CE-25

EE23BTECH11063 - Vemula Siddhartha

Question:
The following function is defined over the interval $[-L, L]$:

$$f(x) = px^4 + qx^5$$

It is expressed as a Fourier series,

$$f(x) = a(0) + \sum_{n=1}^{\infty} \left\{ a(n) \sin\left(\frac{\pi n x}{L}\right) + b(n) \cos\left(\frac{\pi n x}{L}\right) \right\}$$

which options amongst the following are true?

(a) $a(n)$, $n = 1, 2, .., \infty$ depend on p
(b) $a(n)$, $n = 1, 2, .., \infty$ depend on q
(c) $b(n)$, $n = 1, 2, .., \infty$ depend on p
(d) $b(n)$, $n = 1, 2, .., \infty$ depend on q

Solution:

Parameter	Description
$f(x)$	Polynomial function
$2L$	Period of the Polynomial function
$c(n)$	Complex Fourier Coefficients
$a(0)$, $a(n)$, $b(n)$	Trigonometric Fourier Coefficients

TABLE 4
PARAMETERS

The complex exponential Fourier Series of $f(x)$ is,

$$f(x) = \sum_{n=-\infty}^{\infty} c(n) e^{j\frac{\pi n x}{L}} \tag{1}$$

$$\implies c(n) = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-j\frac{\pi n x}{L}} \, dx \tag{2}$$

$$c(n) = \frac{1}{2L} \int_{-L}^{L} \left(px^4 + qx^5\right) e^{-j\frac{\pi n x}{L}} \, dx \tag{3}$$

For $n = 0$,

$$c(0) = \frac{1}{2L} \int_{-L}^{L} \left(px^4 + qx^5\right) \, dx \tag{4}$$

$$= \frac{pL^4}{5} \tag{5}$$

For $n \neq 0$,

$$c(n) = \frac{1}{2L} \int_{-L}^{L} \left(px^4 + qx^5\right) e^{-j\frac{\pi n x}{L}} \, dx \tag{6}$$

$$\begin{aligned} &= \frac{pL^4}{2}\left(e^{j\pi n} - e^{-j\pi n}\right)\left(\frac{1}{j\pi n} + \frac{12}{(j\pi n)^3} + \frac{24}{(j\pi n)^5}\right) \\ &- \frac{pL^4}{2}\left(e^{j\pi n} + e^{-j\pi n}\right)\left(\frac{4}{(j\pi n)^2} + \frac{24}{(j\pi n)^4}\right) \\ &- \frac{qL^5}{2}\left(e^{j\pi n} + e^{-j\pi n}\right)\left(\frac{1}{j\pi n} + \frac{20}{(j\pi n)^3} + \frac{120}{(j\pi n)^5}\right) \\ &+ \frac{qL^5}{2}\left(e^{j\pi n} - e^{-j\pi n}\right)\left(\frac{5}{(j\pi n)^2} + \frac{60}{(j\pi n)^4} + \frac{120}{(j\pi n)^6}\right) \end{aligned} \tag{7}$$

$$\begin{aligned} &= \left(pL^4\right)(-1)^n\left(\frac{4}{(\pi n)^2} - \frac{24}{(\pi n)^4}\right) \\ &- \left(qL^5\right)(-1)^n\left(-\frac{j}{\pi n} + \frac{20j}{(\pi n)^3} - \frac{120j}{(\pi n)^5}\right) \end{aligned} \tag{8}$$

Given,

$$f(x) = a(0) + \sum_{n=1}^{\infty} \left\{ a(n) \sin\left(\frac{\pi n x}{L}\right) + b(n) \cos\left(\frac{\pi n x}{L}\right) \right\} \tag{9}$$

Finding the Fourier Coefficient $a(0)$,

$$a(0) = c(0) \tag{10}$$

$$\implies a(0) = \frac{pL^4}{5} \tag{11}$$

We know,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{12}$$

Finding the Fourier Coefficients $a(n)$,

$$a(n) = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{\pi n x}{L}\right) \, dx \tag{13}$$

$$a(n) = \frac{1}{L} \int_{-L}^{L} f(x) \left(\frac{e^{j\frac{\pi n x}{L}} - e^{-j\frac{\pi n x}{L}}}{2j}\right) \, dx \tag{14}$$

$$= \frac{1}{2Lj} \int_{-L}^{L} f(x) e^{j\frac{\pi n x}{L}} \, dx - \frac{1}{2Lj} \int_{-L}^{L} f(x) e^{-j\frac{\pi n x}{L}} \, dx \tag{15}$$

$$\implies a(n) = \frac{c(-n) - c(n)}{j} \tag{16}$$

$$a(n) = \left(-2qL^5\right)(-1)^n \left(\frac{1}{\pi n} - \frac{2}{(\pi n)^3} + \frac{120}{(\pi n)^5}\right) \tag{17}$$

We know,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{18}$$

Finding the Fourier Coefficients $b(n)$,

$$b(n) = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi n x}{L}\right) dx \tag{19}$$

$$b(n) = \frac{1}{L}\int_{-L}^{L} f(x)\left(\frac{e^{j\frac{\pi n x}{L}} + e^{-j\frac{\pi n x}{L}}}{2}\right) dx \tag{20}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{j\frac{\pi n x}{L}}\, dx + \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-j\frac{\pi n x}{L}}\, dx \tag{21}$$

$$\implies b(n) = c(-n) + c(n) \tag{22}$$

$$b(n) = \left(2pL^4\right)(-1)^n \left(\frac{4}{(\pi n)^2} - \frac{24}{(\pi n)^4}\right) \tag{23}$$

Hence, options (b) and (c) are correct.

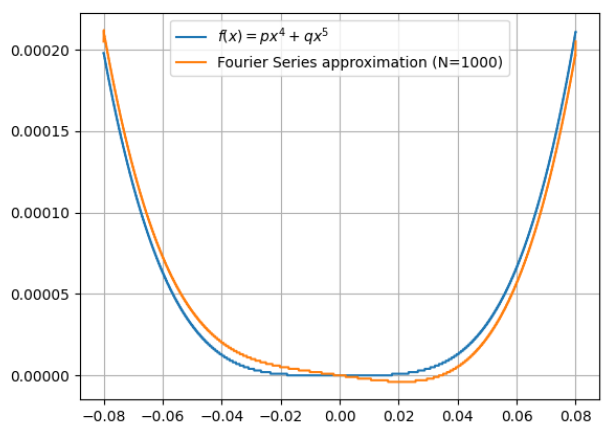

Fig. 4. Fourier Series Approximation of $f(x)$ for $p = 5$, $q = 2$, $L = 0.08$