2012V – 1LM ex

Vejl. løsninger – Lineære modeller V2012

Opg 1

1) + 2)

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} R_3 = R_3 - R_1 \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{pmatrix} \begin{matrix} R_1 = R_1 - R_2 \\ R_3 = R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Heraf ses at u_1 og u_2 er lin. uafh. samt at $u_3 = 2u_1 - 3u_2$.

Så er $\text{span}\{u_1, u_2\} = U$ og $u_3 = (2, -3)$.

3) $L(u_1 + u_2) = Lu_1 + Lu_2 = u_1 + u_2 + Lu_2 = u_3 = 2u_1 - 3u_2$

Så er $Lu_2 = u_1 - 4u_2$, så

$$L \sim \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}$$

4) Da $\det(L) = -5 \neq$ er L regulær og dermed invertibel (bijektiv)

5) $Lu_3 = L(2u_1 - 3u_2) = 2Lu_1 - 3Lu_2$

$= 2(u_1 + u_2) - 3(u_1 - 4u_2) = \underline{\underline{-u_1 + 14u_2}}$

$L^{-1}u_3 = \underline{\underline{u_1 + u_2}}$ da $L(u_1 + u_2) = u_3$

opg 2

Hvis

1) $$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

fås at $A = QDQ^T$ iflg. spektralsætningen.

Da fås

$$A = \begin{pmatrix} \frac{1}{2}+\frac{1}{3} & -\frac{2}{3} & -\frac{1}{2}+\frac{1}{3} \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\ -\frac{1}{2}+\frac{1}{3} & -\frac{2}{3} & \frac{1}{2}+\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & -\frac{2}{3} & -\frac{1}{6} \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\ -\frac{1}{6} & -\frac{2}{3} & \frac{5}{6} \end{pmatrix}$$

2) Da 0 er en egenværdi er A ikke invertibel.

3) $$e^A = Qe^DQ^T = \begin{pmatrix} \frac{1}{3}+\frac{1}{2}e+\frac{1}{6}e^2 & \frac{1}{3}-\frac{1}{3}e^2 & \frac{1}{3}-\frac{1}{2}e+\frac{1}{6}e^2 \\ \frac{1}{3}-\frac{1}{3}e^2 & \frac{1}{3}+\frac{2}{3}e^2 & \frac{1}{3}-\frac{1}{3}e^2 \\ \frac{1}{3}-\frac{1}{2}e+\frac{1}{6}e^2 & \frac{1}{3}-\frac{1}{3}e^2 & \frac{1}{3}+\frac{1}{2}e+\frac{1}{6}e^2 \end{pmatrix}$$

e^A er invertibel da egenværdierne er e^0, e^1, e^2 som alle er $\neq 0$. (Da er $\det e^A = e^0e^1e^2 = e^3 \neq 0$, dvs e^A er regulær og dermed invertibel)

$$Q \underbrace{\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}}_{D} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{a}{\sqrt{3}} & \frac{a}{\sqrt{3}} & \frac{a}{\sqrt{3}} \\ -\frac{b}{\sqrt{2}} & 0 & \frac{b}{\sqrt{2}} \\ \frac{c}{\sqrt{6}} & \frac{-2c}{\sqrt{6}} & \frac{c}{\sqrt{6}} \end{pmatrix}$$

$$\begin{matrix} \frac{1}{3}a + \frac{1}{2}b + \frac{1}{6}c & \frac{1}{3}a - \frac{1}{3}c & \frac{1}{3}a - \frac{1}{2}b + \frac{1}{6}c \\ \frac{1}{3}a - \frac{1}{3}c & \frac{1}{3}a + \frac{2}{3}c & \frac{1}{3}a - \frac{1}{3}c \\ \frac{1}{3}a - \frac{1}{2}b + \frac{1}{6}c & \frac{1}{3}a - \frac{1}{3}c & \frac{1}{3}a + \frac{1}{2}b + \frac{1}{6}c \end{matrix}$$

$a = 0 \quad b = 1 \quad c = 2$

$$\begin{pmatrix} \frac{1}{2} + \frac{1}{3} & -\frac{2}{3} & -\frac{1}{2} + \frac{1}{3} \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\ -\frac{1}{2} + \frac{1}{3} & -\frac{2}{3} & \frac{1}{2} + \frac{1}{3} \end{pmatrix}$$

Opg3

$$\cos(2x)\sin^2(3x) = \frac{e^{i2x}+e^{-i2x}}{2}\left(\frac{e^{i3x}-e^{-i3x}}{2i}\right)^2$$

$$= -\frac{1}{8}\left((e^{i2x}+e^{-i2x})(e^{i6x}+e^{-i6x}-2)\right)$$

$$= -\frac{1}{8}\left(e^{i8x}+e^{-i4x}-2e^{i2x}+e^{i4x}+e^{-i8x}-2e^{-i2x}\right)$$

$$= -\frac{1}{4}\left(\frac{e^{i8x}+e^{-i8x}}{2}+\frac{e^{i4x}+e^{-i4x}}{2}-2\frac{e^{i2x}+e^{-i2x}}{2}\right)$$

$$= -\frac{1}{4}\left(\cos(8x)+\cos(4x)-2\cos(2x)\right)$$

Så er

1) $\int \cos(2x)\sin^2(3x)\,dx = -\frac{1}{4}\left(\frac{1}{8}\sin(8x)+\frac{1}{4}\sin(4x)-2\cdot\frac{1}{2}\sin(2x)\right)$

$$= -\frac{1}{32}\sin(8x)-\frac{1}{16}\sin(4x)+\frac{1}{4}\sin(2x)+k.$$

2) $2z^2-8z+10=0$, $z=\frac{8\pm\sqrt{64-80}}{4}=\underline{\underline{2\pm i}}$

3) $\sum_{n=0}^{3} e^{in\frac{\pi}{2}} = 1+e^{\frac{i\pi}{2}}+e^{i\pi}+e^{i\frac{3\pi}{2}} = 1+i-1-i=\underline{\underline{0}}.$

Opg 4

1) $P_m(x) = \sum_0^\infty e^{-mnx} = \sum_0^\infty (e^{-mx})^n$

Konvergent for $|e^{-mx}| < 1$, dvs
$-mx < 0$
hvorfor $\underline{\underline{x > 0}}$

2) For $x > 0$ er

$$P_m(x) = \frac{1}{1-e^{-mx}}$$

3) Da $P_m'(x) = \frac{-me^{-mx}}{(1-e^{-mx})^2} < 0$

er P_m aftagende, og dermed injektiv.

For $x \to 0^+$ vil $P_m(x) \to +\infty$

For $x \to +\infty$ vil $P_m(x) \to 1^+$

Da er $R(P_m) =]1, \infty[$

4) For $y > 1$ fås

$$\frac{1}{1-e^{-mx}} = y \Leftrightarrow \frac{1}{y} = 1 - e^{-mx}$$

$$\Leftrightarrow \frac{1}{y} - 1 = -e^{-mx} \Leftrightarrow \frac{y-1}{y} = e^{-mx} \Leftrightarrow$$

$$-mx = \ln\left(\frac{y-1}{y}\right) \Leftrightarrow x = -\frac{1}{m}\ln\left(\frac{y-1}{y}\right).$$