Macro C - exam solutions (Jan 7, 2014)

General remarks

Please grade each item of each question between 0 and 10 points. Thus the maximum possible grade of the exam is 100 (since there are two questions and each has five subquestions or items).

Mathematical errors reduce the grade for the item in which they were made, not for subsequent items that carry the mistake (unless the student arrives at an infeasible solution and is unaware of the inconsistency, e.g. stating that money holdings are negative).

1 Problem 1

This is a Sidrauski model with money entering not the households' preferences, but reducing the search cost of buying a bundle of goods.

a) The student gets six points if the budget constraint is derived correctly as

$$\dot{a}_t^i = w_t + r_t a_t^i + z_t^i - c_t^i(1 + e^{-\phi m_t^i}) - (r_t + \pi_t) m_t^i$$

where $a \equiv k + m$ is household wealth in real and per capita terms. The student only gets three points if the budget constraint is derived as:

$$c_t^i(1 + e^{-\phi m_t^i}) + \dot{k}_t^i + \dot{m}_t^i + \pi_t m_t^i = w_t + r_t k_t^i + z_t^i$$

The student must derive this constraint by dividing the original one, given in problem set up, by $N_t P_t$. It does not matter if this is done in one or two steps (i.e. there are no extra points for deriving separately the constraint in per capita terms and in real term, nor there is a penalty for doing everything in one step).

The remaining four points are awarded for a correct answer interpreting the terms related to money balances. This interpretation should note that (two points for each argument): 1) holding money is costly because money does not pay interest as saving in capital, plus money looses purchasing power if there is inflation. Thus the term $r_t + \pi_t$ times money holdings (in real per capita terms) subtracting from the RHS of the budget constraint. This is the nominal interest rate (i.e $i_t \equiv r_t + \pi_t$, no penalty from not identifying nominal interest rate). If the student only derived the budget constraint with money and capital separately but explains correctly this cost then also counts for grade. 2) holding money has the benefit that this reduces the resources that must be spent searching for goods. This is the term $e^{-\phi m_t^i}$ that is decreasing with m^i.

b) Control variables: c and m. State variable: a. Costate variable: multiplier of budget constraint in Hamiltonian. Thus if student does not define wealth as capital plus real money holdings, there will be a mistake in stating the state variable. (two points for correctly stating variables)

Hamiltonian (it is irrelevant if set up as current value or present value, what matters is that the FOC are correct for each setup) (two points for Hamiltonian, four for FOC):

$$
\begin{aligned}
H_t &= \ln c_t^i e^{-\rho t} + \lambda_t \left(w_t + r_t a_t^i + z_t^i - c_t^i(1 + e^{-\phi m_t^i}) - (r_t + \pi_t) m_t^i \right) \\
H_t^c &= \ln c_t^i + \mu_t \left(w_t + r_t a_t^i + z_t^i - c_t^i(1 + e^{-\phi m_t^i}) - (r_t + \pi_t) m_t^i \right)
\end{aligned}
$$

with $\mu_t \equiv \lambda_t e^{\rho t}$.

Student gets full points if stating FOC assuming an interior solution (even if there is no explicit assumption of this, i.e. no penalty from failing to consider corner solution):

$$
\begin{aligned}
\frac{dH_t^c}{dc_t^i} &= \frac{1}{c_t^i} - \mu_t(1 + e^{-\phi m_t^i}) = 0 \\
\frac{dH_t^c}{dm_t^i} &= \mu_t \left(\phi c_t^i e^{-\phi m_t^i} - (r_t + \pi_t) \right) = 0 \\
\dot{\mu}_t &= -\frac{dH_t^c}{da_t^i} + \rho \mu_t = -\mu_t(r_t - \rho) \\
\lim_{t \to \infty} e^{-\rho t} \mu_t a_t^i &= 0
\end{aligned}
$$

Note that the law of motion of the state variable is also a FOC (derivative of Hamiltonian with respect to costate variable λ_t). Not writing it has no penalty. If using H then FOC should be adjusted to that formulation.

There is no unique solution of FOC, just rewriting FOC such that $\mu_t = \frac{1}{c_t^i(1+e^{-\phi m_t^i})}$, i.e. multiplier is marginal value of consumption inclusive of searching costs, and $\phi \frac{e^{-\phi m_t^i}}{1+e^{-\phi m_t^i}} = \mu_t(r_t + \pi_t)$, i.e. marginal benefit of holding more money equal to marginal cost (in reality marginal benefit is $\phi c_t^i e^{-\phi m_t^i}$, above expression uses other FOC to replace for c_t^i). (two points, one for each FOC rewritten and interpreted).

c) The wage and interest rate are determined by imposing equilibrium in factor markets where firms competitively demand labor and capital from households. Thus the student needs to have profit function for firms

$$(K_t^j)^\alpha (L_t^j)^{1-\alpha} - w_t L_t^j - r_t K_t^j$$

where profits are written in real terms (can be written in nominal terms, but solution must be the same). From FOC of firms' problem of maximizing profits we get

$$\begin{aligned}(1-\alpha)(K_t^j)^\alpha (L_t^j)^{-\alpha} &= (1-\alpha)k_t^\alpha = w_t \\ \alpha (K_t^j)^{\alpha-1}(L_t^j)^{1-\alpha} &= \alpha k_t^{\alpha-1} = r_t\end{aligned}$$

where we impose that in equilibrium all firms work with the same capital labor ratio, k.

Government transfers are determined by government budget constraint: all resources generated by printing money are transferred to households. Thus $z_t = \frac{\dot{M}_t}{P_t N_t} = \sigma \frac{M_t}{P_t N_t} = \sigma m_t$ (where we use $\sigma = \frac{\dot{M}_t}{M_t}$ the growth rate of money). (Three points, one for each variable, w, r, z, correctly determined in equilibrium).

Steady state is characterized by $\dot{m}_t = \dot{a}_t = \dot{\mu}_t = 0$. (one point for this). Thus $\dot{m}_t = 0$ implies that $\pi = \sigma$ in a steady state. (one point for this). $\dot{\mu}_t = 0$ implies that $r_t = \rho$. This pins down the steady state capital labor ratio, since interest rate is $\alpha {k^*}^{\alpha-1}$ ($k^* = (\alpha/\rho)^{\frac{1}{1-\alpha}}$) (one point for this). $\dot{a}_t = 0$ implies that $c^*(1+e^{-\phi m^*}) = (k^*)^\alpha$, this comes from law of motion for a replacing transfers and canceling them with term $\pi_t m_t$ and canceling the terms with $r_t m_t$ (one point for this).

From FOC in steady state money demand satisfies

$$\phi \frac{(\alpha/\rho)^{\frac{\alpha}{1-\alpha}}}{1+e^{-\phi m^*}} e^{-\phi m^*} = \rho + \sigma$$

which can be rewritten as

$$\frac{\phi}{1+e^{\phi m^*}} = \frac{\rho+\sigma}{(\alpha/\rho)^{\frac{\alpha}{1-\alpha}}} \tag{1}$$

An increase in either ρ or σ reduces m^*. The intuition for the first is that it implies an increase in the equilibrium interest rate and thus it is more costly to hold money (one point for this). The same happens when σ increases, but in this case because that leads to more inflation (also increasing cost of holding money). (one point for this)

An increase in ϕ has an ambiguous effect on m^*. This can be seen by totally differentiating equation (1) to get

$$\frac{dm}{d\phi} = \frac{1 - m^* e^{\phi m^*} \frac{\rho+\sigma}{(\alpha/\rho)^{\frac{\alpha}{1-\alpha}}}}{\frac{\rho+\sigma}{(\alpha/\rho)^{\frac{\alpha}{1-\alpha}}} e^{\phi m^*}}$$

When $\frac{\phi m^* e^{\phi m^*}}{1+e^{\phi m^*}} < 1 (> 1)$ the derivative is positive (negative). This depends on parameters that determine the value of ϕm^*. The intuition is that an increase in ϕ makes money more effective in reducing search costs. If $m*$ is relatively large it is optimal to reduce money holdings to save on nominal interest rate cost of holding money. But if m^* is relatively low, the optimal response is to increase money holdings and profit from reduction of search costs (one point for this).

d) Money is not superneutral. Although σ does not affect k^*, it does affect c^* as can be seen from equation

$$c^* = \frac{(k^*)^\alpha}{1 + e^{-\phi m^*}}$$

and in c) we have shown that σ affects m^*. (four points for this, two if wrong but correctly states that σ does not affect steady state capital holdings)

Yes, the Friedman rule holds in this economy. Setting $\sigma = -r_t$ such that the marginal cost of holding money is zero is feasible. From FOC for money holdings this implies that $m^* = \infty$ in this case (thus the caveat that in this case wealth growth at a rate that does not violate transversality condition). This is the "optimal quantity of money" since it reduces the cost of holding money and at the same time increases steady state consumption. (four point for this)

Intuition is that money is not superneutral because more money reduces the search cost of consumption. This effect is absent in basic Sidrauski model. Since there is no cost of holding money besides nominal interest rate, when this is zero the holdings of money grow with no bound because this reduces the search cost of consumption to zero at no cost. (two points for this)

e) This changes budget constraint, Hamiltonian and FOC. Now they look like (only relevant equations are written)

$$\begin{aligned}
\dot{a}_t^i &= w_t + r_t a_t^i + z_t^i - c_t^i(1 + e^{-\phi m_t}) - (r_t + \pi_t) m_t^i \\
H_t^c &= \ln c_t^i + \mu_t \left(w_t + r_t a_t^i + z_t^i - c_t^i(1 + e^{-\phi m_t}) - (r_t + \pi_t) m_t^i \right) \\
\frac{dH_t^c}{dc_t^i} &= \frac{1}{c_t^i} - \mu_t (1 + e^{-\phi m_t}) = 0 \\
\frac{dH_t^c}{dm_t^i} &= \mu_t \left(-(r_t + \pi_t) \right) = 0
\end{aligned}$$

Student should note that first FOC (third equation above) implies a positive μ_t. Thus second FOC (fourth equation above) should in general be replaced by corner solution $m_t^i = 0$. The intuition is that if money holdings are perceived to have costs and no benefits then there will be no demand in equilibrium. (four points for correct equations above, i.e. how this assumption changes answers to b)).

Equilibrium condition are unaffected for w, r and z. No penalty from not stating this. Steady state equations are same as before (one point for this), in general $m^* = 0$, and unaffected by changes in parameters, so now $c^* = \frac{(k^*)^\alpha}{2}$ (one point for this).

Money now is superneutral in general, as changes in σ do not affect either k^* nor c^* (two points for this).

Interestingly, the Friedman rule still applies as setting $\sigma = -\rho$ implies that there is no cost in holding money. Since there is no benefit from holding money, for this case money holdings at the individual (and aggregate) level are indeterminate. (two points if correctly explaining that Friedman rule still holds).

2 Problem 2

a) Given that objective function is quadratic and equations that characterize inflation and output growth are linear we know that the optimal policy rule will be linear in the observables. These are θ, ν, ϵ. (two points for this). Because information revealed in the first period has no impact on second period outcomes for inflation and output (the shocks are independent across time), and loss function depends separately on first and second period outcomes, the optimal policy in the second period does not depend on shocks observed in the first period. (two points for correctly stating this).

The candidate policy rule is:

$$m_t = \psi_0 + \psi_\theta \theta_t + \psi_\epsilon \epsilon_t + \psi_\nu \nu_t$$

with parameters ψ_X constant (i.e. the same in both periods), that can be estimated as if the economy lasted for one period with loss function defined on outcomes for that single period.

Since the rule is credible, private inflation expectations are given by:

$$\pi_t^e = E[\pi_t|\theta_t] = E[m_t + \nu_t|\theta_t] = \psi_0 + \psi_\theta \theta_t$$

In equilibrium, inflation and output growth are given by:

$$\begin{aligned} \pi_t &= \psi_0 + \psi_\theta \theta_t + \psi_\epsilon \epsilon_t + (\psi_\nu + 1)\nu_t \\ x_t &= \theta_t + (\psi_\epsilon - 1)\epsilon_t + (\psi_\nu + 1)\nu_t \end{aligned}$$

Plugging these outcomes in the period loss function and minimizing with respect to parameters ψ_X characterizes optimal rule. It is immediate to note that this rule must satisfy (no penalty if derived through maximization instead of using insight):

1) $\psi_\nu = -1$. There is no conflict of interest in stabilizing demand shocks. If the student realizes this and drops demand shocks from this point onwards that is fine. If ν terms are carried on there is no penalty, but in all analysis they should drop in equilibrium.

2) $\psi_0 = 0$ and $\psi_\theta = 0$ because they only show in inflation outcome and target inflation is $\bar{\pi} = 0$.

With this the period expected loss only depends on ψ_ϵ:

$$E[L(\pi, x)] = \frac{1}{2}\left(\psi_\epsilon^2 \sigma_\epsilon + \lambda(\sigma_\theta + (\psi_\epsilon - 1)^2 \sigma_\epsilon)\right)$$

FOC with respect to ψ_ϵ gives

$$\psi_\epsilon + \lambda(\psi_\epsilon - 1) = 0$$

with solution

$$\psi_\epsilon = \frac{\lambda}{1+\lambda}$$

(three points for correctly deriving policy rule).

Equilibrium inflation and output growth follow directly

$$\begin{aligned} \pi_t^C &= \frac{\lambda}{1+\lambda}\epsilon_t \\ x_t^C &= \theta_t - \frac{1}{1+\lambda}\epsilon_t \end{aligned}$$

(three points for correctly deriving equilibrium inflation and output growth).

b) With discretion, a credible policy must satisfy the conditions of being ex post optimal, i.e. that m is chosen after observing shocks to minimize loss function, and that expectations are rational such that on average the private sector is not surprised by ex post incentives from central bank.

Since again there are no feedback effects between first and second period it is sufficient to analyze one period choice. Government chooses m after observing θ, π^e, ϵ and ν. Policy is chosen such that

$$\frac{dL}{dm} = 0$$

(three points if ex post objective is correct) which gives FOC

$$m + \nu + \lambda(\theta + m + \nu - \pi^e - \epsilon - \bar{x}) = 0$$

As we explained above, demand shocks are always fully stabilized with no conflict, so we drop ν shock (no penalty if student carries this shock as long as in equilibrium expression it does not appear). Expected inflation is

$$\pi^e = E[m|\theta] = \lambda(\bar{x} - \theta) > 0$$

(two points if expected inflation is correctly derived, not just written from memory)

Replacing this on FOC and then on output gives equilibrium inflation and output

$$\begin{aligned} \pi_t^D &= \lambda(\bar{x} - \theta_t) + \frac{\lambda}{1+\lambda}\epsilon_t \\ x_t^D &= x_t^C = \theta_t - \frac{1}{1+\lambda}\epsilon_t \end{aligned}$$

(three points if equilibrium is correctly derived)

Intuition for lower welfare comes from no gain in output stabilization and a loss from higher inflation. The student could derive this mathematically by calculating expected loss and comparing with expected loss under commitment. There is no need for math, since in both cases the difference is the inflation bias term $\lambda(\bar{x} - \theta)$ that in expectation gives an additional term in the loss function (which is quadratic) of

$$\frac{1}{2}\lambda^2(\bar{x}^2 + \sigma_\theta)$$

(two points for correct intuition)

c) In this case policy will be still chosen ex post, but by adequately choosing a central banker's preferences it might be possible to improve welfare with respect to the outcome of b). Denoting λ^{CB} the preferences of the chosen CB we know, from point b) that equilibrium inflation and output growth are given by

$$\begin{aligned} \pi_t^{CB} &= \lambda^{CB}(\bar{x} - \theta_t) + \frac{\lambda^{CB}}{1+\lambda^{CB}}\epsilon_t \\ x_t^{CB} &= \theta_t - \frac{1}{1+\lambda^{CB}}\epsilon_t \end{aligned}$$

(three points as correct solution for equilibrium inflation and output growth)

To characterize optimal choice of λ^{CB} we have to look at ex ante loss function. As again there are no feedback effects between first and second period it is sufficient to analyze one period problem.

$$
\begin{aligned}
\max_{\lambda^{CB}} \quad & \frac{1}{2}E\left[\left(\lambda^{CB}(\bar{x}-\theta_t)+\frac{\lambda^{CB}}{1+\lambda^{CB}}\epsilon_t\right)^2+\lambda\left(\theta_t-\frac{1}{1+\lambda^{CB}}\epsilon_t-\bar{x}\right)^2\right] \\
& \frac{1}{2}\left[\lambda^{CB2}(\bar{x}^2+\sigma_\theta)+\left(\frac{\lambda^{CB}}{1+\lambda^{CB}}\right)^2\sigma_\epsilon+\lambda\left((\bar{x}^2+\sigma_\theta)+\left(\frac{1}{1+\lambda^{CB}}\right)^2\right)\sigma_\epsilon\right]
\end{aligned}
$$

The FOC is

$$
\lambda^{CB}(\bar{x}^2+\sigma_\theta)+(\lambda^{CB}-\lambda)\frac{\sigma_\epsilon}{(1+\lambda^{CB})^3}=0
$$

By inspecting this FOC at $\lambda^{CB}=0$ and $\lambda^{CB}=\lambda$ we find that the optimum satisfies $0<\lambda^{CB}<\lambda$. (four points for correctly showing this).

Finally output volatility is

$$
\sigma_\theta+(\frac{1}{1+\lambda^{CB}})^2\sigma_{\epsilon_t}
$$

Thus it is higher than in b). The reason for this result is that by choosing a more conservative central banker, society faces a trade-off of lower inflation bias against lower pass-through of supply shocks to inflation and higher pass-through of supply shocks to output than optimal. This last part of the trade-off explains the higher output volatility. (three points for this)

d) The central banker wants to maximize

$$
\max_m -L(\pi,x)+P(\theta,\pi)
$$

In principle the student would have to set up the problem of finding an optimal contract $p_{c,t}$ and $p_{\pi,t}(\theta_t)$ and from four FOC find that indeed the optimal contract is independent of time. Instead one can check that a properly chosen autonomous contract leads to the first best equilibrium. If this contract attains the first best, then it must be optimal. We proceed in this way (there is no penalty if result is found by considering both periods of time and looking at four FOC).

Given a contract p_c and $p_\pi(\theta)$ the optimal ex post choice of the CB is given by (again we drop ν term)

$$
-(\pi+\lambda(\theta+\pi-\pi^e-\epsilon-\bar{x}))+p_\pi(\theta)=0
$$

Knowing this FOC determines the ex post choice of inflation by CB the private sector forms expectations on inflation

$$
\pi^e=E[\pi|\theta]=p_\pi(\theta)+\lambda(\bar{x}-\theta)
$$

Thus the optimal contract must satisfy $p_\pi(\theta) = -\lambda(\bar{x} - \theta)$, since in this way the CB is given incentives to reduce inflation in such a way that the inflation bias is eliminated.

Given this, the equilibrium inflation and output growth are

$$\begin{aligned} \pi_t^{IT} &= \frac{\lambda}{1+\lambda}\epsilon_t \\ x_t^{IT} &= \theta_t - \frac{1}{1+\lambda}\epsilon_t \end{aligned}$$

Since this is the same outcome as the first best this contract must be the optimal one, and thus is independent of time as conjectured.

(three points for showing, one way or another, that the contract is independent of time, four points for the correct characterization of $p_\pi(\theta)$).

The constant term p_c must be chosen such that the expected gain to the CB of accepting the job is at least $\bar{\omega}$:

$$p_c + E[-\lambda(\bar{x} - \theta_t)\frac{\lambda}{1+\lambda}\epsilon_t] = p_c \geq \bar{\omega}$$

(three points for this)

e) Now, because the CB can be fired at the end of the first period, the problem must be solved for both periods. If the CB is not fired, then a contract that satisfies $p_{\pi,2}(\theta_2) = -\lambda(\bar{x} - \theta_2)$ must be optimal in the second period. This follows because the economy ends at the end of the second period, and thus the CB (if still on the job) behaves according to incentives as in d).

For first period now the CB has the following incentives

$$\max_m -L(\pi_1, x_1) + P(\theta_1, \pi_1) + (a - b\pi_1)\left[p_{c,2} - \bar{\omega}\right]$$

where the second term tells us that if fired the CB looses the surplus from the contractual relation over the alternative from working as a consultant (Note: this presumes that inflation and output outcomes will be the same if policy is set by other CB with a similar contract in the second period. Otherwise the threat of firing the CB would not be credible. No penalty from not considering how policy is chosen when the first period CB is fired).

(two points for correctly finding that second period contract is same as in d)).

If $p_{c,2} = \bar{\omega}$, then the CB is indifferent between holding the job or not and the threat of firing him will not change his incentives for policy in the first period. In this case the contract in the first period is the same as in point d). If $p_{c,2} > \bar{\omega}$, then the CB strictly prefers to keep the job and the threat of being fired will give him more stringent incentives to reduce inflation in the first period.

(two points for identifying second period surplus as source of incentives for CB in first period. This is sufficient for the "comment" part of the question).

Then the optimal ex post choice of the CB is given by:

$$-\left(\pi_1 + \lambda(\theta_1 + \pi_1 - \pi_1^e - \epsilon_1 - \bar{x})\right) + p_{\pi,1}(\theta_1) - b(p_{c,2} - \bar{\omega}) = 0$$

Knowing this FOC determines the ex post choice of inflation by CB the private sector forms expectations on inflation

$$\pi_1^e = E[\pi_1|\theta_1] = p_{\pi,1}(\theta_1) - b(p_{c,2} - \bar{\omega}) + \lambda(\bar{x} - \theta_1)$$

Again it is possible to eliminate the inflation bias by conveniently choosing $p_{\pi,1}(\theta) = b(p_{c,2} - \bar{\omega}) - \lambda(\bar{x} - \theta_1)$.

(four points for correctly finding $p_{\pi,1}(\theta_1)$).

The contract is fully characterized by finding $p_{c,1}$, which must satisfy

$$p_{c,1} + E[-\lambda(\bar{x} - \theta_1)\frac{\lambda}{1+\lambda}\epsilon_1] + E\left[(a - b\pi_1)(p_{c,2} - \bar{\omega})\right] = p_{c,1} + a(p_{c,2} - \bar{\omega}) \geq \bar{\omega}$$

(two points for this).

(Note: since $a < 1$ (since otherwise when inflation is zero the probability of keeping the job is larger than one), using the threat of firing the CB implies loosing money since when $p_{c,1} + a(p_{c,2} - \bar{\omega}) = \bar{\omega}$ it must be the case that $p_{c,1} + p_{c,2} > 2\bar{\omega}$. No penalty from not realizing this)