Solution Macro C (Final exam)

4 January 2012

Academic aim

At the end of the course, the student should be able to demonstrate:
Understanding of the main model frameworks for long-run macroeconomics. This includes the Diamond model with overlapping generations in discrete time and the Ramsey model in continuous time.
Proficiency in the application of the concepts and methods from these frameworks, including competence in dynamic optimization and dynamic analysis in discrete and continuous time.
Understanding of the role of expectations and basic knowledge of macroeconomic models with forwardlooking expectations under both perfect foresight and uncertainty and rational expectations.
Proficiency in the application of the related concepts and methods.
Competence in analyzing a macroeconomic problem, where the above-mentioned concepts and methods are central, that is competence in solving such models and explaining in economic terms the results and implications and how they derive from the assumptions of the model.
The particularly good performance, corresponding to the top mark, is characterized by a complete fulfilment of these learning objectives.

Problem A – The Ramsey model

1)

The problem of the representative household is to choose a consumption path $(c_t)_{t=0}^{\infty}$ in order to maximize the intertemporal utility function:

$$U_0 = \int_0^{\infty} \frac{c_t^{1-\theta} - 1}{1-\theta} \cdot e^{-(\rho - n)\cdot t} dt$$

subject to the evolution of household wealth per worker:

$$\dot{a}_t = (r_t \cdot (1-\tau) - n) \cdot a_t + w_t + v_t - c_t$$

and a No Ponzi game condition (restricting the asymptotic evolution of a_t), taking initial wealth (a_0) as given. The No Ponzi game condition can be ignored when deriving the optimal *growth rate* of consumption over time. We solve the problem using *optimal control theory*.

At first, write up the present value Hamiltonian:

$$\mathcal{H} = \frac{c_t^{1-\theta} - 1}{1-\theta} \cdot e^{-(\rho - n)\cdot t} + \lambda_t \cdot \left((r_t \cdot (1-\tau) - n) \cdot a_t + w_t + v_t - c_t\right)$$

The intratemporal first order condition with respect to the control (c_t) variable is given by:

$$\frac{\partial \mathcal{H}}{\partial c_t} = c_t^{-\theta} \cdot e^{-(\rho-n)\cdot t} - \lambda_t = 0$$

From this first order condition it follows that:

$$\lambda_t = c_t^{-\theta} \cdot e^{-(\rho-n)\cdot t} \Rightarrow ln\lambda_t = -\theta \cdot lnc_t - (\rho - n) \cdot t$$

Differentiating this expression with respect to time we get:

$$\frac{\dot{\lambda}_t}{\lambda_t} = \frac{dln\lambda_t}{dt} = -\theta \cdot \frac{\dot{c}_t}{c_t} - (\rho - n)$$

where $\dot{c}_t$ and $\dot{\lambda}_t$ are the time derivatives.

The intertemporal first order condition with respect to the state variable (a_t) is given by:

$$\frac{\partial \mathcal{H}}{\partial a_t} = -\dot{\lambda}_t \Rightarrow \lambda_t \cdot (r_t \cdot (1-\tau) - n) = -\dot{\lambda}_t$$

from which it follows:

$$\frac{\dot{\lambda}_t}{\lambda_t} = -(r_t \cdot (1-\tau) - n)$$

Equating the two expressions for $\dot{\lambda}_t/\lambda_t$ above we get:

$$-\theta \cdot \frac{\dot{c}_t}{c_t} - (\rho - n) = -(r_t \cdot (1-\tau) - n) \Rightarrow$$

$$\frac{\dot{c}_t}{c_t} = \frac{r_t \cdot (1-\tau) - \rho}{\theta}$$

which is the Keynes Ramsey rule (the Euler equation) describing the optimal evolution of consumption over time. In addition to this Euler condition the optimal solution also contains a transversality condition, which ensures that the level of consumption is consistent with the intertemporal budget constraint (the No Ponzi game condition).

2)

Equation (A.1) is the Keynes Ramsey rule describing the optimal evolution of consumption over time. It contains the main trade-off of the representative household when deciding on the time pattern of consumption:

- On the one hand, the household prefers a smooth consumption pattern over time (since $\theta > 0$). Thus, consumption in any given period will not depend on current income, given total lifetime earnings.

- On the other hand, the household will take advantage of a potential gap between the return to postponing consumption (given by the after tax return on savings: $r_t \cdot (1-\tau)$) and the cost of postponing consumption (given by the rate of impatience: ρ)

Whenever $r_t \cdot (1-\tau) > \rho$ the household will take advantage of the high return to saving, and accept that consumption is growing over time. Conversely, $r_t \cdot (1-\tau) < \rho$ implies that the household will plan with a consumption level, which is falling over time.

A higher value of θ (meaning that consumption smoothing becomes more important) implies that a given deviation between $r_t \cdot (1-\tau)$ and ρ will result in a flatter consumption profile. In the extreme case where $\theta \to \infty$ the household only cares about smoothing out consumption implying that consumption is constant over time.

We can not determine from the Keynes Ramsey rule however, how the level of consumption depends on the after-tax interest rate, due to offsetting income and substitution effects (and an effect on the present value of future labour income which reinforces the substitution effect).

Equation (A.2) is an arbitrage equation stating that the after-tax return to bonds ($r_t \cdot (1-\tau)$) must equal the after-tax return to physical capital (given by $(R_t - \delta) \cdot (1-\tau)$), since financial and physical capital are implicitly assumed to be perfect substitutes and the return to both kinds of assets are taxed.

Equation (A.3) is the condition describing optimal demand for physical capital, stating that firms demand capital to the point where the marginal product of capital ($f'(k_t)$) equals the price of capital (the real rental rate R_t).

Equation (A.4) is the capital accumulation equation, stating that physical capital per worker accumulates according to gross investment, which equals saving (since we consider a closed economy), net of replacement investment (consisting of depreciation and labour force growth). Further saving per worker is given by that part of output (income) per worker which is not consumed.

By inserting equation (A.3) in (A.2) and inserting this in equation (A.1) we get:

$$(1-\tau) \cdot r_t = (1-\tau) \cdot (f'(k_t) - \delta) \Rightarrow$$

$$\frac{\dot{c}_t}{c_t} = \frac{(f'(k_t) - \delta) \cdot (1-\tau) - \rho}{\theta}$$

The second equation is identical to (A.4).

3)

At first we can find the conditions for $\dot{c}_t = 0$ and $\dot{k}_t = 0$:

$$\dot{c}_t = 0 \Rightarrow (f'(k_t) - \delta) \cdot (1-\tau) - \rho = 0$$

(in addition to the trivial solution $c_t = 0$). From this we define the steady state capital intensity as the k^* satisfying:

$$f'(k^*) = \delta + \frac{\rho}{1-\tau}$$

Now, let's consider the evolution of consumption, when the economy is outside steady state.

If $k_t > k^*$ the interest rate is low (compared to the steady state interest rate) due to diminishing returns to capital, and consumption is falling over time (according to the Keynes Ramsey rule). Conversely, if $k_t < k^*$ the interest rate is relatively high and consumption is growing over time.

The other steady steady state requirement ($\dot{k}_t = 0$) implies:

$$\dot{k}_t = f(k_t) - c_t - k_t \cdot (n + \delta) = 0 \Rightarrow$$

$$c_t = f(k_t) - k_t \cdot (n + \delta)$$

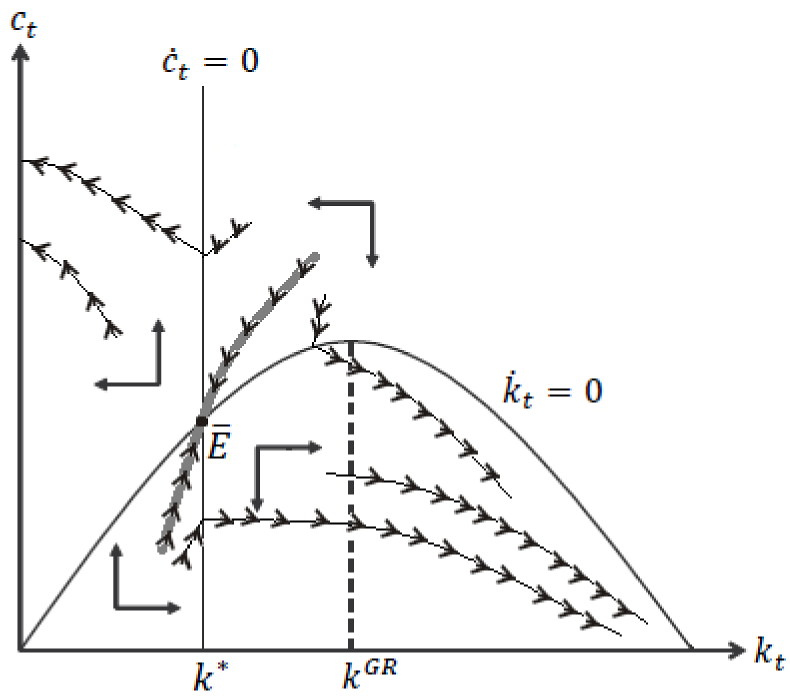
stating that when $\dot{k}_t = 0$ consumption per worker equals the part of output per worker which isn't used for replacement investment. This relationship between k_t and c_t is showed in figure 1.

Notice that if consumption is initially above this curve then saving (and thereby investment) is relatively low and the capital stock is falling over time. If conversely, consumption is initially below this curve then saving is high and the capital stock will increase over time.

The movements of the economy, mentioned above, are illustrated by the arrows in figure 1. Notice that the $\dot{c}_t = 0$ line will always be located to the left of k^{GR} (the capital stock which maximizes consumption per worker in steady state – defined by the condition: $f'(k^{GR}) = n + \delta$). The arrows indicate, that the steady state is saddle path stable, i.e. that the economy will only converge towards steady state *if the economy initially starts out on the saddle path*.

We can rule out paths that start out above the saddle path since in this case the economy will (in finite time) reach a point where $k_t = 0$, whereby the level of consumption will fall (discontinuously) to zero. This can never be compatible with utility maximization. Paths that start below the saddle path can be ruled out since in this case the representative household realizes that it will eventually overaccumulate assets (technically these paths violate the transversality condition).

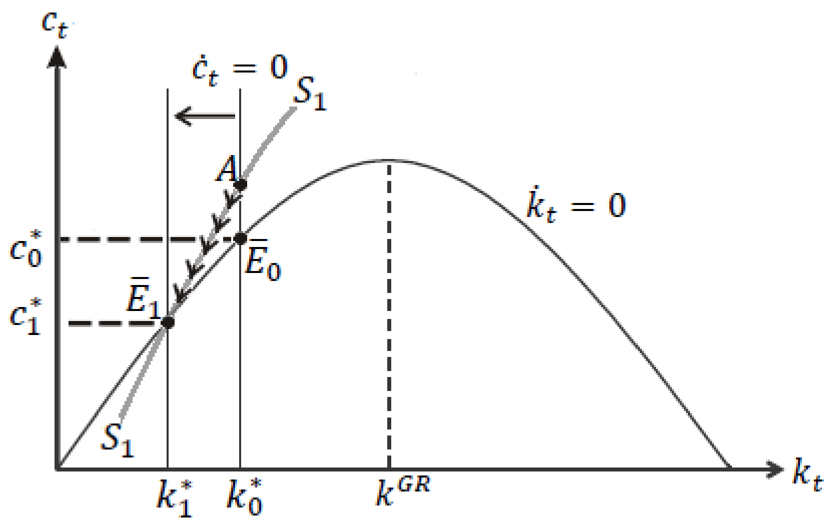
Figure 1: The phase diagram in the Ramsey model



4)

The consequences of an increase in τ are illustrated in figure 2. Notice that the $\dot{k}_t = 0$ curve is unaffected by the increase in τ while the $\dot{c}_t = 0$ line moves to the left. The economic explanation is given below:

Figure 2: An increase in τ



Right at time t_0 the capital stock is predetermined and will only adjust gradually. Consumption is on the other hand a jump variable and responds immediately to the increase in τ by jumping from E_0 to the point A on the new saddle path. The increase in consumption is caused by the intertemporal substitution effect of a lower after-tax interest rate (the lower after-tax return to saving makes saving less attractive). The income effect of the lower after-tax interest rate is neutralized, since the higher tax revenue is transferred back to households through an increase in lump sum transfers.

After time t_0 the economy will move along the new saddle path and converge towards the new steady state, ($\bar{E}_1$). Along this adjustment capital per worker falls due to the initial increase in consumption (which is also a decrease in saving and thereby investment). Also, consumption decreases along the transition towards the new steady state since the after-tax interest rate is now below ρ (which implies a fall in consumption over time according to the Keynes Ramsey rule) until the new steady state is reached.

In the new steady state (at E_1) both the capital stock and the consumption level is lower compared to the initial steady state. The steady state capital stock falls to such an extent, that the after-tax return to saving

(given by $(1-\tau)\cdot(f'(k^*)-\delta)$) is once again equal to ρ.

Problem B – The Blanchard model

1)

Equation (B.1) is the equilibrium condition for the goods market, stating that total output in the economy must equal aggregate demand, which is assumed to depend on an exogenous shock (capturing shifts in fiscal policy, world economic activity, etc.), stock prices and goods prices.

Higher stock prices increase aggregate demand due to the wealth effect on private consumption (households feel wealthier when the market value of their stocks increase and they respond by increasing consumption) and due to the effect on the profitability of investing (higher stock prices increase the market value of firms), the *Tobins-q effect*. The parameter η should be interpreted as measuring the strength of these effects.

Higher goods prices on the other hand decreases aggregate demand by making domestic goods less competitive compared to foreign goods, which reduces net exports and thereby aggregate demand. The parameter β measures the strength of this effect, and should therefore be interpreted as the elasticity of foreign trade.

Equation (B.2) is an arbitrage equation stating that the after-tax return to domestic bonds must equal the after-tax return to foreign bonds since these are perfect substitutes and there is perfect capital mobility and the exchange rate is credibly fixed.

Equation (B.3) is a financial arbitrage equation stating that the return to stocks must equal the after-tax return to bonds since these are perfect substitutes. The total return to stocks consists of dividends and the capital gain on stocks ($\dot{Q}_t$). When dividing the return to stocks by Q_t we get the rate of return to stocks.

Equation (B.4) assumes that dividends positively on output, capturing that profits (and thereby dividends) depends positively on the overall state of the economy.

Equation (B.5) is a simple Philips curve stating that inflation (the rate of change in the price level) depends positively on the output gap (reflecting demand-pull inflation). It is implicitly assumed that the expected inflation rate is equal to zero.

Finally, equation (B.6) states that stock prices are given by the present value of all future dividends. This is the fundamental stock price, and can be derived from the arbitrage condition in equation (B.3) combined with (B.2) and a transversality condition. Notice, that future dividends are discounted with *the after-tax interest rate*.

When inserting equation (B.1) in equation (B.5) we get:

$$\dot{p}_t = \gamma \cdot (z + \eta \cdot Q_t - \beta \cdot p_t - \bar{y}) = \gamma \cdot \eta \cdot Q_t - \gamma \cdot \beta \cdot p_t + \gamma \cdot (z - \bar{y})$$

Further, inserting (B.4), (B.2) and (B.1) in (B.3) we get:

$$\frac{D_t + \dot{Q}_t}{Q_t} = r^f \cdot (1 - \tau) \Rightarrow \dot{Q}_t = r^f \cdot (1 - \tau) \cdot Q_t - D_t =$$

$$r^f \cdot (1 - \tau) \cdot Q_t - \overbrace{\alpha \cdot y_t}^{D_t} = r^f \cdot (1 - \tau) \cdot Q_t - \alpha \cdot \left(\overbrace{z + \eta \cdot Q_t - \beta \cdot p_t}^{y_t} \right) =$$

$$Q_t \cdot (r^f \cdot (1 - \tau) - \alpha \cdot \eta) + \alpha \cdot \beta \cdot p_t - \alpha \cdot z$$

These two differential equations (along with a transversality condition restricting the asymptotic evolution of stock prices) determine the evolution of the economy over time.

2)

The long run equilibrium is defined by the condition that p_t and Q_t are constant over time, i.e. that:

$\dot{p}_t = 0$ and $\dot{Q}_t = 0$. We see immediately from equation (B.5) that:

$$\dot{p}_t = 0 \Rightarrow y^* = \bar{y}$$

since y^* is the long run value of y_t.

Further we can see from equation (B.3) and (B.2) that:

$$\dot{Q}_t = 0 \Rightarrow \frac{D^*}{Q^*} = r^f \cdot (1 - \tau) \Rightarrow Q^* = \frac{D^*}{r^f \cdot (1 - \tau)}$$

Now use (B.4):

$$Q^* = \frac{\alpha \cdot y^*}{r^f \cdot (1 - \tau)} = \frac{\alpha \cdot \bar{y}}{r^f \cdot (1 - \tau)}$$

Finally, we can solve for the steady state price level from equation (B.1).

$$\bar{y} = z + \eta \cdot Q^* - \beta \cdot p^* \Rightarrow$$

$$p^* = \frac{1}{\beta} \cdot (z + \eta \cdot Q^* - \bar{y}) = \frac{1}{\beta} \cdot \left(z + \eta \cdot \frac{\alpha \cdot \bar{y}}{r^f \cdot (1-\tau)} - \bar{y}\right)$$

The expression for Q^* can be derived another way. Insert that in steady state $D_s = \alpha \cdot \bar{y}$ in equation (B.6):

$$Q^* = \int_t^\infty D_s \cdot e^{-r^f \cdot (1-\tau) \cdot (s-t)} ds = \int_t^\infty \alpha \cdot \bar{y} \cdot e^{-r^f \cdot (1-\tau) \cdot (s-t)} ds =$$

$$\alpha \cdot \bar{y} \cdot \left[-\frac{e^{-r^f \cdot (1-\tau) \cdot (s-t)}}{r^f \cdot (1-\tau)} \right]_t^\infty = \frac{\alpha \cdot \bar{y}}{r^f \cdot (1-\tau)}$$

This shows that Q^* can be interpreted as the present value of future dividends when all future dividends are constant and given by their steady state value ($\alpha \cdot \bar{y}$).

3)

We start by noticing that:

$$\dot{p}_t = 0 \Rightarrow \eta \cdot Q_t - \beta \cdot p_t + z - \bar{y} = 0 \Rightarrow Q_t = \frac{\bar{y} - z}{\eta} + \frac{\beta}{\eta} \cdot p_t$$

which defines an upward sloping curve (since an increase in p_t tends to decrease y_t through lower net exports which implies that Q_t is required to increase as long as $y_t = \bar{y}$). Notice that if the economy is initially located above this line (that is Q_t is relatively high) then $\dot{p}_t > 0$ since $y_t > \bar{y}$ initially. Conversely, goods prices will fall if the economy is initially located below the $\dot{p}_t = 0$ line. This is the *stabilizing mechanism* in the Blanchard model.

Further, we notice that:

$$\dot{Q}_t = 0 \Rightarrow Q_t \cdot (r^f \cdot (1-\tau) - \alpha \cdot \eta) + \alpha \cdot \beta \cdot p_t - \alpha \cdot z = 0 \Rightarrow$$

$$Q_t = \frac{\alpha \cdot z}{r^f \cdot (1-\tau) - \alpha \cdot \eta} - \frac{\alpha \cdot \beta}{r^f \cdot (1-\tau) - \alpha \cdot \eta} \cdot p_t$$

which defines a downward sloping curve (since by assumption: $r^f \cdot (1-\tau) - \alpha \cdot \eta > 0$). The reason that the curve is downward sloping is that as long as $\dot{Q}_t = 0$ the arbitrage equation implies:

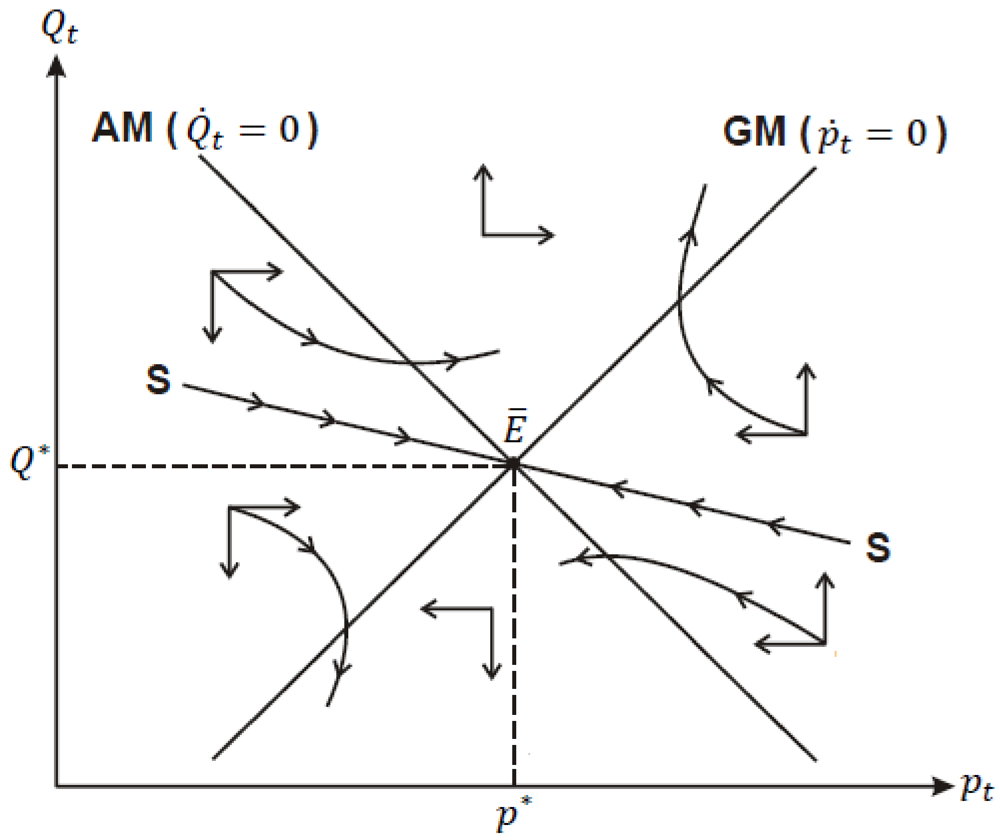
$$\frac{D_t}{Q_t} = r^f \cdot (1-\tau) \Rightarrow D_t = r^f \cdot (1-\tau) \cdot Q_t$$

If Q_t increases by one unit the right hand side of the equation will increase by $r^f \cdot (1-\tau)$ units. The left hand side (dividends) increases by $\alpha \cdot \eta$ units which is less than the required increase (since we have assumed that $r^f \cdot (1-\tau) > \alpha \cdot \eta$ in equation (B.7)). Thus, we need a further increase in dividends, which implies that p_t must fall (in order to stimulate net exports and thereby output and dividends).

Consider now the case where the economy is initially located above the AM-curve. In this case p_t is high (compared to the corresponding point on the AM-curve) implying that net exports are low and thus that output and dividends are low. In order to compensate for the low dividends there must be a capital gain on stock in order for the financial arbitrage equation to be satisfied. This is the *destabilizing element* in the Blanchard model.

The movements of the economy are showed in figure 3.

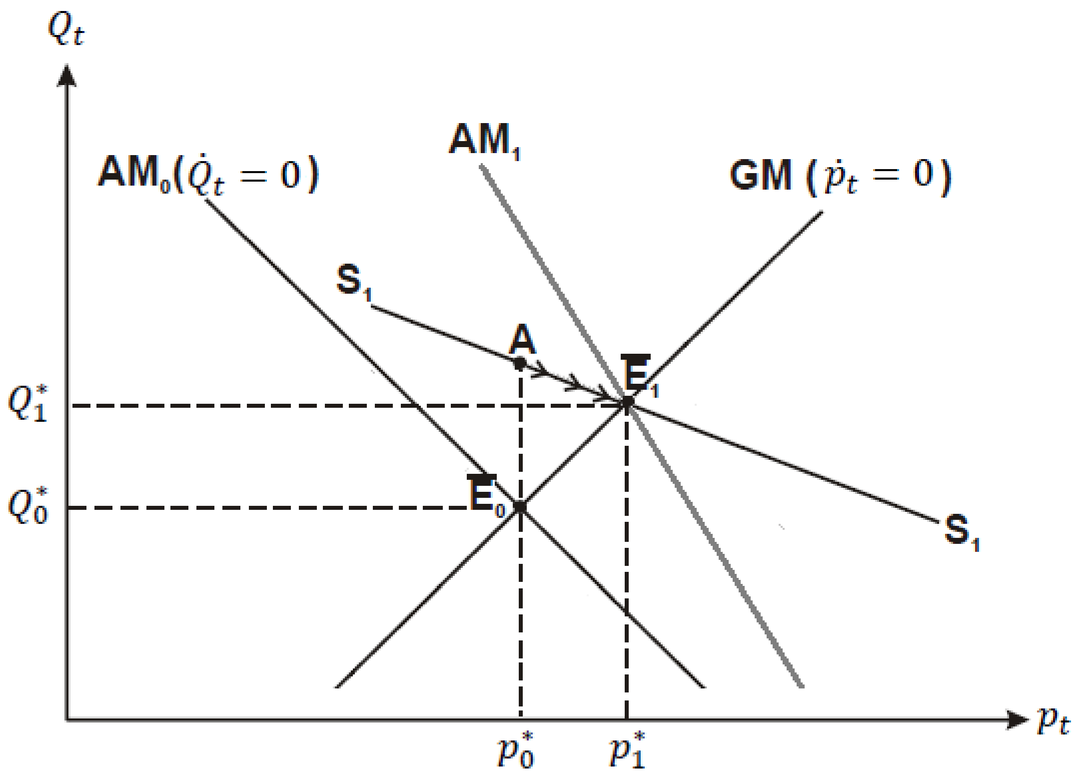
Figure 3: The phase diagram



4)

The consequences of an increase in τ are showed in figure 4. The effects are explained below.

Figure 4: An unanticipated increase in τ



At time t_0 the price level is predetermined and will thus only adjust gradually over time. On the other hand Q_t is a jump variable and jumps *immediately* from $\bar{E}_0$ to A as a response to the higher tax on the return to the bonds. There are two reasons for this increase in Q_t:

First of all, there is a long run increase in Q_t, which can be explained by the fact that future dividends are discounted with a lower after-tax interest rate. Intuitively, the lower after-tax return to bonds implies that financial investors switch from bonds to stocks, which drives up stock prices.

Further, in addition to this long run effect there is also a short run effect: Financial investors realize that the higher stock prices will temporary increase output (i.e. y_t will be above $\bar{y}$ for some time) as a result of the increase in private consumption and investment (since $\eta > 0$ in equation (B.1)). Thus, dividends will increase, according to equation (B.4) and remain above D^* for some time. The increase in future dividends increases stock prices further at time t_0, according to equation (B.6). Due to the short run effect the stock price will initially 'overshoot' its long run value.

At point A output is above its natural level ($y_t > \bar{y}$) due to the increase in stock prices. Thus, goods prices will gradually increase over time (i.e. $\dot{p}_t > 0$) according to equation (B.5). The gradual increase in goods prices will gradually decrease output (through lower net exports) and thereby dividends. The gradual (and fully anticipated) fall in dividends implies that Q_t gradually falls over time, as the economy converges towards the new steady state, where output is once again equal to its long run level ($\bar{y}$) and where goods prices and stock prices have increased (compared to the initial steady state).

5)

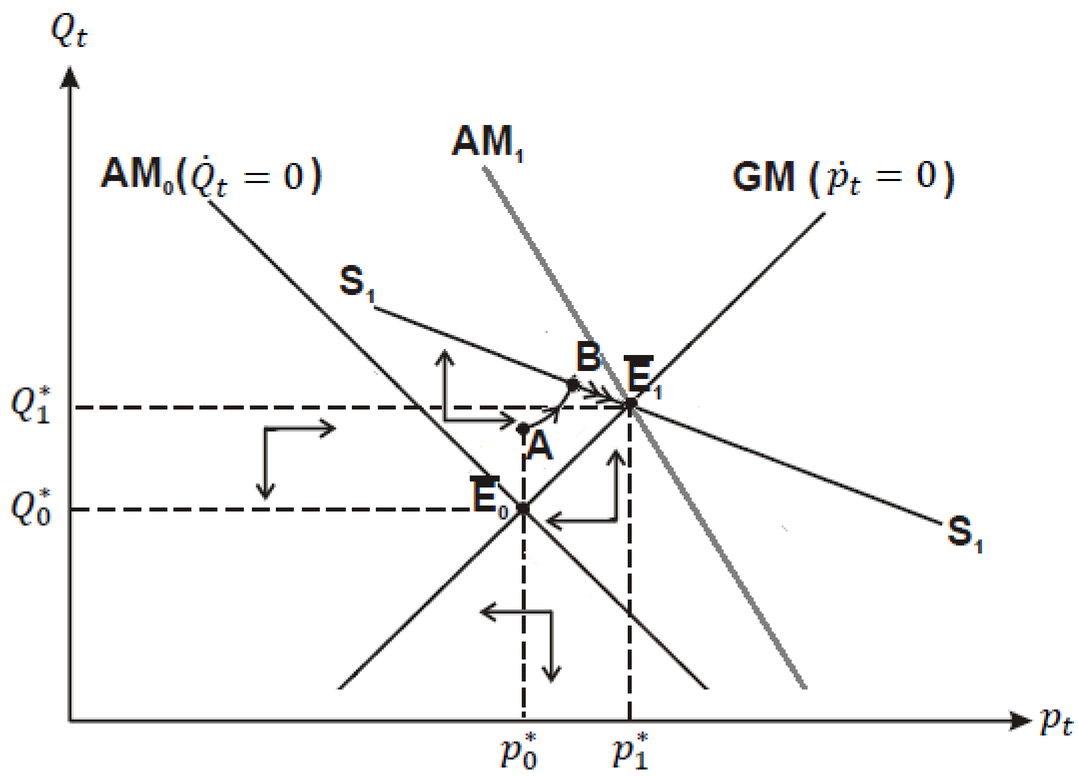
The consequences of an increase in τ which is announced at time t_0 and implemented at time $t_1 > t_0$ are showed in figure 5.

We can determine the evolution of the economy by noticing that:

- Right at time t_0 (when the increase in τ is announced) p_t is predetermined and will thus only adjust gradually over time
- Q_t may change discontinuously at time t_0, but there can be no jumps in Q_t after time t_0.
- At time t_1 (when the increase in τ is actually implemented) the economy *must* be located somewhere on the new saddle path (S_1S_1), in order for the economy to converge to $\bar{E}_1$.

The effects are explained below. The arrows in the figure represent the old dynamics (until time t_1)

Figure 5: An anticipated increase in τ



From the phase diagram we see that right at the announcement Q_t jumps from point $\bar{E}_0$ to point A. There are two reasons for this jump in stock prices (announcement effect).

First of all, a part of the future dividends (those accruing after time t_1) are discounted with a lower after-tax interest rate, which increases stock prices already at time t_0, according to equation (B.6).

Also, financial investors realize that the future increase in τ will increase future stock prices, and that output and dividends will for a while be above their long run levels. The future increases in dividends will further increase stock prices already at time t_0 (according to equation (B.6)).

At point A output is above $\bar{y}$ (due to the announcement effect on stock prices and thereby aggregate demand) implying that goods prices will gradually increase ($\dot{p}_t > 0$). Also Q_t increases from A to B since a gradually increasing part of future dividends is discounted with the low after-tax interest rate (notice that when $t = t_1$ *all* future dividends are discounted with the lower after-tax interest rate).

At time t_1 the economy has reached the point B on the new saddle path and the economy will from thereon converge towards the new steady state, as explained in (B.4).

6)

When both dividends and the return to bonds are taxed at the rate τ the steady state value of Q_t is independent of τ, since:

$$Q^* = \frac{(1-\tau)\cdot D^*}{(1-\tau)\cdot r^f} = \frac{D^*}{r^f} = \frac{\alpha \cdot \bar{y}}{r^f}$$

Also, we see that the steady state value of p_t is independent of τ, since:

$$p^* = \frac{1}{\beta}\cdot(z + \eta \cdot Q^* - \bar{y}) = \frac{1}{\beta}\cdot\left(z + \eta \cdot \frac{\alpha \cdot \bar{y}}{r^f} - \bar{y}\right)$$

that is, an increase in τ will *not* affect the steady state. Thus, if the economy is initially in steady state it will stay there, even though τ is increased, and there are no consequences of an increase in τ. On the one hand, after-tax dividends are lower but they are also discounted with a correspondingly lower after-tax interest rate. Thus, there is no effect on stock prices and thus no effects on the economy in general.