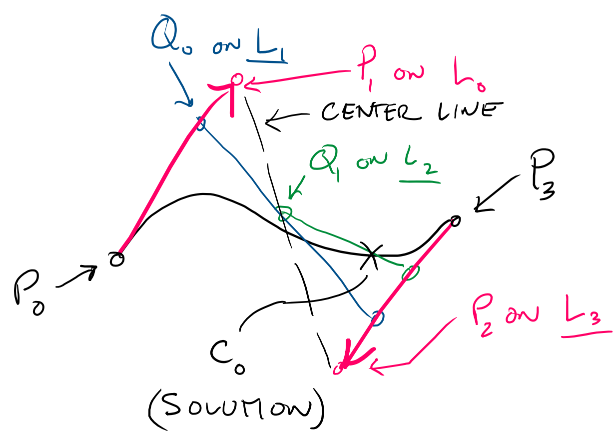
PATH PLANNER — ARM, 6DOF, ANTHROPO. CONFIG.

USING BEZIER CURVES WE WILL:

1. SELECT POINTS IN 3D SPACE
2. CHOOSE A SMOOTH TRAJECTORY BETWEEN POINTS VIA CIRCLES
3. CALCULATE HANDLEBAR POSITIONS WITH TWO STYLES:
   - i) ON CIRCLE POINTS (INTRACIRCULAR)
   - ii) BETWEEN CIRCLE POINTS (INTERCIRCULAR)
4. DRAW THE PATH VIA BEZIER CURVES
5. INTERPOLATE DISCRETE POINTS ON PATH

BACKGROUND:

BEZIER CURVES →



SET POINTS: $\vec{P}_0, \vec{P}_3$

HANDLES: $\vec{P}_1, \vec{P}_2$

LINEAR $\quad 0 \leq t \leq 1$

$$L_0(t) = (1-t)\vec{P}_0 + t\vec{P}_1$$

QUADRATIC

$$L_1(t) = (1-t)\vec{P}_1 + t\vec{P}_2$$

$$\vec{Q}_0(t) = (1-t)L_0(t) + tL_1(t)$$

$$= (1-t)^2\vec{P}_0 + 2(1-t)t\vec{P}_1 + t^2\vec{P}_2$$

CUBIC

$$L_2(t) = (1-t)\vec{P}_2 + t\vec{P}_3$$

$$\vec{Q}_1(t) = (1-t)L_1(t) + tL_2(t)$$

$$\vec{C}_0(t) = (1-t)\vec{Q}_0(t) + tQ_1(t)$$

$$\boxed{\therefore \vec{C}_0(t) = (1-t)^3\vec{P}_0 + 3(1-t)^2 t\vec{P}_1 + 3(1-t)t^2\vec{P}_2 + t^3\vec{P}_3}$$

ON AUTOGENERATING HANDLE POSITIONS:

✓ CONDITION: LET'S BEGIN WITH ONLY 3 SET POINTS FOR A SIMPLE PICK AND PLACE ANIMATION.

✓ 3 POINTS → 2D PLANAR ANIMATION.

✓ THIS MEANS:

1. TRANSFORM POINTS TO RELATIVE $(u, v) \in \mathbb{R}^2$ COORDINATES FROM $(x, y, z) \in \mathbb{R}^3$.
2. SET SMOOTH (NO CUSPS) CURVE BETWEEN POINTS 1 AND 3, VIA 2.
3. TRANSFORM BACK TO ABSOLUTE $(x, y, z) \in \mathbb{R}^3$.

✓ SINCE WE WANT A SMOOTH CURVE WITH NO CUSPS

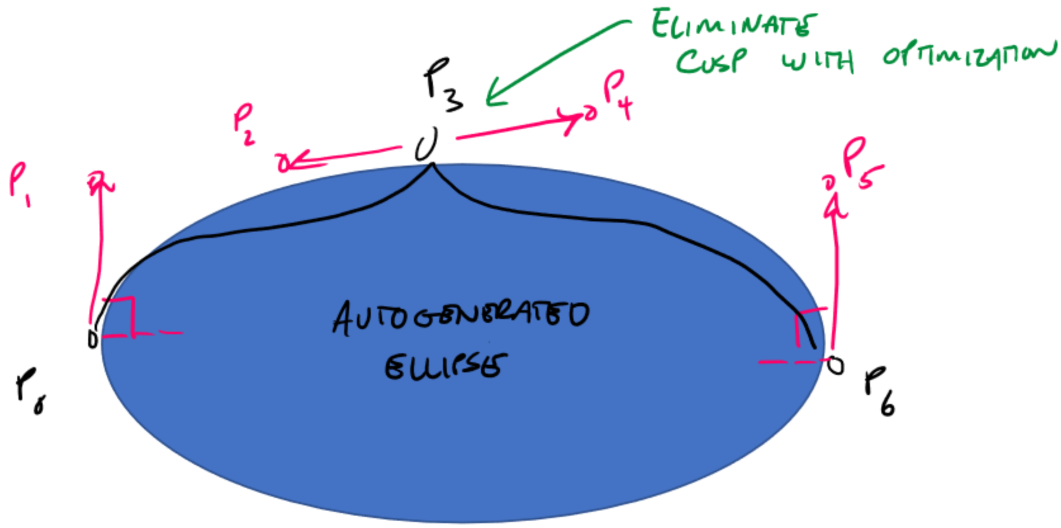
$\longrightarrow m_{\vec{p}_2} = m_{\vec{p}_4}$. HOWEVER $|\vec{p}_2| \neq |\vec{p}_4|$.

INNER HANDLES

- TO FIND $|\vec{p}_2|$ AND $|\vec{p}_4|$, USE AN OPTIMIZATION TO OBTAIN LEAST MAGNITUDES POSSIBLE WITHOUT $\lim_{t \to \vec{p}_3} \ddot{x} = \infty$

OUTER HANDLES

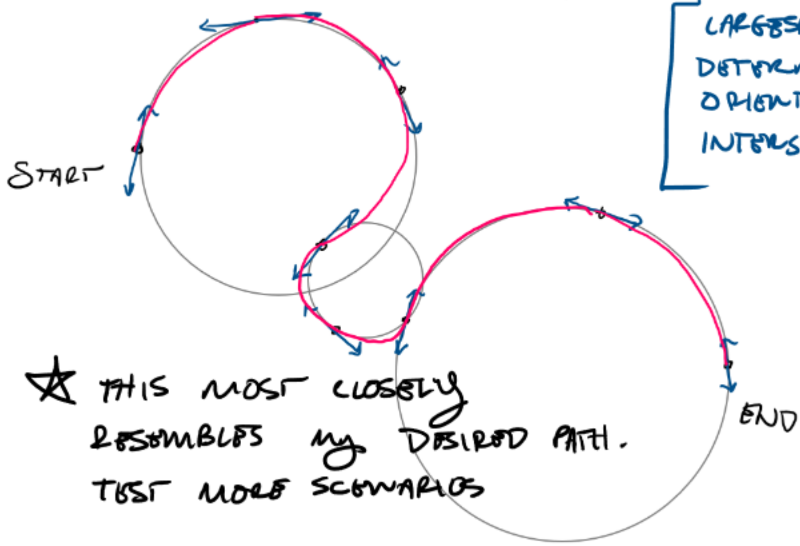
- TO FIND $|\vec{p}_1|$ AND $|\vec{p}_5|$, USE AN OPTIMIZATION TO OBTAIN LEAST MAGNITUDES POSSIBLE



HANDLES SHOULD BE TANGENT TO AUTOGENERATED ELLIPSE. → ANGLES AUTOMATICALLY DETERMINED

ENCIRCLE FIRST 3 / ENCIRCLE NEXT 3 / ENCIRCLE FINAL 1/2 LAST 2/1 → USE DOMINANT CIRCLE

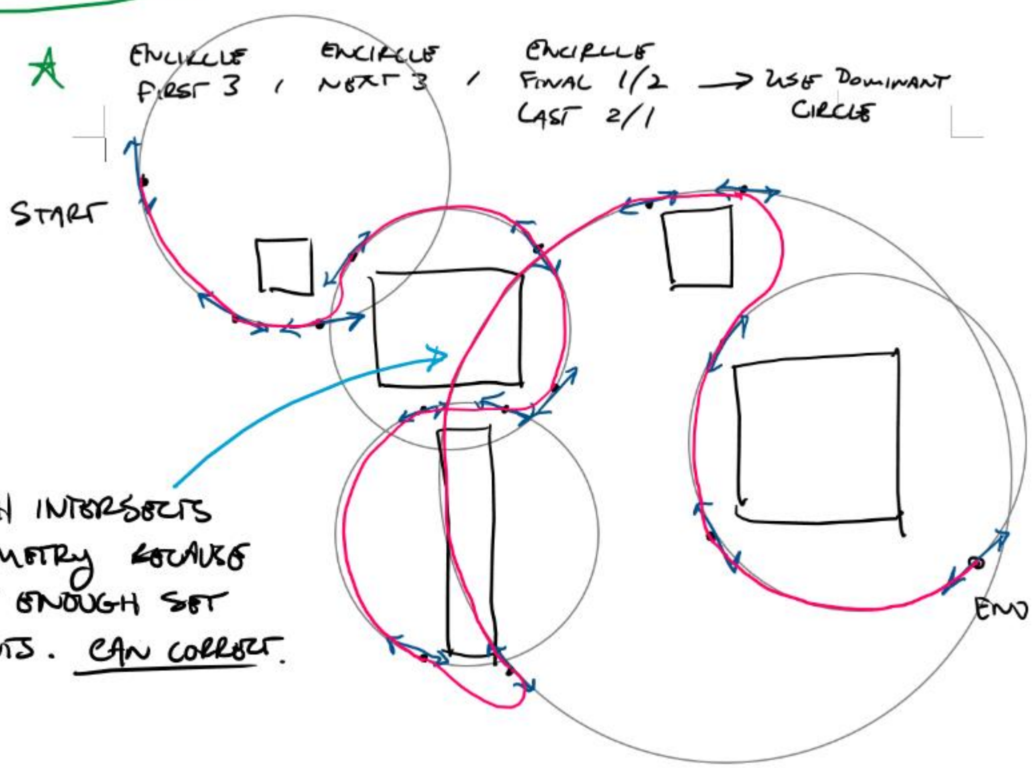
LARGEST CIRCLE DETERMINES ORIENTATION AT INTERSECTIONS



☆ THIS MOST CLOSELY RESEMBLES MY DESIRED PATH. TEST MORE SCENARIOS

RETEST OF B

☆ ENCIRCLE FIRST 3 / ENCIRCLE NEXT 3 / ENCIRCLE FINAL 1/2 LAST 2/1 → USE DOMINANT CIRCLE



PATH INTERSECTS GEOMETRY BECAUSE NOT ENOUGH SET POINTS. CAN CORRECT.

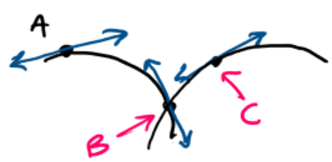
HANDLES BETWEEN POINTS SHARING A CIRCLE.

FOR POINTS SHARING A CIRCLE, USE PAIRS.
EACH PAIR'S HANDLES ARE VECTORS TANGENT TO THE DOMINANT CIRCLE.

THE INTERSECTION OF THESE VECTORS IS THE INTERMEDIATE HANDLE.

THE CURVE SHOULD BE QUADRATIC, NOT CUBIC.

POINTS BETWEEN CIRCLES



THE PATH BETWEEN POINTS B AND C WILL NOT BE PLANAR. THIS MEANS WE SHOULD EXPECT A SPACE CURVE.

FURTHERMORE, THE PATH WILL BE COMPLEX — LIKELY A CUBIC SPACE CURVE.

SO, HOW SHOULD WE MANAGE THE HANDLES BETWEEN?

1. TAKE THE PLANES PARALLEL TO THE CIRCLES IN QUESTION.
2. TAKE THE NORMAL OF THESE PLANES.
3. CROSS THE NORMALS TO GET AN ORTHOGONAL VECTOR. THIS IS OUR BISECTING LINE.
4. THE PLANE BETWEEN THE NORMALS, PARALLEL TO THE ORTHOGONAL VECTOR FORMS OUR BISECTING PLANE.
5. THE INTERSECTION OF OUR TANGENTIAL VECTORS AND THE BISECTING PLANE GIVES US OUR TWO HANDLES.

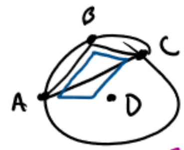
FINDING THE CIRCLES

THIS IS AN OPTIMIZATION WHERE $\vec{AD}$, $\vec{BD}$, $\vec{CD}$ ALL HAVE THE SAME LENGTH AND SHARE POINT D.

EQN. OF CIRCLE

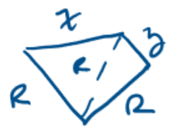
$A = \pi R^2$

$C = 2\pi R$

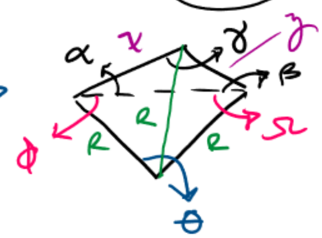


$\pi R_A^2 = \pi R_B^2 = \pi R_C^2$

$2\pi R_A = 2\pi R_B = 2\pi R_C$



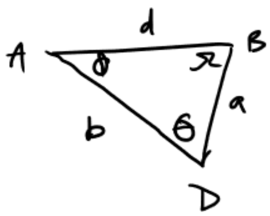
$\longrightarrow$



$\alpha + \beta + \gamma = 180^\circ$

$\Rightarrow$ I $\phi + \Omega + \theta = 180^\circ$

LAW OF COSINES



$b^2 = a^2 + d^2 - 2ad\cos\Omega$

$a^2 = d^2 + b^2 - 2db\cos\phi$

$d^2 = a^2 + b^2 - 2ab\cos\theta$

$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$

$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$

$\rightarrow d^2 = R^2 + R^2 - 2R^2\cos\theta$

$d^2 = 2R^2 - 2R^2\cos\theta$

$d^2 = 2R^2(1 - \cos\theta)$

BOTTOM

II $\dfrac{d^2}{2} = R^2(1 - \cos\theta)$

$\rightarrow \cancel{R^2} = x^2 \cancel{+ R^2} - 2xR\cos(\alpha + \phi)$

$x^2 = 2xR\cos(\alpha + \phi)$

$x = 2R\cos(\alpha + \phi)$

III $\rightarrow \dfrac{x}{2} = R\cos(\alpha + \phi)$ LEFT

IV $\rightarrow \dfrac{y}{2} = R\cos(\beta + \Omega)$ RIGHT

$\rightarrow A_{BOT} = \frac{1}{2}RR\sin\theta = \frac{1}{2}R^2\sin\theta$ V

$\rightarrow A_{LEFT} = \frac{1}{2}Rx\sin(\alpha + \phi)$

$\rightarrow A_{RIGHT} = \frac{1}{2}Ry\sin(\beta + \Omega)$

$\rightarrow A_{TOP} = \frac{1}{2}xy\sin\gamma$

$\rightarrow A_{FULL,1} = A_{TOP} + A_{BOT}$

$= \frac{1}{2}xy\sin\gamma + \frac{1}{2}R^2\sin\theta$

$\sin(\theta + 180)$
$= -\sin(\theta)$

$\sin(-\theta)$
$= -\sin(\theta)$

$\rightarrow A_{FULL_2} = A_{LEFT} + A_{RIGHT}$

$= \frac{1}{2}Rx\sin(\alpha+\phi) + \frac{1}{2}Ry\sin(\beta+\Omega)$

$\rightarrow \gamma + (\alpha+\phi) + (\beta+\Omega) + \theta = 360 = 0$ VIII

$\rightarrow A_{FULL1} = A_{FULL2}$

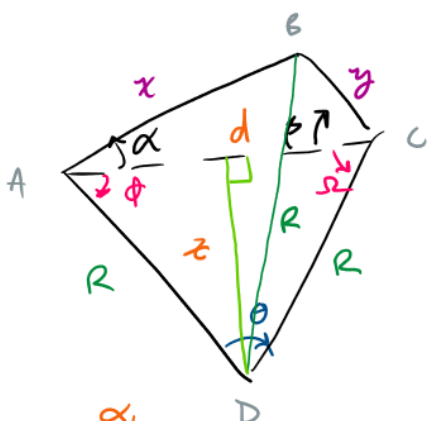
$$\frac{1}{2}xy\sin\gamma + \frac{1}{2}R^2\sin\theta = \frac{1}{2}Rx\sin(\alpha+\phi) + \frac{1}{2}Ry\sin(\beta+\Omega)$$

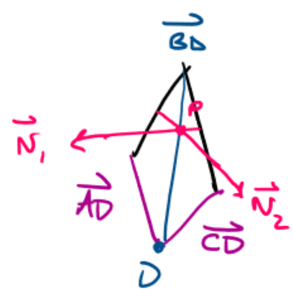
$$xy\sin\gamma + R^2\sin\theta = Rx\sin(\alpha+\phi) + Ry\sin(\beta+\Omega)$$ ~~IX~~ IX

APPROACHES

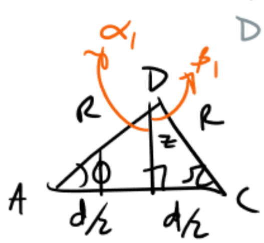
EQUATE AREAS

— LEFT + RIGHT = TOP + BOT





$|\vec{AD}| = |\vec{BD}| = |\vec{CD}|$



$\Rightarrow$

$R\cos\phi = R\cos\Omega$ ~~IX~~ X

$\cos\phi = \cos\Omega$ ~~X~~ XI

$\sin\phi = \sin\Omega$ $\Rightarrow \phi = \Omega$

$\tan\phi = \tan\Omega$

use III & IV

$$\frac{x}{2} = R\cos(\alpha+\phi) \qquad \rightarrow \qquad \text{since } \phi = \Omega$$

$$\frac{y}{2} = R\cos(\beta+\Omega) \qquad\qquad \frac{x}{2} = R\cos(\alpha+\phi)$$

$$\frac{y}{2} = R\cos(\beta+\phi)$$

$$\cos(\varepsilon+\phi) = \cos\varepsilon\cos\phi - \sin\varepsilon\sin\phi$$

$$\longrightarrow \frac{1}{R} = \frac{2}{x}\cos(\alpha+\phi) = \frac{2}{y}\cos(\beta+\phi)$$

$$\overset{A}{\cancel{\frac{2}{x}}}\left[\cos\alpha\cos\phi - \sin\alpha\sin\phi\right] = \overset{B}{\cancel{\frac{2}{y}}}\left[\cos\beta\cos\phi - \sin\beta\sin\phi\right]$$

$$A\cos\alpha\cos\phi - B\cos\beta\cos\phi = A\sin\alpha\sin\phi - B\sin\beta\sin\phi$$

$$\cos\phi\,\underbrace{\left[A\cos\alpha - B\cos\beta\right]}_{F} = \sin\phi\,\underbrace{\left[A\sin\alpha - B\sin\beta\right]}_{G}$$

$$F\cos\phi = G\sin\phi$$

$$\frac{F}{G} = \frac{\sin\phi}{\cos\phi} = \tan\phi$$

~~XI~~ XII

$$\phi = \tan^{-1}\left(\frac{F}{G}\right) = \Omega$$

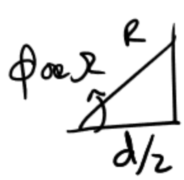
$$\Rightarrow 180 - \phi - \Omega - \Theta = 0$$

$$180 - 2\phi - \Theta = 0$$

~~XII~~ XIII

$$\longrightarrow \Theta = 2\phi - 180$$

$$\boxed{\begin{aligned} F &= \frac{2}{x}\cos\alpha - \frac{2}{y}\cos\beta \\ G &= \frac{2}{x}\sin\alpha - \frac{2}{y}\sin\beta \end{aligned}}$$



T_A^O

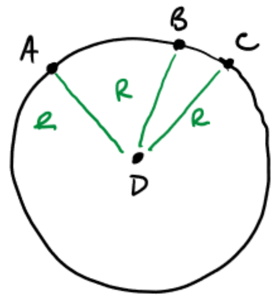
since $T_A^O \rightarrow \tan\phi = \frac{F}{G}$

$\longrightarrow$ opposite $= F$

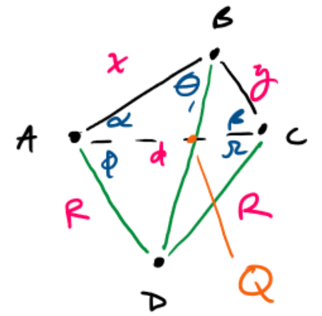
$$\boxed{\therefore\ R = \sqrt{F^2 + \left(\frac{d}{2}\right)^2}}$$

SUCCESS!!

FIND CIRCLE CENTER D



$\longrightarrow$



$\phi = \Omega$

$$\to R^2 = R^2 + x^2 - 2Rx\cos\theta_1$$

$$x^2 = 2Rx\cos\theta_1$$

$$x = 2R\cos\theta_1 \longrightarrow \cos\theta_1 = \frac{x}{2R}$$

(I) $$\boxed{\theta_1 = \cos^{-1}\left(\frac{x}{2R}\right)}$$

FIND α AND β WITHOUT QUATERNIONS

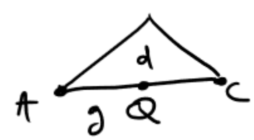
(II) $$\alpha = \cos^{-1}\frac{\vec{AB}\cdot\vec{AC}}{\underbrace{|\vec{AB}|}_{x}\,\underbrace{|\vec{AC}|}_{d}}, \quad \beta = \cos^{-1}\frac{\vec{CB}\cdot\vec{CA}}{\underbrace{|\vec{CB}|}_{y}\,\underbrace{|\vec{CA}|}_{d}}$$ (III)

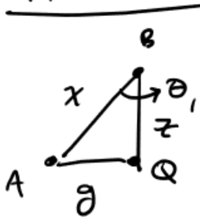
OPTIMIZING R FOR 3 POINTS

(IV) $$R = \sqrt{F^2 + \left(\frac{d}{2}\right)^2}$$

WHERE $F = \frac{2}{x}\cos\alpha - \frac{2}{y}\cos\beta$

FIND Q





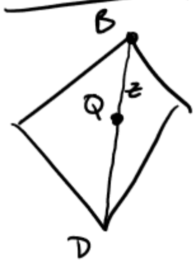
$\sin\theta_1 = \frac{g}{x}$ (V)

$$\rightarrow \boxed{\begin{aligned} g &= x\sin\theta_1 \\ z &= x\cos\theta_1 \end{aligned}}$$

$\rho = \frac{g}{d} \to g$ PERCENT OF d

$$\vec{AQ} = \rho\vec{AC} \longrightarrow \vec{Q} = \vec{AQ} + \vec{A} = (\vec{Q} - \vec{A}) + \vec{A} = \rho\vec{AC} + \vec{A}$$

FIND D



$$\omega = \frac{R}{z} \leftarrow \text{R percent of } z$$

$$\vec{BD} = \omega \vec{BQ}$$

$$\vec{D} = \vec{BD} + \vec{B}$$

$$= (\vec{D} - \vec{B}) + \vec{B}$$

$$= \omega \vec{BQ} + \vec{B}$$

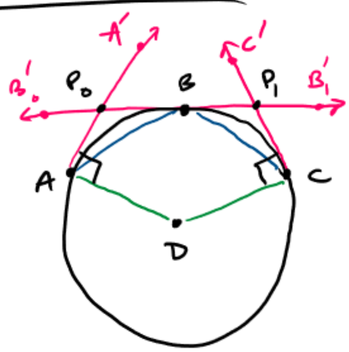
$$= \omega(\vec{Q} - \vec{B}) + \vec{B}$$

$$= \omega\left[(\rho \vec{AC} + \vec{A}) - \vec{B}\right] + \vec{B}$$

VI

$$\boxed{\vec{D} = \frac{R}{z}\left[\left(\frac{q}{d}\vec{AC} + \vec{A}\right) - \vec{B}\right] + \vec{B}}$$

FINDING HANDLES BETWEEN POINTS ON A CIRCLE



$$P_0 = \vec{AA'} \cap \vec{BB'_0}$$

$$P_1 = \vec{CC'} \cap \vec{BB'_1}$$

$$\rightarrow \vec{AA'} = \vec{AD}\cos 90^\circ$$

$$\rightarrow \vec{CC'} = -\vec{CD}\cos 90^\circ$$

$$\rightarrow \begin{cases} \vec{BB'_1} = \vec{BD}\cos 90^\circ \\ \vec{BB'_0} = -\vec{BD}\cos 90^\circ \end{cases}$$

SINCE 3 POINTS FORM A PLANE AND ALL 3 POINTS CAN BE PLACED ON A CIRCLE WITH $\vec{A} \neq \vec{B} \neq \vec{C}$ AND

$$\theta = \cos^{-1}\frac{\vec{DA}\cdot\vec{DC}}{|\vec{DA}||\vec{DC}|}$$

$$0^\circ \leq \theta \leq 180^\circ$$

$\Rightarrow$ CONSECUTIVE TANGENTS COMING FROM $\vec{A}, \vec{B}, \vec{C}$ OFF OF THE CIRCLE D CANNOT BE SKEW OR PARALLEL AND THEREFORE MUST INTERSECT.

PARAMETERIZE TANGENTS

LET $\vec{u}$ AND $\vec{v}$ BE PARAMETERIZED LINES

$$\vec{u} = \vec{A} + t\overrightarrow{AA'} = \begin{bmatrix} \vec{A}_x + t\overrightarrow{AA'}_x \\ \vdots \\ \cdot z \end{bmatrix}$$

$$\vec{v} = \vec{B} + t\overrightarrow{BB'_0} = \begin{bmatrix} \vec{B}_x + t\overrightarrow{BB'}_{0x} \\ \vdots \\ \cdot z \end{bmatrix}$$

DO THE SAME FOR $\overrightarrow{CC'}$ AND $\overrightarrow{BB'_1}$.

INTERSECT TANGENTS

$$\vec{P}_0 = \vec{u} \cap \vec{v}$$

$\vec{u} \cap \vec{v}$: SOLVE FOR $t \rightarrow \vec{u} = \vec{v}$

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \rightarrow x_0 = x_1$$

$$\vec{A}_x + t\overrightarrow{AA'}_x = \vec{B}_x + t\overrightarrow{BB'}_{0x}$$

$$t\overrightarrow{AA'}_x - t\overrightarrow{BB'}_{0x} = \vec{B}_x - \vec{A}_x$$

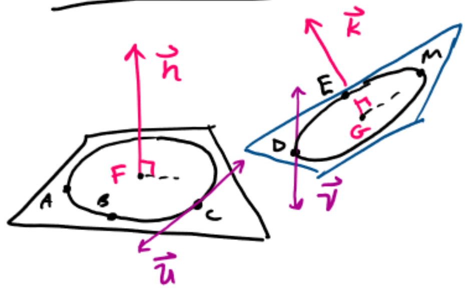
$$t\left(\overrightarrow{AA'}_x - \overrightarrow{BB'}_{0x}\right) = \vec{B}_x - \vec{A}_x$$

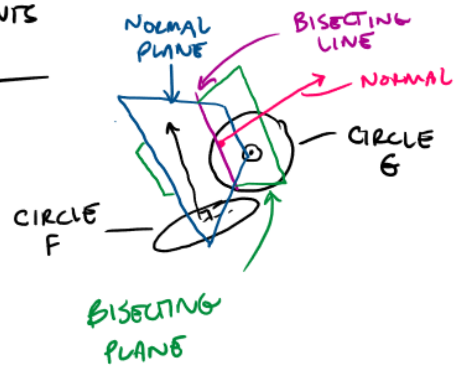
$$\boxed{t = \frac{\vec{B}_x - \vec{A}_x}{\overrightarrow{AA'}_x - \overrightarrow{BB'}_{0x}}}$$

$$\overrightarrow{AP_0} = \vec{A} + t\overrightarrow{AA'} \longrightarrow \boxed{\vec{P}_0 = \overrightarrow{AP_0} + \vec{A}}$$

DO THE SAME FOR $\vec{P}_1$.

FINDING HANDLES BETWEEN POINTS ON DIFFERENT CIRCLES IN $\mathbb{R}^3$



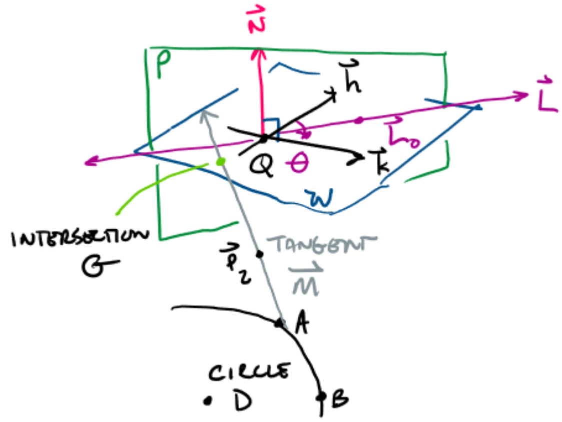


$$\vec{N} = \vec{h} \times \vec{k}$$

$$\theta = \cos^{-1} \frac{\vec{h} \cdot \vec{k}}{|\vec{h}|\,|\vec{k}|}$$

$$\vec{L}_0 = \frac{\vec{k}}{\cos \theta/2}$$

$$\rightarrow \vec{L}_1 = \overrightarrow{QL_0} + t\vec{L}_0$$



PLANE P: $\vec{N} \cdot (\vec{M} - \overrightarrow{QL_0}) = 0$

$$\langle a, b, c \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

INTERSECTION G: $\vec{M} = \vec{A} + t\vec{P}_2$

$$x = \vec{M}_x$$
$$y = \vec{M}_y$$
$$z = \vec{M}_z$$

$$P(\vec{M}) \rightarrow t$$

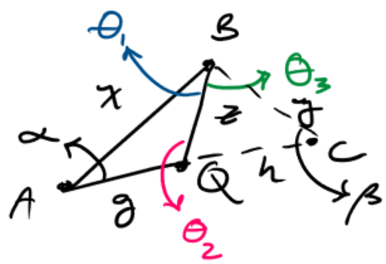
IF $t \in \mathbb{R}$ THEN

$$\boxed{G = \vec{M}(t) - \vec{A}}$$

THERE ARE TWO INTERSECTIONS, ONE FOR EACH CIRCLE. USE A CUBIC BEZIER CURVE BETWEEN THESE POINTS.

FINDING Q

WE KNOW $\triangle ABQ$ IS NOT A RIGHT TRIANGLE



UNKNOWN: $z, g, h, \vec{Q}$

$$\theta_2 = 180 - \alpha - \theta_1$$

USING LAW OF COSINES, SOLVE FOR g AND z

$$z^2 = x^2 + g^2 - 2xg\cos\alpha$$
$$z^2 = y^2 + h^2 - 2yh\cos\beta$$

$$d = g + h$$
$$\rightarrow \boxed{g = d - h}$$

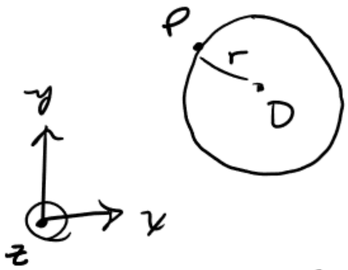
$$\rightarrow x^2 + g^2 - 2xg\cos\alpha = y^2 + h^2 - 2yh\cos\beta$$
$$x^2 + (d-h)^2 - 2x(d-h)\cos\alpha = y^2 + h^2 - 2yh\cos\beta$$
$$x^2 + d^2 - 2dh + \cancel{h^2} - 2xd\cos\alpha + 2xh\cos\alpha = y^2 + \cancel{h^2} - 2yh\cos\beta$$

$$(x^2 + d^2 - y^2) - 2xd\cos\alpha = 2dh - 2xh\cos\alpha - 2yh\cos\beta$$
$$(x^2 + d^2 - y^2) - 2xd\cos\alpha = h\left(2d - 2x\cos\alpha - 2y\cos\beta\right)$$

$$\boxed{h = \frac{(x^2 + d^2 - y^2) - 2xd\cos\alpha}{2d - 2x\cos\alpha - 2y\cos\beta}}$$

$$\boxed{\overrightarrow{AQ} = \frac{g}{d}\,\overrightarrow{AC}}$$

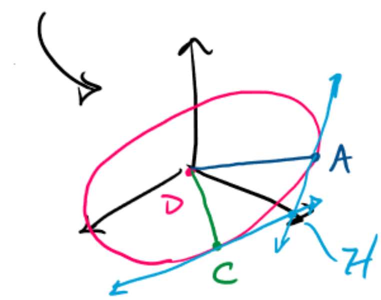
GETTING A TANGENT



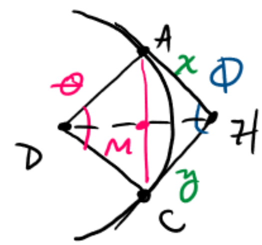
$$x^2 + y^2 + z^2 = r^2$$

$$\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

IN $\mathbb{R}^3$



$\rightarrow$



SINCE ISOSCELES

$x = y$

LAW OF COSINES

$$d^2 = x^2 + y^2 - 2xy\cos\phi$$

$$\rightarrow d^2 = x^2 + x^2 - 2x^2\cos\phi$$

$$d^2 = 2x^2 - 2x^2\cos\phi$$

$$d^2 = 2x^2(1 - \cos\phi)$$

$$2x^2 = \frac{d^2}{1-\cos\phi}$$

$$x^2 = \frac{d^2}{2-2\cos\phi}$$

$$x = \pm\sqrt{\frac{d^2}{2-2\cos\phi}}$$

$$\vec{AM} = \frac{1}{2}\vec{AC} = \vec{M} - \vec{A}$$

$$\vec{M} = \vec{AM} + \vec{A}$$

$$\theta = \cos^{-1}\frac{\vec{DA}\cdot\vec{DC}}{|\vec{DA}|\,|\vec{DC}|}$$

$$\phi = 360 - (180 + \theta)$$

$$d = |\vec{AC}|$$

FIND INTERSECTIONS WITH PLANE P

$$\text{PLANE } P \equiv \vec{N} \cdot (\vec{M} - Q\vec{L}_0) = 0$$

NORMAL OF PLANE W

TANGENT OF CIRCLE D

BISECTING LINE

$$\rightarrow \quad \vec{M} \cap P \equiv \begin{bmatrix} \vec{M}_x \\ \vec{M}_y \\ \vec{M}_z \end{bmatrix} \quad \text{WHERE } \vec{M} = \vec{A} + t\vec{P}_2$$

POINT IN QUESTION FROM CIRCLE D

HANDLEBAR

FIND INTERSECTIONS WITH PLANE P

$$\text{PLANE } P \equiv \vec{N} \cdot (\vec{M} - Q\vec{L}_0) = 0$$

NORMAL OF PLANE W

TANGENT OF CIRCLE D

BISECTING LINE

$$\rightarrow \quad \vec{M} \cap P \equiv \begin{bmatrix} \vec{M}_x \\ \vec{M}_y \\ \vec{M}_z \end{bmatrix} \quad \text{WHERE } \vec{M} = \vec{A} + t\vec{P}_2$$

POINT IN QUESTION FROM CIRCLE D

HANDLEBAR

FIND t ST:

$$\vec{N} \cdot (\vec{m} - Q\vec{L}_0) = 0$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\rightarrow \left. \begin{aligned} &a\left[(\vec{A}_x + t\vec{P}_{2x}) - x_0\right] \\ &+ b\left[(\vec{A}_y + t\vec{P}_{2y}) - y_0\right] \\ &+ c\left[(\vec{A}_z + t\vec{P}_{2z}) - z_0\right] \end{aligned} \right\} = 0$$

$$\rightarrow \left. \begin{aligned} &a\left[(\vec{A}_x + t\vec{P}_{2x}) - x_0\right] \\ &+ b\left[(\vec{A}_y + t\vec{P}_{2y}) - y_0\right] \\ &+ c\left[(\vec{A}_z + t\vec{P}_{2z}) - z_0\right] \end{aligned} \right\} = 0$$

$$\left. \begin{aligned} &(a\vec{A}_x + b\vec{A}_y + c\vec{A}_z) \\ &- (a x_0 + b y_0 + c z_0) \\ &+ t(a\vec{P}_{2x} + b\vec{P}_{2y} + c\vec{P}_{2z}) \end{aligned} \right\} = 0$$

$$\rightarrow \boxed{t = \frac{(a x_0 + b y_0 + c z_0) - (a\vec{A}_x + b\vec{A}_y + c\vec{A}_z)}{(a\vec{P}_{2x} + b\vec{P}_{2y} + c\vec{P}_{2z})}}$$

$\Rightarrow$ HANDLEBAR $\boxed{G = \vec{M}(t) - \vec{A}}$

CROSSING SKEW LINES

$$\left.\begin{aligned} L_1 &\to p = p_0 + \lambda_1 \vec{v}_1 \\ L_2 &\to q = q_0 + \lambda_2 \vec{v}_2 \end{aligned}\right\}$$ LINES IN QUESTION

ORTHOGONAL LINE

$$L_3 = u = u_0 + \lambda_3 \vec{v}_1 \times \vec{v}_2$$

LET US DETERMINE L_3 ST IT PASSES BY L_1 & L_2

$$\longrightarrow \quad p = q + \lambda_3 \vec{v}_1 \times \vec{v}_2 \Rightarrow p_0 + \lambda_1 \vec{v}_1 = q_0 + \lambda_2 \vec{v}_2 + \lambda_3 \vec{v}_1 \times \vec{v}_2$$

MULTIPLY BY $\vec{v}_1, \vec{v}_2$

L_1 $\quad p_0 \cdot \vec{v}_1 + \lambda_1 \vec{v}_1 \cdot \vec{v}_1 = q_0 \cdot \vec{v}_1 + \lambda_2 \vec{v}_2 \cdot \vec{v}_1$ $\quad L_2$

L_1 $\quad p_0 \cdot \vec{v}_2 + \lambda_1 \vec{v}_2 \cdot \vec{v}_2 = q_0 \cdot \vec{v}_2 + \lambda_2 \vec{v}_2 \cdot \vec{v}_2$ $\quad L_2$

SOLVE FOR $\lambda_1 = \lambda_1^*$ $\quad \lambda_2 = \lambda^*$

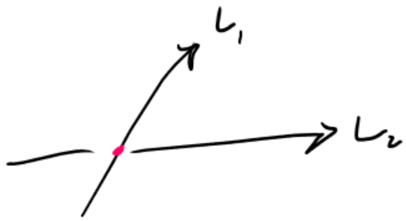
$$L_3 \to p = p_0 + \lambda_1^* \vec{v}_1 + \lambda \vec{v}_1 \times \vec{v}_2$$

OR $\quad L_3 \to p = q_0 + \lambda_2^* \vec{v}_2 + \lambda \vec{v}_1 \times \vec{v}_2$

$$l_1 = L_2$$

$$p_0 + \lambda_1 \vec{v}_1 = q_0 + \lambda_2 \vec{v}_2$$

$$\vec{v}_1 \cdot (p_0 + \lambda_1 \vec{v}_1) = (q_0 + \lambda_2 \vec{v}_2) \cdot \vec{v}_1$$



$L_1 \neq L_2$
SKEW

$$L_1 \to \vec{a}(t) = \vec{\alpha} t + \vec{\sigma}$$

$$L_2 \to \vec{b}(t) = \vec{\beta} t + \vec{\rho}$$

IF $\vec{\alpha} \times \vec{\beta} \neq 0$ THEN $L_1 \times L_2 = L_3$

$$\Leftrightarrow \boxed{L_3 \to \vec{c}(s) = \frac{(\vec{\rho} - \vec{\sigma}) \cdot (\vec{\alpha} \times \vec{\beta})}{(\vec{\alpha} \times \vec{\beta})^2} (\vec{\alpha} \times \vec{\beta}) s + \vec{\sigma}}$$

WHERE $L_3 \cap L_1$ FOR $s = 0$

$L_3 \cap L_2$ FOR $s = 1$

$$L_3 \cap L_1 \Rightarrow \quad a(0) = \sigma + \frac{(\sigma - \rho) \cdot (\alpha \times \beta) \times \beta}{(\alpha \times \beta)^2} \alpha$$

$$L_3 \cap L_2 \Rightarrow \quad b(1) = \rho + \frac{(\sigma - \rho) \cdot (\alpha \times \beta) \times \alpha}{(\alpha \times \beta)^2} \beta$$

SAMPLING

VERTEX VERTEX

ARC LENGTH $= \sum_{i=1}^{n} \left\| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right) \right\|$

SAMPLE: MAX SAMPLE DISTANCE = msd (GIVEN)

$t = \frac{msd}{ARCLEN}$

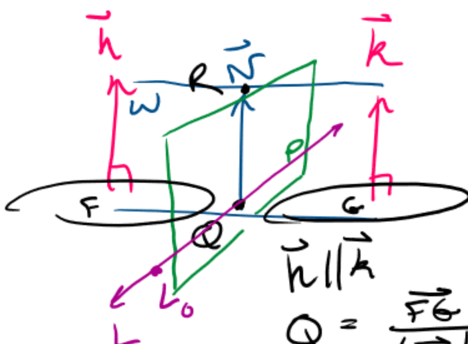
POINT @ $f(jt)$, j IS CURRENT INDEX OF SAMPLE ON CURVE

if $jt > 1$ → RETURN ENDPOINT (BREAK LOOP)

PARALLEL CIRCLES

$h(t) = \vec{F} + t\vec{h}$

$k(t) = \vec{G} + t\vec{k}$



$\vec{Q} = \vec{FQ} + \vec{F}$

$= (\vec{Q} - \vec{F}) + \vec{F}$

$\vec{h} \parallel \vec{k}$

$Q = \frac{\vec{FG}}{|\vec{FG}|}\left(\frac{|\vec{FG}|}{2}\right) + \vec{F}$

$R = \frac{\vec{hk}}{|\vec{hk}|}\left(\frac{\vec{hk}}{2}\right) + \vec{h}$

$\vec{N} = \vec{QR}$

$L_0 = \vec{N} \times \vec{QF}$

TANGENT $\cap$ PLANE

TANGENT LINE $\Rightarrow$ $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$

$$\boxed{\begin{aligned} l &= \cos\alpha = \frac{\vec{v}_x \cdot \hat{x}}{|\vec{v}_x||\hat{x}|} \\ m &= \cos\beta = \frac{\vec{v}_y \cdot \hat{y}}{|\vec{v}_y||\hat{y}|} \\ n &= \cos\gamma = \frac{\vec{v}_z \cdot \hat{z}}{|\vec{v}_z||\hat{z}|} \end{aligned}}$$

PLANE $\quad \vec{N} \cdot (\vec{r} - \vec{g}) = 0$

$$\langle a, b, c \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

INTERSECTION:

$$\frac{x-x_1}{l} = a(x-x_0)$$

$$x - x_1 = alx - alx_0$$

$$x(1-al) = x_1 - alx_0$$

$$x = \frac{x_1 - alx_0}{1-al}$$

$$\boxed{\begin{aligned} x &= \frac{x_1 - alx_0}{1-al} \\ y &= \frac{y_1 - bmy_0}{1-bm} \\ z &= \frac{z_1 - cnz_0}{1-cn} \end{aligned}}$$

CROSSING SKEW LINES

(IN MOST CASES CIRCLES HAVE SKEW NORMALS.)

CIRCLE CENTERS

$$\vec{a}(t) = \vec{h}t + \vec{\sigma}$$

$$\vec{b}(t) = \vec{k}t + \vec{\rho}$$

$\underbrace{\vec{h} \times \vec{k}}_{\text{NORMALS}} = 0$, THEN PARALLEL

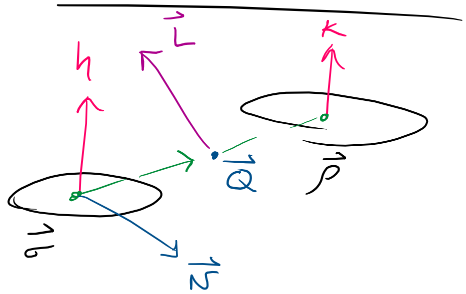
$$0 \le s \le 1 \implies \begin{cases} s = 0 \rightarrow \vec{a} \\ s = 1 \rightarrow \vec{b} \end{cases}$$

$$\boxed{\vec{c}(s) = \frac{(\vec{\rho} - \vec{\sigma}) \cdot (\vec{h} \times \vec{k})}{|\vec{h} \times \vec{k}|^2} (\vec{h} \times \vec{k}) s + \vec{\sigma}}$$

USE $s = 0.5$ FOR MIDPOINT

$$\vec{c}(0.5) = \vec{Q}$$

PARALLEL CIRCLES



$$\vec{Q} = (\vec{\rho} - \vec{\sigma}) \cdot 0.5 = \langle a, b, c \rangle$$

$$\vec{N} = \vec{h} \times \vec{Q}$$

$$\vec{L} = \vec{N} \times \vec{Q}$$

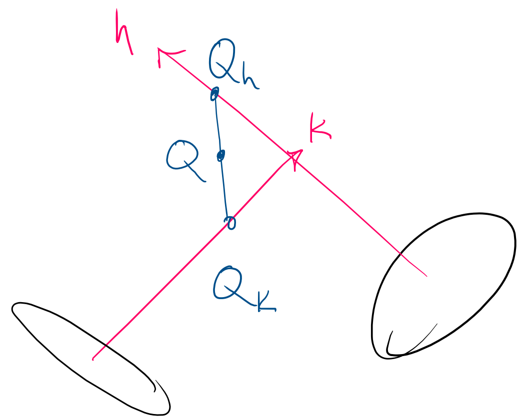
$\vec{Q}$ IS THE NORMAL FOR THE BISECTING PLANE

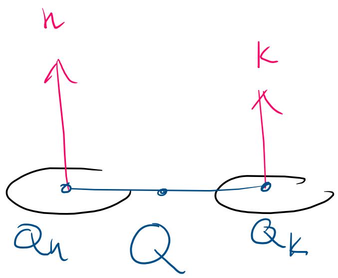
BISECTING PLANE P

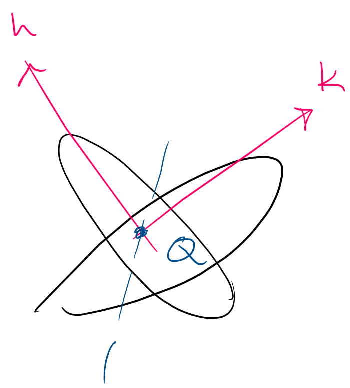
$$\vec{Q} \cdot (\vec{P} - \vec{\sigma}) = 0$$

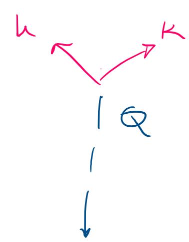
$$\langle a, b, c \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$$

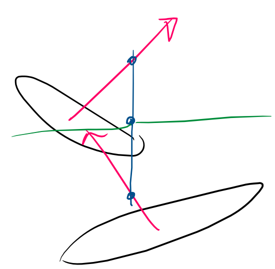
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

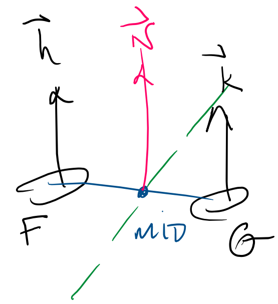












PARALLEL CIRCLES

$\vec{N} \parallel \vec{h}$ or $\vec{k}$

PLANE P:

$\vec{N} = \vec{h}$ or $\vec{k}$

P_0 = MID

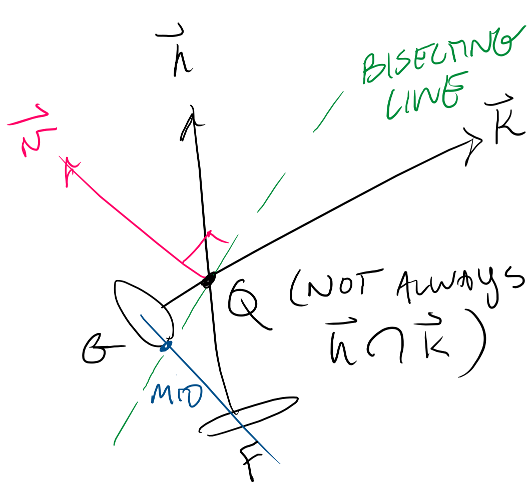
SKEW CIRCLES

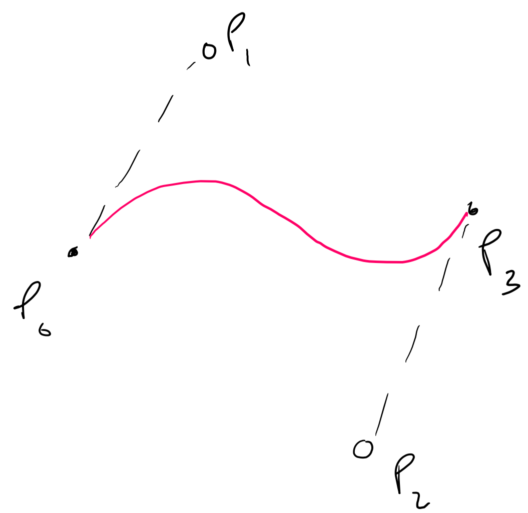
$\vec{N} = \vec{h} \times \vec{k}$

PLANE P:

$\vec{N} = \vec{h} \times \vec{k}$

P_0 = Q or MID





CUBIC

$$\vec{C}(t) = (1-t)^3\vec{P}_0 + 3(1-t)^2 t\,\vec{P}_1 + 3(1-t)t^2\vec{P}_2 + t^3\vec{P}_3$$

SAMPLE

```
length = 0
for (i=0; i<n; i++)
  length = f(i/n) - f((i-1)/n)
array = []
for (j=0; j<n; j++)
  if (j * dist/length > 1): break
  array.push(f(j * dist/length))
RETURN array
```

FUNCTION DEPENDS ON LINE TYPE