24 hours written exam. All questions should be clearly and briefly answered. Calculations and figures should be clear and understandable. Calculations and figures should be explained.

Exercise 1

1.1

The strategy for the expand option is going to be a cut-off strategy

$$Q < \bar{q} \;\; \Rightarrow \;\; wait$$

$$Q \geq \bar{q} \;\; \Rightarrow \;\; expand$$

The strategy for the investment option is going to be a cut-off strategy too, but the cut-off price is going to depend on Q

$$P < P(Q) \;\; \Rightarrow \;\; wait$$

$$P \geq P(Q) \;\; \Rightarrow \;\; invest.$$

1.2

The value of getting P forever is found by CCA with the portfolio: one unit of P forever - n units of the P-output. The dividend rate of the portfolio is

$$\frac{P + \alpha_P P V_P'(P) + 0.5\sigma_p^2 P^2 V_p''(P) - n(\alpha_P + \delta_P)P}{V_P(P) - nP} dt + \frac{\sigma_P P V_P'(P) - n\sigma_P P}{V_P(P) - nP} dz_p$$

where $V_P(P)$ is the value of P forever. Let $n = V'(P)$, then there is no risk and the dividend rate is equal to r:

$$0.5\sigma_P^2 P^2 V_P''(P) + (r - \delta_P) P V_P'(P) - r V_P(P) + P = 0$$

The mathematical solution is

$$V_P(P) = B_1 P^{\beta_1} + B_2 P^{\beta_2} + \frac{P}{\delta_P}$$

where $\beta_1 > 1$ and $\beta_2 < 0$ solve $0.5\sigma_P^2(\beta - 1)\beta + (r - \delta_P)\beta - r = 0$. The economic solution is ("no bubbles" and "$P \to 0 \Rightarrow V_P(P) \to 0$):

$$V_P(P) = \frac{P}{\delta_P}.$$

Similarly the value of getting Q forever is

$$V_Q(Q) = \frac{Q}{\delta_Q}.$$

1.3

The dividend rate of the portfolio is

$$\frac{\alpha_Q Q F_Q'(Q) + 0.5\sigma_p^2 Q^2 F_Q''(Q) - x(\alpha_Q + \delta_Q)Q}{F_Q(Q) - xQ} dt + \frac{\sigma_Q Q F_Q'(Q) - x\sigma_Q Q}{F_Q(Q) - xQ} dz_Q.$$

1.4

Let $x = F_Q'(Q)$, then there is no risk and the dividend rate is equal to r:

$$0.5\sigma_Q^2 Q^2 F_Q''(Q) + (r - \delta_Q) Q F_Q'(Q) - r F_Q(Q) = 0$$

The mathematical solution is

$$F_Q(Q) = A_1 Q^{\gamma_1} + A_2 Q^{\gamma_2}.$$

where $\gamma > 1$ and $\gamma_2 < 0$ solve $0.5\sigma_Q^2(\gamma - 1)\gamma + (r - \delta_Q)\gamma - r = 0$. The economic solution is ("$Q \to 0 \Rightarrow F_Q(Q) \to 0$):

$$F_Q(Q) = A_1 Q^{\beta_1}.$$

The optimal strategy is found by value matching and smooth pasting:

$$\begin{aligned} A_1 \bar{q}^{\beta_1} &= \frac{\bar{q}}{\delta_Q} - I_Q \\ \beta_1 A_1 \bar{q}^{\beta_1 - 1} &= \frac{1}{\delta_Q}. \end{aligned}$$

so $\bar{q} = (\beta_1/(\beta_1 - 1))\delta_Q I_Q$ and A_1 can be determined by one of the equations.

1.5

The value of an active project is

$$V_P(P,Q) = \begin{cases} \dfrac{P}{\delta_P} + A_1 Q^{\beta_1} & \text{for } Q < \bar{q} \\ \dfrac{P}{\delta_P} + \dfrac{Q}{\delta_Q} - I_Q & \text{for } Q \geq \bar{q} \end{cases}$$

1.6

The dividend rate of the portfolio is:

$$\frac{\alpha_P P \dfrac{\partial F_P(P,Q)}{\partial P} + \alpha_Q Q \dfrac{\partial F_P(P,Q)}{\partial Q}}{V_P(P,Q) - yP - zQ} dt$$

$$+ \frac{0.5(\dfrac{\partial^2 F_P(P,Q)}{\partial P^2} + \dfrac{\partial^2 F_P(P,Q)}{\partial Q^2}) + \rho \dfrac{PQ\sigma_P\sigma_Q\partial^2 F_P(P,Q)}{\partial P \partial Q}}{V_P(P,Q) - yP - zQ} dt$$

$$- \frac{y(\alpha_P + \delta_P)P + z(\alpha_Q + \delta_Q)Q}{V_P(P,Q) - yP - zQ} dt$$

$$+ \frac{\sigma_P \dfrac{\partial F_P(P,Q)}{\partial P} - y\sigma_P P}{V_P(P,Q) - yP - zQ} dz_P + \frac{\sigma_Q \dfrac{\partial F_P(P,Q)}{\partial Q} - z\sigma_Q Q}{V_P(P,Q) - yP - zQ} dz_Q$$

1.7

Let $y = \partial F_P(P,Q)/\partial P$ and $z = \partial F_P(P,Q)/\partial Q$, then there is no risk and the dividend rate is equal to r:

$$0.5\sigma_P^2 P^2 \frac{\partial^2 F_P(P,Q)}{\partial P^2} + 0.5\sigma_Q^2 Q^2 \frac{\partial^2 F_P(P,Q)}{\partial Q^2} + \rho\sigma_P\sigma_Q PQ \frac{\partial^2 F_P(P,Q)}{\partial P \partial Q}$$

$$+ (r - \delta_P)\frac{\partial F_P(P,Q)}{\partial P} + (r - \delta^Q)\frac{\partial F_P(P,Q)}{\partial Q} - rF_P(P,Q) = 0$$

The hint, $F_P(P,Q) = A_P P^{\varepsilon_P} + A_Q Q^{\varepsilon_Q}$, satisfies the equation for $\varepsilon_p = \beta_1$ and $\varepsilon_Q = \gamma_1$.

1.8

Value matching and smooth pasting (for $Q < \bar{q}$ and $Q \geq \bar{q}$):

$$\begin{aligned}
A_P P(Q)^{\beta_1} + A_Q Q^{\gamma_1} &= \frac{P(Q)}{\delta_P} + A_1 Q^{\gamma_1} - I_P \\
\beta_1 A_P P(Q)^{\beta_1 - 1} &= \frac{1}{\delta_P} \\
A_P P(Q)^{\beta_1} + A_Q Q^{\gamma_1} &= \frac{P(Q)}{\delta_P} + \frac{Q}{\delta_Q} - I_Q - I_P \\
\beta_1 A_P P(Q)^{\beta_1 - 1} &= \frac{1}{\delta_P}
\end{aligned}$$

There are 4 equations and 3 unknowns, $P(Q)$, A_P and A_Q. However one of the equations is redundant. The equations may be solved by first finding $P(Q)$ as a function of A_Q, then A_P as a function of A_Q and finally A_Q.

Exercise 2

2.1

Intuitively, if the dividend is mean reverting, the value should be mean reverting too. So entry in markets with free entry should be expected to fit as the price in these markets should be expected to be fixed at marginal costs where firms break even.

The optimal strategies for the two sub-projects is probably cut-off strategies with values V^* for the I-project and V' for the J-project. The optimal strategy for the project is probably: V low $\Rightarrow$ wait; V in middle range $\Rightarrow$ K-project for $K \in \{I, J\}$, and V high $\Rightarrow$ K'-project for $K' \in \{I, J\}$.

2.2

The Bellman equation is

$$\rho F_I(V) = \frac{1}{dt} E(dF_I(V))$$

by Ito's lemma

$$0.5\sigma^2 V^2 F_I''(V) + \mu(\bar{V} - V)V F_I'(V) - \rho F_I(V) = 0.$$

2.3

The solution is

$$F_I(V) = AV^\theta H(\frac{2\mu}{\sigma^2}V; \theta, 2\frac{\theta + \mu\bar{V}}{\sigma^2}),$$

where θ solves $0.5\sigma^2(\beta - 1)\beta + \mu\bar{V}\beta - \rho = 0$ - see Dixit & Pindyck p. 161-163.

2.4

The solution has to satisfy value matching and smooth pasting:

$$\begin{aligned} A(V^*)^\theta H(\frac{2\mu}{\sigma^2}V^*; \theta, 2\frac{\theta + \mu\bar{V}}{\sigma^2}) &= E(V^*) - I \\ \theta A(V^*)^{\theta-1} H(\frac{2\mu}{\sigma^2}V^*; \theta, 2\frac{\theta + \mu\bar{V}}{\sigma^2}) & \\ + \frac{2\mu}{\sigma^2} A(V^*)^\theta H_1'(\frac{2\mu}{\sigma^2}V^*; \theta, 2\frac{\theta + \mu\bar{V}}{\sigma^2}) &= E'(V^*) \end{aligned}$$

where $E(V)$ is the expected value of the project at date $t + T$ given the value of project is V at date t. Numerical analysis is needed to find $E(V)$, V^* and A .

2.5

The differential equation is the same as in 2.2.

2.6

The solution is the same as in 2.3.

2.7

The solution has to satisfy value matching and smooth pasting:

$$\begin{aligned} A(V')^{\theta} H(\frac{2\mu}{\sigma^2}V'; \theta, 2\frac{\theta+\mu\bar{V}}{\sigma^2}) &= V' - J \\ \theta A(V')^{\theta-1} H(\frac{2\mu}{\sigma^2}V'; \theta, 2\frac{\theta+\mu\bar{V}}{\sigma^2}) + \frac{2\mu}{\sigma^2}A(V')\theta H_1'(\frac{2\mu}{\sigma^2}V'; \theta, 2\frac{\theta+\mu\bar{V}}{\sigma^2}) &= 1 \end{aligned}$$

2.8

A good answer should rest on numerical analysis. However value matching and smooth pasting are identical in 2.4 and 2.7, except for the right side of value matching. Therefore a discussion based on whether $V^* > I$ or $V^* < I$ is possible.

2.9

The student should be able to make qualitative drawings of $F_I(V)$ and $F_J(V)$ and based on these to discuss the optimal strategy. Alternatively numerical analysis may be used.