Macro III - exam solutions (June 8, 2020)

General remarks

Please grade each item of each question between 0 and 20 points. Thus the maximum possible grade of the exam is 180.

Mathematical errors reduce the grade for the item in which they were made, not for subsequent items that carry the mistake (unless the student arrives at an infeasible solution and is unaware of the inconsistency, e.g. stating that capital holdings are negative).

1 False. Increasing the size of a pay-as-you-go social security system requires higher taxes that, on one hand reduce the young's available income, but at the same time increase their expected future income when retired. Thus, the initial young want to borrow against their future income and the only way to do this is from the rest of the world. Therefore, there is an initial current account deficit.

2 False. In the first best with no borrowing constraints the equilibrium interest rate is $R = \frac{1}{\beta} > 1$ as household smooth consumption perfectly (at different levels). If money is valued, which happens if there are borrowing constraints, the monetary equilibrium must feature $R = 1$ to satisfy market clearing. This implies there is no perfect consumption smoothing and therefore the equilibrium is not the first best.

3 True. If a country has a temptation to default in some states of nature, the borrowing (or insurance) contract must give the country higher consumption in these states to prevent default. This higher consumption must come at the expense of lower consumption in other states of nature. Therefore the country cannot get full consumption insurance.

4 a) The Lagrangian is given by (note that the problem can be solved using the intertemporal budget constraint)

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \lambda_t \left(w_t + R_t k_t - c_t - k_{t+1}\right)\right]$$

The first order conditions are given by

$$\begin{aligned}\frac{d\mathcal{L}}{dc_t} &= 0 \longrightarrow \frac{1}{c_t} - \lambda_t = 0, \\ \frac{d\mathcal{L}}{dk_{t+1}} &= 0 \longrightarrow -\lambda_t + \beta R_{t+1}\lambda_{t+1} = 0\end{aligned}$$

The Euler equation is given by,

$$\frac{1}{c_t} = \beta R_{t+1}\frac{1}{c_{t+1}}.$$

The interpretation is that the household makes consumption saving choices such that the marginal rate of substutition between current and future consumption equals the marginal rate of transformation, R_{t+1}.

From the firms' maximization of profits we know that

$$\begin{aligned}(1-\alpha)K_t^{i\alpha}h_t^{1-\alpha}L_t^{i-\alpha} &= (1-\alpha)K_t^{\alpha}h_t^{1-\alpha} = w_t, \\ \alpha K_t^{i\alpha-1}\left(h_t L_t^i\right)^{1-\alpha} &= \alpha K_t^{\alpha-1}h_t^{1-\alpha} = r_t,\end{aligned}$$

since all firms choose the same capital labor ratio, which given that $L_t = 1$, is given by K_t.

When $\theta = 0$, $h_t = 1$. Replacing in the equations above gives wages and interest rate, $(R_t = 1 + r_t - \delta)$. In steady state, $\beta(1 + \alpha K^{*\alpha-1} - \delta) = 1$ (as $c_t = c_{t+1}$) implying

$$K^* = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{\frac{-1}{1-\alpha}} = \Psi.$$

Note that $K^* = \Psi$, in that sense this was a "convenient normalization". Steady state consumption can be derived from the budget constraint

$$c^* = K^{*\alpha} - K^*.$$

b) With $\theta > 0$, productivity increases with capital. But when $K_t = \Psi$, $h_t = 1$. Therefore the steady state is the *same* as that characterized in a), as the Euler equation is not affected, and the $\dot{k} = 0$ equation is below the one found in a) for $K < \Psi$, above for $K > \Psi$, and crosses it when $K = \Psi$. Given that $K_0 < K^*$, $h_t < 1$ along the transition to steady state, therefore both the wage and interest rate are lower than found in a) (for same level of capital).

Given that households have logarithmic preferences we know that initial consumption is given by the propensity to consume out of wealth times wealth. The former is the same in a) and b) (only determined by β), while the household is unambiguously poorer in b) than in a). this follows since wages are proportional to h_t while the gross interest rate changes less than proportionally since there is the $1-\delta$ term that is unaffected by h_t. Thus, unambiguously, c_0 is lower in b) than in a).

c) The central planner internalizes the effect of capital on labor productivity. This implies that the planner perceives a higher rate of return on capital than the private sector. It will choose to invest up to the point in which the social return to capital equals the inverse of β, where the social return to capital is given by

$$1 + (\alpha + (1-\alpha)\theta)\bar{K}^{\alpha+(1-\alpha)\theta-1}\Psi^{-\theta(1-\alpha)} - \delta, \tag{1}$$

This gives

$$\bar{K} = \left[\frac{\Psi^{\theta(1-\alpha)}}{\alpha+\theta}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{\frac{-1}{1-\alpha-(1-\alpha)\theta}} = \left[\frac{\alpha+\theta}{\alpha}\frac{\Psi^{\frac{1}{1-\alpha}}}{\Psi^{\theta(1-\alpha)}}\right]^{\frac{1}{1-\alpha-(1-\alpha)\theta}},$$

which is larger than K^* when $\theta > 0$. Another way to see this is to evaluate (1) at $K = K^*$ and verify that this gives $R > \frac{1}{\beta}$.

5 a) Characterizing individual saving behavior requires setting up the problem of workers. We start with residents

$$\begin{aligned} \max_{s_t, c_{1t}, c_{2t+1}} \quad & \ln(c_{1t}) + \beta \ln(c_{2t+1}) \\ \text{s.t.} \quad & c_{1t} = w_t(1-\tau) - s_t^r \\ & c_{2t+1} = (s_t^r + \tau w_t) r_{t+1} \end{aligned}$$

Solving this problem and finding the Euler equation, from which

$$c_{2t+1} = \beta r_{t+1} c_{1t} \tag{2}$$

Replacing from period constraints we get residents' savings

$$s_t^r = \frac{\beta}{1+\beta} w_t - \tau w_t. \tag{3}$$

Migrants saving can be characterized in the same way, and clearly follow from making

$\tau = 0$ in (3),

$$s_t^m = \frac{\beta}{1+\beta} w_t.$$

Migrants save more than residents since they do not rely on the social security system.

To get capital accumulation, we first note that average (private plus public) per capita saving in capital is the *same* for residents and migrants. This follows since for residents the social security system saves τw_t so $s_t^r + \tau w_t = s_t^m$. Thus, average saving in capital, and capital accumulation since there is no population growth, is given by (using equilibrium expressions for wage $w = (1-\alpha)Ak^\alpha$)

$$k_{t+1} = s_t = \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta}(1-\alpha)Ak_t^\alpha.$$

Imposing steady state we get

$$k^* = \left[\frac{\beta}{1+\beta}(1-\alpha)A\right]^{\frac{1}{1-\alpha}}. \tag{4}$$

From equation (2) we see that if $\tau > \frac{\beta}{1+\beta}$ desired private savings would be negative. Since this is not possible, then in that case we would have a corner solution with $s_t^r = 0$. To avoid this case is that we impose the restriction $\tau > \frac{\beta}{1+\beta}$. Note that capital accumulation is then independent of τ.

b) The shock is such that in the first period the ratio of workers to the elderly (or holders of capital) is $\frac{1}{1+n}$, and in all subsequent periods is 1 again since there is no population growth (the shock affects the population level). What is different in the setup is that now all workers will contribute, and benefit, from social security. But, since the return of social security is the same as the return from private saving, capital accumulation is independent of τ as seen in a) above.

In the first period, $t = 0$, the wage is higher than in the previous steady state, since now workers have more capital (k_0 becomes $k_0 = (1+n)k^*$). Thus, capital accumulation will be higher than in the initial steady state, $k_1 > k^*$. But the steady state will be the same as the one characterized in a) since nothing changed in (4).

c) Since the absence of immigrants reduces the workforce for a given level of capital in the first period, this reduces the interest rate. This makes the old to be strictly worse off. In b) we argued that $w_0 > w^*$ because of capital deepening. The disposable income of the young residents in the first period is given by $w_0(1-\tau) = (1-\alpha)A\left((1+n)k^*\right)^\alpha(1-\tau) > w^*(1-\tau)$. Thus, the disposable income of the initial young generation of residents is higher than in the steady state.