# 10.5.3.9

EE23BTECH11063 - Vemula Siddhartha

**Question**:
If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first n terms.

**Solution:**
The sum of first $r$ terms of an Arithmetic Progression $(AP)$ $S_r$ , whose first term is $a$ and common difference is $d$ is:

$$S_r = \frac{r}{2}\,(2a + (r-1)\,d) \tag{1}$$

Let the given AP have first term $a$ and common difference $d$.
Given, the sum of first 7 terms of the AP is 49.

$$S_7 = 49 \tag{2}$$
$$49 = \frac{7}{2}\,(2a + (7-1)\,d) \tag{3}$$
$$49 = \frac{7}{2}\,(2a + 6d) \tag{4}$$
$$a + 3d = 7 \tag{5}$$

Also given, the sum of first 17 terms of the AP is 289.

$$S_{17} = 289 \tag{6}$$
$$289 = \frac{17}{2}\,(2a + (17-1)\,d) \tag{7}$$
$$289 = \frac{17}{2}\,(2a + 16d) \tag{8}$$
$$a + 8d = 17 \tag{9}$$

From equations 5 and 9:

$$\begin{pmatrix} 1 & 3 \\ 1 & 8 \end{pmatrix} \begin{pmatrix} a \\ d \end{pmatrix} = \begin{pmatrix} 7 \\ 17 \end{pmatrix} \tag{10}$$
$$\implies \begin{pmatrix} a \\ d \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 17 \end{pmatrix} \tag{11}$$
$$\implies \begin{pmatrix} a \\ d \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 8 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 17 \end{pmatrix} \tag{12}$$
$$\implies \begin{pmatrix} a \\ d \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 56 - 51 \\ -7 + 17 \end{pmatrix} \tag{13}$$
$$\implies \begin{pmatrix} a \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{14}$$
$$\implies a = 1;\ d = 2 \tag{15}$$

The sum of first n terms of the AP is:

$$S_n = \frac{n}{2}\,(2a + (n-1)\,d) \tag{16}$$

Substituting the values of $a$ and $d$:

$$S_n = \frac{n}{2}\,(2\,(1) + (n-1)\,(2)) \tag{17}$$
$$S_n = n\,(1 + n - 1) \tag{18}$$
$$S_n = n^2 \tag{19}$$

The signal corresponding to this will be:

$$x(n) = n^2\,u(n) \tag{20}$$

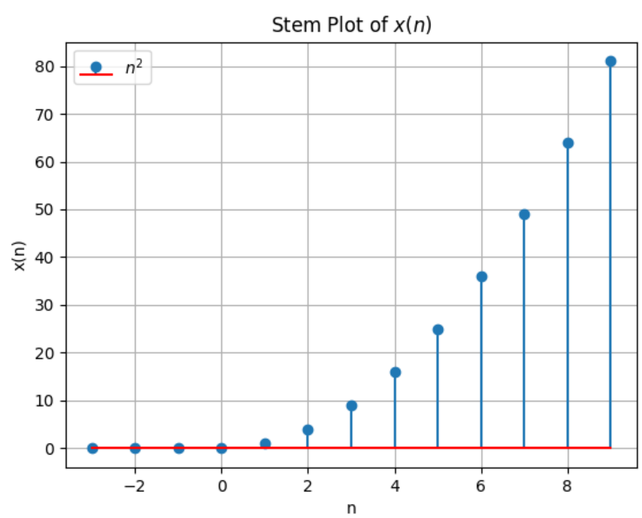


Fig. 1: Stem Plot of x(n)

Applying z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} \left(n^2\,u(n)\right) z^{-n} \tag{21}$$
$$X(z) = 0 + \sum_{n=1}^{\infty} \left(n^2\right)\,z^{-n} \tag{22}$$

For the above series to converge, the limit of the modulus of the ratio of consecutive terms of the series must be less than 1:

$$\implies \lim_{n\to\infty} \left| \frac{(n+1)^2\,z^{-(n+1)}}{n^2\,z^{-n}} \right| < 1 \tag{23}$$

$$\implies \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^2 z^{-1}\right| < 1 \tag{24}$$

$$\implies \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^2 z^{-1}\right| < 1 \tag{25}$$

$$\implies |z| > \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^2\right| \tag{26}$$

$$\implies |z| > 1 \tag{27}$$

Hence, the Region of Convergence (ROC) is $|z| > 1$.

$$X(z) = \left(1^2\right) z^{-1} + \left(2^2\right) z^{-2} + \left(3^2\right) z^{-3} + ... \tag{28}$$

$$X(z) = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + ... \tag{29}$$

Multiplying the equation 19 with $z^{-1}$:

$$z^{-1}X(z) = z^{-2} + 4z^{-3} + 9z^{-4} + 16z^{-5} + ... \tag{30}$$

Subtracting equation 30 from equation 29:

$$X(z)\left(1 - z^{-1}\right) = z^{-1} + 3z^{-2} + 5z^{-3} + 7z^{-4} + ... \tag{31}$$

Multiplying the equation 30 with $z^{-1}$:

$$X(z)\left(z^{-1} - z^{-2}\right) = z^{-2} + 3z^{-3} + 5z^{-4} + 7z^{-5} + ... \tag{32}$$

Subtracting equation 32 from equation 31:

$$X(z)\left(1 - 2z^{-1} + z^{-2}\right) = z^{-1} + 2\left(z^{-2} + z^{-3} + z^{-4} + ...\right) \tag{33}$$

$$X(z)\left(1 - z^{-1}\right)^2 = z^{-1} + 2\frac{\left(z^{-2}\right)}{(1 - z^{-1})} \tag{34}$$

$$X(z) = \frac{z^{-1}\left(1 - z^{-1}\right) + 2z^{-2}}{(1 - z^{-1})^3} \tag{35}$$

$$X(z) = \frac{z^{-1}\left(1 + z^{-1}\right)}{(1 - z^{-1})^3} \tag{36}$$

Generalizing the problem:
Let the sum of first $n_1$ terms of the AP be 49, and the sum of first $n_2$ terms of the AP be 289.
From equation 1:

$$\implies \frac{n_1}{2}\left(2a + (n_1 - 1)\, d\right) = 49 \tag{37}$$

$$\implies a(n_1) + d\,\frac{(n_1 - 1)(n_1)}{2} = 49 \tag{38}$$

Also,

$$\frac{n_2}{2}\left(2a + (n_1 - 1)\, d\right) = 289 \tag{39}$$

$$\implies a(n_2) + d\,\frac{(n_2 - 1)(n_2)}{2} = 289 \tag{40}$$

From equations 38 and 40:

$$\begin{pmatrix} n_1 & \frac{(n_1-1)(n_1)}{2} \\ n_2 & \frac{(n_2-1)(n_2)}{2} \end{pmatrix}\begin{pmatrix} a \\ d \end{pmatrix} = \begin{pmatrix} 49 \\ 289 \end{pmatrix} \tag{41}$$

$$\begin{pmatrix} a \\ d \end{pmatrix} = \begin{pmatrix} n_1 & \frac{(n_1-1)(n_1)}{2} \\ n_2 & \frac{(n_2-1)(n_2)}{2} \end{pmatrix}^{-1}\begin{pmatrix} 49 \\ 289 \end{pmatrix} \tag{42}$$

$$\begin{pmatrix} a \\ d \end{pmatrix} = \frac{1}{n_1\left(\frac{n_2^2-n_2}{2}\right) - \left(\frac{n_1^2-n_1}{2}\right)n_2}\begin{pmatrix} \frac{n_2^2-n_2}{2} & \frac{-n_1^2+n_1}{2} \\ -n_2 & n_1 \end{pmatrix}\begin{pmatrix} 49 \\ 289 \end{pmatrix} \tag{43}$$

$$\begin{pmatrix} a \\ d \end{pmatrix} = \begin{pmatrix} \frac{-n_2+1}{n_1^2-n_1n_2} & \frac{-n_1+1}{n_2^2-n_1n_2} \\ \frac{2}{n_1^2-n_1n_2} & \frac{2}{n_2^2-n_1n_2} \end{pmatrix}\begin{pmatrix} 49 \\ 289 \end{pmatrix} \tag{44}$$

$$\implies a = 49\left(\frac{-n_2+1}{n_1^2 - n_1n_2}\right) + 289\left(\frac{-n_1+1}{n_2^2 - n_1n_2}\right) \tag{45}$$

$$\implies d = 49\left(\frac{2}{n_1^2 - n_1n_2}\right) + 289\left(\frac{2}{n_2^2 - n_1n_2}\right) \tag{46}$$

Substituting the values of a and d in the equation 16:

$$\implies S_n = \frac{n}{2}\left(2\left(49\left(\frac{-n_2+1}{n_1^2 - n_1n_2}\right)\right) + 289\left(\frac{-n_1+1}{n_2^2 - n_1n_2}\right)\right) + (n-1)\left(49\left(\frac{2}{n_1^2 - n_1n_2}\right) + 289\left(\frac{2}{n_2^2 - n_1n_2}\right)\right) \tag{47}$$

| Variable | Description |
|---|---|
| $a$ | First term of the AP |
| $d$ | Common difference of the AP |
| $S_r$ | Sum of $r$ terms of the AP |
| $x(n)$ | General term |
| $X(z)$ | Z- transform of $x(n)$ |

TABLE 1: Variables Used