

11.9.4.7

EE23BTECH11063 - Vemula Siddhartha

Question:

Find the sum to n terms of the series:

$$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$$

Solution:

Variable	Description
$y(n)$	Sum of $n + 1$ terms of the series
$x(n)$	General term

TABLE 0
VARIABLES USED

$$y(n) = 1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots \quad (1)$$

(2)

Let,

$$y(n) = \sum_{k=0}^n x(k) \quad (3)$$

$$y(n) = x(n) * u(n) \quad (4)$$

Then,

$$x(n) = \sum_{k=0}^n (k+1)^2 u(k) \quad (5)$$

$$x(n) = ((n+1)^2 u(n)) * u(n) \quad (6)$$

From equations (??) and (??),

$$X(z) = Z\{(n+1)^2 u(n)\} U(z) \quad (7)$$

$$= \left(z \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} \right) \left(\frac{1}{1-z^{-1}} \right) \quad (8)$$

$$X(z) = \frac{1+z^{-1}}{(1-z^{-1})^4} \quad \{z \in \mathbb{C} : |z| > 1\} \quad (9)$$

From (4),

$$Y(z) = X(z) U(z) \quad (10)$$

$$= \left(\frac{1+z^{-1}}{(1-z^{-1})^4} \right) \left(\frac{1}{1-z^{-1}} \right) \quad (11)$$

$$= \frac{z^4(z+1)}{(z-1)^5} \quad \{z \in \mathbb{C} : |z| > 1\} \quad (12)$$

Taking the Inverse Z transform, from Contour Integration,

$$y(n) = \frac{1}{2\pi j} \oint_C Y(z) z^{n-1} dz \quad (13)$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^4(z+1)}{(z-1)^5} z^{n-1} dz \quad (14)$$

(15)

Using Cauchy's Residual Theorem for 5 repeated roots at $z = 1$,

$$= \frac{1}{(k-1)!} \lim_{z \rightarrow c} \frac{d^{k-1}}{dz^{k-1}} ((z-c)^k f(z)) \quad (16)$$

$$= \frac{1}{4!} \lim_{z \rightarrow 1} \frac{d^4}{dz^4} \left((z-1)^5 \frac{z^4(z+1)}{(z-1)^5} z^{n-1} \right) \quad (17)$$

$$= \frac{1}{24} \lim_{z \rightarrow 1} \frac{d^4}{dz^4} (z^{n+4} + z^{n+3}) \quad (18)$$

$$\Rightarrow y(n) = \frac{(n+1)(n+2)^2(n+3)}{12} u(n) \quad (19)$$

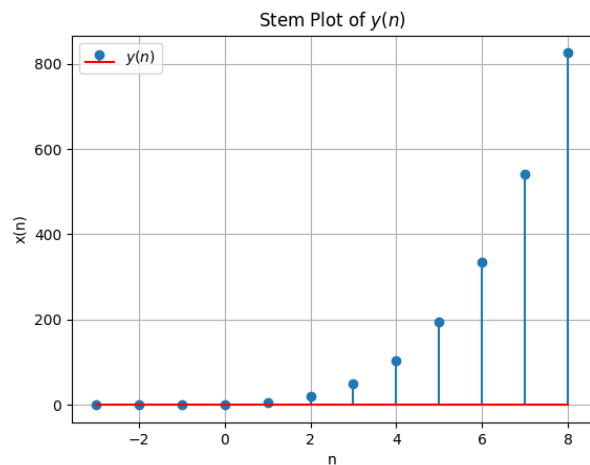


Fig. 0. Stem Plot of $y(n)$