fault. Bo does not know whether it is Anna who guilty or both of them, he only knows that she may be the only guilty one with probability 50%. The accident happened close to a surveillance camera, so if it is reported to the insurance companies of Anna and Bo, the truth will be discovered. Anna offers Bo an "on-the-place" compensation payment of either 2 or 6 (there are only those two options available to Anna). Bo can either accept or reject this payment. If he accepts, then the payment is made, and they do not report the accident to their insurance companies. If Bo rejects, the accident gets reported, and the insurance companies investigate the case. If only Anna is guilty, her insurance premium rises, incurring a cost on her of 6, and on top of it, she has to pay Bo a compensation of 5. If it is found that it is their joint guilt, there is no compensation for Bo, and the insurance premiums rise by 3 for both of them.

   (a) Complete the extensive form of this signalling game, represented on the figure below (i.e., who is the sender? who is the receiver? what are the payoffs when the t_2 type of Sender chooses SIX, and the receiver chooses a in the right information set). How many strategies does each player have, and what are they?

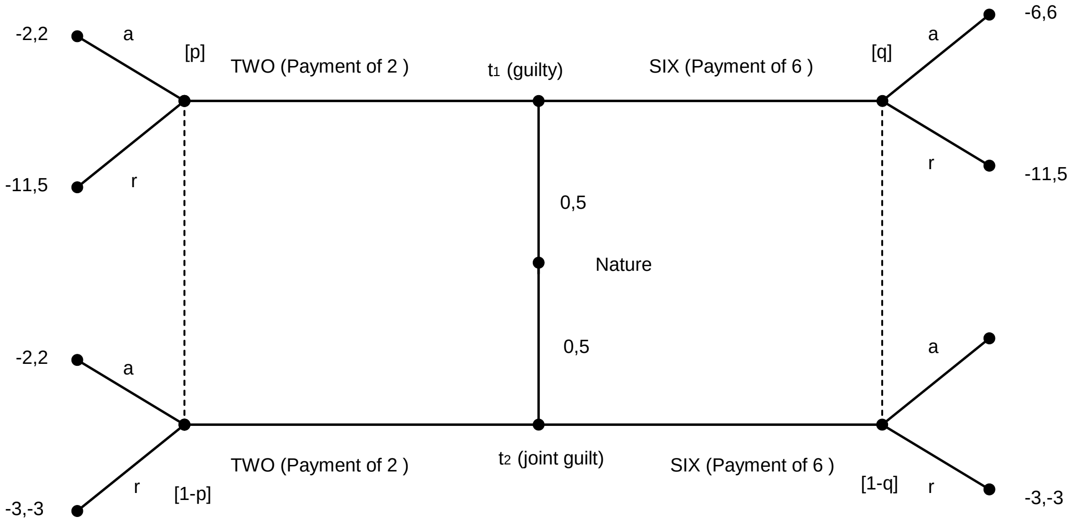
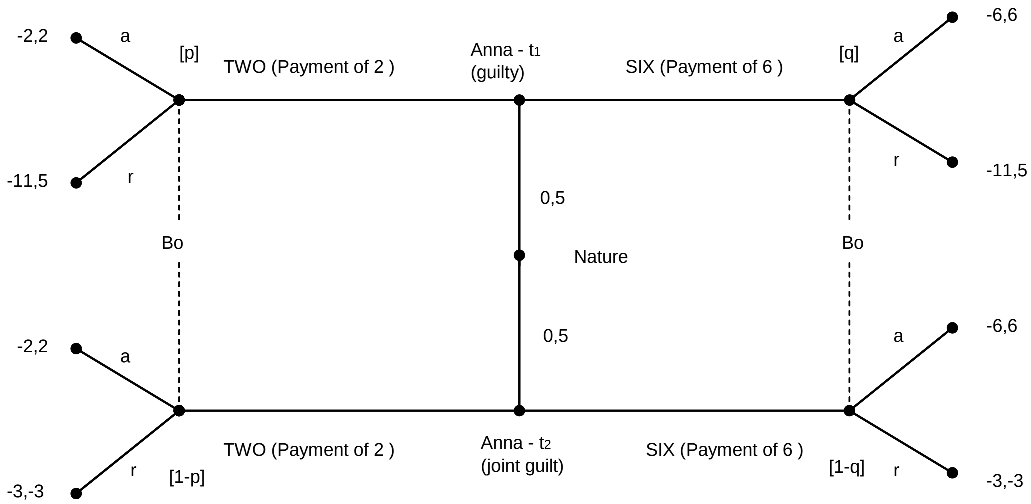


Figure 2.

Solution: Sender is Anna, receiver is Bo, the payoffs when the t_2 type of Sender chooses SIX, and the receiver chooses a in the right information set are (-6,6). Anna has two types, and her strategy should prescribe a signal for each of her types, therefore she has 4 strategies: (TWO, TWO), (TWO, SIX), (SIX, TWO) and (SIX, SIX), where the first action in the pair is for t_1 and the second - for t_2. Bo controls two information sets with two possible actions in each of them, therefore he also has 4 strategies: (aa), (ar), (ra) and (rr), where the first action describes the choice in the left information set, and the second- in the right information set.



(b) Find a pooling Perfect Bayesian equilibrium of this game.

Solution: Assume pooling on "TWO". Then $p = 0.5$, and Bo chooses a in the left info set, as

$$\text{payoff from } a = 2 * \frac{1}{2} + 2 * \frac{1}{2} > 5 * \frac{1}{2} - 3 * \frac{1}{2} = \text{payoff from } r$$

In the right info set Bo gets

$$6q + 6(1-q) = 6$$

if he **a**ccepts, and

$$5q - 3(1-q) = 8q - 3 \leq 8 - 3 = 5$$

if he **r**ejects. So, Bo will always choose r in the right info set, no matter what is the value of q.Notice that whatever Bo chooses in the right information set, Anne will not deviate from (TWO, TWO). Therefore, there is a set of pooling equilibria of a kind $[(TWO, TWO), (a, a), p = 0.5, q]$ for any $q \in [0.1]$.

One can show that there are no other pooling pure strategy perfect Bayesian equilibria in this game

(c) Formulate Signalling Requirement 6 (the intuitive criterion). Does the equilibrium you found in b) survive Signalling Requirement 6? Explain.

Solution: The intuitive criterion (also called signaling requirement 6 by Gibbons) can be found in Gibbons, page 239. It restricts off eq path beliefs, saying that the messages that are equilibrium-dominated should be associated with zero beliefs, *is possible*.

All pooling equilibria proposed in (b) survive the intuitive criterion. The reason is that the signal SIX is equilibrium-dominated for *both* types of sender. So here the "if possible" qualifier comes into play, as it is not possible to have zero beliefs in all nodes of the right information set.

4. Torbjorn, Sven and Nikolaj are working on a project for a design competition. They are free to form groups of any size among themselves. Torbjorn and Sven are both very experienced, so if they both are in the group, the resulting project gets the first prize of DKK 3000 (no matter if it is 2- or 3-person group). If they are not working in the same group, any group of 2 persons which includes either Torbjorn or Sven gets one of the second prizes of DKK 1000. If Torbjorn or Sven work alone, each of them gets one of the 3rd prizes of DKK 500. Finally, Nikolaj is not as experienced as the others, so if he works alone, he cannot get any prize.

(a) Think of this situation as of coalitional game with transferable utilities. Write down the value of each coalition.

Solution: Denote Torbjorn by T, Sven by S and Nikolaj by N. The values for all coalitions are

$$\begin{aligned} v(TSN) &= v(TS) = 3000 \\ v(SN) &= v(TN) = 1000 \\ v(T) &= v(S) = 500 \\ v(N) &= 0 \end{aligned}$$

(b) Find the core of this game.

Solution: Denote the core allocation by (x_T, x_S, x_N). Then in order for it to be in the core it must satisfy the following system

$$\begin{aligned} x_T + x_S + x_N &= v(TSN) = 3000 && (4) \\ x_T + x_S &\geq 3000 && (5) \\ x_S + x_N &\geq v(SN) = 1000 && (6) \\ x_T + x_N &\geq v(TN) = 1000 && (7) \\ x_T &\geq v(T) = 500 && (8) \\ x_S &\geq v(S) = 500 && (9) \\ x_N &\geq v(N) = 0 && (10) \end{aligned}$$

Last 3 inequalities ensure each member's payoff should be nonnegative. Then equations (4) and (5) result in $x_N = 0$. Taking this into account, inequalities (6) and (7) result in

$$\begin{aligned} x_S &\geq 1000 \\ x_T &\geq 1000 \end{aligned}$$

Therefore in the core Nikolaj gets 0, and Torbjorn and Sven share 3000 so that each of them gets at least 1000.

$$\begin{aligned} x_T + x_S &= 3000 \\ x_S &\geq 1000 \\ x_T &\geq 1000 \\ x_N &= 0. \end{aligned}$$