10.5.3.9

EE23BTECH11063 - Vemula Siddhartha

Question:
If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first n terms.

Solution:
The sum of first n terms of an AP S_n , whose first term is $x(0)$ and common difference is d is:

$$S_n = \frac{n}{2}\left(2x(0) + (n-1)\ d\right) \tag{1}$$

Given, the sum of first 7 terms of the AP is 49.

$$S_7 = 49 \tag{2}$$
$$49 = \frac{7}{2}\left(2x(0) + (7-1)\ d\right) \tag{3}$$
$$49 = \frac{7}{2}\left(2x(0) + 6d\right) \tag{4}$$
$$x(0) + 3d = 7 \tag{5}$$

Also given, the sum of first 17 terms of the AP is 289.

$$S_{17} = 289 \tag{6}$$
$$289 = \frac{17}{2}\left(2x(0) + (17-1)\ d\right) \tag{7}$$
$$289 = \frac{17}{2}\left(2x(0) + 16d\right) \tag{8}$$
$$x(0) + 8d = 17 \tag{9}$$

From equations 5 and 9, the augmented matrix is:

$$\begin{pmatrix} 1 & 3 & 7 \\ 1 & 8 & 17 \end{pmatrix} \tag{10}$$
$$R_2 \rightarrow R_2 - R_1$$
$$\begin{pmatrix} 1 & 3 & 7 \\ 0 & 5 & 10 \end{pmatrix} \tag{11}$$
$$R_1 \rightarrow 5R_1 - 3R_2$$
$$\begin{pmatrix} 5 & 0 & 5 \\ 0 & 5 & 10 \end{pmatrix} \tag{12}$$
$$R_1 \rightarrow \frac{1}{5}R_1$$
$$R_2 \rightarrow \frac{1}{5}R_2$$
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{13}$$
$$\implies \begin{pmatrix} x(0) \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{14}$$

$$\implies S_n = \frac{n}{2}\left(2x(0) + (n-1)\ d\right) \tag{15}$$
$$S_n = \frac{n}{2}\left(2(1) + (n-1)(2)\right) \tag{16}$$
$$S_n = n\left(1 + n - 1\right) \tag{17}$$
$$S_n = n^2 \tag{18}$$

The general term of the AP is:

$$x(n) = x(0) + nd \tag{19}$$
$$\implies x(n) = 1 + 2n \quad \forall n \geq 0 \tag{20}$$
$$\implies x(n) = (1 + 2n)\, u(n) \tag{21}$$

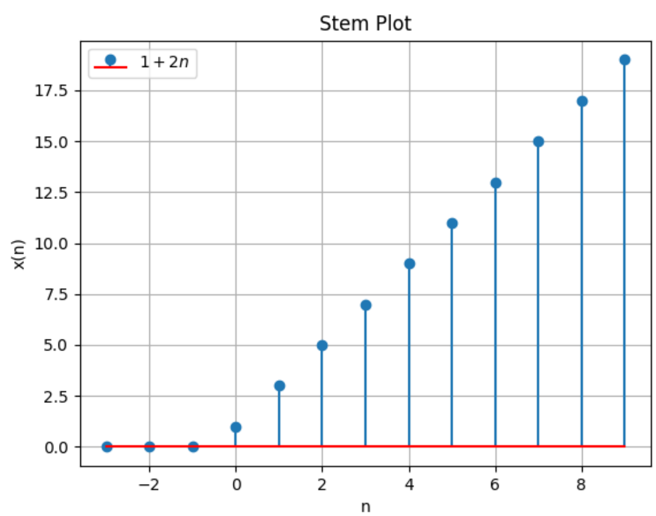

Fig. 1: Stem Plot of x(n)

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{22}$$

$$X(z) = \sum_{n=-\infty}^{\infty} (1+2n)\, u(n)\, z^{-n} \tag{23}$$

$$X(z) = 0 + \sum_{n=0}^{\infty} (1+2n)\, z^{-n} \tag{24}$$

$$X(z) = \sum_{n=0}^{\infty} z^{-n} + 2\sum_{n=0}^{\infty} n z^{-n} \tag{25}$$

$$\text{If: } x(n) \overset{Z}{\longleftrightarrow} X(z)$$

$$\implies n^k x(n) \overset{Z}{\longleftrightarrow} (-1)^k z^k \frac{d^k}{dz^k}(X(z))$$

$$\implies X(z) = \frac{1}{1-z^{-1}} + \frac{2z^{-1}}{(1-z^{-1})^2} \tag{26}$$

$$X(z) = \frac{1+z^{-1}}{(1-z^{-1})^2} \quad \forall \ |z| > 1 \tag{27}$$

Generalizing the problem:
Let the sum of first n_1 terms of the AP be 49, and the sum of first n_2 terms of the AP be 289.
From equation 1:

$$\implies \frac{n_1}{2}\,(2x(0) + (n_1-1)\, d) = 49 \tag{28}$$

$$\implies n_1 x(0) + d\,\frac{(n_1-1)(n_1)}{2} = 49 \tag{29}$$

Also,

$$\frac{n_2}{2}\,(2x(0) + (n_1-1)\, d) = 289 \tag{30}$$

$$\implies x(0)(n_2) + d\,\frac{(n_2-1)(n_2)}{2} = 289 \tag{31}$$

From equations 29 and 31, the augmented matrix is:

$$\begin{pmatrix} n_1 & \frac{(n_1-1)(n_1)}{2} & 49 \\ n_2 & \frac{(n_2-1)(n_2)}{2} & 289 \end{pmatrix} \tag{32}$$

$$R_1 \to \frac{1}{n_1} R_1$$

$$\begin{pmatrix} 1 & \frac{(n_1-1)}{2} & \frac{49}{n_1} \\ n_2 & \frac{n_2(n_2-1)}{2} & 289 \end{pmatrix} \tag{33}$$

$$R_2 \to R_2 - n_2 R_1$$

$$\begin{pmatrix} 1 & \frac{(n_1-1)}{2} & \frac{49}{n_1} \\ 0 & \frac{n_2(n_2-n_1)}{2} & 289 - \frac{49}{n_1} \end{pmatrix} \tag{34}$$

$$R_2 \to \frac{2}{n_2(n_2-n_1)} R_2$$

$$\begin{pmatrix} 1 & \frac{(n_1-1)}{2} & \frac{49}{n_1} \\ 0 & 1 & \frac{2(49n_2-289n_1)}{n_1 n_2 (n_1-n_2)} \end{pmatrix} \tag{35}$$

$$R_1 \to R_1 - \left(\frac{n_1-1}{2}\right) R_2$$

$$\begin{pmatrix} 1 & 0 & \frac{289n_1^2 - 289n_1 - 49n_2^2 + 49n_2}{n_1 n_2 (n_1-n_2)} \\ 0 & 1 & \frac{2(-289n_1+49n_2)}{n_1 n_2 (n_1-n_2)} \end{pmatrix} \tag{36}$$

$$\implies x(0) = 49\left(\frac{-n_2+1}{n_1^2 - n_1 n_2}\right) + 289\left(\frac{-n_1+1}{n_2^2 - n_1 n_2}\right) \tag{37}$$

$$\implies d = 49\left(\frac{2}{n_1^2 - n_1 n_2}\right) + 289\left(\frac{2}{n_2^2 - n_1 n_2}\right) \tag{38}$$

From the equations 15, 37 and 38:

$$\implies S_n = \frac{n}{2}\left(2\left(49\left(\frac{-n_2+1}{n_1^2 - n_1 n_2}\right)\right) + 289\left(\frac{-n_1+1}{n_2^2 - n_1 n_2}\right)\right) + (n-1)\left(49\left(\frac{2}{n_1^2 - n_1 n_2}\right) + 289\left(\frac{2}{n_2^2 - n_1 n_2}\right)\right) \tag{39}$$

The general term of the AP is:

$$x(n) = 49\left(\frac{-n_2+1}{n_1^2 - n_1 n_2}\right) + 289\left(\frac{-n_1+1}{n_2^2 - n_1 n_2}\right) + n\left(49\left(\frac{2}{n_1^2 - n_1 n_2}\right) + 289\left(\frac{2}{n_2^2 - n_1 n_2}\right)\right) \tag{40}$$

$$\forall \ n \geq 0$$

$$\implies x(n) = \left(49\left(\frac{-n_2+1}{n_1^2 - n_1 n_2}\right) + 289\left(\frac{-n_1+1}{n_2^2 - n_1 n_2}\right) + n\left(49\left(\frac{2}{n_1^2 - n_1 n_2}\right) + 289\left(\frac{2}{n_2^2 - n_1 n_2}\right)\right)\right) u(n) \tag{41}$$

Variable	Description
$x(0)$	First term of the AP
d	Common difference of the AP
S_n	Sum of n terms of the AP
$x(n)$	General term
$X(z)$	Z- transform of $x(n)$

TABLE 1: Variables Used