Written Exam for the M.Sc. in Economics 2009 (Spring Term)

Financial Econometrics A: Volatility Modelling

Final Exam: Masters course

17/6-2009 (9-12)

3-hour open book exam.

Notes on Exam: Please note that there are a total of 10 questions which should all be answered. These are divided into Question 1 (Question 1.1-1.6) and Question 2 (Question 2.1-2.4).

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

Exam Question 1:

From the rich literature on ARCH models the absolute value ARCH (A-ARCH) model for log-returns x_t has been proposed and we shall study this model here.

It is given by,

$$x_t = \sigma_t z_t \qquad \sigma_t = \omega + \alpha |x_{t-1}|,$$

with z_t iidN$(0,1)$ for $t = 1, ..., T$ and with x_0 fixed. Moreover, the A-ARCH parameters ω and α are positive, that is, $\omega > 0$ and $\alpha > 0$.

Note that the model differs from the classical ARCH by specifying σ_t as linear in $|x_{t-1}|$ rather than specifying σ_t^2 as linear in x_{t-1}^2.

Question 1.1: Figure 1.1 shows a simulated sample with (true values) $\omega_0 = 1$ and $\alpha_0 = 0.6$. Table 1.1 gives some output from estimation of a (log-)return series.

Comment on Figure 1.1 (ARCH effects) and Table 1.1 (in terms of mis-specification). Why would you think this model could be of interest as opposed to the classic ARCH(1) model.

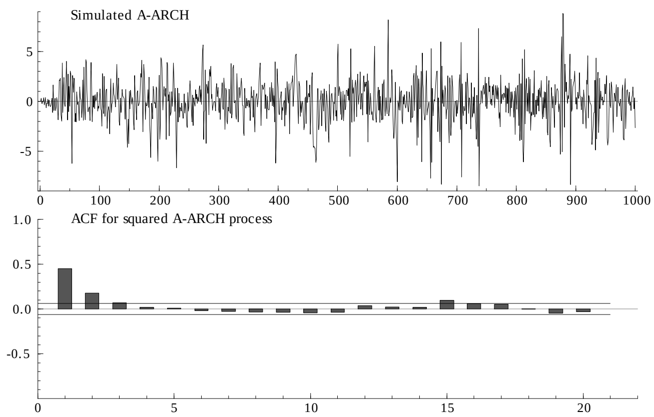

Figure 1.1

Table 1.1	
Parameter estimate:	$\hat{\alpha} = 0.56$ (std.deviation $= 0.04$)
Standarized residuals: $\hat{z}_t = x_t/\hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	1.44 (p-value: 0.22)
LM test for ARCH in $\hat{z}_t$:	1.98 (p-value: 0.37)

Question 1.2: Use the drift function $\delta(x) = 1 + x^2$, and show that for $|x_{t-1}|$ large,

$$E(\delta(x_t)|x_{t-1}) \leq \phi\delta(x_{t-1}),$$

for some $\phi < 1$ if $\alpha < 1$. This implies (do not show this) that x_t is weakly mixing, and hence stationary with $Ex_t^2 < \infty$ if $\alpha < 1$.

Question 1.3: Next, turn to estimation of the parameter α (leaving ω as fixed or known for simplicity). The Gaussian log-likelihood function $\ell_T(\alpha)$ is given by,

$$\ell_T(\alpha) = -\sum_{t=1}^{T}\left(\log\sigma_t + \frac{x_t^2}{2\sigma_t^2}\right).$$

Describe briefly how to obtain the MLE $\hat{\alpha}$ in a programming language such as "ox" which maximizes $\ell_T(\alpha)$. Is there some way by which you can argue that your procedure produces a (local) maximum of $\ell_T(\alpha)$?

Outline 1.4: We know from Theorem III.2 that $\hat{\alpha}$ is consistent and asymptotically Gaussian provided regularity conditions hold. A key condition is (A.1) in Theorem III.2 which states that

$$\frac{1}{\sqrt{T}}\,\partial\ell_T(\alpha)/\partial\alpha\Big|_{\alpha=\alpha_0} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left((x_t/\sigma_t)^2 - 1\right)\frac{|x_{t-1}|}{\sigma_t} \xrightarrow{D} N(0,\Sigma), \quad \text{where}$$

$$\Sigma = 2E\left(\frac{x_{t-1}^2}{\sigma_t^2}\right)$$

Show that this holds. Be specific about which results you use and verify that they hold.

Question 1.5: We can then conclude (do not show this) that

$$\sqrt{T}(\hat{\alpha} - \alpha_0) \xrightarrow{D} N\left(0, \Sigma^{-1}\right).$$

Explain how you would use this result in empirical work.

Sometimes the asymptotic result is reported on the form:

$$\sqrt{T}(\hat{\alpha} - \alpha_0) \xrightarrow{D} N\left(0, \Omega^{-1}\Sigma\Omega^{-1}\right),$$

where $\Sigma \neq \Omega$. Explain why this form is often used.

Question 1.6: The A-ARCH model is applied to the FTSE log-return series studied in lectures for $T = 1516$ daily returns. Table 1.2 gives the output from estimation of the A-ARCH model. Comment on the size of $\hat{\alpha}$ in terms of the theory derived above, and comment on the misspecification tests.

Table 1.2	
Parameter estimate:	$\hat{\alpha} = 0.48$ (std.deviation = 0.02)
Standarized residuals: $\hat{z}_t = x_t/\hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	1074 (p-value: 0.00)
LM test for ARCH in $\hat{z}_t$:	1.05 (p-value: 0.31)

Exam Question 2:

Question 2.1: Figure 2.1 shows a log-returns series x_t with $T = 1200$ observations together with the ACF for $|x_t|$. Furthermore, Table 2.1 shows some output from estimation of a GARCH(1,1) model with these data.

From Figure 2.1, would you expect a GARCH(1,1) to fit the data? Explain the output in Table 2.1.

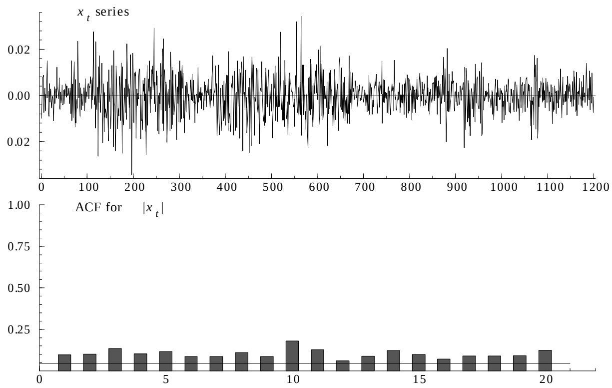

Figure 2.1

Table 2.1	
Parameter estimates in GARCH(1,1):	$\hat{\alpha} + \hat{\beta} = 0.98$
Standarized residuals: $\hat{z}_t = x_t/\hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	48.44 (p-value: 0.00)
LM test for ARCH in $\hat{z}_t$:	1.45 (p-value: 0.15)

Question 2.2: Based on Figure 2.1, we initially consider the following 2-state volatility model for x_t where:

$$x_t = \mu + \sigma_t z_t \text{ with}$$

$$\sigma_t^2 = \begin{cases} \sigma_1^2 & \text{if } s_t = 1 \\ \sigma_2^2 & \text{if } s_t = 2 \end{cases}$$

and z_t iidN$(0,1)$ distributed and $t = 1, ..., T$.

We treat first the variable s_t as an **observed** regime or state variable which values correspond to high and low volatility regimes. For example, if $\sigma_1^2 < \sigma_2^2$ then $s_t = 1$ is the low, and $s_t = 2$ the high volatility regime.

It can be shown that (with μ known) the MLE of σ_1^2, $\hat{\sigma}_1^2$, is given by

$$\hat{\sigma}_1^2 = \frac{\sum_{t=1}^T 1\left(s_t = 1\right)\left(x_t - \mu\right)^2}{\sum_{t=1}^T 1\left(s_t = 1\right)}.$$

Give an intuitive interpretation of this estimator. Use Figure 1 to exemplify.

Question 2.3: Next, we assume that s_t is an **unobserved** two-state Markov-chain as given by the transition matrix,

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{pmatrix},$$

with

$$p_{ij} = P\left(s_t = j | s_{t-1} = i\right)$$

such that $p_{11} = 1 - p_{12}$ and $p_{22} = 1 - p_{21}$.

Set $X = (x_1, ..., x_T)$ and $S = (s_1, ..., s_T)$, and denote the complete likelihood function treating s_t as **observed** (and the initial distribution of s_1 as known) by $\ell_T\left(X, S; \theta\right)$.

With $m_t\left(j\right) = \log \sigma_j^2 + \frac{(x_t - \mu)^2}{\sigma_j^2}$ it can be shown that,

$$L\left(\theta\right) = E\left(\ell_T\left(X, S; \theta\right)|X\right) = -\frac{1}{2}\sum_{t=1}^T \sum_{j=1}^2 p_t^*\left(j\right) m_t\left(j\right) - \sum_{t=2}^T \sum_{i,j=1}^2 p_t^*\left(i, j\right) \log p_{ij},$$

where $p_t^*\left(j\right)$ are the *smoothed* probabilities and $p_t^*\left(i, j\right)$ the *smoothed* transition probabilities.

Maximizing $L\left(\theta\right)$ with respect to σ_1^2 gives $\tilde{\sigma}_1^2$ (keeping $p_t^*\left(j\right)$ and $p_t^*\left(i, j\right)$ fixed).

Find $\tilde{\sigma}_1^2$ and compare it to $\hat{\sigma}_1^2$ from Question 2.2.

Is $\tilde{\sigma}_1^2$ the MLE of σ_1^2 - and if so, why?

Question 2.4: Discuss under which restrictions on the parameters in the transition matrix $\mathbf{P}$ you would expect the MLE of σ_1^2 to be a consistent estimator.

Discuss these conditions in light of the output in Table 2.2 below from estimation of the model using the data in Figure 2.1. Are your findings to be expected – and are they typical for other types of financial data?

Table 2.2
Parameter estimates:
$\hat{\mu} = 0.01$
$\hat{\mathbf{P}} = \begin{pmatrix} 0.98 & 0.01 \\ 0.02 & 0.99 \end{pmatrix}$