Written Exam - Macroeconomics III

(suggested answers)

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Question 1

a FOCs of the profit function in per capita terms are

$$\begin{aligned} R_t &= 1 + r_{t+1} = \alpha A k_t^{\alpha-1} \\ w_t &= (1-\alpha) A k_t^{\alpha} \end{aligned}$$

b The savings problem of a young individual is

$$\max_{c_{1t}, c_{2t+1}} \ln c_{1t} + \frac{1}{1+\rho} \ln c_{2t+1}$$
$$c_{1t} + s_t = w_t(1-\tau)$$
$$c_{2t+1} = s_t\,(1 + r_{t+1}) + \tau w_{t+1}$$

Solving this problem and combining FOCs yields the Euler equation

$$c_{2t+1} = \frac{1 + r_{t+1}}{1+\rho} c_{1t}$$

Replace c_{1t} and c_{2t+1} from the budget constraints to obtain the desired equation describing individual savings behavior:

$$s_t = \frac{1}{2+\rho} w_t (1-\tau) - \tau \frac{1+\rho}{2+\rho} \frac{1}{1+r_{t+1}} w_{t+1}.$$

c To derive the capital accumulation equation we use individual savings and replace $k_{t+1} = s_t$ (there is no population growth term here since by assumption $n = 0$), and use the equilibrium expressions for wages and rental rates to obtain

$$k_{t+1} = \frac{1}{2+\rho}(1-\alpha) A k_t^{\alpha} (1-\tau) - \tau \frac{1+\rho}{2+\rho} \frac{1-\alpha}{\alpha} k_{t+1}.$$

Combine terms with k_{t+1} we get the desired expressions.

d Imposing the steady state we get

$$\bar{k} = \left[\frac{1}{1 + \frac{1+\rho}{2+\rho} \frac{(1-\alpha)}{\alpha} \tau} \left(\frac{1}{2+\rho} (1-\alpha) A (1-\tau) \right) \right]^{\frac{1}{1-\alpha}}$$

e From the previous point we can show that, for $\tau = 0$, the new steady state satisfies

$$\bar{k}_{NEW} = \left[\frac{1}{2+\rho}(1-\alpha)A\right]^{\frac{1}{1-\alpha}} > \bar{k}$$

Recall that capital is predetermined. Thus $k_{T+1} = s_T = \bar{s} = \bar{k}$. As for k_{T+2}, the new law of motion reads as:

$$k_{T+2} = \frac{1}{2+\rho}(1-\alpha)Ak_{T+1}^{\alpha}$$

which implies that k_{T+2} will be greater than the original steady state, $\bar{k}$, as long as

$$\bar{k} < \left[\frac{1}{2+\rho}(1-\alpha)A\right]^{\frac{1}{1-\alpha}} = \bar{k}_{NEW}$$

which is always the case, as we have proved above.

f As $\tau = 0$ becomes effective starting from $T+1$ onwards, the old generation in $T+1$ will be worse-off, as these agents have not had the chance to adapt their consumption/saving profile (given that the government has communicated the decision after they had already optimized), while the contemporaneous young generation will not be covering their social benefits.

g Dynamic efficiency obtains under a situation in which the rate of return is lower than the return on social security, which is implicitly measured by population growth (assumed to be zero, in this case). Thus, as the rate of return is a negative function of the capital stock ($r_t = \alpha A k_t^{\alpha-1} - 1$) and the latter is negatively affected by τ, we infer that a pay-as-you-go social security system has higher chances of depressing capital accumulation and thus rendering the system dynamically efficient.

Question 2

a Given the linear rule $\pi_t = \psi + \psi_\theta \theta_t$, as well as the fact that θ_t is observed by both the public and the policy maker before expectations are formed, output is determined as follows:

$$x_t = \theta_t + \pi_t - \pi_t^e = \theta_t + \psi + \psi_\theta\theta_t - (\psi + \psi_\theta\theta_t) = \theta_t$$

Thus, the expected loss reads as:

$$\begin{aligned}
E\left[L(\pi_t, x_t)\right] &= \frac{1}{2}E\left[\left(\underbrace{\psi + \psi_\theta\theta_t}_{=\pi_t}\right)^2 + \lambda\left(\underbrace{\theta_t}_{=x_t} - \bar{x}\right)^2\right] \\
&= \frac{1}{2}E\left[\psi^2 + 2\psi\psi_\theta\theta_t + \psi_\theta^2\theta_t^2 + \lambda\left(\theta_t^2 - 2\bar{x}\theta_t + \bar{x}^2\right)\right] \\
&= \frac{1}{2}\left[\psi^2 + 2\psi\psi_\theta E\left[\theta_t\right] + \psi_\theta^2 E\left[\theta_t^2\right] + \lambda\left(E\left[\theta_t^2\right] - 2\bar{x}E\left[\theta_t\right] + \bar{x}^2\right)\right]
\end{aligned}$$

Taking the first order conditions of $E\left[L(\pi_t, x_t)\right]$ with respect to ψ and ψ_θ we obtain:

$$\begin{aligned}
\frac{\partial E\left[L(\pi_t, x_t)\right]}{\partial \psi} &= 0 : \psi + \psi_\theta E\left[\theta_t\right] = 0 \\
\frac{\partial E\left[L(\pi_t, x_t)\right]}{\partial \psi_\theta} &= 0 : \psi E\left[\theta_t\right] + \psi_\theta E\left[\theta_t^2\right] = 0
\end{aligned}$$

Thus, the expected loss is minimized by setting $\psi = \psi_\theta = 0$, which implies $\pi_t^C = 0$ and $x_t^C = \theta_t$.

b When the central bank conducts a discretionary policy, the inflation rate is chosen after expectations are formed. Hence, the goal of the central bank is to minimize the loss function, i.e. the monetary policy should be ex post optimal, given π_t^e. Under this assumption, the problem reads as

$$\min_{\pi_t} \frac{1}{2}\left[\pi_t^2 + \lambda\left(\theta_t + \pi_t - \pi_t^e - \bar{x}\right)^2\right]$$

The first order condition for this problem reads as:

$$\frac{\partial L(\pi_t, x_t)}{\partial \pi_t} = 0 : \pi_t + \lambda\left(\theta_t + \pi_t - \pi_t^e - \bar{x}\right) = 0 \Leftrightarrow \pi_t^D = \frac{\lambda}{1+\lambda}\left(\pi_t^e - \theta_t + \bar{x}\right)$$

Thus, the expected rate of inflation is found by taking expectations:

$$E\left[\pi_t^D \middle| \theta_t\right] = \frac{\lambda}{1+\lambda} E_t\left[\pi_t^e - \theta_t + \bar{x}\right] = \frac{\lambda}{1+\lambda}\left(E_t\left[\pi_t^D \middle| \theta_t\right] - \theta_t + \bar{x}\right)$$

which implies $E\left[\pi_t^D \middle| \theta_t\right] = -\lambda\left(\theta_t - \bar{x}\right)$. Therefore:

$$\begin{aligned}
\pi_t^D &= \frac{\lambda}{1+\lambda}\left(\underbrace{-\lambda\left(\theta_t - \bar{x}\right)}_{=\pi_t^e} - \theta_t + \bar{x}\right) = -\lambda\left(\theta_t - \bar{x}\right) \\
x_t^D &= \theta_t
\end{aligned}$$

The excessively high equilibrium inflation associated with the inflation bias problem results from the combination of a lack of commitment and central bank's temptation to temporarily boost the economy beyond its potential level. The latter incentive is embodied by the condition $\bar{x} > \theta$. This makes it clear why raising $\bar{x}$ increases the temptation of the central bank to generate excess inflation in the vain attempt to stimulate real activity.

c Once again, when the central bank conducts a discretionary policy, the inflation rate is chosen after expectations are formed. Hence, the goal of the central bank is to minimize the loss function, i.e. the monetary policy should be ex post optimal, now given π_t^e and θ_t, as the latter is not observed. Under this assumption, the problem reads as

$$\min_{\pi_t} \frac{1}{2}\left[\pi_t^2 + \lambda\left(\theta_t + \pi_t - \pi_t^e - \bar{x}\right)^2\right]$$

The first order condition for this problem reads as:

$$\frac{\partial L(\pi_t, x_t)}{\partial \pi_t} = 0 : \pi_t + \lambda\left(\theta_t + \pi_t - \pi_t^e - \bar{x}\right) = 0 \Leftrightarrow \pi_t^D = \frac{\lambda}{1+\lambda}\left(\pi_t^e - \theta_t + \bar{x}\right)$$

Now, the expected rate of inflation is found by taking unconditional expectations (as θ_t is not observed before expectations are formed):

$$E\left[\pi_t^D\right] = \frac{\lambda}{1+\lambda} E\left[\pi_t^e - \theta_t + \bar{x}\right] = \frac{\lambda}{1+\lambda}\left(E\left[\pi_t^D\right] + \bar{x}\right)$$

which implies $E\left[\pi_t^D\right] = \lambda\bar{x}$. Therefore:

$$\begin{aligned}
\pi_t^{D*} &= \frac{\lambda}{1+\lambda}\left(\underbrace{\lambda\bar{x}}_{=\pi_t^e} - \theta_t + \bar{x}\right) = \frac{\lambda}{1+\lambda}\left[\bar{x}\left(1+\lambda\right) - \theta_t\right] \\
x_t^{D*} &= \theta_t + \frac{\lambda}{1+\lambda}\left[\bar{x}\left(1+\lambda\right) - \theta_t\right] - \lambda\bar{x} = \frac{1}{1+\lambda}\theta_t
\end{aligned}$$

As we set $\lambda = 0$, the policy maker does not face a real activity stabilization objective, so that there is no temptation to inflate the economy to raise output above the target. Thus, no matter the information structure, output will always be equal to θ_t, and thus to its solution under commitment. The same holds true for the rate of inflation.