Written Resit Exam for M.Sc. in Economics

Winter 2010/2011

Advanced Microeconomics

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Master course

3 hours written exam with closed books

Solution

1.1

For p and y (UMP) for consumer i is

$$\begin{aligned} \max_{x_i} \quad & -\textstyle\sum_\ell a_i^\ell e^{-x_i^\ell} \\ \text{s.t.} \quad & p \cdot x_i \leq p \cdot \omega_i \text{ and } x_i \in X_i. \end{aligned}$$

The first-order conditions are

$$\begin{aligned} a_i^1 e^{-x_i^1} - \lambda p_1 &= 0 \\ &\vdots \\ a_i^L e^{-x_i^L} - \lambda p_L &= 0 \\ p \cdot x_i &= p \cdot \omega_i. \end{aligned}$$

Therefore the demand function is

$$x_i(p, p\cdot\omega_i) = \begin{pmatrix} \dfrac{p\cdot\omega_i - \sum_\ell p_\ell((\ln p_1 - \ln p_\ell) - (\ln a_i^1 - \ln a_i^\ell))}{\sum_\ell p_\ell} \\ \vdots \\ \dfrac{p\cdot\omega_i - \sum_\ell p_\ell((\ln p_L - \ln p_\ell) - (\ln a_i^L - \ln a_i^\ell))}{\sum_\ell p_\ell} \end{pmatrix}.$$

1.2

Definition 1 *A* **Pareto optimal** *allocation is an allocation $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_I)$ such that $\sum_i \bar{x}_i = \sum_i \omega_i$ and there is no other allocation x such that $\sum_i x_i = \sum_i \omega_i$ and $u_i(x_i) \geq u_i(\bar{x}_i)$ for all i with at least one "$>$".*

Definition 2 *A* **Walrasian equilibrium** *is a price vector and a allocation $(\bar{p}, \bar{x})$ such that*

- *$\bar{x}_i$ is a solution to the problem of consumer i given $\bar{p}$ for all i.*
- *$\sum_i \bar{x}_i = \sum_i \omega_i$.*

1.3

For $I = 1$ let $\bar{p} \in \mathbb{R}^L_{++}$ be defined by

$$\bar{p} \;=\; Du(\omega)$$

and $\bar{x} = \omega$. Then $(\bar{p}, \bar{x})$ is a Walrasian equilibrium.

1.4

For $L = I = 2$ the Pareto optimal allocations are characterized by collinear gradients and the ressource constraints:

$$\frac{a_1^1}{a_1^2} e^{x_1^2 - x_1^1} \;=\; \frac{a_2^1}{a_2^2} e^{r^2 - x_1^2 - r^1 + x_1^1}$$

where $r = \omega_1 + \omega_2$. This gives

$$x_1^2 \;=\; x_1^1 + \frac{\ln(a_1^1/a_1^2) - \ln(a_2^1/a_2^2) + r^2 - r^1}{2}.$$

2.1

Since the preference relation $\succeq$ is rational and continuous for every consumer i there exists a continuous utility function u that represents $\succeq$. Hence (UMP) for consumer i is

$$\begin{aligned} &\max_x \quad u(x) \\ &\text{s.t.} \quad p \cdot x \leq p \cdot \omega + p \cdot y \text{ and } x \in \mathbb{R}^L_+. \end{aligned}$$

where $y \in Y$ maximize the profit of the firm.

The budget set $\{x \in X | p \cdot x \leq p \cdot \omega + p \cdot y\} = X \cap \{x \in \mathbb{R}^L | p \cdot x \leq p \cdot \omega + p \cdot y\}$ is closed being the intersection of two closed sets. For $p \in \mathbb{R}^L_{++}$ the budget set $\{x \in X | p \cdot x \leq p \cdot \omega + p \cdot y\}$ is bounded from below by 0 and from above by $((p \cdot \omega + p \cdot y)/p_1, \ldots, (p \cdot \omega + p \cdot y)/p_L)$. All in all the budget set is compact. For $p \cdot \omega + p \cdot y \geq 0$ the budget set is non-empty (for simplicity I assume that $0 \in Y$. (UMP) has a solution because a continuous function has a maximum on a non-empty compact set).

Moreover the budget set is convex being the intersection of two convex sets.

Suppose that x and x' are both solutions to (UMP) and $x' \neq x$. Then $0.5x + 0.5x'$ is in the budget set, because the budget set is convex. Since the preference relation is strictly convex $0.5x + 0.5x' \succ x, x'$.

2.2

Definition 3 *A* **feasible allocation** *is a consumption bundle and a production plan (x, y) such that $x \in X$, $y \in Y$ and $x = \omega + y$.*

Definition 4 *A* **Pareto optimal allocation** *is a feasible allocation (x, y) such that there is no other feasible allocation (x', y') such that $x' \succ x$.*

2.3

Clearly (x^*, y^*) is a Pareto optimal allocation if and only if (x^*, y^*) is a solution to

$$\begin{aligned} \max_{(x,y)} \quad & u(x) \\ \text{s.t.} \quad & x = \omega + y, x \in X, y \in Y \end{aligned}$$

or equivalently x^* is a solution to

$$\begin{aligned} \max_{x} \quad & u(x) \\ \text{s.t.} \quad & x \in X \cap (\{\omega\} + Y) \end{aligned}$$

and $y^* = x^* - \omega$.

The set of alternatives is compact because X is closed and $\{\omega\} + Y$ is compact. Therefore the problem has a solution because u is continuous.

The set of alternatives is convex becasue X and $\{\omega\} + Y$ are convex. Hence the problem has at most one solution because $\succeq$ is strictly convex.

2.4

Let $\bar{p} = Du(\bar{x})$ then $(\bar{p}, \bar{x}, \bar{y})$ is a Walrasian equilibrium: The consumer satifies her first-order conditions, the firm maximizes profit and market clears.

3.1

The problem of consumer t is

$$\max_{x,m} \quad x^y + 2\sqrt{x^o}$$

$$\text{s.t.} \quad \begin{cases} p_t x^y + m \leq p_t \omega^y \\ p_{t+1} x^o \leq p_{t+1} \omega^o + m \\ x \in \mathbb{R}^2_+ \text{ and } m \in \mathbb{R} \end{cases}$$

The problem can be solved by using the second budget constraint to eliminate m in the first budget constraint, then maximizing utility using the modified budget constraint and finally finding m by use of the first or second budget constraint.

Using Lagrange the demand function can be found

$$f(p_t, p_{t+1}, p_t\omega^y + p_{t+1}\omega^o) = \begin{pmatrix} \omega^y + \dfrac{p_{t+1}}{p_t}\omega^o - \dfrac{p_t}{p_{t+1}} \\ \left(\dfrac{p_t}{p_{t+1}}\right)^2 \end{pmatrix}$$

m can be found from one of the budget constraints.

3.2

Definition 5 *An* **equilibrium** *is a price system and an allocation* $((\bar{p}_t)_{t\in\mathbb{Z}}, (\bar{x}_t)_{t\in\mathbb{Z}}$ *such that there exists a stock of money M and a sequence of savings $(\bar{m}_t)_{t\in\mathbb{Z}}$ such that*

- $(\bar{x}_t, \bar{m}_t)$ *is a solution to the problem of the consumer t given $\bar{p}_t$ and $\bar{p}_{t+1}$ for all t.*
- $\bar{x}_t^y + \bar{x}_{t-1}^o = \omega^y + \omega^o$ *for all t.*
- $\bar{m}_t = M$ *for all t.*

3.3

Definition 6 *An* **strongly Pareto optimal allocation** $(\tilde{x}_t)_{t\in\mathbb{Z}}$ *is an allocation such that there is no other allocation $(x_t)_{t\in\mathbb{Z}}$ with*

- $x_t^y + x_{t-1}^o = \omega^y + \omega^o$ *for all t.*
- $u_t(x_t) \geq u_t(\tilde{x}_t)$ *for all t with at least one "$>$".*

Suppose that $(x_t)_{t\in\mathbb{Z}}$ is an allocation. Take $a_0 > 0$ of the good from consumer $t = 0$ when young and give it to consumer -1. Then consumer $t = -1$ is better off. Take a_1 from consumer $t = 1$ when young and give it to consumer 0 when old and let a_1 be so large that $x_t^y - a_0 + \sqrt{x_t^o + a_1} > x_t^y + \sqrt{x_t^o}$ - this is possible because consumption when young is unbounded downward. Continue like this, then every consumer from consumer $t = -1$ and forward are better off. The argument rests on consumption when young being unbounded.

3.4

The difference equation is simply

$$f^y(p_t, p_{t+1}, p_t\omega^y + p_{t+1}\omega^o) + f^o(p_{t-1}, p_t, p_{t-1}\omega^y + p_t\omega^o) \;=\; \omega^y + \omega^o.$$

You can fill in the expression for the demands found in 3.1.