Written Exam | Economics | Spring 2020
Econometrics II

Solution Key

Part 1: Hand-In Assignment #2 The Dynamic Relationship Between Stock Market Returns and Trading Volume

The Case: The purpose of Part 1 is to analyze the relationship between weekly US stock market returns and trading volume. Specifically, the aim is to investigate whether it is the returns that drive the trading volume, or if it is the other way around.

The Data: Graphs of the data and relevant transformations must be shown in the exam. Some comments on the stationarity of the time series must be given.

Econometric Theory: The econometric theory must include the following:

(1) A precise definition and interpretation of the vector autoregressive model.

(2) A description of the estimator used.

(3) A description of the sufficient conditions for consistent estimation and valid inference. Given that the estimated models appear to have heteroskedastic errors, one may also describe the notion of robust test statistics (e.g. robust $t-$ or Wald statistics).

(4) The theory must be presented precisely and in a logical order with a consistent and correct notation.

Empirical Results: The empirical results must include the following:

(1) A presentation and discussion of the relevant empirical results, so that the reader is able to understand the steps carried out in the process as well as the conclusions made.

(2) A description of the model selection process.

(3) A discussion of whether the assumptions for consistent estimation and valid inference are satisfied for the estimated models.

(4) A clear conclusion to the main question and a discussion of the limitations of the approach used to reach the conclusion. Specifically, the conclusion regarding the relationship between returns and trading volume.

Part 2: Hand-In Assignment #3 Modelling the Relationship Between Danish Export, International Demand, and Competitiveness

The Case: The purpose of Part 2 is to analyze the relationship between Danish export, international demand, and competitiveness. The analysis is carried out by means of cointegration analysis, and the estimated demand and wage elasticities are compared with their theoretically plausible values.

The Data: Graphs of the data and relevant transformations must be shown in the exam.

Econometric Theory: The analysis can be based on any tool from the co-integration tool-kit, i.e. the two-step Engle-Granger procedure, an analysis based on a single-equation ADL model or a co-integrated VAR. The good solution argues for the choice of approach and discusses advantages and drawbacks of the methods.

Empirical Results: The empirical results must include the following:

(1) A presentation and discussion of the relevant empirical results, so that the reader is able to understand the steps carried out in the process as well as the conclusions made.

(2) A description of the model selection process.

(3) A discussion of whether the assumptions for consistent estimation and valid inference are satisfied for the estimated models.

(4) A conclusion regarding the long-run and short-run relationship between the three variables, as well as a conclusion about the estimated elasticities.

Part 3: The Additional Assignment

(1) It holds that

$$\begin{aligned} E(y_t|\mathcal{I}_{t-1}) &= \phi y_{t-1}, \\ V(y_t|\mathcal{I}_{t-1}) &= \omega + \gamma h(y_{t-1}). \end{aligned}$$

(2) Both models have time-varying conditional heteroskedasticity. For the present model, we note that the conditional variance of y_t dependes on the lagged *level* of y_t. For an ARCH(1), the conditional variance depends on the lagged *error* $\varepsilon_t = z_t\sqrt{\lambda_t}$.

(3) For $\theta_0 = (\frac{1}{2}, 1, 0)'$ the data-generating process is a stationary conditionally homoskedastic AR(1) process. Clearly the OLS estimator is consistent with $\sqrt{T}$-rate of convergence.

(4) For $\theta_0 = (1, 1, 0)'$ the data-generating process is a non-stationary conditionally homoskedastic unit root AR(1) process. From the theory for unit root testing, we have that the OLS estimator is consistent with T-rate of convergence.

(5) The vector θ can be estimated by ML estimation. The log-likelihood contribution for $t = 1, \ldots, T$, is given by

$$l_t(\theta) = \log\left[\frac{1}{\sqrt{2\pi(\omega + \gamma h(y_{t-1}))}} \exp\left(-\frac{(y_t - \phi y_{t-1})^2}{2(\omega + \gamma h(y_{t-1}))}\right)\right].$$

Derivations and explanations should be provided.

(6) We have that

$$\begin{aligned} \tilde{g}(\phi_0) &= E[y_t^2 - (\phi_0 y_{t-1})^2 - \omega_0 - \gamma_0 y_{t-1}^2] \\ &= E[E[y_t^2 - (\phi_0 y_{t-1})^2 - \omega_0 - \gamma_0 y_{t-1}^2 |\mathcal{I}_{t-1}]] \\ &= E[E[y_t^2 - (\phi_0 y_{t-1})^2 |\mathcal{I}_{t-1}] - \omega_0 - \gamma_0 y_{t-1}^2] \\ &= E[E[y_t^2|\mathcal{I}_{t-1}] - (E[y_t|\mathcal{I}_{t-1}])^2 - (\omega_0 + \gamma_0 y_{t-1}^2)] \\ &= E[V(y_t|\mathcal{I}_{t-1}) - V(y_t|\mathcal{I}_{t-1})] \\ &= 0, \end{aligned}$$

where the second equality follows by the law of iterated expectations.

(7) Noting that $\tilde{g}(-\phi) = \tilde{g}(\phi)$, we have that $\tilde{g}(\phi) = 0$ does not imply that $\phi = \phi_0$ (unless $\phi_0 = 0$). [$\tilde{g}(\phi) = 0$ if (but not only if) $\phi = \phi_0$] Hence, if $\phi_0 \neq 0$, the moment condition does not ensure identification, and the moment condition is not useful for method of moments estimation of ϕ.

(8) Using that $v_t \in \mathcal{I}_{t-1}$, we have that

$$\begin{aligned}
E[m(y_t, y_{t-1}, y_{t-2}, \theta_0)|\mathcal{I}_{t-1}] &= \begin{pmatrix} E[(y_t - \phi_0 y_{t-1})v_t|\mathcal{I}_{t-1}] \\ E[(y_t^2 - (\phi_0 y_{t-1})^2 - \omega_0 - \gamma_0 y_{t-1}^2)v_t|\mathcal{I}_{t-1}] \end{pmatrix} \\
&= \begin{pmatrix} E[(y_t - \phi_0 y_{t-1})|\mathcal{I}_{t-1}]v_t \\ E[(y_t^2 - (\phi_0 y_{t-1})^2 - \omega_0 - \gamma_0 y_{t-1}^2)|\mathcal{I}_{t-1}]v_t \end{pmatrix} \\
&= \begin{pmatrix} (E[y_t|\mathcal{I}_{t-1}] - \phi_0 y_{t-1})v_t \\ (E[y_t^2|\mathcal{I}_{t-1}] - (E[y_t|\mathcal{I}_{t-1}])^2 - (\omega_0 + \gamma_0 y_{t-1}^2))v_t \end{pmatrix} \\
&= 0.
\end{aligned}$$

Let $w_t = (y_t, y_{t-1})$ denote the model variables and let v_t denote the instruments, and define

$$f(w_t, v_t, \theta) = m(y_t, y_{t-1}, y_{t-2}, \theta),$$

and

$$g(\theta) = E[f(w_t, v_t, \theta)].$$

Using the derived conditional moment conditions, (5.6), we have that (by the law of iterated expectations),

$$g(\theta_0) = 0.$$

The GMM estimator may be based on the 6 sample moments

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} f(w_t, v_t, \theta).$$

Details should be provided. This includes presenting the optimization problem and a discussion about the choice of (optimal) weight matrix. In particular, the very good answer may note that with $f_t = f(w_t, v_t, \theta_0)$, (5.6) implies that $E[f_t f_s'] = 0$ for $t \neq s$. Hence it is not necessary to apply the HAC (kernel-based) estimator for estimating the optimal weight matrix.

(9) The representation follows directly by recursive substitution.

(10) We have that

$$\phi_t = \phi + \eta_t \sqrt{\gamma},$$

with (η_t) an i.i.d. process with $\eta_t \overset{d}{=} N(0,1)$.
Case 1, $(\phi, \gamma) = (1, 0)$**:** It holds that $\phi_t = 1$, so that $E[\phi_t^2] = 1$ and $E[\log(|\phi_t|)] = 0$. Hence, none of the stationarity conditions are satisfied. This is not surprising, as x_t is a unit root AR(1).
Case 2, $(\phi, \gamma) = (0, 3)$**:** It holds that $\phi_t = \eta_t \sqrt{3}$, so that

$$E[\phi_t^2] = 3 > 1,$$

while

$$E[\log(|\phi_t|)] = \int_{-\infty}^{\infty} \log\left(\left|\eta\sqrt{3}\right|\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\eta^2}{2}\right) d\eta = -0.08587\,5\ldots < 0.$$

Hence, the process is strictly stationary but not covariance stationary. [The qualitative same conclusion is obtained, if one instead uses the dataset to approximate the integral. In fact, with $\{\eta_t : t = 1, \ldots, 10000\}$ the series in the data file, $10000^{-1} \sum_{t=1}^{10000} \log(|\sqrt{3}\eta_t|) \approx -0.085404$] In terms of the first condition, this is not suprising, as x_t is a pure ARCH(1) process (with Gaussian innovations), and we know that the necessary and sufficient condition for covariance stationarity is that $\gamma < 1$.

Case 3, $(\phi, \gamma) = (1, 1)$**:** It holds that $\phi_t = 1 + \eta_t$, so that

$$E[\phi_t^2] = 2 > 1,$$

while

$$E[\log(|\phi_t|)] = \int_{-\infty}^{\infty} \log(|1+\eta|) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\eta^2}{2}\right) d\eta = -0.20850\ldots < 0.$$

Hence, x_t is strictly stationary, but not covariance stationary. [The qualitative same conclusion is obtained, if one instead uses the dataset to approximate the integral. In fact, with $\{\eta_t : t = 1, \ldots, 10000\}$ the series in the data file, $10000^{-1} \sum_{t=1}^{10000} \log(|1 + \eta_t|) \approx -0.22257$] This may not be intuitive, as x_t may look like a (conditionally heteroskedastic) unit root process,

$$\Delta x_t = \varepsilon_t,$$

with

$$\varepsilon_t = x_{t-1}\eta_t\sqrt{\gamma} + \xi_t,$$

satisfying

$$E[\varepsilon_t | x_{t-1}] = 0 \quad \text{and} \quad V(\varepsilon_t | x_{t-1}) = \omega + \gamma x_{t-1}^2.$$

Compared to Case 1, we see that the presence of heteroskedasticity, i.e. $\gamma > 0$, induces stationarity.