11.9.4.7

EE23BTECH11063 - Vemula Siddhartha

Question:
Find the sum to n terms of the series:

$$1^2 + \left(1^2 + 2^2\right) + \left(1^2 + 2^2 + 3^2\right) + \ldots$$

Solution:

Variable	Description
$y(n)$	Sum of $n+1$ terms of the series
$x(n)$	General term

TABLE 0
VARIABLES USED

$$y(n) = 1^2 + \left(1^2 + 2^2\right) + \left(1^2 + 2^2 + 3^2\right) + \ldots \tag{1}$$
$$\tag{2}$$

Let,

$$y(n) = \sum_{k=0}^{n} x(k) \tag{3}$$
$$y(n) = x(n) * u(n) \tag{4}$$

Then,

$$x(n) = \sum_{k=0}^{n} (k+1)^2 u(k) \tag{5}$$
$$x(n) = \left((n+1)^2 u(n)\right) * u(n) \tag{6}$$

From equations (**??**) and (**??**),

$$X(z) = Z\left\{(n+1)^2 u(n)\right\} U(z) \tag{7}$$
$$= \left(z \frac{z^{-1}\left(1+z^{-1}\right)}{(1-z^{-1})^3}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{8}$$
$$X(z) = \frac{1+z^{-1}}{(1-z^{-1})^4} \tag{9}$$

From (4),

$$Y(z) = X(z)\,U(z) \tag{10}$$
$$= \left(\frac{1+z^{-1}}{(1-z^{-1})^4}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{11}$$
$$= \frac{z^4(z+1)}{(z-1)^5} \tag{12}$$

Taking the Inverse Z transform, from Contour Integration,

$$y(n) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{n-1}\, dz \tag{13}$$
$$= \frac{1}{2\pi j} \oint_C \frac{z^4(z+1)}{(z-1)^5}\, z^{n-1} dz \tag{14}$$
$$\tag{15}$$

Using Cauchy's Residual Theorem for 5 repeated roots at $z = 1$,

$$= \frac{1}{(k-1)!} \lim_{z \to c} \frac{d^{k-1}}{dz^{k-1}} \left((z-c)^k f(z)\right) \tag{16}$$
$$= \frac{1}{4!} \lim_{z \to 1} \frac{d^4}{dz^4} \left((z-1)^5 \frac{z^4(z+1)}{(z-1)^5} z^{n-1}\right) \tag{17}$$
$$= \frac{1}{24} \lim_{z \to 1} \frac{d^4}{dz^4} \left(z^{n+4} + z^{n+3}\right) \tag{18}$$
$$\implies y(n) = \frac{(n+1)(n+2)^2(n+3)}{12} u(n) \tag{19}$$

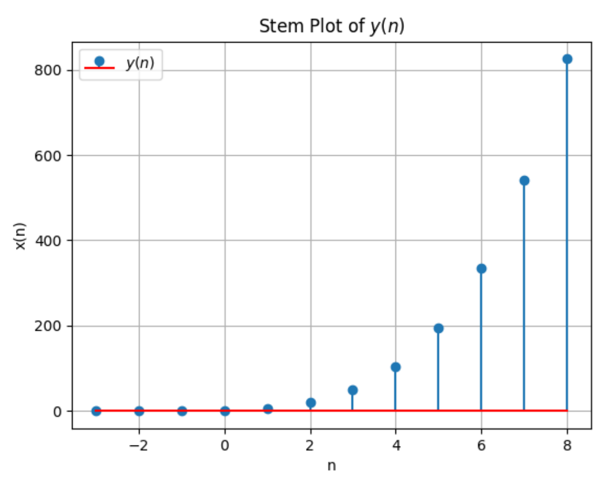

Fig. 0. Stem Plot of y(n)