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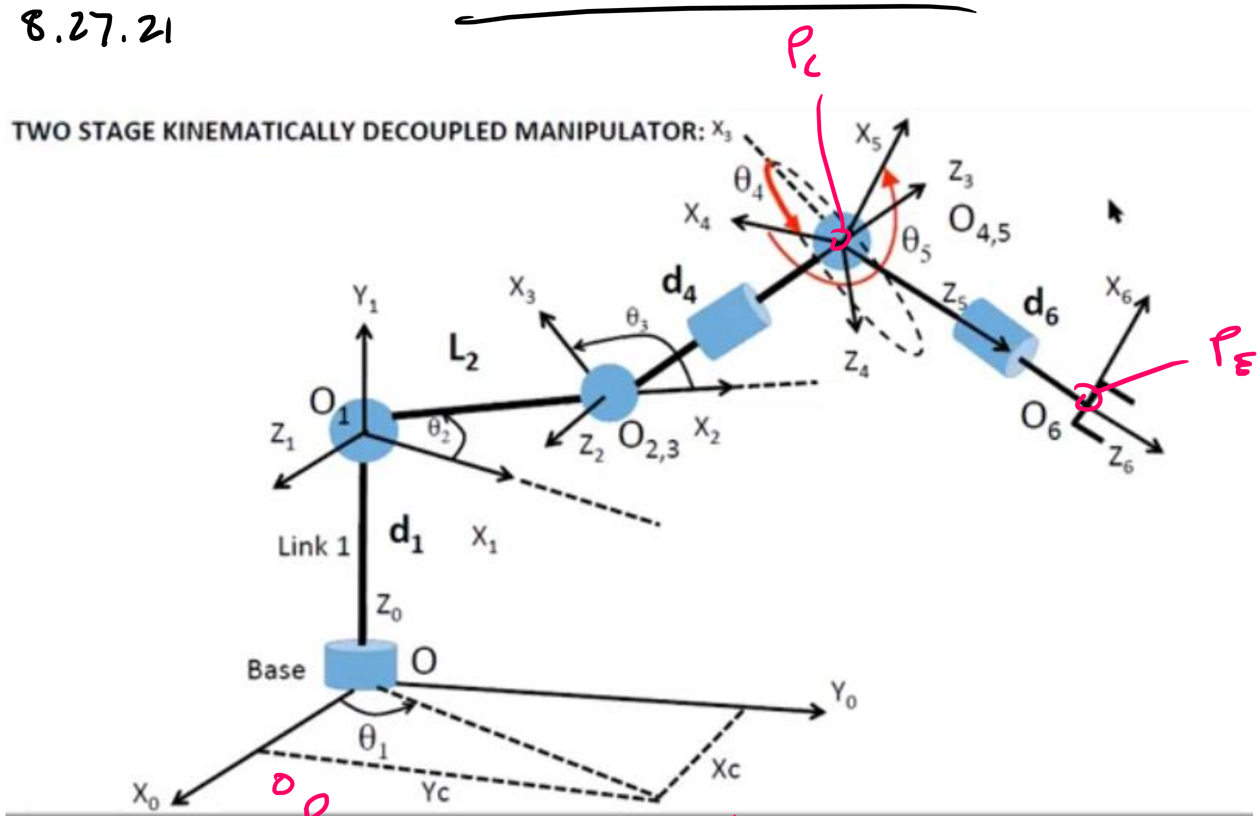
8.27.21

6 DOF Inverse Kinematics

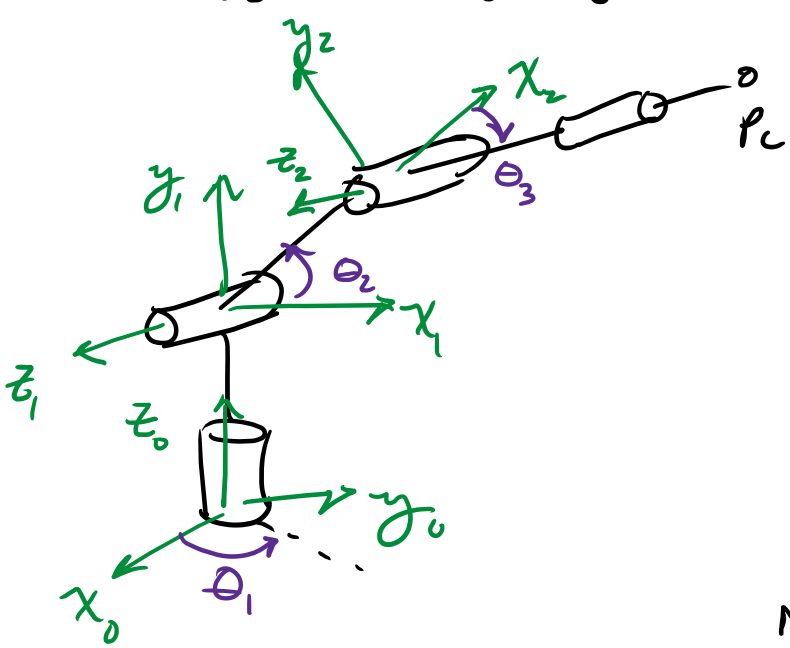


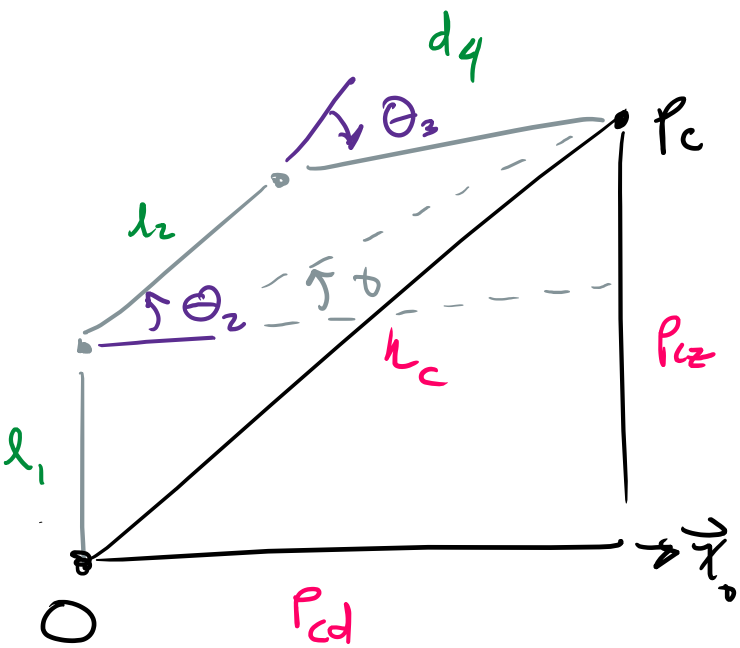
Given

① $${}^0H_6 = \begin{bmatrix} \underbrace{{}^0x_6 \quad {}^0y_6 \quad {}^0z_6}_{{}^0R_6} & \overset{\text{position}}{p_E} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$, where ② ${}^0z_6 = {}^0x_6 \times {}^0y_6$

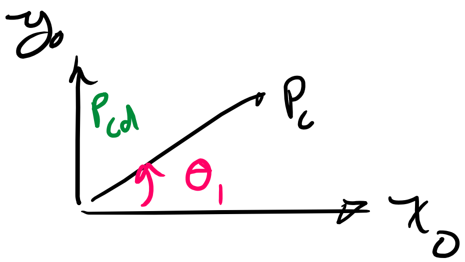
Find p_c, ~~${}^0H_3$, ${}^3H_6$.~~ → ${}^0R_3$, ${}^3R_6$

③ $\vec{p}_c = \vec{p}_E - d_6 \cdot \vec{z}_6$





Notice: p_c can be lower than p_1

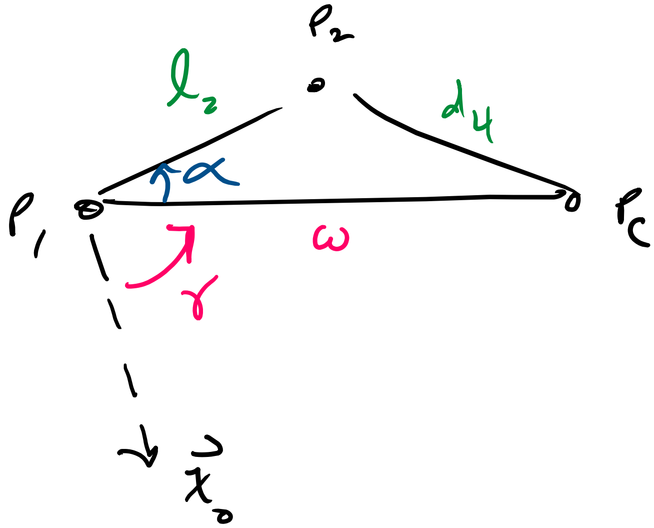


(4) $\rightarrow$ $P_{cd} = \sqrt{P_{cx}^2 + P_{cy}^2}$

(5) $\rightarrow$ $\theta_1 = \tan^{-1}\left(\frac{P_{cy}}{P_{cx}}\right)$

(6) $\rightarrow$ $\omega = \sqrt{P_{cd}^2 + (P_{cz} - l_1)^2}$

(7) $\rightarrow$ $\gamma = \tan^{-1}\left(\frac{P_{cz} - l_1}{P_{cd}}\right)$



Using law of cosines:

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$d_4^2 = l_2^2 + \omega^2 - 2 l_2 \omega \cos\alpha$

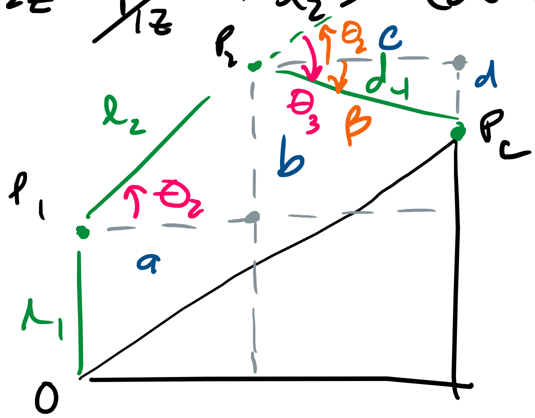
$\rightarrow$ $\frac{d_4^2 - (l_2^2 + \omega^2)}{-2 l_2 \omega} = \cos\alpha$

(8) $\alpha = \cos^{-1}\left[\frac{d_4^2 - (l_2^2 + \omega^2)}{-2 l_2 \omega}\right]$

$\rightarrow$ $P_{2d} = \cancel{P_{1d}}^{\,0} + l_2\cos(\alpha + \gamma)$ (9)

$P_{2z} = \cancel{P_{1z}}^{\,l_1} + l_2\sin(\alpha + \gamma)$ (10)

$\rightarrow$



(11) $a = P_{2d}$

(12) $b = P_{cz} - P_{2z}$

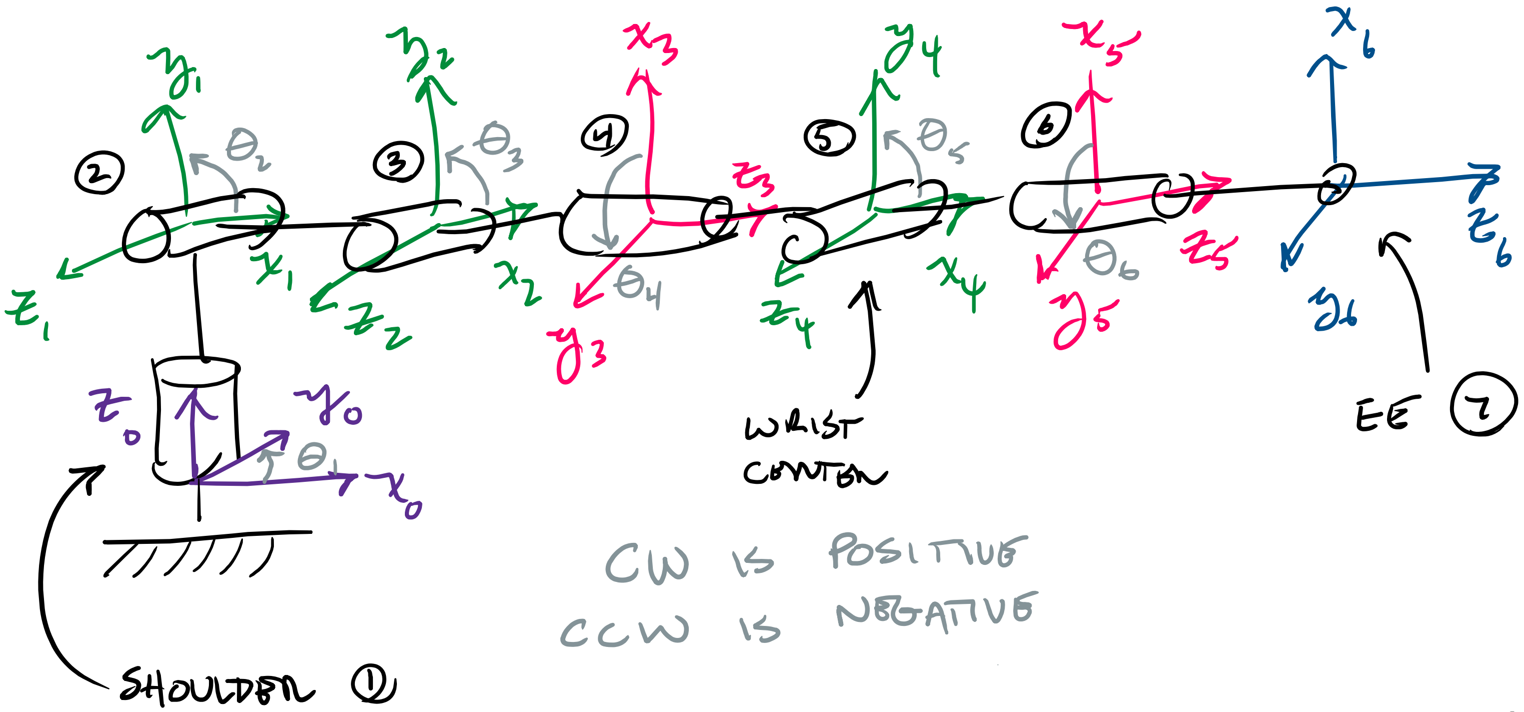
(13) $c = P_{cd} - P_{2d}$

(14) $d = -P_{2z} + P_{cz}$

$\Rightarrow$ $\theta_2 = \tan^{-1}\frac{b}{a}$ (15)

(16)

$\beta = \tan^{-1}\frac{d}{c}$, $\theta_3 = \beta - \theta_2$



CW IS POSITIVE
CCW IS NEGATIVE

$$M_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad M_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_3^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_4^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad M_5^4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_6^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(\theta_x) = \begin{bmatrix} c_x & -s_x & 0 \\ s_x & c_x & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 1 \le x \le 3, \quad x \in \mathbb{Z}$$

$$R_B^A = T(\theta_B)\, M_B^A$$

$$R_3^0 = R_1^0 R_2^1 R_3^2$$

$$R_6^0 = \underline{\text{GIVEN}} \ (\text{EE})$$

$$R_6^0 = R_3^0 R_6^3$$

$$R_6^3 = \left(R_3^0\right)^{-1} R_6^0$$

CALCULATE WITH MATLAB

★ ACTUAL ORIENTATION OF ARM

$\rightarrow$ SOLVE θ_x, $\quad x \in \mathbb{Z}$, $1 \le x \le 3$ (PAGES 1,2)

USING TRIGONOMETRY (KEY IS TO USE LAW OF COSINES)

$$\theta_1 = \tan^{-1}\left(\frac{P_{cy}}{P_{cx}}\right) \qquad \theta_2 = \tan^{-1}\left(\frac{b}{a}\right) \qquad \theta_3 = \beta - \theta_2$$

$\rightarrow$ SOLVE $\theta_4, \theta_5, \theta_6$ BY $R^3_6 = R^3_6$

SYMBOLIC ACTUAL

SYMBOLIC $R^3_6 = R^3_4 \, R^4_5 \, R^5_6$

ACTUAL $R^3_6 = (R^0_3)^{-1} R^0_6$

WHERE $R^0_6 = T(\theta_{ROLL})\, T(\theta_{YAW})\, T(\theta_{PITCH})\, M^0_6$

$$T(\theta_R) = \begin{bmatrix} C_R & -S_R & 0 \\ S_R & C_R & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T(\theta_y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_y & -S_y \\ 0 & S_y & C_y \end{bmatrix}, \quad T(\theta_p) = \begin{bmatrix} C_p & 0 & S_p \\ 0 & 1 & 0 \\ -S_p & 0 & C_p \end{bmatrix}$$

$$M^0_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

MATLAB

I

```
>> syms t1 t2 t3 t4 t5 t6 R01 R12 R23 R45 R56 R03 R36 R06
>> syms T1 T2 T3 T4 T5 T6
>> T1 = [cos(t1) -sin(t1) 0; sin(t1) cos(t1) 0; 0 0 1]

T1 =

[cos(t1), -sin(t1), 0]
[sin(t1),  cos(t1), 0]
[      0,        0, 1]

>> T2 = [cos(t2) -sin(t2) 0; sin(t2) cos(t2) 0; 0 0 1]

T2 =

[cos(t2), -sin(t2), 0]
[sin(t2),  cos(t2), 0]
[      0,        0, 1]

>> T3 = [cos(t3) -sin(t3) 0; sin(t3) cos(t3) 0; 0 0 1]

T3 =

[cos(t3), -sin(t3), 0]
[sin(t3),  cos(t3), 0]
[      0,        0, 1]
```

II

```
>> T4 = [cos(t4) -sin(t4) 0; sin(t4) cos(t4) 0; 0 0 1]

T4 =

[cos(t4), -sin(t4), 0]
[sin(t4),  cos(t4), 0]
[      0,        0, 1]

>> T5 = [cos(t5) -sin(t5) 0; sin(t5) cos(t5) 0; 0 0 1]

T5 =

[cos(t5), -sin(t5), 0]
[sin(t5),  cos(t5), 0]
[      0,        0, 1]

>> T6 = [cos(t6) -sin(t6) 0; sin(t6) cos(t6) 0; 0 0 1]

T6 =

[cos(t6), -sin(t6), 0]
[sin(t6),  cos(t6), 0]
[      0,        0, 1]
```

III

```
>> M01 = [1 0 0; 0 0 -1; 0 1 0]

M01 =

     1     0     0
     0     0    -1
     0     1     0

>> M12 = [1 0 0; 0 1 0; 0 0 1]

M12 =

     1     0     0
     0     1     0
     0     0     1

>> M23 = [0 0 1; 1 0 0; 0 1 0]

M23 =

     0     0     1
     1     0     0
     0     1     0
```

```
>> M34 = [0 1 0; 0 0 1; 1 0 0]

M34 =

     0     1     0
     0     0     1
     1     0     0

>> M45 = [0 0 1; 1 0 0; 0 1 0]

M45 =

     0     0     1
     1     0     0
     0     1     0

>> M56 = [1 0 0; 0 1 0; 0 0 1]

M56 =

     1     0     0
     0     1     0
     0     0     1
```

```
>> R01 = T1 * M01

R01 =

[cos(t1), 0,  sin(t1)]
[sin(t1), 0, -cos(t1)]
[      0, 1,        0]

>> R12 = T2 * M12

R12 =

[cos(t2), -sin(t2), 0]
[sin(t2),  cos(t2), 0]
[      0,        0, 1]

>> R23 = T3 * M23

R23 =

[-sin(t3), 0, cos(t3)]
[ cos(t3), 0, sin(t3)]
[       0, 1,       0]
```

```
>> R34 = T4 * M34

R34 =

[0, cos(t4), -sin(t4)]
[0, sin(t4),  cos(t4)]
[1,       0,        0]

>> R45 = T5 * M45

R45 =

[-sin(t5), 0, cos(t5)]
[ cos(t5), 0, sin(t5)]
[       0, 1,       0]

>> R56 = T6 * M56

R56 =

[cos(t6), -sin(t6), 0]
[sin(t6),  cos(t6), 0]
[      0,        0, 1]
```

IV

```
>> syms tr ty tp Tr Ty Tp
>> Tr = [cos(tr) -sin(tr) 0; sin(tr) cos(tr) 0; 0 0 1]

Tr =

[cos(tr), -sin(tr), 0]
[sin(tr),  cos(tr), 0]
[      0,        0, 1]

>> Ty = [1 0 0; 0 cos(ty) -sin(ty); 0 sin(ty) cos(ty)]

Ty =

[1,       0,        0]
[0, cos(ty), -sin(ty)]
[0, sin(ty),  cos(ty)]

>> Tp = [cos(tp) 0 sin(tp); 0 1 0; -sin(tp) 0 cos(tp)]

Tp =

[ cos(tp), 0, sin(tp)]
[       0, 1,       0]
[-sin(tp), 0, cos(tp)]

>> M06 = [0 0 1; 0 -1 0; 1 0 0]

M06 =

     0     0     1
     0    -1     0
     1     0     0
```

V

```
>> R06 = Tr * Ty * Tp * M06

R06 =

[cos(tr)*sin(tp) + cos(tp)*sin(tr)*sin(ty),  cos(ty)*sin(tr), cos(tp)*cos(tr) - sin(tp)*sin(tr)*sin(ty)]
[sin(tp)*sin(tr) - cos(tp)*cos(tr)*sin(ty), -cos(tr)*cos(ty), cos(tp)*sin(tr) + cos(tr)*sin(tp)*sin(ty)]
[                          cos(tp)*cos(ty),         -sin(ty),                          -cos(ty)*sin(tp)]

>> R03 = R01 * R12 * R23

R03 =

[- cos(t1)*cos(t2)*sin(t3) - cos(t1)*cos(t3)*sin(t2),  sin(t1), cos(t1)*cos(t2)*cos(t3) - cos(t1)*sin(t2)*sin(t3)]
[- cos(t2)*sin(t1)*sin(t3) - cos(t3)*sin(t1)*sin(t2), -cos(t1), cos(t2)*cos(t3)*sin(t1) - sin(t1)*sin(t2)*sin(t3)]
[                     cos(t2)*cos(t3) - sin(t2)*sin(t3),        0,                     cos(t2)*sin(t3) + cos(t3)*sin(t2)]
```

THESE ARE ACTUAL

R06_actual =

cos(tr)*sin(tp) + cos(tp)*sin(tr)*sin(ty)	cos(ty)*sin(tr)	cos(tp)*cos(tr) - sin(tp)*sin(tr)*sin(ty)
sin(tp)*sin(tr) - cos(tp)*cos(tr)*sin(ty)	-cos(tr)*cos(ty)	cos(tp)*sin(tr) + cos(tr)*sin(tp)*sin(ty)
cos(tp)*cos(ty)	-sin(ty)	-cos(ty)*sin(tp)

Where tr is theta_roll, ty is theta_yaw, tp is theta_pitch. These are given by the operator.

R03_actual =

- cos(t1)*cos(t2)*sin(t3) - cos(t1)*cos(t3)*sin(t2)	sin(t1)	cos(t1)*cos(t2)*cos(t3) - cos(t1)*sin(t2)*sin(t3)
- cos(t2)*sin(t1)*sin(t3) - cos(t3)*sin(t1)*sin(t2)	-cos(t1)	cos(t2)*cos(t3)*sin(t1) - sin(t1)*sin(t2)*sin(t3)
cos(t2)*cos(t3) - sin(t2)*sin(t3)	0	cos(t2)*sin(t3) + cos(t3)*sin(t2)

R36_actual = np.dot(np.linalg.inv(R03_actual), R06_actual)

This matrix is calculated with the Python library NumPy.

R36_sym =

cos(t4)*cos(t5)*cos(t6) - sin(t4)*sin(t6)	- cos(t6)*sin(t4) - cos(t4)*cos(t5)*sin(t6)	cos(t4)*sin(t5)
cos(t4)*sin(t6) + cos(t5)*cos(t6)*sin(t4)	cos(t4)*cos(t6) - cos(t5)*sin(t4)*sin(t6)	sin(t4)*sin(t5)
-cos(t6)*sin(t5)	sin(t5)*sin(t6)	cos(t5)

Now we can exploit the fact that R36_sym [2][2] = cos(t5) and solve for t5 by using R36_actual[2][2] such that

$$\mathrm{R36}_{\mathrm{sym}} = \mathrm{R36}_{\mathrm{actual}}$$

$$\theta_5 = \arccos\left(\mathrm{R36}_{\mathrm{actual}[2][2]}\right)$$

$$\theta_6 = \arcsin\left(\frac{\mathrm{R36}_{\mathrm{actual}[2][1]}}{\sin(\theta_5)}\right)$$

$$\theta_4 = \arcsin\left(\frac{\mathrm{R36}_{\mathrm{actual}[1][2]}}{\sin(\theta_5)}\right)$$