Macro III - exam solutions (January 4, 2016)

General remarks

Please grade each item of each question between 0 and 20 points. Thus the maximum possible grade of the exam is 180.

Mathematical errors reduce the grade for the item in which they were made, not for subsequent items that carry the mistake (unless the student arrives at an infeasible solution and is unaware of the inconsistency, e.g. stating that capital holdings are negative).

1 False. Both the Lucas and Calvo models microfound a Philips curve positive relation between output and inflation. An announced change in monetary policy does not have real effects in the economy if all prices are flexible, as happens in the Lucas model. But if there are some nominal rigidities, as in the Calvo model, then a pre-announced change in monetary policy will have effects on output.

2 True. In these models hyperinflations that can be ruled out by having money to be essential because otherwise hyperinflation paths converge to zero real money holdings in finite time, thus violating optimality conditions from the maximum principle (technically $\dot{m}$ would then be strictly negative when $m = 0$).

3 False. In the Meltzer and Richard model only those increases in inequality that make the median voter poorer would lead to an increase in redistribution. In this model the determinant of redistribution is the difference between mean and median income. An increase in inequality (not affecting the mean) can make the median income increase or decrease depending on how incomes become more dispersed.

4 a) The wage and interest rate are determined by imposing equilibrium in factor markets where firms competitively demand labor and capital from households. Thus the student needs to maximize profit function for firms

$$\max_{L_t, K_t^i} {K_t^i}^{\alpha} {L_t^i}^{1-\alpha} - w_t L_t^i - r_t^L K_t^i$$

From FOC of firms' problem of maximizing profits we get

$$\begin{aligned}(1-\alpha)K_t^{i\alpha}L_t^{i-\alpha} &= (1-\alpha)k_t^{\alpha} = w_t \\ \alpha K_t^{i\alpha-1}L_t^{i1-\alpha} &= \alpha k_t^{\alpha-1} = r_t^L\end{aligned}$$

where we impose that in equilibrium all firms work with the same capital labor ratio, k, which must be equal to the ratio of aggregate capital to labor.

Note the relations $d_t = a_t = \frac{k_t}{1-\gamma}$. Market wage, $(1-\alpha)k_t^{\alpha} = (1-\alpha)((1-\gamma)a_t)^{\alpha}$, and deposit rate r_t^D times saving per capita are payments to households (in per capita terms). Thus household income on saving is $a_t r_t^D = a_t\left(\alpha((1-\gamma)a_t)^{\alpha-1}(1-\gamma)\right) = \alpha((1-\gamma)a_t)^{\alpha}$. Partial credit if wrongly derived but intuition is correct.

Control (c) and state variables (a).

b) Hamiltonian (it is irrelevant if set up as current value or present value, what matters is that the FOC are correct for each setup):

$$\begin{aligned}H_t &= \ln(c_t)e^{-(\rho-n)t} + \lambda_t\left(w_t + r_t^D a_t - c_t - na_t\right) \\ H_t^c &= \ln(c_t) + \mu_t\left(w_t + r_t^D a_t - c_t - na_t\right)\end{aligned}$$

with $\mu_t \equiv \lambda_t e^{(\rho-n)t}$.

Student gets full points if stating FOC assuming an interior solution (even if there is no explicit assumption of this, i.e. no penalty from failing to consider corner solution):

$$\begin{aligned}\frac{dH_t^c}{dc_t} &= \frac{1}{c_t} - \mu_t = 0 \\ \dot{\mu}_t &= -\frac{dH_t^c}{da_t} + (\rho-n)\mu_t = -\mu_t(r_t^D - \rho) \\ \lim_{t\to\infty} e^{-\rho t}\mu_t a_t &= 0\end{aligned}$$

Note that the law of motion of the state variable is also a FOC (derivative of Hamiltonian with respect to costate variable λ_t or μ_t).

In the derivation of the Euler equation, or Keynes-Ramsey condition it is important that r^D and not r or r^L or the marginal productivity of capital, be in the Euler equation:

$$\frac{\dot{c}_t}{c_t} = r_t^D - \rho$$

Interpretation is that consumption (in per capita terms) is increasing/falling over time as long as interest rate is above/below rate at which future consumption is discounted, and that for the logarithmic utility function, the instantaneous elasticity of substitution (inverse of coefficient of relative risk aversion), measuring the response of consumption growth rate to a given difference between r_t^D and ρ, is one.

Steady state is characterized by $\dot{a}_t = \dot{c}_t = 0$. Thus $\dot{a}_t = 0$ implies that $c_t = w_t + r_t^D a_t - na_t = (a_t(1-\gamma))^\alpha - na_t$. $\dot{c}_t = 0$ implies that $r_t^D = (1-\gamma)\alpha\,(a_t(1-\gamma))^{\alpha-1} = \rho$. This pins down the steady state saving per capita ($a^* = \frac{1}{1-\gamma}\left(\frac{\alpha(1-\gamma)}{\rho}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}(1-\gamma)^{\frac{\alpha}{1-\alpha}}$)

Phase diagram should show the $\dot{a}_t = 0$ and $\dot{c}_t = 0$ curves and the local dynamics of the variables in the four quadrants they define. The phase diagram should also have the saddle path of convergent dynamics to the steady state (and this correctly identified as the intersection of the $\dot{a}_t = 0$ and $\dot{c}_t = 0$ curves).

c) Denoting the distortionary capital income taxation by τ (assumed constant for simplicity, needed later for steady state) the new dynamic BC is given by

$$\dot{a}_t = w_t + r_t^D(1-\tau)a_t - c_t - na_t + z_t,$$

where z_t are the lump sum transfers receive from the government (in equilibrium it must be the case that $z_t = \tau r_t^D k_t$, but households do not know this). Since resources from taxes are returned to the households the $\dot{a}_t = 0$ is not affected by taxation. But taxes discourage savings since the $\dot{c}_t = 0$ is now given by $r_t^D(1-\tau) = \rho$. So steady state capital per capita is lower ($k^{*\prime} = \left(\frac{\alpha(1-\gamma)(1-\tau)}{\rho}\right)^{\frac{1}{1-\alpha}}$)

Consumption jumps up on impact as households respond to the lower after tax return by reducing savings (and thus have to increase consumption). From the new level consumption will embark on a downward trajectory consistent with the Euler equation, up to reaching the new steady state with lower capital and lower consumption.

5 a) Characterizing individual saving behavior requires setting up the problem of workers.

$$\begin{aligned}
\max_{c_{1t}, c_{2t+1}} \quad & \ln(c_{1t}) + \frac{1}{1+\rho}\ln(c_{2t+1}) \\
\text{s.t.} \quad & c_{1t} = w_t(1-\tau) - s_t \\
& c_{2t+1} = s_t(1+r_{t+1}) + \tau w_{t+1}
\end{aligned}$$

Solving this problem and finding the Euler equation, from which

$$c_{2t+1} = \frac{1+r_{t+1}}{1+\rho} c_{1t}$$

Replacing from period constraints we get individual savings

$$s_t = \frac{1}{2+\rho} w_t(1-\tau) - \left(\frac{1+\rho}{2+\rho}\right) \frac{1}{(1+r_{t+1})} \tau w_{t+1}$$

To get capital accumulation we replace individual savings with next period capital per worker $k_{t+1} = s_t$ (note there is no $\frac{1}{1+n}$ term since there is no population growth), and we use equilibrium expressions for wage and interest rates, $w = (1-\alpha)Ak^{\alpha}$, $r = \alpha A k^{\alpha-1}$)

NOTE: there was a mistake in the setup. It should have been assumed that capital fully depreciates. Under that assumption the following would have been straigthforward as:

$$k_{t+1} = \left[\frac{1}{2+\rho}(1-\alpha)Ak_t^{\alpha}(1-\tau) - \left(\frac{1+\rho}{2+\rho}\right)\frac{(1-\alpha)k_{t+1}}{\alpha}\tau\right]$$

Combining terms with k_{t+1}

$$k_{t+1} = \frac{1}{\left[1 + \frac{1+\rho}{2+\rho}\frac{1-\alpha}{\alpha}\tau\right]} \frac{1}{2+\rho}(1-\alpha)Ak_t^{\alpha}(1-\tau)$$

From here imposing steady state we get the following

$$k^* = \left[\frac{1}{\left[1 + \frac{1+\rho}{2+\rho}\frac{1-\alpha}{\alpha}\tau\right]} \frac{1}{2+\rho}(1-\alpha)A(1-\tau)\right]^{\frac{1}{1-\alpha}}.$$

Instead *since capital does not depreciate* the equation for capital accumulation only gives k_{t+1} as an implicit function of k_t:

$$k_{t+1} = \left[\frac{1}{2+\rho}(1-\alpha)Ak_t^{\alpha}(1-\tau) - \left(\frac{1+\rho}{2+\rho}\right)\frac{(1-\alpha)Ak_{t+1}^{\alpha}\tau}{1+\alpha A k_{t+1}^{\alpha-1}}\right]$$

And it is not possible to get a close-form solution for k^*. Students should get full points if either stating this or solving under assumption of full depreciation.

b) The government chooses contribution rate τ' such that the old receive the same benefits. Since the presence of immigrants increases the workforce for a given level of capital (initial steady state level k^*) this reduces the wage. But the contribution base for social security is larger by a factor $1+n$ (benefits are now $(1+n)\tau' w_t$). Combining these effects the new contribution rate is determined by

$$(1+n)\tau' w_t = (1+n)\tau'(1-\alpha)A\left(\frac{k^*}{1+n}\right)^{\alpha} = (1+n)\tau'\frac{w^*}{(1+n)^{\alpha}} = \tau w^*$$

Thus,

$$\tau' = \frac{\tau}{(1+n)^{1-\alpha}}.$$

The effect on consumption depends on parameters (see c) below). First period consumption is given by

$$c_{1t} = w_t(1-\tau') - s_t = \frac{1+\rho}{2+\rho}\left[w^*\left(\frac{1}{(1+n)^\alpha} - \frac{\tau}{1+n}\right) + (1+n)\frac{\tau}{(1+n)^{1-\alpha}}\frac{w_{t+1}}{1+r_{t+1}}\right]$$

Because, as will be shown, $k_{t+1} < k^*$ the second term will be lower with an increase in n if $k_{t+1} \leq k_t$ (since $(1+n)^\alpha w_t = w^*$, it can be rewritten as $\tau w^* \frac{w_{t+1}}{(1+r_{t+1})w_t}$). And the effect of an increase in n on the first term depends on parameters α and τ (see c) below).

The new steady state level of capital is given by

$$k^{*'} = \frac{1}{1+n}\left[\frac{1}{2+\rho}(1-\alpha)A{k^{*'}}^\alpha\left(1 - \frac{\tau}{(1+n)^{1-\alpha}}\right) - \left(\frac{1+\rho}{2+\rho}\right)\frac{(1-\alpha)A{k^{*'}}^\alpha \frac{(1+n)\tau}{(1+n)^{1-\alpha}}}{1+\alpha A{k^{*'}}^{\alpha-1}}\right]$$

Even though there is no close-form solution for $k^{*'}$ we can see that it must be the case that $k^{*'} < k^*$ since the second term in brackets becomes more negative with n, and the positive effect of n in the first term is less than linear and thus does not compensate the effect from population growth that dilutes the savings onto a larger labor force (the $\frac{1}{1+n}$ term in front of the left bracket). Because the shock lowers initial capital per capita, $k_t < k^*$, and $k^{*'} < k^*$, then $k_{t+1} < k^*$.

c) Since the presence of immigrants increases the workforce for a given level of capital, this increases the interest rate. This makes the old to be strictly better off since they have the same level of benefits but a higher capital income. The disposable income of the young in the first period is given by $w_t(1-\tau') = (1-\alpha)A\left(\frac{k^*}{1+n}\right)^\alpha(1 - \frac{\tau}{(1+n)^{1-\alpha}}) = w^*\left(\frac{1}{(1+n)^\alpha} - \frac{\tau}{1+n}\right)$. As established in b), the effect of n on the term in parenthesis is ambiguous. More precisely its derivative with respect to n is given by

$$\left(\frac{-\alpha}{(1+n)^{1+\alpha}} + \frac{\tau}{(1+n)^2}\right)$$

E.g. a sufficient condition for this to be negative is $\tau \leq \alpha$. Basically the intuition is that the negative effect of immigation on wages depends on α, the elasticity of wages to capital, and the positive effect of a reduction in contributions (positive since the economy is dynamicaly efficient) is proportional to τ, the size of social security.