LM August 2020

① 1) $L: \mathbb{R}^5 \to \mathbb{R}^2$, så $n=5, m=2$.

2) $\underline{a=0}$ Da $n>m$ er L aldrig injektiv.

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \leadsto \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Frie: $x_2 = r$, $x_3 = s$, $x_5 = t$

$x_4 = -x_5 = -t$, $x_1 = -2x_3 = -2s$

$$N(L): \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}_1} + s \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}_2} + t \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\underline{v}_3}, \quad s,t,r \in \mathbb{R}$$

3) For $\underline{a=0}$ udgør vektorerne ovenfor fra sp2) en basis for $N(L)$.

For $\underline{a \neq 0}$ fås

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 2 \\ 0 & 0 & a & 1 & 1 \end{bmatrix} \xrightarrow[\frac{1}{a}R_2]{R_1 - \frac{2}{a}R_2} \begin{bmatrix} 1 & 0 & 0 & 2-\frac{2}{a} & 2-\frac{2}{a} \\ 0 & 0 & 1 & \frac{1}{a} & \frac{1}{a} \end{bmatrix}$$

Frie $x_2 = r$, $x_4 = s$, $x_5 = t$

$x_3 = -\frac{1}{a}x_4 - \frac{1}{a}x_5 = -\frac{1}{a}s - \frac{1}{a}t$, $x_1 = -(2-\frac{2}{a})x_4 - (2-\frac{2}{a})x_5$
$= (\frac{2}{a}-2)s + (\frac{2}{a}-2)t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{w}_1} + s \underbrace{\begin{bmatrix} \frac{2}{a}-2 \\ 0 \\ -1/a \\ 1 \\ 0 \end{bmatrix}}_{\underline{w}_2} + t \underbrace{\begin{bmatrix} \frac{2}{a}-2 \\ 0 \\ -1/a \\ 0 \\ 1 \end{bmatrix}}_{\underline{w}_3}, \quad s,t,r \in \mathbb{R}$$

Her er $\underline{w}_1, \underline{w}_2, \underline{w}_3$ en basis for $N(L)$.

4) L er altid surjektiv (2 lin. uafh. søjler)

5) Her må vi skelne mellem $a=0$ og $a \neq 0$

For $a=0$:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 2 & 2 & y_1 \\ 0 & 0 & 0 & 1 & 1 & y_2 \end{array}\right] \overset{R_1-2R_2}{\longleftrightarrow} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & y_1-2y_2 \\ 0 & 0 & 0 & 1 & 1 & y_2 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y_1-2y_2 \\ 0 \\ 0 \\ y_2 \\ 0 \end{bmatrix} + r\underline{v}_1 + s\underline{v}_2 + t\underline{v}_3, \quad s,t,r \in \mathbb{R}$$

$\underline{v}_1, \underline{v}_2, \underline{v}_3$ angivet i sp 2).

For $a \neq 0$ fås

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 2 & 2 & y_1 \\ 0 & 0 & a & 1 & 1 & y_2 \end{array}\right] \begin{matrix} R_1-\frac{2}{a}R_2 \\ \frac{1}{a}R_2 \end{matrix} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2-\frac{2}{a} & 2-\frac{2}{a} & y_1-\frac{2}{a}y_2 \\ 0 & 0 & 1 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a}y_2 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y_1-\frac{2}{a}y_2 \\ 0 \\ \frac{1}{a}y_2 \\ 0 \\ 0 \end{bmatrix} + r\underline{w}_1 + s\underline{w}_2 + t\underline{w}_3, \quad s,t,r \in \mathbb{R}$$

$\underline{w}_1, \underline{w}_2, \underline{w}_3$ angivet i sp 3).

② 1/ Vi kan vælge $\underline{v}_3 = (0,0,1)$, så er $\underline{v}_1, \underline{v}_2, \underline{v}_3$ indb. ortogonale.

2/ $\det(A-\lambda E) = \det \begin{bmatrix} 1-\lambda & & \\ & 4-\lambda & \\ & & 9-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda)(9-\lambda)$

$= -\lambda^3 + 14\lambda^2 - 49\lambda + 36 = P_A(\lambda).$

3) Da $0 \notin \sigma(A)$ er A invertibel

4) Hvis $B^2 = A$ skal $D_B = \pm\sqrt{D_A} = \begin{bmatrix} \pm 1 & & \\ & \pm 2 & \\ & & \pm 3 \end{bmatrix}$

så er $B = Q D_B Q^T$. Vi vælger + alle steder.

$$B = Q \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 2/\sqrt{2} & 2/\sqrt{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+1 & -\frac{1}{2}+1 & 0 \\ -\frac{1}{2}+1 & \frac{1}{2}+1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

5) $B(\underline{v}_1 + \underline{v}_2 + \underline{v}_3) = 1\underline{v}_1 + 2\underline{v}_2 + 3\underline{v}_3 = \text{------}$

3) $\int \cos(mx)\sin(2x)\sin(3x)\,dx$

$$= \int \left(\frac{e^{imx}+e^{-imx}}{2}\right)\left(\frac{e^{i2x}-e^{-i2x}}{2i}\right)\left(\frac{e^{i3x}-e^{-i3x}}{2i}\right)dx$$

$$= -\frac{1}{8}\int \left(e^{imx}+e^{-imx}\right)\left(e^{i5x}-e^{-ix}-e^{ix}+e^{-i5x}\right)dx$$

$$= -\frac{1}{8}\int e^{i(5+m)x} - e^{i(m-1)x} - e^{i(m+1)x} + e^{i(m-5)x}$$

$$+ e^{i(5-m)x} - e^{-i(m+1)x} - e^{-i(m-1)x} + e^{-i(m+5)x}\,dx$$

$$= -\frac{1}{8}\int \left(e^{i(m+5)x}+e^{-i(m+5)x}\right) - \left(e^{i(m-1)x}+e^{-i(m-1)x}\right)$$

$$-\left(e^{i(m+1)x}+e^{-i(m+1)x}\right) + \left(e^{i(m-5)x}+e^{-i(m-5)x}\right)dx$$

$$= -\frac{1}{4}\int \cos(m+5)x - \cos(m-1)x - \cos(m+1)x + \cos(m-5)x\,dx$$

$$= -\frac{1}{4}\left(\frac{1}{m+5}\sin(m+5)x - \frac{1}{m-1}\sin(m-1)x - \frac{1}{m+1}\sin(m+1)x\right.$$

$$\left. + \frac{1}{m-5}\sin(m-5)x\right) + k.$$

for $m \in \mathbb{N} \setminus \{1,5\}$.

Hvis $m \in \{1,5\}$ er $\cos = 1$ og stamfkt. er x.

(3)

2) $(z - i10)(1+i) = i8(1-i)$

$$z - i10 = \frac{i8(1-i)}{1+i}$$

$$z - i10 = \frac{i8(1-i)^2}{(1+i)(1-i)} = \frac{i8(1-1-2i)}{2}$$

$$z - i10 = i8(-i) = 8$$

$$\underline{\underline{z = 8 + i10}}$$

(4)

$$f(x) = \sum_{n=0}^{\infty} (x\ln(x) - x)^n.$$

1) Her må $x > 0$ til en start.

Vi løser $|x\ln(x) - x| < 1$.

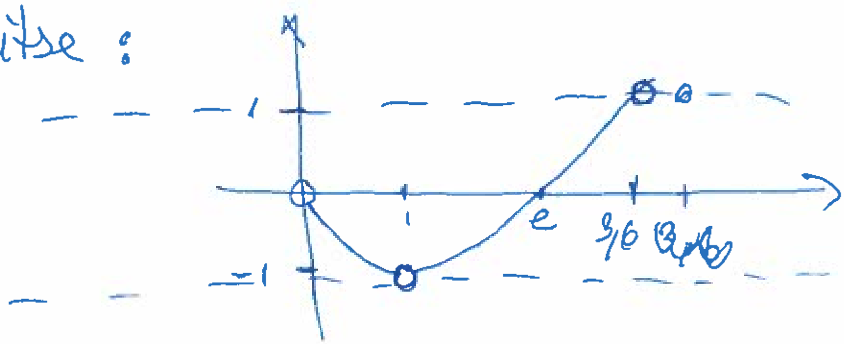
$x\ln(x) - x = 1$ løses af $x = 3{,}6$

$x\ln(x) - x = -1$ løses af $x = 1$ (oplagt).

Da $(x\ln(x) - x)' = \ln(x)$, hvorfor $x = 1$ er et lok. min for $x\ln(x) - x$, ser vi at f er veldefineret for

$$x \in M = \,]0;1[\, \cup \,]1;3{,}6[$$

skitse:



$$f(x) = \frac{1}{1 - (x\ln(x) - x)}, \quad x \in M.$$

3)

x	0		1		$3{,}6$
g	≀	÷	≀	$+$	≀
f	≀	↘	≀	↗	≀

4) For $x \to 0^+$ vil $f(x) \to 1$

For $x \to 1^-$ vil $f(x) \to \frac{1}{2}$

For $x \to 1^+$ vil $f(x) \to \frac{1}{2}$

For $x \to 3{,}6^-$ vil $f(x) \to \infty$

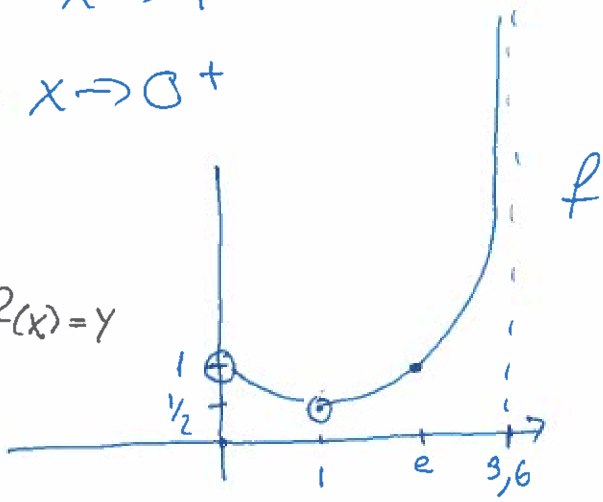
idet $x\ln(x) - x \to 1$ for $x \to 3{,}6^-$

og $x\ln(x) - x \to -1$ for $x \to 1^\pm$

og $x\ln(x) - x \to 0$ for $x \to 0^+$

$Vm(f) = \left]\frac{1}{2}, \infty\right[$

ej injektiv (åbenlyst), da $f(x) = y$ har 2 løsn. for $\frac{1}{2} < y < 1$.



5) For $y \geq 1$ har $f(x) = y$ kun 1 løsning.