(1)

LM Juni 17

Opg 1

Vi løser $Lx=y$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 1 & 0 & 1 & y_2 \\ 0 & 1 & 1 & y_3 \\ 0 & 1 & 0 & y_4 \end{array}\right] \begin{array}{l} R_1-R_3 \\ R_2-R_1 \\ R_3-R_4 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & y_1-y_3 \\ 0 & -1 & 0 & y_2-y_1 \\ 0 & 0 & 1 & y_3-y_4 \\ 0 & 1 & 0 & y_4 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ R_4+R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & y_1-y_3 \\ 0 & 1 & 0 & y_1-y_2 \\ 0 & 0 & 1 & y_3-y_4 \\ 0 & 0 & 0 & y_4+y_2-y_1 \end{array}\right]$$

1) Heraf ses, at $N(L)=\{\bar{0}\}$, og L er injektiv, samt at

2) $R(L)=\text{span}\{\text{søjlerne}\}=\text{span}\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}\right\}$

hvor søjlerne er lin. uafh. og udgør en basis for $R(L)$

L er <u>ikke</u> surjektiv da $\dim R(L)=3<4$.

Dim. sætn.: $3-0=3$.

3) $x = (y_1 - y_3,\ y_1 - y_2,\ y_3 - y_4)$

hvor $y_4 + y_2 - y_1 = 0$ for at $y \in R(L)$

4) Da $2+4-6=0$ vil $y \in R(L)$.

Ligningen er

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 5 \\ 2 \end{bmatrix}$$

med totalmatrix

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

Fra 3) ses

løsningen at være

$$(\alpha_1, \alpha_2, \alpha_3) = (6-5,\ 6-4,\ 5-2) = \underline{\underline{(1,2,3)}}$$

som er koordinaterne.

Opg 2

1) Da $Av_1 = (0,0,0) = 0v_1$,

$$Av_2 = (0,1,0) = 1v_2 \quad \text{og}$$

$$Av_3 = (2,0,2) = 2v_1$$

er egenværdierne $\underline{\underline{0, 1, 2}}$.

2) $A^4(A-E) = A^5 - A^4$. Egenværdiere er

$0^5 - 0^4 = \underline{\underline{0}}$, $1^5 - 1^4 = \underline{\underline{0}}$, og $2^5 - 2^4 = \underline{\underline{16}}$.

3) $f(A) = Q f(D) Q^T$, hvor

$$f(D) = \begin{bmatrix} f(0) & & \\ & f(1) & \\ & & f(2) \end{bmatrix} \quad \text{og}$$

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

$$f(A) = \begin{bmatrix} \frac{1}{2}f(0) + \frac{1}{2}f(2) & 0 & -\frac{1}{2}f(0) + \frac{1}{2}f(2) \\ 0 & f(1) & 0 \\ -\frac{1}{2}f(0) + \frac{1}{2}f(2) & 0 & \frac{1}{2}f(0) + \frac{1}{2}f(2) \end{bmatrix}$$

4) $f(A)(v_1+v_2+v_3) = f(A)v_1 + f(A)v_2 + f(A)v_3$

$= f(0)v_1 + f(1)v_2 + f(2)v_3$

$= (-f(0), 0, f(0)) + (0, f(1), 0) + (f(2), 0, f(2))$

$= (f(2)-f(0); f(1); f(0)+f(2))$

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opg 3

1) $\int \sin^2(bx)\cos((a+b)x)dx =$

$$\int \left(\frac{e^{i2bx}+e^{-i2bx}-2}{-4}\right)\left(\frac{e^{i(a+b)x}+e^{-i(a+b)x}}{2}\right)dx$$

$$-\frac{1}{8}\int e^{i(a+3b)x}+e^{-i(a+3b)x}+e^{i(a-b)x}+e^{-i(a-b)x}$$

$$+2(e^{i(a+b)x}+e^{-i(a+b)x})\,dx =$$

$$-\frac{1}{4}\int \cos((a+3b)x)+\cos((a-b)x+2\cos((a+b)x)\,dx$$

Hvis ingen af argumenterne er 0 fås så

$$= -\frac{1}{4}\left(\frac{1}{a+3b} \sin((a+3b)x) + \frac{1}{a-b} \sin((a-b)x) \right.$$

$$\left. + 2 \frac{1}{a+b} \sin((a+b)x) \right) + k.$$

Hvis f.eks. $a+3b=0$ er

$\cos((a+3b)x) = 1$, med stamfunktionen

x, som så skal erstatte første led.

Analogt for de øvrige muligheder.

2)

$$iz^2+1=0 \iff -z^2+i=0 \iff z^2=i.$$

Skriv $z=x+iy$, med $x,y \in \mathbb{R}$.

Så er $z^2 = x^2-y^2+i2xy$, hvorfor

$$x^2-y^2=0 \text{ og } 2xy=1.$$

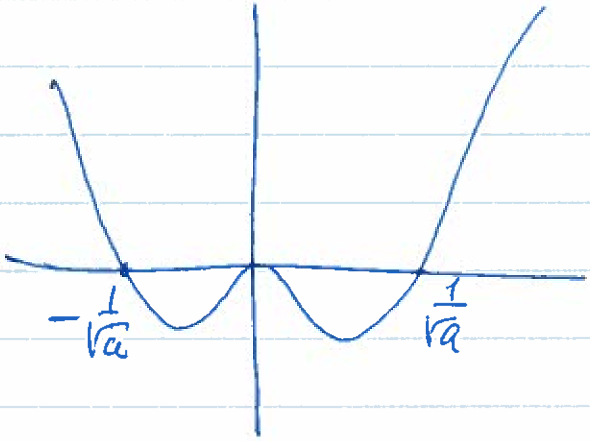
Så er $y=\frac{1}{2x} \Rightarrow x^2 - \frac{1}{4x^2} = 0$

$$\iff 4x^4-1=0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}.$$

Så fås $z = \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right).$

Opg 4 For $a > 0$ er $ax^4 - x^2 = x^2(ax^2 - 1)$

lige, med udseendet



Så skal $-1 < ax^4 - x^2 < 1$ for $x \in]-2, 2[$.

For $x = \pm 2$ fås $a \cdot 16 - 4 = 1 \Leftrightarrow a = \frac{5}{16}$.

$ax^4 - x^2$ har ~~vem~~. ekstremum for $4ax^3 - 2x = 0 \Leftrightarrow$
$x(4ax^2 - 2) = 0$

dvs $x = 0$ eller $4ax^2 - 2 = 0 \Leftrightarrow$

$$x^2 = \frac{1}{2a} \Leftrightarrow$$

$$x = \pm\sqrt{\frac{1}{2a}} \ .$$

Værdien er $a\left(\sqrt{\frac{1}{2a}}\right)^4 - \left(\sqrt{\frac{1}{2a}}\right)^2$

$$= a\frac{1}{4a^2} - \frac{1}{2a} = \frac{1}{4a} - \frac{1}{2a} = -\frac{1}{4a}$$

altså skal $-\frac{1}{4a} > -1 \Leftrightarrow 4a > 1$

$$\underline{a > \frac{1}{4}}$$

Heraf ses, at $a=\frac{5}{16}$.

2) $f(x)=\frac{1}{1-g(x)}=\frac{1}{1-\left(\frac{5}{16}x^4-x^2\right)}$, $-2<x<2$.

3) $4\cdot\frac{5}{16}x^3-2x=0 \Leftrightarrow$

$x\left(\frac{5}{4}x^2-2\right)=0 \Leftrightarrow$

$x=0$ eller $x^2=2\cdot\frac{4}{5}=\frac{8}{5}$ så

$x=0,\ x=-\sqrt{\frac{8}{5}},\ x=\sqrt{\frac{8}{5}}$

x	-2		$-\sqrt{\frac{8}{5}}$		0		$\sqrt{\frac{8}{5}}$		2
f'		$-$	0	$+$	0	$-$	0	$+$	
f		$\searrow$	lok. min	$\nearrow$	lok max	$\searrow$	lok min	$\nearrow$	

4) Da $\frac{5}{16}\left(\sqrt{\frac{8}{5}}\right)^4-\left(\sqrt{\frac{8}{5}}\right)^2=\frac{5}{16}\cdot\frac{64}{25}-\frac{8}{5}=-\frac{4}{5}$

Da er $f\left(\sqrt{\frac{8}{5}}\right) = \frac{1}{1+\frac{4}{5}} = \frac{5}{9}$.

$Vm(f) = \left[\frac{5}{9}, \infty\right[$

5) +6) $f(x) = y$, $y \geq \frac{5}{9}$.

$$\frac{1}{1-g(x)} = y \iff \frac{1}{y} = 1 - g(x) \iff$$

$$g(x) = 1 - \frac{1}{y} = \frac{y-1}{y}.$$

$$\frac{5}{16}x^4 - x^2 = \frac{y-1}{y} \iff$$

$$\frac{5}{16}x^4 - x^2 + \frac{1-y}{y} = 0 \iff$$

$$x^2 = \frac{1 \pm \sqrt{1 - 4 \cdot \frac{5}{16} \cdot \frac{1-y}{y}}}{\frac{10}{16}}$$

$$= \left(1 \pm \sqrt{1 - \frac{5}{4}\left(\frac{1-y}{y}\right)}\right) \cdot \frac{16}{10}.$$

Løsningerne er generelt

$$x = \pm\sqrt{\left(1 \pm \sqrt{1 - \frac{5}{4}\left(\frac{1-y}{y}\right)}\right) \cdot \frac{8}{5}}$$

og der er

2 løsninger for $y = \frac{5}{9}$

4 løsninger for $\frac{5}{9} < y < 1$

3 løsninger for $y = 1$

2 løsninger for $y > 1$.