Written Exam for M.Sc. in Economics 2008-II

Investment Theory

12. August 2010

Master course

Corrections

Exercise 1.

1.a It could be entry in a market. Perhaps a market with competition between network goods: either our product becomes the standard ($d = 160$) or not; and, if not either the market is split between the different networks ($d = 120$) or one of the other networks becomes the standard ($d = -40$).

1.b The exit option is only relevant at date 2 for $d = -40$. The NPV at date $t = 2$ for $d = -40$ is

$$-\sum_{n=0}^{\infty} \frac{40}{1.05^n} \quad = \quad -\frac{40}{1 - 1/1.05} \quad = \quad -40 \cdot 21 \quad = \quad -840.$$

For $E \leq 840$ the optimal strategy for the exit option is

$$\begin{cases} d \leq -40 & \text{exit} \\ \\ d > -40 & \text{continue.} \end{cases}$$

For $E > 840$ the optimal strategy for the exit option is not to use it. The value of the exit option at date $t = 0$ is

$$NPV_E \quad = \quad \frac{1}{4} \cdot \frac{1}{1.05^2} \cdot \max\{840 - E, 0\}.$$

1.c The NPV of the strategy is

$$\begin{aligned} NPV_0 \quad &= \quad -I + \frac{1}{2} \sum_{n=0}^{\infty} \frac{100}{1.05^n} + NPV_E \\ &= \quad -I + 2100 + \frac{1}{2^2} \cdot \frac{1}{1.05^2} \cdot \max\{840 - E, 0\}. \end{aligned}$$

1.d The NPV of the strategy is

$$\begin{aligned} NPV_1 \quad &= \quad \frac{1}{2} \cdot \frac{1}{1.05} \left(-I + \sum_{n=0}^{\infty} \frac{160}{1.05^n} \right) \\ &= \quad \frac{1}{2} \cdot \frac{1}{1.05} (-I + 3360). \end{aligned}$$

Obviously the exit option is not used so it has no value.

1.e The NPV of the strategy is

$$NPV_2 = \frac{1}{2^2} \cdot \frac{1}{1.05^2} \left(-I + \sum_{n=0}^{\infty} \frac{120}{1.05^n} \right)$$
$$= \frac{1}{2^2} \cdot \frac{1}{1.05^2}(-I + 2520).$$

Obviously the exit option is not used so it has no value.

1.f The four relevant strategies are: enter at date $t = 0$; enter at date $t = 1$ if and only if $d = 160$; enter at date $t = 1$ if and only if $d = 160$ and enter at date $t = 2$ if and only if $d = 120$; and, don't enter. All other strategies are easily seen to be dominated by at least one of the four relevant strategies.

NPV of "enter at date $t = 1$" is

$$NPV_0 \;=\; -1000 + 2100 + \frac{1}{2^2} \cdot \frac{1}{1.05^2} \cdot 300 \;\approx\; 1168.$$

NPV of "enter at date $t = 1$ for $d = 160$" is

$$NPV_1 \;=\; \frac{1}{2} \cdot \frac{1}{1.05}(-1000 + 3360) \;\approx\; 1123.$$

NPV of "enter at date $t = 1$ for $d = 160$ or enter at date $t = 2$ for $d = 120$" is

$$\begin{aligned} NPV_{12} &= NPV_1 + NPV_2 \\ &= \frac{1}{2} \cdot \frac{1}{1.05}(-1000 + 3360) + \frac{1}{2^2} \cdot \frac{1}{1.05^2}(-1000 + 2520) \\ &\approx 1123 + 345 \\ &= 1468. \end{aligned}$$

Obviously the NPV of no entry is 0.

The optimal strategy is "enter at date $t = 1$ if and only if $d = 160$ and enter at date $t = 2$ if and only if $d = 120$". The value of the entry option is $NPV_{12} - NPV_0 \approx 300$ and the value of the exit option is zero because it is not used.

Exercise 2.

2.a It could be building a mine. There is a cost of getting the right to extract and building the mine I. P is the revenue from selling the mineral and C is the cost of extracting the mineral. S is the cost of suspending – sending workers home etc. and R is the cost of reactivating – hiring workers etc.

2.b There are three options in the project: entry; suspension; and, reactivation. The strategies for all options could be cut-off strategies.

If the project is not started the entry option is relevant:

$$\begin{cases} P < P^* & \text{wait} \\ P \geq P^* & \text{invest} \end{cases} \qquad F(P) = \begin{cases} ? & \text{for } P < P^* \\ V_A(P) - I & \text{for } P \geq P^* \end{cases}$$

If the project is active the suspension option is relevant:

$$\begin{cases} P \leq P_S & \text{suspend} \\ P > P_S & \text{continue.} \end{cases} \qquad V_A(P) = \begin{cases} V_S(P) - S & \text{for } P \leq P_S \\ ? & \text{for } P > P_S \end{cases}$$

If the project is suspended the reactivate option is relevant:

$$\begin{cases} P < P_R & \text{continue} \\ P \geq P_R & \text{reactivate.} \end{cases} \qquad V_S(P) = \begin{cases} ? & \text{for } P < P_R \\ V_A(P) - R & \text{for } P \geq P_R \end{cases}$$

It is implicitly assumed that $P^* > P_S$ and $P_R > P_S$: it doesn't make sense to invest in a project and then suspend it, because $I, S > 0$; and, it doesn't make sense to suspend and then reactivate because $S, R > 0$.

The functions should satisfy value matching (VM) + smooth pasting (SP) + $p \to 0 \Rightarrow F(P), V_S(P) \to 0$ + "no bubbles" for F and V_A.

We need to find P^*, P_S, P_R and the three "?".

2.c For V_A for $P > P_S$ consider the portfolio consisting of an active project and $-n$ units of the asset. Then the dividend rate of the portfolio is

$$\frac{P - C + dV_A(P) - ndQ}{V_A(P) - nQ}$$

which by use of Ito's Lemma becomes

$$\frac{P - C + \frac{1}{2}\sigma^2 P^2 V_A''(P) + \alpha P V_A'(P) - n(\alpha + \delta)Q}{V_A(P) - nQ} dt + \frac{\sigma P V_A'(P) - \sigma Q}{V_A(P) - nQ} dz.$$

Let $n = PV_A'(P)/Q$. Then there is no uncertainty about the dividend rate for the portfolio. Therefore the dividend rate is equal to r – otherwise there would be an arbitrage opportunity. Rewriting the dividend rate gives the following differential equation

$$\frac{1}{2}\sigma^2 P^2 V_A''(P) + (r - \delta) P V_A'(P) - r V_A'(P) + P - C \ = \ 0.$$

For V_S for $P < P_R$ the steps done for V_A gives the following differential equation

$$\frac{1}{2}\sigma^2 P^2 V_S''(P) + (r - \delta) P V_S'(P) - r V_S'(P) \ = \ 0.$$

2.d The solutions to second-order linear differential equations consists of a particular solution and the solutions to the homogenous part of the differential equations.

For V_A for $P > P_S$ the form of the mathematical solution is

$$V_A(P) \ = \ \frac{P}{\delta} - \frac{C}{r} + B_1 P^{\beta_1} + B_2 P^{\beta_2}$$

where $B_1, B_2 \in \mathbb{R}$ and $\beta_1 > 1$ and $\beta_2 < 0$ are solutions to

$$\frac{1}{2}\sigma^2(\beta - 1)\beta + (r - \delta)\beta - r \ = \ 0.$$

The form of the economically relevant solutions is

$$V_A(P) \ = \ \frac{P}{\delta} - \frac{C}{r} + B_2 P^{\beta_2}$$

where $B_1 = 0$ because of "no bubbles".

For V_S for $P < P_R$ the form of the mathematical solution is

$$V_S(P) \;=\; D_1 P^{\beta_1} + D_2 P^{\beta_2}$$

where $D_1, D_2 \in \mathbb{R}$. The form of the economically relevant solutions is

$$V_S(P) \;=\; D_1 P^{\beta_1}$$

where $D_2 = 0$ because of $P \to 0 \Rightarrow V_S(P) \to 0$.

2.e The undetermined constants are P_S, P_R, B_2, C_1. The equations are VM + SP.

VM for V_A

$$\frac{P_S}{\delta} - \frac{C}{r} + B_2 P_S^{\beta_2} \;=\; D_1 P_S^{\beta_1} - S.$$

SP for V_A

$$\frac{1}{\delta} + \beta_2 B_2 P_S^{\beta_2 - 1} \;=\; \beta_1 D_1 P_S^{\beta_1 - 1}.$$

VM for V_S

$$D_1 P_R^{\beta_1} \;=\; \frac{P_R}{\delta} - \frac{C}{r} + B_2 P_R^{\beta_2} - R.$$

SP for V_S

$$\beta_1 D_1 P_R^{\beta_1 - 1} \;=\; \frac{1}{\delta} + \beta_2 B_2 P_R^{\beta_2 - 1} - R.$$

It is not possible to obtain an analytical solution.

2.f For $V_A(P)$ for $P > P_S$: P/δ – NPV of P forever; C/r – NPV of C forever, the difference between the denominators for P and C differ because P is stochastic and C is fixed; and, $B_2 P^{\beta_2}$ – NPV of suspension (and reactivation and suspension and...), so we expect $B_2 > 0$. Note that the value of the option to suspend is decreasing in P, which reflects that the higher P is, the longer it is expected to take until suspension becomes relevant.

For $V_S(P)$ for $P < P_R$: $D_1 P^{\beta_1}$ – NPV for reactivation (and suspension and reactivation and...), so we expect $D_1 > 0$.

2.g For $F(P)$ for $P < P^*$ the steps in 2.c gives the differential equation

$$\frac{1}{2}\sigma^2 P^2 F''(P) + (r-\delta)PF'(P) - rF'(P) \;=\; 0.$$

This is identical to the equation for V_S.

The form of the mathematical solutions is

$$F(P) \;=\; A_1 P^{\beta_1} + A_2 P^{\beta_2}$$

where $A_1, A_2 \in \mathbb{R}$. The form of the economically relevant solutions is

$$F(P) \;=\; A_1 P^{\beta_1}$$

where $A_2 = 0$ because of $P \to 0 \Rightarrow F(P) \to 0$.

2.h the undetermined constants are P^* and A_1. The equations are VM + SP.

VM for F

$$A_1(P^*)^{\beta_1} \;=\; \frac{P^*}{\delta} - \frac{C}{r} + B_2(P^*)^{\beta_2} - I.$$

SP for F

$$\beta_1 A_1 (P^*)^{\beta_1 - 1} \;=\; \frac{1}{\delta} + \beta_2 B_2 (P^*)^{\beta_2 - 1}.$$

It is not possible to obtain an analytical solution.