Written Resit Exam for M.Sc. in Economics

Winter 2010/2011

Investment Theory

4. January 2011

Master course

3 hours written exam with closed books

Corrections

Exercise 1.

In the exercise the distribution of the dividend at date $t+1$ depends on the dividend at date $t-1$ only. Therefore the evolution of dividends is described by two independent Markov chains. However solving the exercise assuming that the distribution of the dividend at date $t+1$ depends on the dividend at date t only should be considered equally correct. Indeed below the exercise is solved assuming that.

(a) There are two NPVs depending. They are found by solving the two Bellman equations in the two unknowns V_L, V_H

$$\begin{aligned} V_L &= D_L + \frac{\pi_L V_L + (1-\pi_L)V_H}{1+\rho} \\ V_H &= D_H + \frac{(1-\pi_H)V_L + \pi_H V_H}{1+\rho}. \end{aligned}$$

The solution is

$$\begin{pmatrix} V_L \\ V_H \end{pmatrix} = \frac{1+\rho}{(2-\pi_L-\pi_H+\rho)\rho} \begin{pmatrix} 1-\pi_H+\rho & 1-\pi_L \\ 1-\pi_H & 1-\pi_L+\rho \end{pmatrix} \begin{pmatrix} D_L \\ D_H \end{pmatrix}.$$

(b) There are three possible strategies: (1). never invest; (2) invest in cast the dividend is D_H; and, (3) always invest. The value of (1) is zero. The value of (2) is found by solving the following two Bellman equations in the two unknowns F_L, F_H

$$\begin{aligned} F_L &= \frac{\pi_L F_L + (1-\pi_L)(V_H - I)}{1+\rho} \\ F_H &= V_H - I \end{aligned}$$

The solution is

$$\begin{pmatrix} F_L \\ F_H \end{pmatrix} = \begin{pmatrix} \frac{1-\pi_L}{1-\pi_L+\rho}(V_H - I) \\ V_H - I \end{pmatrix}.$$

The value of (3) is $G_L = V_L - I$ and $G_H = V_H - I$.

(1) is the optimal strategy for $G_H = F_H < 0$. (1), (2) and (3) are the optimal strategies for $F_H = 0$. (2) is the optimal strategy for $F_L > G_L$ and $F_H > 0$. (2) and (3) are the optimal strategies for $F_L = G_L$ and $F_H > 0$. (3) is the optimal strategy for $F_L < G_L$ and $F_H > 0$.

(c) At date t there are two NPVs. They are found by solving the two Bellman equations in the two unknowns P_L, V_H where P_L is the value of a passive project in case of D_L

$$\begin{aligned} P_L &= \frac{\pi_L P_L + (1-\pi_L)(V_H - R)}{1+\rho} \\ V_H &= D_H + \frac{(1-\pi_H)(P_L - E) + \pi_H V_H}{1+\rho} \end{aligned}$$

The solution is

$$\begin{pmatrix} P_L \\ V_H \end{pmatrix} = \frac{1}{(2-\pi_L-\pi_H+\rho)\rho} \begin{pmatrix} 1-\pi_H+\rho & 1-\pi_L \\ 1-\pi_H & 1-\pi_L+\rho \end{pmatrix} \begin{pmatrix} -(1-\pi_L)R \\ (1+\rho)D_H - (1-\pi_H)E \end{pmatrix}.$$

(d) The NPVs are found by solving the two Bellman equations in two unknowns V_L, V_H

$$\begin{aligned} V_L &= -E \\ V_H &= D_H + \frac{(1-\pi_H)V_L + \pi_H V_H}{1+\rho}. \end{aligned}$$

The solution is

$$\begin{aligned} V_L &= -E \\ V_H &= \frac{(1+\rho)D_H - (1-\pi_H)E}{1-\pi_H+\rho}. \end{aligned}$$

(e) There are three suspension and reactivation strategies: (1) never suspend and reactivate; (2) Suspend in case of D_L, never reactivate; and, (3) Suspend in case of D_L, reactivate in case of D_H. We have found NPV of (1) in (a), (2) in (d), and (3) in (c). Theses expressions can be compared though it is a mess.

(f) The value of the option to invest is found by using the the NPVs for the project with optimal suspension and reactivation strategies as in (b).

(g) By filling in the numbers in the different expressions in (a), (d) and (c). First for the suspension and reactivation strategies the optimal strategy is the strategy is not to use suspension and reactivation. The NPVs are $V_L = 150$ and $V_H = 450$. Since the investment costs are 500, the optimal strategy is never to invest.

Exercise 2.

(a) An example could be buying a machine - perhaps a slot machine - that produces n units of a good. The investment cost is the cost of the machine, P/n is the price of good, death corresponds to the machine breaking down and the exit cost is simply the cost of getting rid of the machine. Another example could be entry in a market. The investment cost is the cost of establishing a brand in the market, P is the profit of being in the market, death corresponds to collaps of the market, perhaps because of the introduction of another superior good and the exit cost is the damage to brand.

(b) There is only one strategy, namely the entry strategy. A possible cut-off strategy is

$$\begin{cases} P < P^* & \text{wait (with the investment)} \\ P \geq P^* & \text{invest.} \end{cases}$$

The strategy implies that

$$F(P) = \begin{cases} ? & \text{for } P < P^* \\ V(P) - I & \text{for } P \geq P^* \end{cases}$$

where $V(P)$ has to satisfies $P \to 0 \Rightarrow V(P) \to$ *expected payment*, no bubbles and $F(P)$ has to satisfy $P \to 0 \Rightarrow F(P) \to 0$, value matching (VM) and smooth pasting (SP).

(c) For $V(P)$ the Bellman equation is

$$\rho V(P) = P + \frac{1}{dt} T E(dq) + \frac{1}{dt} E(dV(P))$$

where $dq = 0$ with probability $1 - \lambda dt$ and $dq = -1$ with probability λdt. Ito's Lemma gives

$$dV(P) = (\alpha P V'(P) + \frac{1}{2}\sigma^2 P^2 V''(P))dt + \sigma P V'(P) dz + (0 - V(P))dq.$$

Therefore

$$E(dV(P)) = (\alpha P V'(P) + \frac{1}{2}\sigma^2 P^2 V''(P) - \lambda V(P))dt$$

Using the expression for $E(dV(P))$ in the Bellman equation and rearrangering gives a differential equation in V

$$\frac{1}{2}\sigma^2 P^2 V''(P) + \alpha P V'(P) - (\rho + \lambda) V(P) + P - \lambda T = 0.$$

Similarly the differential equation in F for $P < P^*$ is

$$\frac{1}{2}\sigma^2 P^2 F''(P) + \alpha P F'(P) - \rho F(P) = 0.$$

(d) The mathematical solution to the differential equation in V is

$$B_1 P^{\beta_1'} + B_2 P^{\beta_2'} + \frac{P}{\rho + \lambda - \alpha} + \frac{\lambda}{\rho + \lambda} T$$

where $\beta_1' > 1$ and $\beta_2' < 0$ are the two roots in

$$\frac{1}{2}\sigma^2 (\beta' - 1)\beta' + \alpha \beta' - (\rho + \lambda) = 0.$$

The solution is only meaningful for $\alpha < \rho + \lambda$. Similarly the mathematical solution to the differential equation in F is

$$A_1 P^{\beta_1} + A_2 P^{\beta_1}$$

where $\beta_1 > 1$ and $\beta_2 < 0$ are the two roots in

$$\frac{1}{2}\sigma^2(\beta-1)\beta + \alpha\beta - \rho \;=\; 0.$$

The solutions is only meaningful for $\alpha < \rho$.

(e) By use of $P \to 0 \Rightarrow V(P) \to$ *expected payment* and no bubbles the economic solution for V is found

$$V(P) \;=\; \frac{1}{\rho+\lambda-\alpha}P - \frac{\lambda}{\rho+\lambda}T.$$

This is simply the expected discounted value of getting P until the project dies minus the expected discounted value of cleaning up when the project dies.

(f) By use of no bubbles the economic solution for F is found

$$F(P) \;=\; A_1 P^{\beta_1}.$$

This is the value of the option to invest in case P is below P^*, so the optimal strategy is to wait.

(g) (VM) gives

$$A_1(P^*)^{\beta_1} \;=\; \frac{1}{\rho+\lambda-\alpha}P^* - \frac{\lambda}{\rho+\lambda}T - I.$$

(SP) gives

$$\beta_1 A_1(P^*)^{\beta_1-1} \;=\; \frac{1}{\rho+\lambda-\alpha}.$$

Dividing (VM) with (SP) gives

$$\frac{P^*}{\beta_1} \;=\; P^* - (\rho+\lambda-\alpha)(I + \frac{\lambda}{\rho+\lambda}T)$$

so

$$P^* \;=\; \frac{\beta_1}{\beta_1-1}(\rho+\lambda-\alpha)(I + \frac{\lambda}{\rho+\lambda}T).$$

A_1 can be found by using the expression for P^* in (VM) or (SP) and isolating A_1.