LM August 2016 - Sommerskole
Vejl. Løsninger.

Opg 1

1) Da $\dim(V)=5$ viser vi at de fem vektorer er lin. uafh.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

oplagt lin. uafh og dermed en basis.

2)

$$Tx=0 \Leftrightarrow x_1 - x_2 - x_3 - x_4 - x_5 = 0$$

$$\begin{bmatrix} 1 & -1 & -1 & -1 & -1 \end{bmatrix} \quad x_2 = s,\ x_3 = t,\ x_4 = r,\ x_5 = u$$

$$x_1 = x_2 + x_3 + x_4 + x_5 = s + t + r + u$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{N(T)} \qquad s,t,r,u \in \mathbb{R}$$

De fire angivne vektorer udgør en basis for $N(T)$. Da $N(T) \neq \{\bar{0}\}$ er T ikke injektiv.

3) $Tx = y$. Jfr 2) fås

$$x_1 = y + x_2 + x_3 + x_4 + x_5 \text{ , dvs}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$s, t, r, u \in \mathbb{R}$.

4) $\alpha_1 u_1 + \alpha_2 v + \alpha_3 w + \alpha_4 u_4 + \alpha_5 u_5 = u_3 - u_4$

Mht basen u_1, u_2, u_3, u_4, u_5 har vi koordinater som giver

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ R_2 + R_3 \\ -R_3 \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

koordinaterne er $\underline{\underline{(0, 1, -1, -1, 0)}}$ mht u_1, v, w, u_4, u_5.

Opg 2

1) $$Av = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix} = \sqrt{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Heraf ses det ønskede, egenværdien er $\sqrt{2}$.

2) $$\det(A-\lambda E) = (\sqrt{2}-\lambda)\Big((1-\lambda)(-1-\lambda)-1\Big) = 0$$

$$\lambda = \sqrt{2} \text{ eller } \lambda^2-2=0, \text{ dvs}$$

$$\lambda = \sqrt{2} \text{ eller } \lambda = -\sqrt{2}, \text{ med}$$

$am(\sqrt{2}) = gm(\sqrt{2}) = 2$ (da A symmetrisk).
$am(-\sqrt{2}) = gm(-\sqrt{2}) = 1$.

3) Jfr. spektralsætn. er

$$A = QDQ^T, \text{ med } D = \begin{bmatrix} \sqrt{2} & & \\ & -\sqrt{2} & \\ & & \sqrt{2} \end{bmatrix}$$

og Q ortogonal.

Så er $A^4 = QD^4Q^T$. Her er $D^2 = 2E$, hvorfor $D^4 = 4E$ så

$$A^4 = Q4EQ^T = 4QEQ^T = \underline{\underline{4E}}$$

4) $$(A^2-A)(A^2+A) = A^4 - A^2 = 4E - 2E = 2E = A^2$$

5) $A^{2k} = (A^2)^k = (2E)^k = 2^k E$, dvs

$\det(A^{2k}) = \det(2^k E) = (2^k)^3 = 8^k$.

6) $A^{-1}v = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

7) $A^{2k+1} = A A^{2k} = A(2^k E) = 2^k A$

så $A^{2k+1} v = 2^k A v = 2^k \sqrt{2}\, v = (0, 0, 2^k \sqrt{2})$.

Opg 3 $\int \sin^2((a+b)x)\cos(bx)\,dx =$

$\int \left(\frac{e^{i(a+b)x} - e^{-i(a+b)x}}{2i}\right)^2 \left(\frac{e^{ibx} + e^{-ibx}}{2}\right) dx =$

$-\frac{1}{8}\int \left(e^{i2(a+b)x} + e^{-i2(a+b)x} - 2\right)\left(e^{ibx} + e^{-ibx}\right) =$

$-\frac{1}{8}\int e^{i(2a+3b)x} + e^{-i(2a+3b)x} + e^{i(2a+b)x} + e^{-i(2a+b)x}$
$- 2(e^{ibx} + e^{-ibx})\ dx$

$-\frac{1}{4}\int \cos(2a+3b)x + \cos(2a+b)x - 2\cos(bx)\ dx$

$= -\frac{1}{4}\left(\frac{\sin(2a+3b)x}{2a+3b} + \frac{\sin(2a+b)x}{2a+b} - 2\frac{\sin(bx)}{b}\right) + k.$

3

2) $$\frac{z}{2} + \frac{i}{z} = 1+i \iff$$

$$\frac{1}{2}z^2 + i = (1+i)z \iff$$

$$\frac{1}{2}z^2 - (1+i)z + i = 0.$$

Andengradsligu - diskriminanten er

$$B^2 - 4AC = \left(-(1+i)\right)^2 - 4 \cdot \frac{1}{2} \cdot i = 2i - 2i = 0$$

$$z = \frac{-B \pm \sqrt{0}}{2a} = \frac{(1+i)}{2 \cdot \frac{1}{2}} = \underline{\underline{1+i}}.$$

4

$$\sum_{n=0}^{\infty} (g(x))^n, \quad g(x) = a^2x^2 - 2ax + 1 = (ax-1)^2$$

1) konvergent for $|g(x)| < 1$, dvs

$$-1 < (ax-1)^2 < 1 \quad \text{~~vare~~}, \text{ dvs}$$

$$(ax-1)^2 < 1 \iff$$

$$-1 < ax - 1 < 1$$

$0 < ax$ og $ax - 1 < 1$

$\underline{\underline{0 < x}}$ $\quad ax < 2$

$x < \frac{2}{a}$

$$\underline{\underline{x \in \left]0, \frac{2}{a}\right[}}$$

2) Sumfunktionen er $f(x) = \dfrac{1}{1-(ax-1)^2}$, $x \in M$.

3)

f har monotoniforhold som g.

$g'(x) = 2(ax-1) \cdot a = 0 \Leftrightarrow x = \frac{1}{a}$

x	0		$\frac{1}{a}$		$\frac{2}{a}$
f'		$-$	0	$+$	
f		$\searrow$	lok. min.	$\nearrow$	

f er aftagende i $]0, \frac{1}{a}]$
f er voksende i $[\frac{1}{a}, \frac{2}{a}[$

4) $f(\frac{1}{a}) = \dfrac{1}{1-(1-1)^2} = 1$

$f(x) \to \infty$ for $x \to 0^+$
$f(x) \to \infty$ for $x \to \frac{2}{a}^-$

$Vm(f) = [1, \infty[$.

f er <u>ikke</u> injektiv da f har min i et indre punkt.

5) $f(x) = y \Leftrightarrow x = \frac{1}{a}\left(1 \pm \sqrt{\frac{y-1}{y}}\right)$, $\underline{y \geq 1}$.