Suggestions for solutions Advanced Microeconomics exam 21FEB2014

3 hours closed book exam

Problem A

(a) Give a graphic example of a production set $Y \subset \mathbb{R}^2$ satisfying P1, except the convexity part, and prices, $(p_1, p_2) \in \mathbb{R}^2_{++}$ such that there are precisely two solutions to the Producer Problem.

SOLUTION: See Figure 1.

(b) Let a consumer have $\mathbb{R}^2_+$ as consumption set and lexicographic preferences. Define such preferences.

SOLUTION: $x \succsim \bar{x}$ if either $x_1 > \bar{x}_1$ or if $x_1 = \bar{x}_1$ and $x_2 \geq \bar{x}_2$

(c) For the consumer from (b) find the indifference class containing $x = (1, 1)$

SOLUTION: The indifference class contains only the consumption $x = (1, 1)$.

(d) Assume that Arrow's assumptions A1 to A3 for a SWF are satisfied and that Schedule 1 below is mapped to a ranking for society with b above a. What can be concluded about society's ranking of a and b for Schedule 2?

Schedule 1			Schedule 2		
c	b	a	c	a	c
b	a	c	b	b	a
a	c	b	a	c	b

SOLUTION: Schedule 1and Schedule 2 do not have the same $a-b$ pattern so Independence of Irrelevant Alternatives can not be applied. Individual 1 and 2 have the same preferences over a, b so one of them is a dictator.

(e) Let $\mathcal{E} = (\mathbb{R}^2_+, u^i, \omega^i)_{i\in\{a,b\}}$ be a pure exchange economy with private ownership. Define what is meant by a Walras equilibrium for $\mathcal{E}$.

SOLUTION: See MWG or NotesWa

(f) Let the consumption set be $X = \mathbb{R}^2_+$ and consider a consumer with homothetic preferences $\succsim$. Assume that $x \sim \bar{x}$. Can it be the case that $2x \succ 2\bar{x}$? Illustrate in a diagram.

SOLUTION: No, it must be the case that $2x \sim 2\bar{x}$. See Figure 2 and MWG

(g) Show by a graphic example, in $\mathbb{R}^2$, that when a consumer faces a price-wealth pair (p, w), $p \in \mathbb{R}^2_{++}$ such that $\mathrm{w} = \min\{px \mid x \in X\}$ then there might be $\hat{x} \in X$ such that $\hat{x}$ is a solution to the expenditure minimization problem (for utility level $\hat{u} = u(\hat{x})$ at prices p) but $\hat{x}$ does not solve the Utility Maximization Problem.

SOLUTION: See Figure 3 or NotesOpt.

(h) In an economy the production sector has 2 producers, a and b, with production sets Y^a and Y^b. What is the (total, aggregate) production set for the production sector.

SOLUTION: $Y^a + Y^b$ see NotesThPr.

(i) Let $(\bar{x}, \bar{y}, p)$, $p = (1, 1)$, be a Walras equilibrium for an economy $\mathcal{E} = \{(u, \omega), Y, \alpha\}$ (one consumer, one producer) where $\alpha = 1$ (the consumer owns the producer). State the conditions that $(\bar{x}, \bar{y}, p)$ must satisfy. Be careful when defining the consumer´s wealth. Illustrate in a diagram.

SOLUTION: See Figure 4. $\bar{y}$ solves PMP at prices p, $\bar{x}$ solves UMP at prices p and wealth $\mathrm{w} = p\bar{x} = p\bar{y} + p\omega$, markets balance so $\bar{x} = \bar{y} + \omega$.

(j) Assume that the preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is represented by the non-negative strictly monotone utility function u with the property $u(\alpha x) = \alpha^2 u(x)$ for $\alpha \geq 0$ (homogenous of degree 2). Is $\succsim$ a homothetic preference relation?

SOLUTION: $\succsim$ is a homothetic preference relation if and only if there is a utility function homogenous of degree 1 representing $\succsim$. The function $t \longrightarrow \varphi(t) = t^{\frac{1}{2}}$ is increasing and defined on $\mathbb{R}_+$. Thus $\varphi \circ u$ with values $\varphi \circ u(x) = \left(u(x)^{1/2}\right)$ represents $\succsim$, But $\varphi \circ u(\alpha x) = \left(\alpha^2 u(x)\right)^{\frac{1}{2}} = \alpha \left(u(x)\right)^{\frac{1}{2}} = \alpha \cdot \varphi \circ u(x)$ so $\varphi \circ u$ is a representation of $\succsim$ which is homogenous of degree 1.

Problem B

(a) Consider an economy with commodity space $\mathbb{R}^L$ having precisely two producers, a and b, with production sets $Y^a \subset \mathbb{R}^L$ and $Y^b \subset \mathbb{R}^L$. Let the price vector $p \in \mathbb{R}^L_{++}$ be given and let $\bar{y}^a$ and $\bar{y}^b$ be solutions to the individual producer problems. Show that $\bar{y}^a+ \bar{y}^b$ solves the aggregate (total) profit maximization problem.

SOLUTION: Let $y \in Y^a + Y^b$ then $y = y^a + y^b$ with $y^a \in Y^a$ and $y^b \in Y^b$. Then

$$\begin{aligned} p\bar{y}^a &\geq py^a \\ p\bar{y}^b &\geq py^b \end{aligned}$$

and thus $p\left(\bar{y}^a + \bar{y}^b\right) \geq p\left(y^a + y^b\right) = py$.

(b) Let $\mathcal{E} = \left(\mathbb{R}^2_+, u^i, \omega^i\right)_{i\in\{a,b\}}$ be an exchange economy with private ownership where $\omega^a = \omega^b$ and $\left(\hat{x}^a, \hat{x}^b, p\right)$ a Walras equilibrium for $\mathcal{E}$. Show that the equilibrium allocation is a fair allocation

SOLUTION: Since $\omega^a = \omega^b$ both consumers have the same wealth, $\mathrm{w}^a = \mathrm{w}^b = p\omega^a = pw^b$. Consumer a could have chosen b ´s consumption. Hence $u^a(\hat{x}^a) \geq u^a(\hat{x}^b)$. The same reasoning applies to b so that $u^b\left(\hat{x}^b\right) \geq u^b\left(\hat{x}^a\right)$ which shows that the equilibrium allocation is fair.

Problem C

Consider a private ownership pure exchange economy with commodity space $\mathbb{R}^3$. The economy has two consumers, a and b, with consumption sets $X^a = X^b = \mathbb{R}^3_+$ initial endowments $\omega^a, \omega^b \in \mathbb{R}^3_{++}$ and preferences given by utility functions $U^a : \mathbb{R}^3_+ \longrightarrow \mathbb{R}$ and $U^b : \mathbb{R}^3_+ \longrightarrow \mathbb{R}$. where furthermore,

$$\begin{aligned} U^a(x_1, x_2, x_3) &= v^a(x_1) + v^a(x_2) + v^a(x_3) \\ U^b(x_1, x_2, x_3) &= v^b(x_1) + v^b(x_2) + v^b(x_3) \end{aligned}$$

and where, for $i = a, b$, the function $v^i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ has positive first derivative (differentiably monotone), $Dv^i > 0$, and negative second derivative (differentiably concave), $D^2 v^i < 0$ on $\mathbb{R}_{++}$. Furthermore $\lim_{y \to 0} Dv^i(y) = \infty$.

(a) Does $f : \mathbb{R} \longrightarrow \mathbb{R}$ with values $f(t) = 2t^{\frac{1}{2}}$ satisfy the conditions assumed for v^a?

SOLUTION: $Df(t) = t^{-\frac{1}{2}}$ and $D^2 f(t) = -\dfrac{1}{2} t^{-\frac{3}{2}}$ from which is seen that f satisfies the assumptions for v^a.

(b) Is U^a a quasi-concave function? **Hint:** U^a is the sum of the functions $(x_1, x_2, x_3) \longrightarrow v^a(x_1), (x_1, x_2, x_3) \longrightarrow v^a(x_2)$ and $(x_1, x_2, x_3) \longrightarrow v^a(x_3)$.

SOLUTION: Let $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Then for $x, y \in \mathbb{R}^3_+$

$$\begin{aligned} \alpha x + \beta y &\longrightarrow v^a(\alpha x_1 + \beta y_1) \geq \alpha v^a(x_1) + \beta v^a(y_1) \\ \alpha x + \beta y &\longrightarrow v^a(\alpha x_2 + \beta y_2) \geq \alpha v^a(x_2) + \beta v^a(y_2) \\ \alpha x + \beta y &\longrightarrow v^a(\alpha x_3 + \beta y_3) \geq \alpha v^a(x_3) + \beta v^a(y_3) \end{aligned}$$

with at least one strict inequality if $x \neq y$. Hence if $x \neq y$

$$\begin{aligned} U(\alpha x + \beta y) &= v^a(\alpha x_1 + \beta y_1) + v^a(\alpha x_2 + \beta y_2) + v^a(\alpha x_3 + \beta y_3) > \\ &\quad \alpha U(x) + \beta U(y) \end{aligned}$$

which shows that U is a strictly concave function and hence a strictly quasi-concave function.

(c) Let $p \in \mathbb{R}^3_{++}$ and income (wealth) $\mathrm{w}^a > 0$. State consumer a´s problem (the UMP). Does it have a solution?

SOLUTION: The UMP is

$$Max\ U^a(x) \text{ subject to } x \in \mathbb{R}^3_+ \text{ and } px \leq \mathrm{w}^a$$

Since the budget set is compact (and U^a continuous) there is a solution. Since U^a is strictly quasi-concave the solution is unique.

(d) Assume that $\bar{x}^a$ is a solution to the problem from (c). Derive the marginal conditions that $\bar{x}^a$ will satisfy.

SOLUTION: If $\bar{x}^a$ is a solution then $\bar{x}^a$ solves the UMP with equality in the budget restriction.

Then there is $\lambda \in \mathbb{R}$ such that

$$\begin{array}{rcccl} Dv^a(\bar{x}^a_1) & - & \lambda p_1 & = & 0 \\ Dv^a(\bar{x}^a_2) & - & \lambda p_2 & = & 0 \\ Dv^a(\bar{x}^a_3) & - & \lambda p_3 & = & 0 \end{array}$$

which also shows that $\lambda > 0$. Note that the condition $\lim_{y \to 0} Dv^i(y) = \infty$ excludes boundary solutions.

(e) Assume that $p_1 < p_2 < p_3$ and let λ be the multiplier from (d). Make a qualitatively correct plot of the first derivative of $(1/\lambda)\, v^a$. Indicate in your diagram the location of pairs $\left(\frac{1}{\lambda} Dv^a(\bar{x}^a_h), \bar{x}^a_h\right) = (p_h, \bar{x}^a_h)$, $h = 1, 2, 3$ where λ is the multiplier from (d).

SOLUTION: See Figure 5.

(f) Assume in the sequel that total initial endowment $\omega = \omega^a + \omega^b$ is such that $\omega_1 > \omega_2 > \omega_3$. Let $(\hat{x}^a, \hat{x}^b, \hat{p})$ be a Walras equilibrium. State the market balance conditions for commodity 1 and 2. Using these show that $\hat{x}^a_1 > \hat{x}^a_2$ or $\hat{x}^b_1 > \hat{x}^b_2$.

SOLUTION: Assume that $\hat{x}^a_1 \leq \hat{x}^a_2$ and $\hat{x}^b_1 \leq \hat{x}^b_2$. Then $\omega_1 = \hat{x}^a_1 + \hat{x}^b_1 \leq \hat{x}^a_2 + \hat{x}^b_2 = \omega_2$ contradicting that $\omega_1 > \omega_2$.

(g) Show that the result in (f) implies $p_1 < p_2$. **Hint:** Apply the reasoning from (e).

SOLUTION: Assume from (f) that $\hat{x}_1^a > \hat{x}_2^a$ is (part of) a solution to consumer a ´s problem. From Figure 5 it is seen that this can be the case only if $p_1 < p_2$.

(h) Thus in an exchange economy with preferences like U^a and U^b (separable preferences) there is a relation between the ordering of the total endowment and Walras equilibrium prices. Does this extend also to the ordering of the equilibrium consumptions?

SOLUTION: Yes, clearly prices are increasing if the initial endowment is decreasing. But with increasing prices both consumers choose consumptions which are decreasing.

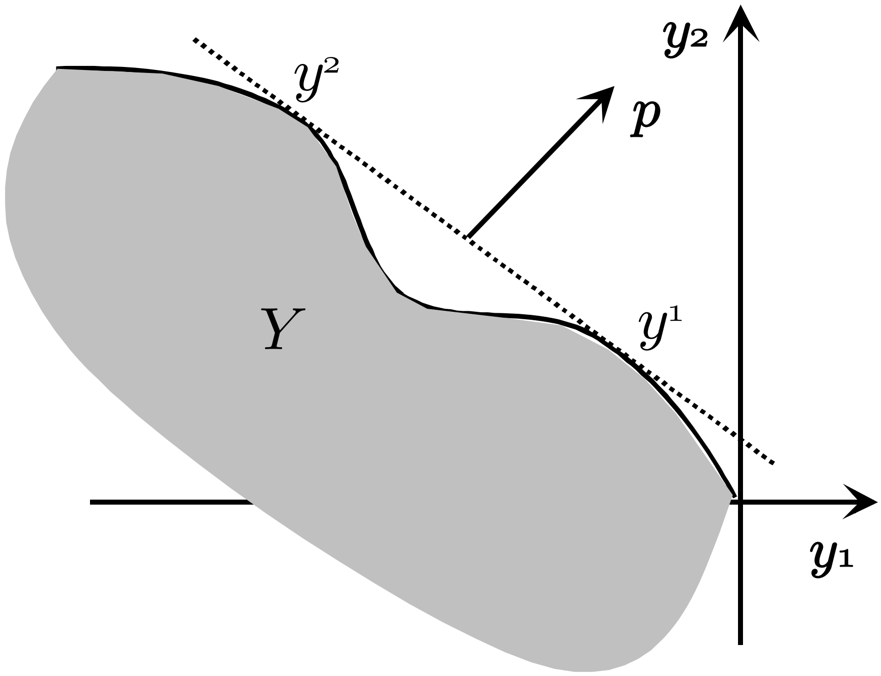

Figure 1: At prices p the PMP has precisely two solutions; y^1 and y^2

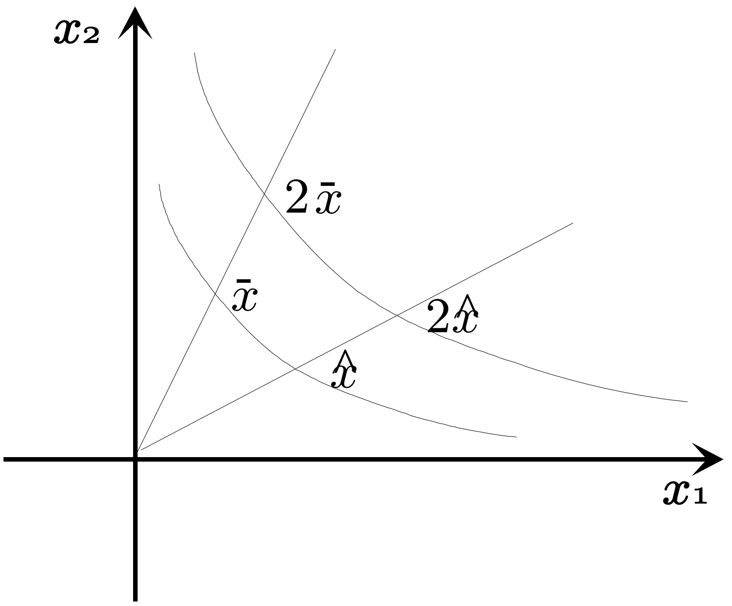

Figure 2: $\bar{x} \sim \hat{x}$ implies $2\bar{x} \sim 2\hat{x}$. Let $I(x)$ be the indifference class containing x. We have $I(2\bar{x}) = 2\ I(\bar{x})$.

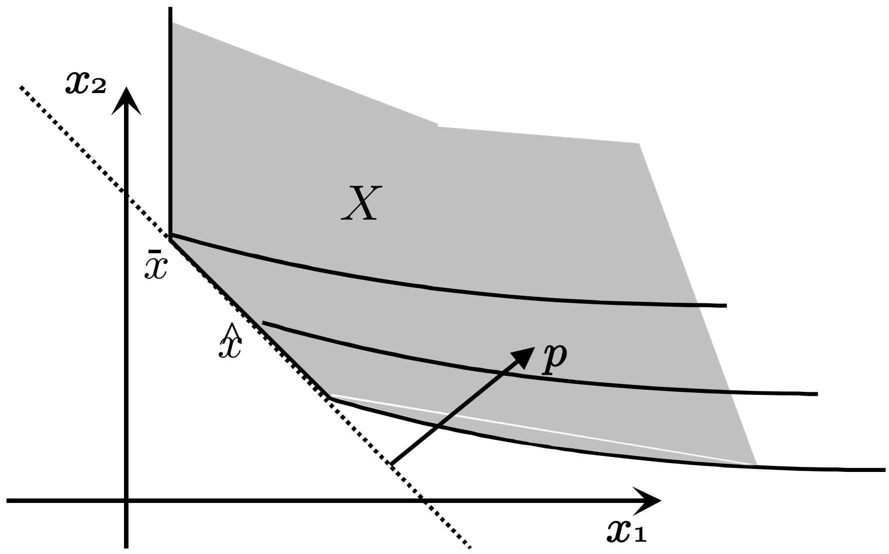

Figure 3: $\hat{x}$ is a solution to the EMP (at prices p and utility level $u(\hat{x})$) but $\bar{x}$ is the unique solution to the UMP at prices p and wealth $p\bar{x} = p\hat{x}$

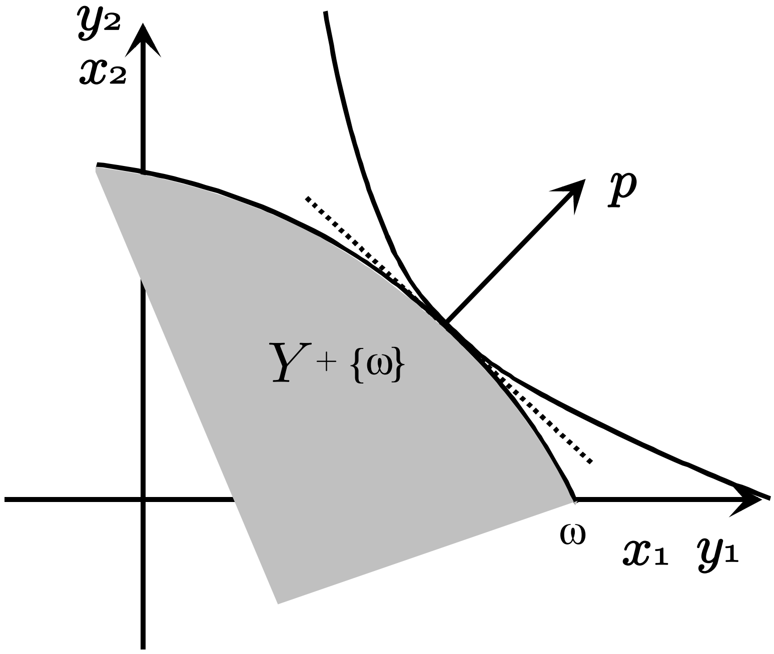

Figure 4: As initial endowment is $\omega = (\omega_1, 0)$ with $\omega_1 > 0$ the consumptions $x = y + \omega$ are available to the consumer, In order to realize the consumption the consumer needs wealth $\mathrm{w} = px = py + p\omega$

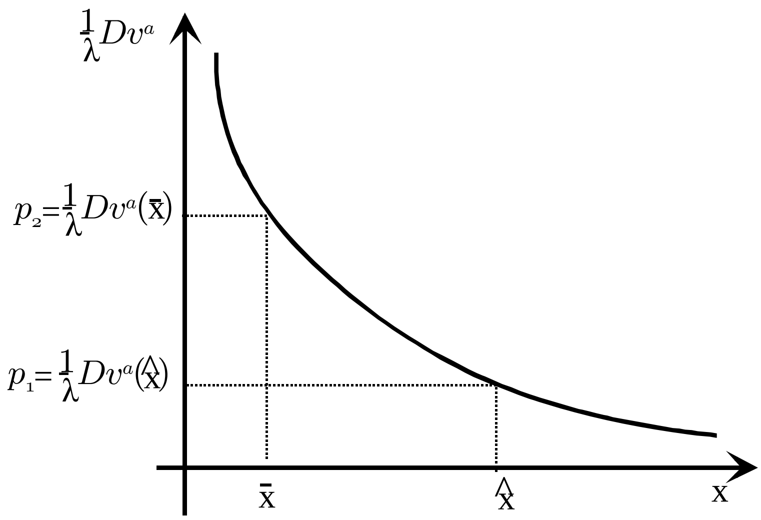

Figure 5: For λ fixed the function $(1/\lambda)\, Dv^a$ is decreasing since $D^2v^a < 0$. Consumer a demands a larger quantity, $\hat{x}$, of the commodity with the low price.