Macro C - exam solutions (Feb 17, 2014)

General remarks

Please grade each item of each question between 0 and 10 points. Thus the maximum possible grade of the exam is 100 (since there are two questions and each has five subquestions or items).

Mathematical errors reduce the grade for the item in which they were made, not for subsequent items that carry the mistake (unless the student arrives at an infeasible solution and is unaware of the inconsistency, e.g. stating that capital holdings are negative).

1 Problem 1

a) The wage and interest rate are determined by imposing equilibrium in factor markets where firms competitively demand labor and capital from households. Thus the student needs to have profit function for firms

$$\left(\frac{K_t^T}{L_t^T}\right)^{\gamma}(K_t^j)^{\alpha}(L_t^j)^{1-\alpha} - w_t L_t^j - r_t K_t^j$$

From FOC of firms' problem of maximizing profits we get

$$\begin{aligned}\left(\frac{K_t^T}{L_t^T}\right)^{\gamma}(1-\alpha)(K_t^j)^{\alpha}(L_t^j)^{-\alpha} &= (1-\alpha)k_t^{\alpha+\gamma} = w_t \\ \left(\frac{K_t^T}{L_t^T}\right)^{\gamma}\alpha(K_t^j)^{\alpha-1}(L_t^j)^{1-\alpha} &= \alpha k_t^{\alpha-1+\gamma} = r_t\end{aligned}$$

where we impose that in equilibrium all firms work with the same capital labor ratio, k, which must be equal to the ratio of aggregate capital to labor.

Student gets two points for correctly stating objective function of firms. Two points for each FOC correctly derived (partial credit if wrongly derived but there is some intuition). Two points for correctly identifying market wage and interest rate times capital holdings as payments to households (in per capita terms).

Two points for correct intuition on the term $\left(\frac{K_t^T}{L_t^T}\right)^{\gamma}$. This tells that, when $\gamma > 0$, there are positive externalities from aggregate capital intensity into individual firms' productivity. Thus the higher $\left(\frac{K_t^T}{L_t^T}\right)^{\gamma}$, the higher is the output that a given firm obtains from its use of labor and capital. Because this is an effect that takes place in the aggregate, individual firms do not internalize that by using more capital they are having a positive effect on the productivity of other firms, that is why this is an externality (and a positive one).

b) One point for the no-Ponzi game condition (both expressions, with total assets or assets per capita, are correct):

$$\lim_{t\to\infty} e^{-\int_0^t (r_s-n)ds} a_t = \lim_{t\to\infty} e^{-\int_0^t r_s ds} A_t \geq 0$$

One point for correct intuition: we rule out schemes in which one household issues debt and rolls it over forever. This allows for debt, but total debt cannot grow at a rate faster than the interest rate (thus it cannot be rolled over entirely). Equivalently debt in per capita terms cannot grow at a rate faster than $r-n$.

One point for correctly stating each of control (c) and state variables (a).

Hamiltonian (it is irrelevant if set up as current value or present value, what matters is that the FOC are correct for each setup) (one point for Hamiltonian, three for FOC):

$$\begin{aligned}
H_t &= \frac{c_t^{1-\theta}}{1-\theta} e^{-(\rho-n)t} + \lambda_t \left(w_t + r_t a_t - c_t - n a_t\right) \\
H_t^c &= \frac{c_t^{1-\theta}}{1-\theta} + \mu_t \left(w_t + r_t a_t - c_t - n a_t\right)
\end{aligned}$$

with $\mu_t \equiv \lambda_t e^{(\rho-n)t}$.

Student gets full points if stating FOC assuming an interior solution (even if there is no explicit assumption of this, i.e. no penalty from failing to consider corner solution):

$$\begin{aligned}
\frac{dH_t^c}{dc_t} &= c_t^{-\theta} - \mu_t = 0 \\
\dot{\mu}_t &= -\frac{dH_t^c}{da_t} + (\rho-n)\mu_t = -\mu_t(r_t-\rho) \\
\lim_{t\to\infty} e^{-\rho t}\mu_t a_t &= 0
\end{aligned}$$

Note that the law of motion of the state variable is also a FOC (derivative of Hamiltonian with respect to costate variable λ_t or μ_t). Not writing it has no penalty. If using H then FOC should be adjusted to that formulation.

One point for deriving (not showing from memory) the Euler equation, or Keynes-Ramsey condition:

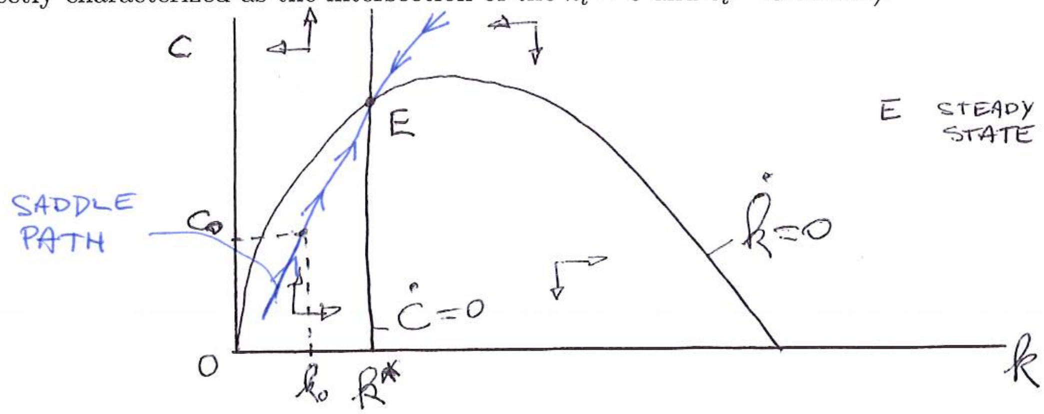
$$\frac{\dot{c}_t}{c_t} = \frac{r_t-\rho}{\theta}$$

And one point for correct interpretation. This is that consumption (in per capita terms) is increasing/falling over time as long as interest rate is above/below rate at which future consumption is discounted, and that $\frac{1}{\theta}$, the instantaneous elasticity of substitution (inverse of coefficient of relative risk aversion), measures the response of consumption growth rate to a given difference between r_t and ρ.

c) In equilibrium, and given that all households are identical, aggregate debt must be equal to zero always (not just in steady state) because there is no reason why a given household will borrow from another (e.g. if borrowing is optimal, then nobody lends). This implies that the dynamics of the aggregate economy in equilibrium can be described by having $a_t = k_t$, i.e. capital is the relevant state variable. Three points for correctly explaining this.

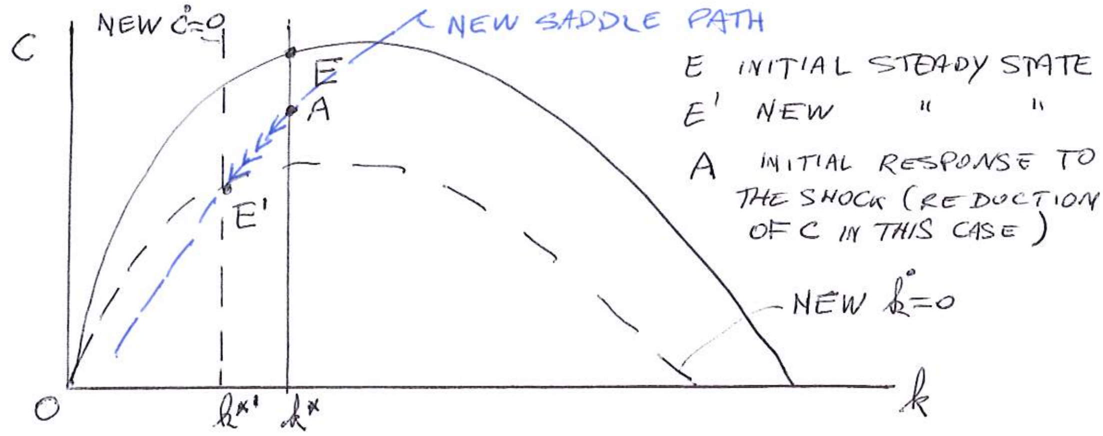
Steady state is characterized by $\dot{k}_t = \dot{c}_t = 0$. (one point for this). Thus $\dot{k}_t = 0$ implies that $c_t = w_t + r_t k_t - n k_t = k_t^{\alpha+\gamma} - n k_t$. (one point for this). $\dot{c}_t = 0$ implies that $r_t = \alpha {k_t}^{\alpha-1+\gamma} \rho$ (one point for this). This pins down the steady state capital labor ratio, $\rho/\alpha)^{\frac{1}{\alpha+\gamma-1}}$ (one point for this, even if only writing $r^* = f'(k^*)$ as long as it is explained, and not just written out of the blue).

Three points for correct phase diagram showing the $\dot{k}_t = 0$ and $\dot{c}_t = 0$ curves and the local dynamics of the variables in the four quadrants they define. The phase diagram should also have the saddle path of convergent dynamics to the steady state (and this correctly characterized as the intersection of the $\dot{k}_t = 0$ and $\dot{c}_t = 0$ curves).



Credit also if diagram uses multiplier instead of c in the vertical axis (but only if using current value Hamiltonian).

d) Here five points for correct graphical description of shock. This requires some analytical analysis. First to note that a decrease in γ lowers the $\dot{k}_t = 0$ curve, one point for this (note that this is not a parallel shift downwards as when a lump sum tax is imposed, if student draws a parallel shift then no point). Then a decrease in γ decreases steady state capital per capita ($k^* = (\rho/\alpha)^{\frac{1}{\alpha+\gamma-1}}$), since we are told that initially $k^* > 1$ (one point for this). One point for correctly showing where new steady state is, one point for correctly showing that the adjustment implies a jump of consumption to the new saddle path, and one point if correctly has no jump in capital when the shock takes place (credit also if this is not explicitly stated but graphically there is no change in capital initially).



Two point for correctly stating that it is not possible to say whether consumption jumps upwards or downwards at the time of the shock. This depends on parameters.

Thee points for explanation that should include the following (partial credit for partial explanation): a) shock reduces productivity thus the interest rate is reduced at initial steady state, this means that capital has to decrease to return interest rate to level of ρ, b) consumption must jump as is the only variable that can be adjusted discreetly when there is a shock, its new level depends on the relative effects of the decrease in the $\dot{k}_t = 0$ curve which tends to make consumption jump downwards to adjust to production possibilities, and the shift to the left of the $\dot{c}_t = 0$, which tends to make consumption jump upwards to adjust capital to new desired lower level.

e) Two points for correctly stating that the planner would choose a different allocation of resources because he/she internalizes the externality and thus would have more resources devoted to investment than the decentralized economy (i.e. steady state capital would be higher for the planner). Two points for saying that this implies that the decentralized economy is inefficient since welfare can be increased by adopting the central planner's allocation of resources. Two points for discussion, that should mention the failure of the first welfare theorem, which implies that decentralized equilibrium is not Pareto efficient, and/or some of the previous arguments of this paragraph.

Two points for describing the tax/subsidy policy can be implemented in the decentralized equilibrium to restore efficiency. This requires a subsidy to the return to capital financed by a lump sum tax. The size of the subsidy should be such that the steady state capital per capita is equal to the level chosen by planner. Two points for explanation (no need to show with math, but the relevant equation here is $(1+\tau)\alpha k^{\alpha-1+\gamma}$, where τ

is the subsidy) that the subsidy, by increasing marginal return to capital, gives incentive to invest. If subsidy is chosen at the right level this incentive will lead to replicate the allocation of the social planner.

2 Problem 2

a) Given that objective function is quadratic and equations that characterize inflation and output growth are linear we know that the optimal policy rule will be linear in the observables. These are θ, and ϵ. (one point for this).

The candidate policy rule is:

$$m = \psi_0 + \psi_\theta \theta + \psi_\epsilon \epsilon$$

Since the rule is credible, private inflation expectations are given by:

$$\pi^e = E[\pi|\theta] = E[m|\theta] = \psi_0 + \psi_\theta \theta$$

In equilibrium, inflation and output growth are given by:

$$\begin{aligned} \pi &= \psi_0 + \psi_\theta \theta + \psi_\epsilon \epsilon \\ x &= \theta + (\psi_\epsilon - 1)\epsilon \end{aligned}$$

Plugging these outcomes in the loss function and minimizing with respect to parameters ψ_X characterizes optimal rule. It is immediate to note that this rule must satisfy (no penalty if derived through maximization instead of using insight):

$\psi_0 = 0$ and $\psi_\theta = 0$ because they only show in inflation outcome and target inflation is $\bar{\pi} = 0$.

With this the expected loss only depends on ψ_ϵ:

$$E[L(\pi, x)] = \frac{1}{2}\left(\psi_\epsilon^2 \sigma_\epsilon + \lambda^D(\sigma_\theta + (\psi_\epsilon - 1)^2 \sigma_\epsilon)\right)$$

FOC with respect to ψ_ϵ gives

$$\psi_\epsilon + \lambda^D(\psi_\epsilon - 1) = 0$$

with solution

$$\psi_\epsilon = \frac{\lambda^D}{1 + \lambda^D}$$

(two points for correctly deriving policy rule).

Equilibrium inflation and output growth follow directly

$$\begin{aligned}\pi^C &= \frac{\lambda^D}{1+\lambda^D}\epsilon \\ x^C &= \theta - \frac{1}{1+\lambda^D}\epsilon\end{aligned}$$

(two points for correctly deriving equilibrium inflation and output growth).

With discretion, a credible policy must satisfy the conditions of being ex post optimal, i.e. that m is chosen after observing shocks to minimize loss function, and that expectations are rational such that on average the private sector is not surprised by ex post incentives from central bank.

Government chooses m after observing θ, π^e, and ϵ. Policy is chosen such that

$$\frac{dL}{dm} = 0$$

(one point if ex post objective is correct) which gives FOC

$$m + \lambda^D(\theta + m - \pi^e - \epsilon - \bar{x}) = 0$$

Expected inflation is

$$\pi^e = E[m|\theta] = \lambda^D(\bar{x} - \theta) > 0$$

(one point if expected inflation is correctly derived, not just written from memory)

Replacing this on FOC and then on output gives equilibrium inflation and output

$$\begin{aligned}\pi^D &= \lambda^D(\bar{x} - \theta) + \frac{\lambda^D}{1+\lambda^D}\epsilon \\ x^D &= x^C = \theta - \frac{1}{1+\lambda^D}\epsilon\end{aligned}$$

(two points if equilibrium is correctly derived)

One point if explains that difference comes from discretion creating an incentive to create surprise inflation (as long as, and this is assumed, $\bar{x} > \theta$). This incentive is correctly internalized by private sector (they have rational expectations), and leads to expect high inflation. Given this expectation, the monetary authority finds optimal to validate it and have higher inflation, although this leads to lower welfare since there is no gain in output stabilization and a loss from higher inflation.

b) If the krone is pegged to the euro, then money growth, m, and with it inflation, will be given by europe's policy and inflation. In particular this means that $\pi^P = \pi^{eu} = \frac{1}{1+\lambda^{eu}}\epsilon^{eu}$ (superscript P is for peg). Since inflation expectations are formed before observing π^{eu}, this means that

$$\pi^E = E[\pi^P|\theta] = 0$$

Two points for correctly finding this. Now, plugging π^P and expected inflation into equations that characterize behavior of Danish economy gives equilibrium inflation and output growth rates

$$\begin{aligned}
\pi^P &= \pi^{eu} = \frac{1}{1+\lambda^{eu}}\epsilon^{eu} \\
x^P &= \theta + \frac{1}{1+\lambda^{eu}}\epsilon^{eu} - \epsilon
\end{aligned}$$

Three points for correctly finding this. Only one point if answer is $\pi^P = 0$ and $x = \theta - \epsilon$, and this only if "derived" and not just written from memory.

Three points if correctly finding, using math, that expected loss under the peg is not always lower than under discretion, and that this depends on parameters. In particular if λ^D is very high, then society dislikes so much output volatility that would not prefer the peg. (partial credit if comparison of loss functions but not providing a clear answer)

Two points if reason that there is a trade-off with the peg that reduces inflation relative to discretion, but at the cost of higher output volatility (coming both from full exposure to local supply shocks, and from volatility in Europe's inflation rate). Thus the welfare effects of a peg relative to discretion depend on the valuation that society gives to policy objectives.

c) Under a simple rule such that $m = a + b\theta$, expected inflation would be

$$\pi^E = E[a + b\theta|\theta] = a + b\theta$$

One point for this. And plugging this result into equations that characterize behavior of Danish economy gives inflation and output growth rates (superscript S for simple rule)

$$\begin{aligned}
\pi^S &= a + b\theta \\
x^S &= \theta - \epsilon
\end{aligned}$$

Plugging equilibrium in expected loss function gives

$$E[L(\pi, x)] = \frac{1}{2}\left(a^2 + b^2\sigma_\theta + \lambda^D(\bar{x}^2 + \sigma_\theta + \sigma_\epsilon)\right)$$

Clearly optimal policy calls for $a = b = 0$, thus equilibrium is given by

$$\begin{aligned} \pi^S &= 0 \\ x^S &= \theta - \epsilon \end{aligned}$$

Three points for correct derivation (does not need to set up expected loss function if reasoned correctly that a and b only show in inflation, but not on output growth rate.

Three points for correctly showing that this rule is always preferable to the peg, since it has lower inflation and lower output volatility than the peg. Obviously the answer does not depend on volatilities and other parameters.

Three points for correct explanation that should note that the optimal rule manages to pin down inflation expectations to the target level of zero. Given this, output volatility is high (relative to discretion), but always lower than with the peg, since in that case output volatility reflected volatility in inflation in the eurozone, and that is (at least weakly) positive. Since inflation and output volatility are both lower, this rule should always be (at least weakly) preferred to peg.

d) In this case policy will be chosen ex post, but by adequately choosing a central banker's preferences it might be possible to improve welfare with respect to the discretion outcome. Denoting λ^{CB} the preferences of the chosen CB we know, from point a) that equilibrium inflation and output growth are given by

$$\begin{aligned} \pi^{CB} &= \lambda^{CB}(\bar{x} - \theta) + \frac{\lambda^{CB}}{1 + \lambda^{CB}}\epsilon \\ x^{CB} &= \theta - \frac{1}{1 + \lambda^{CB}}\epsilon \end{aligned}$$

(three points for correct solution for equilibrium inflation and output growth)

To characterize optimal choice of λ^{CB} we have to look at ex ante loss function.

$$\begin{aligned} \max_{\lambda^{CB}} \quad & \frac{1}{2}E\left[\left(\lambda^{CB}(\bar{x} - \theta) + \frac{\lambda^{CB}}{1 + \lambda^{CB}}\epsilon\right)^2 + \lambda^D\left(\theta - \frac{1}{1 + \lambda^{CB}}\epsilon - \bar{x}\right)^2\right] \\ = \quad & \frac{1}{2}\left[\lambda^{CB^2}(\bar{x}^2 + \sigma_\theta) + \left(\frac{\lambda^{CB}}{1 + \lambda^{CB}}\right)^2 \sigma_\epsilon + \lambda^D\left((\bar{x}^2 + \sigma_\theta) + \left(\frac{1}{1 + \lambda^{CB}}\right)^2\right)\sigma_\epsilon\right] \end{aligned}$$

The FOC is

$$\lambda^{CB}(\bar{x}^2 + \sigma_\theta) + (\lambda^{CB} - \lambda^D)\frac{\sigma_\epsilon}{(1 + \lambda^{CB})^3} = 0$$

By inspecting this FOC at $\lambda^{CB} = 0$ and $\lambda^{CB} = \lambda^D$ we find that the optimum satisfies

$0 < \lambda^{CB} < \lambda^D$. (four points for correctly showing this).

Three points for economic intuition of result, which relates to the fact that a more conservative central banker will lead to an equilibrium with lower inflation bias and lower inflation volatility, but higher inflation volatility. When evaluating the FOC at $\lambda^{CB} = \lambda^D$ the second term vanishes because discretionary policy reflected optimal inflation output volatility trade-off. But there is the gain from lower inflation bias, the term $\bar{x}^2 + \sigma_\theta$, that results from choosing somebody more conservative. This explains why $\lambda^{CB} < \lambda^D$. When FOC is evaluated at $\lambda^{CB} = 0$, there is no gain from reducing the inflation bias, since an extremely conservative central banker pegs inflation at zero. But the output inflation trade-off is very distorted, making this choice not optimal. This is reflected in term, $-\lambda^D \frac{\sigma_\epsilon}{(1+\lambda^{CB})^3}$. This explains why $0 < \lambda^{CB}$.

e) Here we only need to show that welfare is higher under an independent central banker than under the simple rule of c), since we already showed that this rule is preferable to the peg in b). But since we showed that the optimal choice of λ^{CB} implies $0 < \lambda^{CB}$, this means that welfare under the independent central bank is higher than under the simple rule of b), because the equilibrium in that case corresponds to the one that we would get under an extremely conservative central banker with $\lambda^{CB} = 0$ (see equilibrium equations above and compare to equilibrium equations for c)). Six points for clear explanation that can be a narrative like the one I presented, or using some math.

Four points for discussion that should note that, as stated above, it is optimal to have a more conservative central banker since this reduces inflation bias, but not an extremely conservative one since this loses all benefits from output stabilization, which is what happens when the simple rule of c) is followed.